Unsteady Slender Rivulet-Flow down an Inclined Porous Plane

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Declaration

I declare that this dissertation is my own unaided work. It is being submitted for the degree of Masters of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other university.

Angela Lowry-Corry

Abstract

The unsteady three-dimensional flow of a thin slender rivulet of incompressible Newtonian fluid down an inclined porous plane is investigated. The leak-off velocity is not specified in the model but is determined in the process of deriving the invariant solution. A second order nonlinear partial differential equation in two spatial variables and time and containing the leak-off velocity is derived for the height of the thin slender rivulet. Using Lie group analysis it is found that the partial differential equation can be reduced in two steps to an ordinary differential equation provided the leak-off velocity satisfies a first order linear partial differential equation in three variables. An exact analytical solution with a dry patch in the central region is derived for a special leak-off velocity. Two models are considered, one with the leak-off velocity proportional to the height of the rivulet and the other with leak-off velocity proportional to the cube of the height. Numerical solutions are obtained for the height of the rivulet using a shooting method which also determines the two-dimensional boundary of the rivulet on the inclined plane. The effect of fluid leak-off on the height and width of the rivulet is investigated numerically and compared in the two models. The conservation laws for the partial differential equation with no fluid leak-off are investigated. Two conserved vectors are derived, the elementary conserved vector and a new conserved vector. The Lie point symmetry of the partial differential equation associated with each conserved vector is obtained. Each associated Lie point symmetry is used to perform a double reduction of the partial differential equation, but the solutions obtained are not physically significant.

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Contents

Chapter 1

INTRODUCTION

1.1 Introduction

A rivulet is a thin three-dimensional fluid film which flows down an inclined plane under gravity. The fluid is generally considered Newtonian, but it could be non-Newtonian. A slender rivulet is a rivulet in which the rate of change in the lateral direction is greater than the rate of change in the longitudinal direction. A rivulet in unsteady flow implies that it changes over time.

Rivulets occur commonly in a wide range of geophysical, biological and industrial contexts, ranging from the flow of lava to the complex fluids in industrial devices such as condensers and heat exchangers [20].

There have been numerous studies on both the steady and unsteady state solutions of thin slender rivulets. The original study was done by Smith [18] who obtained a similarity solution describing flow of a rivulet emanating from a point source on the plane, when surface tension is neglected.

It is proposed to investigate the solution of a rivulet flowing down an inclined porous plane under gravity. The rivulet is taken to be a thin three-dimensional Newtonian fluid which is incompressible, under unsteady flow and symmetric about the line of greatest slope down the plane.

1.2 Literature review

The problem of flow down an inclined plane has been studied for numerous years. An example is the book by Batchelor [3] in 1967. These early studies were limited to two-dimensions or axial symmetry. In recent years studies have been done which extend into three-dimensional problems. Smith [18] found a similarity solution for the distribution of layer thickness in viscous source flow down an inclined plane. He concluded that the asymptotic layer-thickness profile is parabolic and that the flow spreads according to an $x^{\frac{3}{7}}$ power law down the plane. Lister [11] looked at flow from a point source and again used similarity solutions to gain insight into the dynamics of the model. Yatim *et al* [20] looked at unsteady flow of a thin slender rivulet of Newtonian fluid driven by gravity on an inclined plane. They concluded that sessile (fixed point rivulets) rivulets exhibit a finite-time blow-up, becoming singular everywhere at some instant.

The role of conservation laws in this problem are important as they could potentially provide some highly useful insights into what is happening in special cases. Naz *et al* [15] provided a comprehensive approach to conservation laws and the numerous methods available to find them. In [14] Mason and Anthonyrajah derived an invariant solution by associating a Lie point symmetry with a conserved vector. In [13] Maluleke and Mason used Lie point symmetry generators to derive new conserved vectors from known conserved vectors and to simplify the derivation of known conserved vectors for a nonlinear wave equation.

We will use Lie group analysis to reduce in two steps the partial differential equation for the rivulet height in three independent variables to an ordinary differential equation. Standard texts on Lie group analysis are "Lie Group Analysis of Differential Equations" Volume 1 [2] and Volume 2 [1], Symmetries and Differential Equations [4] and [5], Applications of Lie Groups to Differential Equations [16], Introduction to Symmetry Analysis [6], and the Progress in Industry report "Similarity Solutions for Unsteady Rivulets" [19].

Chapter 2

DERIVATION OF THE 3-DIMENSIONAL MODEL

2.1 Introduction

In this chapter we will first define the model and then use the Navier-Stokes equation and thin fluid film theory to derive the defining partial differential equation.

The rivulet is a thin three-dimensional Newtonian fluid, which is incompressible and in unsteady flow. The x-axis is down the line of greatest slope, the y-axis is the cross-slope axis and the z-axis is normal to the plane. Thus we expect a partial differential equation in three variables, x, y and t for the surface of the rivulet.

The surface of the rivulet is

$$z = h(x, y, t). \tag{2.1.1}$$

The flow is restricted to being symmetric about the x-axis ($y = 0$), thus $h(x, y, t)$ is an even function of y. The rivulet is bounded by the curves

$$y = \pm a(t, x). \tag{2.1.2}$$

2.2 Navier-Stokes equation

The model, as outlined, enables us to use the Navier-Stokes equation which is one vector equation. It splits into the following three components

x-component:

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g \sin \alpha, \tag{2.2.1}$$

y-component:

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right), \tag{2.2.2}$$

z-component:

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) - \rho g \cos \alpha. \tag{2.2.3}$$

We consider the case in which the rivulet is slender. It varies much more slowly in the x-direction than in the transverse y-direction. The conditions for the rivulet to be thin and slender are

$$H \ll D \ll L \tag{2.2.4}$$

where

H is the characteristic distance in the z-direction,

D is the characteristic distance in the y-direction (the transverse direction $D \sim 2a$),

L is the characteristic distance in the x-direction.

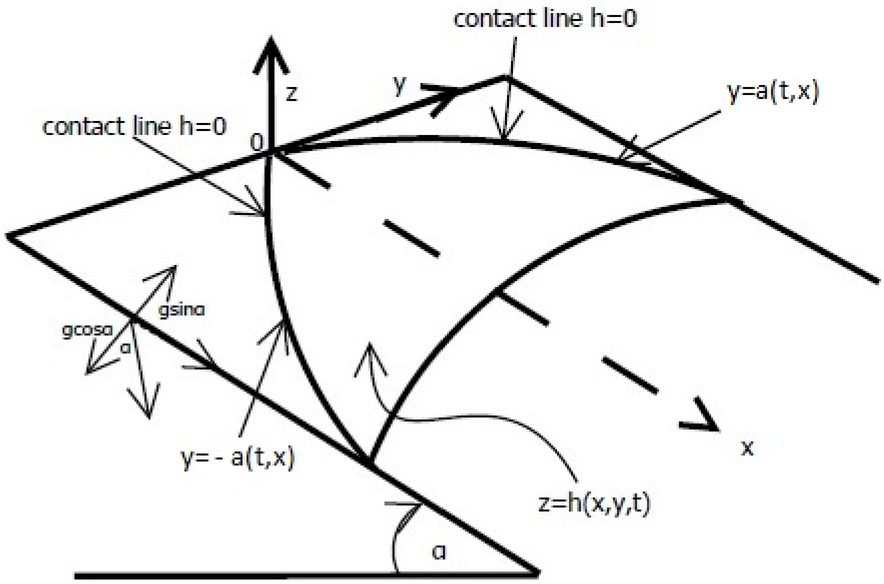

Figure 2.2.1: *Rivulet flow down an inclined plane*

2.3 The continuity equation

The continuity equation for an incompressible fluid is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0. \tag{2.3.1}$$

We define:

U to be the characteristic fluid velocity in the x-direction,

V to be the characteristic fluid velocity in the y-direction,

W to be the characteristic fluid velocity in the z-direction.

Thus the continuity equation becomes:

$$\frac{U}{L} + \frac{V}{D} + \frac{W}{H} \simeq 0. \tag{2.3.2}$$

In order to create a full 3-dimensional flow we assume all three terms in (2.3.2) have the same order of magnitude. Thus

$$V \sim \frac{D}{L}U, \quad W \sim \frac{H}{L}U, \tag{2.3.3}$$

where we have written V and W in terms of U.

The Reynolds number for the flow down the plane is given by

$$Re = \frac{inertia\, force}{viscous\, force} = \frac{UL\rho}{\mu}. \tag{2.3.4}$$

In the thin fluid film approximation it is assumed that

$$Re\left(\frac{H}{L}\right)^2 \ll 1. \tag{2.3.5}$$

2.4 Characteristic time

The characteristic time is

$$T = \frac{L}{U}. \tag{2.4.1}$$

The same value for the characteristic time is obtained if the characteristic length and velocity in the y and z directions are used. For

$$T_2 = \frac{D}{V} = D\frac{L}{DU} = \frac{L}{U}, \tag{2.4.2}$$

$$T_3 = \frac{H}{W} = H\frac{L}{HU} = \frac{L}{U}. \tag{2.4.3}$$

The three definitions for characteristic time agree, which shows that the scalings for velocity are consistent.

2.5 Characteristic pressure

In the x-component of the Navier-Stokes equation the pressure gradient could balance either the viscous term or the body force due to gravity. In the y-component the pressure gradient must balance the viscous term because there is no gravity component. Therefore we determine the scaling for pressure, P, by balancing the pressure gradient with the viscous term in the y-direction of the Navier-Stokes equation:

$$\frac{\partial p}{\partial y} \simeq \mu \left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right]. \tag{2.5.1}$$

In terms of magnitudes,

$$\frac{P}{D} \simeq \mu \left[\frac{V}{L^2} + \frac{V}{D^2} + \frac{V}{H^2} \right] \tag{2.5.2}$$

and therefore

$$\frac{P}{D} \simeq \mu \frac{V}{H^2} \left[\left(\frac{H}{L}\right)^2 + \left(\frac{H}{D}\right)^2 + 1 \right]. \tag{2.5.3}$$

Now, $H \ll L$ and $H \ll D$, and hence the characteristic pressure is

$$P = \mu \frac{LU}{H^2} \left(\frac{D}{L}\right)^2. \tag{2.5.4}$$

2.6 Dimensionless variables

We make the Navier-Stokes equations dimensionless using the dimensionless variables

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{D}, \quad \bar{z} = \frac{z}{H}, \quad \bar{p} = \frac{p}{P}, \quad \bar{t} = \frac{t}{T},$$
$$\bar{v_x} = \frac{v_x}{U}, \quad \bar{v_y} = \frac{v_y}{V}, \quad \bar{v_z} = \frac{v_z}{W}. \tag{2.6.1}$$

The three components of the Navier-Stokes equation become:

x-component (2.2.1):

$$Re \left(\frac{H}{L}\right)^2 \left[\frac{\partial \bar{v_x}}{\partial \bar{t}} + \bar{v_x}\frac{\partial \bar{v_x}}{\partial \bar{x}} + \bar{v_y}\frac{\partial \bar{v_x}}{\partial \bar{y}} + \bar{v_z}\frac{\partial \bar{v_x}}{\partial \bar{z}} \right]$$

$$= -\left(\frac{D}{L}\right)^2 \frac{\partial \bar{p}}{\partial \bar{x}} + \left(\frac{H}{L}\right)^2 \frac{\partial^2 \bar{v}_x}{\partial \bar{x}^2} + \left(\frac{H}{D}\right)^2 \frac{\partial^2 \bar{v}_x}{\partial \bar{y}^2} + \frac{\partial^2 \bar{v}_x}{\partial \bar{z}^2} + \frac{\rho g L \sin\alpha}{P}\left(\frac{D}{L}\right)^2, \tag{2.6.2}$$

y-component (2.2.2):

$$Re\left(\frac{H}{L}\right)^2 \left[\frac{\partial \bar{v}_y}{\partial \bar{t}} + \bar{v}_x \frac{\partial \bar{v}_y}{\partial \bar{x}} + \bar{v}_y \frac{\partial \bar{v}_y}{\partial \bar{y}} + \bar{v}_z \frac{\partial \bar{v}_y}{\partial \bar{z}}\right] = -\frac{\partial \bar{p}}{\partial \bar{y}} + \left(\frac{H}{L}\right)^2 \frac{\partial^2 \bar{v}_y}{\partial \bar{x}^2} + \left(\frac{H}{D}\right)^2 \frac{\partial^2 \bar{v}_y}{\partial \bar{y}^2} + \frac{\partial^2 \bar{v}_y}{\partial \bar{z}^2}, \tag{2.6.3}$$

z-component (2.2.3):

$$Re\left(\frac{H}{L}\right)^2 \left(\frac{H}{D}\right)^2 \left[\frac{\partial \bar{v}_z}{\partial \bar{t}} + \bar{v}_x \frac{\partial \bar{v}_z}{\partial \bar{x}} + \bar{v}_y \frac{\partial \bar{v}_z}{\partial \bar{y}} + \bar{v}_z \frac{\partial \bar{v}_z}{\partial \bar{z}}\right]$$

$$= -\frac{\partial \bar{p}}{\partial \bar{z}} + \left(\frac{H}{D}\right)^2 \left[\left(\frac{H}{L}\right)^2 \frac{\partial^2 \bar{v}_z}{\partial \bar{x}^2} + \left(\frac{H}{D}\right)^2 \frac{\partial^2 \bar{v}_z}{\partial \bar{y}^2} + \frac{\partial^2 \bar{v}_z}{\partial \bar{z}^2}\right] - \frac{\rho g H \cos\alpha}{P}. \tag{2.6.4}$$

Now

$$Re\left(\frac{H}{L}\right)^2 \ll 1, \quad \left(\frac{H}{L}\right)^2 \ll 1, \quad \left(\frac{H}{D}\right)^2 \ll 1, \quad \left(\frac{D}{L}\right)^2 \ll 1. \tag{2.6.5}$$

Therefore the three components of the Navier-Stokes equation reduce to

$$0 = \frac{\partial^2 \bar{v}_x}{\partial \bar{z}^2} + \frac{\rho g L}{P}\left(\frac{D}{L}\right)^2 \sin\alpha, \tag{2.6.6}$$

$$0 = -\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\partial^2 \bar{v}_y}{\partial \bar{z}^2}, \tag{2.6.7}$$

$$0 = -\frac{\partial \bar{p}}{\partial \bar{z}} - \frac{\rho g H}{P} \cos\alpha. \tag{2.6.8}$$

Expressing (2.6.6), (2.6.7) and (2.6.8) in dimensional form gives

$$0 = \mu \frac{\partial^2 v_x}{\partial z^2} + \rho g \sin\alpha, \tag{2.6.9}$$

$$0 = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v_y}{\partial z^2}, \tag{2.6.10}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g \cos\alpha. \tag{2.6.11}$$

The continuity equation is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0. \tag{2.6.12}$$

2.7 Boundary conditions

The rivulet satisfies boundary conditions at the base $z = 0$ and at the free surface $z = h(x, y, t)$.

$\mathbf{z = 0}$:

$$v_x(x, y, 0, t) = 0, \qquad (no \;\; slip) \tag{2.7.1}$$

$$v_y(x, y, 0, t) = 0, \qquad (no \;\; slip) \tag{2.7.2}$$

$$v_z(x, y, 0, t) = -w(x, y, t), \qquad (porous \;\; substrate) \tag{2.7.3}$$

where w is the leak-off velocity at which fluid is flowing into the porous substrate.

$\mathbf{z = h(x, y, t)}$:

Normal stress: In the lubrication approximation

$$p(x, y, h, t) = p_0, \tag{2.7.4}$$

where p_0 is the atmospheric pressure.

Tangential stress: In the lubrication approximation

$$\tau_{zx} = \mu \frac{\partial v_x}{\partial z} = 0, \quad \tau_{zy} = \mu \frac{\partial v_y}{\partial z} = 0 \tag{2.7.5}$$

and therefore since the tangential stress vanishes on the free surface

$$\frac{\partial v_x}{\partial z}(x, y, h, t) = 0, \tag{2.7.6}$$

$$\frac{\partial v_y}{\partial z}(x, y, h, t) = 0. \tag{2.7.7}$$

Kinematic condition: A fluid particle on the surface remains on the surface, which can be expressed as

$$\frac{D}{Dt}\left(z - h(x,y,t)\right)|_{z=h} = 0, \tag{2.7.8}$$

where $\frac{D}{Dt}$ denotes the material or convective time derivative. Hence

$$v_z(x,y,h,t) = \frac{\partial h}{\partial t} + v_x(x,y,h,t)\frac{\partial h}{\partial x} + v_y(x,y,h,t)\frac{\partial h}{\partial y}. \tag{2.7.9}$$

2.8 Deriving the model

Now we can derive the partial differential equation for the height, h, of the rivulet.

Firstly $v_z(x,y,h,t)$ is obtained by integrating the continuity equation, (2.6.12), with respect to z from $z=0$ to $z=h(x,y,t)$ and imposing the boundary condition (2.7.3) on v_z at $z=0$:

$$v_z(x,y,h,t) = -w(x,y,t) - \int_0^h \frac{\partial v_x}{\partial x}\,dz - \int_0^h \frac{\partial v_y}{\partial y}\,dz. \tag{2.8.1}$$

Now using the formula for differentiation under the integral sign [7], equation (2.8.1) becomes,

$$\begin{aligned} v_z(x,y,h,t) = &- w(x,y,t) - \frac{\partial}{\partial x}\int_0^h v_x(x,y,z,t)\,dz \\ &+ v_x(x,y,h,t)\frac{\partial h}{\partial x} - \frac{\partial}{\partial y}\int_0^h v_y(x,y,z,t)\,dz + v_y(x,y,h,t)\frac{\partial h}{\partial y}. \end{aligned} \tag{2.8.2}$$

From the kinematic condition (2.7.9), this simplifies to

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h v_x(x,y,z,t)\,dz + \frac{\partial}{\partial y}\int_0^h v_y(x,y,z,t)\,dz = -w(x,y,t). \tag{2.8.3}$$

Equation (2.8.3) is in the form of a conservation equation with a sink term.

We now need to solve for v_x and v_y. They are obtained from the Navier-Stokes equation, equations (2.6.9), (2.6.10) and (2.6.11) and from the boundary conditions (2.7.1), (2.7.2), (2.7.4), (2.7.6) and (2.7.7). Solving these equations gives

$$v_x(x,y,z,t) = \frac{\rho g z}{2\mu}\sin\alpha\,(2h - z)\,, \tag{2.8.4}$$

and

$$v_y(x,y,z,t) = \frac{\rho g z}{2\mu}\cos\alpha\frac{\partial h}{\partial y}\,(z - 2h)\,. \tag{2.8.5}$$

Substituting equations (2.8.4) and (2.8.5) into equation (2.8.3) and evaluating the integrals gives

$$\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu}\cos\alpha\frac{\partial}{\partial y}\left(h^3\frac{\partial h}{\partial y}\right) - \frac{\rho g}{3\mu}\sin\alpha\frac{\partial}{\partial x}(h^3) - w(x,y,t). \tag{2.8.6}$$

Now we make equation (2.8.6) dimensionless using the transformations

$$t^* = \frac{t}{T},\quad x^* = \frac{x}{X},\quad y^* = \frac{y}{Y},\quad h^* = \frac{h}{H},\quad w^* = \frac{w}{H}T \tag{2.8.7}$$

where T, X, Y and H have yet to be specified. Equation (2.8.6) becomes, in its dimensionless form:

$$\frac{\partial h^*}{\partial t^*} = \frac{\rho g}{3\mu}\cos\alpha\frac{H^3T}{Y^2}\frac{\partial}{\partial y^*}\left(h^{*3}\frac{\partial h^*}{\partial y^*}\right) - \frac{\rho g}{3\mu}\sin\alpha\frac{H^2T}{X}\frac{\partial}{\partial x^*}(h^{*3}) - w^*(x,y,t). \tag{2.8.8}$$

We therefore choose

$$\frac{\rho g}{3\mu}\cos\alpha\frac{H^3T}{Y^2} = 1, \tag{2.8.9}$$

$$\frac{\rho g}{3\mu}\sin\alpha\frac{H^2T}{X} = 1. \tag{2.8.10}$$

Now H, T, Y and X are unknown and ρ, g, μ, $\cos\alpha$ and $\sin\alpha$ are known. So we have two equations and four unknowns we must choose two and calculate the other two based on our assumption. This leads to two cases.

Case 1: Change of variables

We choose $H = 1$ and $T = 1$, then

$$X = \frac{\rho g}{3\mu}\sin\alpha \tag{2.8.11}$$

and

$$Y = \left(\frac{\rho g}{3\mu} \cos \alpha \right)^{\frac{1}{2}}. \tag{2.8.12}$$

We substitute (2.8.11) and (2.8.12) into equation (2.8.8) to obtain

$$\frac{\partial h^*}{\partial t^*} = \frac{\partial}{\partial y^*} \left(h^{*3} \frac{\partial h^*}{\partial y^*} \right) - \frac{\partial}{\partial x^*} \left(h^{*3} \right) - w^*(x, y, t). \tag{2.8.13}$$

Case 2:

We specify X to be the characteristic length in the x-direction and T to be the characteristic time. Then

$$H = \left(\frac{3\mu}{\rho g \sin \alpha} \frac{X}{T} \right)^{\frac{1}{2}}, \quad Y = \left(\frac{3\mu \cos^2 \alpha X^3}{\rho g \sin^3 \alpha T} \right)^{\frac{1}{4}}, \tag{2.8.14}$$

and equation (2.8.8) reduces to

$$\frac{\partial h^*}{\partial t^*} = \frac{\partial}{\partial y^*} \left(h^{*3} \frac{\partial h^*}{\partial y^*} \right) - \frac{\partial}{\partial x^*} (h^{*3}) - w^*(x, y, t). \tag{2.8.15}$$

2.9 Concluding remarks

We have now derived the defining equation (2.8.15) and as expected it is a partial differential equation in two spatial variables and the time variable. The leak-off w has been left unspecified. Its form will be determined from the condition that (2.8.15) has an invariant solution for $h(x, y, t)$ and $w(x, y, t)$.

Chapter 3

LIE GROUP ANALYSIS

3.1 Introduction

In this chapter we will reduce equation (2.8.15) from a partial differential equation in three independent variables to an ordinary differential equation. We will then investigate the analytical and numerical solution of the ordinary differential equation in subsequent chapters.

The reduction of the partial differential equation to an ordinary differential equation is a two step process. We will first reduce equation (2.8.15) to a partial differential equation in two variables and then we will repeat the process and reduce it to and ordinary differential equation.

Lie group analysis of the partial differential equations will be used to perform the reductions. A linear combination of the Lie point symmetries of the partial differential equations (2.8.15) will be used in the analysis. Throughout the dissertation we will consider the general case in which the constants in the linear combination do not take special values. Special cases for the constants do arise but they will not be investigated. It is not the aim of the dissertation to perform a group classification of the solutions of the partial differential equation. The aim is to perform a general reduction of the partial differential equation to an ordinary differential equation for the general range of values of the parameters and investigate analytical and numerical solutions of the ordinary differential equation obtained.

3.2 First stage in the Lie point symmetry reduction of the partial differential equation

We first expand the derivatives in (2.8.15)

$$\frac{\partial h}{\partial t} - 3h^2\left(\frac{\partial h}{\partial y}\right)^2 - h^3\frac{\partial^2 h}{\partial y^2} + 3h^2\frac{\partial h}{\partial x} + w(x,y,t) = 0. \tag{3.2.1}$$

We write the derivatives in subscript notation so that (3.2.1) is expressed as

$$h_t - 3h^2h_y^2 - h^3h_{yy} + 3h^2h_x + w(x,y,t) = 0. \tag{3.2.2}$$

Equation (3.2.2) is of the form

$$F\left(t,x,y,h,h_t,h_x,h_y,h_{yy},w\right) = 0. \tag{3.2.3}$$

Now since F contains derivatives of h up to second order, we need the second prolongation of X, which for the derivatives in (3.2.3) is

$$X^{[2]} = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \xi^3\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial h} + \zeta_1\frac{\partial}{\partial h_t} + \zeta_2\frac{\partial}{\partial h_x} + \zeta_3\frac{\partial}{\partial h_y} + \zeta_{33}\frac{\partial}{\partial h_{yy}} \tag{3.2.4}$$

where $\xi^1 = \xi^1(t,x,y,h)$, $\xi^2 = \xi^2(t,x,y,h)$, $\xi^3 = \xi^3(t,x,y,h)$ and $\eta = \eta(t,x,y,h)$.

