ASYMPTOTICS OF GENERAL ORTHOGONAL POLYNOMIALS FOR MEASURES ON THE UNIT CIRCLE AND $[-1, 1]$.

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Declaration

I declare that this dissertation is my own unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

SBDamel

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17 day of May 1993

Abstract

The primary aim of this dissertation is to survey the subject of asymptotics of general orthogonal and orthonormal polynomials for different measures on the unit circle and $[-1,1]$ with particular emphasis on recent developments in the theory of non–Szegő asymptotics. Although the classicial Szegő theory for weights on the unit circle and real line is well developed, a corresponding theory for more general weights is far from complete.

Ratio, weak, uniform and comparative results of Nevai, Máté, Totik, Rakhmanov and Lopez are stated, proved and compared. Analogues are also constantly drawn to the classical theory and open research problems are raised.

Acknowledgements

It is fitting that I extend first and foremost my sincere thanks to my supervisor, Professor Doron Lubinsky for his never ending enthusiasm, support and for all he has taught me in the four years that I have known him. I hope that this dissertation is of a sufficiently high standard to pay tribute to his remarkable dedication to his work and his students. My thanks also go to the Foundation of Research and Development for their funding during 1991 and 1992, to the staff of the Mathematics Department at Wits who enabled me to hold a Wits Senior Bursary during 1991 and 1992 and to Ingrid Eitzen for her excellent typing of this manuscript. Finally, I wish to thank my wonderful family, Dad, Mom and Len and my best friend in the world, Shanki, for giving me the opportunity and encouragement to reach and pursue this goal.

To my family and fiancée

Dad, Mom, Len and Sara.

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Introduction

Recent years have seen a great deal of progress in the field of orthogonal polynomials. The origins of the subject can be traced back to the investigation of certain types of continued fractions bearing the names of Gauss, Jacobi, Christoffel and Stieltjies. Notwithstanding this historic relationship, orthogonal polynomials soon acquired their own perspective and respectability. Historically, certain classes of orthogonal polynomials were first discovered and then studied. In this connection, we mention names such as Legendre, Hermite, Chebyshev and Lagurre. These classes were found to have wide applications in areas such as Padé approximation, continued fractions, numerical analysis, scattering theory, nuclear and solid state physics, digital signal processing and electrical engineering.

Orthogonal polynomials have been found to yield very important connections with special and basic hypergeometric functions. These contributions have been made by many researchers, including Al-Salam, Allaway, Andrews, Askey and Foata. Mathematicians also began to study general systems of orthogonal polynomials for more general classes of weights. Directions that have been found here include counterexamples to Steklov's conjecture, Cesaro boundedness of orthogonal polynomials in the Szegõ's class, extensions of the Freudian conjectures and exponential weights, matrix valued orthogonal polynomials and Toeplitz matricies, non Szegõ asymptotics. It is to the latter, that we now turn our attention.

One of the major areas in the study of orthogonal polynomials is their behaviour as the degree of the polynomial is allowed to become very large. It transpired that this behaviour or as it is most commonly known, asymptotic behaviour, was first investigated by Szegõ in the early 1920's. In a series of papers starting in 1915 he developed asymptotics for general orthogonal polynomials that satisfied a certain strong integrability condition known today, as Szegõ's criterion. This theory, which describes the asymptotics of orthogonal polynomials on the unit circle and $[-1,1]$ is very much complete and gives us a very good description of the asymptotic behaviour of orthogonal polynomials in this setting.

In recent years, attempts have been made to try and develop a complete theory for more general weights, that is non Szegõ asymptotics. Most of the ideas, and research in this area originates with Paul Nevai and his collaborators with major contributions from E.A. Rakhmanov and Lopez. It is the purpose of this dissertation to survey these recent developments.

Chapters one and two provide a solid framework for our study introducing orthogonal polynomials for sigma finite Borel measures and the classical Szegõ theory. Chapter three extends the well known asymptotic $\lim_{n\to\infty} \frac{\varphi_{n+1}(z)}{\varphi_n(z)} = z$ to measures satisfying Erdös, Turan type conditions. The proofs of this result were given independently by Nevai et al and Rakhmanov with generalizations by Lopez.

Comparative and uniform asymptotics are then introduced with results on and off the unit circle stated and proved in the settings of Nevai et al and Rakhmanov.

We then consider the natural transition to $[-1,1]$. The famous classical asymptotic formula of Szegõ is considered. We give the proof of the weaker ratio form for non Szegõ weights found first by Rakhmanov and simplified by Nevai et al with generalizations by Lopez. We state its extended form in the complex plane found by Nevai et al using the ideas of the recurrence relations of the orthogonal polynomials as well as considering the very recent strong asymptotic form of the above for exponential weights due to Lubinsky et al.

Finally we give proofs of other analogues in $[-1,1]$ including generalizations of Lopez. Open research problems are raised whenever appropriate.

