GRADE 10 LEARNER ERRORS WHEN OPERATING ON INTEGERS

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DECLARATION OF ORIGINALITY

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I, Sheldon S. Naidoo declare that this research study is my own work, and no part has been copied from another source (unless indicated as a quote). It should be noted that this study is an add-on to my Honours research project, hence some information is similar. All phrases, sentences and paragraphs taken directly from other works have been cited and the reference recorded in full in the reference list. This research study is being submitted to meet the requirements of the course Research Project for the degree of Masters in Education in Mathematics in Education at the University of Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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ABSTRACT

This study focuses on learners' errors in integers and explores the changes in their errors during their grade 10 year. The data utilised in this study was collected by the WMCS project in 2018. The data was collected from learners of 15 low-performing schools in Gauteng, South Africa. Part of this study consisted of a comparison of two groups of learner responses. The first group comprised of learners of teachers who participated in a professional development course offered by the project in 2016/2017 (TM group) and the second group are learners whose teachers did not attend the course (CNT group). These schools had no prior relationships with the Wits Maths Connect Secondary (WMCS) project. I looked at the responses that grade 10 learners made when ordering and operating on integers at two different points of a school year.

The tests were written in February and September of the same year and were analysed from a random sample of 196 grade 10 learners from 40 low-performing secondary schools in Gauteng, South Africa. Test items dealt with ordering of integers and operating on integers. Working from a Vygotskian perspective, a framework dealing with concept formation of negative numbers was developed to analyse learners' test responses. In this study, I focused solely on the responses given by the learners of the two groups and explored the persistency of errors within each group as well as the type of errors found across the groups. Having this data point per learner allowed for further investigations regarding persistent errors and errors which were remediated and waned during the year. Furthermore, it provided an opportunity to detect new errors arising which would impact the learning of other topics.

This framework made it possible to identify seven types of errors. The findings show that while learners' performance improved from pre-test to post-test, many grade 10 leaners still have difficulty with the concept of negative numbers.

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May each of you be blessed.

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CHAPTER 1: PURPOSE AND PLAN

1.1 Introduction

This study focuses on the errors grade 10 learners make when ordering and operating on two term integer items. The key component of this study was to develop a framework that allows for an error analysis. This framework would then be used to identify the thought processes a learner might have on the minus symbol when operating on integers and their possible understanding of the concept, negative numbers. The decision to focus on the minus symbol and negative numbers was based on my previous study, which is discussed in greater detail, later in this chapter. In my previous study, the findings revealed the poor results of grade 9 learners, from low-performing schools, on the integer items. This led me to question the performance of grade 10 learners in the same test. I quickly realised that this choice was not as simple as anticipated, as I encountered numerous challenges, which compelled me to reflect on what I was doing.

The idea of building on my previous study became my greatest challenge; the mindset I had which was not commensurate with the level of a Masters' student. The major factor that influenced how I analysed errors was the understanding of error analysis, and what it meant for research. Borasi, (1987) propounds that, teachers' use of error analysis can assist them to develop strategies that could lead to remediation of such errors for future lessons. In research, such as this, error analysis on learners' responses can be used to gather insightful information (Borasi, 1987). Both the teacher and researcher can use errors to determine the development of a concept and what should be done to rectify these errors.

Through error analysis, I looked at the persistency of integer errors over an academic year. As this research progressed, there was a shift from a Piagetian perspective to a Vygotskian socio-culturalist approach to learning. This was a major turning point for me (described in Chapter 2). In brief, the Piagetian perspective focuses on the idea that an error is the product of a misconception (Olivier, 1989) the shift was to now look at an error because of a learner lingering in their development of truly understanding the concept.

Integers is a topic that is fundamental to secondary phase mathematics, being introduced in grade 7 and continues through to grade 9 as a topic on its own in the curriculum but then appear in almost all topics in some manner. For example, a basic numerical question that focuses on integers, such as $3 - 4$, can be found in a similar form in algebra, such as simplifying the expression: $3x - 4x$. Another example is working with coordinates off a Cartesian place

$(-4;3)$ and $(2;\ -5)$ and asking learners to find the distance or gradient of the two points. Integers consists of both positive and negative numbers, however for this study the focus is on the negative numbers.

1.2 Background of the Study

Having tutored learners from various grades, I noticed the frequent occurrence of learners having trouble with dealing with the minus symbol in various topics. This could be due to the difficulties in understanding the different functions of the minus symbol, and not being able to distinguish between sign and operation. Having the opportunity to do this study under the Wits Maths Connect Secondary (WMCS) project, allowed me to gain significant insights into this topic.

This study looked at the responses that grade 10 learners made when ordering and operating on integers at two different points of a school year. This data was collected by the WMCS project in 2018, from learners of 15 low-performing schools in Gauteng, South Africa. Part of this study consisted of a comparison of two groups of learner responses. The first group comprised of learners of teachers who participated in a professional development course offered by the project in 2016/2017 (TM group) and the second group are learners whose teachers did not attend the course (CNT group). These schools had no prior relationships with the WMCS project.

In this study, I focused solely on the responses given by the learners of the two groups and explored the persistency of errors within each group as well as the type of errors found across the groups. Having this data point per learner allowed for further investigations regarding persistent errors and errors which were remediated and waned during the year. Furthermore, it provided an opportunity to detect new errors arising which would impact the learning of other topics.

The errors identified would also allow for predictions on the development of understanding negative numbers. I did not focus on the professional development course in this study. However, the comparison was to see if there were any differences in the types of responses by the different learner groups and to offer guidance to the project in developing a modus operandi of aiding these low-performing schools concerning the teaching and learning of integers.

1.3 My Previous Research

In 2019 I had conducted a similar study under the WMCS project. In this study, I analysed the errors that grade 9 learners made when operating on integers, by perusing the responses given by 58 learners. The test items studied were the same items used in this study (Appendix A). The grade 9 learners had written this test twice in the same academic year (February and September). The first item focused on ordering integers, and the next five items focused on operating on integers. Table 1.1presents the important findings from the study, the first row represents percentage of learners that had got all items correct and the other rows represent the percentage of the most common error given for each item.

Table 1.1: Correct response and most common responses made by grade 9 learners.

Item	**Responses**	**February Test**	**September Test**
	Correct responses	33.6%	31.8%
[redacted]			
1.	$-2; -10; -35; -500; 4; 30$	22.4%	15.5%
3a.	$5 - 7 = 2$	25.9%	17.2%
3b.	$5 + (-7) = 12$	25.9%	19%
	$5 + (-7) = -12$	22.4%	32.8%
3c.	$-5 + 7 = -12$	27.6%	29.3%
3d.	$6 - (-10) = -4$	29.3%	29.3%
3f.	$-7 - 5 = -2$	36.2%	32.8%

Less than 34% of grade 9 learners were able to order and operate on integers correctly, this then decreased further to 31.8% in the test written in September. Fewer learners gave answers that were positive numbers by the end of the year, however more learners used the minus symbol as an object that can be detached and re-attached to the final answers. This can be seen in the common responses given for Items 3b (second response), 3c, 3d and 3f. Learners focused more on the operation than the minus symbol attached to a number.

My previous research opened a window to opportunity for more intense and dedicated study to be undertaken and this was one of the driving forces to my current research. Since I know, the same tests were given to grade 10 learners, the motivation and objective was to see the processes grade 10 learners were implementing when operating with integers, and whether they perform better or worse during an academic year. The other motivating factor was to gain greater insight to the possible reasoning of learners that resulted in the errors made. This diagnostic of their reasoning can be used by teachers in understanding learners' responses when marking any form of assessments. The reasoning and codes used in my previous research was at a dilettante level and thus more work was needed to improve its strength of validity.

The final motivating factor for this study was to perform research that could be beneficial to the WMCS project. The comparison of two groups of learners gives the project insightful information regarding the impact the professional development course might have had on the teachers who subsequently conveyed their expertise on the learners in the transition group (TM). However, there is no evidence that there is a direct relationship between the course and the outcome of the responses from the learners in this group and the aim of this study was not to establish a causal relationship between the two. Interviews with the test subjects were not possible since these tests were written in 2018. To enhance the study, it would have been useful to get other grade 10 learners in similar backgrounds, but the fairness would not be achievable since every person thinks differently. In this study the test items will be the same (as seen in Appendix A), however there will be an increase in the sample size and additional research questions are raised to allow in-depth research to take place.

1.4 Research Problem

Based on what was seen in grade 9 from my Honours research, I was curious to see what the grade 10 learners from the same schools might do. Whereas the honours research focused on learners' errors and the possible reasoning, this study has extended the work substantially but building a theoretical framework and analytical tools to provide greater insight and more explanatory power to the research, with greater focus on concept formation and just errors. Grade 10 learners should be proficient in the topic of integers, however if learners at the end of grade 9 are still not grasping this concept, then looking at what the grade 10 learners are doing with this topic is vital. Since integers is one of the fundamental concepts in the understanding of numbers, it is important to see how grade 10 learners operate on this topic.

This research focused more on the results and the errors of the two groups of grade 10 learners. This information will be used for the WMCS project in enlightening them on whether the workshops they provided to teachers were beneficial or not. The project provided professional development for teachers from the low-performing schools in Gauteng. The course included algebra, functions and relationships and integers. It is important to remember that this study is not going to be used to determine the professional development of the teachers that took part of this study.

This study was driven by two main research questions, each with three sub-questions:

1. To what extent are there differences in learner performances between the comparison (CNT) and transition (TM) groups?

a) Are there differences in the performance of each group on the test written in February?
b) Are there differences in the performance of each group on the test written in September?
c) What are the differences in the persistent errors?

2. What errors do learners make when responding to test items on integers?

a) What errors do learners make in the test written in February?
b) What errors do learners make in the test written in September?
c) Which errors persist from the test written in February to the test written in September and what insights can be gained regarding learning the concept of negative numbers?

1.5 Outline and Conclusion

I present this study in six chapters. The following paragraph gives a brief outline of each chapter. In Chapter 2, I discuss the theoretical framework that this study was built on by discussing the possible reasoning behind the errors made by the learners based on prior research. In addition, I link the possible reasoning with the theoretical framework to create a framework that can aid me in interpreting the errors seen. Chapter 3 explains how the study was conducted, it describes the process of the coding system and the trustworthiness of the findings, as well as the interpretations made. Chapter 4 presents the overall performance of the learners by focusing on the correct responses. The use of statistical tools strengthens the comparisons made about each group and each test. Chapter 5 gives an in-depth look at the errors made in each item. These errors are then linked to the conceptual framework developed in Chapter 2. The final chapter presents the answers to the research questions, as well as a general discussion about the grade 10 learners when ordering and operating on integers. The limitations found in this study and further recommendations for another research are also discussed.

CHAPTER 2: A CONCEPTUAL FRAMEWORK FOR LEARNERS' INTERPRETATION OF NEGATIVE NUMBERS.

2.1 Introduction

In this chapter I propose a framework that can be used to analyse and interpret the data. The framework draws together the theoretical tools I have selected and the concept of negative numbers. The journey to develop the framework was challenging and involved many iterations and revisions. I present the final versions in this chapter without providing the details of earlier versions. I have named it '*A conceptual framework for learners' interpretation of negative numbers.* As noted in Chapter 1, this framework is a substantial extension of my previous study and has enabled me to achieve a depth of analysis that I could not reach in my Honours research.

To build this framework, I began with the stages of concept formation based on the work of Vygotsky (1986) and further elaborated by Berger (2004a and 2004b) which are discussed below. Thereafter, I discuss the concept of negative numbers defined by Vlassis (2004, 2008) and Gallardo (2008) and the types of integer errors seen in literature by Vlassis (2004) and Bofferding (2010). The framework summarised in Table 2.2 assisted me in locating the errors being analysed based on the literature. This process allowed me to have a better understanding of how learners might have interpreted negative numbers and where they are in the progression of concept formation.

2.2 Theoretical Framework

The study is founded on a socio-cultural perspective of learning based on Vygotsky (1986) idea of learning and the influence society has on it. To better understand Vygotsky's theory of concept formation in relation to mathematics, I drew on the work of Berger's (2004a; 2004b; 2005) interpretation of the theory of concept formation which aided me in understanding the phases within concept formation that Vygotsky (1986) discussed in relation to mathematics. Berger (2005) explains that a framework is required in which there is a link between the construction of an individual's concept and their social knowledge, to develop mathematical knowledge. Integrating the theory of concept formation with the concept of negative numbers provided a theoretical foundation for me, enabling me to analyse the responses given by the learners. It further enabled me to make claims about where learners are (their level of competency) in their understanding of negative numbers. I make inferences about the erroneous thinking that led to learners' errors, and I also suggest where learners are in their development of this concept.

I have chosen a socio-cultural approach (Vygotsky, 1986) rather than a Piagetian approach since Vygotsky (1986) describes understanding concepts in two ways: spontaneous and scientific concepts. Vygotsky (1986) also looks at the functional use of signs in developing concepts, this is important in this study as it focuses on the concept of integers and the use of the minus symbol. Vygotsky (1986) elaborates more on the formation of concepts and the process that is involved in this formation. Since the participants of this study are grade 10 learners, it would be more beneficial to look at the formation of the concept that the learners already have. The learners have had many influences in their understanding of integers up until this point. These influences may have come from the school but then influences came from their community and their social interactions with them. Therefore, a socio-cultural approach is more appropriate.

2.2.1 Spontaneous Concept verses Scientific Concept.

The theory of concept formation focuses on the development of a concept according to the individual characteristics that an object holds when compared with a group of objects (Vygotsky, 1986). The inclusion of the role of language, social regulations, and social constitution found within the mathematical domain is important in concept formation (Vygotsky, 1986). For instance, the position of a minus symbol in an object could then be referred to as 'subtraction' or 'negative', however through social influences the symbol is most often referred to as 'minus' then differentiation in concepts is not made, for example: -7 (minus seven) or $2-5$ (two minus five), 'subtraction' and 'negative' could then be regarded as the same. The term *object/s*, as used by Vygotsky (1986) referred to the different shapes that were presented to the child, whereby the child then focused on specific characteristics from one shape. In mathematics, an object can refer to mathematical questions like $7-5=$, where the learner then focuses on either the symbols ($-$ and $=$) or the numbers or both.

Vygotsky (1986) distinguishes between spontaneous and scientific concepts. A spontaneous concept is developed through the child's everyday experience whereas a scientific concept is developed through instruction/mediation of some kind (Vygotsky, 1986). For instance: a learner is exposed to ordering positive numbers from smallest to biggest correctly since $2<3$, however when ordering negative numbers, the learner could still say $-2<3$ but $3<-5$ based on the absolute values and not the value of the negative number. Based on this, learners then understand negative numbers as a spontaneous concept due to not differentiating between negative numbers and positive numbers.

To differentiate between negative and positive numbers, a learner would need to understand negative numbers, this means that negative numbers are regarded as a concept on its own. Negative numbers have their own set of definitions and procedures to follow in mathematics (this is discussed in detail later in this chapter) and when it is treated like positive numbers, it is then regarded as a spontaneous concept. Hence, I integrate spontaneous and scientific concepts in the theory of concept formation. Although grade 10 learners have been exposed to the concept of negative numbers, the concept may not have been fully developed. Thus, the concept of negative numbers may be understood at a spontaneous level and still needs to be developed to a scientific concept level (Vygotsky, 1986). This development can occur through the theory of concept formation.

2.2.2 Functional Use of Signs.

The functional use of signs plays a vital role in the formation of concepts (Vygotsky, 1986). Berger (2005) adds on to this, postulating that to understand the theory of concept formation in the mathematical domain, it needs to be linked with the functional use of signs. I refer to the functional use of signs as the role of signs presented by (Sfard, 2000). The appropriate use of signs in the mathematical domain affects the outcome in dealing with mathematical objects (Sfard, 2000). To navigate the way mathematical objects are understood we need to understand the signs that are being represented. Sfard (2000) presents the terminologies *signs and signifiers* to describe symbols. Signs are "anything…experienced as meaningful, whether it is a spoken word, a written symbol, or any other artifact used for communication" (Sfard, 2000, p. 48). Vlassis (2008) elaborates on this and states that signs could refer to either mathematical language or symbols and that the use of these has an influence in the conceptualisation of negative numbers. Based on this, I use the word *symbol* instead of *signs*, hence the symbol ' $-$ 'is then just seen as a minus symbol. Signifiers are the relation a person has with signs and must have a form that relates to the object (Sfard, 2000). Signifiers could represent 'subtraction' or 'negative' dependent on the object.

There are two types of signifiers that Sfard, (2000) introduces, namely, operational, and structural signifiers. An operational signifier denotes the necessity to act, it signals the action that a person needs to perform based on the mathematical object, for instance in $7-5$ the minus symbol is seen as an operational signifier to instruct the learner to subtract 5 from 7 or to get the difference of 5 and 7. A structural signifier, according to Sfard (2000, p.49) “refer[s] to those mathematical symbols that appear in propositions dealing with objects …”, this means that structural signifiers are used to describe objects and they are distinguished based on how the language is used, for instance, -3 is read as negative three, it gives the object (3) meaning, 3 units below zero resulting in -3. Differentiating the minus symbol as an operational signifier or a structural signifier is vital when developing the concept of negative numbers.

The functional use of signs can then be described as the use of signs (spoken words or written symbols) in the mathematical domain without necessarily understanding the full concept (Berger, 2005). For instance, a child can use a word, such as *minus* to refer to the symbol in a mathematical equation such as $-5 + 7$(an object), it could be read as “minus five add seven”, without having a full grasp of negative numbers. Yet the child uses it in the manner s/he sees their teacher using it or how it is used in a textbook, both of which are external influences from society (Vygotsky, 1986).