The unknown functions are ξ^1, ξ^2, ξ^3 and η. We first calculate ζ_1, ζ_2, ζ_3 and ζ_{33} by using

$$\zeta_i = D_i\left(\eta\right) - h_kD_i\left(\xi^k\right), \quad i = 1,2,3\ldots \tag{3.2.5}$$

$$\zeta_{ij} = D_j\left(\xi_i\right) - h_{ki} D_j\left(\xi^k\right), \quad i, j = 1, 2, 3 \ldots \tag{3.2.6}$$

where

$$D_1 = D_t = \frac{\partial}{\partial t} + h_t \frac{\partial}{\partial h} + h_{tt} \frac{\partial}{\partial h_t} + h_{xt} \frac{\partial}{\partial h_x} + h_{yt} \frac{\partial}{\partial h_y} \ldots \tag{3.2.7}$$

$$D_2 = D_x = \frac{\partial}{\partial x} + h_x \frac{\partial}{\partial h} + h_{tx} \frac{\partial}{\partial h_t} + h_{xx} \frac{\partial}{\partial h_x} + h_{yx} \frac{\partial}{\partial h_y} \ldots \tag{3.2.8}$$

$$D_3 = D_y = \frac{\partial}{\partial y} + h_y \frac{\partial}{\partial h} + h_{ty} \frac{\partial}{\partial h_t} + h_{xy} \frac{\partial}{\partial h_x} + h_{yy} \frac{\partial}{\partial h_y} \ldots \tag{3.2.9}$$

Therefore

$$\zeta_1 = \zeta_t = \frac{\partial \eta}{\partial t} + h_t \frac{\partial \eta}{\partial h} - h_t \frac{\partial \xi^1}{\partial t} - h_t^2 \frac{\partial \xi^1}{\partial h} - h_x \frac{\partial \xi^2}{\partial t} - h_t h_x \frac{\partial \xi^2}{\partial h} - h_y \frac{\partial \xi^3}{\partial t} - h_t h_y \frac{\partial \xi^3}{\partial h}, \tag{3.2.10}$$

$$\zeta_2 = \zeta_x = \frac{\partial \eta}{\partial x} + h_x \frac{\partial \eta}{\partial h} - h_t \frac{\partial \xi^1}{\partial x} - h_t h_x \frac{\partial \xi^1}{\partial h} - h_x \frac{\partial \xi^2}{\partial x} - h_x^2 \frac{\partial \xi^2}{\partial h} - h_y \frac{\partial \xi^3}{\partial x} - h_x h_y \frac{\partial \xi^3}{\partial h}, \tag{3.2.11}$$

$$\zeta_3 = \zeta_y = \frac{\partial \eta}{\partial y} + h_y \frac{\partial \eta}{\partial h} - h_t \frac{\partial \xi^1}{\partial y} - h_t h_y \frac{\partial \xi^1}{\partial h} - h_x \frac{\partial \xi^2}{\partial y} - h_x h_y \frac{\partial \xi^2}{\partial h} - h_y \frac{\partial \xi^3}{\partial y} - h_y^2 \frac{\partial \xi^3}{\partial h}, \tag{3.2.12}$$

and

$$\begin{aligned}
\zeta_{33} = \zeta_{yy} = {} & \frac{\partial^2 \eta}{\partial y^2} + h_{yy} \frac{\partial \eta}{\partial h} + 2h_y \frac{\partial^2 \eta}{\partial y \partial h} + h_y^2 \frac{\partial^2 \eta}{\partial h^2} - 2h_{ty} \frac{\partial \xi^1}{\partial y} - 2h_{ty} h_y \frac{\partial \xi^1}{\partial h} - h_t \frac{\partial^2 \xi^1}{\partial y^2} \\
& - h_t h_{yy} \frac{\partial \xi^1}{\partial h} - 2h_t h_y \frac{\partial^2 \xi^1}{\partial y \partial h} - h_t h_y^2 \frac{\partial^2 \xi^1}{\partial h^2} - 2h_{xy} \frac{\partial \xi^2}{\partial y} - 2h_{xy} h_y \frac{\partial \xi^2}{\partial h} \\
& - h_x \frac{\partial^2 \xi^2}{\partial y^2} - h_x h_{yy} \frac{\partial \xi^2}{\partial h} - 2h_x h_y \frac{\partial^2 \xi^2}{\partial y \partial h} - h_x h_y^2 \frac{\partial^2 \xi^2}{\partial h^2} - 2h_{yy} \frac{\partial \xi^3}{\partial y} \\
& - 2h_{yy} h_y \frac{\partial \xi^3}{\partial h} - h_y \frac{\partial^2 \xi^3}{\partial y^2} - h_y h_{yy} \frac{\partial \xi^3}{\partial h} - 2h_y^2 \frac{\partial^2 \xi^3}{\partial y \partial h} - h_y^3 \frac{\partial^2 \xi^3}{\partial h^2}.
\end{aligned} \tag{3.2.13}$$

So the prolongation of X, (3.2.4), becomes

$$
\begin{aligned}
X^{[2]} =& \xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial x}+\xi^3\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial h}+\left[\frac{\partial \eta}{\partial t}+h_t\frac{\partial \eta}{\partial h}-h_t\frac{\partial \xi^1}{\partial t}-h_t^2\frac{\partial \xi^1}{\partial h}-h_x\frac{\partial \xi^2}{\partial t}-h_th_x\frac{\partial \xi^2}{\partial h}\right.\\
&\left.-h_y\frac{\partial \xi^3}{\partial t}-h_th_y\frac{\partial \xi^3}{\partial h}\right]\frac{\partial}{\partial h_t}+\left[\frac{\partial \eta}{\partial x}+h_x\frac{\partial \eta}{\partial h}-h_t\frac{\partial \xi^1}{\partial x}-h_th_x\frac{\partial \xi^1}{\partial h}-h_x\frac{\partial \xi^2}{\partial x}-h_x^2\frac{\partial \xi^2}{\partial h}\right.\\
&\left.-h_y\frac{\partial \xi^3}{\partial x}-h_xh_y\frac{\partial \xi^3}{\partial h}\right]\frac{\partial}{\partial h_x}+\left[\frac{\partial \eta}{\partial y}+h_y\frac{\partial \eta}{\partial h}-h_t\frac{\partial \xi^1}{\partial y}-h_th_y\frac{\partial \xi^1}{\partial h}-h_x\frac{\partial \xi^2}{\partial y}-h_xh_y\frac{\partial \xi^2}{\partial h}\right.\\
&\left.-h_y\frac{\partial \xi^3}{\partial y}-h_y^2\frac{\partial \xi^3}{\partial h}\right]\frac{\partial}{\partial h_y}+\left[\frac{\partial^2 \eta}{\partial y^2}+h_{yy}\frac{\partial \eta}{\partial h}+2h_y\frac{\partial^2 \eta}{\partial y\partial h}+h_y^2\frac{\partial^2 \eta}{\partial h^2}-2h_{ty}\frac{\partial \xi^1}{\partial y}-2h_{ty}h_y\frac{\partial \xi^1}{\partial h}\right.\\
&-h_t\frac{\partial^2 \xi^1}{\partial y^2}-h_th_{yy}\frac{\partial \xi^1}{\partial h}-2h_th_y\frac{\partial^2 \xi^1}{\partial y\partial h}-h_th_y^2\frac{\partial^2 \xi^1}{\partial h^2}-2h_{xy}\frac{\partial \xi^2}{\partial y}-2h_{xy}h_y\frac{\partial \xi^2}{\partial h}-h_x\frac{\partial^2 \xi^2}{\partial y^2}\\
&-h_xh_{yy}\frac{\partial \xi^2}{\partial h}-2h_xh_y\frac{\partial^2 \xi^2}{\partial y\partial h}-h_xh_y^2\frac{\partial^2 \xi^2}{\partial h^2}-2h_{yy}\frac{\partial \xi^3}{\partial y}-2h_{yy}h_y\frac{\partial \xi^3}{\partial h}-h_y\frac{\partial^2 \xi^3}{\partial y^2}-h_yh_{yy}\frac{\partial \xi^3}{\partial h}\\
&\left.-2h_y^2\frac{\partial^2 \xi^3}{\partial y\partial h}-h_y^3\frac{\partial^2 \xi^3}{\partial h^2}\right]\frac{\partial}{\partial h_{yy}}. \qquad (3.2.14)
\end{aligned}
$$

We now calculate

$$X^{[2]}F|_{F=0}=0, \qquad (3.2.15)$$

where $X^{[2]}$ is given by (3.2.14) and F is given by

$$F=h_t-3h^2h_y^2-h^3h_{yy}+3h^2h_x+w(t,x,y). \qquad (3.2.16)$$

We evaluate (3.2.15) at $F=0$, by substituting

$$h_{yy}=\frac{1}{h^3}\left[\frac{h_t}{h}-3hh_y^2+3hh_x+\frac{w(t,x,y)}{h}\right], \qquad (3.2.17)$$

to obtain

$$
\begin{aligned}
&-6hh_y^2\eta - 3\frac{h_t}{h}\eta + 9hh_y^2\eta - 9hh_x\eta - 3\frac{w}{h}\eta + 6hh_x\eta + \frac{\partial\eta}{\partial t} + h_t\frac{\partial\eta}{\partial h} + 3h^2\frac{\partial\eta}{\partial x} \\
&+ 3h^2h_x\frac{\partial\eta}{\partial h} - 6h^2h_y\frac{\partial\eta}{\partial y} - 6h^2h_y^2\frac{\partial\eta}{\partial h} - h^3\frac{\partial^2\eta}{\partial y^2} - h_t\frac{\partial\eta}{\partial h} + 3h^2h_y^2\frac{\partial\eta}{\partial h} - 3h^2h_x\frac{\partial\eta}{\partial h} \\
&- w\frac{\partial\eta}{\partial h} - 2h^3h_y\frac{\partial^2\eta}{\partial y\partial h} - h^3h_y^2\frac{\partial^2\eta}{\partial h^2} + w_t\xi^1 - h_t\frac{\partial\xi^1}{\partial t} - h_t^2\frac{\partial\xi^1}{\partial h} - 3h^2h_t\frac{\partial\xi^1}{\partial x} \\
&- 3h^2h_th_x\frac{\partial\xi^1}{\partial h} + 6h^2h_th_y\frac{\partial\xi^1}{\partial y} + 6h^2h_th_y^2\frac{\partial\xi^1}{\partial h} + 2h^3h_{ty}\frac{\partial\xi^1}{\partial y} + 2h^3h_{ty}h_y\frac{\partial\xi^1}{\partial h} + h^3h_t\frac{\partial^2\xi^1}{\partial y^2} \\
&+ h_t^2\frac{\partial\xi^1}{\partial h} - 3h^2h_th_y^2\frac{\partial\xi^1}{\partial h} + 3h^2h_th_x\frac{\partial\xi^1}{\partial h} + wh_t\frac{\partial\xi^1}{\partial h} + 2h^3h_th_y\frac{\partial^2\xi^1}{\partial y\partial h} + h^3h_th_y^2\frac{\partial^2\xi^1}{\partial h^2} \\
&+ w_x\xi^2 - h_x\frac{\partial\xi^2}{\partial t} - h_th_x\frac{\partial\xi^2}{\partial h} - 3h^2h_x\frac{\partial\xi^2}{\partial x} - 3h^2h_x^2\frac{\partial\xi^2}{\partial h} + 6h^2h_xh_y\frac{\partial\xi^2}{\partial y} + 6h^2h_xh_y^2\frac{\partial\xi^2}{\partial h} \\
&+ 2h^3h_{xy}\frac{\partial\xi^2}{\partial y} + 2h^3h_{xy}h_y\frac{\partial\xi^2}{\partial h} + h^3h_x\frac{\partial^2\xi^2}{\partial y^2} + h_th_x\frac{\partial\xi^2}{\partial h} - 3h^2h_xh_y^2\frac{\partial\xi^2}{\partial h} + 3h^2h_x^2\frac{\partial\xi^2}{\partial h} \\
&+ wh_x\frac{\partial\xi^2}{\partial h} + 2h^3h_xh_y\frac{\partial^2\xi^2}{\partial y\partial h} + h^3h_xh_y^2\frac{\partial^2\xi^2}{\partial h^2} + w_y\xi^3 - h_y\frac{\partial\xi^3}{\partial t} - h_th_y\frac{\partial\xi^3}{\partial h} \\
&- 3h^2h_y\frac{\partial\xi^3}{\partial x} - 3h^2h_xh_y\frac{\partial\xi^3}{\partial h} + 6h^2h_y^2\frac{\partial\xi^3}{\partial y} + 6h^2h_y^3\frac{\partial\xi^3}{\partial h} + 2h_t\frac{\partial\xi^3}{\partial y} - 6h^2h_y^2\frac{\partial\xi^3}{\partial y} + 6h^2h_x\frac{\partial\xi^3}{\partial y} \\
&+ 2w\frac{\partial\xi^3}{\partial y} + 2h_th_y\frac{\partial\xi^3}{\partial h} - 6h^2h_y^3\frac{\partial\xi^3}{\partial h} + 6h^2h_xh_y\frac{\partial\xi^3}{\partial h} + 2wh_y\frac{\partial\xi^3}{\partial h} + h^3h_y\frac{\partial^2\xi^3}{\partial y^2} \\
&+ h_th_y\frac{\partial\xi^3}{\partial h} - 3h^2h_y^3\frac{\partial\xi^3}{\partial h} + 3h^2h_xh_y\frac{\partial\xi^3}{\partial h} + wh_y\frac{\partial\xi^3}{\partial h} + 2h^3h_y^2\frac{\partial^2\xi^3}{\partial y\partial h} + h^3h_y^3\frac{\partial^2\xi^3}{\partial h^2} = 0.
\end{aligned}
\tag{3.2.18}
$$

Since ξ^1, ξ^2, ξ^3 and η are independent of the derivatives of h we can separate (3.2.18) by derivatives of h and determine expressions for ξ^1, ξ^2, ξ^3 and η. This gives

$$\xi^1 = \xi^1(t,x), \quad \xi^2 = \xi^2(t,x), \quad \xi^3 = \xi^3(t,x,y), \quad \eta = \eta(t,x,y,h)$$

and

$$
\begin{aligned}
h_y^2 : & \qquad 3\eta - 3h\frac{\partial \eta}{\partial h} - h^2\frac{\partial^2\eta}{\partial h^2} = 0, && (3.2.19)\\
h_y : & \qquad 6h^2\frac{\partial \eta}{\partial y} + 2h^3\frac{\partial^2\eta}{\partial y\partial h} + \frac{\partial \xi^3}{\partial t} + 3h^2\frac{\partial \xi^3}{\partial x} - h^3\frac{\partial^2\xi^3}{\partial y^2} = 0, && (3.2.20)\\
h_x : & \qquad 3h\eta + \frac{\partial \xi^2}{\partial t} + 3h^2\frac{\partial \xi^2}{\partial x} - 6h^2\frac{\partial \xi^3}{\partial y} = 0, && (3.2.21)\\
h_t : & \qquad 3\eta + h\frac{\partial \xi^1}{\partial t} + 3h^3\frac{\partial \xi^1}{\partial x} - 2h\frac{\partial \xi^3}{\partial y} = 0, && (3.2.22)\\
R : & \quad \xi^1\frac{\partial w}{\partial t} + \xi^2\frac{\partial w}{\partial x} + \xi^3\frac{\partial w}{\partial y} = \left[3\frac{\eta}{h} + \frac{\partial \eta}{\partial h} - 2\frac{\partial \xi^3}{\partial y}\right]w - \frac{\partial \eta}{\partial t} - 3h^2\frac{\partial \eta}{\partial x} + h^3\frac{\partial^2\eta}{\partial y^2}. && (3.2.23)
\end{aligned}
$$

Since equation (3.2.19) is equi-dimensional, we look for a solution for η of the form

$$\eta(t, x, y, h) = Ah^r, \qquad (3.2.24)$$

where $A = A(t, x, y)$. By direct substitution of (3.2.24) into (3.2.19) it is found that $r = -3$ and $r = 1$ and therefore

$$\eta(t, x, y, h) = \frac{A(t, x, y)}{h^3} + B(t, x, y)h. \qquad (3.2.25)$$

Substituting (3.2.25) into equations (3.2.20), (3.2.21) and (3.2.22) enables us to separate these equations by powers of h since ξ^1, ξ^2 and ξ^3 are independent of h. This gives

$$
\begin{aligned}
\xi^1(t) &= c_1 t + c_4, && (3.2.26)\\
\xi^2(x) &= c_2 x + c_5, && (3.2.27)\\
\xi^3(y) &= \frac{1}{4}(3c_2 - c_1)y + c_6, && (3.2.28)\\
\eta(h) &= \frac{1}{2}h(c_2 - c_1), && (3.2.29)
\end{aligned}
$$

Hence

$$X = (c_1 t + c_4)\frac{\partial}{\partial t} + (c_2 x + c_5)\frac{\partial}{\partial x} + \left(\frac{1}{4}(3c_2 - c_1)y + c_6\right)\frac{\partial}{\partial y} + \frac{1}{2}(c_2 - c_1)h\frac{\partial}{\partial h} \qquad (3.2.30)$$

and the Lie point symmetries of the partial differential equations (3.2.1) are

$$X_1 = t\frac{\partial}{\partial t} - \frac{1}{4}y\frac{\partial}{\partial y} - \frac{1}{2}h\frac{\partial}{\partial h}, \tag{3.2.31}$$

$$X_2 = x\frac{\partial}{\partial x} + \frac{3}{4}y\frac{\partial}{\partial y} + \frac{1}{2}h\frac{\partial}{\partial h}, \tag{3.2.32}$$

$$X_3 = \frac{\partial}{\partial t}, \tag{3.2.33}$$

$$X_4 = \frac{\partial}{\partial x}, \tag{3.2.34}$$

$$X_5 = \frac{\partial}{\partial y}, \tag{3.2.35}$$

provided that the remaining condition (3.2.23) is satisfied. When (3.2.26) to (3.2.29) are substituted into (3.2.23) it becomes the following first order linear partial differential equation for the leak-off velocity $w(t,x,y)$:

$$(c_1 t + c_4)\frac{\partial w}{\partial t} + (c_2 x + c_5)\frac{\partial w}{\partial x} + \left[\frac{1}{4}(3c_2 - c_1)\,y + c_6\right]\frac{\partial w}{\partial y} = \frac{1}{2}(c_2 - 3c_1)\,w. \tag{3.2.36}$$

If $c_1 \neq 0$ and $c_2 \neq 3c_1$, the general solution to (3.2.36) is

$$w = (c_1 t + c_3)^{\frac{1}{2}\left(\frac{c_2}{c_1} - 3\right)}\,U(\phi,\psi), \tag{3.2.37}$$

where U is an arbitrary function and

$$\phi = \frac{c_2 x + c_5}{(c_1 t + c_4)^{\frac{c_2}{c_1}}}, \tag{3.2.38}$$

$$\psi = \frac{\frac{1}{4}(3c_2 - c_1)\,y + c_6}{(c_1 t + c_4)^{\frac{3c_2 - c_1}{4c_1}}}. \tag{3.2.39}$$

Now $h = \Phi(t,x,y)$ is a group invariant solution of the partial differential equation provided

$$X\left(h - \Phi(t,x,y)\right)|_{h=\Phi} = 0, \tag{3.2.40}$$

that is, provided

$$(c_1 t + c_4)\frac{\partial \Phi}{\partial t} + (c_2 x + c_5)\frac{\partial \Phi}{\partial x} + (\frac{1}{4}(3c_2 - c_1)\,y + c_6)\frac{\partial \Phi}{\partial y} = \frac{1}{2}(c_2 - c_2)\Phi. \tag{3.2.41}$$

Equation (3.2.41) is a first order linear partial differential equation for $\Phi(t, x, y)$. The form of the left hand side of (3.2.36) and (3.2.41) is the same which means that w and Φ will depend on the same similarity variables, ϕ and ψ. Since $h = \Phi(t, x, y)$, the general solution for h is

$$h = (c_1 t + c_4)^{\frac{1}{2}\left(\frac{c_2}{c_1}-1\right)} G(\phi, \psi), \tag{3.2.42}$$

where ϕ and ψ are defined by (3.2.38) and (3.2.39).

Substituting (3.2.37) and (3.2.42) into the defining partial differential equation, (2.8.15) and simplifying we obtain

$$\begin{aligned} &\frac{9}{16} c_1 \left(\frac{c_2}{c_1} - \frac{1}{3}\right)^2 \frac{\partial}{\partial \psi}\left(G^3 \frac{\partial G}{\partial \psi}\right) + \frac{c_2}{c_1}\phi \frac{\partial G}{\partial \phi} + \frac{3}{4}\left(\frac{c_2}{c_1} - \frac{1}{3}\right)\psi \frac{\partial G}{\partial \psi} \\ &\quad - \frac{c_2}{c_1}\frac{\partial}{\partial \phi}\left(G^3\right) - \frac{1}{2}\left(\frac{c_2}{c_1} - 1\right) G - \frac{1}{c_1} U(\phi, \psi) = 0. \end{aligned} \tag{3.2.43}$$

The original partial differential equation has now been reduced from an equation in three variables to one in two variables. We now reduce this partial differential equation in two variables to an ordinary differential equation. This requires a second Lie point symmetry reduction.

3.3 Second stage in the Lie point symmetry reduction of the partial differential equation

The next step requires us to work from equation (3.2.43) and repeat the Lie group reduction. We therefore first expand the derivatives in (3.2.43) and write the derivatives in subscript notation. We consider the general case in which $c_1 \neq 0$. This gives

$$F\left(\phi, \psi, G, G_\phi, G_\psi, G_{\psi\psi}, U\right) = 0, \tag{3.3.1}$$

where

$$F=\frac{27}{16}c_1(\alpha-\frac{1}{3})^2G^2G_\psi^2+\frac{9}{16}c_1(\alpha-\frac{1}{3})^2G^3G_{\psi\psi}+\alpha\phi G_\phi+\frac{3}{4}(\alpha-\frac{1}{3})\psi G_\psi-3\alpha G^2G_\phi-\frac{1}{2}(\alpha-1)G-\frac{1}{c_1}U \tag{3.3.2}$$

and $\alpha=\frac{c_2}{c_1}$.

In order to find the Lie point symmetries of (3.3.1) we need to calculate the second prolongation of X, of the form

$$X^{[2]}=\xi^1\frac{\partial}{\partial\phi}+\xi^2\frac{\partial}{\partial\psi}+\eta\frac{\partial}{\partial G}+\zeta_1\frac{\partial}{\partial G_\phi}+\zeta_2\frac{\partial}{\partial G_\psi}+\zeta_{22}\frac{\partial}{\partial G_{\psi\psi}}, \tag{3.3.3}$$

where $\xi^1=\xi^1(\phi,\psi,G)$, $\xi^2=\xi^2(\phi,\psi,G)$, and $\eta=\eta(\phi,\psi,G)$.
We first calculate ζ_1, ζ_2 and ζ_{22} from the definitions

$$\zeta_i=D_i\left(\eta\right)-h_kD_i\left(\xi^k\right), \quad i=1,2,3\dots \tag{3.3.4}$$

$$\zeta_{ij}=D_j\left(\xi_i\right)-h_{ki}D_j\left(\xi^k\right), \quad i,j=1,2,3\dots \tag{3.3.5}$$

where

$$D_1=D_\phi=\frac{\partial}{\partial\phi}+G_\phi\frac{\partial}{\partial G}+G_{\phi\phi}\frac{\partial}{\partial G_\phi}+G_{\phi\psi}\frac{\partial}{\partial G_\psi}\dots \tag{3.3.6}$$

$$D_2=D_\psi=\frac{\partial}{\partial\psi}+G_\psi\frac{\partial}{\partial G}+G_{\phi\psi}\frac{\partial}{\partial G_\phi}+G_{\psi\psi}\frac{\partial}{\partial G_\psi}\dots \tag{3.3.7}$$

Hence

$$\begin{aligned}\zeta_1&=\zeta_\phi\\&=\frac{\partial\eta}{\partial\phi}+G_\phi\frac{\partial\eta}{\partial G}-G_\phi\frac{\partial\xi^1}{\partial\phi}-G_\phi^2\frac{\partial\xi^1}{\partial G}-G_\psi\frac{\partial\xi^2}{\partial\phi}-G_\phi G_\psi\frac{\partial\xi^2}{\partial G},\end{aligned} \tag{3.3.8}$$

$$\begin{aligned}\zeta_2&=\zeta_\psi\\&=\frac{\partial\eta}{\partial\psi}+G_\psi\frac{\partial\eta}{\partial G}-G_\phi\frac{\partial\xi^1}{\partial\psi}-G_\phi G_\psi\frac{\partial\xi^1}{\partial G}-G_\psi\frac{\partial\xi^2}{\partial\psi}-G_\psi^2\frac{\partial\xi^2}{\partial G},\end{aligned} \tag{3.3.9}$$

and

$$\begin{aligned}
\zeta_{22} &= \zeta_{\psi\psi} \\
&= \frac{\partial^2\eta}{\partial\psi^2} + G_{\psi\psi}\frac{\partial\eta}{\partial G} + 2G_{\psi}\frac{\partial^2\eta}{\partial\psi\partial G} + G_{\psi}^2\frac{\partial^2\eta}{\partial G^2} - 2G_{\phi\psi}\frac{\partial\xi^1}{\partial\psi} \\
&\quad - 2G_{\phi\psi}G_{\psi}\frac{\partial\xi^1}{\partial G} - G_{\phi}\frac{\partial^2\xi^1}{\partial\psi^2} - G_{\phi}G_{\psi\psi}\frac{\partial\xi^1}{\partial G} - 2G_{\phi}G_{\psi}\frac{\partial^2\xi^1}{\partial\psi\partial G} - G_{\phi}G_{\psi}^2\frac{\partial^2\xi^1}{\partial G^2} \\
&\quad - 2G_{\psi\psi}\frac{\partial\xi^2}{\partial\psi} - 3G_{\psi}G_{\psi\psi}\frac{\partial\xi^2}{\partial G} - G_{\psi}\frac{\partial^2\xi^2}{\partial\psi^2} - 2G_{\psi}^2\frac{\partial^2\xi^2}{\partial\psi\partial G} - G_{\psi}^3\frac{\partial^2\xi^2}{\partial G^2}.
\end{aligned} \tag{3.3.10}$$

The second prolongation, (3.3.3) becomes

$$\begin{aligned}
X^{[2]} &= \xi^1\frac{\partial}{\partial\phi} + \xi^2\frac{\partial}{\partial\psi} + \eta\frac{\partial}{\partial G} + \zeta_1\frac{\partial}{\partial G_{\phi}} + \zeta_2\frac{\partial}{\partial G_{\psi}} + \zeta_{22}\frac{\partial}{\partial G_{\psi\psi}} \\
&= \xi^1\frac{\partial}{\partial\phi} + \xi^2\frac{\partial}{\partial\psi} + \eta\frac{\partial}{\partial G} + \left[\frac{\partial\eta}{\partial\phi} + G_{\phi}\frac{\partial\eta}{\partial G} - G_{\phi}\left(\frac{\partial\xi^1}{\partial\phi} + G_{\phi}\frac{\partial\xi^1}{\partial G}\right) - G_{\psi}\left(\frac{\partial\xi^2}{\partial\phi}\right.\right. \\
&\quad \left.\left.+G_{\phi}\frac{\partial\xi^2}{\partial G}\right)\right]\frac{\partial}{\partial G_{\phi}} + \left[\frac{\partial\eta}{\partial\psi} + G_{\psi}\frac{\partial\eta}{\partial G} - G_{\phi}\left(\frac{\partial\xi^1}{\partial\psi} + G_{\psi}\frac{\partial\xi^1}{\partial G}\right) - G_{\psi}\left(\frac{\partial\xi^2}{\partial\psi} + G_{\psi}\frac{\partial\xi^2}{\partial G}\right)\right]\frac{\partial}{\partial G_{\psi}} \\
&\quad + \left[\frac{\partial^2\eta}{\partial\psi^2} + G_{\psi\psi}\frac{\partial\eta}{\partial G} + 2G_{\psi}\frac{\partial^2\eta}{\partial\psi\partial G} + G_{\psi}^2\frac{\partial^2\eta}{\partial G^2} - 2G_{\phi\psi}\left(\frac{\partial\xi^1}{\partial\psi} + G_{\psi}\frac{\partial\xi^1}{\partial G}\right) - G_{\phi}\left(\frac{\partial^2\xi^1}{\partial\psi^2}\right.\right. \\
&\quad \left.+G_{\psi\psi}\frac{\partial\xi^1}{\partial G} + G_{\psi}\frac{\partial^2\xi^1}{\partial\psi\partial G}\right) - 2G_{\psi\psi}\left(\frac{\partial\xi^2}{\partial\psi} + G_{\psi}\frac{\partial\xi^2}{\partial G}\right) - G_{\psi}\left(\frac{\partial^2\xi^2}{\partial\psi^2} + G_{\psi\psi}\frac{\partial\xi^2}{\partial G} + G_{\psi}\frac{\partial^2\xi^2}{\partial\psi\partial G}\right) \\
&\quad \left.-G_{\phi\psi}\left(\frac{\partial^2\xi^1}{\partial\psi\partial G} + G_{\psi}\frac{\partial^2\xi^1}{\partial G^2}\right) - G_{\psi}^2\left(\frac{\partial^2\xi^2}{\partial\psi\partial G} + G_{\psi}\frac{\partial^2\xi^2}{\partial G^2}\right)\right]\frac{\partial}{\partial G_{\psi\psi}}.
\end{aligned} \tag{3.3.11}$$

We now calculate the determining equation

$$X^{[2]}F|_{F=0} = 0, \tag{3.3.12}$$

where $X^{[2]}$ is given by (3.3.11) and F is given by (3.3.2).