Chapter 1

Existence, uniqueness and fundamental properties of orthonormal polynomials on the unit circle

Notation

Throughout $\varphi_n(d\mu, z)$ will mean the orthonormal polynomial of degree n with respect to the measure $d\mu$. We will write $\varphi_n(z)$ to mean $\varphi_n(d\mu, z)$. π_n will mean a polynomial of degree $\leq n$, and "meas E" will mean the linear Lebesgue measure of a set E. $\mathbb{R}$ will denote the real numbers, $\mathbb{C}$ the complex numbers, $\mathbb{N}$ the natural numbers and $\tau = \{z: |z| = 1\}$ the unit circle. If $P(z) = \sum_j b_j z^j$ then by $\overline{P}(z)$ we mean $\sum_j \overline{b}_j z^j$. We also introduce o and O notation. Given two sequences (a_n) and (b_n) we say that $a_n = o(b_n)$ if $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = 0$ and $a_n = O(b_n)$ if there exists an absolute constant C such that $a_n \leq Cb_n$ as $n \to \infty$. Finally "condition" a.e. (measure) will mean that the condition holds except on a set of zero measure with respect to that measure.

Definition 1.1

Let $E \subseteq \mathbb{C}$. Denote by $B(E)$ the set of all Borel sets on E, that is the sigma algebra containing all compact $K \subseteq E$. A set function $\mu: B(E) \to [0, \infty)$ will be called a Borel measure on E if it is countably additive. A set function $\mu: B(\mathbb{C}) \to [0, \infty)$ will be called a sigma finite Borel measure if for each compact $K \subseteq \mathbb{C}$, μ restricted to K is a Borel measure on K. We say that μ has all moments finite, if

$$\int_{\mathbb{C}} |z|^j d\mu(z) < \infty, \; j = 0, 1, 2, \ldots .$$

The support of μ denoted $supp\,\mu$ will be $\{z : \mu(S \cap K) > 0\}$ for all open sets S such that $z \in S$.

Theorem 1.2 (Existence and Uniqueness)

Let $d\mu$ be a sigma finite Borel measure with all moments finite. Further let $supp(d\mu)$ be infinite. In this case, $d\mu$ will always be called a distribution. Then there exists polynomials

$$\varphi_n(z) = \gamma_n z^n + \ldots, \; \gamma_n > 0$$

and of full degree n such that

$$\int_{\mathbf{C}} \varphi_n(z)\overline{\varphi_m(z)}d\mu(z) = \delta_{mn}, \; \delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}.$$

Moreover, the φ_n are uniquely determined for each $n \geq 0$.

Proof: Assume $n \geq 1$. Let

$$\mu_{j,k} = \int_{\mathbf{C}} z^j \overline{z^k} d\mu(z), \; j,k = 0,1,2,\ldots .$$

Then as $d\mu$ has all moments finite, $\mu_{j,k}$ converges for all j,k. Now consider the following determinant

$$B = \begin{vmatrix} \mu_{o,o} & \cdots & \mu_{n,o} \\ \vdots & & \vdots \\ \mu_{o,n-1} & \cdots & \mu_{n,n-1} \\ 1 & \cdots & z^n \end{vmatrix} \text{ and let } P(z) = B.$$

Then $P(z)$ is a polynomial of degree $\leq n$. Also then,

$$\begin{aligned} \int_{\mathbf{C}} P(z)\overline{z^j}d\mu(z) &= \int_{\mathbf{C}} B\overline{z^j}d\mu(z) \\ &= \det \begin{bmatrix} \mu_{o,o} & \cdots & \mu_{n,o} \\ \vdots & & \\ \mu_{o,j} & \cdots & \mu_{n,j} \end{bmatrix} \\ &= \det(A_j), \; 0 \leq j \leq n \\ &= 0 \text{ if } 0 \leq j \leq n-1 \end{aligned}$$

by the well known result of determinants. Thus taking finite linear combinations we obtain,

$$\int_{\mathbf{C}} P(z)\overline{\pi_{n-1}(z)}\,d\mu(z) = 0 \tag{1.1}$$

for every π_{n-1} of degree $\leq n-1$.

Next

$$\int_{\mathbf{C}} P(z)\overline{z^n}\,d\mu(z) = \det(A_n) = D_n$$

say. We must show that $D_n > 0$ for each $n \geq 1$. To this end, we must show that A_n is positive definite. So let

$$z = (\alpha_0, \ldots, \alpha_n)^T \in \mathbf{C}^{\,n+1} \backslash (0, \ldots, 0)^T$$

be given, then

$$\overline{z}^T A_n z = \sum_{j,k=0}^{n} \overline{\alpha_k}\mu_{j,k}\alpha_j = \int_{\mathbf{C}} \sum_{j,k=0}^{n} \overline{\alpha_k} z^j \overline{z^k}\alpha_j\,d\mu(z)$$

$$= \int_{\mathbf{C}} \left(\sum_{j=0}^{n} \alpha_j z^j\right)\left(\overline{\sum_{k=0}^{n} \alpha_k z^k}\right) d\mu(z) = \int_{\mathbf{C}} \left|\sum_{j=0}^{n} \alpha_j z^j\right|^2 d\mu(z) > 0,$$

as $supp(d\mu)$ is infinite. Thus if λ is an eigenvalue of A_n with eigenvector $z \neq (0, \ldots, 0)^T$ then,

$$0 < \overline{z}^T A_n z = \overline{z}^T(\lambda z) = \lambda \overline{z}^T z = \lambda \sum_{j=0}^{n} |\alpha_j|^2.$$