The way Vygotsky (1986) defines the term *word* is based on generalisations which give rise to a concept. For example, *negative* is used to describe the minus symbol in front of a number which can give rise to negative numbers being its own concept. The same can apply in the mathematical domain in terms of the use of signs (Berger, 2005). The more a sign is appropriately used in a social setting the greater the chance that the learner will develop the related concept. Vygotsky (1986) argues that a child cannot just develop a concept but rather the development is dependent on interactions in the social world. However, in the end, the child will make their own meaning of that concept and the use of the signs (Vygotsky, 1986). One of the goals of teaching is that the child’s meaning, and use of signs, will fit with the social conventions of its meaning and use.

2.2.3 Vygotsky’s Theory of Concept Formation

The formation of concepts occurs in three phases; and within each phase there are stages of development (Vygotsky, 1986). It is important to note that the process of concept formation is not linked to a child’s biological age. It is a process that can occur in any grade where he/she is introduced to a new concept and over time the concept develops to achieve full understanding, which Vygotsky calls a *true concept* (Vygotsky, 1986). Similarly, since I am focusing on concept formation in a classroom, then the *true concept* is when the *spontaneous concept* and *scientific concept* grow towards each other to become one (Vygotsky, 1986). Vygotsky (1986) presents the phases of concept formation in a general sense, whereas Berger

(2004a; 2004b) links these phases to mathematical learning although her focus is at university level. The focus for this study is on the first two phases of concept formation because the study focuses on the errors which stem from the first two phases, Figure 2.1 and Figure 2.2 are presented on the next page, which will be discussed in further details.

A. Phase One of Concept Formation

Figure 2.1 shows the first phase and the stages found within the phase according to (Vygotsky, 1986), and Berger (2004a; 2004b). The initial/primary source is (Vygotsky, 1986), and this is complemented with Berger's work. I will still include the extensions that Berger (2004a; 2004b) made when applying this theory to undergraduate students in the mathematical domain, when introduced to new concepts. Not all the stages in this phase are applicable to my study and therefore I do not discuss them all in depth.

Figure 2.1: Phase one of concept formation

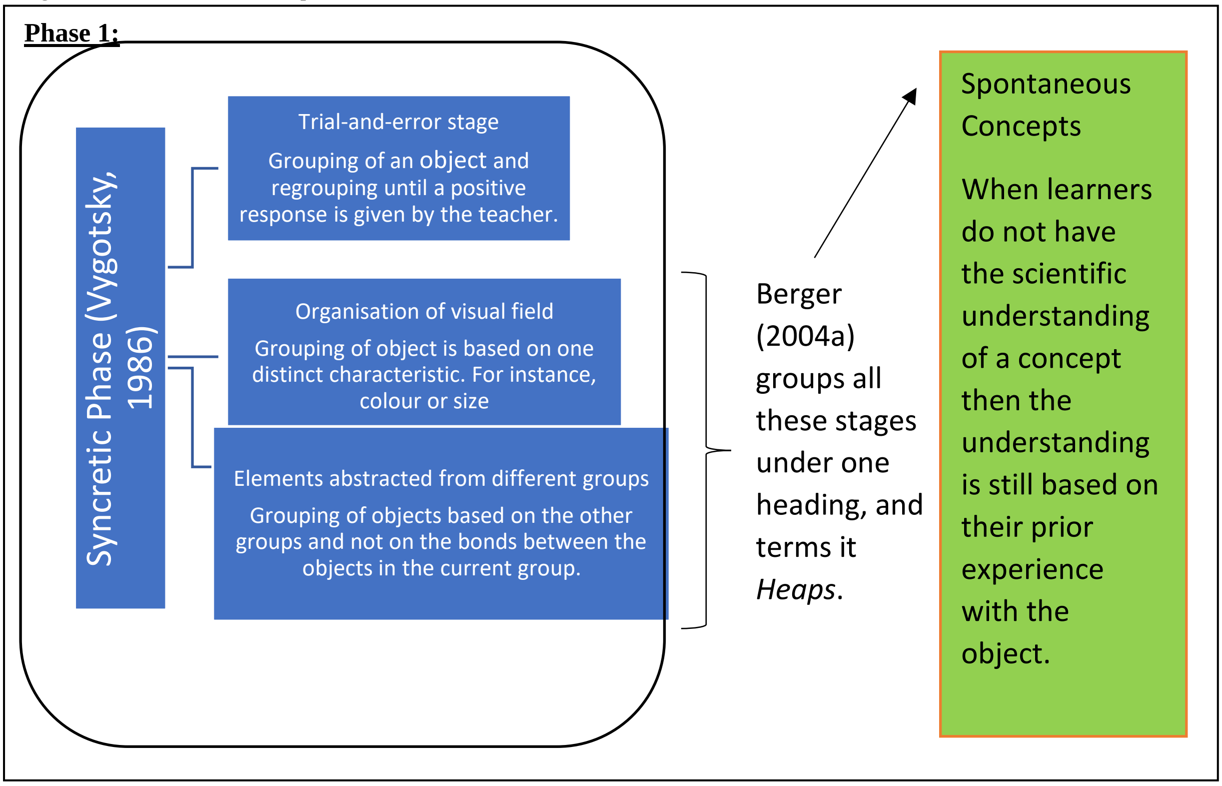

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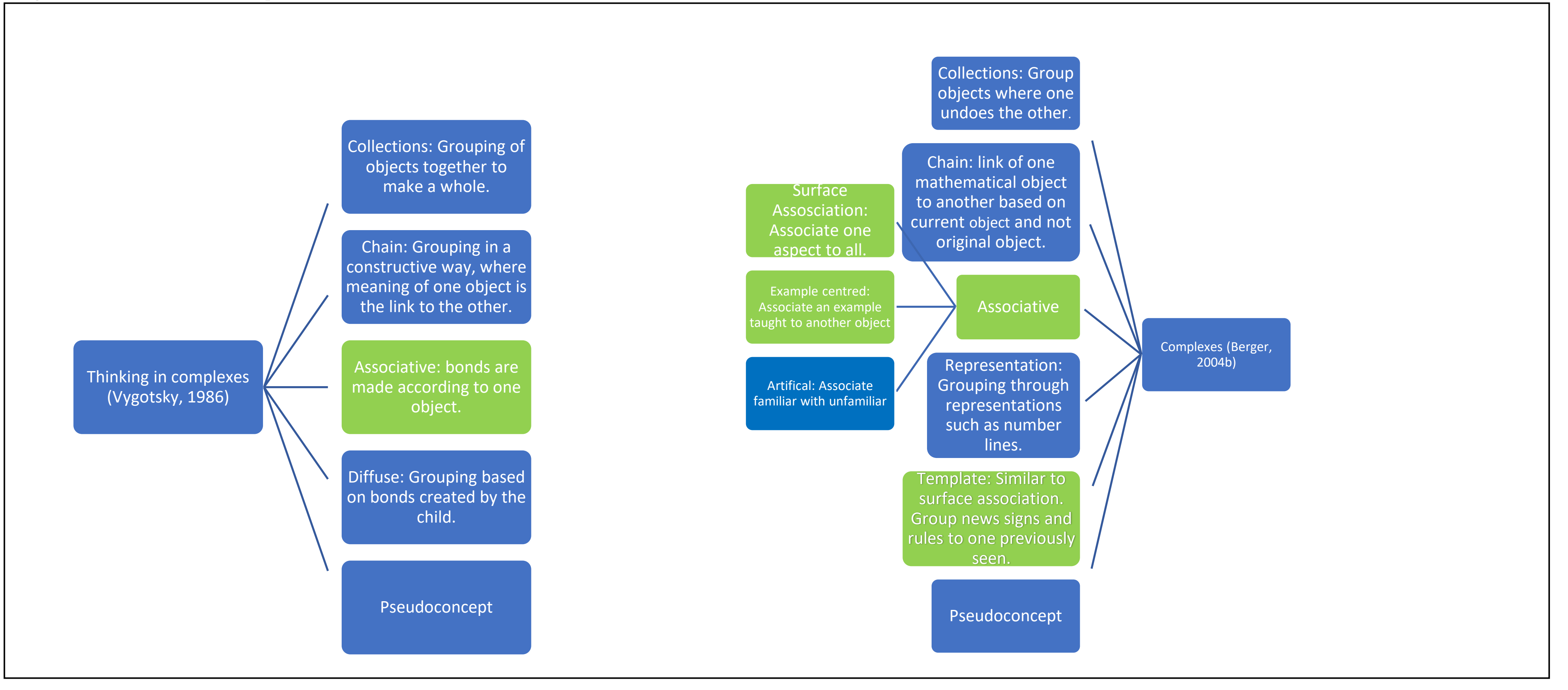
Figure 2.1: Phase two of concept formation



The first phase of concept formation that Vygotsky (1986) presents is the *syncretic phase*. Before experiencing any form of social interactions with a teacher or adult, a learner would group objects in an unorganised manner with no factual basis or relating factors between these set of objects, Vygotsky (1986) called these groups *heaps*, which is how Berger (2004a) titles this phase. Learners still understand objects based on their spontaneous concept. There are no logical bonds between objects. Berger (2004a) posits that in the mathematical domain, a learner is in the heap stage when they associate signs with each other based on what is seen, instead of what is meant (structural or operational). For example, the minus symbol will be seen as both a subtraction and a negative symbol.

The main outcome for the first phase is to allow the learner to group objects that are more relatable to one another and create their own meaning for this group of objects, a syncretic heap. Thinking in heaps is spontaneous and incoherent. There are no logical bonds that can be seen between objects (Vygotsky, 1986). Within this phase Vygotsky (1986) identifies three stages: trial-and-error stage, organising a visual field and elements abstracted from other groups. For this study, I am going to refer learners in this phase, as learners having a *spontaneous concept.* Even though Vygotsky (1986) states that *heaps* are not concepts, it is assumed that the grade 10 learners already have an idea of the minus symbol, and though it may not be mathematically correct, it is still their understanding, thus it could be a *spontaneous concept*.

A. Phase Two of Concept Formation

The second and most important phase for my study is what Vygotsky (1986) termed as *thinking in complexes*. Where thinking in heaps is spontaneous and lacks coherence, thinking in complexes is objective and has coherence with the objects. However, the connections between objects are through bonds that are factual but lack logic, thus there can be multiple diverse types of bonds that are made between objects. For instance, in a school of fish, there can be multiple fish, different in colour and sizes. A child would group a shark and goldfish together because they both have fins, yet they are completely different fish, but since the focus is on the look, then grouping the sharks with goldfish makes sense to the child.

Learners move from phase one to phase two by grouping objects coherently and not at 'random'. The groups they make in this phase have more relation to the concept than the groups made in phase 1 (Vygotsky, 1986). Both Vygotsky (1986) and Berger (2004b) described stages within this phase. (Vygotsky, 1986) identified five stages, whereas Berger (2004b) proposed six. *Complex thinking* differs to concepts as the bonds found in concepts are more abstract

(Vygotsky, 1986). These bonds in this phase are characteristics that are similar between objects. For instance, in a group of different shapes a learner could group all the triangles together based on a bond of a shape having three sides. Berger (2004b) states that the different forms of thinking that Vygotsky (1986) proposed are inadequate for characterising signs in the mathematical domain and thus she adjusted some of the stages in this phase. I will discuss the stages that have been highlighted in green in Figure 2.2.

i. Associative Type

In this type of complex thinking, learners create bonds according to one main object (known as the nucleus); the name given to the nucleus will then represent all the objects it is then bonded too. The bonds do not have to be exact but can be similar, for instance the main object could be a blue triangle, therefore the bonds could be either 3-sided shapes or the colour blue (Vygotsky, 1986). Berger (2004b) states in mathematics the associative complex occurs when a learner focuses on one mathematical sign (the nucleus) and associates the other mathematical signs to the nucleus, in integers one can then say that the symbol '$-$', would then be associated to both a sign and an operation and would be referred to as a minus symbol regardless of the object the symbol is found in. Berger (2004) further states that a mathematical object could also be found in the nucleus and similar objects would be related to the nucleus to form a complex. For instance, the object $-7+5=$ could be a nucleus and a similar question $-7-5=$ could be bonded to the nucleus due to the leading term being a negative number, therefore the learner would use the same procedure in answering the first object for the second object.

Berger (2004b) introduces a further three sub-stage that can be seen in the associative stage. The first is *surface association.* In this stage, learners form a nucleus based on a specific aspect that can be seen in the mathematical expression and associates this aspect (signifier) with new signs. Berger (2004b) adds that when learners focus on the aspect, they tend to ignore the other signifiers that are seen in the expression. For instance, in the expression $5-7=$, the learner could focus on the operation and not be concerned about the minuend or subtrahend and could read the expression as $7-5=$.

The next sub-stage, which is seen most often in the mathematical domain is when a concept is constructed through a nucleus that is an example; this stage is titled *example-centred* (Berger, 2004). Learners would use the example offered by social interactions (a teacher or textbook) and appropriate it to another mathematical object. According to Berger (2004) the learner will associate certain aspects of a given example to the new mathematical object, for example $7-(-5)=7+5$ a learner could take this example and apply it to $-7-5=7+5$ because there are two minus symbols, then the operation should change based on the multiplicative rule of integers: negative and a negative is positive. The placement of the signs is not taken into consideration. The last sub-stage Berger (2004b) introduced was the artificial association of familiar with the unfamiliar. This category is not relevant to my study and therefore will not be discussed.

ii. Template-orientation Complex

One of the other stages that Berger (2004b) introduces is the template-orientation complex. This stage is like *surface association* because this stage focuses on the use of signs as a signifier. When a learner is introduced to a new mathematical sign, they would associate that sign to a sign that is more familiar because there are similarities in the template in which the familiar sign is seen (Berger, 2004b). The learner would transfer the properties of the old sign to the new sign to have a sense of familiarity (Berger, 2004b). For instance, learners may have been taught that brackets represent multiplication (e.g., distributive law) and so they apply the multiplication template to an item such as $5 - (-7)$ thus getting an answer of -35.

iii. Pseudoconcept

The pseudoconcept is the bridge between complexes and concepts (Vygotsky, 1986). It has similar traits of a concept due to the ready-to-made definitions but not have the in-depth understanding (Vygotsky, 1986). This stage is important to progress from phase two to phase three in developing a concept. In mathematics, it is difficult to detect when a learner is in the pseudoconcept phase because they can communicate and use mathematics, as if they fully understand it (Berger, 2004b). For instance, a child could gather all the objects such as squares based on a sample: blue square, the gathering of the objects could have been based on the concept of squares but was more likely gathered based on the sample (a form of associative form of thinking), thus the process of grouping of objects is a different form of conceptual thinking that is required. It may seem that the words the learner uses might be correct, however the in-depth understanding of these words are still not fully developed, therefore the words used are based on generalisations (Vygotsky, 1986).

This stage is crucial in moving from complexes to concepts where it has potential to be a concept but still has the characteristics of complexes. In mathematics, the pseudoconcept is pervasive. The use of a pseudoconcept is evident when learners use and communicate mathematics as if they have a full understanding of it (Berger, 2004b). Using the pseudoconcept allows the learner to engage with the mathematical community and this engagement allows learners to build their generalisation process (Berger, 2004b). According to Berger (2004b), detecting a learner in the pseudoconcept stage can be difficult and requires a range of activities, which means that at first a learner may show understanding of a concept in one activity but immediately thereafter does another activity on the concept, being inconsistent in answering the questions. For example, a learner may order and compare integers correctly' but when it comes to operating on them, they struggle. Another example, a learner would be able to get that $7-(-5)=12$ but then also say that $-7-5=12$.

In my study, I have one test written twice by the same learners at different times of the year. The time-gap between the writing of the two tests is approximately a semester, however if a learner obtained a correct answer in the test written in February and then an incorrect answer in the test written in September, then the understanding of the concept is not stable and based on the response, I could see if other concepts influenced the response. However, I also needed to keep in mind that the response in the test written in September could also be a minor error such as a calculation error, for instance $5-7=-3$.

B. *Phase 3 of Concept Formation*

The final phase, known as the *Abstraction phase,* includes two stages, namely: *grouping maximally similar objects* and the *potential concept* stages (Vygotsky, 1986). A learner would go through these stages to progress from *thinking in complexes* to conceptual thinking. As discussed above, the main outcome of the complex stage is to form bonds between objects. When progressing to the abstraction phase, the focus is on single elements and their distinguishable characteristics from the concrete whole, thus generalisation and abstraction are intertwined (Vygotsky, 1986).

For instance, if a learner had to group objects such as $7-5$ and $-7+5$ based on the associative stage in the *complex phase* due the generalisations made of the minus symbol then to move to the next phase the learner would abstract a single element such as -7 and investigate the reason this is different from the generalised group of objects. However, I agree with Berger (2004b) that in mathematics it is almost impossible to differentiate potential concepts from the mathematical complexes and since some mathematical complexes, such as collection, are very closely related to the characteristics of abstraction, the two run concurrently with each other to develop mathematical concepts.

Mathematical concepts, as referred to by Berger (2004b) are ideas that occur mentally about a mathematical object. The links between the properties of a mathematical object are consistent and logical, i.e., the concept can be applied to all mathematical objects of that nature (Berger, 2004b). It involves being able to move between the sign and operation appropriately depending on the context in which the minus symbol is used. When a learner has progressed to the conceptual stage in mathematics, they are able to deal with the entire mathematical objects and not focus on parts of it, which is facilitated by the prior knowledge of other concepts (Berger, 2004b). For instance, a learner can see that $7 - 5$ and $-5 + 7$ is the same without having to calculate the answer for both. The next section discusses the concept of negative numbers and the minus symbol.

2.3 Thinking and Accepting Negative Numbers

In this literature review I discuss the concept of negative numbers by analysing how learners use the minus symbol, hence the way the minus symbol is used can then reveal the level in which learners accept negative numbers. The key literature on integers includes Vlassis (2004; 2008) and Gallardo, (2008). This literature describes the minus symbol and reasons behind the errors learners make when working with the minus symbol and how this relates to learners' understanding of negative numbers.

The true concept of negative numbers is being able to work flexibly with the three natures of the minus symbols, meaning that one can identify the three natures and shift between them as well as accepting negative numbers at the highest form. By linking these two ideas together with the theoretical framework, it creates a conceptual framework that is used in this study.

