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Number pattern: developing a sense of structure with primary school teachers

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Chapter 2

Review of the Literature

Early algebra can help to prepare learners for more complex mathematics, to cultivate habits of mind that attend to the deeper underlying structure of mathematics and to improve their algebra understanding.

(Blanton & Kaput, 2005: 412).

Introduction

This chapter provides a review of the literature, focusing particularly on the triadic notion of pattern, structure and algebraic reasoning in the research literature. This takes place against the backdrop of research related to EA and the use of structure at FP. The initial focus will be on what EA is and what the implications are for mathematics teaching at FP. Thereafter the focus will shift to the symbiotic relationship between algebra and arithmetic which EA uses to advance a relational view of mathematics learning at an early age. This then leads to the notion of generalising in FP mathematics and an exploration of the possibilities it opens up for a structural view of mathematics in CA 2. Structure is then discussed as a means to *algebrafy* arithmetic, to decompose mathematical objects of learning and also to develop a sound relational understanding of mathematics. This is followed by a discussion of the concepts and trajectories related to pattern, referring in particular to repeating pattern as a cognitive tool, the notion of repeat and growth in patterns, the structure and complexity of the repeating pattern and growth patterns and then the broader notion of sequencing as a pedagogical tool.

The next area of focus will be generalising patterns and the body of research into layers of generality. Generalisation as an outcome of structural extension will be examined. Lastly, the literature review will focus on teacher knowledge and the teaching of algebra by examining teacher subject matter knowledge, as well as pedagogical content knowledge. This will include an exploration of teachers' conceptualisations of algebra and their experiences in learning to become teachers of algebra. The chapter concludes with a brief recapitulation.

Understanding Early Algebra in the Foundation Phase

The field of EA has been researched since the early eighties, and has subsequently been incorporated into the teaching of mathematics at FP. In more recent decades, the proposal that algebraic thinking should be introduced in the primary grades has been more widely accepted, with even more recent emphasis on patterning and early algebraic reasoning in the preschool and early grades (Mulligan, Cavanagh and Keanan-Brown, 2012). The focus of EA is a relational approach to learning mathematics, which supports Warren's (2011) view that early algebraic thinking refers to studying number from a structural perspective. Structure is extracted through exploring number and space relationally, which in turn initiates reasoning that focuses not on the object, but on the underlying associations the object possesses. Relational thinking represents a fundamental shift from an arithmetical focus (calculating answers) to an algebraic focus (examining relations) (Jacobs, Franke, Carpenter, Levi and Battey, 2007, p.3), which depends on critical relations that give meaning to both arithmetic and algebra. In arithmetic these 'relations are often left unarticulated' (Ibid, p. 4). Jacobs et. al. (2007) maintain that a relational view integrates concepts and skills in arithmetic with the enhanced concepts and skills needed to learn algebraic reasoning skills effectively, and more formal algebra later.

Mulligan et.al.(2012) assert that problems which learners experience with algebra learning in the senior phases at school, 'stem from a lack or an inadequate study of the relational aspects of number right from the pre-school context' (p. 2). Many researchers support this argument (Carpenter and Franke, 2001; Warren and Cooper, 2001; Warren, 2002). There is, furthermore, general agreement amongst researchers that EA comprises two central features: (i) generalizing, i.e. identifying, expressing and justifying mathematical structure, properties and relationships and (ii) reasoning and actions based on the forms of generalisations (Lins and Kaput, 2004; Kaput, 2007). In practice EA has focused on (i) the use of arithmetic as a domain for expressing and forming generalisations (generalised arithmetic) and (ii) generalising numeric and special patterns to describe functional relationships (functional thinking) (Blanton, Schifter, Inge, Lofgren, Willis, Davis and Confrey, 2007, p.2).

EA is not an 'add-on' to the existing curriculum where teachers will teach EA after they have focused on arithmetical procedures and skills. Blanton et.al. (2007) argue that EA is a 'way of thinking that brings meaning, depth and coherence to learners' mathematical understanding

by delving more deeply into concepts already being taught so that there is opportunity to generalise relationships and properties in mathematics' (p.3). The goal of EA is that learners learn to reason algebraically and use the symbolic language of algebra for expressing and justifying their mathematical understanding (Ibid, p.4), without doing algebraic manipulations. A teaching focus which enhances early algebraic thinking advances the numeracy outcomes of young learners (Taylor-Cox, 2003, Mason, 1996; Murray, 2010).

The algebraic reasoning that one would expect young learners to develop in their early grades is not the same as that which one would anticipate high school students to possess. At FP the focus is on reasoning with the relational attributes embedded in mathematical objects through focusing on the structural extension of basic arithmetic, whereas high school learners manipulate formalised algebraic representations. Algebraic reasoning at FP level involves many forms of cognitive process, including extending patterns, doing and undoing (Driscoll, 1999; Carpenter, Franke and Levi, 2003), functional thinking (Kaput, 2000), understanding equivalence, solving unknowns (Warren, 2005) and working with indeterminate quantities, writing a generalisation for a regularity observed in a sequence activity or through arithmetic exploration (Radford, 2012) and working with various structural forms (Hallagan, Rule and Carlson, 2009). Algebra, as a formal body of knowledge, is acquired and developed over a long span of time. Researchers are increasingly arguing that algebraic reasoning is possible even in pre-primary school mathematical experiences (Blanton and Kaput, 2005; Carraher and Schliemann, 2007; Warren and Cooper, 2008; Yeap and Kaur, 2008; Papic, 2007; Papic, Mulligan and Mitchelmore, 2011). This study focuses on the development of a relational approach to sequencing in the FP curriculum, and is founded on the understanding that EA forms a cohesive approach to primary mathematical exploration which enhances learners' algebraic reasoning skills.

Early Algebraic Thinking – a Symbiotic Relationship Between Arithmetic and Algebra

A symbiotic relationship is predicated upon the principle that both parties benefit mutually through their interaction. This biological concept serves as a powerful analogy to describe the reciprocal relationship between arithmetic and algebra. Malara and Navarra (2003) distinguish between arithmetical thinking as thinking that focuses on 'product' and algebraic thinking that focuses on 'process'. They argue that the power of elementary mathematics lies in the co-existence of algebraic and arithmetical thinking, and maintain, along with Mason (2006) and Warren (2010), that these two streams of thought augment one another, and over

time develop more numerate learners of mathematics. Blanton and Kaput (2005) distinguish different categories of algebraic reasoning in the FP classroom as: generalised arithmetic, functional relationships, properties of numbers and operations and the algebraic treatment of number relations. Warren (2010) argues that EA is embedded in two core approaches, namely relations and change. EA is an approach to teaching arithmetic through exploring variance in a relational manner, using more sophisticated forms of arithmetical exploration. EA is an approach to the foundational learning of mathematics. It is a system that mirrors the structural features of arithmetic (Usiskin, 1988) and is characterised by the indeterminacy of mathematical objects, analytical thinking and symbolic ways of designating objects (Radford, 2006; Warren, 2009). Researchers have demonstrated that working with unknown quantities is possible at a young age, as learners use the unknown (variable) number line to establish relationships between quantities, and use these relationships to solve problems (Carraher, Schliemann and Brizuela, 1999) by developing a range of different strategies (Brito Lima and da Rocha Falcão, 1997).