We expand (3.3.12) and evaluate it at $F = 0$ by replacing $G_{\psi\psi}$ using, (3.3.2),

$$\begin{aligned}
c_1\left(\alpha - \frac{1}{3}\right)^2 G_{\psi\psi} = \frac{16}{9G^3}&\left[-\frac{27}{16}c_1\left(\alpha - \frac{1}{3}\right)^2 G^2G_{\psi}^2 - \alpha\phi G_{\phi} - \frac{3}{4}\left(\alpha - \frac{1}{3}\right)\psi G_{\psi}\right. \\
&\left.+3\alpha G^2G_{\phi} + \frac{1}{2}(\alpha - 1)G + \frac{1}{c_1}U(\phi,\psi)\right].
\end{aligned} \tag{3.3.13}$$

The determining equation becomes

$$
\begin{aligned}
&\alpha\xi^1 G_\phi - \frac{1}{c_1}\xi^1 U_\phi + \frac{3}{4}(\alpha-\frac{1}{3})\psi G_\psi \xi^2 - \frac{1}{c_1}U_\psi \xi^2 + \frac{27}{8}c_1(\alpha-\frac{1}{3})^2 G G_\psi^2 \eta \\
&+ 3\eta\left[-\frac{27}{16}c_1(\alpha-\frac{1}{3})^2 G G_\psi^2 - \frac{\alpha\phi}{G}G_\phi - \frac{3}{4G}(\alpha-\frac{1}{3})\psi G_\psi + 3\alpha G G_\phi + \frac{1}{2}(\alpha-1) + \frac{1}{c_1 G}U\right] \\
&- 6\alpha G G_\phi \eta - \frac{1}{2}(\alpha-1)\eta + \alpha\phi\left[\frac{\partial\eta}{\partial\phi} + G_\phi\frac{\partial\eta}{\partial G} - G_\phi\frac{\partial\xi^1}{\partial\phi} - G_\phi^2\frac{\partial\xi^1}{\partial G} - G_\psi\frac{\partial\xi^2}{\partial\phi} - G_\phi G_\psi\frac{\partial\xi^2}{\partial G}\right] \\
&- 3\alpha G^2\left[\frac{\partial\eta}{\partial\phi} + G_\phi\frac{\partial\eta}{\partial G} - G_\phi\frac{\partial\xi^1}{\partial\phi} - G_\phi^2\frac{\partial\xi^1}{\partial G} - G_\psi\frac{\partial\xi^2}{\partial\phi} - G_\phi G_\psi\frac{\partial\xi^2}{\partial G}\right] \\
&+ \frac{27}{8}c_1(\alpha-\frac{1}{3})^2 G^2 G_\psi\left[\frac{\partial\eta}{\partial\psi} + G_\psi\frac{\partial\eta}{\partial G} - G_\phi\frac{\partial\xi^1}{\partial\psi} - G_\phi G_\psi\frac{\partial\xi^1}{\partial G} - G_\psi\frac{\partial\xi^2}{\partial\psi} - G_\psi^2\frac{\partial\xi^2}{\partial G}\right] \\
&+ \frac{3}{4}(\alpha-\frac{1}{3})\psi\left[\frac{\partial\eta}{\partial\psi} + G_\psi\frac{\partial\eta}{\partial G} - G_\phi\frac{\partial\xi^1}{\partial\psi} - G_\phi G_\psi\frac{\partial\xi^1}{\partial G} - G_\psi\frac{\partial\xi^2}{\partial\psi} - G_\psi^2\frac{\partial\xi^2}{\partial G}\right] \\
&+ \frac{9}{16}c_1(\alpha-\frac{1}{3})^2 G^3\left[\frac{\partial^2\eta}{\partial\psi^2} + 2G_\psi\frac{\partial^2\eta}{\partial\psi\partial G} + G_\psi^2\frac{\partial^2\eta}{\partial G^2} - 2G_{\phi\psi}\frac{\partial\xi^1}{\partial\psi} - 2G_\psi G_{\phi\psi}\frac{\partial\xi^1}{\partial G}\right. \\
&\left.-G_\phi\frac{\partial^2\xi^1}{\partial\psi^2} - 2G_\phi G_\psi\frac{\partial^2\xi^1}{\partial\psi\partial G} - G_\phi G_\psi^2\frac{\partial^2\xi^1}{\partial G^2} - G_\psi\frac{\partial^2\xi^2}{\partial\psi^2} - 2G_\psi^2\frac{\partial^2\xi^2}{\partial\psi\partial G} - G_\psi^3\frac{\partial^2\xi^2}{\partial G^2}\right] \\
&- \frac{27}{16}c_1(\alpha-\frac{1}{3})^2 G^2 G_\psi^2\frac{\partial\eta}{\partial G} - \alpha\phi G_\phi\frac{\partial\eta}{\partial G} - \frac{3}{4}(\alpha-\frac{1}{3})\psi G_\psi\frac{\partial\eta}{\partial G} + 3\alpha G^2 G_\phi\frac{\partial\eta}{\partial G} + \frac{1}{2}(\alpha-1)G\frac{\partial\eta}{\partial G} \\
&+ \frac{1}{2}U\frac{\partial\eta}{\partial G} + \frac{27}{16}c_1(\alpha-\frac{1}{3})^2 G^2 G_\phi G_\psi^2\frac{\partial\xi^1}{\partial G} + \alpha\phi G_\phi^2\frac{\partial\xi^1}{\partial G} + \frac{3}{4}(\alpha-\frac{1}{3})\psi G_\phi G_\psi\frac{\partial\xi^1}{\partial G} \\
&- 3\alpha G^2 G_\phi^2\frac{\partial\xi^1}{\partial G} - \frac{1}{2}(\alpha-1)G G_\phi\frac{\partial\xi^1}{\partial G} - \frac{1}{2}U G_\phi\frac{\partial\xi^1}{\partial G} + \frac{27}{8}c_1(\alpha-\frac{1}{3})^2 G^2 G_\psi^2\frac{\partial\xi^2}{\partial\psi} + 2\alpha\phi G_\phi\frac{\partial\xi^2}{\partial\psi} \\
&+ \frac{3}{2}(\alpha-\frac{1}{3})\psi G_\psi\frac{\partial\xi^2}{\partial\psi} - 6\alpha G^2 G_\phi\frac{\partial\xi^2}{\partial\psi} - (\alpha-1)G\frac{\partial\xi^2}{\partial\psi} - U\frac{\partial\xi^2}{\partial\psi} + \frac{81}{16}c_1(\alpha-\frac{1}{3})^2 G^2 G_\psi^3\frac{\partial\xi^2}{\partial G} \\
&+ 3\alpha\phi G_\phi G_\psi\frac{\partial\xi^2}{\partial G} + \frac{9}{4}(\alpha-\frac{1}{3})\psi G_\psi^2\frac{\partial\xi^2}{\partial G} - 9\alpha G^2 G_\phi G_\psi\frac{\partial\xi^2}{\partial G} - \frac{3}{2}(\alpha-1)G G_\psi\frac{\partial\xi^2}{\partial G} - \frac{3}{2}U G_\psi\frac{\partial\xi^2}{\partial G} = 0.
\end{aligned}
\tag{3.3.14}
$$

Since ξ^1, ξ^2 and η are independent of the derivatives of G we can separate (3.3.14) by derivatives of G and solve for ξ^1, ξ^2 and η.

We consider the general case in which

$$\alpha \neq \frac{1}{3} \ \ \textit{and} \ \ \alpha \neq 0. \tag{3.3.15}$$

By separating according to the derivatives $G_\psi G_{\phi\psi}$ and $G_{\phi\psi}$ it can be shown that

$$\xi^1 = \xi^1(\phi) \tag{3.3.16}$$

and by separating according to $G_\phi G_\psi$ we obtain

$$\xi^2 = \xi^2(\phi, \psi). \tag{3.3.17}$$

Also,

$$G_\psi^2 : \quad G^2 \frac{\partial^2 \eta}{\partial G^2} + 3G\frac{\partial \eta}{\partial G} - 3\eta = 0, \tag{3.3.18}$$

$$\begin{aligned} G_\psi : \quad & \frac{3}{4}(\alpha - \frac{1}{3})\xi^2 - \frac{9}{4}(\alpha - \frac{1}{3})\frac{\psi}{G}\eta - \alpha\phi\frac{\partial \xi^2}{\partial \phi} + 3\alpha G^2 \frac{\partial \xi^2}{\partial \phi} + \frac{27}{8}c_1(\alpha - \frac{1}{3})^2 G^2 \frac{\partial \eta}{\partial \psi} \\ & + \frac{3}{4}(\alpha - \frac{1}{3})\psi\frac{\partial \xi^2}{\partial \psi} + \frac{18}{16}c_1(\alpha - \frac{1}{3})^2 G^3 \frac{\partial^2 \eta}{\partial \psi \partial G} - \frac{9}{16}c_1(\alpha - \frac{1}{3})^2 G^3 \frac{\partial^2 \xi^2}{\partial \psi^2} = 0, \end{aligned} \tag{3.3.19}$$

$$G_\phi : \quad \xi^1 - 3\frac{\phi\eta}{G} + 3\eta G - \phi\frac{d\xi^1}{d\phi} + 3G^2\frac{d\xi^1}{d\phi} + 2\phi\frac{\partial \xi^2}{\partial \psi} - 6G^2 \frac{\partial \xi^2}{\partial \psi} = 0, \tag{3.3.20}$$

$$\begin{aligned} R : \quad & -\frac{1}{c_1}\xi^1 U_\phi - \frac{1}{c_1}\xi^2 U_\psi + (\alpha - 1)\eta + \frac{3U}{c_1 G}\eta + \alpha\phi\frac{\partial \eta}{\partial \phi} - 3\alpha G^2 \frac{\partial \eta}{\partial \phi} + \frac{3}{4}(\alpha - \frac{1}{3})\psi\frac{\partial \eta}{\partial \psi} \\ & + \frac{9}{16}c_1(\alpha - \frac{1}{3})^2 G^3 \frac{\partial^2 \eta}{\partial \psi^2} + \frac{1}{2}(\alpha - 1)G\frac{\partial \eta}{\partial G} + \frac{1}{c_1}U\frac{\partial \eta}{\partial G} - (\alpha - 1)G\frac{\partial \xi^2}{\partial \psi} - U\frac{\partial \xi^2}{\partial \psi} = 0. \end{aligned} \tag{3.3.21}$$

Now (3.3.18) is an equi-dimensional differential equation. Therefore we look for a solution of the form

$$\eta\,(\phi, \psi, G) = AG^r, \tag{3.3.22}$$

where $A = A(\phi, \psi)$ and r have to be determined. By substituting (3.3.22) into (3.3.18) we find that $r = -3$ and $r = 1$. Thus

$$\eta = A(\phi, \psi)G + \frac{B(\phi, \psi)}{G^3}. \tag{3.3.23}$$

We then substitute this result for η into equations (3.3.19), (3.3.20) and (3.3.21). This eliminates η in these equations, and we can then separate by powers of G in equations (3.3.19) and (3.3.21) and solve for ξ^1, ξ^2 and η. This gives

$$\xi^1 = A\phi, \quad \xi^2 = \frac{3}{4}A\psi, \quad \eta = \frac{1}{2}AG, \tag{3.3.24}$$

where A is now a constant. When we substitute the solutions for ξ^1, ξ^2 and η into equation (3.3.20) we obtain

$$\phi\frac{\partial U}{\partial \phi}+\frac{3}{4}\psi\frac{\partial U}{\partial \psi}=\frac{1}{2}U. \tag{3.3.25}$$

The Lie point symmetry of (3.2.43) is therefore

$$X=\phi\frac{\partial}{\partial \phi}+\frac{3}{4}\psi\frac{\partial}{\partial \psi}+\frac{1}{2}G\frac{\partial}{\partial G}, \tag{3.3.26}$$

provided $U(\phi,\psi)$ satisfies (3.3.25).

Equation (3.3.25) is a first order linear partial differential equation for $U(\phi,\psi)$. It can be solved by integrating the differential equations of its characteristic curves. The general solution of (3.3.25) is

$$U=\phi^{\frac{1}{2}}V(\eta), \quad \eta=\frac{\psi}{\phi^{\frac{3}{4}}}, \tag{3.3.27}$$

where V is an arbitrary function. The similarity variable η defined in (3.3.27) is not to be confused with the function η in the Lie point symmetry (3.2.4).

Now $G=\Phi(\phi,\psi)$ is a group invariant solution of the partial differential equation (3.2.43) provided

$$X(G-\Phi)|_{G=\Phi}=0, \tag{3.3.28}$$

which expands to

$$\phi\frac{\partial G}{\partial \phi}+\frac{3}{4}\psi\frac{\partial G}{\partial \psi}=\frac{1}{2}G. \tag{3.3.29}$$

Equation (3.3.29) is a first order linear partial differential equation for $G(\phi,\psi)$. The left hand side of (3.3.29) has the same form as the left hand side of (3.3.25) for $U(\phi,\psi)$ and the similarity variable η is therefore the same for the two solutions. The general solution of (3.3.29) is

$$G=\phi^{\frac{1}{2}}H(\eta), \tag{3.3.30}$$

where η is defined in (3.3.27) and H is an arbitrary function.

We now substitute G and U into (3.2.43) and simplifying we obtain

$$\frac{9}{16}c_1\left(\alpha-\frac{1}{3}\right)^2\frac{d}{d\eta}\left(H^3\frac{dH}{d\eta}\right)+\frac{3}{4}\alpha\frac{d}{d\eta}\left(\eta H^3\right)-\frac{1}{4}\frac{d}{d\eta}\left(\eta H\right)+\frac{3}{4}H\left(1-3\alpha H^2\right)-\frac{1}{c_1}V=0. \tag{3.3.31}$$

Equation (3.2.43) was an equation for G and U in the two variables ϕ and ψ while (3.3.31) is an equation for H and V in one variable, η. Finally we express the similarity variable η and h and w in terms of the original variables t, x, y. We use (3.2.37) and (3.2.42) from stage 1 of the reduction and (3.3.27) and (3.3.30) from stage 2. This gives

$$\eta=\frac{\left[\frac{1}{4}\left(3c_2-c_1\right)y+c_6\right]\left(c_1t+c_4\right)^{\frac{1}{4}}}{\left(c_2x+c_5\right)^{\frac{3}{4}}}, \tag{3.3.32}$$

$$h(t,x,y)=\frac{\left(c_2x+c_5\right)^{\frac{1}{2}}}{\left(c_1t+c_4\right)^{\frac{1}{2}}}H(\eta), \tag{3.3.33}$$

and

$$w(t,x,y)=\frac{\left(c_2x+c_5\right)^{\frac{1}{2}}}{\left(c_1t+c_4\right)^{\frac{3}{2}}}V(\eta). \tag{3.3.34}$$

3.4 Boundary of the rivulet

The boundary of the rivulet is given by

$$y=\pm a(t,x). \tag{3.4.1}$$

Thus, on the curves $y=\pm a(t,x)$ the height of the rivulet vanishes which implies that

$$h(t,x,\pm a(t,x))=0 \tag{3.4.2}$$

for all $t\geq 0$ and $x\geq 0$.

Hence from (3.3.32) and (3.3.33),

$$h(t,x,\pm a(t,x))=\frac{\left(c_2x+c_5\right)^{\frac{1}{2}}}{\left(c_1t+c_4\right)^{\frac{1}{2}}}H(\eta(t,x,\pm a(t,x)))=0 \tag{3.4.3}$$

and therefore

$$H(\eta(t,x,\pm a(t,x))) = 0. \tag{3.4.4}$$

Consider first the interface $y = +a(t,x)$. Since (3.4.2) holds for all $t \geq 0$ and all $x \geq 0$,

$$\frac{\partial h}{\partial t}(t,x,a(t,x)) = 0, \quad \frac{\partial h}{\partial x}(t,x,a(t,x)) = 0. \tag{3.4.5}$$

But

$$\frac{\partial h}{\partial t}(t,x,a(t,x)) = -\frac{c_1(c_2x+c_5)^{\frac{1}{2}}}{2(c_1t+c_4)^{\frac{3}{2}}}H(t,x,a(t,x)) + \frac{(c_2x+c_5)^{\frac{1}{2}}}{(c_1t+c_4)^{\frac{1}{2}}}\frac{dH}{d\eta}\frac{\partial \eta}{\partial t}(t,x,a(t,x)) = 0. \tag{3.4.6}$$

Now, in general,

$$\frac{dH}{d\eta} \neq 0 \tag{3.4.7}$$

and using also (3.4.4) it follows from (3.4.6) that

$$\frac{\partial \eta}{\partial t}(t,x,a(t,x)) = 0. \tag{3.4.8}$$

Similarly it can be shown that

$$\frac{\partial \eta}{\partial x}(t,x,a(t,x)) = 0 \tag{3.4.9}$$

and therefore

$$\eta(t,x,a(t,x)) = A, \tag{3.4.10}$$

where A is a constant. From (3.4.4) it follows that

$$H(A) = 0. \tag{3.4.11}$$

Similarly by considering the interface $y = -a(t,x)$ and from symmetry it can be shown that

$$\eta(t,x,-a(t,x)) = -A \tag{3.4.12}$$

and

$$H(-A) = 0. \tag{3.4.13}$$

Now, using (3.3.32) for η, equation (3.4.10) becomes

$$\frac{\left[\frac{1}{4}\left(3c_2 - c_1\right) a(t,x) + c_6\right] \left(c_1 t + c_4\right)^{\frac{1}{4}}}{\left(c_2 x + c_5\right)^{\frac{3}{4}}} = A \tag{3.4.14}$$

and therefore

$$\frac{1}{4}\left(3c_2 - c_1\right) a(t,x) = \frac{\left(c_2 x + c_5\right)^{\frac{3}{4}}}{\left(c_1 t + c_4\right)^{\frac{1}{4}}} A - c_6. \tag{3.4.15}$$

But

$$3c_2 - c_1 = 3c_1(\alpha - \frac{1}{3}) \neq 0 \tag{3.4.16}$$

since we are assuming that $\alpha = \frac{c_2}{c_1} \neq \frac{1}{3}$. Thus (3.4.15) gives

$$a(t,x) = \frac{4}{\left(3c_2 - c_1\right)} \frac{\left(c_2 x + c_5\right)^{\frac{3}{4}}}{\left(c_1 t + c_4\right)^{\frac{1}{4}}} A - \frac{4c_6}{\left(3c_2 - c_1\right)}. \tag{3.4.17}$$

Similarly, we find from (3.4.12) that

$$a(t,x) = \frac{4\left(c_2 x + c_5\right)^{\frac{3}{4}} A}{\left(3c_2 - c_1\right)\left(c_1 t + c_4\right)^{\frac{1}{4}}} + \frac{4c_6}{\left(3c_2 - c_1\right)}. \tag{3.4.18}$$

Subtracting (3.4.17) from (3.4.18) gives $c_6 = 0$ and therefore

$$a(t,x) = \frac{4A}{\left(3c_2 - c_1\right)} \frac{\left(c_2 x + c_5\right)^{\frac{3}{4}}}{\left(c_1 t + c_4\right)^{\frac{1}{4}}}. \tag{3.4.19}$$

We choose $c_5 = 0$ so that $a(t,x) = 0$ when $x = 0$. Thus the boundary of the rivulet is $y = \pm a(x,t)$ where

$$a(t,x) = \frac{4Ac_2^{\frac{3}{4}} x^{\frac{3}{4}}}{(3c_2 - c_1)(c_1 t + c_4)^{\frac{1}{4}}} \tag{3.4.20}$$

and

$$H(A) = 0, \quad H(-A) = 0. \tag{3.4.21}$$

Since $c_5 = 0$ and $c_6 = 0$, equations (3.3.32), (3.3.33) and (3.3.34) for η, h and w simplify to

$$\eta = \frac{(3c_2 - c_1)(c_1 t + c_4)^{\frac{1}{4}}}{4c_2^{\frac{3}{4}}} \frac{y}{x^{\frac{3}{4}}}, \tag{3.4.22}$$

$$h(t,x,y) = \frac{c_2^{\frac{1}{2}} x^{\frac{1}{2}}}{(c_1 t + c_4)^{\frac{1}{2}}} H(\eta), \tag{3.4.23}$$

$$w(t,x,y) = \frac{c_2^{\frac{1}{2}} x^{\frac{1}{2}}}{(c_1 t + c_4)^{\frac{3}{2}}} V(\eta). \tag{3.4.24}$$

3.5 Transformation of variables

In order to simplify the problem we consider the transformation from η, H, V to ξ, K, W where

$$\xi = \lambda n \quad H = \mu K \quad V = \beta W \tag{3.5.1}$$

and the constants λ, μ and β have still to be specified. The ordinary differential equation (3.3.31) becomes, after simplification,

$$\frac{9}{16} c_1 \left(\alpha - \frac{1}{3}\right)^2 \lambda^2 \mu^3 \frac{d}{d\xi}\left(K^3 \frac{dK}{d\xi}\right) + \frac{3}{4}\alpha\mu^2 \frac{d}{d\xi}\left(\xi K^3\right) - \frac{1}{4}\frac{d}{d\xi}\left(\xi K\right) + \frac{3}{4}K - \frac{9}{4}\alpha\mu^2 K^3 - \frac{1}{c_1}\frac{\beta}{\mu} W = 0. \tag{3.5.2}$$

We choose

$$\frac{9}{16} c_1 \left(\alpha - \frac{1}{3}\right)^2 \lambda^2 \mu^3 = 1, \quad \alpha\mu^2 = 1, \quad \frac{\beta}{c_1 \mu} = 1 \tag{3.5.3}$$

and solving for λ, μ and β we obtain

$$\lambda = \frac{4\alpha^{\frac{3}{4}}}{3 c_1^{\frac{1}{2}} \left(\alpha - \frac{1}{3}\right)}, \quad \mu = \frac{1}{\sqrt{\alpha}}, \quad \beta = \frac{c_1}{\sqrt{\alpha}}. \tag{3.5.4}$$

The ordinary differential equation (3.5.2) becomes

$$\frac{d}{d\xi}\left(K^3 \frac{dK}{d\xi}\right) + \frac{3}{4}\frac{d}{d\xi}\left(\xi K^3\right) - \frac{1}{4}\frac{d}{d\xi}\left(\xi K\right) + \frac{3}{4}K\left(1 - 3K^2\right) - W = 0. \tag{3.5.5}$$

Now $\alpha = \frac{c_2}{c_1}$. Using (3.4.22), (3.4.23) and (3.4.24) we obtain

$$\xi = \left(t + \frac{c_4}{c_1}\right)^{\frac{1}{4}} \frac{y}{x^{\frac{3}{4}}}, \tag{3.5.6}$$

$$h(t,x,y) = \frac{x^{\frac{1}{2}}}{\left(t + \frac{c_4}{c_1}\right)^{\frac{1}{2}}} K(\xi), \tag{3.5.7}$$

$$w(t,x,y) = \frac{x^{\frac{1}{2}}}{\left(t + \frac{c_4}{c_1}\right)^{\frac{3}{2}}} W(\xi). \tag{3.5.8}$$

At the boundary interface, $y = \pm a(t,x)$, the height of the rivulet vanishes and we find as before

$$\xi(t,x,\pm a(t,x)) = \pm A \tag{3.5.9}$$

where A is a constant that satisfies

$$K(\pm A) = 0. \tag{3.5.10}$$

Hence from (3.5.9),

$$a(t,x) = \frac{x^{\frac{3}{4}} A}{\left(t + \frac{c_2}{c_1}\right)^{\frac{1}{4}}}. \tag{3.5.11}$$

3.6 Concluding remarks

We have reduced the defining partial differential equation to an ordinary differential equation. This required a two step Lie group analysis. We can now investigate the analytical and numerical solution of the reduced model.

Chapter 4

ANALYTICAL AND NUMERICAL SOLUTIONS FOR A RIVULET ON A POROUS SUBSTRATE

4.1 Introduction

We have reduced equation (2.8.15) from a partial differential equation in three variables to equation (3.5.5), an ordinary differential equation. We can now look for analytical and numerical solutions.

Equation (3.5.5) is a nonlinear second-order ordinary differential equation for $K(\xi)$ and contains the function $W(\xi)$ which has not yet been specified. Since $W(\xi)$ is an independent function of ξ we can choose its form to enable us to find analytical and numerical solutions.

4.2 Mathematical formulation

The problem is to solve the ordinary differential equation (3.5.5) for $K(\xi)$ for some choice for $W(\xi)$ subject to boundary conditions. At the interface $y = \pm a(t,x)$ the boundary condition is

$$K(\pm A) = 0, \tag{4.2.1}$$

where A is a constant that has to be determined. A second boundary condition on $K(\xi)$ is obtained from the condition that since the line $y = 0$ is an axis of symmetry,

$$V_y(x, 0, z, t) = 0. \tag{4.2.2}$$

Now from (2.8.5),

$$V_y = \frac{\rho g z}{2\mu} \cos\alpha \frac{\partial h}{\partial y} (z - 2h), \tag{4.2.3}$$

and condition (4.2.2) is therefore satisfied provided

$$\frac{\partial h}{\partial y}(x, 0, z, t) = 0. \tag{4.2.4}$$

But using (3.5.6) and (3.5.7) it follows that

$$\frac{\partial h}{\partial y}(x, y, z, t) = \frac{1}{x^{\frac{1}{4}} \left(t + \frac{c_4}{c_1}\right)^{\frac{1}{2}}} \frac{dK}{d\xi}, \tag{4.2.5}$$

and therefore condition (4.2.4) is satisfied provided

$$\frac{dK}{d\xi}(0) = 0. \tag{4.2.6}$$

One further condition has to be satisfied by the rivulet, that is the mass flux across the contact line $y = \pm a(t,x)$, is zero. Let

$$Q_x(x, y, t) = \int_0^h v_x(x, y, z, t) dz, \tag{4.2.7}$$

$$Q_y(x,y,t) = \int_0^h v_y(x,y,z,t)dz, \tag{4.2.8}$$

and using (2.8.4) and (2.8.5) for v_x and v_y we obtain

$$Q_x(x,y,t) = \frac{\rho g}{3\mu}\sin\alpha h^3, \tag{4.2.9}$$

$$Q_y(x,y,t) = \frac{-\rho g}{3\mu}\cos\alpha h^3 \frac{\partial h}{\partial y}. \tag{4.2.10}$$

Consider first the mass flux across the segment $y = +a(t,x)$. Now

$$mass\ flux\ across\ the\ contact\ line\ y = +a(t,x) \propto Q_y(x, a(t,x))\cos\alpha - Q_x(x, a(t,x))\sin\alpha, \tag{4.2.11}$$

where α is the angle the tangent to the contact line makes with the x-axis. But

$$Q_x(x, a(t,x), t) = \frac{\rho g}{3\mu}\sin\alpha h^3(x, a(t,x), t) = 0. \tag{4.2.12}$$

Since $h = 0$ on the contact line. Thus, the mass flux across the contact line $y = +a(t,x)$ vanishes provided

$$Q_y(x, a(t,x), t) = 0, \tag{4.2.13}$$

that is provided

$$h^3 \frac{\partial h}{\partial y}|_{y=+a(t,x)} = 0. \tag{4.2.14}$$

Using (3.5.7) for $h(t,x,y)$, condition (4.2.14) becomes

$$K^3(A)\frac{dK}{d\xi}(A) = 0. \tag{4.2.15}$$

Similarly it can be shown that the mass flux across the contact line $y = -a(t,x)$ vanishes provided

$$K^3(-A)\frac{dK}{d\xi}(-A) = 0. \tag{4.2.16}$$

Although $K(\pm A) = 0$, the product in (4.2.15) and (4.2.16) may not necessarily vanish if the magnitude of $\frac{dK}{d\xi}$ tends to infinity as ξ tends to $\pm A$.

The problem is therefore to solve the differential equation

$$\frac{d}{d\xi}\left(K^3\frac{dK}{d\xi}\right)+\frac{3}{4}\frac{d}{d\xi}\left(\xi K^3\right)-\frac{1}{4}\frac{d}{d\xi}\left(\xi K\right)+\frac{3}{4}K\left(1-3K^2\right)-W=0, \tag{4.2.17}$$

for $K(\xi)$ for some choice of $W(\xi)$ subject to the boundary conditions

$$K(\pm A)=0,\ \ \frac{dK}{d\xi}(0)=0, \tag{4.2.18}$$

where the constant A has to be determined. The solution for $K(\xi)$ must satisfy the condition

$$K^3(\pm A)\frac{dK}{d\xi}(\pm A)=0. \tag{4.2.19}$$

Once $K(\xi)$ and A have been derived, $h(t,x,y)$, $w(t,x,y)$ and $a(t,x)$ are obtained from

$$h(t,x,y)=\frac{x^{\frac{1}{2}}}{\left(t+\frac{c_4}{c_1}\right)^{\frac{1}{2}}}K(\xi), \tag{4.2.20}$$

$$w(t,x,y)=\frac{x^{\frac{1}{2}}}{\left(t+\frac{c_4}{c_1}\right)^{\frac{3}{2}}}W(\xi), \tag{4.2.21}$$

$$a(t,x)=\frac{Ax^{\frac{3}{4}}}{\left(t+\frac{c_4}{c_1}\right)^{\frac{1}{4}}}, \tag{4.2.22}$$

where the similarity variable ξ is

$$\xi=\left(t+\frac{c_4}{c_1}\right)^{\frac{1}{4}}\frac{y}{x^{\frac{3}{4}}}. \tag{4.2.23}$$

The mean velocity averaged over the height of the fracture is

$$v_x^*(x,y,t)=\frac{1}{h}\int_0^h v_x(x,y,z,t)dz=\frac{\rho g}{3\mu}\sin\alpha h^2, \tag{4.2.24}$$

$$v_y^*(x,y,t)=\frac{1}{h}\int_0^h v_y(x,y,z,t)dz=\frac{-\rho g}{3\mu}\cos\alpha h^2\frac{\partial h}{\partial y}. \tag{4.2.25}$$

Expressed in dimensionless form and using (2.8.9) and (2.8.10), the mean velocities are

$$v_x^*(x,y,t) = h^2, \tag{4.2.26}$$

$$v_y^*(x,y,t) = -h^2 \frac{\partial h}{\partial y}. \tag{4.2.27}$$

Using (3.5.7) for h, (4.2.26) and (4.2.27) become

$$v_x^*(x,y,t) = \frac{x}{\left(t + \frac{c_4}{c_1}\right)} K^2(\xi), \tag{4.2.28}$$

$$v_y^*(x,y,t) = \frac{x^{\frac{3}{4}}}{\left(t + \frac{c_4}{c_1}\right)^{\frac{5}{4}}} K^2(\xi) \frac{dK}{d\xi}. \tag{4.2.29}$$

The mean velocity can therefore be calculated once $K(\xi)$ has been found.

We will choose $\frac{c_4}{c_1} = 0$ although values of $\frac{c_4}{c_1} > 0$ can be considered if the solution is not well behaved at $t = 0$. From (3.2.30) the Lie point symmetry which generates the solution is

$$X = t\frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x} + \frac{1}{4}(3\alpha - 1)\, y \frac{\partial}{\partial y} + \frac{1}{2}(\alpha - 1)\, h \frac{\partial}{\partial h}, \tag{4.2.30}$$

where $\alpha = \frac{c_2}{c_1}$.