2.3.1 The Concept of Negative Numbers

To understand integers learners are required to understand the concept of positive numbers and negative numbers. As discussed above, negative numbers are its' own concept as it requires its own understanding to make differentiations of the minus symbol, it requires its' own set of knowledge to work with these types of numbers. To be proficient in the use of the minus symbol, the concept of negative numbers needs to be well developed. Vlassis (2004; 2008) proposes that there are three natures of the minus symbol and through the understanding of these three natures a learner will be able to understand negative numbers.

A. Triple Nature of the Minus Symbol

In describing the natures of the minus symbol, Vlassis (2004; 2008) discusses two aspects: the first is function and the second is their signifier role (Sfard, 2000). In acknowledging learning as part of a social and cultural activity, the role of the use of signs and its importance in the mathematical classroom is vital (Vlassis, 2008). The focus of this study is not on determining whether learners have difficulty in conceptualising negative numbers but rather on the difficulties learners have in using negative numbers, and more specifically what form of procedure they followed based on the positionings of the minus symbol/s in relation to other symbols and operations which then would require the minus symbol to take on different natures.

Vlassis (2008) identifies the minus symbol in three different natures depending on the context:

1. Unary nature – The minus symbol is a structural signifier; it describes the number as negative and "…corresponds to the sign as a predicate." (Vlassis, 2004, p. 472). Simply stated, it is the minus symbol that is attached to a number to create what we call *negative numbers* and can be seen as a definitive answer (Vlassis, 2008), For example, -2 which is read as *negative two*.
2. Binary nature – The minus symbol is seen as an operational signifier. The minus symbol is seen as an instruction of differences between two terms. According to Vlassis (2008), there are two main functions of the minus symbol as an operational signifier: arithmetical subtraction (subtraction between numbers: $10 - 8$) and algebraic subtraction (subtraction between a variable and a constant: $x - 4$). For this study, the focus is strictly on seeing the minus symbol regarding arithmetical subtraction. That means that subtraction is seen in one of three ways: taking away, the difference between, and counting on (Penlington, 2018). This does not take into consideration negative answers. However, the way in which Vlassis (2004) describes the binary nature is at an elementary level which is how Penlington (2018) defines subtraction. Table 2.1 shows three ways to describe subtraction.

Table 2.1: Three ways to describe subtraction (Penlington,2018)

Item	$12 - 4 = 8$
Taking away	12 take away 4 leaves 8
The difference between	The difference between 12 and 4 is 8. 12 is the minuend, 4 is the subtrahend and 8 is the difference.
Counting on	4 and 8 make 12. Therefore 12 subtract 4 is 8.

3. Symmetrical nature – The nature of the minus symbol signals the learner to take "…the opposite of a number or of a sum" (Vlassis, 2008, pg. 561). Vlassis (2004) states that the symmetrical nature "treats" the minus symbol as an operational signifier (Sfard, 2000). For example, $5 - (-7) = 5 + 7 = 12$. Stephan and Akyuz, (2012) elaborate on this nature, in saying that the first minus symbol signifies an operation, this operation is the opposite sign of the second minus symbol, and therefore the operation is addition. This nature is only applicable when there is at least one minus symbol seen in the question.

B. Levels of Accepting Negative Numbers

Gallardo (2008) proposes four levels of accepting negative numbers. These levels provide a means of indicating where a learner is in the conceptualising of negative numbers.

Gallardo's (2008) levels are described below:

1. Subtrahend: The idea of number which is related to size. Thus, $a - b$, where $a > b$, results in positive answers only. When there are two minus symbols next to each other such as $7 - (-5)$, then if the learner is at this level of acceptance, they would then ignore the sign and focus on the operation, -5 is the subtrahend, and is then read as $7 - 5$. The binary nature of the minus symbol is emphasised in this level but to the extent of the subtrahend must be smaller than the minuend.
2. Relative or directed number: Negative numbers are seen as opposite yet symmetrical quantities to the positive numbers, for example in temperature 32°C is very hot and −32°C is very cold, these quantities are opposite to each other in the spectrum. This idea of opposite quantities is bounded in context in the discrete domain (Gallardo, 2008). Idea of symmetry appears and is also seen in the continuous domain, such as ordering and comparing of numbers for example: $-50;\ -15; -5;\ 0;\ 5;\ 15;\ 50$. There is symmetry in ordering of the numbers. This would be the first step in acknowledging the unary nature of the minus symbol.
3. Isolated number: The result or solution to the mathematical problem or equation, such as $5 - 7 = -2$. A learner operating at this level will accept that a negative number can be a solution to an equation. Learners can also substitute negative numbers in algebraic expressions correctly and operate on them thereafter. This level can then also be known as *negative solution*. At this level, learners should be comfortable with the unary nature of the minus symbol.

4. Formal negative number: Having a full understanding of the mathematical concept of negative numbers that belong to the vaster concept (integers). At this level, a learner would not see negative numbers as a separate entity but rather part of the bigger idea. Learners can work with positive and negative numbers but deal with negative numbers differently. All three natures of the minus symbol are acquired at this level.

Gallardo (2008) states that to fully accept negative numbers, learners should reach the formal level of acceptance, which involves working with the minus symbol and negative numbers in both the discrete and continuous domains. In the next section, I discuss the link between Gallardo, (2008) and Vlassis, (2004) in depth. This process of conceptualisation is then related to the theoretical framework. When the conceptualisation of negative numbers is not developed correctly, then errors can arise. The next section will discuss potential errors made and how that is then related to the formation of concept process. This forms the conceptual framework of my study.

2.4 Conceptual Framework

This section presents the conceptual framework that I have developed. It describes possible reasons of the errors that learners made and relates it to the theory mentioned in the sections above. This part of the chapter is vital in my research. Aspects of this section are used for the coding of the data. It links the literature to the theory so that strong inferences can be made about my data. I have called this framework: *A conceptual framework for learners' interpretation of negative numbers.*

It is important to note that when looking at the responses from the learners, the reasoning I associate to the response will be an assumption of the possible procedure the learner followed because in many cases, learners only wrote down final answers to the test items. Since this study is based on socio-cultural learning theory. I veer away from the term 'misconception' due to the link the term has with constructivist learning theory. Instead, I discuss the progression of development of the concept with references to learners' test responses.

2.4.1 Negative Numbers as a Spontaneous Concept

Before a learner encounters the different natures of the minus symbol, they have their own ideas of the symbol and the meaning it holds for them. This meaning could have been based on interactions with family members, television, and the world around them and not with someone who is a specialist in mathematics (teacher) or mathematical resources (textbook).

The development of the spontaneous concept to scientific concept runs parallel to the progress of concept formation.

The symbol could bring a negative connotation such as taking away or removing or could simply be just a sign with no meaning but rather action. Fragkiadaki, (2020) investigated pre-school children's understanding of the minus symbol and showed that the learners recognised the sign and related it to their shopping experiences (discounts on items). However, they did not know the name for the sign or how it is used (Fragkiadaki, 2020).

To relate these terms back to Vygotsky's (1986) idea of heap, it is the organising objects in a disorganised manner based on the child's view of the object. Learners would group all objects that hold both the minus and plus symbol together based on some spontaneous concept that they believe is true. Since the learners, that my research is based on, are in grade 10, I can be certain that they have had exposure to the scientific concept of integers and operations on them. However, based on the responses received, when learners do not differentiate the minus symbol from operation to sign, then it also shows that learners could possibly still be stuck on their ideas, intimating that they have based their working out of the equation on their spontaneous concept knowledge and therefore do not have a developed understanding of the true concept. Their responses could also suggest that the teaching they have received thus far has not explicitly differentiated operation from sign either.

Learners in this phase of concept formation only accept negative numbers as a *subtrahend* (Gallardo, 2008), likely because that is what they have been exposed to the most, inside and outside school and therefore they expect that all answers to subtraction problems to be positive. Such as $5-7=7-5=2$. Consequently, given a question such as $5\ +\ (-7)$, learners would ignore the negative sign that belongs to the 7 and add: $5+7\ =\ 12$.

In terms of ordering of integers such as ordering the following from biggest to smallest: $2;-5;\ 12;-10;-15;20$, learners in this stage would order integers according to their absolute value, $20;\ -15;\ 12;-10;-5;2$. Bofferding, (2010) describes this as *whole number mental model*. The ordering of negative numbers is mixed with positive numbers based on their absolute value. The assumption is that the negative numbers are seen as the same as positive numbers (Bofferding, 2010), $-15\ >\ 12$ because the absolute value of -15 (which is 15) is greater than 12.

Learners can be at a higher level of accepting a negative number but still understand negative numbers at a spontaneous concept level. This is the beginning of understanding the unary nature of the minus symbol that Vlassis (2004) describes. It looks at negative numbers at a more concrete level through the relating it to everyday concepts and therefore learners accept negative numbers, which Vlassis (2004) describes as *relative numbers*. When learners order negative numbers, they apply the ordering they do for positive numbers and therefore, learners can distinguish between positive values and negative values (Bofferding, 2010). For instance, the ordering would be $-2; -5; -20; 3; 15; 30$. However, when ordering the negative numbers, the numbers are ordered according to their absolute value, thus $-9 > -3$ but $-9 < 3$ (Bofferding, 2010). Bofferding (2010) names this type of ordering as *magnitude mental model*.

2.4.2 Thinking in Complexes of Negative Numbers.

The *complex thinking phase* of concept formation is the phase that is most relevant to this study due to the nature of the data. There are three key stages in this phase that can be seen as a form of reasoning when linked to the work of Bofferding, (2010) and Vlassis, (2004). I have sectioned this chapter according to the stages in the *complex thinking* phase of concept formation and discuss its' relation to the types of errors seen.

A. Surface Association Type with the Nucleus Being Absolute Values.

Berger (2004) identifies this form of complex thinking under the stage of *association type* that Vygotsky (1986) describes. This sub-stage focuses more on a mathematical aspect where learners tend to ignore the signs/symbols involved in the object (Berger, 2004). In this case the main aspect would be the properties of absolute/natural numbers and learners would deal with the objects as they would with absolute values.

The minus symbol in front of the number is taken into consideration but it is something that can move around and is not seen as part of a number, by removing the symbol and adding it back to the answer. This relates to Vlassis' (2004) form of reasoning called *bracket reasoning*, this is described as when there is a placement of invisible brackets which will not be visible on page (idea of grouping), to create familiarity in the question, for example: $3 + 6 - 5 - 1$, learners might then write $(3 + 6) - (5 - 1) = 9 - 4 = 5$ (Vlassis, 2004). According to Vlassis (2004), in the equation $-5 - 1$, the learners would place imaginary brackets around the 5 and 1, for this to be more familiar. Since there is no assurance that these grade 10 learners have done that, I then used this idea and related it to that of Linchevski and Herscovics, (1996) who describe a similar process but terms it as *detachment of the minus symbol*. This is when learners would operate the numbers but ignore the negative sign and thereafter re-attach the negative sign to the answer. For example: $-5 - 1 = 5 - 1 = 4 = -4$. Based on this description by Linchevski and Herscovics (1996), the minus symbol is ignored, and the focus is on the part of the object that is familiar to them. The idea that the learners have not accepted negative numbers is due to the misuse of the minus symbol and them ignoring it.

B. Example-centred Focused on the Multiplicative Rule.

The final sub-stage that Berger (2004b) developed under Vygotsky (1986) *associative type* of complex thinking is *example-centred*, whereby learners will apply a rule they have seen in an example to any type of object that seems familiar to the object they are currently working with. In this case the example would be the *symmetrical nature* of the minus symbol (Vlassis, 2004), $6-(-10) = 6+10$. Therefore, the learners are exposed to the unary nature and now starting with the symmetrical nature of the minus symbol, they would then associate this rule to other objects, where the symbols are not directly next to each other learners would still apply the rule, for example $-7-5 = 7+5$. Vlassis (2004) describes this form of error as the *signs rule* where the error is based on the use of a rule learnt but not used correctly, for example: *a minus and minus gives a plus*. This form of reasoning is used mostly when a question has a multiple number of terms, but relates to my data, as per the example used above. I have called this error *applying the symmetrical nature.* Learners have accepted negative numbers as a solution to an equation or problem, but they are still not at that formal stage of accepting negative numbers (Gallardo, 2008).

C. Template-oriented with the Focus on Multiplication

Like the *associative complex*, Berger (2004b) presents the *template-orientated complex*. In this stage, learners focus on symbols that are most familiar to them based on the way the object is seen. In this study, the focused symbols were brackets which would have alerted learners to multiply numbers to each other, the other symbols around are then seen as structural signifiers and not operational signifiers. The aspect of *symmetrical nature*, that Vlassis (2004) discusses is implies that a negative multiplied by a negative is positive, however this is now applied to multiplication of integers (positive and negative numbers) due to brackets being present. Learners take an item such as $6-(-10)$, and supposedly focus on the brackets which alerts multiplication as the focal operation and treat the other symbols as structural signifiers to then get $-6(-10) = 60$. The idea of multiplying the numbers, due to brackets, could have been due to process of distribution that the learners were exposed to in the classroom, such as $x(-x) = -x^2$. This template is then related to the question as it is familiar to the learners. Since learners can work with the minus symbols, they would accept negative numbers at the third level (Gallardo, 2008) because they could get a negative solution if required. This error is called *multiplicative rule signalled by brackets.*

2.5 Conclusion

All the sections discussed forms the *conceptual framework for learners' interpretation of negative numbers*, which is summarised in Table 2.2. This framework is used to interpret the findings from the data. Based on the solution the learner gave, an assumption is made on the level of acceptance (Gallardo, 2008). The next two columns focus on the possible procedures that the learner may have followed in giving the solution. The possible procedures were informed by constructs from Bofferding, (2010) and Vlassis (2004). Bofferding's (2010) notion of mental models informs my analysis of responses to Item 1.

Vlassis (2004) notion of possible reasonings informs my interpretation of responses to understand Items 3a-f. The items are discussed in the next chapter in Section 3.5. The final column in the table links the possible procedures to the possible understanding learners may have on integers based on the theory of concept formation (Vygotsky, 1986) and the interpretation that Berger (2004a; 2004b) discussed. In the proceeding chapter, I will discuss the methodology of this study.

Table 2.2: Framework describing the concept formation of negative numbers.

Levels of accepting a negative number	Triple nature of the minus symbol and possible reasons	Bofferding (2010) - Mental Models	Theory of Concept Formation (Vygotsky, 1986, Berger, 2004a; 2004b)	Examples
A. Subtrahend Number is subordinated to magnitude. $a - b;\ a > b$ Positive answer only. When two minus symbols are next to each other, ignore the sign and focus on the operation.	Binary Nature: It is limited to the subtraction of natural numbers to give a positive answer only. Right to left reasoning can be seen for the answer to be positive when you have a-b and a<b.	Models link to this when they are unable to differentiate between negative and positive numbers and combine the numbers when ordering them.	With regards to grade 10 learners, this is then the Heap phase of concept formation, however I term it as learners' understanding negative numbers as their spontaneous concept because learners have been exposed to multiple symbols but still focus on one aspect (operation). Learners are still focused on the spontaneous concept of the minus symbol, with the idea of getting only a positive answer.	$5 - 7 = 7 - 5$ $-5 + 7 = 5 + 7$ $6 - (-10) = 6 - 10 = 10 - 6$ $5 + (-7) = 5 + 7$ $-7 - 5 = 7 - 5$ $-2; 4; 7; -10; 15; -25$
B. Relative/Directed Idea of opposite quantities in relation to a quality arises in the discrete domain (Gallardo, 2008). Idea of symmetry appears in the continuous domain.	Beginning of Unary Nature: Learners recognise the negative sign, and this gives them an idea of what it could mean in a contextual setting. (Debit/Credit, Increase/Decrease).	Models link to this with regards to the ordering and comparing of numbers. Learners will order the negative numbers in the same way they would order positive numbers.	Heap: Not being able to distinguish between positive and negative numbers. Learners will arrange negative numbers the same way they do positive numbers, based on the spontaneous concept of negative numbers that is related to context only. Associative type - nucleus being positive numbers, Bergers' (2004a) notion of 'Surface association', where learners will associate numbers according to the absolute value and not signs.	$-2;\ -4;\ -6;\ 2;\ 4;\ 6$
C. Negative Solution: Result of an operation/ solution to a problem/ equation. Allowing a solution to be a negative number.	Full exposure to all three natures of the minus symbol. Differentiation of the symbols is where errors will start arising and thus certain rules will combine with each other.	N/A	Associative types: Learners will associate certain rules due to certain symbols or objects. The nucleus can differ based on the object and learner. Nuclei: Operation or Sign. Surface association: relate what they did to one object with another based on a similar aspect seen in the objects. Template-orientated types: Applying a procedure done based on a focused symbol and apply the rules for that symbol.	Examples may vary depending on what is being focused on.
D. Formal Negative number. Having a clear distinguishable factor of negative numbers in comparison to positive numbers under the main heading of Integers.	Flexibility in all 3 natures of the minus symbol.	N/A	In accordance with Berger (2004b), learners develop the mathematical concept when the links between objects are logical and consistent, and when the focus is not a particular aspect but rather on the object which would contain concepts that learners should distinguish.	

CHAPTER 3 – RESEARCH DESIGN AND METHODOLOGY

3.1 Introduction

In this chapter I discuss the processes undertaken in selecting my sample and the process of coding used for this study. I also discuss the trustworthiness of the interpretations made in the latter chapters based on the coding systems. This chapter gives a more cogent understanding of the data that was selected and the process of capturing the data. It is reiterated that I did not administer the tests, but I am analysing the responses from tests administered by the WMCS project in 2018.

3.2 Overall Design of the Study

The goal of this study was to see the processes grade 10 learners undertake when ordering and operating with integers and compare two groups of learners. Based on the objective of this study, I utilised a mixed-method approach. Almeida (2018) claims that the purpose of a mixed-method study, which contains aspects of both a quantitative and qualitative study, is to give a wider and clearer idea of the problem that I aim to explore. To support the claims with regards to the responses given by the learners, quantitative methods were used. To understand the different errors that were made by the learners, qualitative methods were used.