Studies on early algebraic experience have found that learners who are in Grade 1 can operate on and with an unknown quantity (Bodanskii, 1991; Carraher and Schliemann, 2004, 2005, 2006, 2007). Bodanskii (1991) worked with young learners on writing equations to solve verbal problems, and found that young learners who were introduced to algebra problems from first grade performed significantly better than their peers, who only worked on algebra from Grade 6 and later. Research conducted by Carraher, Schliemann and Brizuela examined the use of algebraic reasoning by elementary school learners. They worked with two groups from Grades 3 to 5: the first group was taught early algebra by the researchers and the second group was taught the traditional curriculum by the classroom teachers. Learners in the first group performed significantly better than the control group on written assessments developed by the researchers, as well as on standard assessment items (Carraher, Schliemann and Brizuela, 2011). There is therefore evidence that arithmetic and algebra can be treated as complementary, co-existing and interwoven strands of mathematical activity, and that this symbiosis can start at the outset of mathematical learning and produce better achievement in current standardised assessments and future learning of algebra.

Activating Algebraic Thinking and Generalising at Foundation Phase

The literature reveals that generalisation is a crucial skill required to activate algebraic thinking. Mason (1996) presents a strong argument for generalisation at the elementary level

of mathematics learning. Young learners bring with them an innate tendency to generalise and justify (Mason, 2008) and EA enhances this thinking (Blanton, 2008). Teaching algebra in the early grades promotes the development of algebraic thinking, while focusing on algebra as it flows into the categories of *generalised arithmetic* and *functional thinking* (Murray, 2010). In generalised arithmetic, learners' reasoning will become apparent as they recognise and extrapolate patterns that emerge and express these patterns using forms of symbolic notation. Learners at a young age begin their arguments by testing numeric cases (Blanton, 2008). These reasoning practices can be developed into sophisticated forms of reasoning using the context of problems and employing various representations of cases. This leads to their engaging a variety of pedagogic and semiotic resources (Radford, 2010) and actively using previously established generalisations to access the items that lie outside of the perceptual. Generalisation lies at the heart of algebraic reasoning (Carraher and Schliemann, 2002; Mason, 1996).

When developing conceptual mathematical tools, it is important to ground the learning of new concepts in familiar cognitive territory. Mason (1985, 1996) describes number pattern as a route towards expressing generality, and thus regularity, across mathematical objects. Mathematics lessons without the opportunity for learners to express a generality are, in fact, not truly lessons in mathematics (Mason, Graham and Johnstone-Wilder, 2005, p.297). Facility in using algebraic symbols is an integral part of becoming proficient in applying algebra to solve problems, but it is not sufficient in itself (Seeley, 2004). Working with symbols, without a foundation in thinking algebraically, is likely to lead to frustration and possible failure (Ibid, p.1). Providing learners with algebra as a *ready-made* package of mathematics and mathematical information, as is often done in schools, does not enhance the relational understanding of algebra as a conceptual tool. Learners may need to develop these generalisations through using pedagogical tools such as 'number lines, function tables, symbolic notation and drawings' (Murray, 2010, p.75). These tools are pathways to extracting the relational aspects of mathematical objects, and naturally enhance a structural view of mathematics.

Algebraic thinking does not appear spontaneously, but emerges from a lengthy process of conceptualisation and re-conceptualisation, which is embedded in historical and cultural forms of mathematical thinking (Radford, 2012, p.5). An essential element of algebraic reasoning is the focus on learners' reasoning and the discourse that will enable them to identify the connections between concepts and to build on these connections when forming

generalisations (Murray, 2010). It is important to note that the learners' discourse does not occur naturally; instead it is as a result of a well-articulated plan that has been developed by a teacher. A teacher in the early grades, therefore, needs to understand the underlying algebraic aspects of the content (Ibid, p.75) which s/he is teaching, as well as the pedagogical path that will lead to the relational and the general, which is needed to build strong algebraic bonds between mathematical objects.

Enhancing Structural Exploration in the Primary Mathematics Classroom

Within the structure of mathematical objects, one decomposes the relational aspects through considering that which varies from one item to the next, and that which remains invariant. It is necessary that teachers understand the process of reasoning algebraically and also how to prompt the appropriate thinking among learners to advance such cognitive extensions. When one focuses on relational thinking as a skill that enhances learners' abilities to explore structure, a clearly planned pedagogical path is required to elicit the necessary thinking which enhances a sense of structure and, at the same time, develops number constructs. The limited exploration of structure and relationships in the early grades contributes to learners' difficulties in operating on a symbolic level (Sfard, 1991). Blanton and Kaput (2011), Kieran (1990) and Cooper (2008) all argued that functional thinking (co-variance) should be used as an organising theme in the primary curriculum. Du Plessis and Setati (2007) showed how decomposition of number in base ten arithmetic, can lead to an understanding of the algebraic process of polynomial division, from a structural perspective, using the structural extension of arithmetical long division on number. Kaput (1995) calls this process 'algebrafying arithmetic'. Blanton and Kaput (2001) remark that teachers in the elementary school, especially, need to grow 'algebra eyes and ears' (p. 91) in order to see and make use of opportunities where they can *algebrafy* arithmetic.

A pedagogical plan has been suggested by Mark Driscoll (1999). He posits three algebraic habits of mind which have at their heart (a) doing and undoing (reversibility and undoing of mathematical processes), (b) building rules that represent functions (recognizing patterns, organizing data and establishing relationships) and (c) abstraction from computation (thinking of computations independently from numbers). According to Driscoll, these habits foster the development of algebraic reasoning skills and should inform pedagogical decisions made by teachers. Developing these algebraic habits of mind requires teachers to move around mathematical territory in a flexible manner and ask appropriate questions that allow

learners to start thinking and exploring relationally and structurally. Algebraic thinking includes recognizing and analysing patterns (seeking structure), studying and representing relationships (formalising structure), making generalisations (abstracting structure) and analysing how phenomena change.

Research findings increasingly support the idea that young learners can be taught to seek and recognise mathematical structure, and that the effect on their overall mathematics achievement can be substantial (Mulligan, Prescott, Papic, and Mitchelmore, 2006; Scliemann, Carraher and Brizuela, 2011). Studies which focused on imagery and arithmetic showed that learners who recognise and work with the structure of mathematical processes acquire a deeply conceptual cognition of mathematics (Gray, Pitta and Tall, 2000; Pitta-Pantazzi, Gray and Christou, 2004). Learners with lower levels of achievement resorted to descriptive and idiosyncratic methods, founded on non-mathematical aspects and surface characteristics based on visual cues (Mulligan and Mitchelmore, 2009). These research studies maintain that learners who worked with structure had an advantage over their peers and performed better on items related to algebraic reasoning, as well as on standard items.

Structural extension is one way to elevate cognitive activity towards the relational aspects of arithmetical exploration, and in turn, enhances algebraic reasoning skills from an early age. It relies on certain aspects of basic procedural knowledge to provide entry to structural exploration. Mason, Stephens and Watson (2009) maintain that structural extension helps to bridge the 'mythical chasms' (p.1) between conceptual and procedural approaches to teaching and learning mathematics. One of the examples they cite is the sequence: RB; RBB; RBBB; RBBBB; ... where the R stays the same and the B shows growth in every consecutive term. They argue that good teaching practice combines processes and concepts in a mutually beneficial manner that enhances the learning of mathematics, and affords young learners opportunities to engage with such sequences at a more sophisticated level by focusing on what stays the same and what changes. These basic procedures are based on recognition of surface structure in sequenced items, and then using these structures to explore and express relationships by extending the systemic structural characteristics embedded in the set of sequenced items.