This implies that $\lambda > 0$ so that $D_n = \det(A_n) = \prod$ eigenvalues > 0, $n \geq 1$. Now $P(z) = z^n D_{n-1} +$ lower powers $= D_{n-1}(z^n + S(z))$ where degree $S \leq n-1$, $D_{n-1} > 0$. Thus we must have degree $P = n$ and (1.1) becomes

$$\int_{\mathbf{C}} P(z)\overline{\pi_{n-1}(z)}\,d\mu(z) = 0 \tag{1.2}$$

degree $P = n$, degree $\pi_{n-1} \leq n-1$. Now let

$$\varphi_n(z) = P(z) \cdot (D_n D_{n-1})^{-1/2}$$

a polynomial of degree n with leading coefficient

$$\gamma_n = \left(\frac{D_{n-1}}{D_n}\right)^{1/2} > 0.$$

Now by (1.2)

$$\int_{\mathbf{C}} \varphi_n(z)\overline{\pi_{n-1}(z)}d\mu(z) = 0 \tag{1.3}$$

degree $\pi_{n-1} \leq n-1$. Also

$$\begin{aligned}\int_{\mathbf{C}} \varphi_n(z)\overline{\varphi_n(z)}d\mu(z)&\\ &= \int_{\mathbf{C}} \frac{P(z)\overline{P(z)}}{D_n D_{n-1}}d\mu(z)\\ &= (D_n D_{n-1})^{-1}\int_{\mathbf{C}} P(z)D_{n-1}\overline{\left[z^n + S(z)\right]}d\mu(z)\\ &= D_n D_n^{-1} = 1\end{aligned}$$

by (1.3). Thus

$$\int_{\mathbf{C}} \varphi_n(z)\overline{\varphi_m(z)}d\mu(z) = \delta_{mn}.$$

Finally we show uniqueness. Let S be another polynomial of degree n,

$$S(z) = \delta_n z^n + \cdots,\ \delta_n > 0,\ \int_{\mathbf{C}} S(z)\overline{S(z)}d\mu(z) = 1$$

and

$$\int_{\mathbf{C}} S(z)\overline{\pi_{n-1}(z)}d\mu(z) = 0. \tag{1.4}$$

We show that $\varphi_n(z) = S(z)$. Let $T(z) = \varphi_n(z) - \gamma_n(\delta_n)^{-1}S(z)$ a polynomial of degree $\leq n-1$. Then,

$$\begin{aligned}o &\leq \int_{\mathbf{C}} |T(z)|^2 d\mu(z)\\ &= \int_{\mathbf{C}} (\varphi_n(z) - \gamma_n(\delta_n)^{-1}S(z))\overline{T(z)}d\mu(z) = 0\end{aligned}$$

by (1.3) and (1.4). As $supp(d\mu)$ is infinite, $T \equiv 0$. Thus

$$\varphi_n = \gamma_n(\delta_n)^{-1}S(z).$$

But

$$1 = \int_{\mathbb{C}} |\varphi_n|^2 d\mu(z) = \left(\gamma_n(\delta_n)^{-1}\right)^2 \int_{\mathbb{C}} |S|^2 d\mu(z) = \left(\gamma_n(\delta_n)^{-1}\right)^2.$$

Thus $\gamma_n = \delta_n$ for all n and thus $\varphi(z) = S(z)$. ∎

<u>Definition 1.3.</u> Throughout this chapter, let $d\mu$ denote a finite positive Borel measure on τ with $supp(d\mu)$ infinite. Such a $d\mu$ will be represented by a non decreasing function μ on $[0, 2\pi]$. Also for such $d\mu$, the corresponding orthonormal polynomials will be

$$\varphi_n(z) = \gamma_n z^n + \cdots, \ \gamma_n > 0$$

with

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(z)\overline{\varphi_m(z)}d\mu(z) = \delta_{mn}, \ z = e^{i\theta}.$$

<u>Remarks:</u>

(a) By the Radon–Nikodym theorem we will often write

$$d\mu(x) = \mu'(x)dx + d\mu_s(x) \tag{1.5}$$

where μ' is the absolutely continuous part of $d\mu$ and $d\mu_s$ the singular part of $d\mu$ consisting of a pure jump function and a singularly continuous part.

(b) For the sake of clarity, $d\mu_s$ is a singular Borel measure with respect to "meas" if there exists a Borel set E with meas $(E) = 0$ such that

$$\mu_s(E^c) = \int_{E^c} d\mu_s = 0.$$

Equivalently, the Radon–Nikodym derivative

$$\frac{d\mu_s}{d\,meas} = 0 \ a.e.(meas).$$

(c) If μ is absolutely continuous, $d\mu(x) = \mu'(x)dx$ and μ' will be called a weight function and often will be denoted by $w(\theta)$ or $f(\theta)$.