The analysis of the data is divided into two chapters. Chapter 4 uses quantitative data analysis techniques to support the claims made. The purpose of quantitative analysis is to make generalisations on the greater population (Fairbrother, 2018). This study used hypothesis testing through statistical measures such as t-testing and other measures of spread. Fairbrother (2018) claims that the purpose of quantitative study is to make deductions from the data based on the theory and literature that the study is founded on.

Chapter 5 uses qualitative analysis techniques. The aim of a qualitative study is to understand the perceptions (understandings) brought forward by humans through data such as written scripts, in this case test script (Geertz, 1973). Sharma (2013) builds on this stating that a form of collecting data in a qualitative study is through the analysis of documents from the original source as it provides evidence of human understanding. In this study, the documents are in the form of test scripts that were provided to the learners in which learners gave their own response based on their knowledge. Although the responses can vary, this being mathematical research, there is a certain range of responses that learners can give.

By looking at the same data but from two different views: quantitative and qualitative, then this form of mixed-method design is known as the *Triangulation design* (Almeida, 2018).

3.3 WMCS Project and Test

The data used for this study is part of a larger study that was conducted by the WMCS project. In 2018, the WMCS project tested grade 9 and 10 learners in 40 low performing schools in the Gauteng province. Learners were tested twice on the same test, first in February/March and then repeated in September/October. The purpose of the testing was to investigate the gains made by learners across the school year. The larger study involved a comparison of learners taught by teachers who had participated in a WMCS professional development course (Transition Maths 1), with learners in schools where teachers had not participated in the course. I, therefore, name the groups as the CNT group (learners whose teachers did not attend the course) and the TM group (learners whose teachers did attend the course).

3.4 Random Sample

To make inferences about the population, Zieffer, Garfield, DelMas and Reading (2008) claim that a random sample reveals patterns that can be seen in the real world, whereas a general sample creates generalisations about the population that at times cannot be true for all. To obtain the random sample for this study a selection process was performed. The selection process began by focusing on the grade 10 learners who wrote both tests in 2018, be it learners from the CNT group or TM group. The second filter was to focus on the middle group of learners. To obtain this group, I focused on learners who achieved between $15 - 35$ out of the total 45 marks for the entire test. This was done for both tests. The outcome of this selection process culminated in 119 learners in the CNT group and 77 learners in the TM group, this in total gave 196 learners, each having written the same test twice in one year. I then had a total of 392 test scripts to analyse. There is a gap between the number of learners from each group, however, to make the analysis equitable, I used percentages instead of the number of learners.

3.5 Test Items

The test designed by WMCS members had a total of 45 questions based on the grade 8 curriculum. The project members captured and labelled the responses given by the learners according to three labels: correct, incorrect, or blank. Blank responses meant that the learner did not answer that question. A calculator was not allowed to be used in the paper, and the learners had one hour to write the test. Out of the 45 items in the test script, only two questions, comprising of seven items, dealt with integers only. One of the items was very difficult to analyse (Item 3e) and therefore was omitted from this research. This left me with six items per script (Appendix A), therefore a total of 2 352 responses were analysed. presents the six items the study focuses on.

Table 3 1: Items analysed.

Question number	Instruction	Item
1.	Write these numbers in order from smallest to largest.	$30;\ -35;\ -2;\ -500;\ -10;\ 4$
3.	Work out the answers	a) $5-7=$
		b) $5+(-7)=$
		c) $-5+7=$
		d) $6-(-10)=$
		f) $-7-5=$

Question 1 focused on the ordering of six integers from smallest to biggest. Question 3 focused on operating on integers. The items analysed in Question 3 all had two terms that learners had to operate on Four of these items used the same digits (5 and 7). Items 3c and 3f are similar since both first terms are negative numbers. Items 3b and 3d are similar because in both items the second term was a negative number in brackets. Having variation in the items allowed me to make better inferences, which is examined in the proceeding chapters. Even though the digits were the same for four out of five items, the arrangements and inclusion of the negative sign then created responses that varied. Item 3e, was significantly challenging as it contained four numbers in the item that learners had to operate on, creating a vast number of responses, hence this item was omitted.

3.6 Test Responses

When coding Items 3a-d and 3f, the coding was focused solely on the final answer; this was due to seeing minimal procedures in these items, hence the decision was made in the interest of a fair analysis. Although I focused primarily on the final answer, the minimal procedures used by the learners, was used as validity in coding, to ensure that procedures should match the inferences made. When learners gave two answers for one item, I then focused on the first option only, however if the first option was the correct response, then I would use the second option. The reasoning was that the optional answer was an indication of the learner's uncertainty. This irresolute response was evidenced thrice in the entire study.

3.7 Coding

The coding system used in my study is highly influenced by Table 2.2. An inductive approach was used in the coding systems. Thomas (2006) explains that an inductive approach in coding is common in qualitative studies, and qualitative analysis. Inductive coding starts with the in depth reading of literature and theory (Thomas, 2006). This literature and theory were discussed in Chapter 2. Inductive coding is based on certain aspects from literature and theory that hold meaningful information so that inferences can be made about the data (Thomas, 2006).

For example: $5 - 7 \; = \; 2$, since the answer is positive, it can be assumed that the learners are in the first level of acceptance (Gallardo, 2008). The response of 2, means that learners operated from right to left (Vlassis, 2004) by reading the item as $7 - 5$. This then relates to the theoretical framework, inferring that learners are applying their spontaneous understanding of the concept because they are subtracting a smaller number from a bigger number to obtain a positive answer only. This means that learners are still in the first phase of concept formation. This is an example of inductive coding based on the idea that Thomas (2006) proposed as it takes important aspects of the literature and relates it to data. Table 3.1 summarises the codes used in the analysis of my data; this was discussed in detail in Chapter 2.4.

Code name	Description	Example	Explanation of example
Codes applicable for Item 1 only.			
Magnitudes	Separate positive numbers from negative numbers. The positive numbers are ordered correctly. Negative numbers are arranged according to their absolute values (Bofferding, 2010).	$-2; -10; -35; -500; 4; 30$	Learners recognise that positive numbers are different from negative numbers, however they order negative numbers the same way as positive numbers.
Whole numbers	There is no separation of positive and negative numbers. All the numbers are arranged according to their absolute values (Bofferding, 2010).	$-2; 4; 10; 30; -35; -500$	Learners regard all numbers as positive numbers and order them accordingly.
Codes applicable for Item 3a-f only.			
Operating from right to left.	This code is applicable for Items 3a-3f. Operate right to left instead of left to right so that the item is familiar, and the difference is positive (Vlassis, 2004).	$5-7=7-5=2$	Learners focused on the idea that only a smaller number can be subtracted from a bigger number and the result should be always positive.
Detachment of the minus symbol	Learners would detach the minus symbol from the first number. They perform the operation between the numbers and then re-attach the symbol to the answer (Linchevski & Herscovics, 1996).	$-7-5=7-5=-2$	Learners detached the minus symbol from the first term, focused on the subtraction of the absolute values and then re-attached the minus symbol to the answer
Ignoring the minus symbol	Learners ignore the minus symbol in front of a number and t focus on the operation. When this occurs, it could result in either a positive or negative number. When it is a positive result then learners are then not able to accept the negative sign as according to Gallardo (2008). When learners get a negative result then learners are at the third level according to Gallardo (2008)	A. $-7-5=7-5=2$ $5+(-7)=5+7=12$ B. $5+(-7)=35$ C. $-5-7=5-7=$ $7-5=2$	Level 1: A. Learners ignore the negative and focus on the operation to obtain a positive result. B. In this case the focus was on the brackets which could have led to the multiplication of the numbers. C. This can also be combined with operating from right to left.
		$5-(-7)=5-7=-2$ $-5-7=5-7=-2$	Level 3: Learners ignore the negative sign and focus on the operation. When the operation is subtraction of a smaller number by a bigger number it then results in a negative solution.
Applying the symmetrical nature	Learners would apply the symmetrical nature of a minus symbol inappropriately (Vlassis, 2004). This is done to get the operation of the item; the items would contain two signs that are not next to each other.	$-7-5=7+5=12$ $-7+5=7-5=2$	In this example, learners simplified the signs to determine the operation of the question. The inappropriate multiplicative rule of the minus symbols gave the operation of the item.
Multiplicative rule signalled by brackets	Learners would apply the multiplicative rule (negative times a negative gives positive) based on the symmetrical nature of the minus symbol, however due to the brackets being present, they overgeneralise the rule to the numbers as well as the signs. The symbols are seen as structural signifiers rather than one operational signifier and one structural signifier.	$7+(-5)=-35$	Learners could focus on the brackets as multiplication and use the multiplicative rule of integers to obtain their answer. The other symbols are seen as structural signifiers; thus, the question could be read as $7(-5)=-35$

Table 3.2: Summary of Codes

3.8 Trustworthiness in Research

Lincoln and Guba (1985) strode to the forefront by proposing a shift focused on the trustworthiness of the study as this encompasses validity and reliability. Even though the study is a mixed-method design, a very small portion of the study is quantitative whereas the main section is qualitative. Due to this, I have focused more on trustworthiness rather than discussing inference quality. As stated above, trustworthiness includes both reliability and validity. Trustworthiness focuses on four aspects: *credibility; transferability; dependability and confirmability* (Lincoln & Guba, 1985). Each aspect will be discussed below.

3.8.1 Credibility

Stahl and King, (2020) claim that credibility questions the findings congruent to the reality. Credible research is research that evokes confidence in the truth of the findings because *credibility* requires certain activities to take place, which look at the interpretations and findings to be dependable (Lincoln & Guba, 1985). Shenton (2004) suggests that one of the activities in obtaining credibility is through the process of *triangulation. Triangulation* is the use of multiple sources of information and procedures that are related to the study to create a pattern that is identifiable (Stahl & King, 2020). *Triangulation* was obtained through the analysis of two sets of answers given by the same learners at different points of a school year against reasonings that were made by other researchers, i.e., Vlassis (2004) and Gallardo (2008). The claims that have been made are based on these two responses per student against literature that links to this study. Triangulation is also gained using statistical methods and qualitative analysis of learners' responses to make final claims. Having both quantitative and qualitative methods improve the trustworthiness of the claims.

Alongside *triangulation,* the data that was collected was from grade 10 learners which is in sync with the topic of this study. As discussed in sections 3.3 and section 3.5, the data collected was done in a controlled environment and was fair across all schools. This means that the learners were monitored by a person from the project to ensure that everyone follows the instructions correctly. The framework developed had undergone a long process of development, so that it could be utilised to analyse the responses. The use of various research, theories, and investigators (that worked with similar data) was used to develop this framework describing the concept formation of negative numbers. Credibility requires the researcher to refine their study where the coding processes and analysis are tested and re-tested with other people before presenting the findings in the research (Lincoln & Guba, 1985).

Initially, when the test scripts were coded, post-graduate students of the WMCS project checked 15% of the scripts that were coded as incorrect or correct and blank. There were less than 2% of errors made and these were dealt with by the post-graduate students. I used a spreadsheet as a tool that enabled the counting of the different types of errors and made it easier to categorise errors, mitigating the incidence of human error, thereby strengthening the accuracy of the data presented. The actual response given by each learner was captured, and with the aid of my supervisor we agreed on the correct codes for the responses.

3.8.2 Transferability

Transferability is creating the possibility for other researchers or readers to apply the research study findings to applicable contexts. This possibility is increased using a *thick description* (Lincoln & Guba, 1985). Stahl and King (2020) posit that a *thick description* should include a clear description of the way the data was collected, and the participants involved in the data, discussed in sections 3.3 and section 3.4. The use of random samples is important so that generalisations can be made from the sample to a similar population. However, boundaries such as the type of schools, the grade of the learners and the area in which the learners live are important to note. The findings, both qualitative and quantitative, of this study should be transferable to schools of similar backgrounds.

3.8.3 Dependability

Where *credibility* and *transferability* focus on the validity of the study, *dependability* represents the reliability of the study (Lincoln & Guba, 1985). The *overlap method* that Lincoln & Guba (1985) described is most fitting to this study. The *overlap method* is employed to obtain findings that are dependable, with the interpretation of the data through multiple methods, such as statistical tools and deductive reasoning. Using qualitative methods which increased dependability. Working with the data directly and having clear methodology for each step of analysis creates a paper trail that allows for scrutiny from my peers. This forms the second aspect of establishing dependability through the process of auditing. Stahl and King (2020) claim that scrutiny from peers and other researchers (in this case my supervisor and other Masters' students) allowed for feedback of the study for improvement before it is made public.

3.8.4 Confirmability

Confirmability is demonstrated when the findings are objectively portrayed. during data collection and analysis, (Stahl & King, 2020). When this is done correctly, i.e., procedures for checking and rechecking the data throughout the research and has been properly managed (where all processes have been documented) then the final aspect, *confirmability*, can also be established (Lincoln & Guba, 1985). Having a detailed description of the codes with examples, allows for other researchers to reuse the codes the same way as I did. During this research study the framework was discussed, with my supervisor, with peers and other academics, implemented, changed, and eventually finalised after much trial and error. However, the limitations experienced, and assumptions made must also be discussed, so that when the codes are re-used by other researchers, they have a holistic and true notion. Limitations are discussed in Chapter 6 of this study.

3.9 Ethical Consideration

The protocol number that I received for this study is 2020ECE014M (Appendix B). Since I used data that was originally collected by the WMCS project, the protocol number H17/01/01 (Appendix C) was obtained at the time. The WMCS research team distributed and gathered consent forms to the parents as well as learners who participated in this study. The principals and Department of Education also granted consent to use the data on the basis that all information about the schools and learners remained anonymous and confidential. Anonymity was achieved by giving each learner a unique ID code and their names remained unknown to any reader. Confidentiality was maintained by the initial team as well as myself, as I did not share test scripts with anyone outside of the WMCS project.

3.10 Conclusion

This chapter focused on the selection process of obtaining the data, information about the random sample obtained, and the codes that were used in the coding of the data. Trustworthiness of the research document was discussed and the stringent criteria of credibility, dependability, confirmability, and transferability (Lincoln & Guba, 1985), including the ethical considerations, were adhered to.

CHAPTER 4: OVERVIEW OF LEARNER PERFORMANCE

4.1 Introduction

In this chapter I discuss learners' overall performance by focusing on the overall correct responses in both tests. I compare the percentages of learners of each group who gave the correct response for every item, as well as the percentages between the tests. The number of learners in each group differs, therefore percentages are used to make comparisons. The comparison of learners of each group and each test is described using statistical measures like averages, standard deviations, and t-tests.

Table 4.1 presents the abbreviations used for this analysis, for instance TwF-CNT [1] would refer to the learners who wrote the test in February and the teacher who was not a participant the course hosted by the WMCS project.

Table 4.1: Number of learners in each group

Learners	TwF	TwS
CNT	119	119
TM	77	77
Combined (C)	196	196

4.2 Overall Correct Responses

I have separated this discussion into two sections. Firstly, I discuss the overall correct responses of all grades 10 learners' scripts by comparing the Test written in February (TwF) and Test written in September (TwS). This gives an indication of how the learners performed. The second section involves a comparison of the two groups (CNT and TM), as well as the changes in their performance from the TwF to the TwS for each test item analysed. Appendix DA gives the accurate percentages and number of learners for all figures presented in this chapter.

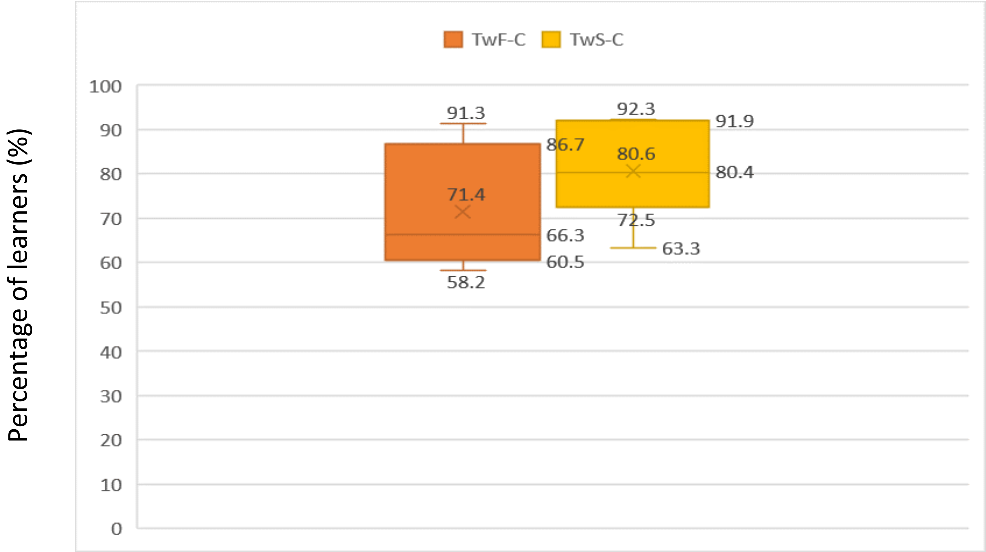
4.2.1 Overview of the Two Groups Combined.

Figure 4.1 compares the spread of the data for each of the tests written. There are two box-and-whisker plots, the orange represents the TwF and yellow represents the TwS. The plot presents the five number summary (minimum, quartile one, median, quartile three and maximum). The

[1] TWF: Test written in February, TWS: Test written in September, TM: Transition group, CNT: Comparison group.