Structure – The Key to Developing Relational Understanding with Young Learners

As I have argued previously, structure is central to the development of relational understanding in young learners. Mulligan and Mitchelmore (2009) developed more explicit descriptors of the structural features of learners' representations which focused on the 'pictorial, iconic and symbolic' characteristics of the representations. They argue that these descriptors are applicable to a wide mathematical base. Mulligan and Mitchelmore have conducted extensive research on the awareness of mathematical structure. They have proposed a new construct that they call the Awareness of Mathematical Pattern and Structure (AMPS,) which 'generalised across mathematical objects, can be measured reliably, and is correlated with mathematical understanding' (Ibid, p.1). They consider AMPS to possess two inter-dependent components: a cognitive component (which signals knowledge of structure) and a meta-cognitive component (which signals the habit of seeking and analysing underlying patterns) (Ibid, p.6). Research was conducted to ascertain whether AMPS could be described and measured as a general construct, with applicability across a range of mathematical concepts and processes. They used an assessment tool derived from a comprehensive research focus (Wright, 1994; Mulligan and Wright, 2000; Howell and Kemp, 2005; Doig, 2005; Clements and Sarama, 2007) on early mathematics learning and assessment. This

1.	*Pre-structural stage*	Representations lack any evidence of numerical or spatial structure. Most examples show idiosyncratic features.
2.	*Emergent (inventive-semiotic) stage*	Representations show some relevant elements of the given structure, but their numerical or spatial structure is not represented.
3.	*Partial structural stage (extended stage)*	Representations show most relevant aspects of numerical or spatial structure, but the representation is incomplete.
4.	*Stage of structural development*	Representations correctly integrate numerical and spatial structural features.

Table 2.1(a): The stages of structural development as proposed by Mulligan and Mitchelmore.

tool was named the Pattern and Structure Assessment (PASA) tool. Their findings showed that the PASA tool is a reliable and valid tool to assess the conceptual understanding of mathematics. In their formulation of the stages of structural development, Mulligan and Mitchelmore identify four categories of structural development in learners' work. These categories and their descriptors are given in Table 2.1(a). After follow-up studies were

conducted with the top eight achievers in their study, based on their PASA scores, it was ascertained that they had exceeded AMPS to such an extent that a new category had to be constructed to represent this progress. This category was named the Advanced Structural stage (Mulligan and Mitchelmore, 2009). There are no descriptors provided for this category but one can reasonably assume that at this stage learners use structure to extend beyond the perceptual, based on the illustration in Table 2.1(b). It shows the five stages in the development of structural thinking among a sample of 316 Kindergarten learners. Their working with spatial features was observed over a 2 year period (Mulligan, English, Mitchelmore, Welsby and Crevensten, 2011, p.9):

					
Pre-Structural	**Emergent Structural**		**Partial Structural**	**Structural**	**Advanced Structural (generalizing pattern)**

Table 2.1(b): Learner responses to a task focusing on spatial structure of triangular numbers.

Mulligan and Mitchelmore (2009) propose several mechanisms that can be used to explain why learners with high AMPS developed more advanced mathematical thinking than those with low levels of AMPS (p.12):

- Students with high levels of AMPS would become knowledgeable about spatial structures such as rectangular arrays that incorporate equal sized units at an earlier age.
- Through their awareness of these structures (in particular, equal grouping and unitising) they would more easily learn basic properties of number, space and measurement.
- They would have a tendency to look for patterns and to explore similarities and differences between them. They would hence learn new structures more easily (Mason, 1996).
- They would be aware of what constitutes a mathematical pattern and would hence ignore irrelevant features when learning new concepts.

The Pattern and Structure Mathematics Awareness Program (PASMAP) has investigated the development of patterning and early algebraic reasoning over a series of studies with four- to eight-year olds. The researchers involved in this program found that an awareness of mathematical pattern and structure enabled real mathematical thinking and simple forms of

generalisation from an early age. Multiplicative structure, including unitising and partitioning, and spatial structuring were found to be critical to the development of an awareness of pattern and structure (Mulligan, Mitchelmore and Prescott, 2006; Milligan, Mitchelmore, English and Robertson, 2010; Mulligan, Mitchelmore, English and Crevensten, 2012, 2014). Radford (2012) maintains that a structural understanding is a 'crucial aspect of the emergence of algebraic thinking' (p.7). There is evidence that teaching, and hence learning, that focuses on developing an awareness of mathematical structure and pattern improves learners' relational understanding of mathematical processes and develops their ability to discern what constitutes a mathematical pattern and what information to treat as irrelevant (Mulligan and Mitchelmore, 2009).

A pedagogy that builds on learners' intuitive understanding of structure can be explored and developed by teachers in the domain of arithmetic (Radford, 2012). This will help enhance a conceptual view of the algebraic behaviour of generalized structures, particularly when working with number (Warren and Cooper, 2005; Threlfall, 1999). The notion of sequencing items provides the opportunity to explore structurally with young learners, and then guide them to use this explicated structure to create a new generalised object that behaves in the same way across other items that have similar structural extensions (Radford, 2012). The enhancement of the arithmetical experience, through a focus on structure within and between numbers, will serve to develop in learners the skill of reasoning algebraically (Carraher, Schliemann and Brizuela, 2002; Subramaniam and Banerjee, 2004; Warren, 2005; Kaput, 2006). Abstraction of patterns is the 'basis of structural knowledge, the goal of (relational) mathematics learning' (Warren, 2005, p.4). Mason, Stephens and Watson (2009) argue that through structural thinking an individual will develop a deeper awareness of the way in which relational mathematical properties are used, explained and connected. In order to gain this deep awareness of the mathematical properties, learners will need to develop a thorough understanding of both the arithmetic and algebraic concepts embedded in the context of sequencing. Subramaniam and Banerjee (2004) maintain that learners' understanding of arithmetic and algebraic expressions are inter-connected and, as a result, many of the difficulties which learners face when learning algebra arise out of a poor understanding of relational arithmetic knowledge. An extension from arithmetic to algebra requires the learner to understand the three-fold meaning of arithmetic expressions. These are:

- Process: procedure and conceptual ideas used when working with arithmetic expressions

- Product: the meaning of the final solution produced at the end of the process
- Relation: the structural way in which the numbers and symbols relate to one another.

It is thus appropriate to consider ways in which the teaching of primary mathematics could take on a pedagogical approach that enhances the use of structure to develop prowess in relational awareness. This study seeks to understand how the use of structure in the domain of sequencing enhances the relational approach to the teaching of mathematics at FP level.

Concepts Related to Pattern

The awareness of pattern and structure is integral to the development of subsequent mathematical understanding which builds toward later learning of formal algebra (Mulligan et.al., 2011). This view takes pattern to a level that is more than just a sequence of items, held together by some surface regularity; it becomes the search for mathematical variation, regularity and structure (Clements and Sarama, 2009) that reaches across domains in mathematics. Pattern, in this sense, addresses the underlying presence of some regularity that gives meaning to the relational aspects embedded in the object of learning. This deliberate search for pattern develops logical thinking processes that work with mathematical objects and sequenced items in a relational way. In the FP, the opportunity to develop the basic skills to seek relationally that which gives mathematical objects meaning and shape, lies in structurally exploring in the domain of sequencing in CA 2. In the following sections I will discuss concepts that are related to the notion of sequencing in general, and the principles that inform these ideas about pattern.

Pattern Through Rhythm and Regularity – the Repeating Pattern as a Cognitive Tool

From an early age learners are sensitive to patterns in actions, behaviours, visual displays, musical tunes and so on (Clements and Sarama, 2009; Liljedahl, 2004). Patterns for young learners are at first experienced and perceived as rhythm and rhyme. This rhythm is sustained by the repeated nature of pattern (Zazkis and Liljedahl, 2002; Clements and Sarama, 2009), sometimes also referred to as its cyclic structure (Liljedahl, 2004). The recognition of rhythm enables learners from a young age to continue the pattern and also to design their own patterns (Threlfall, 1999). In a study conducted by Radford, Bardini and Sabena (2006), it was demonstrated how a Grade 9 student and her group used rhythm as a subtle semiotic device which helped learners project regularities which in turn helped them to work towards

generalising ideas in mathematics. Rhythm creates the expectation of a forthcoming event (You, 1994) and constitutes a 'crucial semiotic means of objectification to make apparent the feeling of an order that goes beyond the particular' (Radford, 2010, p.14). Rhythm regularity is thus explicated through learners' application of this regularity to express the generality in the set of sequenced events, far beyond the perceptual (Radford, 2006; Radford, Bardini and Sabena, 2006).