4.3 Analytical solution

Consider a leak-off velocity of the form

$$W(\xi) = \frac{3}{4} K \left(1 - 3K^2\right). \tag{4.3.1}$$

For leak-off into the substrate, $W(\xi) > 0$. Thus $K(\xi)$ lies in the range $0 < K(\xi) < \frac{1}{\sqrt{3}}$. Equation (4.2.17) reduces to

$$\frac{d}{d\xi}\left(K^3 \frac{dK}{d\xi}\right) + \frac{3}{4}\frac{d}{d\xi}\left(\xi K^3\right) - \frac{1}{4}\frac{d}{d\xi}\left(\xi K\right) = 0. \tag{4.3.2}$$

Equation (4.3.2) can be integrated immediately once with respect to ξ to obtain

$$K^3\frac{dK}{d\xi}+\frac{3}{4}\xi K^3-\frac{1}{4}\xi K+a=0, \tag{4.3.3}$$

where a is a constant. By imposing either the boundary conditions (4.2.18) and (4.2.19) at $\xi = A$ or the boundary condition (4.2.18) at $\xi = 0$ it can be shown that $a = 0$. Hence

$$K^2\frac{dK}{d\xi}=\frac{1}{4}\xi\left(1-3K^2\right). \tag{4.3.4}$$

which is variable separable and can be written as

$$\frac{K^2dk}{(1-3K^2)}=\frac{\xi}{4}d\xi. \tag{4.3.5}$$

Integration of (4.3.5) gives

$$\frac{1}{2\sqrt{3}}\ln\left[\frac{1+\sqrt{3}K}{1-\sqrt{3}K}\right]-K=\frac{3}{8}\xi^2+b, \tag{4.3.6}$$

where b is a constant. But since $K(\pm A) = 0$,

$$b=\frac{-3}{8}A^2 \tag{4.3.7}$$

Hence

$$\frac{1}{2\sqrt{3}}\ln\left[\frac{1+\sqrt{3}K}{1-\sqrt{3}K}\right]-K=\frac{3}{8}\left(\xi^2-A^2\right), \tag{4.3.8}$$

which is an exact implicit solution for $K(\xi)$. The constant A is not determined and can be given arbitrary positive values.

In order to investigate the properties of the solution consider the expansion of (4.3.8) for small values of K. Since at the contact line, $K = 0$, this expansion should be applicable near the contact line. Now if $|\xi| < 1$,

$$\ln\left(1+\epsilon\right)=\epsilon-\frac{\epsilon^2}{2}+\frac{\epsilon^3}{3}-\frac{\epsilon^4}{4}+\frac{\epsilon^5}{5}+O\left(\epsilon^6\right), \tag{4.3.9}$$

as $\epsilon \to 0$. Using (4.3.9) to expand (4.3.8) we obtain

$$K^3 + \frac{9}{5}K^5 + = \frac{3}{8}\left(\xi^2 - A^2\right), \tag{4.3.10}$$

as $K \to 0$. Since the left hand side of (4.3.10) is positive we conclude that $\xi^2 > A^2$ and therefore that the rivulet flows in the region $\xi < -A$ and $\xi > A$. The range of the rivulet at a given value of x and time t is therefore

$$-\infty < y < \frac{-Ax^{\frac{3}{4}}}{t^{\frac{1}{2}}}, \quad \frac{Ax^{\frac{3}{4}}}{t^{\frac{1}{2}}} < y < \infty. \tag{4.3.11}$$

There is therefore a dry patch in the flow defined by

$$\frac{-Ax^{\frac{3}{4}}}{t^{\frac{1}{2}}} < y < \frac{Ax^{\frac{3}{4}}}{t^{\frac{1}{2}}}. \tag{4.3.12}$$

For a fixed value of time t the width of the dry patch increases as x increases down the inclined plane. The dry patch is a result of fluid leak-off into the porous substrate. Dry patches in rivulets due to other mechanisms have been found. For example, Holland, Wilson and Duffy [8] found slender dry patches in rivulets on an inclined plane that is either heated or cooled and includes surface tension effects.

From (4.3.10) and to lowest order in K,

$$K(\xi) = \left(\frac{3}{8}\right)^{\frac{1}{3}} \left(\xi^2 - A^2\right)^{\frac{1}{3}}, \tag{4.3.13}$$

and therefore

$$\frac{dK}{d\xi} = \frac{\xi}{3^{\frac{2}{3}}\left(\xi^2 - A^2\right)^{\frac{2}{3}}}. \tag{4.3.14}$$

Hence

$$\frac{dK}{d\xi} \to -\infty \;\; as \;\; \xi \to -A, \quad \frac{dK}{d\xi} \to \infty \;\; as \;\; \xi \to A. \tag{4.3.15}$$

Also

$$\frac{\partial h}{\partial y} = \frac{x^{\frac{1}{2}} y}{3^{\frac{2}{3}} t^{\frac{1}{2}} \left(y - \frac{Ax^{\frac{3}{4}}}{t^{\frac{1}{4}}}\right)\left(y + \frac{Ax^{\frac{3}{4}}}{t^{\frac{1}{4}}}\right)}, \tag{4.3.16}$$

thus

$$\frac{\partial h}{\partial y} \to -\infty \quad as \quad y \to \frac{-Ax^{\frac{3}{4}}}{t^{\frac{1}{4}}}, \quad \frac{\partial h}{\partial y} \to \infty \quad as \quad y \to \frac{Ax^{\frac{3}{4}}}{t^{\frac{1}{4}}} \tag{4.3.17}$$

The magnitude of the gradient $\frac{\partial h}{\partial y}$ is infinite at the edge of the dry patch. The lubrication approximation, $\frac{H}{D} << 1$, therefore breaks down at the edge of the dry patch.

From (4.2.20), the height of the rivulet is given by

$$h(t, x, y) = \left(\frac{x}{t}\right)^{\frac{1}{2}} K(\xi). \tag{4.3.18}$$

From (4.3.4), $K(\xi)$ is an increasing function of ξ for $0 < K < \frac{1}{\sqrt{3}}$ and from (4.3.4), $K(\xi) \to \frac{1}{\sqrt{3}}$ as $\xi \to \pm\infty$.

The leak-off velocity $w(t, x, y)$ is obtained from (4.2.21) and (4.3.1):

$$w(t, x, y) = \frac{3}{4}\frac{x^{\frac{1}{2}}}{t^{\frac{3}{2}}} K(\xi)\left(1 - 3K^2(\xi)\right). \tag{4.3.19}$$

It increases from zero at the edge of the dry patch where $K = 0$, and reaches a maximum value for given x and t of

$$w_{max} = \frac{1}{6}\frac{x^{\frac{1}{2}}}{t^{\frac{3}{2}}}, \tag{4.3.20}$$

when $K = \frac{1}{3}$ and tends to zero as $\xi \to \pm\infty$.

4.4 Numerical method

In this section we outline the numerical method. In order to find the constant A, which occurs in the equation of the contact line and the boundary condition

$$K(\pm A) = 0, \tag{4.4.1}$$

we let

$$K(0) = b, \tag{4.4.2}$$

where b is a constant to be determined. For a given choice of $W(\xi)$ we seek a solution for the pair, b and A. This converts the boundary value problem into an initial value problem. The initial value problem is as follows

$$\frac{d}{d\xi}\left(K^3\frac{dK}{d\xi}\right)+\frac{3}{4}\frac{d}{d\xi}\left(\xi K^3\right)-\frac{1}{4}\frac{d}{d\xi}\left(\xi K\right)+\frac{3}{4}K\left(1-3K^2\right)-W\left(\xi\right)=0, \tag{4.4.3}$$

$$\frac{dK(0)}{d\xi}=0, \tag{4.4.4}$$

$$K(0)=b, \tag{4.4.5}$$

where we seek b such that $K(\pm A)=0$ has a solution for some A.

Since $W(\xi)$ is an arbitrary function of ξ, we can choose the form it takes. This gives rise to two interesting cases, when $W(\xi)$ is proportional to $K(\xi)$ and when it is proportional to $K(\xi)^3$.

The numerical solutions were found using a shooting method. The in-built ordinary differential equation solver in Mathematica 7 was used. Mathematica was chosen instead of Matlab. Although Matlab treats the infinite derivative of $K(\xi)$ at $\xi=\pm A$ better than Mathematica, the graphs produced by Mathematica were easier to read and interpret. In addition the use of the manipulate function in Mathematica enables us to investigate more than one solution and acquire insight for the solution without running the code numerous times.

4.5 Numerical solution for leak-off velocity proportional to height of the rivulet

Consider

$$W(\xi)=\beta K(\xi) \tag{4.5.1}$$

where β is a positive constant. Then from (4.2.20) and (4.2.21), with $c_4 = 0$,

$$W(\xi) = \beta \frac{h(t,x,y)}{t}. \tag{4.5.2}$$

The leak-off velocity is therefore proportional to the height $h(t,x,y)$ of the rivulet. It depends on time explicitly and is inversely proportional to t. The initial value problem (4.4.3), (4.4.4) and (4.4.5) becomes

$$\frac{d}{d\xi}\left(K^3 \frac{dK}{d\xi}\right) + \frac{3}{4}\frac{d}{d\xi}\left(\xi K^3\right) - \frac{1}{4}\frac{d}{d\xi}\left(\xi K\right) + \frac{3}{4}\left(1 - \frac{4}{3}\beta - 3K^2\right) K = 0, \tag{4.5.3}$$

$$K(0) = b, \tag{4.5.4}$$

$$\frac{dK}{d\xi}(0) = 0, \tag{4.5.5}$$

where we seek b such that

$$K\left(\pm A\right) = 0 \tag{4.5.6}$$

has a solution for some value of A.

The similarity variable, ξ, is given by (4.2.23) with $c_4 = 0$:

$$\xi = \frac{ty}{x^{\frac{3}{4}}}. \tag{4.5.7}$$

When we are investigating the dependence of h on y the other variables, x and t, are set equal to unity. When we are considering the dependence of h on x we set $y = 0$ and $t = 1$.

Consider the solution for a range of values of the leak-off parameter β. We start with no leak-off, $\beta = 0$, and then consider increasing values, $\beta = 0.1$, $\beta = 0.2$, $\ldots$. We can expect that there will be an upper limit to the value of β because the rivulet will not flow down the plane if the the leak-off is too strong. Now $b = h\left(1,1,0\right)$, is the height of the rivulet at $t = 1$, $x = 1$ and $y = 0$. To illustrate the procedure consider $\beta = 0$ and $b = 0.2$. The value of A is increased from $A = 0$ until the boundary conditions $K(\pm A) = 0$ are satisfied which determines A. It is only for this specific value $A = 0.094$ that the boundary condition is satisfied when $\beta = 0$ and $b = 0.2$. The cross section of the rivulet at $x = 1$, $t = 1$ for $\beta = 0$ and $b = 0.2$ is shown in Figure 4.5.1.

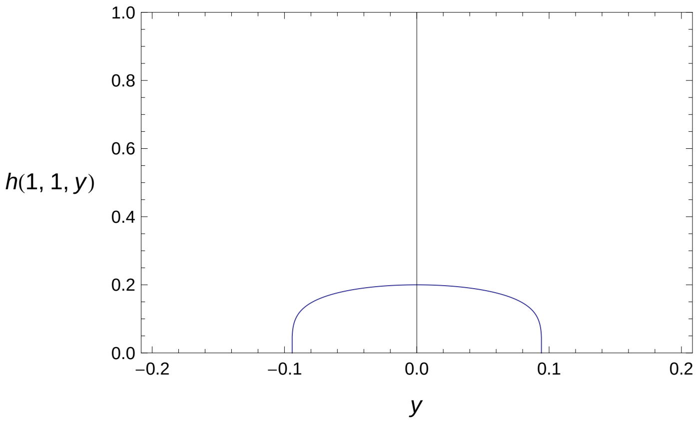

Figure 4.5.1: *Leak-off parameter* $\beta = 0$. *A cross-section of the rivulet at* $x = 1$, $t = 1$, *for* $b = 0.2$. *The value obtained for* A *was* $A = 0.094$

.

Keeping $\beta = 0$, a range of positive values of b are considered and in each case the corresponding value of A is obtained. It is found that as b increase the value of A increases and the rivulet becomes higher and broader as illustrated in Figure 4.5.2 for $\beta = 0$ and $b = 0.4$.

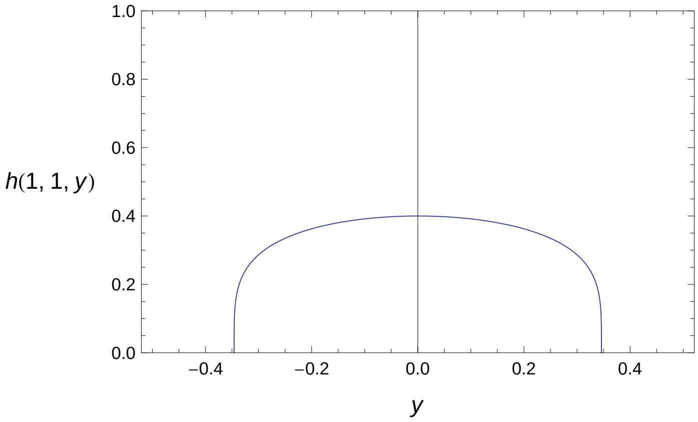

Figure 4.5.2: *Leak-off parameter* $\beta = 0$. *A cross-section of the rivulet at* $x = 1$, $t = 1$ *for* $b = 0.4$. *The value obtained for* A *was* $A = 0.346$

.

For $b = 0.57$ the solution of the boundary conditions $K(\pm A) = 0$ is $A = \infty$ and the rivulet becomes infinitely wide. This is illustrated in Figure 4.5.3.

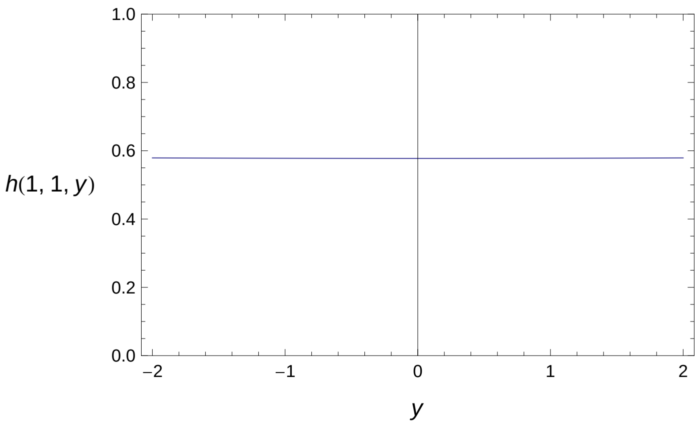

Figure 4.5.3: *Leak-off parameter* $\beta = 0$. *A cross-section of the rivulet at* $x = 1$, $t = 1$ *for* $b = 0.57$. *The value obtained for* A *was* $A = \infty$

.

For $b > 0.57$ the rivulet remains infinitely wide and the leak-off becomes stronger in the central region in the region of the axis of symmetry. This is illustrated in Figure 4.5.4 for $\beta = 0$ and $b = 0.65$.

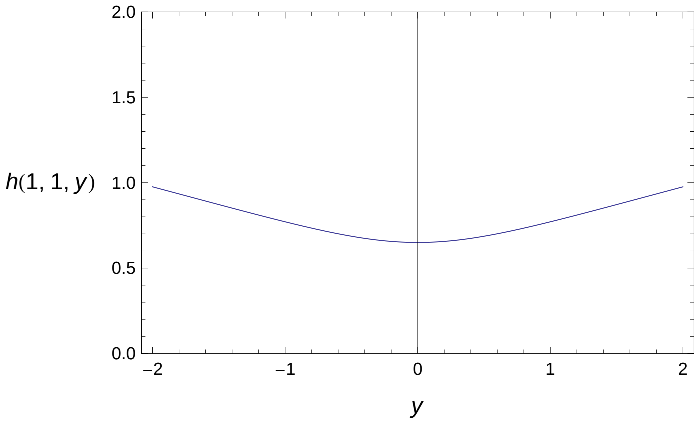

Figure 4.5.4: *Leak-off parameter* $\beta = 0$. *A cross-section of the rivulet at* $x = 1$, $t = 1$ *for* $b = 0.65$. *The value obtained for* A *was* $A = \infty$

.

The relation between the height b and the boundary value A for $\beta = 0$ is shown in Figure 4.5.5.

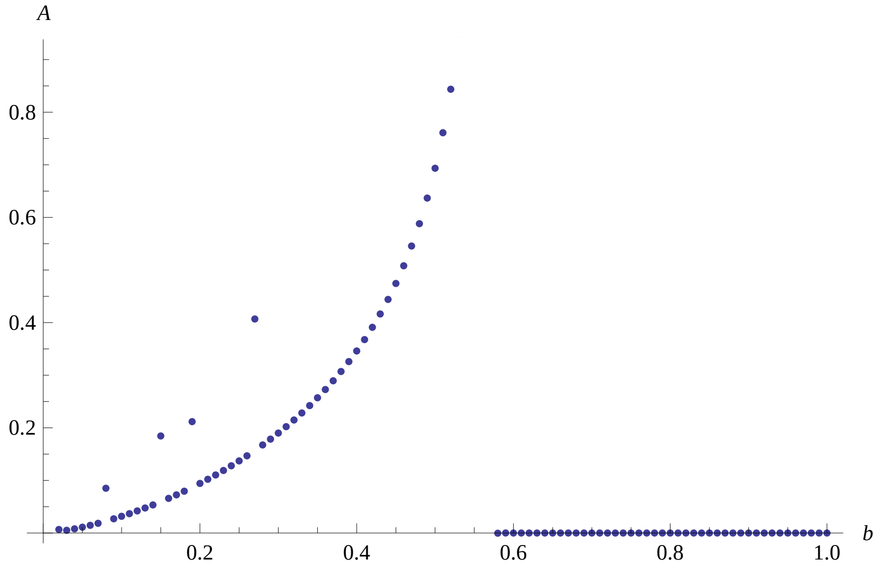

Figure 4.5.5: *Leak-off parameter* $\beta = 0$. *The boundary value* A *plotted against the height,* $b = K(0)$. *The maximum value of* b *is* $b_{max} = 0.57$

.

There are some points in Figure 4.5.5 not on the main locus of points but this is a feature of the shooting method for which there can be outlying points. We see that, for $\beta = 0$, as b increases from $b = 0$ to $b = 0.57$ the corresponding value of A increases steadily from zero and that $A \to \infty$ as $b \to 0.57$.

The value $\beta = 0.1$ is then chosen. It is found that the range of b is decreased. As b increases from $b = 0$ to $b = 0.51$ the corresponding value of A increase from zero to infinity. This is illustrated in Figure 4.5.6.

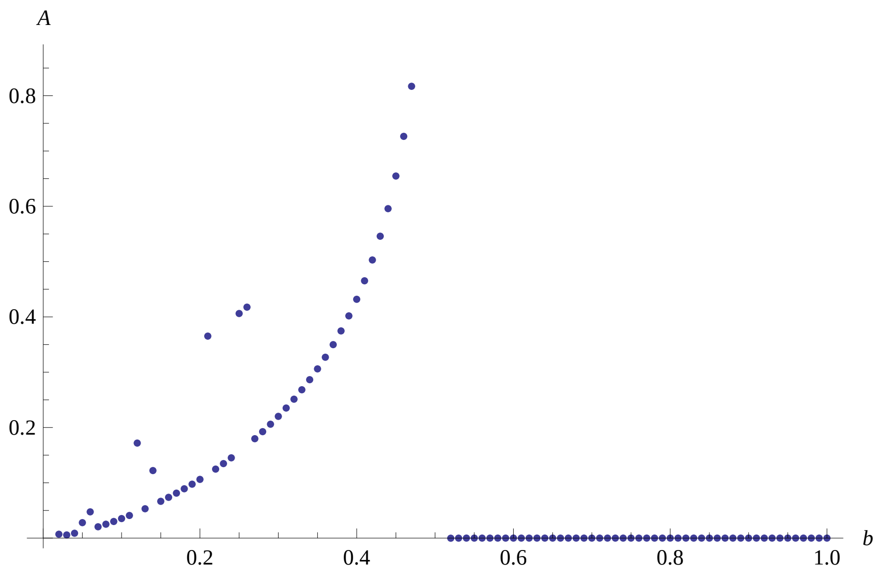

Figure 4.5.6: *Leak-off parameter* $\beta = 0.1$. *The boundary value* A *plotted against the height,* $b = K(0)$. *The maximum value of* b *is* $b_{max} = 0.51$

.

For $\beta = 0.2$, $\beta = 0.3$ and $\beta = 0.4$ the range of b steadily decreases from $0 < b < 0.44$ to $0 < b < 0.36$ and finally to $0 < b < 0.25$ as illustrated in Figures 4.5.7 to 4.5.9.

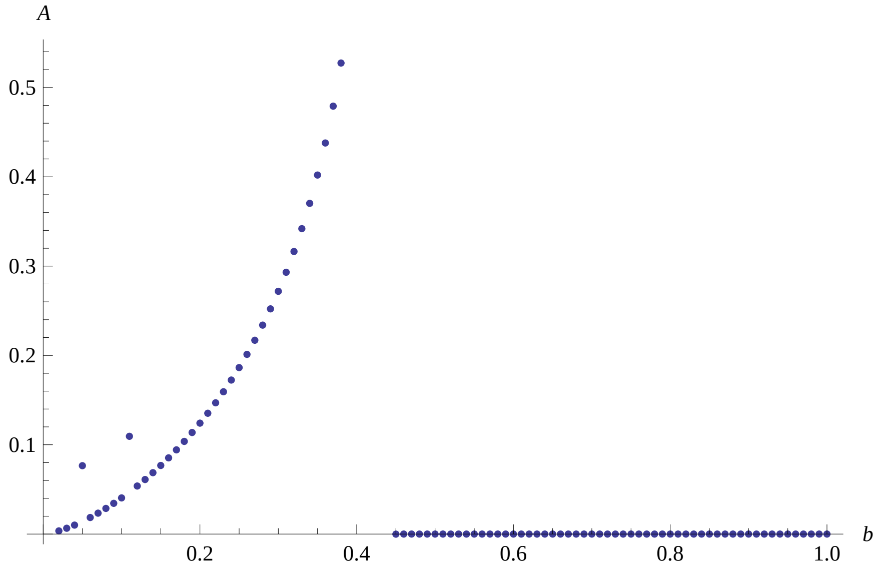

Figure 4.5.7: *Leak-off parameter* $\beta = 0.2$*. The boundary value* A *plotted against the height,* $b = K(0)$ *where* $b_{max} = 0.44$

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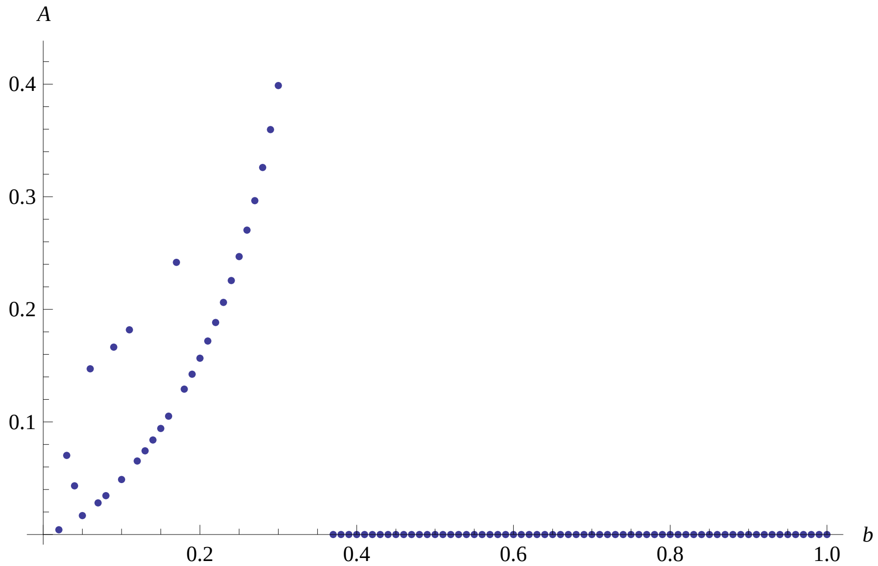

Figure 4.5.8: *Leak-off parameter* $\beta = 0.3$*. The boundary value* A *plotted against the height,* $b = K(0)$ *where* $b_{max} = 0.36$

.

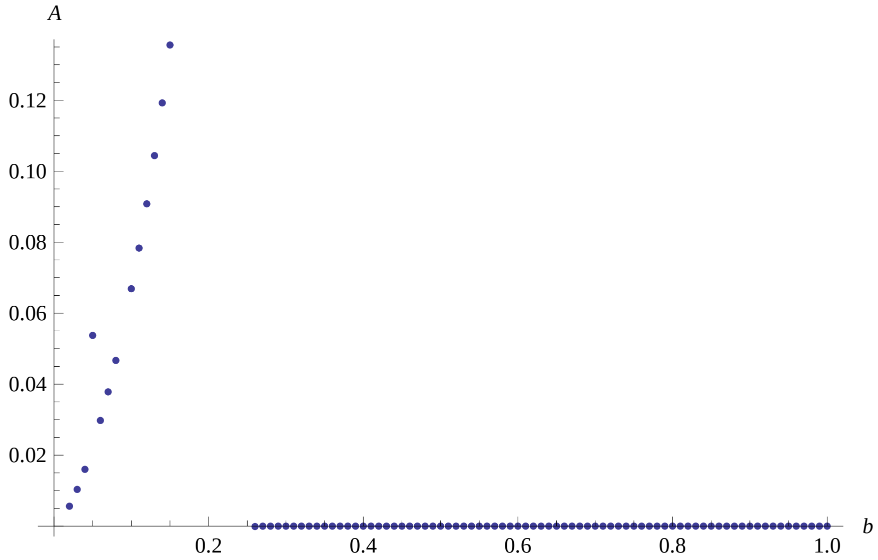

Figure 4.5.9: *Leak-off parameter* $\beta = 0.4$. *The boundary value* A *plotted against the height,* $b = K(0)$ *where* $b_{max} = 0.25$

.

When $\beta = 0.487$ there are no values of $b > 0$ for which $K(\pm A)$ has a finite solution. The value $\beta = 0.487$ is therefore the limiting value of the leak-off parameter for the rivulet flow to exist. The values of the leak-off parameter β and the corresponding maximum values of b are listed in Table 4.5.1. We see that the leak-off parameter cannot exceed the maximum value $\beta = 0.487$. For values of β in the range $0 < \beta < 0.487$ and for any given value of A in the range $0 < A < \infty$, there is a solution for the height $b = K(0) = h(1, 1, 0)$ and therefore a solution to the problem exists. As A increases and therefore as the width increases, the height b increases and therefore the depth of the rivulet increases. If the height b is given then a solution to the rivulet problem will only exist for a certain range $0 \leq b < b_{max}$ where b_{max} depends on the leak-off parameter β and decreases as β increases.

Consider now the properties of the solution. We first fix the initial height $b = K(0)$ at a range of values and for each value of b we investigate how the width of the rivulet changes

Table 4.5.1: The maximum value of $b = K(0)$ for values of the leak-off parameter β

β	maximum b
0	0.57
0.1	0.51
0.2	0.44
0.3	0.36
0.4	0.25
0.487	0

as the strength β of the leak-off velocity is increased from $\beta = 0$. Consider first $b = 0.2$, in Figure 4.5.10 the cross-section of the rivulet is plotted at $x = 1$ and $t = 1$ with $b = 0.2$ and for $\beta = 0.1,\ 0.429$ and 0.4399.