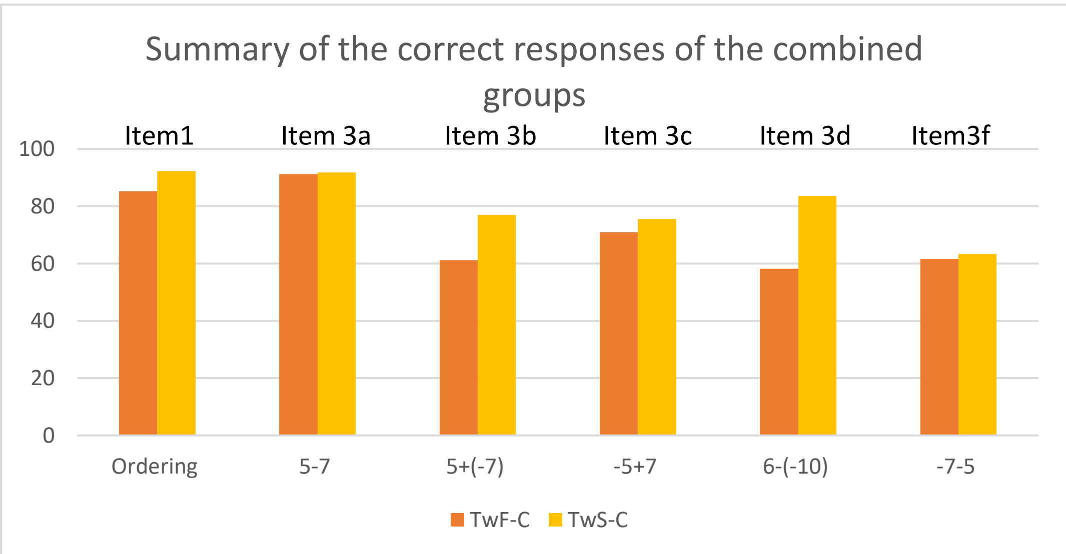
'X' in the box represents the mean. The plot is sectioned off into four parts, each section representing 25% of the learners.

Figure 4.1: Box-and-whisker diagram presenting the spread of correct response in both tests.



The TwS reveals that the spread of the percentages is slightly higher than the TwF. The box-and-whisker diagram in Figure 4.1 for TwS demonstrates that the mean is almost equal to the median, thus the spread of percentages has a symmetrical distribution as compared to the spread of the percentages for the TwF. In the TwF the mean is greater than the median, thus the data is skewed to the right. This means that for barely over 50% of the items, the percentage of correct responses was below the mean of 67.5%. Figure 4.2 summarises the correct responses of the grade 10 learners for each item. The orange bars represent the TwF, and the yellow bars represent the TwS. These are the percentages of 196 learners.

Figure 4.2: Bar graph displaying the percentage of correct responses for both tests.



A paired-sample t-test was done to compare the learner performances on the tests. Table 4.2 presents the statistical results calculated. In the table, I have presented the statistics for the inclusion of Item 1 and when it is not. The choice of excluding Item 1 was to see the statistical information on the items that focused on operating on integers.

Table 4.2: Statistical measures on the percentage of correct responses of the entire grade 10's

	TwF	TwS
p-value	< 0.05	
t-value (including Item 1)	0.03384	
t-value (excluding Item 1)	0.058383	
Mean (including Item 1)	71.4%	80.6%
Mean (excluding Item 1)	68.7%	78.3%
Standard deviation (including Item 1)	13.8	11.0
Standard deviation (excluding Item 1)	13.5	10.6

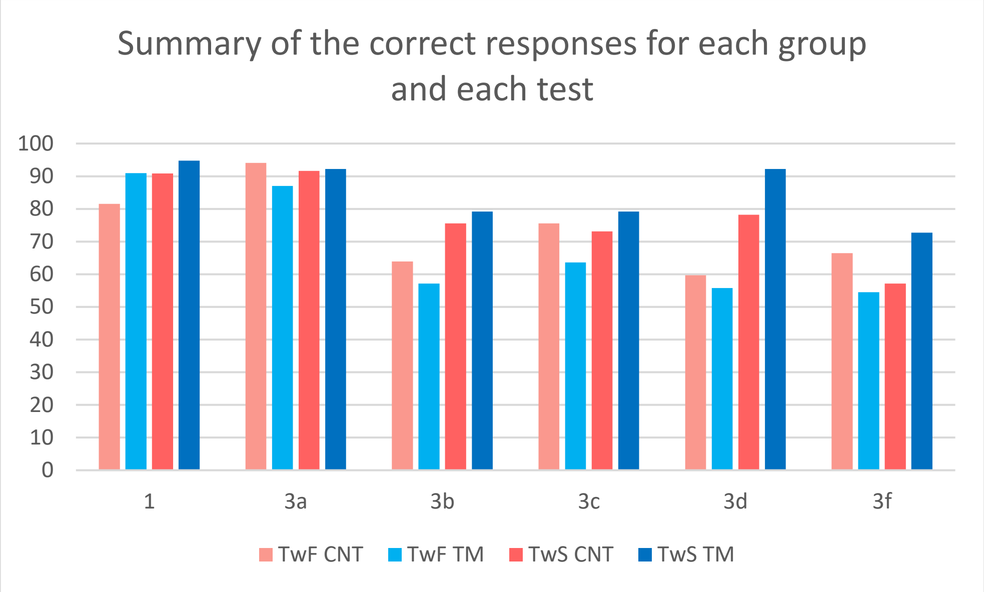
The t-test revealed that the change in performance from TwF to TwS was statistically significant at the 95% level when Item 1 is included. However, when I excluded Item 1, then the change was not statistically significant at the 95% level. As perceived in Figure 4.2, the best answered were Items 1 and 3a. Based on these results, it can be deduced that majority of

the learners can order and compare integers correctly and can carry out a basic operation of $5 - 7$ (Item 3a). Also evident was that Items 3b and 3d had substantial increases from the TwF to the TwS in terms of percentage of correct responses where the percentage increased from 61.2% to 77% and for Item 3d and in Item 3b it was 58.2% to 83.7%.

4.2.2 Overview of the Separate Groups.

This discussion is going to be split further into two sections. The first comparison is on the performance between each test per group, the second comparison is on the groups themselves. The information is presented in Figure 4.3. The red bars represent the percentages from the CNT group and the blue bars represent the TM group. The lighter bars represent the TwF, whereas the darker bars are the percentages from the TwS.

Figure 4.3: Bar graph displaying the correct responses for each group for each test.



A. Comparison of the Performance of Each Group in the Two Tests.

i. Transition Group (TM)

Based on the statistical tools, the following information was gathered:

Table 4. 3: Statistical measures on the percentage of correct responses from the TM group.

	TwF	TwS
p-value	<0.05	
t-value (including Item 1)	0.0090	
t-value (excluding Item 1)	0.0091	
Mean (including Item 1)	68.2%	85.1%
Mean (excluding Item 1)	63.6%	83.1%
Standard deviation (including Item 1)	16.5	9.1
Standard deviation (excluding Item 1)	13.5	8.7

A paired-sample t-test was conducted to compare the TwF to the TwS for the TM learners. Although Item 1 was removed, to focus solely on the operation items, it was found that the gains are statistically significant at the 95% level. The difference in the mean (including Item 1) is 16.9 p.p[2]. When the focus is solely on Items 3a-f, the t-score was like the latter and the difference in means is higher. This means that more learners in the TM group obtained the correct response for the operating questions. As reflected in Figure 4.2, the correct responses

[2] p.p – this refers to percentage point which is the difference between the two percentage for that row.

in Item 3d increased from the TwF to the TwS. There were increases across all items especially with Items 3b; 3c; 3d; 3e and 3f, which reflected significantly higher increases than the other items.

ii. Comparison Group (CNT)

Based on statistical tools, the following information was calculated and presented in Table 4.4:

Table 4.4: Statistical measures on the percentage of correct responses from the CNT group.

	TwF	TwS
p-value	<0.05	
t-value (including Item 1)	0.1876	
t-value (excluding Item 1)	0.2847	
Mean (including Item 1)	73.5%	77.7%
Mean (excluding Item 1)	71.9%	75.1%
Standard deviation (including Item 1)	12.9	12.8
Standard deviation (excluding Item 1)	13.7	12.4

A paired-sample t-test was conducted to compare TwF-CNT and TwS-CNT. The results indicate that improvement was not statistically significant at the 95% level between the tests written. This can also be seen with the comparison of the averages, from the TwF to the TwS, the average increased by only 2.1p.p., when Item 1 was included. It was seen from Figure 4.3 that three of the six items (Items 3a, 3c and 3f) had a decrease in the percentage of correct responses, indicating that as the year progressed these learners' performance did not improve and therefore their understanding of negative numbers could be seen as a pseudoconcept.

B. Comparison of the Two Groups.

With regards to the TwF, the learners in the CNT group performed better than the learners in the TM group apart from Item 1, as exhibited in Figure 4.3. From the TwF, Item 3a produced the most correct responses in the CNT group, where over 90% of learners obtained the correct response. In the TM group this finding differed in the TwF as over 90% of learners answered Item 1 correctly. The worst item in the TwF for both groups, was Item 3d (less than 60%) of the TM group had correct responses. Item 3b did not fare well either. In the TwS, Item 1 and Item 3a had the best overall performance with both groups having more than 80% of learners giving the correct response for both tests, however there was a slight decrease in percentage (2.5p.p.) with the CNT group for Item 3a. An improvement in all the items from the TM group was noted, whereas in the CNT group improvements were only evident in Item 1, Item 3b and Item 3d for the learners. In the proceeding chapters the possible reasonings resulting in the decrease of correct responses in the other items.

Majority of the learners were able to order and compare integers (Item 1) and subtract a minuend that is smaller than the subtrahend (Item 3a). The observation revealed that at least 50% of the learners were able to operate with integers in simple questions that contained two numbers, which increased to at least 70% of learners in the TwS. An exception was Item 3f for the CNT learners.

Based on Figure 4.3, for both Items 3b and 3d, the CNT group in the TwF performed better than the TM group, for Item 3b the CNT group acquired 63.9% whereas the TM group got 57.1% and Item 3d the CNT group got 59.7% and the TM group got 55.8%. In the TwS the TM group performed better, for Item 3b the TM group obtained 79.2% and the CNT group 75.6%. For Item 3d, the TM group attained 92.2% and the CNT group 78.2%.

The same is seen for Items 3c and 3f. When looking closely at Items 3c and 3f, the learners in the CNT group had a decrease in the overall achievements from the TwF to the TwS (Item 3c: 75.6% to 73.1% and Item 3f: 66.4% to 57.1%). More of these percentages are seen in Appendix D. Both items had a minuend that was a negative number which then can suggest that learners in the CNT group struggled with these types of questions in the TwS In the chapters to follow I will unravel the type of responses seen.

4.3 Conclusion

This chapter gave an insight on the performance of grade 10 learners on the topic of integers, by perusing their correct response for each item. This chapter highlighted that the TM group performed better during one academic year compared to the CNT group. In the next chapter, each item is examined in depth with the focus on the prevalent errors made in each item for each group, and the possible reasoning for the errors occurring. The purpose of this rigorous examination into the errors was tackled to garner a better understanding of why learners make certain errors and whether the errors are persistent. I will also discuss patterns seen between related items and assumptions that can be made about these patterns.

CHAPTER 5: ANALYSIS AND INTERPRETATION OF LEARNERS' INCORRECT RESPONSES

In Chapter 4, I discussed the overall performance of each item by referring to the correct responses given. Chapter 4 formed the quantitative method of this study whereas this chapter will use qualitative methods to analyse the data. In this chapter, the discussion focuses on the incorrect responses. The chapter is divided into two sections. The first section discusses the possible reasonings behind the responses based on the coding presented in Chapter 3. The second section focuses on linking the responses to the theories and concepts that were discussed in Chapter 2. A discussion on how I linked the responses of simple integer questions to the theory also ensues. Table 5.1 summarises important information regarding the participants of the study.

Table 5. 1: Reminder of important information

Grade 10 learners	Groups	Number of learners		Approximate percentage of one learner	
Combined (C)	Teachers who did not attend the course (CNT)	119	196	0.8%	0.5%
	Teachers who did attend the course	77		1.6%	

The table summarises the number of learners per group, the percentage weight of one learner per group (0.8% - CNT and 1.6% - TM), as well as the percentage weighting of one student (0.5%) compared to the entire grade 10 learner group. As a reminder to the reader, there were no interviews conducted in this study, due to the collection of the data occurred in 2018. The purpose of this study is purely focused on analysing errors that learners make and give possible reasons for the errors.

5.1 Interpretation of Responses

In this section, I discuss the common incorrect responses found in the six integer items analysed. The focus of this section is to present the most frequent incorrect responses and the possible reasoning behind the response based on the codes presented in Chapter 3. In the responses there were two responses, which though minor, need to be acknowledged, namely, *other* and *calculation error.*

5.1.1 Other Responses and the Calculation Error

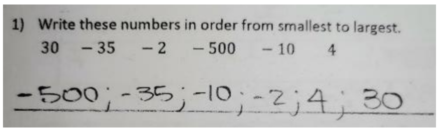
There were certain responses that were made by a small number of learners (fewer than 3% of learners in each item) that could not be easily interpreted. These responses were coded as *other*. For instance, in Item 3b, there were responses such as -54 and -8, which learners gave to $5 + (-7)$. Each response was made by one learner and therefore collated under the *other* code. Since these responses were made by minimal learners, they have negligible influence on the analysis.

Learners were not permitted to use calculators in the test, hence slips in calculating were inevitable. For example, $5 - 7 = -3$, the learner subtracted correctly but instead of getting -2, s/he got -3. The *calculation error* was not influenced by the misuse of the minus symbol but rather it arises from poor calculation skills. These types of errors are grouped with the general response that they should have obtained if they had calculated correctly. Throughout the analysis I will give examples of certain *calculation error/s* that was evidenced in the responses of the learners. These *calculation errors* did not negatively impact the validity of my interpretations, as it was made by less than 3% of the sample population and the actual integer procedures that my study focuses on was still achieved.

5.1.2 Ordering of Integers

The first item focused on the ordering of integers. Figure 5.1contains Item 1 and a correct response.

Figure 5.1: Response given by learners T02 1001 024



As discussed in the previous chapter, majority of learners revealed competency in this item, with 85.2% correct responses for the TwF which then increased to 92.3% for the TwS. There was a greater increase with the CNT group, however overall, the TM learners still performed better. Table 5.2 summarises the responses and the percentage of learners.

Table 5.2: Summary of responses in Item 1

Reponses	**Combined (%)**		**CNT (%)**		**TM (%)**	
	TwF	**TwS**	**TwF**	**TwS**	**TwF**	**TwS**
Correct	85.2	92.3	81.5	90.8	90.9	94.8
Magnitudes	9.1	6.6	11.8	7.6	5.2	5.2
Combination	1.5	1.0	2.4	1.6	N/A	N/A
Other	4.0	N/A	4.8	N/A	3.2	N/A

There were two prevalent errors. The first is the most common error of arranging the negative numbers incorrectly. The second is an error where learners combined both positive and negative numbers. As majority of learners got this item correct, the assumption that the learners can order and compare integers was made.

The error of '$-2;\ -10;\ -35;\ -500;\ 4;\ 30$' is the most frequent error seen. This response is referred to as *magnitudes.* This form of error reveals that learners can distinguish between the positive and negative numbers, however when it comes to the ordering of negative numbers the learners are ordering the numbers according to the absolute values in saying that $-2 < -10$ (because 2 is less than 10). The percentage of TM learners giving this response stayed the same. Only one learner scribed the same response for the two tests, the other learners who gave this incorrect response in TwF, then achieved a correct response in TwS. There were also learners who got the item correct for TwF and then made this error in TwS, this could be due to uncertainty in the understanding of the concept.

The response of *combination* is where the learners did not distinguish between positive and negative numbers but rather ordered the numbers according to their absolute values, thus $-3 > 2$ since $3 > 2$. This error was only seen in the CNT group and not the TM group. When a learner is unable to order integers correctly, they think of negative numbers as a *spontaneous concept*, which will be elaborated on in Chapter 5.2.

The *other* response includes responses where learners had ordered positive numbers incorrectly, but the ordering of the negative numbers was presented correctly or ordering the numbers from largest to smallest. This is the only item where the percentage of *other* responses was higher than the threshold set out. This does not affect the validity of the interpretations made on ordering of integers since majority of the learners were able to order the numbers correctly.

5.1.3 Operating on Integers with Two Terms.

I present the findings of five items: Items 3a, 3b, 3c, 3d, 3f, (excluding 3e). The items presented all consisted of two terms with the minus symbol at different positions. For each of the items, I will discuss the common responses and their potential reasoning according to the *conceptual framework for learners' interpretation of negative numbers*.

A. Item 3a: $5 - 7 =$

This item focused on the subtraction of a bigger number from a smaller number, which resulted with an answer of -2. With the inclusion of *calculation errors* in the correct responses, the data shows that there was an overall percentage decrease. This percentage decrease means that the learners obtained the correct answer in the TwF then got the wrong answer in the TwS. In the previous chapter this was true for only the CNT group as I did not include *calculation errors* in the previous chapter.

As seen in Table 5.3 there was one prevalent error in response to this item. This error was a response of 2, which comes from subtracting five from seven and thus has been given the code of *operating from right to left* (Vlassis, 2004).

Table 5.3: Summary of responses in Item 3a

Response	**Combined (%)**		**CNT (%)**		**TM (%)**	
	TwF	**TwS**	**TwF**	**TwS**	**TwF**	**TwS**
Correct	96.4	93.9	96.6	94.1	96.1	93.5
Operating from right to left	3.1	4.6	2.5	5.0	3.9	3.9
Other	0.5	1.5	0.8	0.8	N/A	2.6

This error was made by less than 5% of the grade 10 learners. More learners in the CNT group gave this response in the TwS, whereas for the TM group the percentage remained the same. The increase in the CNT group suggests that more learners in the TwS are *operating from right to* left, this increase is what influenced the increase of the percentage of the combined grade 10 learners. None of the learners in the CNT group gave the same incorrect response twice. The learners in the TwS for the CNT group for this error all had the correct response in the TwF. The same occurred for the TM group. All the learners from the TM group who submitted this incorrect response in the TwF, gave the correct response in the TwS. This indicates that there is still uncertainty with some learners when it comes to subtracting a bigger number from a smaller number.