Learners are able to see and use the rhythm of pattern to create similar patterns, long before algebraic relationships and structures are identified (Zazkis and Liljedahl, 2004). Warren and Millar (2010) researched cognition of patterns amongst indigenous learners through the facilitation of tasks that required learners to copy, continue, complete and create their own patterns. They found that in order for learners to be able to recognise the structure of a repeating pattern they were first required to be capable of 'identifying the rhythm of the pattern and breaking the rhythm into the repeating component' (p.600). This allows for rhythm patterns to be used as a cognitive tool[10]. This identification with rhythm allows learners to identify and continue sequenced sets (or actions) at an early age (Papic, 2007). They identify the underlying feature of these patterns as cyclicality (Liljedahl, 2004), whether it is a clap – stomp – clap – clap – stomp, clap – stomp – clap – clap – stomp... or an array of manipulatives red – blue – red – red – blue, red – blue – red – red – blue ... or the letters ABAABABAAB... or arrays of objects that repeat or grow with identical regularity which is preserved across items. Radford et.al. (2006) argue that rhythm is an important semiotic device through which learners make 'apparent the perception of an order that goes beyond the particular figures' (p.4). Teachers can gain access to this ability (Papic, Mulligan and Mitchelmore, 2011; Mason, Stephens and Watson, 2009) of young learners and to build towards a habit of mind that supports the search for regularity and generalisation in mathematics through applying rhythm.

Deconstructing Pattern in Sequenced Sets or Items: Repeat and Growth

The literature on pattern shows that there are two main types of sequenced sets that are studied in the context of elementary mathematics classrooms (Threlfall, 1999; Liljedahl, 2004; Zaskis and Liljedahl 2004; Papic, 2004). The two broadly defined categories, according to Liljedahl's (2004) work, are repeating pattern and number pattern. In sequenced

[10] When technology or a phenomenon (in this case, rhythm) is used to assist learners' learning and problem solving, they become cognitive tools (Lui and Bera, 2005).

sets, based on repeating patterns, Liljedahl (2004) identifies two principles at play when identifying the next element in the pattern: (i) the equality between every element in the pattern and one of the first *n* elements and (ii) the element *n* positions prior to it (p.2). These two principles represent the enactment of cyclic structure using the core of the repeating pattern. In this sense the repeated pattern ABABAB... has a core (unit length) of two elements AB and in this case the form of the pattern will be AB which is then isomorphic to all other patterns with the same form[11] (eg. 121212...or clap-stomp; clap-stomp; clap-stomp). Liljedahl proposes a definition for sequences that considers repeat to be an organising principle, as a 'sequence that possesses a cyclic structure that can be generated by the repeated application of a smaller portion of the pattern' (Ibid, p.3). Threlfall (1999) terms this smaller portion a 'unit of repeat'. He cautions that it should not just be the rhythmic nature, but also the unit length that is repeated, which should comprise the primary cognitive tool. The utilisation of both the rhythm and the unit structure will develop the conceptual understanding of repeated patterns and give learners the necessary tools to think about these patterns.

Liljedahl's (2004) definition of number sequences emphasizes the numerical value:

> 'In a number pattern the numerical value of the elements is important and it can therefore not transfer to a non-numeric context without loss of some crucial property of the pattern' (p.2).

Papic (2007) refers to sequences that are formed by not applying a cyclic structure as patterns that can be classified as *growth* patterns. These growth sequences can be formed by using shapes, objects (see Table 2.2) or numbers and can be explored in a uni-variate manner, which supports the notion of thinking recursively about sequenced items and the patterns embedded in them. Growth sequences can also be explored through explicating the co-variant relationships embedded in the underlying patterns that exist in the sequence, and this search for co-variance leads naturally towards developing functional thinking in the domain of sequencing (Blanton, 2008; Kieran, 2004; Driscoll, 1999)[12]. Radford, Bardini and Sabena (2006) illustrate these two uses of sequencing (recursion and co-variance) by means of an example derived from their empirical study conducted with Grade 9 learners. In this particular instance one of the learners searched for a generalisation of the sequence presented

[11] More examples of sequences with the same form are given in Table 2.3.
[12] More examples of such patterns are found in Chapter 7, p. 234 -Figure 7.2 (co-variance) and Figure 7.8, p.268 (recursion).

during the study (see Table 2.2 for details) by looking for repeat and growth in the given sequence. This study will work with both growth and repeating patterns, and will explore these using shapes, objects and numbers in sequenced sets. The use of variance in complexity, and also the way in which sequenced sets are investigated structurally to elicit underlying patterns, will be examined. The next few sections provide some insight into the structural aspects that are embedded in repeating and growth sequences.

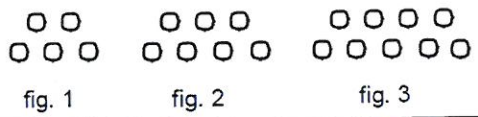
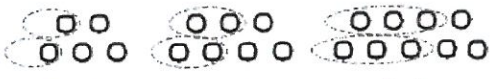
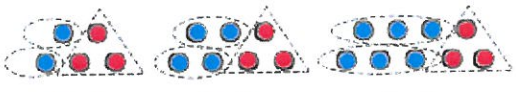
The Given Figure		
Searching for regularity		Looking across the consecutive figures, grouping the last three dots together since they appear to repeat across the figures in exactly the same way. Then grouping the remainder of the dots together in two rows as they show a constant growth from one figure to the next across the items. This is done recursively, from one figure to the next.
Searching for change		
Expressing the generalised sequence structure		Now the general sequence can be expressed by looking at the relational features of figure number and growth item: one, one plus three; two, two plus 3; three, three, plus three…

Table 2.2: Working relationally towards a generalised expression using recursion and co-variance as the tools that express the structural aspects of the sequenced dots.

Structure and Complexity of the Repeating Pattern

A sequence with an embedded repeated pattern cannot be described as a growing sequence because its structural aspects are cyclical. Zazkis and Liljedahl (2004) define the pattern ▲■●▲■●▲■●... as a repeating pattern using shapes with a unit repeat length of three. This type of pattern is referred to in the CAPS (2010) as a geometric pattern, as it uses geometric shapes in the set of sequenced items. The exploration of the given pattern can be extended beyond the surface level to include the structural aspects of *core* and *form* of the pattern. The core of a repeating pattern is the shortest string of elements that repeat. This core is always fully repeated and never partially shown. The form of a repeating pattern is the component structure and runs across patterns with the same number of core elements which repeat in the

same way. Form thus indicates the various elements in the core pattern and how they are arranged. We use letter symbols to exemplify the form. The core of the above repeating pattern can thus be articulated as ▲■● (identified by looking at the equal elements in the pattern and the elements that preceded them) and the form of this pattern can be conveyed using the letter symbols ABC (identifying the string that repeats in all elements across the sequence in any position). Similar patterns can now be created using this form, and the pattern 123123123, with core elements 123, and ☼☺♥☼☺♥, with core elements ☼☺♥, are examples of two patterns with the same form as ABC but different core elements.