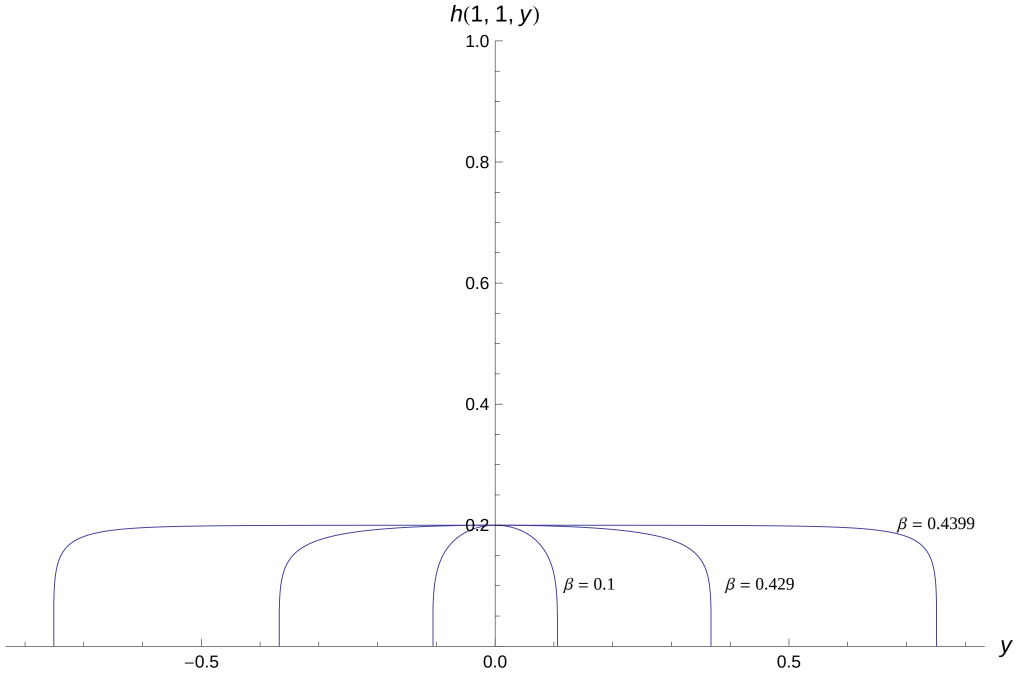

Figure 4.5.10: *Cross-section of the rivulet at* $x = 1$, $t = 1$, *for initial height* $b = 0.2$ *and leak-off parameter* $\beta = 0.1$, 0.429 *and* 0.4399

.

We see that if the initial height b remains fixed then the rivulet becomes wider as the strength β of the leak-off velocity increases.

For $\beta = 0.44$ the cross-section of the rivulet becomes infinite as shown in Figure 4.5.11 and in agreement with Table 4.5.1, while for $\beta = 0.441$ the rivulet is infinitely wide but the drainage is greater at the centre as shown in Figure 4.5.12. The drainage at the centre becomes stronger as β increases beyond $\beta = 0.441$.

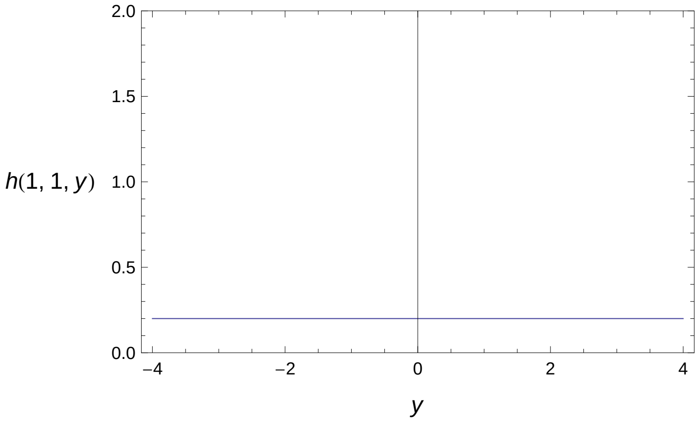

Figure 4.5.11: *Cross-section of the rivulet at* $x = 1$, $t = 1$, *for initial height* $b = 0.2$ *and leak-off parameter* $\beta = 0.44$

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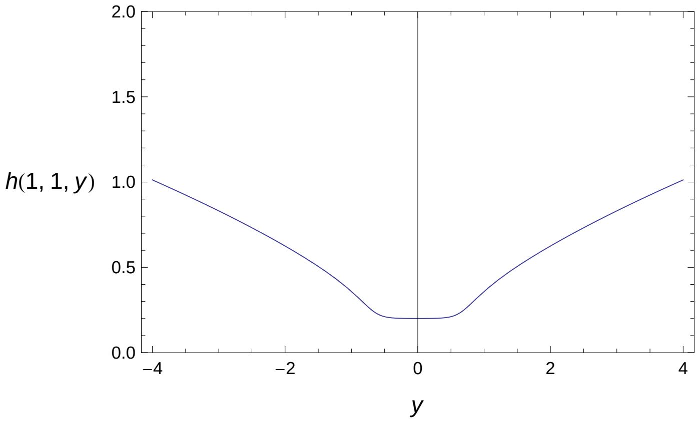

Figure 4.5.12: *Cross-section of the rivulet at* $x = 1$, $t = 1$, *for initial height* $b = 0.2$ *and leak-off parameter* $\beta = 0.441$

.

Consider next the initial height $b = 0.4$. In Figure 4.5.13 the cross-section of the rivulet is plotted at $x = 1$, $t = 1$ with $b = 0.4$ and $\beta = 0$, 0.15 and 0.23. We again see that if the initial height b is kept fixed then the rivulet becomes wider as the strength β of the leak-off velocity increases.

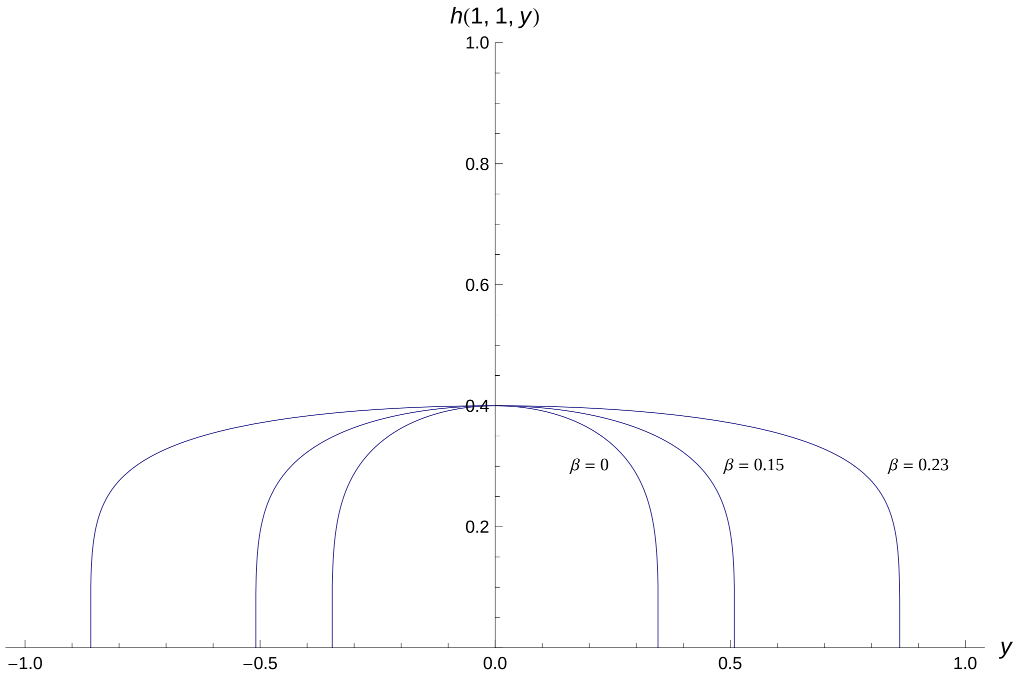

Figure 4.5.13: *Cross-section of the rivulet at* $x = 1,\ t = 1,$ *for initial height* $b = 0.4$ *and leak-off parameter* $\beta = 0,\ 0.15$ *and* 0.23

.

For $\beta = 0.2597$ the width of the cross-section is still finite. We see from Figure 4.5.14 that for $\beta = 0.26$ the cross-section of the rivulet becomes infinite. For $\beta = 0.261$ the rivulet continues to be infinitely wide but the drainage becomes strongest at the centre as shown in Figure 4.5.15.

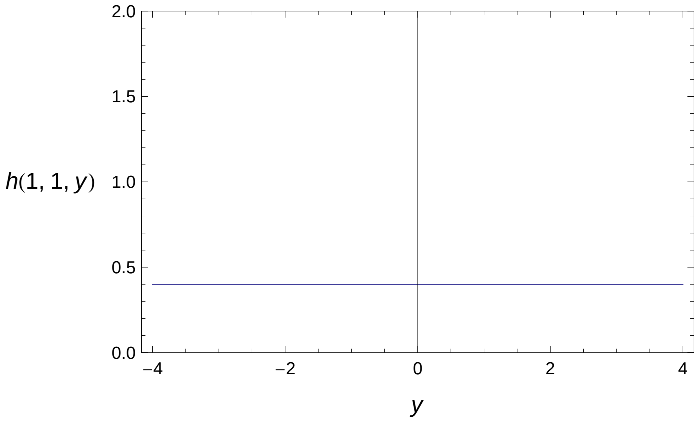

Figure 4.5.14: *Cross-section of the rivulet at $x = 1$, $t = 1$, for initial height $b = 0.4$ and leak-off parameter $\beta = 0.26$*

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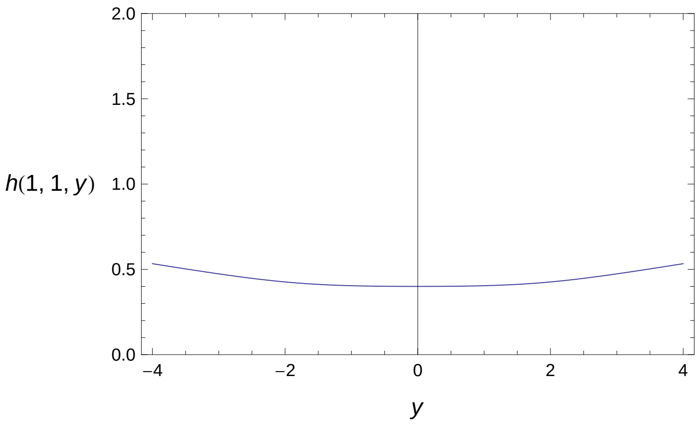

Figure 4.5.15: *Cross-section of the rivulet at $x = 1$, $t = 1$, for initial height $b = 0.4$ and leak-off parameter $\beta = 0.261$*

.

We know from Table 4.5.1 that the maximum value of b for the leak-off $\beta \geq 0$ is $b = 0.57$. We therefore consider next the initial height $b = 0.5$. In Figure 4.5.16 the cross-section of the rivulet is plotted at $x = 1$, $t = 1$ with $b = 0.5$ and $\beta = 0,\ 0.05$ and 0.115.

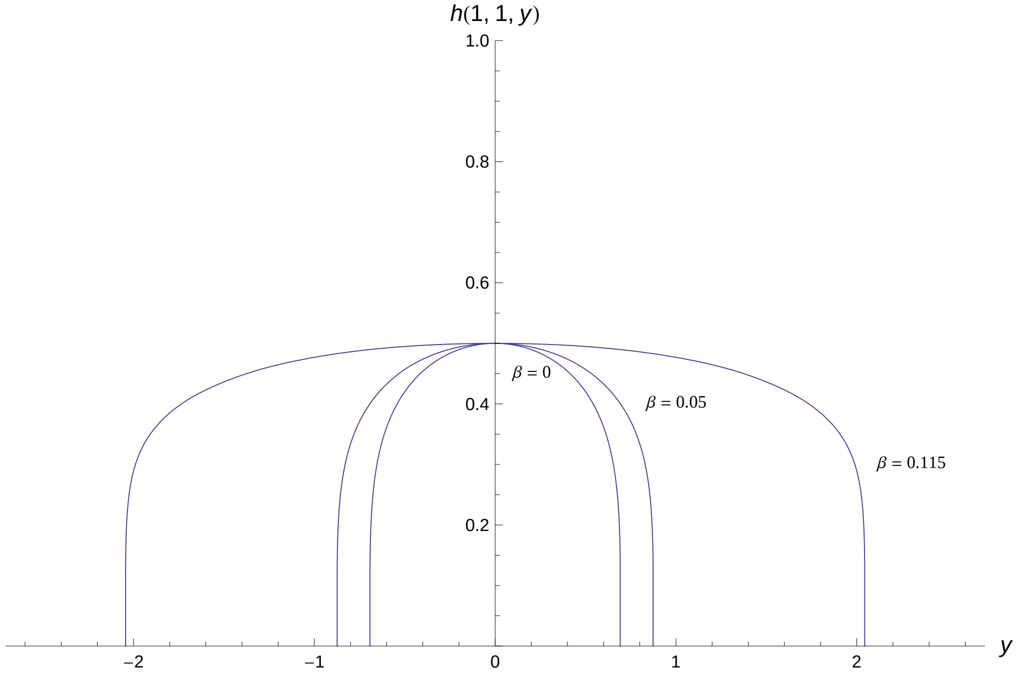

Figure 4.5.16: *Cross-section of the rivulet at* $x = 1$, $t = 1$, *for initial height* $b = 0.5$ *and leak-off parameter* $\beta = 0,\ 0.05$ *and* 0.115

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If the initial height is fixed at $b = 0.5$ we see again that the width of the rivulet increase as the strength β of the leak-off velocity increases. For $\beta = 0.123$ the width of the cross-section is still finite but from Figure 4.5.17 we see that the cross-section of the rivulet becomes infinite when $\beta = 0.125$. For larger values of β the cross-section of the rivulet remains infinite but with stronger drainage at the centre as illustrated in Figure 4.5.18 for $\beta = 0.126$. The maximum value of β for a finite cross-section is less than when $b = 0.4$.

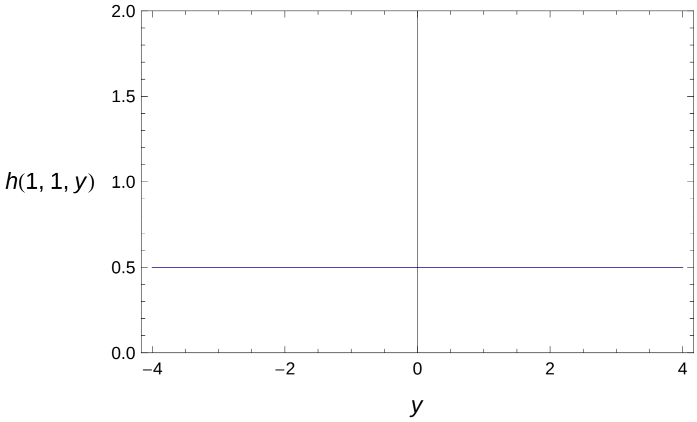

Figure 4.5.17: *Cross-section of the rivulet at $x = 1$, $t = 1$, for initial height $b = 0.5$ and leak-off parameter $\beta = 0.125$*

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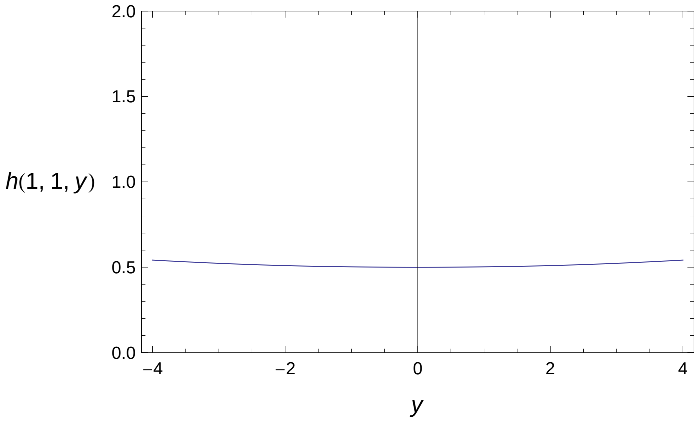

Figure 4.5.18: *Cross-section of the rivulet at* $x = 1$, $t = 1$, *for initial height* $b = 0.5$ *and leak-off parameter* $\beta = 0.126$

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As we have observed, from Table 4.5.1, the maximum value of b for $\beta \geq 0$ is $b = 0.57$, when we increased b to $b = 0.6$ we found that for all values of $\beta \geq 0$ the cross-section of the rivulet was infinite with strongest leak-off at the centre. The same occurred for $b = 0.8$ and $b = 1$.

In summary, when the initial height b is fixed and β is increased from $\beta = 0$ the rivulet becomes wider. For a given value of $b < 0.57$ there is a maximum value of $\beta \geq 0$ for the cross-section of the rivulet to remain finite as shown in Table 4.5.2. The maximum value of β decreases as the initial height b increases and decreases to $\beta = 0$ as b increases to $b = 0.57$. We conclude that for a given value of the initial height b a wider rivulet is needed to maintain a stronger leak-off velocity. As b increase the range of values of β, for leak-off in a rivulet with finite cross-section, decreases.

Consider now how b and β vary for a fixed value of the half width A. In Figure 4.5.19 the boundary value A is fixed at $A = 0.7$. As the initial height b is increased the leak-off param-

Table 4.5.2: The maximum value of the leak-off parameter β for a rivulet with finite cross-section for values of the initial height $h = K(0)$

.

b	maximum β
0	0.487
0.2	0.44
0.4	0.26
0.5	0.125
0.578	0

eter β decreases to maintain the boundary value $A = 0.7$. When b increases to $b = 0.502$, the leak-off parameter reduces to $\beta = 0$. The results of Figure 4.5.19 are as expected. They show that as the strength of the leak-off increases the height of the rivulet decreases. The stronger the leak-off velocity the thinner the rivulet.

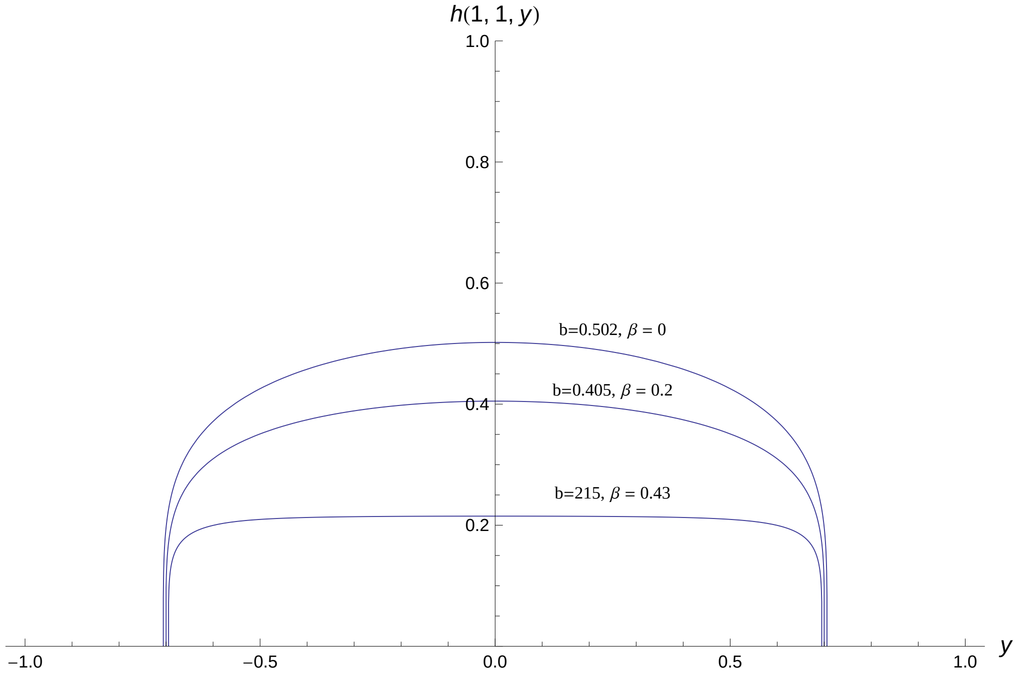

Figure 4.5.19: *Cross-section of the rivulet at* $x = 1$*,* $t = 1$ *with* $A = 0.7$ *and a range of heights* $b = H(1, 0, 1)$ *and leak-off parameter* β

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We next investigate how the rivulet evolves with time. In Figures 4.5.20, 4.5.21 and 4.5.22, b and β are kept fixed and time t is given a range of values. The values chosen for b are $b = 0.2$, 0.4 and 0.5 which are the same as in Figures 4.5.10, 4.5.13 and 4.5.16. For each value of b the value chosen for β is the limiting value for the rivulet to have a finite cross-section, $\beta = 0.439$, $\beta = 0.2597$ and $\beta = 0.123$. The boundary $y = a(t, x)$, given by (4.2.22), is plotted against y for $t = 0.1$, 1 and 10. The value of A is obtained from the boundary condition $K(A) = 0$. We see that as time increases the width of the rivulet decreases and it becomes narrower as it flows down the inclined plane. This is consistent with the time dependence in (4.2.22) for $a(t, x)$. The calculated value of A increases across the three figures causing the width of the rivulet to be greater at a given time.

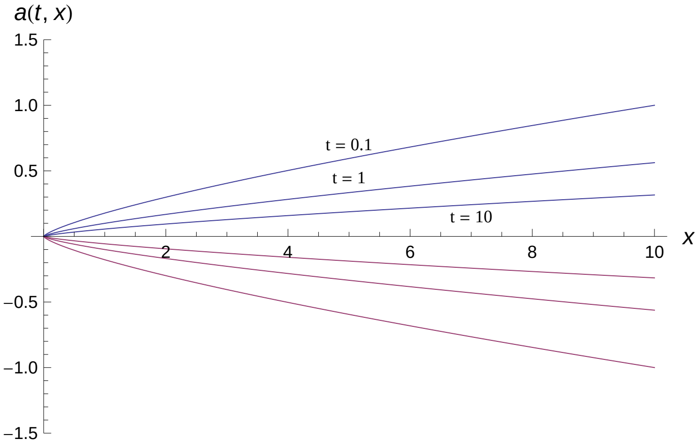

Figure 4.5.20: *Rivulet for* $b = 0.2$, $\beta = 0.439$ *and* $A = 0.1$ *at times* $t = 0.1,\ 1$ *and* 10

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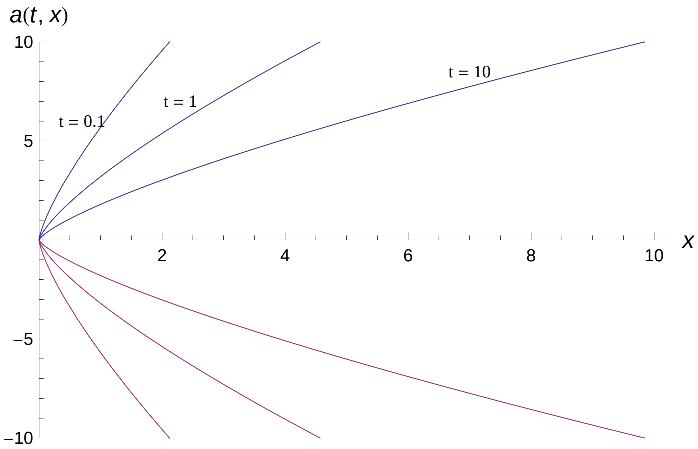

Figure 4.5.21: *Rivulet for* $b = 0.4$, $\beta = 0.2597$ *and* $A = 3.2$ *at times* $t = 0.1,\ 1$ *and* 10

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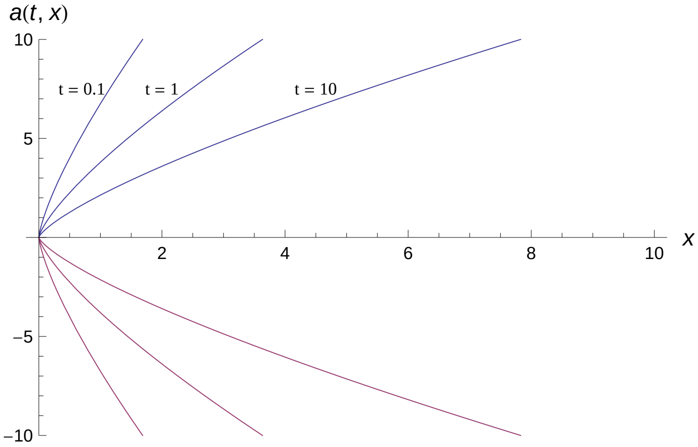

Figure 4.5.22: *Rivulet for* $b = 0.5$*,* $\beta = 0.123$ *and* $A = 3.8$ *at times* $t = 0.1,\ 1$ *and* 10

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Finally we investigate how the rivulet evolves as the leak-off parameter β is increased. In Figures 4.5.23, 4.5.24 and 4.5.25, b and t are kept fixed and β is given a range of values. We again choose $b = 0.2,\ 0.4$ and 0.5 and take $t = 1$. In each figure the values of β range from zero to the maximum value for the cross-section of the rivulet to be finite. For each pair of values of b and β the constant A is obtained from the boundary condition $K(A) = 0$. The boundary $y = a(t, x)$, given by (4.2.22), is plotted against x for $t = 1$. We see again that if b and t are kept fixed then the rivulet becomes wider as the leak-off parameter β is increased in agreement with Figures 4.5.10, 4.5.13 and 4.5.16.

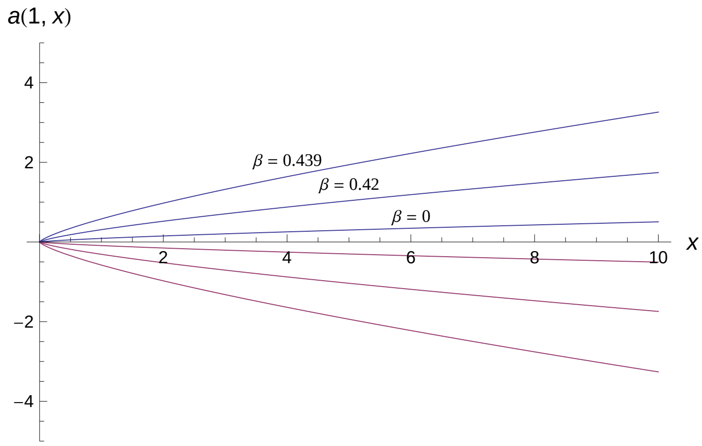

Figure 4.5.23: *Rivulet for* $b = 0.2,\ t = 1$ *and* $\beta = 0,\ 0.42$ *and* 0.439

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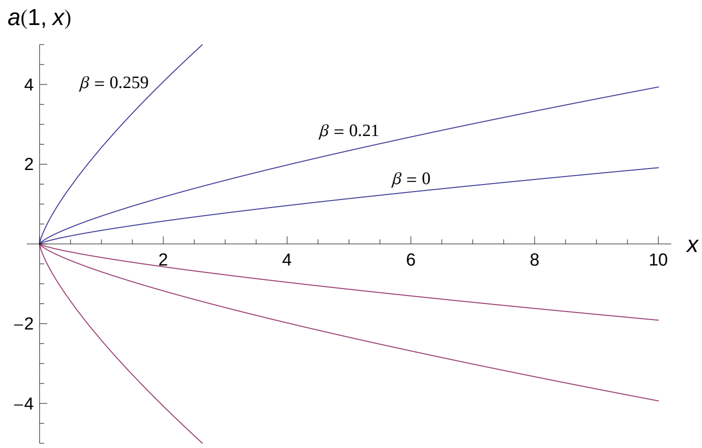

Figure 4.5.24: *Rivulet for* $b = 0.4,\ t = 1$ *and* $\beta = 0,\ 0.21$ *and* 0.259

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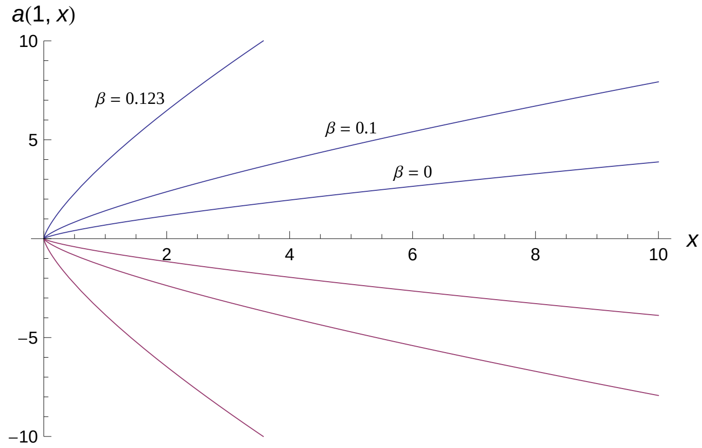

Figure 4.5.25: *Rivulet for* $b = 0.5,\ t = 1$ *and* $\beta = 0,\ 0.1$ *and* 0.123

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4.6 Numerical solutions for the leak-off velocity proportional to the cube of the height of the rivulet

Consider next

$$W(\xi) = \beta K^3(\xi) \tag{4.6.1}$$

where, for leak-off, β is a positive constant. It follows from (4.2.20) and (4.2.21) that

$$w(t, x, y) = \frac{\beta h^3(t, x, y)}{x}. \tag{4.6.2}$$

The leak-off velocity is proportional to the cube of the height and inversely proportional to the distance x down the plane. Equation (4.6.2) compares with the leak-off velocity (4.5.2) considered in Section 4.5. Using (4.6.1) the initial value problem (4.4.3), (4.4.4) and (4.4.5) becomes

$$\frac{d}{d\xi}\left(K^3\frac{dK}{d\xi}\right) + \frac{3}{4}\frac{d}{d\xi}\left(\xi K^3\right) - \frac{1}{4}\frac{d}{d\xi}\left(\xi K\right) + \frac{3}{4}\left[1 - \left(3 + \frac{4}{3}\beta\right)K^2\right]K = 0, \tag{4.6.3}$$

$$K(0) = b, \quad \frac{dK}{d\xi}(0) = 0, \tag{4.6.4}$$

where b is to be found such that

$$K(\pm A) = 0 \tag{4.6.5}$$

has a solution for some positive constant A. The similarity variable is

$$\xi = \frac{t^{\frac{1}{4}} y}{x^{\frac{3}{4}}}. \tag{4.6.6}$$

As in Section 4.5 when we are considering the dependence of h on y we set $x = 1$, $t = 1$ and when we are considering the dependence of h on x we set $y = 1$, $t = 1$. The structure of Section 4.6 is the same as Section 4.5 so that the two models, $W = \beta K$ and $W = \beta K^3$ can be compared more easily. The two models give the same result for $\beta = 0$ when they reduce to rivulet flow with no leak-off.