B. Item 3b: $5 + (-7) =$

Item 3b is the sum of two terms, the first addend is a positive number and the second is a negative number in brackets. The sum of the numbers is -2. There was an overall increase in the number of correct responses. The TM group had a greater percentage increase than the CNT group. There were four prevalent incorrect responses to this item. These are *multiplicative rule signalled by brackets*, *detachment of the minus symbol*, *operating from right to left* and *ignoring the minus symbol*. Some *calculation errors* were also evident in these responses. Table

5.4 is a summary of the codes of the prevalent responses and the percentage of learners who gave these responses. Each response, except for the other response, will be discussed in its own sub-section.

Table 5.4: Summary of responses in Item 3b

Codes	**Combined (%)**		**CNT (%)**		**TM (%)**	
	Feb.	**Sep.**	**Feb.**	**Sep.**	**Feb.**	**Sep.**
Correct	61.7	77.0	63.9	75.6	58.4	79.2
Multiplicative rule signalled by brackets	14.3	2.6	13.4	1.7	15.6	3.9
Detachment of the minus symbol	11.7	7.7	11.8	8.4	11.7	6.5
Operating from right to left	6.6	5.1	6.7	2.5	6.5	9.1
Ignoring the minus symbol	4.6	6.6	4.2	10.0	5.2	1.3
Other	1.0	1.0	N/A	1.6	2.6	N/A

Multiplicative Rule Signalled by Brackets.

This was the most frequent error in the TwF. The codes indicate that since there are brackets, it is assumed that learners would multiply 5 and -7 to get the answer of -35. In the TwF, over 14% of the grade 10 learners had given this response which then decreased to 2.6%. Included in this error was 1.5% of learners who ignored the minus symbol and just focused on multiplication, they multiplied 5 and 7 to get 35. This error was seen in the TwF and not in the TwS. Between the CNT and TM groups, the CNT group had a greater decrease.

Detachment of the Minus Symbol

In this error, it is assumed that learners detached the minus symbol, focused on the operation, and then re-attached the symbol to the answer. It is assumed based on the codes that learners added 7 to 5 to get 12 and re-attached the minus symbol to get -12. In the TwF, just over 11% of learners made this error, which then dropped in the TwS. This was the most common error in the TwS.

Operating from Left to Right.

In allocating these error codes, I assumed learners did the following: $5 + (-7) = 5 - 7$, however learners could have subtracted 5 from 7 to get 2. Learners applied the symmetrical nature to the signs correctly to obtain $5 - 7$, but then operated from right to left to get the answer of 2. 6.6% of grade 10 learners gave this response in the TwF. More learners in the TM group gave this response in the TwS, demonstrated by an increase of 2.6 p.p. Two learners in the TM group gave the same response in both tests, another two learners had the correct response in the TwF and then gave this response in the TwS. The remainder had different responses in both tests.

Ignoring the Minus Symbol

In this response the focus was on the addition of the two numbers and the minus symbol was ignored. It is assumed that learners added 5 to 7 and got 12. This error was made by only 4.6% of grade 10 learners in the TwF but increased to 6.6% in the TwS. The CNT group was the only group that had an increase of 5.8p.p. from the TwF to the TwS. From the 10% of learners, 50% of them attained the correct answer in the TwF and then made this error in the TwS, whereas the other half had another error in the TwF. No learner gave the same error twice.

C. Item 3c: $-5+7=$

This item focused on the sum of a negative number and positive number that resulted in the answer of 2. Overall, the learners performed better on this item than the previous item. When comparing the two groups, the learners in the CNT group performed better than the learners from the TM group in the TwF, however this changed in the TwS. The CNT had a decrease of 1.7p.p. whereas the TM group had an increase of 15.6p.p. as presented in Table 5.5.

There were three prevalent errors seen in this item: *applying the symmetrical nature* (included a *calculation error), detachment of the minus symbol*, and lastly *ignoring the negative sign* (included a *calculation error* of 11). Table 5.5 summarises the responses seen. Each error will be discussed in detail.

Table 5. 1: Summary of Responses in Item 3c

Codes	**Combined (%)**		**CNT (%)**		**TM (%)**	
	TwF	**TwS**	**TwF**	**TwS**	**TwF**	**TwS**
Correct	71.4	76.5	75.6	73.9	64.9	80.5
Applying the symmetrical nature	13.8	10.2	13.4	11.7	14.3	7.8
Detachment of the minus symbol	9.7	7.6	6.7	10.1	14.3	3.9
Ignoring the minus symbol	5.1	5.1	4.2	4.2	6.5	6.5
Other	N/A	0.5	N/A	N/A	N/A	1.3

The first error seen is coded as *applying the symmetrical nature*. It is assumed that learners applied the multiplicative integer rule inappropriately to obtain the operation. Based on the code it is assumed that learners subtracted 7 from 5, this operation was determined by applying the symmetrical nature of the minus symbol in front of 5 with the addition operation. In the

CNT group, 13.4% of the learners gave this answer in the TwF, which decreased to 11.7% in the TwS. The TM group showed great improvement with this error, by having a decrease of 6.5p.p. from the TwF to the TwS. Only two learners from the TM group gave the same error in both tests.

The next error was *detachment of the minus symbol*, it is assumed that the learners added 7 to 5 and got 12, and then re-attached the minus symbol from -5 to the answer to get -12. There was a percentage increase from the TwF to the TwS for the CNT group of 3.4p.p. This means that some learners had the correct response in the TwF and then gave this incorrect response in the TwS. The TM group had a decrease from 14.3% to 3.9%. From the 3.9% in the TwS of the TM group, all the learners had a different error in the TwF and made this error in the TwS.

The last error labelled as *ignoring the negative sign* assumed that learners added 5 to 7 to get 12. This is the only error where the TM percentage (6.5%) is higher than the CNT percentage (4.2%) in TwS. There was no percentage change from the TwF to the TwS for both groups. In the TM group, no learner made this error twice. Two learners had the correct response in the TwF and gave this response in the TwS and the remainder of learners had different incorrect responses.

D. Item 3d: $6-(-10)=$

This is the only item that had different numbers. To get the correct answer, learners should have done the following: $6-(-10)=6+10=16$. This item had the lowest percentage of learners (58.2%) who submitted the correct response in the TwF. Overall, the TM group (55.8%) performed worse to the CNT group (59.7%) in the TwF. However, the TM group had the better increase of the two groups, denoted by an increase of 36.4p.p. whereas the CNT group had an increase of only 18.5p.p.

There were four prevalent errors in this item, each with errors of different focuses. These responses were: *multiplicative rule signalled by brackets*, *detachment of the minus symbol* and *ignoring the minus symbol* (included the *calculation error* of -5). The fourth error is a combination of two errors: *ignoring the minus symbol and operating from right to left,* Table 5.6 summarises the data.

Table 5.6: Summary of responses in Item 3d

Response	**Combined (%)**		**CNT (%)**		**TM (%)**	
	TwF	**TwS**	**TwF**	**TwS**	**TwF**	**TwS**
Correct	58.2	83.7	59.7	78.2	55.8	92.2
Multiplicative rule signalled by brackets	15.8	5.6	16.0	5.8	12.6	5.2
Detachment of the minus symbol	7.7	5.1	6.7	6.7	9.1	2.6
Ignoring the minus symbol.	11.2	3.1	10.9	5.0	11.7	N/A
Ignoring the minus symbol and operating from right to left.	4.6	2.6	3.4	4.2	6.5	N/A
Other	2.6	N/A	3.4	N/A	1.3	N/A

Multiplicative Rule Signalled by Brackets.

It is assumed that learners did the following: $6 - (-10) \rightarrow -6(-10) \rightarrow 60$. This was the most common error in the TwF, made by 12.2% of grade 10 learners. It is assumed that learners overgeneralised the multiplicative rule of integers and since there are brackets presented in the item, learners multiplied the terms and the minus symbols resulting in a positive answer. As we can see in Table 5.6, fewer learners made this error in both groups in the *TwS*. There were a handful of learners who multiplied -6 to 10 and got -60 as their response, the focus was on multiplication and the minus symbol attached to the number was ignored. This was made by a handful of learners (3.6% of learners) in the TwF and an even smaller percentage (0.5%) of learners in the TwS.

Detachment of the Minus Symbol

The next error made in this response focused on addition. The assumed procedure is as follows: $6 - (-10) \rightarrow -(6 + 10) \rightarrow -16$. Learners have followed the multiplicative rule of integers correctly; however, they then re-attached the minus symbol to the answer. From the two groups, only the TM group had a decrease in percentage (6.5p.p.) from the TwF to the TwS, whereas the CNT group had no change in percentage (6.7%). Only two learners in the CNT group gave the same response in both tests.

Ignoring the Minus Symbol

This error focused on subtraction. It is assumed that the learners performed the following procedure: $6 - (-10) \rightarrow 6 - 10 \rightarrow -4$. More learners in the TM group gave this response in the TwF compared to the CNT group, however in the TwS, the TM group had 0 responses whereas the CNT group had 5% of learners giving this response. Two learners had the correct response in the TwF and then gave this response in the TwS.

Ignoring the Minus Symbol and Operating from Right to Left.

This last error of 4 is likely a combination of two responses discussed earlier. The assumed procedure is: $6-(-10) \rightarrow 6-10 \rightarrow 10-6 \rightarrow 4$. Learners could have ignored the minus symbol of -10 and operated from right to left to obtain a positive answer. This error was not common in the TwS. In the TwF, this was the third most common error in the TM group.

E. Item 3f: $\boldsymbol{-7-5} =$

This item involved subtracting a positive number from a negative number which resulted in an answer of -12. The percentage of the correct response for grade 10 learners was 63.8%. However, there was a decrease in the CNT group and an increase in the TM group in the TwS. In the TwS, the percentage of correct responses (64.3%) was the lowest in comparison to all the other items of the combined grade 10 learners.

This item had a total of three prevalent errors: *detachment of the minus symbol*, *applying the symmetrical nature*, and *ignoring the minus* symbol. Table 5.7summarises the information seen from the data collected.

Table 5.7: Summary of responses in Item 3f

Response	**Combined (%)**		**CNT (%)**		**TM (%)**	
	TwF	**TwS**	**TwF**	**TwS**	**TwS**	**TwS**
Correct	63.8	64.3	68.1	58.8	57.1	72.7
Detachment of the minus symbol	16.3	10.2	16.0	10.9	16.9	9.1
Applying the symmetrical nature	12.8	19.9	10.9	21	15.6	18.2
Ignoring the minus symbol	9.2	4.6	3.4	7.6	9.1	N/A
Other	1.5	1.0	1.7	1.7	1.3	N/A

The first and most common error in the TwF is the *detachment of the minus symbol*, the focus was on subtraction. It is assumed that learners had completed the equation as follows: $-7-5 \rightarrow -(7-5) \rightarrow -2$. About 16% of learners had given this response and this dropped to about 10%. In the CNT group, a handful of learners gave the same response in both tests, the rest either had different responses or went from the correct response to this. The same applied for the TM group.

The most common error of the TwS was *applying the symmetric nature*. It is assumed that the learners followed the following procedure: $-7 - 5 = 7 + 5 = 12$. Learners applied the symmetrical nature of the minus symbol inappropriately. More learners in both groups gave this response in the TwS than the TwF. Majority of the learners in both groups who made this error in the TwS, had the correct response in the TwF.

The last error being analysed is assumed to have followed the procedure as: $-7 - 5 \rightarrow 7 - 5 \rightarrow 2$. I gave this error the code of *ignoring the minus symbol* with the focus being on subtraction. This is another response where the CNT group had a percentage increase from the TwF to the TwS of 4.2p.p, whereas the TM group had no learners that gave this answer in the TwS. The TM group had a higher percentage in the TwF and the biggest decrease in the TwS.

The next section is a discussion of the link between the responses and the theory presented in Chapter 2.

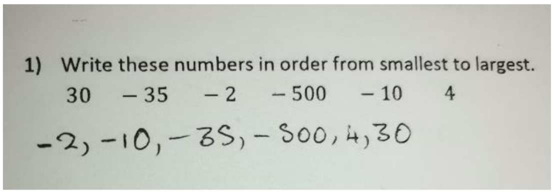
5.2. Linking the incorrect response to the framework describing the concept formation of negative numbers.

Each response presented above will be linked to the framework discussed in Chapter 2. The *conceptual framework for learners' interpretation of negative numbers* is discussed in greater detail in Chapter 2.4. The framework links theory of concept formation to the concept of negative numbers and the three natures of the minus symbol. The framework is designed in a way that apprises a learner's competency level in the phases of concept formation based on their errors made. The focus is on the first three levels of accepting a negative number and is highlighted in yellow. Majority of the learners are found within these levels in red, based on the errors they made. Based on the responses, the link to theory is discussed in the sub-sections to follow.

5.2.1 Magnitudes

Magnitudes suggests that learners are accepting negative numbers on a *relative/directed* level (Gallardo, 2008), because the ordering of negative numbers is done the same way as the ordering of positive numbers. Figure 5.2. displays this error made by a learner from the study.

Figure 5.2: Learner T20 1001 041 magnitudes response

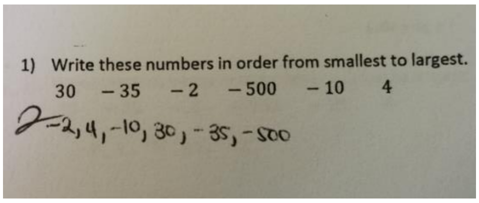


Being able to make a differentiation between positive and negative numbers then suggests that learners are in the first phase of concept formation, particularly that they understand negative numbers as a *spontaneous concept*. This assumption is made as they can differentiate between positive and negative numbers, however the learners order the negative numbers in the same way they would order positive numbers, since $5 > 2$, then $-5 > -2$. This was seen commonly in Item 1.

5.2.2 Combination

Combination reveals that learners are still in the *subtrahend level of acceptance* (Gallardo, 2008). This error was not made as often as the previous one, however Figure 5.3 shows what process the learners followed in answering Item 1.

Figure 5.3: Example of a combination response by learner C15 1001 031



As seen in Figure 5.3 the learner treated the negative numbers the same way as positive numbers and therefore still see the minus symbol according to their own *spontaneous concept* thus, $-2 < 4 < -10$ (Vygotsky, 1986). Learners focus on the absolute value or whole number instead of the integer.

Table 5.2: A conceptual framework for learners' interpretation of negative numbers, highlighting parts used in this study.

Levels of accepting a negative number	Triple nature of the minus symbol and possible reasons	Bofferding (2010) - Mental Models	Theory of Concept Formation (Vygotsky, 1986, Berger, 2004a; 2004b)	Examples
a) Subtrahend Number is subordinated to magnitude. $a - b;\ a > b$ Positive answer only. When two minus symbols are next to each other, ignore the sign and focus on the operation.	Binary Nature: It is limited to the subtraction of natural numbers to give a positive answer only. Right to left reasoning can be seen for the answer to be positive when you have a-b and a<b.	Models link to this when they are unable to differentiate between negative and positive numbers and combine the numbers when ordering them.	With regards to grade 10 learners, this is then the Heap phase of concept formation, however I term it as learners' understanding negative numbers as their spontaneous concept because learners have been exposed to the multiple symbols but still focus on one aspect (operation). Learners are still focused on the spontaneous concept of the minus symbol, with the idea of getting only a positive answer.	$5 - 7 = 7 - 5$ $-5 + 7 = 5 + 7$ $6 - (-10) = 6 - 10$ $= 10 - 6$ $5 + (-7) = 5 + 7$ $-7 - 5 = 7 - 5$ $-2; 4; 7; -10; 15; -25$
b) Relative/Directed Idea of opposite quantities in relation to a quality arises in the discrete domain (Gallardo, 2008). Idea of symmetry appears in the continuous domain.	Beginning of Unary Nature: Learners recognize the negative sign, and this gives them an idea of what it could mean in a contextual setting. (Debit/Credit, Increase/Decrease).	Models link to this with regards to the ordering and comparing of numbers. Learners will order the negative numbers in the same way they would order positive numbers.	Heap: Not being able to distinguish between positive and negative numbers. Learners will arrange negative numbers the same way they do positive numbers, based on the spontaneous concept of negative numbers that is related to context only. Associative type - nucleus being positive numbers, Bergers' (2004a) notion of 'Surface association', where learners will associate numbers according to the absolute value and not signs.	$-2;\ -4;\ -6;\ 2;\ 4;\ 6$
c) Negative Solution: Result of an operation/ solution to a problem/ equation. Allowing a solution to be a negative number.	Full exposure to all three natures of the minus symbol. Differentiation of the symbols are where errors will start arising and thus certain rules will mix with each other.	N/A	Associative types: Learners will associate certain rules due to certain symbols or objects. The nucleus can differ based on the object and learner. Nuclei: Operation or Sign. Surface association: relate what they did to one object with another based on a similar aspect seen in the objects. Template-orientated types: Applying a procedure done based on a focused symbol and apply the rules for that symbol.	Examples may vary depending on what is being focused on.
d) Formal Negative number. Having a clear distinguishable factor of negative numbers in comparison to positive numbers under the main heading of Integers.	Flexibility in all 3 natures of the minus symbol.	N/A	According to Berger (2004b), learners develop the mathematical concept when the links between objects are logical and consistent and when the focus is not a particular aspect but rather on the object which would contain concepts that learners should distinguish.	