Liljedahl (2004) describes the form of a repeating pattern as the creation of an isomorphism between repeating patterns which converge in the expression of form. The pattern 123123123 is not seen as a number pattern, but rather as a repeating pattern using numbers. Repeating patterns change in complexity through the addition of variability in single or multiple layers. The complexity increases as more items and variability are added to the core of the pattern, where variability may consist of colour, orientation or shape. Table 2.3 shows a breakdown of the attributes that define the complexity and the variance of repeated patterns.

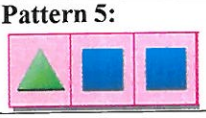
Pattern Core	Complexity (Simple/Complex)	Variability (Single/Multi)	Form
Pattern 1: 	**Simple**: Two elements in the core of the pattern	**Single**: Same colour/different shape	**AB**
Pattern 2: 	**Simple**: Two elements in the core of the pattern	**Single**: Same shape/different colour	**AA**
Pattern 1: 	**Simple:** Three elements in the core	**Single:** Same colour/different shapes	**ABB**
Pattern 4: 	**Complex:** Two elements in the core of the pattern.	**Multi:** Different shape/different colour	**AB**
Pattern 5: 	**Complex**: Three elements in the core of the pattern	**Multi**: Different shape/different colour	**ABB**
Pattern 6: 	**Complex**: Three elements in the core of the pattern	**Multi**: Different shape/different colour	**ABC**

Table 2.3: Illustrating form and the levels of complexity and variance in repeated pattern items.

A pattern consisting of two or three core elements with single variability, is considered to be a simple pattern. Patterns with two core elements and multi-variance is considered a complex pattern. Complex patterns thus start with two core elements and multi-variance, and continue

to three core elements with multi-variance. Patterns that have four or more elements in the core, irrespective of variance, are considered complex sequences.

The important constructs that can be made available through teaching are the notion of the *core* elements of a repeated pattern and the *form* of the repeated pattern. These two mathematically grounded concepts in repeating patterns enable the emergence and use of cyclic structure to extend learner activity beyond merely copying and repeating a collection of shapes or objects. They afford learners the opportunity to create their own cyclic mathematical patterns based on a sound understanding of key mathematical constructs in the domain of repeating pattern, and provide significance and mathematical depth to the conception of a repeating pattern. In turn, this develops the algebraic character of repeating patterns as 'patterns can be symbolised and represented in different ways' (Papic, 2007, p.2) and resultantly foster the process of algebraic thinking since every pattern is a type of generalization in that it involves a relationship that is 'everywhere the same' (Papic, Mulligan and Mitchelmore, 2011, p.3).

Structure and the Growth Pattern

A sequence that possesses a growth structure, consisting of shapes that increase in number, is not a repeating pattern, despite the fact that the shapes are repeated, since the primary attribute is the growth which is evident in every newly formed consecutive item in the sequence. Sequences that are formed by a growth pattern can represent variation of one data set and can also be representative of a co-variant relationship between two data sets (Warren, 2005, Warren and Cooper, 2006; Blanton, 2008). Growth patterns can possess an additive structure (constant growth sequences: 2; 5; 8; ...) where the growth can be positive (constantly increasing sequence items) or negative (constantly decreasing sequence items). Growth patterns can also be based on a multiplicative structure (sequenced items that double as the pattern grows: 4; 8; 16; ...) and can also include sequenced items based on shapes that change or vary using growth as a defining principle. Sequences formed by growth patterns using shapes are illustrated in Table 2.4 on the following page.

Sequenced Items	Description of Growth
Pattern 1: 	With each consecutive item, there are another two congruent shapes added to the array; one to the left of the bottom row, and one on top of the right column.
Pattern 2: 	With each consecutive item, the number of shapes in the array is increased by adding one congruent shape for each of the shapes.
Pattern 3: 	With each consecutive item, the squares grow in side length which increases their area with each new item.

Table 2.4: Illustrating the use of geometric shapes in growth patterns.

Sequenced items that consist of shapes that grow according to some regularity possess *spatial structure*. Battista (1999, p.418) defines the act of spatial structuring as:

> The mental operation of constructing an organisation or form for an object or set of objects. It determines the object's nature, shape, or composition by identifying its spatial components, relating and combining these components, and establishing interrelationships between components and the new object.

Spatial reasoning involves being able to locate and move objects, either physically or mentally, in space (National Research Council, 2006). Spatial structure and reasoning engage three main components of mathematical cognition: spatial visualisation, geometry (shapes) and spatial orientation (space) (Van Ness and De Lange, 2007, p.6). A learner's ability to comprehend and work with spatial structure is essential for the development of basic numeracy skills such as ordering, comparing, generalising and classifying (Papic and Mulligan, 2005).

In the FP classroom, the growth patterns that are most commonly used are number sequences that branch off and maturate from counting activity. These sequences consist of constant growth patterns that are additive in structure and, more often than not, rely on recognition from counting activity. They conflate counting and pattern, do not build towards a structural notion of number and consequently fail to encourage generalisation based on structural extension. Sequencing activities that resemble (skip) counting, more often than not, rely on cues that are activated through proficiency in skip counting and counting-on activities, and do not work with the relational aspects that foreground structure (Liljedahl, 2004). Learners can, however, learn to recognize the relationship between different representations of the same

pattern, which is crucial to generalising and identifying structural characteristics through number sequencing (Clements and Sarama, 2009; Papic, 2007). An example of such generalisations that rely on different conceptions of the structural features, is the example taken from Radford (2006) on page 35, Table 2.2, and illustrated in Table 2.5 below:

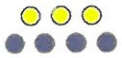
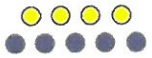
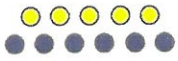
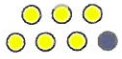
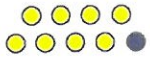
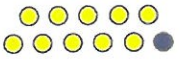
Figure 1	Figure 2	Figure 3	Figure 4	
				The starting number of dots is referenced, followed by the growth number added recursively to each consecutive item.
5	5 + 2	5 + 2 + 2	5 + 2 + 2 + 2	
				Each row is calculated, and then a growth of one is added to each row recursively as progress is made between the consecutive terms.
2 + 3	3 + 4	4 + 5	5 + 6	
				Here the number of dots in the first row is referenced in the second row, which has one more dot than the first row. This can be arrived at by applying either multiplicative or additive reasoning, or both.
2 + 2 + 1	3 + 3 + 1	4 + 4 + 1	5 + 5 + 1	
2 x 2 +1	2 x 3 +1	2 x 4 +1	2 x 5 +1	

Table 2.5: Different structural decompositions of the same pattern, all leading to the generalisation of the same sequence.

Sequencing as a Pedagogical Resource

A *pedagogical resource* is a resource or a concept that is used to shape and develop a mathematical idea. An awareness of the sequencing principles that define a repeating pattern and growth pattern is important. Liljedahl (2004) argues that pedagogically and mathematically the distinction between the two sequencing principles is blurred when activities that are repeating in nature, and at the same time operating in number, are used in class. One such example would be counting in fives and using rhythmic repetition to enhance the pattern – 1,2,3,4,5 (clap) 6,7,8,9,10 (clap)...(Ibid, p.5). He points out that the rhythmic clapping is a repeated pattern by nature, whilst the numbers are not. Liljedahl (2004) advises that learners' attention must be drawn to the distinction between repeating pattern and growth pattern, in order to avoid blurring the two types of pattern.