We first consider the solution for a range of values of the leak-off parameter β, from $\beta = 0$ to the limiting value of β for a solution with leak-off to exist. The method is the same as in Section 4.5 where it was illustrated in Figures 4.5.1 to 4.5.5 for $\beta = 0$ which again applies here. A graph of the constant A, which is the half-width at $x = 1$, $t = 1$ plotted against b for $\beta = 0$ is shown in Figure 4.5.5.

The value $\beta = 0.25$ is next chosen. The range of b is found to be smaller than when $\beta = 0$. As b increases from $b = 0$ to $b = 0.54$, the constant A increases from zero to infinity as shown in Figure 4.6.1.

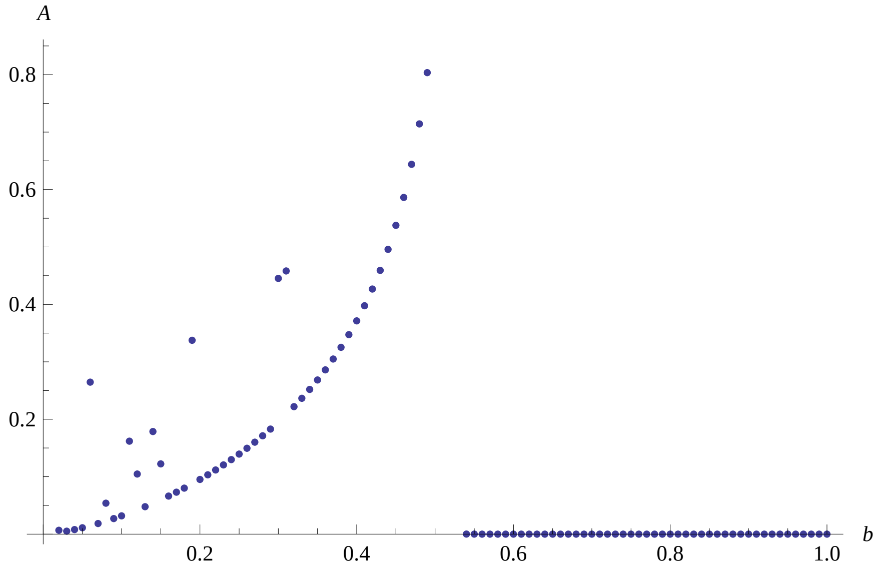

Figure 4.6.1: *Leak-off parameter* $\beta = 0.25$ *the boundary value* A *plotted against the height* $b = K(0)$. *The maximum value of* b *is* $b_{max} = 0.53$

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As β increases the range of values of b steadily decreases, as shown in Figures 4.6.2 to 4.6.8 for values of β from $\beta = 0.5$ to $\beta = 8$.

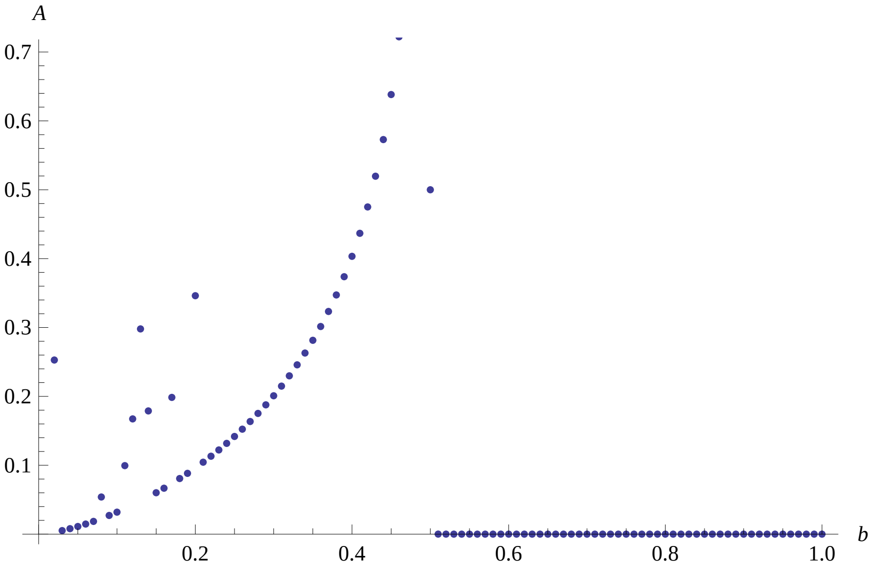

Figure 4.6.2: *Leak-off parameter* $\beta = 0.5$. *The boundary value* A *plotted against the height* $b = K(0)$ *where* $b_{max} = 0.5$

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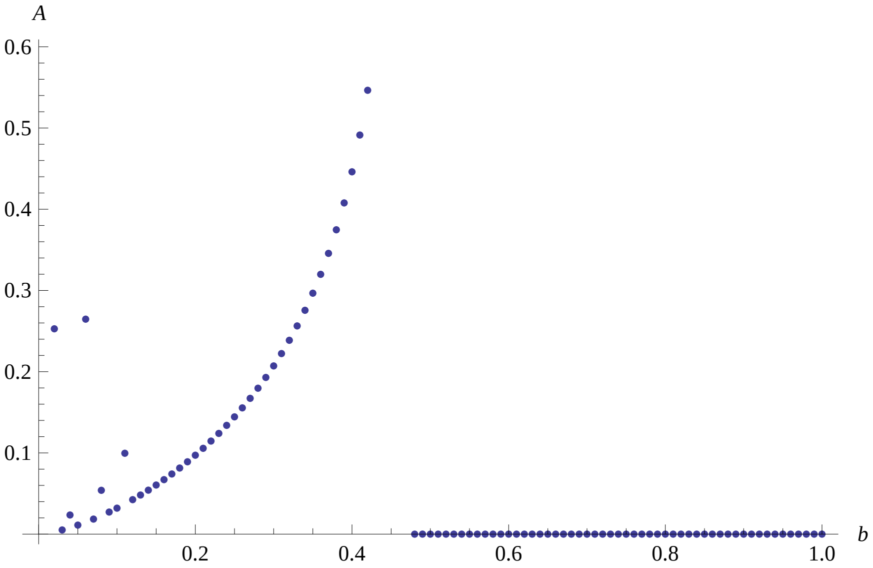

Figure 4.6.3: *Leak-off parameter* $\beta = 0.75$. *The boundary value* A *plotted against the height* $b = K(0)$ *where* $b_{max} = 0.47$

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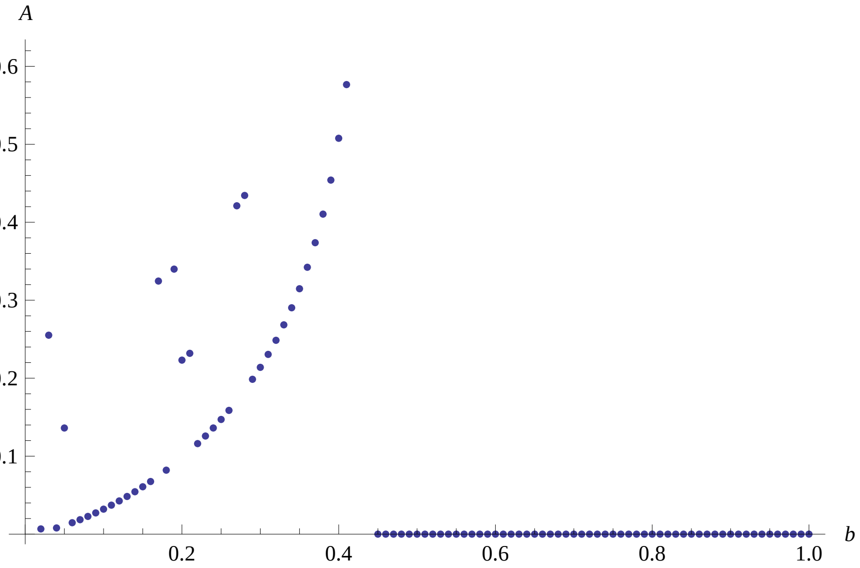

Figure 4.6.4: *Leak-off parameter* $\beta = 1$. *The boundary value* A *plotted against the height* $b = K(0)$ *where* $b_{max} = 0.44$

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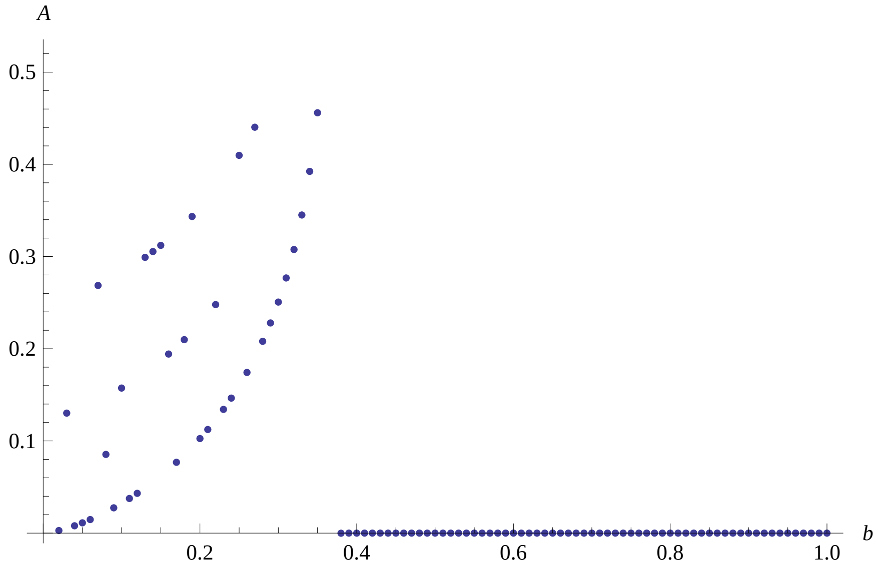

Figure 4.6.5: *Leak-off parameter* $\beta = 2$. *The boundary value* A *plotted against the height* $b = K(0)$ *where* $b_{max} = 0.37$

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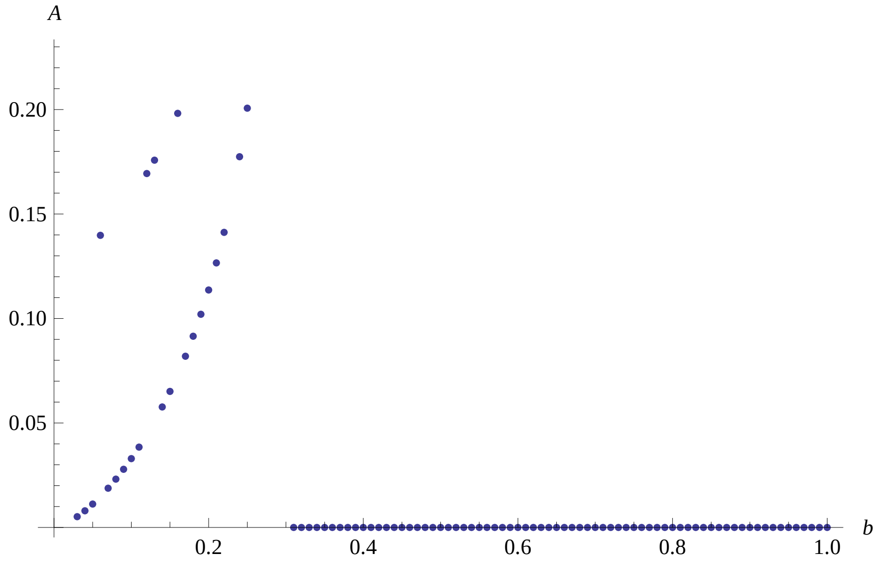

Figure 4.6.6: *Leak-off parameter* $\beta = 4$. *The boundary value* A *plotted against the height* $b = K(0)$ *where* $b_{max} = 0.3$

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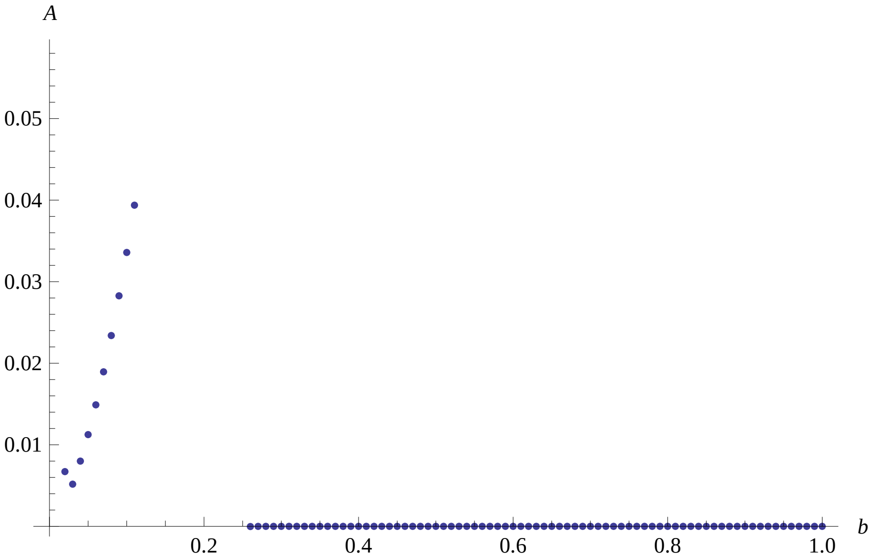

Figure 4.6.7: *Leak-off parameter* $\beta = 6$. *The boundary value* A *plotted against the height* $b = K(0)$ *where* $b_{max} = 0.25$

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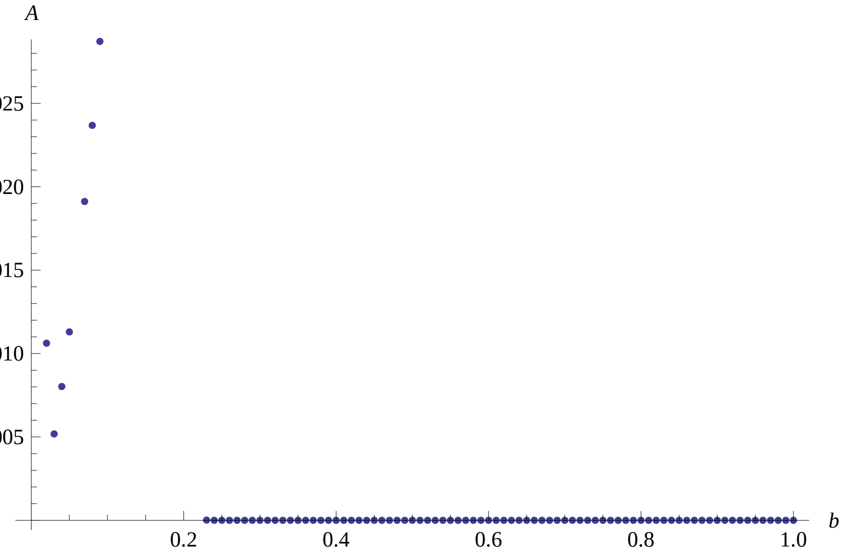

Figure 4.6.8: *Leak-off parameter* $\beta = 8$. *The boundary value* A *plotted against the height* $b = K(0)$ *where* $b_{max} = 0.22$

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As β increase above 10 the range of b for which a solution exists decrease to lie between zero and 0.2. When β is significantly large, approximately 48, there are no values of $b > 0$ for which $K(\pm A) = 0$ has a finite solution for A. However for all values of $\beta > 10$ the maximum values of b for a solution to exist are very close and so $\beta = 11$ can be viewed as the limiting value of β and therefore the strongest leak-off for the rivulet flow to exist. In Table 4.6.1, the values of the leak-off parameter β and the corresponding maximum values of b are listed.

There is therefore an upper limit, $\beta = 11$, on the leak-off parameter β for the rivulet flow with finite cross-section to exist. The solution of the initial value problem exists with leak-off for $0 < \beta < 11$. From the graphs of A against b we see that A is an increasing function of b and therefore wider rivulets are thicker rivulets. If the initial value of $K(0) = b$ is given a solution will exist only if $b < b_{max}$ where b_{max} depends on β and decreases as β increases. We next investigate the properties of the solution. We fix the initial height $b = K(0)$ at a

Table 4.6.1: The maximum value of $b = K(0)$ for values of the leak-off parameter β where $W = \beta K^3$

β	maximum b
0	0577
0.25	0.5345
0.5	0.5
0.75	0.4715
1	0.44725
2	0.37797
4	0.301511
6	02581989
8	0.2294158
10	0.208851442
11	0

range of values and for each choice of b we investigate how the width of the rivulet changes as the leak-off parameter β is increased from $\beta = 0$. We choose the same values for b, 0.2, 0.4 and 0.5, as used in Section 4.5.

In Figure 4.6.9 the cross-section of the rivulet is plotted at $x = 1$, $t = 1$, with $b = 0.2$ and for $\beta = 0.541,\ 5.5$ and 10. We see that if b is kept fixed the rivulet becomes wider as the leak-off parameter β increases. For $\beta = 11$, the cross-section of the rivulet becomes infinite as shown in Figure 4.6.10. For $\beta = 11.5$ the width of the rivulet remains infinite and the drainage is strongest on the central region as shown in Figure 4.6.11.

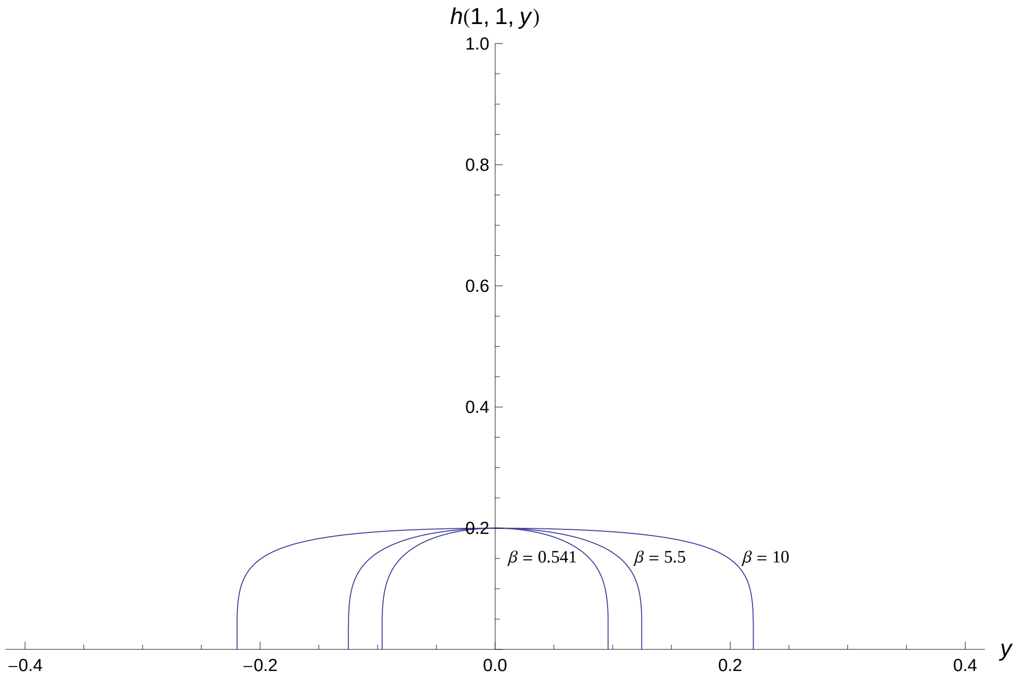

Figure 4.6.9: *Cross-section of the rivulet at* $x = 1$, $t = 1$ *for initial height* $b = 0.2$ *and leak-off parameter* $\beta = 0.541,\ 5.5$ *and* 10

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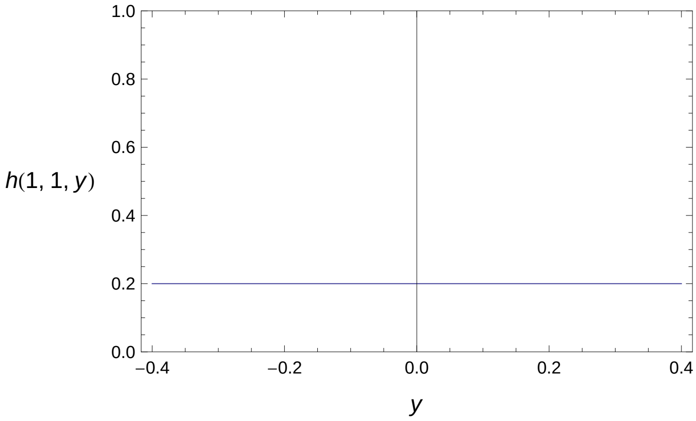

Figure 4.6.10: *Cross-section of the rivulet at $x = 1$, $t = 1$ for initial height $b = 0.2$ and leak-off parameter $\beta = 11$*

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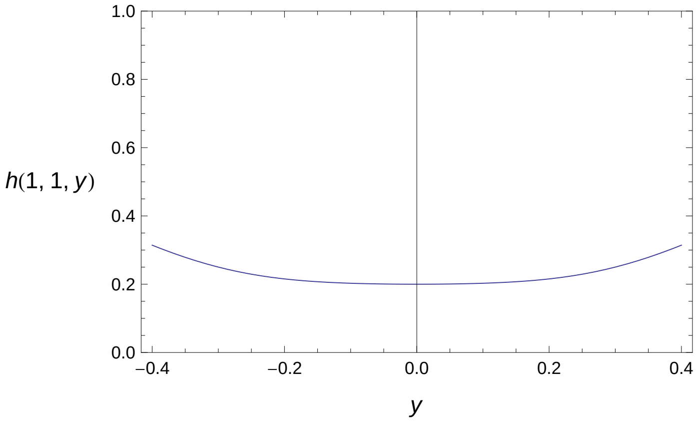

Figure 4.6.11: *Cross-section of the rivulet at $x = 1$, $t = 1$ for initial height $b = 0.2$ and leak-off parameter $\beta = 11.5$*

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In Figure 4.6.12 cross-sections of the rivulet are plotted at $x = 1$, $t = 1$, for $b = 0.4$ and $\beta = 0.95,\ 1.35$ and 1.6. The rivulet again becomes wider if the initial height $b = K(0)$ is kept fixed and the strength of the leak-off is increased. For $\beta = 1.625$ the width of the rivulet becomes infinite as illustrated in Figure 4.6.13 and for $\beta = 2$ the cross-section is infinite with strongest leak-off at the central region as shown in Figure 4.6.14.

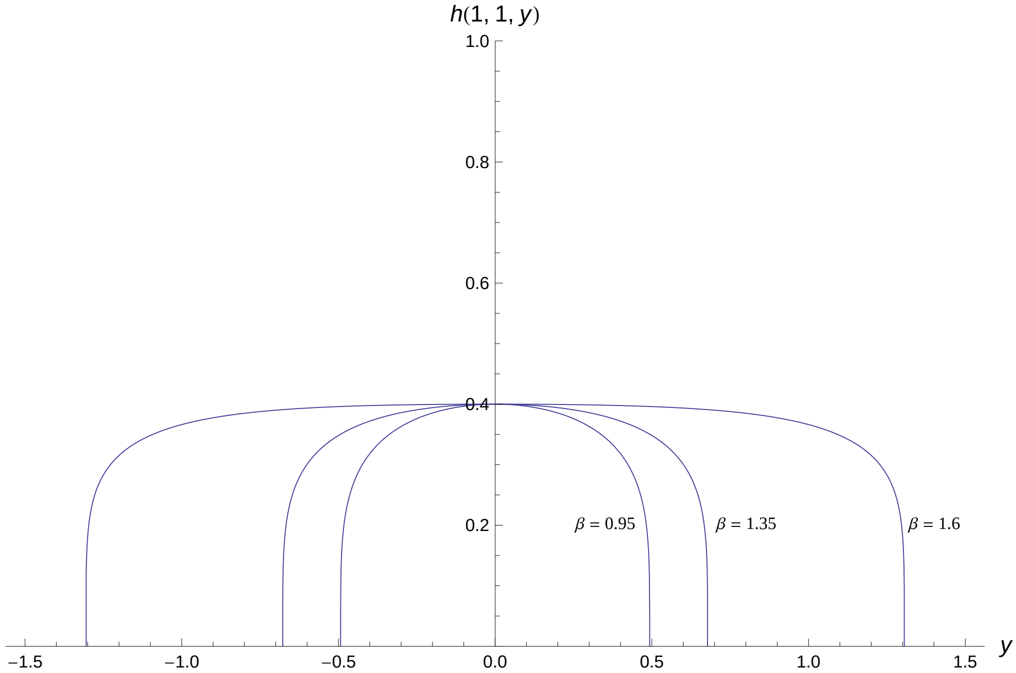

Figure 4.6.12: *Cross-section of the rivulet at* $x = 1$, $t = 1$ *for initial height* $b = 0.4$ *and leak-off parameter* $\beta = 0.95,\ 1.35$ *and* 1.6

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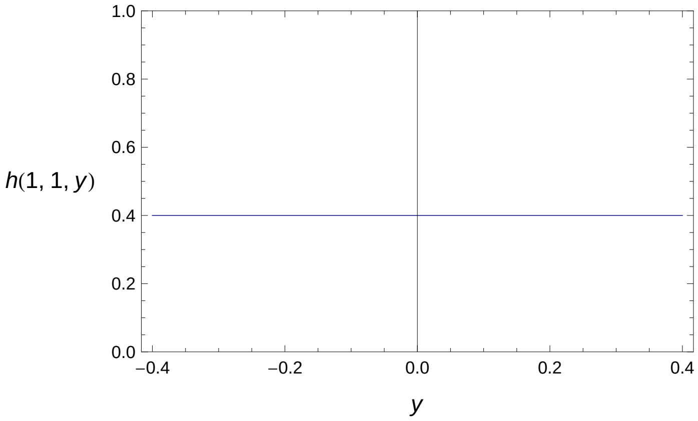

Figure 4.6.13: *Cross-section of the rivulet at* $x = 1$, $t = 1$ *for initial height* $b = 0.4$ *and leak-off parameter* $\beta = 1.625$

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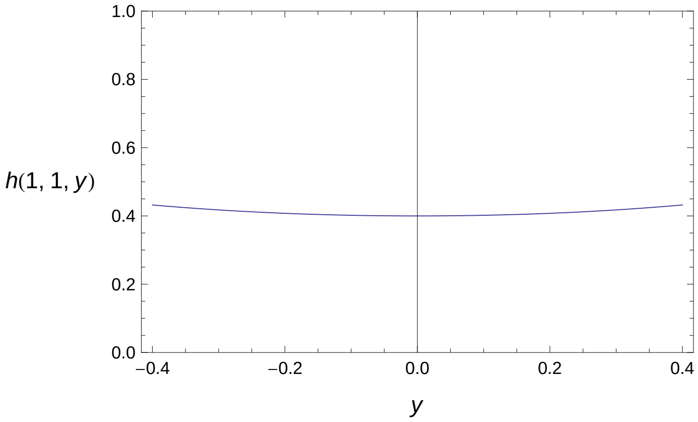

Figure 4.6.14: *Cross-section of the rivulet at* $x = 1$, $t = 1$ *for initial height* $b = 0.4$ *and leak-off parameter* $\beta = 2$

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In Figure 4.6.15 the cross-section of the rivulet at $x = 1$, $t = 1$ for $b = 0.5$ is plotted for $\beta = 0.1,\ 0.44$ and 0.49. Again we see that with b kept fixed the rivulet becomes wider as the leak-off parameter β is increased. The width of the cross-section becomes infinite at $\beta = 0.5$

as shown in Figure 4.6.16. When $\beta = 0.54$ the cross-section is infinite and the drainage is strongest in the central region as shown in Figure 4.6.17.