5.2.3 Operating from Right to Left.

This response was seen in Items 3a, 3b and 3d. For example, learners gave the following responses:

- Item 3a: $5 - 7 \rightarrow 2$
- Item 3b: $5 + (-7) \rightarrow 2$
- Item 3d: $6 - (-10) \rightarrow 4$

In each of these responses given by the learners, the focus was on subtraction, however learners subtracted from right to left as it was more familiar to them. These responses suggest that learners are at a *subtrahend level* (level 1) of accepting a negative number (Gallardo, 2008). This is based on the idea that learners are not yet comfortable in getting a negative solution. Learners manipulate the item to get an answer that is positive; this could include combing two reasonings such as *ignoring a minus symbol* and *operating from right and left* like in Item 3d. By accepting negative numbers as subtrahends, the focus is more on the subtraction. Learners understanding of the minus symbol is at a *spontaneous concept* level, thus they would be in phase 1 of concept formation. This assumption is due to the idea that subtraction can only occur when a bigger number is subtracted by a smaller number, thus operating from right to left and obtaining a positive answer.

With regards to Item 3b, where learners applied the symmetrical nature of the minus symbol correctly but then operated from right to left to obtain their answer. This does not mean that learners who obtained the operation understand the minus symbol based on the *triple nature of the minus symbol* (Vlassis, 2004). It just means that the multiplicative rule of integers was applied, but the minus symbol was not acknowledged, therefore learners are still in the first level of accepting a negative number.

The error of *operating from right to left* suggests that learners are applying rules but do not have a fully developed understanding of the concept of negative numbers. Since the focus was on the signs and not the numbers or the entire item, it also reveals that the learners could be thinking of integers in a *surface association* way, associating signs to operation and not concerned about the position of the numbers (Berger, 2004b).

5.2.4 Detachment of the Minus Symbol

This response was seen in Items 3b, 3c, 3d, and 3f. In these responses learners obtained a negative solution. The following responses were seen for each item:

- Item 3b: $5 + (-7) \rightarrow -12$
- Item 3c: $-5 + 7 \rightarrow -12$
- Item 3d: $6 - (-10) \rightarrow -16$
- Item 3f: $-7 - 5 \rightarrow -2$

Obtaining an answer that is negative suggests that learners are in the second level of accepting a negative number (Gallardo, 2008). I suggest the second level, since learners do recognise that there is this minus symbol in front of a number, but it is used incorrectly. The focus was on the operation for Items 3b, 3c and 3d was addition and for Item 3f it was subtraction, Vygotsky (1986) terms the focus as the nucleus, it is what drives the question, any other symbols are then regarded as an afterthought, thus learners could detach and re-attach these symbols as they want. Based on this, learners are in the second phase of concept formation, more specifically, the stage of *surface association* (Berger, 2004b).

5.2.5 Ignoring the Minus Symbol.

This response was seen in Items 3b, 3c, 3 and 3f. Learners obtained a positive answer and thus are in the first level of acceptance (Gallardo, 2008). The following responses of Items 3b, 3c and 3f were noticed., Item 3d will be discussed further:

- Item 3b: $5 + (-7) \rightarrow 12$
- Item 3c: $-5 + 7 \rightarrow 12$
- Item 3f: $-7 - 5 \rightarrow 2$

Learners accept negative numbers at the *subtrahend level* which suggests that these learners also understand the concept of integers in the same manner of *spontaneous concepts* (Vygotsky, 1986). This is a consequence of the learners, regarding -7 and 7 as the same and that is why they can ignore the negative sign and continue with the operation. The negative sign has no meaning to the learner; thus, the *unary nature of minus symbol* is also disregarded (Vlassis, 2004).

The response in Item 3d ($6 - (-10) \rightarrow 4$) is a combination of *ignoring the minus symbol and operating from right to left*. This response means that learners are still basing their understanding of negative numbers as a *spontaneous concept* (Vygotsky, 1986). The two minus symbols next to each other could have been read as one minus symbol, thus not accepting negative numbers and being at level 1 (Gallardo, 2008). There is a response that is an exception. In Item 3d, the response of -4, learners got a negative solution, however they are still grappling with the *symmetrical nature of a minus symbol*. This means that the learners in this group would be in the second level of accepting a negative sign. Learners focused on other operations, thus are found in phase two of concept formation under *surface association* (Berger, 2004b).

5.2.6 Multiplicative Rule Signalled by Brackets.

This code was based on the stage of *template-orientated* (Berger, 2004b). In this stage of concept formation, found in the second phase, the focus is on a specific symbol. In this case the symbol in focus was the brackets which served as a signal for the learners to multiply the numbers. This could be based on prior knowledge from different topics such as algebra, where $x(-x) = -x^2$. The errors for this were seen in Items 3b and 3d. The responses were as follows:

- Item 3b: $5 + (-7) \rightarrow -35 \; or \; 35$
- Item 3d: $6 - (-10) \rightarrow 60 \; or - 60$

For both items, these responses were the most common errors made. With the focus being on multiplication, the *multiplicative rule of integers* was also applied to get the sign of the answer. This means that learners accept negative numbers at the third level described by Gallardo (2008). This is due to the idea that solutions can be negative.

The learners who did not consider the minus symbol attached to the number, would then accept the negative number on the stage two level, where the meaning of negative numbers is only affiliated to quantities and not to numbers. In such cases learners would give the incorrect product of 35 or -60 because the focus was on the brackets that led to multiplication and the signs was integrated with the rules linked to multiplication, i.e., *a negative multiplied with a negative is a positive*.

5.2.7 Applying the Symmetrical Nature.

This response was seen in both Items 3b and 3f, where the leading term was a negative number. The responses to these items were as follows:

- Item 3b: $-5 + 7 \rightarrow -2$
- Item 3f: $-7 - 5 \rightarrow 12$

Since learners can obtain a negative solution in the response of *applying the symmetrical nature*, it can be deduced that the learners are in the third level of acceptance, the *Negative solution* level (Gallardo, 2008). However, the procedure the learner went about obtaining the response suggests that learners are thinking in an *example-centered* type of way (Berger, 2004b). This is a type of association complex that Berger (2004b) presented. Learners focused on aspects that could have been presented in an example to them. Learners should have been exposed to the idea that when there are two signs next to each other then they would follow the *symmetrical nature* of a minus symbol to obtain the operation, for instance, $2 + (-7) = 2 - 7$. Based on this example, learners would associate the example to the item ($-5 + 7 \rightarrow 5 - 7$).

5.3 Conclusion

This chapter discussed the findings from the analysis of the data analysed. My interpretations were based on the codes discussed in Chapter 3.7. The findings reveal the most common errors seen for each item. From the interpretations of the errors, I was able to link the errors to the theory, using the *conceptual framework for learners' interpretation of negative numbers*. This link allowed me to see where learners are regarding forming the concept of negative numbers and at what level they accept negative numbers.

Two important findings were made. The first significant finding is that when there are multiple symbols, the focus will be on one operation and most times the minus symbol attached to a number is negated. There were responses where the minus symbol was taken into consideration which resulted in either the *detachment of the minus symbol, multiplicative rule signalled by brackets* or *applying the symmetrical nature*. The second important finding was that learners have still not grasped the concept of negative numbers and most of these learners do not accept negative numbers, when it comes to operating with integers in a two-term item. Majority of the learners can order and compare integers, but this is not an assurance the learners fully grasp the quantifying characteristics of negative numbers. In the next chapter, I will discuss responses to the research questions I presented in Chapter 1, as well as limitations faced in this study and recommendations for further studies.

CHAPTER 6: CONCLUSION

6.1 Introduction

This study investigated the errors grade 10 learners make when ordering and operating on integers. To do this, I developed a framework called: *A conceptual framework for learners' interpretation of negative numbers*, which is underpinned by Vygotskian theory (Vygotsky, 1986). This chapter answers the research questions and summarises the findings presented in Chapter 4 and Chapter 5. The limitations faced in this study are discussed. Recommendations for further research, recommendations for teachers when dealing with this topic and recommendations to the WMCS project are presented.

6.1.1 Data and Sample

The study focused on the responses of grade 10 learners to test items on ordering and operating on integers. The data had already been collected by the WMCS project in 2018. Learners wrote the same test twice in an academic year (February and September). The learners came from low performing schools in Gauteng. In Chapter 3, I discussed the sample selection process which enabled me to randomly select two groups from the project's larger data set. The first group was the comparison (CNT) group that comprised of 119 learners. The teachers of these learners had not participated in WMCS professional development as described in Chapter 3. The second group was the Transition Maths 1, or TM group, and it comprised of 77 learners who had been taught by teachers who participated in the WMCS professional development course.

The test that the learners wrote consisted of 45 questions, and I chose six questions which focused on integers. One item focused on the ordering of integers and the other five items focused on operating on two numbers (e.g., $5 - 7$, Appendix A). In Chapter 4 and Chapter 5, I presented findings on learners' performance on these items. Chapter 4 provided a comparison of the performance of the two groups. Chapter 5 focused on the common errors. These findings are summarised in Chapter 6.3. Table 6.1 is a summary of the test items that was used for this study, more details of the test items is seen in Chapter 3.5.

Table 6.1: Summary of items used for the study.

Question number	Item
1. Arrange from smallest to biggest	$30;\ -35;\ -2;\ -500;\ -10;\ 4$
3. Operate:	a) $5-7=$
	b) $5+(-7)=$
	c) $-5+7=$
	d) $6-(-10)=$
	f) $-7-5=$

6.1.2 Framework used to Interpret the Findings.

Interpreting the findings of the analysis, comprised of two facets, the first being the theoretical framework which was based on the *theory of concept formation* that Vygotsky (1986) presented and the interpretation of this theory by Berger (2004a and 2004b). The focus here was on the first two phases of concept formation, i.e., the *heap* phase and *thinking in complexes phase*, based on the descriptions that both Vygotsky (1986) and Berger (2004a and 2004b) had provided, I focused on specific stages within phase two and interpreted phase one in relation to my data. The focus thus was on *spontaneous concepts* which replaced phase one and the stages on phase two were: *surface association, example-centered,* and *template-oriented*. These were discussed in depth in Chapter 2.2.3.

The second key facet is the concept of negative numbers with particular focus on the *triple nature of the minus symbol* (Vlassis, 2004) and the *levels of accepting a negative number* (Gallardo, 2008). The focus here was on the different natures that the minus symbol (*unary nature, binary nature, and symmetrical nature*) has and possible reasoning for errors when the three natures are not fully grasped (Vlassis, 2004). The other focus was to establish the level a learner would be categorised under, based on the way the negative numbers were handled (Gallardo, 2008). This was part of the foundation of the codes used for this study and details were presented in Chapter 2.3.1.

Having considered both theory and concept, I designed a framework to aid in the interpretation of the errors. The title given to the framework is: *A conceptual framework for learners' interpretation of negative numbers*. The framework links the two key facets and the errors that could possibly arise within each of the level of accepting a negative number (represented by the rows in the table), these levels were presented by Gallardo (2008). Greater detail about the table, as well as the table itself is seen in Chapter 2.4, Table 2.2.

6.2 Research Questions

The research questions which guided this study were:

1. To what extent are there differences in learner performances between the CNT and TM groups?

a) Are there differences in the performance of each group on the test written in February?
b) Are there differences in the performance of each group on the test written in September?
c) What are the differences in the persistent errors?

2. What errors do learners make when responding to test items on integers?

a) What errors do learners make in the test written in February?
b) What errors do learners make in the test written in September?
c) Which errors persist from the test written in February to the test written in September and what does this mean regarding the concept formation of negative numbers?

6.2.1 Differences in Learner Performance

There were three sub-questions (a, b, and c above) that were presented under question 1, the focus is to see the differences between the two groups. The findings for the first question above were presented in Chapter 4. I will give a response to each sub-question and thereafter will give an overall response to the entire question 1.

A. *Are there differences in the performance of each group on the test written in February?*

When comparing the overall performances of the two groups (CNT group and TM group) regarding the TwF, it was recorded in Chapter 4.2.2, that the CNT group performed better in the operating questions (Items 3a, 3b, 3c, 3d, and 3f) and the TM group performed better in Item 1 which focused on the ordering of integers. This can be viewed in Figure 4.3, (the lighter coloured bars). The analysis showed that 73.5% of the learners in the CNT group had obtained a correct response in the February test, whereas only 68.2% of the TM learners obtained a correct response. There is a 5.3p.p difference between the two groups.

B. *Are there differences in the performance of each group on the test written in September?*

Based on the findings presented in Chapter 4.2.2 and Figure 4.3, the following differences were seen. The TM group performed better than the CNT group in all six items. The TM group had an average of 85.1% whereas the CNT group had an average of 77.7%. A 7.4p.p. difference was noted, which was much greater than the difference in the February test. The t-test revealed that the gains achieved by the TM group between the two tests were statistically significant at the 95% level, whereas this did not apply to the CNT group. There were three items, where the CNT group had a lower percentage of correct responses in the TwS compared to the TwF. These items were 3a, 3c and 3f, with Item 3f having the biggest decrease compared to 3a and 3c.

C. What are the differences in the persistent errors.

When surveying the overall performances of both groups, (Figure 4.2), there were no items where all (100%) of the learners got an item correct, indicating that there are errors that are persistently being made. It was heartening to observe that the percentage of errors made from the TwF to the TwS decreased across all items, meaning that there was a decrease in persistency in all errors in the entire grade 10 sample. However, when we hone on the performance of each group a different picture unfolds. In the TM group there was a decrease in the persistence of errors, however with the CNT group, there were three items (3a, 3c and 3f) where the percentage of errors increased from February to September, Item 3f, had the greatest difference in percentage suggesting that the errors were more persistent in this item for the CNT group, whereas in the other items the difference was not substantial. Items 3c and 3f are the only two items where the leading term is a negative number, thus the errors seen in these items were similar. The differences are small for the TM group, but for the CNT group there were considerable differences for Item 3f and decreased for Items 3c and 3a.

Regarding the principal question: *To what extent are there differences in learner performances between the CNT and TM groups?* The TM group had an increase in percentage of correct responses in all the items whereas the CNT group only had an increase in percentage for three out of the six items. Based on the findings in Chapter 4, the TM group had fewer persistent errors compared to the CNT group. For the CNT group, Item 3f was the only item where there was a considerable decrease from the TwF to the TwS regarding the percentage of achieving a correct response, meaning that the errors made in the TwF were persistent in the TwS. The errors made in all items will be discussed in responding to the second question.

6.2.2 Errors on the Test Items

There were three sub-questions that was presented under the second research question to be considered and these are presented in separate sections. The responses of these questions are based on the findings presented in Chapter 5. Table 6.1 summaries the errors made, according to their code for both tests. The comment of *same* means that for both groups the error was seen in the TwS and the comment of *only seen in the CNT group* means that this error was made by only the CNT in the TwS. The 'X' placed under the group columns refer to that group making the error.

Table 6. 2: Types of errors made in the TwF and TwS

Items	**Error codes**	**Grade 10 groups**		**Comment on frequency in TwS**
		CNT	**TM**	
Item 1: Ordering of Integers	Magnitudes	X	X	Same
	Combination	X	Not seen	Same
Item 3a: $5 - 7 =$	Operating from right to left	X	X	Same
Item 3b: $5 + (-7) =$	Multiplicative rule signalled by brackets	X	X	Same
	Detachment of the minus symbol	X	X	Same
	Operating from right to left	X	X	Same
	Ignoring the minus symbol	X	X	Same
Item 3c: $-5 + 7 =$	Applying the symmetrical nature	X	X	Same
	Detachment of the minus symbol	X	X	Same
	Ignoring the minus symbol	X	X	Same
Item 3d: $6 - (-10)$	Multiplicative rule signalled by brackets	X	X	Same
	Detachment of the minus symbol	X	X	Same
	Ignoring the minus symbol	X	X	Only seen in the CNT group
	Ignoring the minus symbol and operating from right to left	X	X	Only seen in the CNT group
Item 3f: $-7 - 5 =$	Detachment of the minus symbol	X	X	Same
	Applying the symmetrical nature	X	X	Same
	Ignoring the minus symbol	X	X	Only seen in the CNT group

A. *What errors do learners make in the test written in February?*

The errors discussed are the most frequent errors seen in the responses to each item. Table 6.1 summarises the errors made, according to their code, in the TwF and TwS. In this discussion I will focus on the errors seen only in the TwF.

To a large extent, both groups of learners made the same errors except for error *combination* (discussed in Chapter 5.1.2), in this error, it was seen that learners combined both positive and negative numbers when ordering the numbers from smallest to biggest, it was assumed that learners did not differentiate between both negative and positive numbers, i.e. $-2 < 3$ but also

$3 < -5$, which only occurred with the CNT learners. The most common error that was seen in Item 1 (ordering of integers) was magnitudes, (discussed in Chapter 5.1.2). This is where learners order negative numbers according to their absolute values, i.e., $-2 < -3$, since $2 < 3$.

Regarding the operating Items 3a, 3b, 3c, 3d and 3f), there were five types of errors that were made in this test, each error will be discussed briefly in the sub-sections below. I will give a brief description of the error and the responses that were given in each error for each item, greater details about these errors were discussed in Chapter 5.1.3.

Operating from Right to Left

This was the most frequent error in Item 3a, but much less frequent in 3b and 3d. This error focused on subtraction as the main operation, however learners operated from right to left to avoid getting negative answers. The following incorrect responses are typical for each item:

- Item 3a: $5 - 7 \rightarrow 2$
- Item 3b: $5 + (-7) \rightarrow 2$
- Item 3d: $6 - (-10) \rightarrow 4$

Detachment of the Minus Symbol

In this error, learners detach the minus symbol that is joined to the number, focus on the operation in the item and then re-attach the minus symbol to their answer. This error was seen in four out of the five items (3b, 3c, 3d, and 3f). It was the most frequent error in Item 3f. The second most frequent error in the other three items are:

- Item 3b: $5 + (-7) \rightarrow -12$
- Item 3c: $-5 + 7 \rightarrow -12$
- Item 3d: $6 - (-10) \rightarrow -16$
- Item 3f: $-7 - 5 \rightarrow -2$

As seen in Item 3d, learners worked out the operation by first using the *symmetrical nature of the minus symbol* (Vlassis, 2004) and then re-attached the minus symbol to their final answer. In Chapter 5 under each item, more detail was discussed about this error.