Definition 1.4

For each φ_n, we associate the monic orthogonal polynomial Φ_n defined by

$$\Phi_n(z) = (\gamma_n)^{-1}\varphi_n(z).$$

Also we define

$$\varphi_n^*(z) = z^n\overline{\varphi_n((\bar{z})^{-1})}$$

where φ_n^* is the reciprocal polynomial of φ_n and $*$ is called the star transform.

Remarks:

Notice that we have

(1.6) $$\left|\frac{\varphi_n(z)}{\varphi_n^*(z)}\right| = \left|\frac{\Phi_n(z)}{\Phi_n^*(z)}\right| \begin{cases} \leq 1, & |z| < 1, \\ = 1, & |z| = 1, \\ \geq 1, & |z| > 1; \end{cases}$$

(1.7) $$\gamma_n = \varphi_n^*(0) = \overline{\gamma_n}.$$

Lemma 1.5

The following relations hold for φ_n and Φ_n:

(1.8) $$\pi_n(z) = \sum_{k=0}^{n} a_k\varphi_k(z) \ \text{ iff } \ a_k = \frac{1}{2\pi}\int_0^{2\pi} \pi_n(z)\overline{\varphi_k(z)}d\mu(\theta)$$

$k = 0, \ldots, n$, for all π_n of degree n.

(1.9) $$\gamma_n \varphi_{n+1}(z) = \gamma_{n+1} z \varphi_n(z) + \varphi_{n+1}(0) \varphi_n^*(z);$$

(1.10) $$\gamma_n \varphi_{n+1}^*(z) = \gamma_{n+1} \varphi_{n+1}^*(z) + \overline{\varphi_{n+1}(0)} z \varphi_n(z);$$

(1.11) $$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{a_n} \Phi_n^*(z),$$

where $a_n = -\overline{\Phi_{n+1}(0)}$,

(1.12) $$\frac{\gamma_n}{\gamma_{n+1}} \varphi_{n+1}(z) = z\varphi_n(z) - \overline{a_n} \varphi_n^*(z).$$

Proof:

To prove (1.8) multiply both sides by $\overline{\varphi_k}$ for each k, integrate with respect to $d\mu(\theta)$ and use orthonormality to get the a_k, $k = 0, \ldots, n$. We now prove (1.9): We write

(1.13) $$\frac{\varphi_n(z) - \gamma_n z^n}{z^{n-1}} = \sum_{j=0}^{n-1} a_k^{(n)} \overline{\varphi_k}\left(\frac{1}{z}\right).$$

Now multiply both sides of (1.13) by $\varphi_k(z)$, set $z = e^{i\theta}$ and integrate to obtain for $k = 0, 1, 2, \ldots, n-1$

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi_n(e^{i\theta}) \varphi_k(e^{i\theta}) e^{i(1-n)\theta} d\mu(\theta) -$$

(1.14) $$- \frac{\gamma_n}{2\pi} \int_0^{2\pi} e^{i\theta} \varphi_k(e^{i\theta}) d\mu(\theta) = a_k^{(n)}.$$

Set

(1.15) $$I_k = \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} \varphi_k(e^{i\theta}) d\mu(\theta), \; k = 0, 1, 2, \ldots.$$

Then by orthonormality (1.14) becomes $a_k^{(n)} = -\gamma_n I_k$ so that (1.13) becomes

$$\frac{\varphi_n(z)}{\gamma_n} = z^n - z^{n-1} \sum_{j=0}^{m-1} I_k \overline{\varphi_j}\left(\frac{1}{z}\right)$$

which implies that

$$(1.16) \qquad \frac{\varphi_{n+1}(z)}{\gamma_{n+1}} - \frac{z\varphi_n(z)}{\gamma_n} = -I_n\varphi_n^*(z)$$

for all z. Set $z = 0$ then

$$I_n = \frac{-\varphi_{n+1}(0)}{\gamma_n\gamma_{n+1}}$$

and substituting back into (1.16) proves (1.9). To prove (1.10) replace z by $\frac{1}{z}$ in (1.9) and take the star transform. To prove (1.11) use the definition of Φ_n, a_n and (1.9). (1.12) follows immediately from (1.9). ∎