Watson (2000) points to pattern activities that read *with the grain* or read *across the grain.* When reading with the grain, one focuses on the relationship between two consecutive terms – a recursive view that foregrounds the change that moves you from one item in the sequence to the next. When looking at number patterns across the grain, one is foregrounding a vertical relationship based on the position of a term in the sequenced set, generalizing by looking

across the direction of the pattern, and thus using co-variance which develops functional thinking. It is important to develop an understanding of both approaches to sequencing activity, with and across the grain, in order to develop conceptually the structural extension of pattern activities. In the primary years both repeating and growth patterns are studied as 'with the grain' tasks (Liljedahl, 2004) and it is essential that the focus also shifts to a functional view of sequencing at FP level. In a more recent empirical study involving ten teachers and their senior primary school learners, Wilkie and Clarke (2016) researched the development of learners' functional thinking by visualising the structure of a growth pattern in different ways using different cognitive tools, and extending this visual experience to generalise the functional relationship between two changing quantities. Wilkie and Clarke (2016) adapted a framework from Markwoth (2010) and formulated the learning progression framework for the development of functional thinking with growth patterns as follows:

1. Extend a growth pattern by identifying its physical structure, features that change, and features that remain the same
2. Identify quantifiable aspects of items that vary in a geometric growth pattern.
3. Articulate the linear functional relationship between quantifiable aspects of a growing pattern by identifying the change between successive items in the sequence (recursively)
4. Generalise the linear functional relationship between aspects of a growth pattern by:
   4.1 describing the relationship between a quantifiable aspect of an item and its position in the sequence (covariance)
   4.2 using symbols or letters to represent variables; or
   4.3 representing the generalisation of a linear function in a full, symbolic equation.
5. Apply an understanding of linear functional relationships between variables to further pattern analysis and multiple representations.

(Ibid, p.2)

The illustration in Table 2.5 (p.38) is an example of such multiple representations which use colour as a cognitive tool to elicit the structural relationships embedded in the array of dots. The colour that is used in this way indicates pathways that can be followed to work towards the general in the sequence of dots by following the framework above.

Observing Regularity and Change: Generalising Mathematical Patterns

Radford (2012) provides insight into how the linkage of spatial and numerical structures constitutes an important aspect in the development of algebraic thinking. Learners are guided towards objectifying the numerical-spatial regularity and then using this new object to deal

with items beyond the perceptual[13]. Regularity gains visibility through carefully sequenced instruction and exploration under the guidance of the teacher, who draws attention to the need for structural extension and access to those sequenced items beyond the perceptual. Radford (2012) maintains that algebraic thinking does not appear in ontogeny[14] by chance, nor does it appear as the necessary consequence of cognitive maturation. The conceptual development of algebraic thinking relies on the learner's ability to co-ordinate numeric and spatial structures, which entails a complex relationship between speech, forms of visualisation and imagination, gesture and activity on signs (Ibid, p.14).

Layers and Types of Generality

The ability to generalise can be inculcated by working with sequenced items and investigation through structural extension, or by taking a functional view of number relations, especially in the domain of sequencing. It is structure that allows ideas to be generalised and expressed as patterns. Mulligan and Mitchelmore (2009) and Mason (1996) argue that young children, who have experienced and used regularity and change, develop a habit of mind with regard to the structural aspects of sequencing. This formation of habits advances the numeracy agenda in mathematics and leads the search for similarities and differences when new patterns are encountered. It is this focus on variation, in and across patterns, that leads young learners to generalisations across items that behave the same.

Radford (2010, p.6) suggests the following definition for the concept of generalisation:

> Generalising a pattern algebraically rests on the capability of grasping a commonality noticed on some elements of a sequence S, being aware that this commonality applies to all the terms of S and being able to use it to provide a direct expression of whatever term of S.

A study reported in Radford (2010) involved activities that were designed to work on three levels. Firstly, learners had to conduct an arithmetical investigation by extending sequences based on given information and then extending this to items that were beyond the perceptual realm (for example items 15 or 25 or 100). Thereafter, the learners had to express some generality using natural language (in the form of a message to someone) and, lastly, this generalisation had to be expressed using standard algebraic symbolism. This study gave rise

[13] Table 2.5 on p.38 illustrates how colour is used to search for variation amongst the items in the sequenced set and then turning these special features into numerical instantiations of the physical array of dots.
[14] Emerging over time in the course of development.

to a classification of the levels and types of generality that emerged from learners' working on sequencing activities that have an embedded spatial structure. Radford (2010) distinguishes between induction and generalisation, when learners deal with sequenced items, and warns that the two processes are different. He termed the induction *naïve* when learners' attempts at generalising rest on a rule that is formed by guessing. He proceeded to classify generalisations into two broader categories. The first is arithmetical generalisation, where generalisation happens in the realm of number and is based on observing a commonality, without providing an expression that gives information about any random term in the sequence. An arithmetical generalisation rests on the formation of a genus. The second classification is that of algebraic generalisation, which rests on the genus evolving into a schema; hence the formation of a rule that is expressed using algebraic symbolism. An arithmetic generalisation is formulated by looking recursively (with the grain) at the sequenced set while working with covariance (across the grain) can lead to the formulation of an algebraic generalisation.

Generalisation and Objectification as outcomes of Structural Extension

Schliemann, et. al. (2008), in a longitudinal classroom study of young learners' thinking and representations as they engaged with early algebra activities, found that learners are capable of learning and understanding elementary algebraic ideas and representations as part of the early mathematics curriculum. At the outset, the reasons and justifications for forming and manipulating expressions should be a major emphasis of instruction (Kaput et al., 2008). Booth (1988) has suggested that algebraic generalisations should be integral to the arithmetic curriculum, while Carraher, Schliemann and Brizuela (2002), in a study of Grade 3 and 4 learners, found that 'opportunities for generalising and for thinking about functions and variables and using algebraic notation, abound in elementary mathematics' (p.10).

The CAPS (2010, p.14) for FP CA2 uses the concepts of 'algebra as a language', 'communicating mathematics', 'study of functions', 'relationships between variables', 'manipulative skills', 'describing relationships', 'using symbolic expressions', 'identification and analysis', 'regularity and change' and 'patterns and relationships' to describe the general content focus. All of these concepts necessitate a focus upon both the relational and the general in the mathematics class at FP. The relational and objectification processes are operationalised by the actions 'copy', 'extend', 'describe' and 'create' through a focus on 'number and geometric pattern', working particularly with the notions of *grasping* and

understanding the *logic* of the pattern. The language and the descriptors that are used to elaborate on the FP specific content focus for patterns, functions and algebra draw attention to the use of structure (logic[15]) and more abstract representations of arithmetic to objectify arithmetic processes. This is to be achieved through choosing pedagogical tools that assist with identifying, analysing and expressing both regularities and change when working with sequenced items.

The generalisation of sequenced items as a route to algebra relies on the 'natural correspondence between algebraic thinking and generalisation' (Radford, 2006, p.4). There is a developing body of research that addresses the different ways to generalise. Lannin (2005) proposes two types of generalisation: 'recursive' which involves a single variable and works with the grain, and also 'explicit' which describes a co-variant relationship which is generated by looking across the grain. Generalisation is then further expanded by Radford (2010) who formulated layers of generality by identifying stages of learner progress as they engage with activities that lead to generalisation. These layers are:

- *factual* generalisations where learners engage mainly in gestures, words and perceptual activities and do not move onto generalising outside of the perceptual.
- *contextual* generalisations where learners reduce the signs through semiotic contraction to move towards a greater expression of meaning and generalising outside of the perceptual.
- *symbolic* generalisation where a further semiotic contraction progresses learners to using symbols, such as letters, to express the generality and thus replacing words with symbols.

This study will make use of these layers of generality and also the stages of structural development (Mulligan and Mitchelmore, 2009) to investigate the teaching of pattern in the FP class.