Multiplicative rule signalled by brackets.

This error only applies to Items 3b and 3d, both of which contained brackets. The error was the most frequent error in both items. The learners multiplied the numbers together. The minus symbol attached to the answer may or may not have been considered when learners multiplied the numbers. This is explained in more detail in Chapter 5.2.6. Two responses for each item were seen, these were:

- Item 3b: $5 + (-7) \rightarrow -35\ or\ 35$
- Item 3d: $6 - (-10) \rightarrow 60\ or - 60$

Ignoring the Minus Symbol

In this error, it is assumed that learners ignored the minus symbol that is attached to the number and then worked out the question with the focus being on the operation. This error was seen in four out of five items, as with the error above, but it was the least recurring error for all the items, apart from Item 3d. It was then the second least frequent error. The responses were as follows:

- Item 3b: $5 + (-7) \rightarrow 12$
- Item 3c: $-5 + 7 \rightarrow 12$
- Item 3d: $6 - (-10) \rightarrow -4$
- Item 3f: $-7 - 5 \rightarrow 2$

Applying the Symmetrical Nature

This was the final error seen in the TwF. This error is only applicable to the questions that had two operating symbols that were not next to each other (Items 3c and 3f). It is assumed that learners applied the *symmetrical nature of the minus symbol* inappropriately. The appropriate way is when there are two symbols next to each other (for example: $5 + (-7)$), one would then apply the symmetrical nature: an operational signifier is inverted when next to a minus symbol (Vlassis, 2004) to get the operation (i.e., $5 - 7$). However, learners incorrectly applied this to these items and got the following results:

- Item 3c: $-5 + 7 \rightarrow -2$
- Item 3f: $-7 - 5 \rightarrow 12$

These are all the common errors seen in the TwF. Majority of the errors were made by both CNT and TM learners. In the next section I will discuss the errors seen in the TwS.

B. What errors do learners make in the test written in September?

Based on Table 6.1, majority of the errors made in the TwS were the same as the errors in the TwF, the only difference is that the TM learners did not make certain errors in Items 3d and 3f. These errors were: *Ignoring the minus symbol* and *ignoring the minus symbol and operating from right to left*. This shows that these errors were not as persistent as the others. What is also worth noting is that in Item 3b ($5 + (-7) =$), the error of, *multiplicative rule signalled by brackets*, the TM group only had the response of -35 and not 35. The CNT group had both errors. The TM group were less persistent with certain errors compared to the CNT group. In the next section, I focus on the persistent errors and how the errors relate to the concept formation of negative numbers.

C. *Which errors persist from the TwF to the TwS and what does this mean regarding the concept formation of negative numbers.*

The five errors discussed in the above sections are errors that persist not only between the two tests but also persist through the items. With the focus being on Items 3a, 3b, 3c, 3d and 3f, the most persistent errors are *detachment of the minus symbol* and *ignoring the minus symbol*, and they occur in four of the five items. Less persistent errors are operating *from right to left* and then the final two, *multiplicative rule signalled by brackets* (only with items that had brackets) and *applying the symmetrical nature* (only with items where the leading term was a negative number). Regarding the ordering of integers, the *magnitudes* errors are more consistent than the error of *combining numbers.*

6.3. Linking Errors to the Framework

Details about the errors analysed was given, in this discussion I will now link the errors to the framework: *conceptual framework for learners' interpretation of negative numbers,* which I developed. I will highlight some key insights from Chapter 5.2 regarding links between the errors and concept formation. This section is divided according to the levels of accepting a negative number, which were developed by Gallardo (2008). The levels are discussed in detail in Chapter 2.3.1 B. Table 6.2 summarises the errors made and the link it has with the levels of accepting a negative number and phases/stages of concept formation, where 'L' refers to the level and 'P' refers to the phase.

Table 6.3: Types of errors linked to the theory discussed in Chapter 2

Type of error	**Items**	**Level of accepting a negative number**	**Phase/stage of concept formation**
Combination	Item 1	Subtrahend level (L1)	Spontaneous concept (P1)
Magnitudes	Item 1	Relative/directed level (L2)	Spontaneous concept (P1)
Detachment of the minus symbol	Item 3b, c, d, and f	Subtrahend level (L1)	Surface association (P2)
Operating from right to left	Item a and d	Subtrahend level (L1)	Spontaneous concept (P1)
	Item 3b		Surface association (P2)
Ignoring the minus symbol	Item 3b, c and f	Subtrahend level (L1)	Spontaneous concept (P1)
	Item 3d	Relative/directed level (L2)	Surface association (P2)
Multiplicative rule signalled by brackets	Item 3b and d: -35 or 60	Relative/directed (L2)	Template-oriented (P2)
	Item 3b and d: 35 or -60	Negative solution (L3)	
Applying the symmetrical nature	Item 3c and f	Negative solution (L3)	Example-centered (P2)

Subtrahend Level

The first level of accepting a negative number, either includes positive responses that were given through manipulating the item to obtain a positive response, or the ordering of the integers in a way that the positive and negative numbers were combined. The included errors were *combination, detachment of the minus symbol, operating from right to left* and *ignoring the minus symbol*. Of the four errors, three are seen as a *spontaneous concept* because learners ordered or operated on these items based on their knowledge of negative numbers. The focus was still on treating the minus symbol as only subtraction and not seeing the minus symbol as a negative symbol.

Originally the first phase was presented as the *heap phase* (Berger, 2004a). However, the learners being in grade 10, have been exposed to negative numbers, thus according to theory it cannot be in the heap phase. Thus, learners understand negative numbers as a spontaneous concept, the similarities between heap and spontaneous concept are discussed in Chapter 2.2.3 A. From the description of a *spontaneous concept*, the following errors fell under this heading, namely: *combination, operating from right to left* (Items 3a and 3d) and *ignoring the minus symbol.*

Operating from right to left and *detachment of the minus symbol* belonged to phase two of concept formation, specifically *surface association*. It is in this stage that learners focus on one specific symbol, such as an operation and associate procedures based on that focused symbol and disregard any other symbol such as other structural signifiers like the negative symbol (Vygotsky, 1986). For Item 3b where learners operated from right to left ($5 + (-7) \rightarrow 2$), the focus was on applying the symmetrical nature and then the learner operated according to what is familiar to them. With respect to the error of *detachment of the minus symbol,* the focus was on the operation. Learners would focus on the operation and ignore the negative sign, once they have worked out their answer, they would then re-attach the negative sign to the answer.

Relative/directed Level.

In this level, negative numbers are accepted as quantities and not numbers that can be operated on. Learners differentiate between positive and negative numbers, thus the minus symbol in front of a number has some association to a spontaneous concept to the learner (Gallardo, 2008). The errors seen in this level were: *magnitudes, ignoring the minus symbol* (Item 3d) and *multiplicative rule signalled by brackets* (certain responses). Each of these errors belongs to a different phase or stage of concept formation.

The error of *magnitudes* is seen as an error due to the learner understanding negative numbers as a *spontaneous concept*. Even though learners can distinguish between the positive and negative numbers, they still order negative numbers according to their understanding and not the scientific understanding, thus seeing $-2 < -3$. The error of *ignoring the minus symbol* for Item 3d only where the following was seen: $6 - (-10) \rightarrow -4$. This error was placed under the stage, *surface association,* which is part of phase two of concept formation, since it is assumed that the focus was on subtraction and the minus symbol in front of the 10 was ignored. The last error in this level is *multiplicative rule signalled by brackets*. This error is also part of phase two of concept formation, however in a stage named *template-oriented thinking*. In this stage, learners follow rules or procedures according to a symbol and format that is familiar to them, for example: $x(x) = x^2$. However, there were certain responses where the minus symbol attached to the number was not considered and learners just focused on the rule of multiplication. These responses were: $5 + (-7) \rightarrow 35$ and $6 - (-10) \rightarrow -60$.

Negative Solution Level

In this level, learners take negative numbers into consideration. The errors in this level are more complex. These errors were: *multiplicative rule signalled by brackets* and *applying the symmetrical nature.* Both errors fall in the second phase of concept formation but under various stages. The error of *multiplicative rule signalled by brackets,* as above, is considered as *template-oriented,* the difference being that the responses here take the minus symbol attached to the number into account, thus the responses are: $5 + (-7) \rightarrow -35$ and $6 - (-10) \rightarrow 60$. Learners are multiplying both numbers and symbols to obtain their answers.

The second error, *applying the symmetrical nature,* as discussed earlier is when the learners apply the symmetrical nature inappropriately. This error is under phase two of concept formation, specifically under the stage of *example centred.* In this stage, the error is made based on an example that learners were exposed to such as $5 + (-7) = 5 - 7$. The assumption is that the learners apply the symmetrical nature to questions where the two symbols are not directly next to each other. The operation is determined from the example that the learners were exposed to. This is seen in Item 3c $(-5 + 7)$ or Item 3f $(-7 - 5)$. The procedure of the example is then applied to the question.

Regarding the ordering of integers there were two persistent errors: *magnitudes and combination.* For items involving operating of integers there were five main errors which were: *operating from right to left, detachment of the minus symbol, ignoring the minus symbol, multiplicative rule signalled by brackets* and *applying the symmetrical nature.* Each of these errors play a different role in forming the concept of negative numbers and accepting negative numbers.

6.4 Reflections

Initially the study was going to focus only on learners' responses and classifying their errors. As the study progressed, I realised that it was crucial to delve into the reasoning behind the errors which led me to the theory of concept formation (Vygotsky, 1986). This process of reworking initial ideas helped me realise and appreciate that there is a magnitude of considerations, then simply classifying errors. This study challenged me to go beyond a superficial error analysis, to explore possible reasons behind the errors. This level of theorizing opens the possibility of tackling the cause and not just the symptom of learners' incorrect thinking. Using the research works of the numerous authors assisted in creating a solid foundation to develop my framework which allowed me to analyse and interpret my findings in a significantly more meaningful way, using my theoretical tools to gain new insights into my data.

Having taught and tutored grade 10 learners, my learners and I often became overwhelmed with other topics such as *functions* and *trigonometry*. When learners make errors in these topics, we do not go back to fundamentals such as integers. This study revealed that there are grade 10 learners who still cannot perform basic integer operations. I was not taken aback by this, However, the type of errors that were seen in this study took me by surprise. Grade 10 learners are still avoiding the minus symbol in front of a number which suggests that they are operating with negative numbers at a very basic level.

6.5 Limitations Faced in this Study.

The lack of interview data and opportunities for interviewing participants was the greatest limitation that I faced in this study. As a result, I had to make assumptions and inferences relating to their thinking, based on written responses. This is one of the reasons I had to develop a suitable framework that could be used to analyse their written responses. The second limitation was that when analysing the operating items, there was minimal written work to analyse because often learners only wrote their answer which was a single number. Therefore, the procedures which learners likely used were assumed from prior literature such as Vlassis (2004) to make inferences about the responses.

Although this study is linked to a larger study on the impact of a professional development course, I did not focus on the pedagogical application, hence I cannot make claims about better teaching methodologies of integers by the TM teachers, although the results suggest that the course helped them in their teaching of integers. This finding is important for the WMCS project.

6.6 Recommendations for Teachers, Researchers and the WMCS Project

Having seen how grade 10 learners operate on integers, identifying their errors as well as revealing the stages of concept formation through the literature reviewed, this study may contribute to enhancing teachers' understanding of how learners learn and what the errors mean. Integers is first introduced to learners in grade 7 and then revised in grades 8 and 9 (DBE, 2011). It is seen in this study that learners in grade 10 are still struggling with this topic, therefore I recommend that teachers should take the time in grade 8 and 9 to consolidate this topic. From the errors related to negative numbers which learners make in other topics, mathematics teachers should become aware of learners' understanding of negative numbers. When teachers speak with their learners about errors, it may provide an opportunity to address the cause of the error and thereby assist in remediation. This has the potential to reduce the number of integer-related errors in other topics. From grade 7 onwards, the words used by the teacher in describing the various actions of the minus symbol are important and teachers should be mindful of this. This attention to language and the meaning of the minus symbol must be consolidated in the subsequent grades.

There is a need to explore learner performance on more complex integer items such as $5 + (-7) - 2 + 3$. The complex items can also include items that allow for basic algebraic simplification such as $7x + 5y - 9x + (-8y)$ or equation solving like $7x = -14$. By having more complex questions that relate to the framework developed, research can be done to determine more accurately the level of acceptance the learner is at, and more procedures will allow the research to have stronger reasoning that would link to the errors that were discussed in Chapter 5. More complex questions may give a clearer picture on the development of negative number concept. Although the lack of interview opportunities was a limitation in this study, it would be a recommendation for future studies.

6.7 Conclusion

The journey to get to this point in the research has been extremely challenging, yet worthwhile. Having done a similar study in my previous degree, I did not think that there would be major differences between this study and the previous one. However, even though there were similar outcomes in the types of responses, the reasonings behind the errors in this study is more in detailed and gives the reader a better understanding of the errors the learner makes regarding concept formation of negative numbers. This is due to the framework that was developed to analyse such errors. The positive outcome when comparing this research to my prior research is that even though grade 10 learners are still struggling with operating on integers, it is not as bad as compared to the grade 9 learners. There is some progression in learner achievement, however the progression is not where we would like it to be for grade 10 learners.

Based on this study, it can be deduced that many grade 10 learners are uncomfortable with the minus symbol, and although many of the learners can order integers, most learners struggle to operate on them, as seen in the five errors which have been highlighted. More learners struggle in questions when there are multiple symbols in a question, such as brackets or more than one minus symbol. The *conceptual framework for learners' interpretation of negative numbers,* that I developed can be a useful tool to use in understanding learners' ideas of negative numbers. In this study, I made clear links between the errors and the relation to the theory of concept formation (Vygotsky, 1986) and the levels of accepting negative numbers (Gallardo, 2008). The fact that the TM group performed better in the tests, would hopefully provide motivation for teachers to attend professional development to be able to offer the best version of themselves to their learners. In the end, teachers are the ones that can remediate these errors and therefore as researchers we want to assist teachers to recognize the errors, especially with topics that we take for granted.

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APPENDICES

Appendix A - Test Items

1) Write these numbers in order from smallest to largest.

$30 \quad -35 \quad -2 \quad -500 \quad -10 \quad 4$

3) Work out the answers.

a) $5 - 7 =$

b) $5 + (-7) =$

c) $-5 + 7 =$

d) $6 - (-10) =$

f) $-7 - 5 =$

Appendix B – Ethics Approval

WITS SCHOOL OF EDUCATION

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

SCHOOL OF EDUCATION ETHICS COMMITTEE

CONSTITUTED UNDER THE UNIVERSITY HUMAN RESEARCH ETHICS COMMITTEE (NON-MEDICAL)

CLEARANCE CERTIFICATE	PROTOCOL NUMBER: 2020ECE014M
PROJECT TITLE	Grade 10 learners' errors when operating on integers.
INVESTIGATOR	Sheldon Naidoo
SCHOOL/DEPARTMENT OF INVESTIGATOR	WITS SCHOOL OF EDUCATION
DATE CONSIDERED	17 June 2020
DECISION OF THE COMMITTEE	Approved unconditionally
EXPIRY DATE	Date of submission of the project report

ISSUE DATE OF CERTIFICATE 29 June 2020

CHAIRPERSON (Dr Paul Goldschagg)

cc: Supervisor: Prof Craig Pournara

DECLARATION OF INVESTIGATOR

To be completed in duplicate and ONE COPY emailed to the Ethics Office: Matsie.Mabeta@wits.ac.za .

I fully understand the conditions under which I am authorized to carry out the abovementioned research and I guarantee to ensure compliance with these conditions. Should any departure be contemplated from the research procedure as approved I/we undertake to resubmit the protocol to the Committee.

S Naidoo

Signature

30 June 202

Date

PLEASE QUOTE THE PROTOCOL NUMBER ON ALL ENQUIRIES

Appendix C – Ethics approval of WMCS

Research Office

HUMAN RESEARCH ETHICS COMMITTEE (NON-MEDICAL)
R14/49 Pournara

CLEARANCE CERTIFICATE	**PROTOCOL NUMBER: H17/01/01**
PROJECT TITLE	Investigating the impact of teacher participation in the transition Maths 1 professional development course on grade 9 learner attainment in mathematics
INVESTIGATOR(S)	Dr C Pournara
SCHOOL/DEPARTMENT	Wits School of Education/
DATE CONSIDERED	16 January 2017
DECISION OF THE COMMITTEE	Approved
EXPIRY DATE	24 January 2020

DATE 25 January 2017 **CHAIRPERSON** J. Knight

(Professor J Knight)

cc: Supervisor : N/A

DECLARATION OF INVESTIGATOR(S)

To be completed in duplicate and **ONE COPY** returned to the Secretary at Room 10004, 10th Floor, Senate House, University. Unreported changes to the application may invalidate the clearance given by the HREC (Non-Medical)

I/We fully understand the conditions under which I am/we are authorized to carry out the abovementioned research and I/we guarantee to ensure compliance with these conditions. Should any departure to be contemplated from the research procedure as approved I/we undertake to resubmit the protocol to the Committee. **I agree to completion of a yearly progress report.**

S. Naidoo
Signature

03 / 02 / 2023
Date

PLEASE QUOTE THE PROTOCOL NUMBER ON ALL ENQUIRIES

Appendix D – Percentages of learners for all the figures in Chapter 4.

Item	Test written in February (TwF)			Test written in September (TwS)		
	Whole group (C)	Controlled group (CNT)	Transition group (TM)	Whole group (C)	Controlled group (CNT)	Transition group (TM)
1	85.2	81.5	90.9	92.3	90.8	94.8
3a	91.3	94.1	87.0	91.8	91.6	92.2
3b	61.2	63.9	57.1	77.0	75.6	79.2
3c	70.9	75.6	63.6	75.5	73.1	79.2
3d	58.2	59.7	55.8	83.7	78.2	92.2
3f	61.7	66.4	54.5	63.3	57.1	72.7