<u>Lemma 1.6</u>

$$(1.17) \qquad \frac{1}{\gamma_{n+1}} = \frac{1}{2\pi}\int_0^{2\pi} e^{i(-n-1)\theta}\overline{\varphi_n(e^{i\theta})}\,d\mu(\theta);$$

$$(1.18) \qquad \gamma_{n+1}^2 - \gamma_n^2 = |\varphi_{n+1}(0)|^2,\ n = 0, 1, 2, \dots ;$$

$$(1.19) \qquad \frac{\gamma_n^2}{\gamma_{n+1}^2} = 1 - |a_n|^2,\ |a_n| < 1.$$

Let

$$\lambda_n(d\mu, z) = \min_{\pi_{n-1} \text{ degree } \le n-1} \frac{1}{2\pi}\int_0^{2\pi} \frac{|\pi_{n-1}(t)|^2 d\mu(t)}{|\pi_{n-1}(z)|^2}$$

be the nth Christoffel function for $d\mu$. Then

$$(1.20) \qquad \begin{aligned} \frac{1}{\gamma_n^2} &= \lambda_{n+1}(d\mu, 0) = \left(\sum_{j=0}^{n}|\varphi_j(0)|^2\right)^{-1} \\ &= \min_{\substack{\text{monic } \pi_n \\ \text{degree } n}} \frac{1}{2\pi}\int_0^{2\pi}|\pi_n(z)|^2 d\mu(\theta),\ z = e^{i\theta} \end{aligned}$$

$$(1.21) \qquad \lim_{n\to\infty}\gamma_n = \gamma \le \infty.$$

Proof:

To prove (1.17),

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(-n-1)\theta}\varphi_{n+1}(e^{i\theta})d\mu(\theta)$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\varphi_{n+1}(e^{i\theta})}{\gamma_{n+1}} - \pi_n(e^{i\theta})\right]\cdot\overline{\varphi_{n+1}(e^{i\theta})}d\mu(\theta) = \frac{1}{\gamma_{n+1}}$$

by orthonormality. To prove (1.18) we use (1.9). Multiply by z^{-n-1} on both sides, write $z = e^{i\theta}$, and integrate to yield

$$\frac{\gamma_n}{2\pi}\int_0^{2\pi} e^{-i(n+1)\theta}\varphi_{n+1}(e^{i\theta})d\mu(\theta) = \frac{\gamma_{n+1}}{2\pi}\int_0^{2\pi} e^{-in\theta}\varphi_n(e^{i\theta})d\mu(\theta) +$$
$$+\frac{\varphi_{n+1}(0)}{2\pi}\int_0^{2\pi} e^{-i\theta}\overline{\varphi_n(e^{i\theta})}d\mu(\theta).$$

Use now the definition of $\overline{I_n}$, (1.15) and (1.17) to deduce

$$\frac{\gamma_n}{\gamma_{n+1}} = \frac{\gamma_{n+1}}{\gamma_n} + \varphi_{n+1}(0)\overline{I_n} = \frac{\gamma_{n+1}}{\gamma_n} - \frac{|\varphi_{n+1}(0)|^2}{\gamma_n\gamma_{n+1}}$$

which implies the result. To prove (1.19) use the definition of a_n and (1.18). We remark that we will show in Lemma (1.8) that $\varphi_n(z)$ has all its zeros in $|z| < 1$, thus we may write

$$\varphi_{n+1}(z) = \prod_j (z - z_j)$$

where z_j are the zeros of φ_n with $|z_j| < 1$, that is $|\varphi_{n+1}(0)| < 1$, that is, $|a_n| < 1$. To prove (1.20) we write

$$\pi_n(z) = \sum_{j=0}^{n} C_j\varphi_j(z),\ C_n = \frac{1}{\gamma_n}$$

by (1.8). Then,

$$|\pi_n(z)|^2 = \left|\sum_{j=0}^{n} C_j\varphi_j(z)\right|^2 = \sum_{j,k=0}^{n} C_j\varphi_j\overline{C_k\varphi_k(z)}.$$

Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi}|\pi_n(z)|^2 d\mu(\theta) = \frac{1}{2\pi}\int_0^{2\pi}\sum_{j,k=0}^{n} C_j\varphi_j\overline{C_k\varphi_k(z)}\,d\mu(\theta)$$

$$(1.22)\qquad = \sum_{j=0}^{n}|C_j|^2 = \frac{1}{\gamma_n^2} + \sum_{j=0}^{n}|C_j|^2 > \frac{1}{\gamma^2}$$

with minimum at

$$\pi_n(z) = \frac{1}{\gamma_n}\varphi_n(z).$$

Also, we can write

$$\pi_{n-1}(z) = \sum_{j=0}^{n-1} C_j\varphi_j(z)$$

so that

$$|\pi_{n-1}(z)| = \left|\sum_{j=0}^{n-1} C_j\varphi_j(z)\right| \le \left(\sum_{j=0}^{n-1}|C_j|^2\right)^{1/2}\left(\sum_{j=0}^{n-1}|\varphi_j(z)|^2\right)^{1/2}$$

$$(1.23)\qquad = \left(\frac{1}{2\pi}\int_0^{2\pi}|\pi_{n-1}(t)|^2 d\mu(t)\right)^{1/2}\left(\sum_{j=0}^{n-1}|\varphi_j(z)|^2\right)^{1/2}$$

using (1.22) and the Cauchy Schwarz inequality.