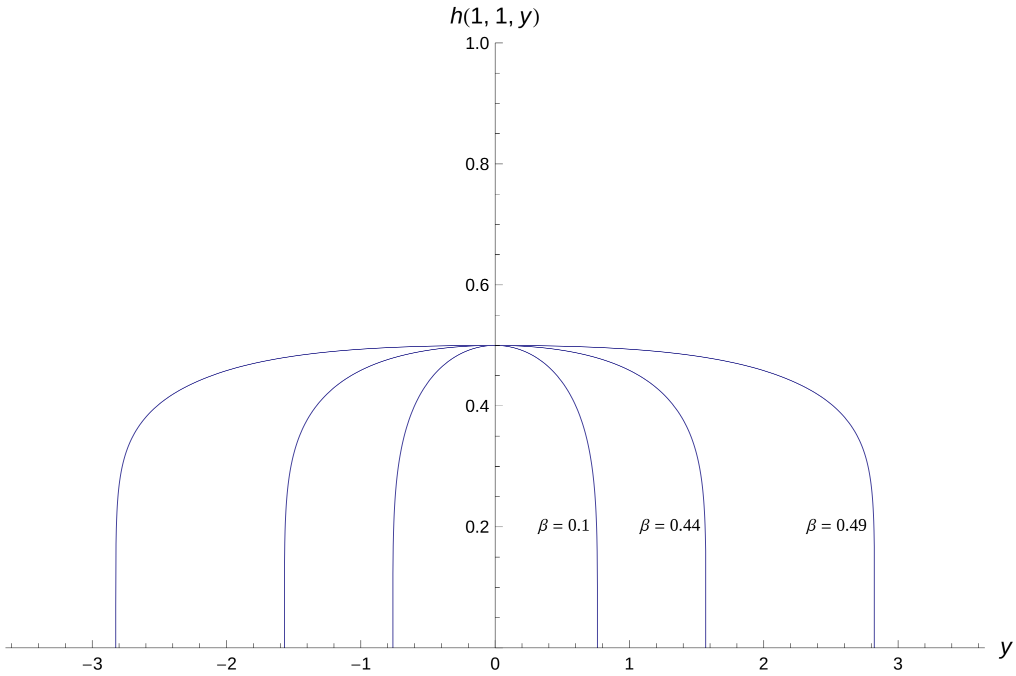

Figure 4.6.15: *Cross-section of the rivulet at* $x = 1,\ t = 1$ *for initial height* $b = 0.5$ *and leak-off parameter* $\beta = 0.1,\ 0.44$ *and* 0.49

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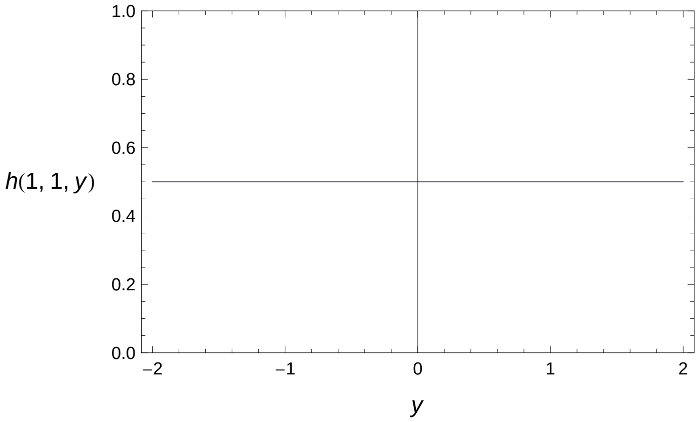

Figure 4.6.16: *Cross-section of the rivulet at $x = 1$, $t = 1$ for initial height $b = 0.5$ and leak-off parameter $\beta = 0.5$*

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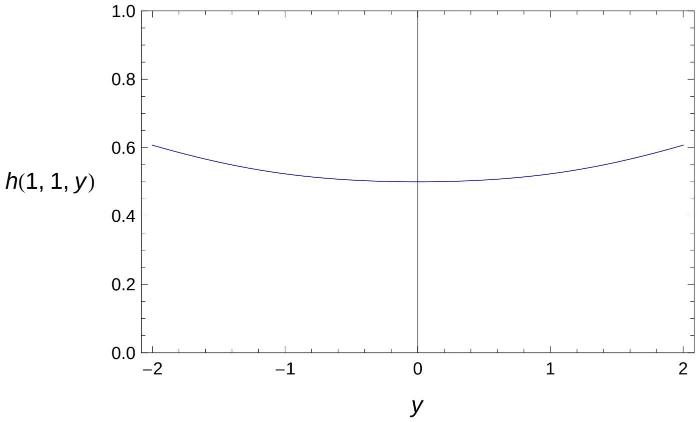

Figure 4.6.17: *Cross-section of the rivulet at $x = 1$, $t = 1$ for initial height $b = 0.5$ and leak-off parameter $\beta = 0.54$*

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From Table 4.6.1 the maximum value of b for positive values of β is $b = 0.577$. When b was increased to $b = 0.6$ we found that for all $\beta \geq 0$ the cross-section of the rivulet is infinite with strongest leak-off in the central region and the same occurred for $b = 0.8$ and $b = 1.0$.

We have found that when the initial height b is fixed the rivulet becomes wider as β is increased from zero. For a given value of $b < 0.577$ there is a maximum positive value of β for the cross-section of the rivulet to remain finite. A selection of the maximum values of β are listed in Table 4.6.2. The maximum value of β decreases to $\beta = 0$ as b increases to $b = 0.577$. The results are similar to those obtained in Section 4.5 for $W = \beta K$.

Table 4.6.2: The maximum value of the leak-off parameter β for a rivulet with finite cross-section for values of the initial height $b = K(0)$ when $W = \beta K^3$

b	maximum β
0.2	11
0.4	1.625
0.5	0.5
0.577	0

We next consider how b and β vary for a fixed value of the half-width A. In Figure 4.6.18, A is held fixed at $A = 0.7$. We see that as the height $b = h(1, 1, 0)$ is increased the leak-off parameter β decreases to maintain the half-width at $A = 0.7$. When b increases to $b = 0.501$ the leak-off parameter reduces to zero. The graphs in Figure 4.6.18 show that as the strength of the leak-off increases the height of the rivulet decreases as expected. We see again that the greater the leak-off velocity the thinner the rivulet.

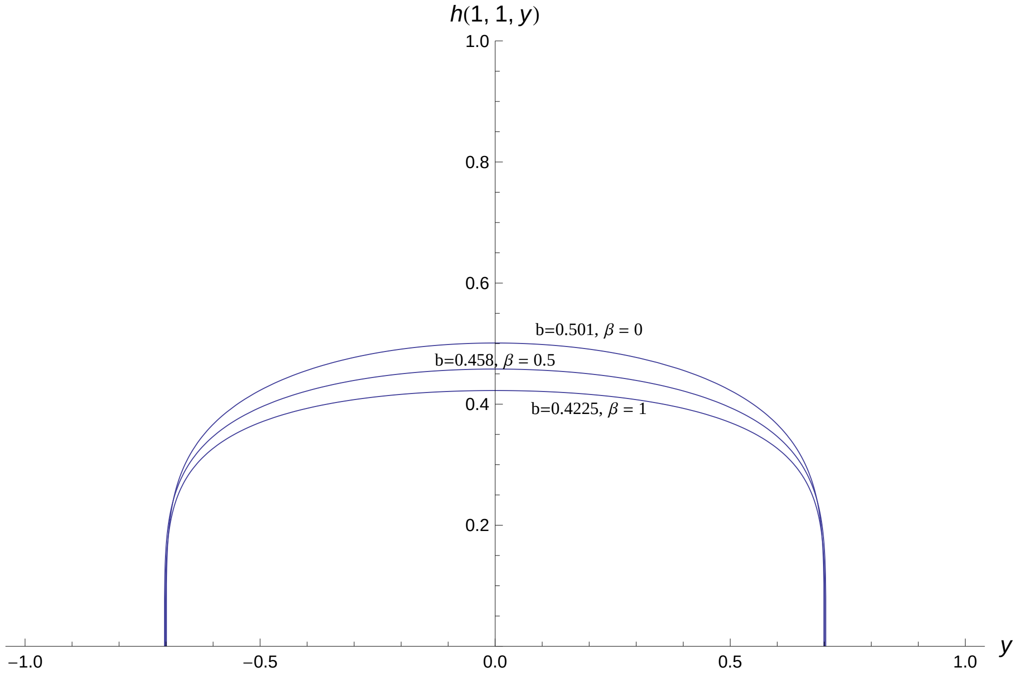

Figure 4.6.18: *Cross-section of the rivulet at* $x = 1$*,* $t = 1$ *with* $A = 0.7$ *and a range of height* b *and leak-off parameter* β

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Consider next the evolution of the rivulet with time. In Figures 4.6.19, 4.6.20 and 4.6.21 b and β are kept fixed and time is given a range of values. We take $b = 0.2$, 0.4 and 0.5 which are the same values as in Figures 4.6.9, 4.6.12 and 4.6.15 and also in Figures 4.5.20 to 4.5.22 which are the corresponding figures in Section 4.5. For each value of b the value chosen for β is the limiting value for the rivulet to have a finite cross-section. The boundary $y = \pm a(t, x)$, where $a(t, x)$ is given by (4.2.22), is plotted against x for $t = 0.1$, 1 and 10. The half width A is obtained from the boundary condition $K(A) = 0$. The three figures show that as time increases the width if the rivulet decreases and it becomes narrower.

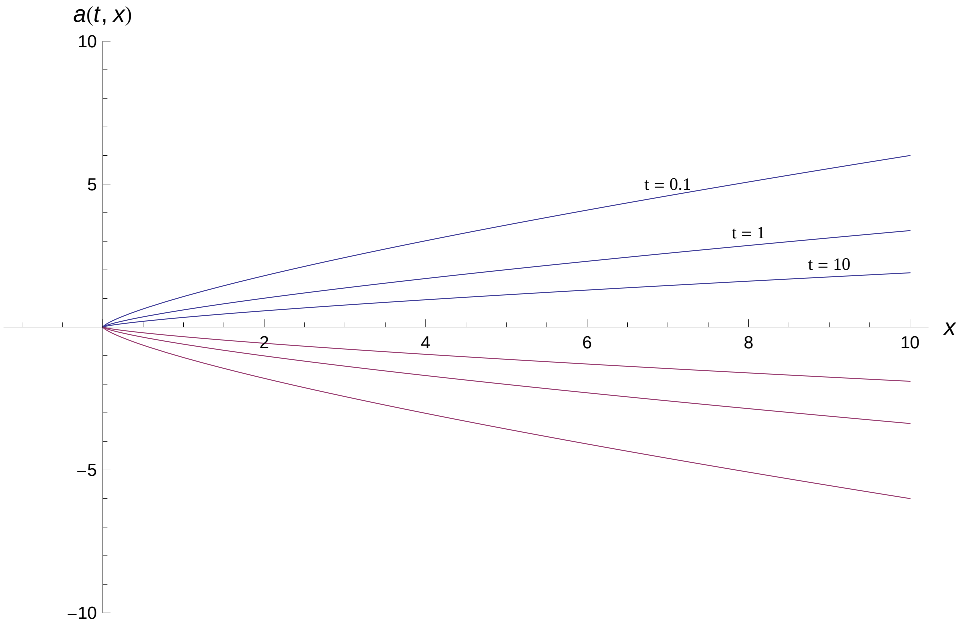

Figure 4.6.19: *Rivulet for* $b = 0.2$, $\beta = 10.998$ *and* $A = 0.6$ *at times* $t = 0.1$, 1 *and* 10

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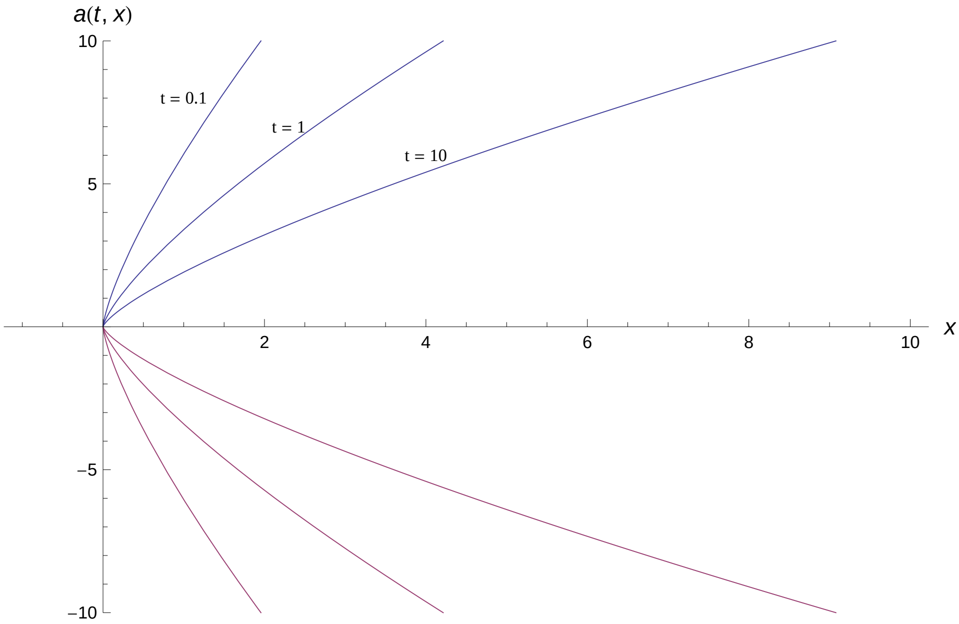

Figure 4.6.20: *Rivulet for* $b = 0.4$, $\beta = 1.6249$ *and* $A = 3.4$ *at times* $t = 0.1$, 1 *and* 10

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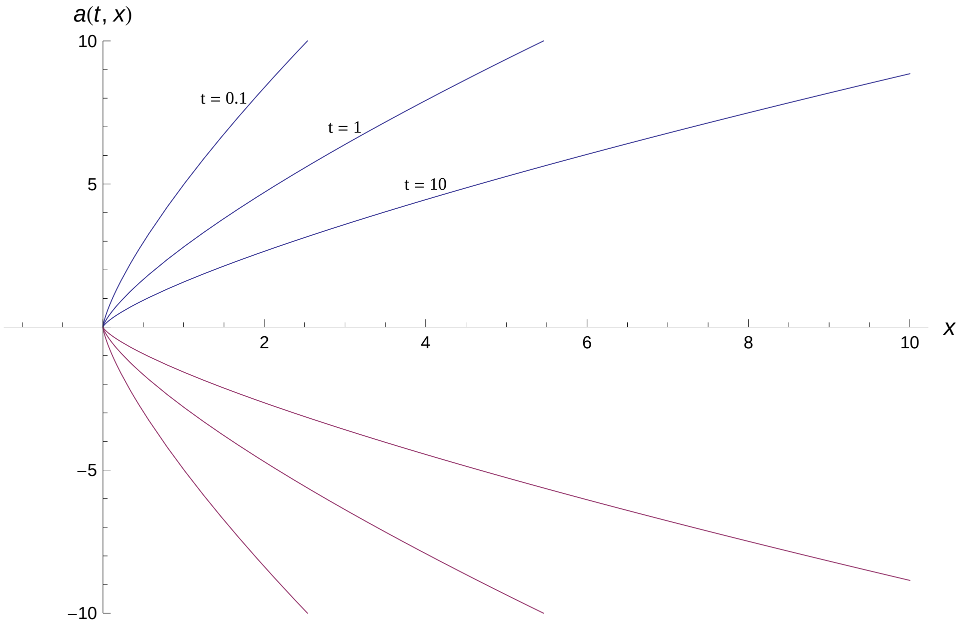

Figure 4.6.21: *Rivulet for* $b = 054$, $\beta = 0.49$ *and* $A = 2.8$ *at times* $t = 0.1$, 1 *and* 10

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Lastly, we investigate the dependence of the profile of the rivulet on the leak-off parameter β. In Figures 4.6.22 to 4.6.24, b and t are kept fixed and β is given a range of values. We again choose $b = 0.2$, 0.4 and 0.5 and take $t = 1$. In each figure the values of β range from zero to close to the maximum value for the cross-section to be finite. For each pair, b and β, the constant A is obtained from the boundary condition $K(A) = 0$. The boundary $y = \pm a(1, x)$ is given by (4.2.22) and is plotted against x for $t = 1$. We see again from the three figures that if b and t are kept fixed then the width of the rivulet increases as the leak-off parameter β increases, in agreement with Figures 4.6.9, 4.6.12 and 4.6.15.

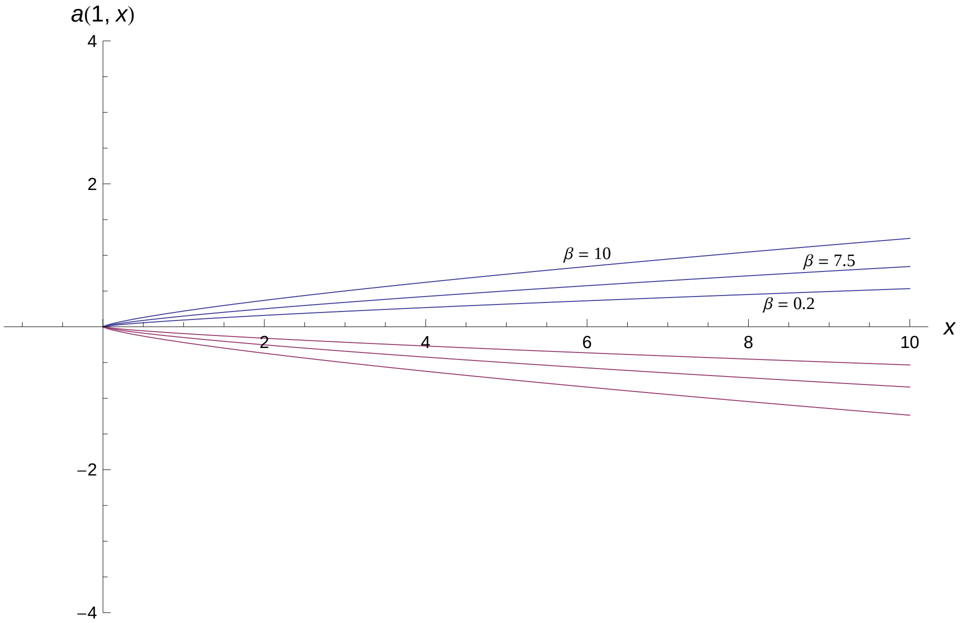

Figure 4.6.22: *Rivulet for* $b = 0.2$ *and* $\beta = 0.2,\ 7.5$ *and* 10

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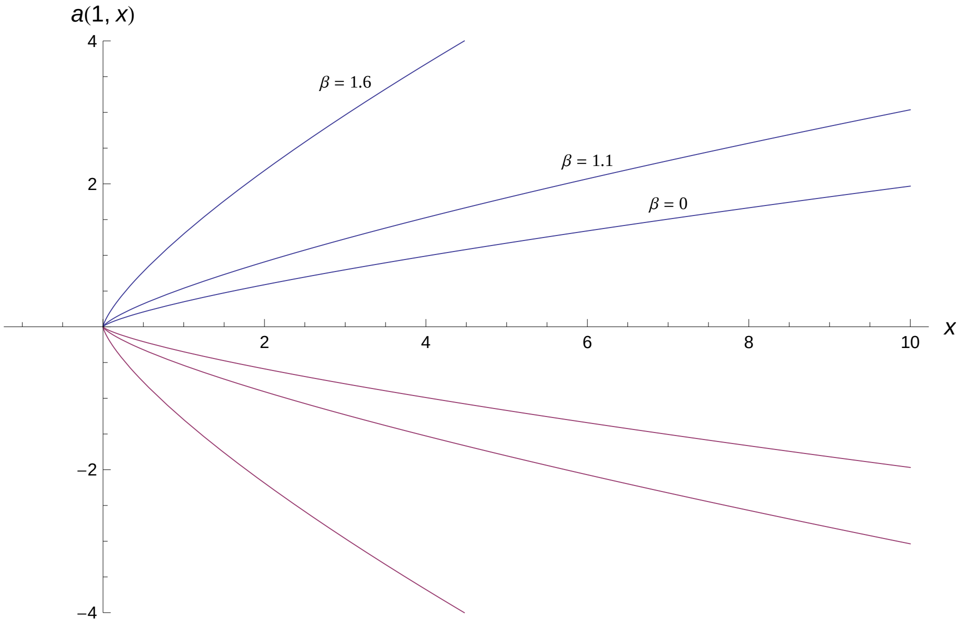

Figure 4.6.23: *Rivulet for* $b = 0.4,\ t = 1$ *and* $\beta = 0,\ 1.1$ *and* 1.6

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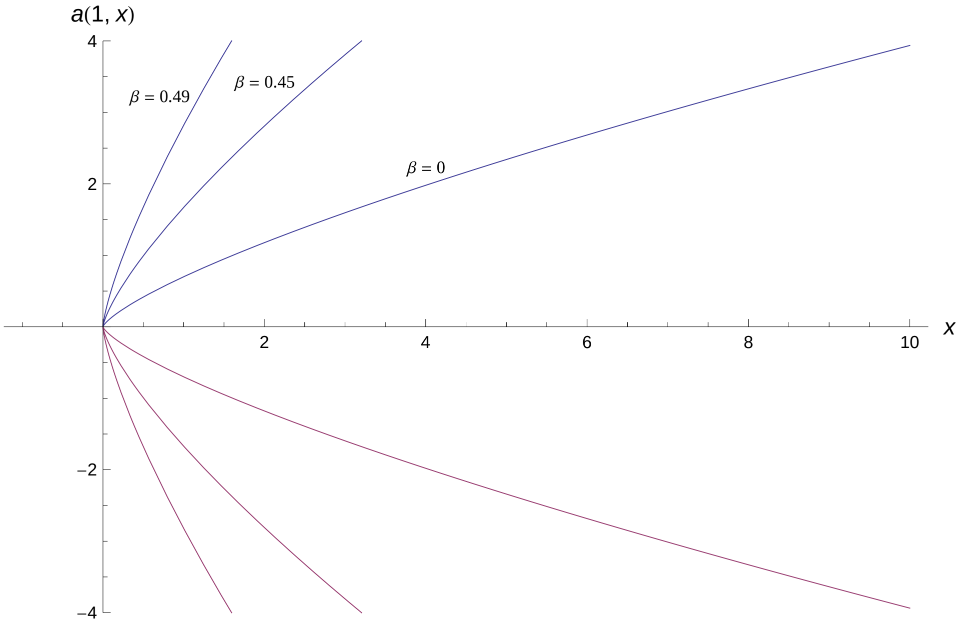

Figure 4.6.24: *Rivulet for* $b = 0.5,\ t = 1$ *and* $\beta = 0,\ 0.45$ *and* 0.49

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4.7 Comparison of the numerical solution of the two leak-off models

In the first leak-off model for the rivulet,

$$W(\xi) = \beta K(\xi), \quad w(t,x,y) = \beta \frac{h(t,x,y)}{t} \tag{4.7.1}$$

and in the second model

$$W(\xi) = \beta K^3(\xi), \quad w(t,x,y) = \beta \frac{h^3(t,x,y)}{x}. \tag{4.7.2}$$

In the first model the leak-off velocity is proportional to the height h and inversely proportional to the time t. It decreases like $\frac{1}{t}$ and there is also time dependence in $h(t,x,y)$. In the second model the leak-off velocity is proportional to the cube of the height and inversely proportional to x. It decreases like $\frac{1}{x}$ as x increases down the plane and there is also dependence on x in $h(t,x,y)$. In the first model the leak-off velocity decreases as time increases while in the second model it decreases as x increases down the inclined plane. The two models behave in the same way although there are quantitative differences.

The shooting method was a successful numerical method for both models. From Tables 4.5.1 and 4.6.1, there was a maximum value of the initial condition $b = h(1,1,0)$ for a given value of β in both models. The range of values of the leak-off parameter β for solution with finite cross section to exist was an order of magnitude greater in the second model. Initially A increased slowly as b increased from zero but it increased rapidly and tended to infinity as b approached its maximum value.

We found that for both models if the height $b = h(1,1,0)$ is kept fixed then as the leak-off parameter is increased the width of the rivulet increases. This is illustrated in Figures 4.5.10, 4.5.13 and 4.5.16 for the first model and in Figures 4.6.9, 4.6.12 and 4.6.15 for the second model. The values $b = 0.2$, 0.4 and 0.5 were used in both sets of figures. For each value of b there was a maximum value of β. The maximum values of β are listed in Tables 4.5.2 and 4.6.2. The range of values of b is the same in both models because the maximum value of the height b corresponds to no leak-off, $\beta = 0$. The range of values of β is an order of magnitude greater for the second model.

From (4.2.22) the constant A is the half-width of the rivulet at $x = 1$, $t = 1$. The way the b and β vary for a constant value of A, $A = 0.7$ for example, is illustrated in figures 4.5.19 and 4.6.18. The maximum value of b corresponds to no leak-off, $\beta = 0$, and is the same in both models. As the leak-off parameter increases from $\beta = 0$ the height decreases in both models as expected. In the first model, $\beta \to 0.487$ as $b \to 0$ while in the second model,

$\beta \to 11$ as $b \to 0$.

The evolution of the boundary profile of the rivulet with time is shown in Figures 4.5.20, 4.5.21 and 4.5.22 in the first model and in Figures 4.6.19, 4.6.20 and 4.6.21 for the second model. In both sets of figures, $b = 0.2$, 0.4 and 0.5 and the values chosen for β are close to the maximum values for a rivulet with a finite cross-section to exist. In both models the rivulet becomes narrower as the time increases. The thickness of the rivulet increases as b increases and we see that in both models the thicker rivulets are wider rivulets.

Finally, the dependence of the boundary profile on the leak-off parameter β was illustrated for the first model in Figures 4.5.23 to 4.5.25 and for the second model in Figures 4.6.22 to 4.6.24. The range of b, $0 \leq b \leq 0.577$ is the same in both models. The values for b where again $b = 0.2$, 0.4 and 0.5 and β ranged from zero to close to the maximum value for the rivulet to exist with finite cross-section. In both models, when b is fixed, the width of the rivulet increases when β was increased in agreement with previous observations.

4.8 Conclusion

There are few known analytical solutions for the rivulet flow down an inclined plane. The analytical solution which we obtained has leak-off velocity

$$w\left(t, x, y\right) = \frac{3h}{t} - \frac{9}{4}\frac{h^3}{x}, \tag{4.8.1}$$

which is a linear combination of the two leak-off velocities considered numerically. The analytic solution is interesting because it has a dry patch in the central region. The dry patch is due to leak-off into the porous substrate. For the solutions in the literature with a dry patch, the dry patch is caused by thermal and surface tension effects.

The shooting method was an effective numerical method for solving the boundary value problem (4.2.17) and (4.2.18) for the two models of leak-off velocity considered. We were able to obtain intervals for the values of the parameters and determine the essential properties of the two solutions.

Other models for the leak-off velocity could be considered as future work. In the models considered so far the leak-off velocity depended explicitly on t and x. A model in which it depends explicitly on y is

$$W(\xi) = \beta \frac{K^4(\xi)}{\xi^2} w(t, x, y) = \beta \frac{h^4}{y^2}. \tag{4.8.2}$$

The leak-off velocity (4.8.2) is an even function of y as required and becomes larger in the central region. It may represent a solution with a dry patch in the central region due to the strong drainage in the central region.

Chapter 5

CONSERVATION LAWS

5.1 Introduction

In this section we will investigate conservation laws for the partial differential equation (2.8.15) with zero leak-off velocity. We will use three different methods to find the conservation laws and compare the results obtained from each method. Finally we will associate the conserved vectors with a Lie point symmetry and discuss the physical significance of the conserved vectors.

A **conserved vector** for the m^{th} order partial differential equation for $u(x^1, x^2, ..., x^n)$,

$$E(x, u, u_{(1)}, u_{(2)}, ..., u_{(m)}) = 0, \tag{5.1.1}$$

where $x = (x^1, x^2, ..., x^n)$ and $u_{(k)}$ denotes the k^{th} order partial derivatives of u, is defined as an n-tuple $T = (T^1, T^2, ..., T^n)$ such that

$$D_i T^i = 0, \tag{5.1.2}$$

where D_i is the total derivative with respect to x^i, holds for all solutions of the partial differential equation (5.1.1). Equation (5.1.2) is called a local **conservation law**.