Teacher Knowledge and the Teaching of Algebra

The transition from a procedural approach in arithmetic to the development of a structural understanding in EA does not come easily (Kieran, 1992). Research has shown that

[15] There is no elaboration provided in the CAPS with regard to the *logic* of a pattern. It is taken to mean the relational features embedded in a sequence.

elementary mathematics teachers generally do not have a deep understanding of the mathematics that they are required to teach (Van Es and Conroy, 2009; Phillip et al., 2007; NRC, 2001; Ball 1990). Elementary teachers need their own experiences with richer and more connected algebra, and require an understanding of how to create opportunities with, and for, their learners (Blanton and Kaput, 2003). Without the pre-requisite content and pedagogical content knowledge on the part of elementary teachers, meeting the objectives of EA for learners is unlikely. To meet the goals of teaching algebraic reasoning in the primary (elementary) school curriculum, we need to understand more about how teachers are prepared for this undertaking (Hallagan, Rule and Carlson, 2009). In order to produce an environment where learners develop skills that help them to generalise their mathematical thinking, express and justify the generality, teachers have to *algebrafy* instructional materials, identify possibilities and find pedagogies and activities to develop and support algebraic reasoning among their learners (Blanton and Kaput, 2003, p.2). This calls for a specialised knowledge of algebra, algebraic reasoning, structural extension and the conceptual teaching of algebra (NCTM, 2000; RAND, 2003). Teachers' knowledge and practices and their preparation for the teaching of algebra have gone largely unexamined in the research literature (Doerr, 2004; Kieran, 2006; Stein, Kaufman, Sherman, and Hillen, 2011). Doerr (2004) argues that:

> The scarcity of research on teachers' beliefs and cognitions about algebra is accompanied by a lack of knowledge on how teachers interpret and adapt materials in the textbooks from which they teach and on their understandings of students' cognitions related to algebra.
>
> (Ibid, p.268).

Maat and Zakaria (2014) conclude that 'teachers may have more procedural knowledge than conceptual knowledge or PCK, but lack a deep understanding of the knowledge that will impact their teaching' (p. 5). To be an effective teacher of mathematics, an educator must know the content that they will teach in a thorough and connected way (Ball, Thames and Phelps, 2008; Ma, 1999). Researchers appear to be unanimous that the conceptual knowledge of subject matter required for teaching includes much more than a familiarity with, and understanding of, the mathematical concepts, and that a profoundly deeper relational perspective on these mathematical objects is required. Further research into understanding teachers' conceptualisations of algebra, and their interpretations of the concomitant instantiation of algebra in curricular materials, is required, especially at primary school level and also in the South African context. There is a dearth of research that provides insight into

how teachers use the notion of structural extension in the teaching of arithmetic and, since this notion is closely linked to algebraic reasoning, it is necessary to gain insight into what teachers know and believe about algebra, structure and patterns and how these areas develop the notion of EA. A fundamental aspect of EA is developing a sound conceptual understanding of operations and relations instead of the rote acquisition of skills and procedures.

Teachers' Conceptualisations of Algebra

Research over the years (Usiskin, 1988; Bednarz, Kieran and Lee, 1996) has advocated the conception of algebra as (i) generalised arithmetic, (ii) a means to solve certain problems, (iii) a study of relations and (iv) structure. Doer and colleagues (2004) warn that the conceptions of algebra held by academics, curriculum developers and mathematics educators do not correspond with those that are held by teachers. Teachers express difficulty in discussing the nature of algebra and their conception of algebra with colleagues (Johnson, 2001), even after years of experience with the subject (Menzel, 2001; Gadanidis, 2001). Teachers also tend to emphasise the procedural, as the core of algebra, when working with equations that are expressed algebraically (Bishop and Stump, 2000; Haimes, 1996; Menzel and Clarke,1998). The elementary school teachers in Bishop and Stump's (2000) study viewed pattern generalisation tasks largely as problem-solving activities instead of generalising the familiar patterns of arithmetic. In a study conducted by Haines (1996) teachers believed that the difference between algebra as generalised arithmetic and the functions-based approach to algebra was that the latter is burdened by repetition which can easily be omitted. This suggests that the teachers held a diminished view of the conceptual use of graphical representations, using tables to organise and develop covariant relations and also using meaningful contexts. Teachers also reveal a disconnection between function-based activities and the development of learners' algebraic reasoning skills. Kieran and Sfard (1999) caution that these disjunctures, enacted by teacher activity, lead to an impoverished experience of the use of symbolic expressions as tools for meaningful mathematical communication.

Chazan, Larriva and Sandow (1999) draw attention to the necessity for teachers to be involved in professional development programs focusing on building a deeper conceptualisation of algebra. Their study described the knowledge of one pre-service teacher, who advanced a functions-based approach to algebra, and a second teacher, whose conceptions appeared ambiguous. The first pre-service teacher was able to articulate clearly

the tensions she experienced in concurrently understanding the two notions of variable as an *unknown quantity* and as a *quantity that varies* (Leikin, Chazan and Yerushalmy, 2001). A study conducted by Gray, Loud, and Sokolowski (2005) also found that college students of algebra and calculus found it difficult to work with a variable as a generalised number. This continuous interplay between the notions of variables plays an important part when solving algebraic equations and systems of equations formed by combining separate algebraic equations. Developing the concept of 'variable' is a major component in the learning of algebra, and has proven to be problematic when developing the relational aspects embedded in the notion of using an indeterminate[16] to objectify the relational (Koedinger and Nathan, 2004; Moses and Cobb, 2001; Usiskin, 1988). When developing algebraic reasoning skills with young learners in the context of EA, it is the relational aspects that find expression through focusing on structure and making structural relations explicit (Gray et.al., 2005).

A study conducted by Bishop and Stump (2000) found that many pre-service teachers did not understand what distinguishes arithmetic from algebra. The difference between arithmetic and algebraic problem-solving strategies used by teachers in their own problem-solving behaviour and in their evaluations of learners' work was the focus of a study by Schmidt and Bednarz (1997). The study involved both elementary and secondary school pre-service teachers who were divided into three categories: elementary, secondary and learning difficulties teachers. The results were categorised in terms of the academic preparation of the three groups of pre-service student teachers. Across the three groups, the students with more mathematical preparation in their academic programmes, referred to as a structure profile, were more inclined to use algebraic methods to solve problems than those with less mathematical preparation, referred to as an arithmetic profile. The students with an arithmetic profile characterised algebra as a repertoire of arbitrary rules over which one has little or no control, and recognised the sophistication of algebraic solutions over arithmetic solutions, including algebraic solutions for arithmetic problems. The students with a structure profile viewed arithmetic as a more effective tool for solving arithmetical problems, and saw algebra as useful for solving more complex problems.

Dooren, Verschaffel, and Onghena (2002) found similar results when they measured pre-service teachers' problem-solving behaviour at the beginning and at the end of their teacher

[16] An indeterminate is taken to mean an unknown quantity; something whose value is not specified. The term 'indeterminate' is sometimes used as a synonym for unknown or variable (Becker and Weispfenning 1993, p. 188). I use the term to indicate a quantity that is not given and can be represented by a variable.

training. They found that teachers' preparation programs were successful in improving the skills that pre-service teachers use to solve problems, but that their preference for solution strategies did not change. In the same study, they also examined the evaluations that pre-service teachers made about learner solutions to arithmetical and algebraic problems, and found that these reflected the way in which teachers solved the problems themselves. The secondary pre-service teachers gave the highest scores to algebraic solutions and referred to the overall superiority of algebraic methods, regardless of the nature of the problems. The primary teachers gave higher evaluations to arithmetic solutions and tended to notice the arithmetical nature of the problems to justify their approaches to the solutions. Interestingly, about half of the pre-service primary teachers could not solve the more complicated problems on the test items due to their 'lack of understanding of algebra' (Ibid, p.9). They experienced difficulty in comprehending learners' algebraic work and thus evaluated their work negatively. This shows that the pre-service primary teachers did not possess the readiness or the disposition to prepare primary learners in EA.