Thus (1.23) implies that

$$\frac{\frac{1}{2\pi}\int_0^{2\pi}|\pi_{n-1}(t)|^2 d\mu(t)}{|\pi_{n-1}(z)|^2} \ge \left(\sum_{j=0}^{n-1}|\varphi_j(z)|^2\right)^{-1}$$

with minimum at

$$\pi_{n-1}(t) = \sum_{j=0}^{n-1}\varphi_j(t)\overline{\varphi_j(z)}.$$

Finally,

$$\frac{1}{\gamma_n^2} = \min_{\substack{\pi_n \text{ monic} \\ \text{degree } n}} \frac{1}{2\pi}\int_0^{2\pi} \left|\pi_n(z)\right|^2 d\mu(\theta), \ (z = e^{i\theta})$$

$$= \min_{\substack{\pi_n^* \text{ degree } \leq n \\ \pi_n^*(0)=1}} \frac{1}{2\pi}\int_0^{2\pi} \left|\pi_n^*(z)\right|^2 d\mu(\theta)$$

$$= \min_{Q \text{ degree } \leq n} \frac{1}{2\pi}\int_0^{2\pi} \frac{\left|Q(e^{i\theta})\right|^2 d\mu(\theta)}{\left|Q(0)\right|^2} = \lambda_{n+1}(0)$$

$$= \left(\sum_{j=0}^{n} \left|\varphi_j(0)\right|^2\right)^{-1}.$$

Finally, to prove (1.21) we note that as

$$\frac{1}{\gamma_n^2} = \left(\sum_{j=0}^{n} \left|\varphi_j(0)\right|^2\right)^{-1},$$

this implies that the γ_n are increasing with n and therefore $\lim_{n\to\infty} \gamma_n = \gamma \leq \infty$. ■

Remarks:

We note that (1.21) will play a crucial role in what follows as it is the difference between $\gamma = \infty$ or $\gamma < \infty$ which will determine the asymptotic behaviour of the sequence $(\varphi_n)_{n=0}$ on and off τ.

(1.25) $$K_n(\xi, z) = \sum_{k=0}^{n-1} \overline{\varphi_k(\xi)}\varphi_k(z)$$

and thus K_n is a polynomial of degree $\leq n-1$;

(1.26) $$\lambda_n(z) = \frac{1}{K_n(z,z)}, \ \pi_{n-1}(z) = \frac{1}{2\pi}\int_0^{2\pi} \overline{K_n(z,t)}\pi_{n-1}(t)d\mu(t).$$

In particular $\frac{1}{\gamma_n^2} = \frac{1}{K_n(0,0)}$;

(1.27) $$|K_n(\xi, z)| \leq 2|\xi - z|^{-1}|\varphi_n(\xi)||\varphi_n(z)|, \ |z| = |\xi| = 1,$$

(1.28) $$\begin{aligned}&(K_n(z,z))^{-1}\\ &= \min_{\pi_{n-1} \text{ of degree } \leq n-1} \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{\pi_{n-1}(u)}{\pi_{n-1}(z)}\right|^2 d\mu(t), \ u = e^{i\theta}.\end{aligned}$$

<u>Proof</u> By Lemma 1.6,

(1.29) $$\frac{1}{\gamma_n^2} = \lambda_{n+1}(0) = \min_{\pi_n} \frac{1}{2\pi}\int_0^{2\pi} \frac{|\pi_n(z)|^2 d\mu(\theta)}{|\pi_n(0)|^2}, \ z = e^{i\theta}$$

where the minimum is attained at

$$\pi_n(z) = \sum_{j=0}^{n} \varphi_j(z)\overline{\varphi_j(0)}.$$

Then

$$S(z) = \frac{\pi_n(z)}{\pi_n(0)}$$

is a polynomial of degree $\leq n$ with $S(0) = 1$. Thus,

$$T(z) = z^n\overline{S((\bar{z})^{-1})}$$

is a monic polynomial of degree n and for $|z| = 1$,

$$|T(z)| = |S((\bar{z})^{-1})| = |S(z)| = \left|\frac{\pi_n(z)}{\pi_n(0)}\right|.$$

This implies by (1.29)

$$\frac{1}{2\pi}\int_0^{2\pi}|T(z)|^2 d\mu(\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{|\pi_n(z)|^2}{|\pi_n(0)|^2}d\mu(\theta),\ (z = e^{i\theta})$$

$$= \lambda_{n+1}(d\mu, 0) = \gamma_n^{-2}(d\mu). \tag{1.30}$$

Now by Lemma 1.6, $(\gamma_n)^{-1}\varphi_n(z)$ is the unique polynomial where the minimum is attained in (1.29). Thus

$$\gamma_n^{-1}\varphi_n(z) = T(z) = z^n \overline{S((\bar{z})^{-1})}$$

by (1.30). Therefore we have

$$\gamma_n^{-1}\varphi_n^*(z) = S(z) = \gamma_n^{-2}\sum_{j=0}^{n}\varphi_j(z)\overline{\varphi_j(0)}$$

which implies that

$$\gamma_n\varphi_n^*(z) = \sum_{j=0}^{n}\overline{\varphi_j(0)}\varphi_j(z)$$

and this in turn implies

$$\gamma_n\varphi_n^*(z) - \gamma_{n-1}\varphi_{n-1}^*(z) = \overline{\varphi_n(0)}\varphi_n(z)$$

so

$$\gamma_{n-1}\varphi_{n-1}^*(z) = \gamma_n\varphi_n^*(z) - \overline{\varphi_n(0)}\varphi_n(z) = K_n(0, z)$$

by definition of K_n. So we have (1.24).

Next we proceed to prove (1.25). We observe that $K_n(\xi, z)$ is a polynomial of degree $n-1$ in z, thus we can write by (1.8)

$$K_n(\xi, z) = \sum_{j=0}^{n-1}B_j\varphi_j(z)$$

where

$$B_j = \frac{1}{2\pi}\int_0^{2\pi}K_n(\xi, z)\overline{\varphi_j(z)}d\mu(\theta).$$

Now let $\pi_{n-1}(z)$ be an arbitrary polynomial of degree $n-1$ in z. Write

$$\pi_{n-1}(z) = \pi_{n-1}(\xi) + (z-\xi)\pi_{n-2}(z).$$

Then we have,

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \overline{K_n(\xi,z)}(z-\xi)\pi_{n-2}(z)d\mu(\theta) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\varphi_n^*(\xi)\overline{\varphi_n^*(z)} - \varphi_n(\xi)\overline{\varphi_n(z)}\right] \cdot \frac{1}{1-\bar{\xi} z^{-1}}(z-\xi)\pi_{n-2}(z)d\mu(\theta) \\
&= \varphi_n^*(\xi)\frac{1}{2\pi}\int_0^{2\pi} \overline{\varphi_n^*(z)} z\pi_{n-2}(z)d\mu(\theta) - \varphi_n(\xi)\int_0^{2\pi} \overline{\varphi_n(z)} z\pi_{n-2}(z)d\mu(\theta) \\
&= 0
\end{aligned}$$

by orthonormality. Therefore,

$$\int_0^{2\pi} \overline{K_n(\xi,z)}(z-\xi)\pi_{n-2}(z)d\mu(\theta) = 0. \tag{1.31}$$

Also, using the definition of $\pi_{n-1}(z)$ and (1.31) we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \overline{K_n(\xi,z)}\pi_{n-1}(z)d\mu(\theta) \\
&= \pi_{n-1}(\xi)\frac{1}{2\pi}\int_0^{2\pi} \overline{K_n(\xi,z)}d\mu(\theta) =: \eta(\xi)\pi_{n-1}(\xi).
\end{aligned} \tag{1.32}$$

Now let $\pi_{n-1} = \varphi_v$, $0 \le v \le n-1$ in (1.32). Then

$$B_v = \frac{1}{2\pi}\int_0^{2\pi} K_n(\xi,z)\overline{\varphi_v(z)}d\mu(\theta) = \overline{\eta(\xi)}\varphi_n(\xi).$$

This implies that

$$K_n(\xi,z) = \overline{\eta(\xi)}\sum_{j=0}^{n-1} \overline{\varphi_j(\xi)}\varphi_j(z). \tag{1.33}$$

Also as $\overline{K_n(\xi,z)} = K_n(z,\xi)$ we have

$$K_n(\xi,z) = \overline{K_n(z,\xi)} = \eta(z)\sum_{j=0}^{n-1} \overline{\varphi_j(\xi)}\varphi_j(z). \tag{1.34}$$

By (1.28) and (1.29) $\overline{\eta(\xi)} = \eta(z)$ for all z and ξ which implies that $\eta =$ constant C say. Therefore

$$K_n(\xi, z) = C \sum_{j=0}^{n-1} \overline{\varphi_j(\xi)}\varphi_j(z).$$

Now put $\xi = 0$, use the definition of $K_n(\xi, z)$ and (1.24) to get $C = 1$. Thus (1.25) is proved. (1.27) follows from Lemma 1.6. Next, writing $\pi_{n-1}(z) = \sum_{j=0}^{n-1} c_j\varphi_j(z)$, $c_j = \frac{1}{2\pi}\int_0^{2\pi} \pi_{n-1}(t)\overline{\varphi_j(t)}d\mu(t)$, $0 \le j \le n-1$ by (1.8). We have,

$$\pi_{n-1}(z) = \sum_{j=0}^{n-1} \frac{1}{2\pi}\int_0^{2\pi} \pi_{n-1}(t)\overline{\varphi_j(t)}\varphi_j(z)d\mu(t)$$

which implies

$$\pi_{n-1}(z) = \frac{1}{2\pi}\int_0^{2\pi} \overline{K_n(z,t)}\pi_{n-1}(t)d\mu(t).$$