A study conducted by Nathan and Koedinger (2000) provides insight into teachers' conceptualisation of algebra and their perceptions of learner performance. They found that textbooks exert a particularly strong influence on teachers' perceptions of the difficulties involved in learning algebra, and that this is particularly evident in the way that textbooks sequence the learning of concepts in algebra. The textbooks hold a 'symbol-precedence' view of learners' mathematical development wherein arithmetic reasoning strictly precedes algebraic reasoning, and 'symbolic problem-solving develops prior to verbal reasoning' (Ibid, p.1). The textbooks that are used in class are central to the teaching of EA and to developing practices that engage learners mathematically through fostering a relational understanding in mathematics. Stump (1999) argues that if the textbooks (and other instructional resources) present topics as unrelated concepts, then it is to be expected that teachers will teach topics in a similar manner. Different textbooks implicitly suggest different teaching activities, which reflect varying conceptions of mathematical content (Mesa, 2001), as well as the pedagogical beliefs and practices suggested both explicitly and implicitly by the authors of such texts. Teachers, therefore, need to develop a specialist gaze to discern texts and support materials that enhance their teaching in ways that are conceptually sound, relationally rich and do not develop ideas in mathematics in a disconnected way.

Teachers' Learning to Become Teachers of Early Algebra

Understanding how a novice teacher of algebra becomes an expert provides useful insights into the teaching of algebraic reasoning at primary level. The preparation of teachers to teach algebraic reasoning in the context of EA has not been widely researched (Stein, Kaufman, Sherman, and Hillen, 2011). The few studies available focus mainly on pre-service teachers and teachers at a secondary level. A focus on enhancing teachers' ability to think flexibly and to value mathematical connections is necessary (McGowen and Davis, 2001). This can be achieved by preparing elementary teachers for the teaching of algebra through learning about working with indeterminates, generalisations, problem solving, modelling and the study of functions (Stump and Bishop, 2001; Kaput and Blanton, 2001). The notion of generalisation can be developed by extending arithmetical conversations into algebraic ones, adapting resources for this purpose and enriching lessons to build in connectedness and cohesively extend algebraic themes (Kaput and Blanton, 2001; Schliemann, Carraher, Brizuela, Goodrow and Pered, 2003; Warren, 2005). Changes in pedagogy that reflect a deeper conceptual approach to the teaching of algebra (away from procedural manipulation) are only possible if teachers possess strong subject matter knowledge (Lloyd and Wilson, 1998; Buschang, Chung, Delacruz and Baker, 2012).

With a function-based approach to algebra gaining momentum across the research, it is essential to understand the notion of function and the pedagogical possibilities it holds for studying algebra conceptually. The professional development experiences of teachers should address their subject matter knowledge and their pedagogical content knowledge in relation to the function concept at FP. Several studies (Haimes, 1996; Wilson, 1994; Heid, Blume, Zbiek and Edwards, 1999) have reported that a limitation in the understanding of the function concept can constrain the choice of tasks that teachers use, the complexity of questions that teachers pose and the connections that are made within the curriculum. Although these studies focused on secondary school algebra teaching, the lack of a deeply conceptual understanding in the FP of algebraic concepts, such as function and the concomitant covariant relationships, needs more research. A conceptual approach to functions in EA emphasises co-variance, as well as the notion of dependence, and uses graphs and tables to understand patterns of co-variation embedded in mathematical objects. A relational view also uses multiple representations flexibly to increase understanding and promotes meaningful mathematical discussions with learners that encourage deeper exploration as opposed to a shallow focus on surface features (Lloyd and Wilson, 1998; Chazan, 2000). Co-variance is

also a very powerful pedagogical tool with which to study processes of generalisation and objectification. In addition, tables and T diagrams serve as useful pedagogical instruments to organise and display co-variance in a structured manner.

Teachers' self-conception as teachers of mathematics was found to play a role in the development of their expertise in teaching the notion of an indeterminate in ways that enhanced learner understanding (Agudelo-Valderrama, 2000). Teachers' learning through their instructional practices was also described by researchers Smith (1997) and Miller (1992), who observed how teachers learn through planning, actual instruction, reflection on the instruction and using writing prompts in class which make learners' conceptions available to the teacher. Without the prerequisite content knowledge on the part of (pre-service) elementary teachers, meeting the objectives of EA is unlikely (Hallagan, Rule and Carlson, 2009). Ongoing professional development of in-service teachers can ideally use their classroom experience as a cognitive foundation and build new learning experiences that enhance their understanding of their own practices and beliefs.

Summary

Research has shown that pattern as a cognitive and pedagogical tool can provide teachers with a useful way in which to develop and organise early mathematical thinking. Particularly EA can provide young learners with opportunities to think structurally through using rhythm, core and forms of cyclic pattern that repeat themselves. The notion of regularity can be cultivated through carefully designed activities that enhance the need to identify regularity and to describe the emergence of the general. Teachers can tap into learners' natural inclination to seek patterns and regularity, which can support through a focus on structure in arithmetical exploration. The literature describes various notions of structure (surface, systemic, cyclic, spatial, extension, decomposition, form) and describes structure sense and knowledge at varying levels of awareness of mathematical form. Teachers are advised that an exploration of structure and relationships in the early grades contributes to learners' being able to operate with an indeterminate and use and understand symbolic representation. The need to *algebrafy* arithmetic through focusing on structure in arithmetic is foregrounded, together with algebraic habits of mind, as these serve as essential pedagogical tools in EA.

Research with young learners has shown that an emphasis on structure develops conceptual knowledge, advances numeracy levels and improves learners' mathematical performance

throughout their schooling. Structure is shown to bridge the gaps between conceptual and procedural approaches to teaching and learning. The four stages of structural development describe the structural landscape in which learning takes place. Teachers need to be aware of the structural activity that learners engage in, as a focus on structure is positively correlated to the promotion of thinking that leads to generalisation.

Functional thinking and generalised arithmetic, together with the study of relations and change, are suggested as vehicles to advance algebraic reasoning at an early age. The research all points to the need to advance the triad of pattern, structure and algebraic reasoning. Teachers need to be supported to become aware of how learners work with the general and which types of activity lead conceptually to a generalised object of learning. This includes working with an indeterminate, being aware of the layers and levels of generality and the semiotic resources that are used in the process of objectification of knowledge. Resources must contain activities that enhance structure in pattern and operationalise such structural decompositions and actively seek to generalise relational attributes. This will require that the authors of instructional texts understand that EA is possible at FP level.

It is in this context that my study focuses on the FP and the development of mathematical habits of mind that support the move towards using structure, in the context of pattern (CA 2), to promote the goals of EA in Grade 2 and develop the algebraic reasoning skills of young learners. Research on the use of structure by teachers in the FP class in South African schools is uncharted territory. While the current body of research in this field has focused on using structure in Grades 6 and 7 (Liebenberg et.al.,1998) and on curriculum specifications and EA (Vermeulen, 2007; Roberts, 2010), my study seeks to investigate how FP teachers, through their pedagogical approaches to sequencing and their use of structure, can advance EA in the South African classroom.

In the next chapter, the research design and methodology is described in detail. Details of the participating teachers, the lesson data and the documents under review, as well as the series of intervention workshops designed to advance a structural approach to the teaching of sequencing in FP, against the background of EA, will be given.