Now (1.28) follows from the definition of K_n and Lemma 1.6 respectively. ∎

Lemma 1.8

Let z_j, $1 \le j \le n$ be the zeros of $\varphi_n(z)$ then

(1.35) $$z_j \in \{z : |z| < 1\} \text{ for all } 1 \le j \le n.$$

Also, let $\pi_n(z)$ be any arbitrary polynomial of degree $\le n$. Then

(1.36) $$\frac{1}{2\pi}\int_0^{2\pi} \frac{\pi_n(z)}{|\varphi_n(z)|^2}d\theta = \frac{1}{2\pi}\int_0^{2\pi} \pi_n(z)d\mu(\theta).$$

Proof: We first prove (1.35). By Lemma 1.7

$$|\varphi_n^*(z)|^2 = (1-|z|^2)\sum_{j=0}^{n-1}|\varphi_j(z)|^2 + |\varphi_n(z)|^2 \ge |\varphi_0(z)|^2(1-|z|^2)$$

(1.37) $$= \gamma_0^2(1-|z|)(1+|z|).$$

Now if φ_n^* has zeros in $|z| < 1$ we obtain a contradiction to (1.37). Also if φ_n^* has zeros in $|z| = 1$, say at $z = e^{i\theta}$, then

$$\varphi_n^*(re^{i\theta}) - \varphi_n^*(e^{i\theta})$$
$$= \int_r^1 \frac{d}{ds}[\varphi_n^*(se^{i\theta})]\,ds = O(1-r),\ 0 < r < 1.$$

This implies

$$|\varphi_n^*(re^{i\theta}|^2 = O(1-r)^2$$

which contradicts (1.37) again. So φ_n^* has all its zeros in $|z| > 1$ and thus φ_n has all its zeros in $|z| < 1$. We now turn to the proof of (1.36). We first show

(a)
$$\int_0^{2\pi} \frac{\overline{\pi_n(z)}\varphi_n(z)}{|\varphi_n(z)|^2}\,d\theta = \int_0^{2\pi} \overline{\pi_n(z)}\varphi_n(z)\,d\mu(\theta).$$

By (1.35), since $\varphi_n^*(z)$ has no zeros in $|z| \leq 1$ we may apply Cauchy's integral formula to obtain for every π_{n-1}

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\overline{\pi_{n-1}(z)}\varphi_n(z)}{|\varphi_n(z)|^2}\,d\theta = \frac{2}{2\pi i}\oint \frac{\overline{\pi_{n-1}(z)}\,dz}{\overline{\varphi_n(z)}z}$$
$$= \frac{1}{2\pi i}\oint \frac{\pi_{n-1}^*(z)}{\varphi_n^*(z)}\,dz = 0 = \frac{1}{2\pi}\int_0^{2\pi} \overline{\pi_{n-1}(z)}\varphi_n(z)\,d\mu(\theta),$$

so that in the special case of $\pi_n = \varphi_n$ (a) is satisfied. Now notice that any π_n can be written as $\pi_n = a\varphi_n + \pi_{n-1}$ using (1.8). So (a) is proved.

(b) Next let

$$\mu_n(t) = \int_0^t \frac{d\theta}{|\varphi_n(z)|^2}.$$

With this notation (a) becomes

$$\int_0^{2\pi} \overline{\pi_n(z)}\varphi_n(z)\left[d\mu_n(\theta) - d\mu(\theta)\right] = 0$$

and taking conjugates on both sides we get

$$\int_0^{2\pi} \pi_n(z)\overline{\varphi_n(z)}\left[d\mu_n(\theta) - d\mu(\theta)\right] = 0.$$

So substituting π_n^* for π_n we obtain

$$\int_0^{2\pi} \pi_n^*(z)\overline{\varphi_n(z)}[d\mu_n(\theta)-d\mu(\theta)] = \int_0^{2\pi} \overline{\pi_n(z)}\varphi_n^*(z)[d\mu_n(\theta)-d\mu(\theta)] = 0$$

whence we obtain for $v = n$ and for arbitrary polynomials $\pi_v^{(1)}$, $\pi_v^{(2)}$ of degree v,

$$\int_0^{2\pi} \left[\overline{\pi_v^{(1)}(z)}\varphi_v(z) + \overline{\pi_v^{(2)}(z)}\varphi_v^*(z)\right][d\mu_n(\theta) - d\mu(\theta)] = 0. \tag{1.38}$$

Now we recall from the proof of (1.24) that we derived

$$\gamma_n\varphi_n^*(z) - \overline{\varphi_n(0)}\varphi_n(z) = \gamma_{n-1}\varphi_{n-1}^*(z) \tag{1.39}$$

and taking the star transform

$$\gamma_n\varphi_n(z) - \varphi_n(0)\varphi_n^*(z) = \gamma_{n-1}z\varphi_{n-1}(z). \tag{1.40}$$

So we see that if (1.38) is valid for a value v then by (1.39) and (1.40) it is valid for $v-1$, so proceeding thus we see that (1.38) is valid for every non negative integer $v \leq n$. Now by putting $\pi_v^{(1)} = 1$, $\pi_v^{(2)} = 0$ in (1.38) we see that (1.36) is valid for $\pi_v = \varphi_v$, $v = 0, 1, \ldots, n$. The general case is proved by expressing linearly the arbitrary π_n by the φ_v. ■