5.2 Elementary conserved vector

The partial differential equation (2.8.15) with leak-off velocity $w = 0$ can be written as

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h^3\right) + \frac{\partial}{\partial y}\left(-h^3\frac{\partial h}{\partial y}\right) = 0. \tag{5.2.1}$$

When expanded, (5.2.1) is

$$h_t + 3h^2 h_x - 3h^2 h_y^2 - h^3 h_{yy} = 0. \tag{5.2.2}$$

We regard $t, x, y, h, h_t, h_x, h_y$ and all higher partial derivatives of h with respect to t, x and y as independent. Then (5.2.2) becomes

$$D_1(h) + D_2(h^3) + D_3(-h^3 h_y) = 0, \tag{5.2.3}$$

where

$$D_1 = D_t = \frac{\partial}{\partial t} + h_t\frac{\partial}{\partial h} + h_{tt}\frac{\partial}{\partial h_t} + h_{xt}\frac{\partial}{\partial h_x} + h_{yt}\frac{\partial}{\partial h_y} + \ldots, \tag{5.2.4}$$

$$D_2 = D_x = \frac{\partial}{\partial x} + h_x\frac{\partial}{\partial h} + h_{tx}\frac{\partial}{\partial h_t} + h_{xx}\frac{\partial}{\partial h_x} + h_{yx}\frac{\partial}{\partial h_y} + \ldots, \tag{5.2.5}$$

$$D_3 = D_y = \frac{\partial}{\partial y} + h_y\frac{\partial}{\partial h} + h_{ty}\frac{\partial}{\partial h_t} + h_{xy}\frac{\partial}{\partial h_x} + h_{yy}\frac{\partial}{\partial h_y} + \ldots. \tag{5.2.6}$$

Equation (5.2.3) has the same form as (5.2.1). From (5.2.3),

$$T^1 = h, \quad T^2 = h^3, \quad T^3 = -h^3 h_y \tag{5.2.7}$$

is a conserved vector for the partial differential equation (5.2.1). It is referred to as the **elementary conserved vector**.

We investigate if there are other conserved vectors for the partial differential equation (5.2.1). In order to look for possible conserved vectors for (5.2.1) we will use three different methods, namely the direct method, the multiplier method and the partial Lagrangian method.

5.3 Direct method

Consider the partial differential equation (5.2.1). We now look for a conserved vector with components of the form

$$T^1 = T^1\left(t, x, y, h, h_t\right), \quad T^2 = T^2\left(t, x, y, h, h_x\right), \quad T^3 = T^3\left(t, x, y, h, h_y\right). \tag{5.3.1}$$

Conserved vectors of this form are more general than the elementary conserved vector (5.2.7). For (5.3.1) to be a conserved vector the components must satisfy

$$D_1 T^1 + D_2 T^2 + D_3 T^3|_{PDE} = 0, \tag{5.3.2}$$

where the total derivatives D_1, D_2 and D_3 are defined by (5.2.4) to (5.2.6). Substituting (5.3.1) into (5.3.2) gives

$$\frac{\partial T^1}{\partial t} + h_t \frac{\partial T^1}{\partial h} + h_{tt} \frac{\partial T^1}{\partial h_t} + \frac{\partial T^2}{\partial x} + h_x \frac{\partial T^2}{\partial h} + h_{xx} \frac{\partial T^2}{\partial h_x} + \frac{\partial T^3}{\partial y} + h_y \frac{\partial T^3}{\partial h} + h_{yy} \frac{\partial T^3}{\partial h_y}|_{PDE} = 0. \tag{5.3.3}$$

Now from equation (5.2.2) we have

$$h_t = 3h^2 h_y^2 + h^3 h_{yy} - 3h^2 h_x. \tag{5.3.4}$$

Equation (5.3.4) is substituted into equation (5.3.3) to obtain

$$\begin{aligned} &\frac{\partial T^1}{\partial t} + \frac{\partial T^1}{\partial h}\left[3h^2 h_y^2 + h^3 h_{yy} - 3h^2 h_x\right] + h_{tt} \frac{\partial T^1}{\partial h_t} + \frac{\partial T^2}{\partial x} + h_x \frac{\partial T^2}{\partial h} + h_{xx} \frac{\partial T^2}{\partial h_x} + \frac{\partial T^3}{\partial y} \\ &\quad + h_y \frac{\partial T^3}{\partial h} + h_{yy} \frac{\partial T^3}{\partial h_y} = 0. \end{aligned} \tag{5.3.5}$$

Since T^1 depended on h_t it now depends on h_{yy}. We cannot therefore separate (5.3.5) by h_{yy}. However, T_1, T_2 and T_3 are independent of h_{tt} and h_{xx} and we can therefore separate (5.3.5) by the second derivatives h_{tt} and h_{xx}

$$h_{tt}: \qquad \frac{\partial T^1}{\partial h_t} = 0, \tag{5.3.6}$$

$$h_{xx}: \qquad \frac{\partial T^2}{\partial h_x} = 0, \tag{5.3.7}$$

$$R: \frac{\partial T^1}{\partial t} + 3h^2 h_y^2 \frac{\partial T^1}{\partial h} - 3h^2 h_x \frac{\partial T^1}{\partial h} + h^3 h_{yy} \frac{\partial T^1}{\partial h} + \frac{\partial T^2}{\partial x} + h_x \frac{\partial T^2}{\partial h} + \frac{\partial T^3}{\partial y} + h_y \frac{\partial T^3}{\partial h} + h_{yy} \frac{\partial T^3}{\partial h_y} = 0. \tag{5.3.8}$$

From (5.3.6) we obtain

$$T^1 = T^1 (t, x, y, h) \tag{5.3.9}$$

and from (5.3.7)

$$T^2 = T^2 (t, x, y, h) . \tag{5.3.10}$$

Hence T^1 does not depend on h_{yy} and we can now separate (5.3.8) by the second derivative h_{yy}.

$$h_{yy}: \qquad h^3 \frac{\partial T^1}{\partial h} + \frac{\partial T^3}{\partial h_y} = 0. \tag{5.3.11}$$

Integrating (5.3.11) with respect to h_y we obtain

$$T^3(t, x, y, h, h_y) = -h^3 h_y \frac{\partial T^1}{\partial h} + A(t, x, y, h). \tag{5.3.12}$$

We now substitute (5.3.12) into (5.3.8) to obtain

$$\begin{aligned} &\frac{\partial T^1}{\partial t} + 3h^2 h_y^2 \frac{\partial T^1}{\partial h} - 3h^2 h_x \frac{\partial T^1}{\partial h} + \frac{\partial T^2}{\partial x} + h_x \frac{\partial T^2}{\partial h} - h^3 h_y \frac{\partial^2 T^1}{\partial y \partial h} + \frac{\partial A}{\partial y} - h^3 h_y^2 \frac{\partial^2 T^1}{\partial h^2} \\ &- 3h^2 h_y^2 \frac{\partial T^1}{\partial h} + h_y \frac{\partial A}{\partial h} = 0. \end{aligned} \tag{5.3.13}$$

Since T^1, T^2 and A are independent of the first derivatives of h we can therefore separate (5.3.13)by first derivatives.

$$h_y^2: \qquad \frac{\partial^2 T^1}{\partial h^2} = 0, \tag{5.3.14}$$

$$h_y: \qquad -h^3 \frac{\partial^2 T^1}{\partial y \partial h} + \frac{\partial A}{\partial h} = 0, \tag{5.3.15}$$

$$h_x: \qquad -3h^2 \frac{\partial T^1}{\partial h} + \frac{\partial T^2}{\partial h} = 0, \tag{5.3.16}$$

$$R: \qquad \frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} + \frac{\partial A}{\partial y} = 0. \tag{5.3.17}$$

We integrate (5.3.14) twice with respect to h to obtain

$$T^1 = B(t, x, y)h + C(t, x, y) \tag{5.3.18}$$

and substituting (5.3.18) into (5.3.16) and integrating with respect to h we obtain

$$T^2 = B(t,x,y)h^3 + D(t,x,y). \tag{5.3.19}$$

We now substitute (5.3.18) into (5.3.15) which gives

$$\frac{\partial A}{\partial h} = h^3 \frac{\partial B}{\partial y} \tag{5.3.20}$$

and since $B(t,x,y)$ is independent of h we can integrate (5.3.20) with respect to h to obtain

$$A(t,x,y,h) = \frac{1}{4}h^4 \frac{\partial B}{\partial y} + E(t,x,y). \tag{5.3.21}$$

We now substitute (5.3.21) and (5.3.18) into (5.3.12) which gives

$$T^3(t,x,y,h,h_y) = -h^3 h_y B + \frac{1}{4}h^4 \frac{\partial B}{\partial y} + E. \tag{5.3.22}$$

Finally substituting (5.3.18), (5.3.19) and (5.3.21) into (5.3.17) we obtain

$$h\frac{\partial B}{\partial t} + \frac{\partial C}{\partial t} + h^3 \frac{\partial B}{\partial x} + \frac{\partial D}{\partial x} + \frac{1}{4}h^4 \frac{\partial^2 B}{\partial y^2} + \frac{\partial E}{\partial y} = 0. \tag{5.3.23}$$

Now, since B, C, D and E are all independent of h we can separate (5.3.23) by powers of h.

$$h^4 : \qquad \frac{\partial^2 B}{\partial y^2} = 0, \tag{5.3.24}$$

$$h^3 : \qquad \frac{\partial B}{\partial x} = 0, \tag{5.3.25}$$

$$h : \qquad \frac{\partial B}{\partial t} = 0, \tag{5.3.26}$$

$$R : \qquad \frac{\partial C}{\partial t} + \frac{\partial D}{\partial x} + \frac{\partial E}{\partial y} = 0. \tag{5.3.27}$$

From equations (5.3.25) and (5.3.26) we can deduce that $B = B(y)$. We can therefore integrate (5.3.24) with respect to y to obtain

$$B(y) = c_2 y + c_1. \tag{5.3.28}$$

Substituting (5.3.28) into equations (5.3.18), (5.3.19) and (5.3.22) it follows that

$$T^1 = c_2 yh + c_1 h + C(t, x, y), \tag{5.3.29}$$

$$T^2 = c_2 yh^3 + c_1 h^3 + D(t, x, y), \tag{5.3.30}$$

$$T^3 = -c_2 yh^3 h_y - c_1 h^3 h_y + \frac{1}{4} c_2 h^4 + E(t, x, y). \tag{5.3.31}$$

In equations (5.3.29) to (5.3.31) when c_1 and c_2 are both set to zero, we obtain

$$T^1 = C(t, x, y), \;\; T^2 = D(t, x, y), \;\; T^3 = E(t, x, y), \tag{5.3.32}$$

where C, D and E must satisfy (5.3.27).

Since (5.3.27) is satisfied without imposing the partial differential (5.2.1), the components (5.3.32) form a trivial conserved vector. Hence C, D and E are set to zero.

Setting $c_1 = 1$ and $c_2 = 0$, we obtain the elementary conserved vector

$$T^1 = h, \;\; T^2 = h^3, \;\; T^3 = -h^3 h_y. \tag{5.3.33}$$

Setting $c_1 = 0$ and $c_2 = 1$ we obtain a second conserved vector

$$T^1 = yh, \;\; T^2 = yh^3, \;\; T^3 = -yh^3 h_y + \tfrac{1}{4} h^4. \tag{5.3.34}$$

Equations (5.3.33) and (5.3.34) are the components of the first two conserved vectors for the partial differential equation (5.2.1) obtained by the direct method. We will now investigate the conserved vectors that are derived using the multiplier method and the partial Lagrangian method.

5.4 Multiplier method

Consider again the partial differential equation

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial y}\left(h^3 \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial x}(h^3) = 0 \tag{5.4.1}$$

which can be expanded as

$$h_t - 3h^2 h_y^2 - h^3 h_{yy} + 3h^2 h_x = 0. \tag{5.4.2}$$

A multiplier Λ for the equation (5.4.1) has the property that [15]

$$\Lambda \left(h_t - 3h^2 h_y^2 - h^3 h_{yy} + 3h^2 h_x\right) = D_t T^1 + D_x T^2 + D_y T^3 \tag{5.4.3}$$

for all functions $h(t, x, y)$ where the total derivatives are defined by (5.2.4) to (5.2.6). The right hand side of equation (5.4.3) is a divergence expression and T^1, T^2 and T^3 are the components of the conserved vector $T = (T^1, T^2, T^3)$ [15].

We will consider a multiplier of the form

$$\Lambda = (t, x, y, h, h_t, h_x, h_y). \tag{5.4.4}$$

We operate on (5.4.3) with the Euler operator defined by

$$\begin{aligned} E_h =& \frac{\partial}{\partial h} - D_t \frac{\partial}{\partial h_t} - D_x \frac{\partial}{\partial h_x} - D_y \frac{\partial}{\partial h_y} + D_t^2 \frac{\partial^2}{\partial h_{tt}} + D_t D_x \frac{\partial^2}{\partial h_{tx}} \\ &+ D_t D_y \frac{\partial^2}{\partial h_{ty}} + D_x^2 \frac{\partial^2}{\partial h_{xx}} + D_x D_y \frac{\partial^2}{\partial h_{xy}} + D_y^2 \frac{\partial^2}{\partial h_{yy}} - \ldots \end{aligned} \tag{5.4.5}$$

Equation (5.4.3) becomes

$$E_h \left[\Lambda \left(h_t - 3h^2 h_y^2 - h^3 h_{yy} + 3h^2 h_x\right)\right] = E_h \left[D_t T^1 + D_x T^2 + D_y T^3\right]. \tag{5.4.6}$$

Now the Euler operator has the property that it annihilates divergence expressions. Hence

$$E_h \left[D_t T^1 + D_x T^2 + D_y T^3\right] = 0 \tag{5.4.7}$$

and (5.4.6) reduces to

$$E_h\left[\Lambda\left(h_t - 3h^2h_y^2 - h^3h_{yy} + 3h^2h_x\right)\right] = 0. \tag{5.4.8}$$

Equation (5.4.8) is the determining equation for the multiplier Λ. The expansion of (5.4.8) gives rise to the following equation,

$$\begin{aligned}
&\frac{\partial\Lambda}{\partial h}\left(h_t - 3h^2h_y^2 - h^3h_{yy} + 3h^2h_x\right) + \Lambda\left[-6hh_y^2 - 3h^2h_{yy} + 6hh_x\right]\\
&- D_t\left[\frac{\partial\Lambda}{\partial h_t}\left(h_t - 3h^2h_y^2 - h^3h_{yy} + 3h^2h_x\right) + \Lambda\right] - D_x\left[\frac{\partial\Lambda}{\partial h_x}\left(h_t - 3h^2h_y^2\right.\right.\\
&\left.\left.-h^3h_{yy} + 3h^2h_x\right) + 3h^2\Lambda\right] - D_y\left[\frac{\partial\Lambda}{\partial h_y}\left(h_t - 3h^2h_y^2 - h^3h_{yy} + 3h^2h_x\right) - 6h^2h_y\Lambda\right]\\
&- D_y^2\left[h^3\Lambda\right] = 0.
\end{aligned} \tag{5.4.9}$$

Expanding the total derivatives in (5.4.9) and simplifying gives

$$
\begin{aligned}
&\left(h_t - 3h^2h_y^2 - h^3h_{yy} + 3h^2h_x\right)\frac{\partial\Lambda}{\partial h} + \Lambda\left[-6hh_y^2 - 3h^2h_{yy} + 6hh_x\right] - \frac{\partial\Lambda}{\partial t} + (-h_t\\
&+3h^2h_y^2 + h^3h_{yy} - 3h^2h_x\right)\frac{\partial^2\Lambda}{\partial t\partial h_t} + \left(-h_t + 3h^2h_y^2 + h^3h_{yy} - 3h^2h_x\right)h_t\frac{\partial^2\Lambda}{\partial h\partial h_t}\\
&+\left(6hh_y^2 + 3h^2h_{yy} - 6hh_x\right)h_t\frac{\partial\Lambda}{\partial h_t} - h_t\frac{\partial\Lambda}{\partial h} + \left(-h_t + 3h^2h_y^2 + h^3h_{yy} - 3h^2h_x\right)h_{tt}\frac{\partial^2\Lambda}{\partial h_t^2}\\
&-2h_{tt}\frac{\partial\Lambda}{\partial h_t} + \left(-h_t + 3h^2h_y^2 + h^3h_{yy} - 3h^2h_x\right)h_{tx}\frac{\partial^2\Lambda}{\partial h_x\partial h_t} - 3h^2h_{tx}\frac{\partial\Lambda}{\partial h_t} - h_{tx}\frac{\partial\Lambda}{\partial h_x}\\
&+\left(-h_t + 3h^2h_y^2 + h^3h_{yy} - 3h^2h_x\right)h_{ty}\frac{\partial^2\Lambda}{\partial h_y\partial h_t} + 6h^2h_yh_{ty}\frac{\partial\Lambda}{\partial h_t} - h_{ty}\frac{\partial\Lambda}{\partial h_y} + h^3h_{yyt}\frac{\partial\Lambda}{\partial h_t}\\
&-3h^2\frac{\partial\Lambda}{\partial x} + \left(-h_t + 3h^2h_y^2 + h^3h_{yy} - 3h^2h_x\right)\frac{\partial^2\Lambda}{\partial x\partial h_x} + \left(-h_t + 3h^2h_y^2 + h^3h_{yy}\right.\\
&\left.-3h^2h_x\right)h_x\frac{\partial^2\Lambda}{\partial h\partial h_x} - 3h^2h_x\frac{\partial\Lambda}{\partial h} + \left(6hh_y^2 + 3h^2h_{yy} - 6hh_x\right)h_x\frac{\partial\Lambda}{\partial h_x} - 6hh_x\Lambda - 3h^2h_{xt}\frac{\partial\Lambda}{\partial h_t}\\
&+\left(-h_t + 3h^2h_y^2 + h^3h_{yy} - 3h^2h_x\right)h_{xt}\frac{\partial^2\Lambda}{\partial h_t\partial h_x} - h_{xt}\frac{\partial\Lambda}{\partial h_x} - 3h^2h_{xx}\frac{\partial\Lambda}{\partial h_x} + \left(-h_t + 3h^2h_y^2\right.\\
&\left.+h^3h_{yy} - 3h^2h_x\right)h_{xx}\frac{\partial^2\Lambda}{\partial h_x^2} - 3h^2h_{xx}\frac{\partial\Lambda}{\partial h_x} - 3h^2h_{xy}\frac{\partial\Lambda}{\partial h_y} + \left(-h_t + 3h^2h_y^2 + h^3h_{yy}\right.\\
&\left.-3h^2h_x\right)h_{xy}\frac{\partial^2\Lambda}{\partial h_y\partial h_x} + 6h^2h_yh_{xy}\frac{\partial\Lambda}{\partial h_x} + h^3h_{yyx}\frac{\partial\Lambda}{\partial h_x} + 6h^2h_y\frac{\partial\Lambda}{\partial y} + \left(-h_t + 3h^2h_y^2\right.\\
&\left.+h^3h_{yy} - 3h^2h_x\right)\frac{\partial^2\Lambda}{\partial y\partial h_y} + 12hh_y^2\Lambda + 6h^2h_y^2\frac{\partial\Lambda}{\partial h} + \left(-h_t + 3h^2h_y^2 + h^3h_{yy} - 3h^2h_x\right)h_y\frac{\partial^2\Lambda}{\partial h\partial h_y}\\
&+\left(6hh_y^2 + 3h^2h_{yy} - 6hh_x\right)h_y\frac{\partial\Lambda}{\partial h_y} + 6h^2h_yh_{yt}\frac{\partial\Lambda}{\partial h_t} + \left(-h_t + 3h^2h_y^2 + h^3h_{yy}\right.\\
&\left.-3h^2h_x\right)h_{yt}\frac{\partial^2\Lambda}{\partial h_t\partial h_y} - h_{yt}\frac{\partial\Lambda}{\partial h_y} + 6h^2h_yh_{yx}\frac{\partial\Lambda}{\partial h_x} + \left(-h_t + 3h^2h_y^2 + h^3h_{yy} - 3h^2h_x\right)h_{yx}\frac{\partial^2\Lambda}{\partial h_x\partial h_y}\\
&-3h^2h_{yx}\frac{\partial\Lambda}{\partial h_y} + 6h^2h_{yy} + 6h^2h_yh_{yy}\frac{\partial\Lambda}{\partial h_y} + \left(-h_t + 3h^2h_y^2 + h^3h_{yy} - 3h^2h_x\right)h_{yy}\frac{\partial^2\Lambda}{\partial h_y^2}\\
&+6h^2h_yh_{yy}\frac{\partial\Lambda}{\partial h_y} + h^3h_{yyy}\frac{\partial\Lambda}{\partial h_y} - h^3\frac{\partial^2\Lambda}{\partial y^2} - 3h^2h_y\frac{\partial\Lambda}{\partial y} - h^3h_y\frac{\partial^2\Lambda}{\partial y\partial h} - h^3h_{yt}\frac{\partial^2\Lambda}{\partial y\partial h_t}\\
&-h^3h_{yx}\frac{\partial^2\Lambda}{\partial y\partial h_x} - h^3h_{yy}\frac{\partial^2\Lambda}{\partial y\partial h_y} - 3h^2h_y\frac{\partial\Lambda}{\partial y} - h^3h_y\frac{\partial^2\Lambda}{\partial y\partial h} - 6hh_y^2\Lambda - 3h^2h_y^2\frac{\partial\Lambda}{\partial h} - 3h^2h_y^2\frac{\partial\Lambda}{\partial h}
\end{aligned}
$$

$$
\begin{aligned}
&- h^3 h_y^2 \frac{\partial^2 \Lambda}{\partial h^2} - 3h^2 h_y h_{yt} \frac{\partial \Lambda}{\partial h_t} - h^3 h_y h_{yt} \frac{\partial^2 \Lambda}{\partial h \partial h_t} - 3h^2 h_y h_{yx} \frac{\partial \Lambda}{\partial h_x} - h^3 h_y h_{yx} \frac{\partial^2 \Lambda}{\partial h \partial h_x} \\
&- 3h^2 h_y h_{yy} \frac{\partial \Lambda}{\partial h_y} - h^3 h_y h_{yy} \frac{\partial^2 \Lambda}{\partial h \partial h_y} - h^3 h_{yt} \frac{\partial^2 \Lambda}{\partial h_t \partial y} - 3h^2 h_y h_{yt} \frac{\partial \Lambda}{\partial h_t} - h^3 h_y h_{yt} \frac{\partial^2 \Lambda}{\partial h_t \partial h} \\
&- h^3 h_{yt}^2 \frac{\partial^2 \Lambda}{\partial h_t^2} - h^3 h_{yx} h_{yt} \frac{\partial^2 \Lambda}{\partial h_t \partial h_x} - h^3 h_{yt} h_{yy} \frac{\partial^2 \Lambda}{\partial h_t \partial h_y} - h^3 h_{yx} \frac{\partial^2 \Lambda}{\partial h_x \partial y} - 3h^2 h_y h_{yx} \frac{\partial \Lambda}{\partial h_x} \\
&- h^3 h_y h_{yx} \frac{\partial^2 \Lambda}{\partial h_x \partial h} - h^3 h_{yt} h_{yx} \frac{\partial^2 \Lambda}{\partial h_x \partial h_t} - h^3 h_{yx}^2 \frac{\partial^2 \Lambda}{\partial h_x^2} - h^3 h_{yy} h_{yx} \frac{\partial^2 \Lambda}{\partial h_x \partial h_y} - h^3 h_{yy} \frac{\partial^2 \Lambda}{\partial h_y \partial y} \\
&- 3h^2 h_{yy} \Lambda - 3h^2 h_y h_{yy} \frac{\partial \Lambda}{\partial h_y} - h^3 h_{yy} \frac{\partial \Lambda}{\partial h} - h^3 h_y h_{yy} \frac{\partial^2 \Lambda}{\partial h_y \partial h} - h^3 h_{yt} h_{yy} \frac{\partial^2 \Lambda}{\partial h_y \partial h_t} - h^3 h_{yx} h_{yy} \frac{\partial^2 \Lambda}{\partial h_y \partial h_x} \\
&- h^3 h_{yy}^2 \frac{\partial^2 \Lambda}{\partial h_y^2} - h^3 h_{yyt} \frac{\partial \Lambda}{\partial h_t} - h^3 h_{yyx} \frac{\partial \Lambda}{\partial h_x} - h^3 h_{yyy} \frac{\partial \Lambda}{\partial h_y} = 0. \qquad (5.4.10)
\end{aligned}
$$

Now t, x, y, h, and the partial derivatives of h are regarded as independent. Also the multiplier Λ does not depend on the second and third partial derivatives of h. We can therefore separate (5.4.10) by the second and third partial derivatives of h. Consider first the third derivatives of h.

$$h_{yyt}: \qquad -h^3 \frac{\partial \Lambda}{\partial h_t} + h^3 \frac{\partial \Lambda}{\partial h_t} = 0, \qquad (5.4.11)$$

$$h_{yyx}: \qquad -h^3 \frac{\partial \Lambda}{\partial h_x} + h^3 \frac{\partial \Lambda}{\partial h_x} = 0, \qquad (5.4.12)$$

$$h_{yyy}: \qquad -h^3 \frac{\partial \Lambda}{\partial h_y} + h^3 \frac{\partial \Lambda}{\partial h_y} = 0. \qquad (5.4.13)$$

Equations (5.4.11), (5.4.12) and (5.4.13) are all identically satisfied and therefore give no information on the form of Λ.

We next separate equation (5.4.10) by the second derivatives of h.

$$h_{tt}: \quad \frac{\partial \Lambda}{\partial h_t} = 0, \tag{5.4.14}$$

$$h_{xx}: \quad \frac{\partial \Lambda}{\partial h_x} = 0, \tag{5.4.15}$$

$$h_{yt}: \quad \frac{\partial \Lambda}{\partial h_y} + h^3 \frac{\partial^2 \Lambda}{\partial y \partial h_t} = 0, \tag{5.4.16}$$

$$h_{yy}: \quad 2\frac{\partial \Lambda}{\partial h} - \frac{\partial^2 \Lambda}{\partial t \partial h_t} - \frac{\partial^2 \Lambda}{\partial x \partial h_x} + \frac{\partial^2 \Lambda}{\partial y \partial h_y} = 0, \tag{5.4.17}$$

$$h_{tx}: \quad \frac{\partial \Lambda}{\partial h_t} = 0, \tag{5.4.18}$$

$$h_{xy}: \quad 3\frac{\partial \Lambda}{\partial h_y} + h \frac{\partial^2 \Lambda}{\partial y \partial h_x} = 0. \tag{5.4.19}$$

From (5.4.14) and (5.4.15), Λ is independent of h_t and h_x. Thus (5.4.16) reduces to

$$\frac{\partial \Lambda}{\partial h_y} = 0 \tag{5.4.20}$$

and therefore Λ is also independent of h_y. Equation (5.4.17) reduces to

$$\frac{\partial \Lambda}{\partial h} = 0. \tag{5.4.21}$$

Equations (5.4.18) and (5.4.19) are identically satisfied. Thus

$$\Lambda = \Lambda(t, x, y). \tag{5.4.22}$$

The determining equation (5.4.10) reduces to

$$\frac{\partial \Lambda}{\partial t} + 3h^2 \frac{\partial \Lambda}{\partial x} + h^3 \frac{\partial^2 \Lambda}{\partial y^2} = 0. \tag{5.4.23}$$

Since Λ is independent of h, equation (5.4.23) can be separated by powers of h:

$$h^0: \quad \frac{\partial \Lambda}{\partial t} = 0, \tag{5.4.24}$$

$$h^2: \quad \frac{\partial \Lambda}{\partial x} = 0, \tag{5.4.25}$$

$$h^3: \quad \frac{\partial^2 \Lambda}{\partial y^2} = 0. \tag{5.4.26}$$

From (5.4.24) and (5.4.25) it follows that $\Lambda = \Lambda(y)$. Integrating (5.4.26) twice with respect to y we obtain

$$\Lambda(y) = c_1 + c_2 y, \tag{5.4.27}$$

where c_1 and c_2 are constants.

From (5.4.3) and (5.4.27) we obtain by elementary manipulations

$$\begin{aligned}(c_1 + c_2 y)\left[h_t + 3h^2 h_x - 3h^2 h_y^2 - h^3 h_{yy}\right] &= D_t\left[c_1 h + c_2 y h\right] + D_x\left[c_1 h^3 + c_2 y h^3\right] \\ &\quad + D_y\left[c_1\left(-h^3 h_y\right) + c_2\left(-y h^3 h_y + \frac{1}{4}h^4\right)\right],\end{aligned} \tag{5.4.28}$$

for arbitrary functions $h(t, x, y)$. When $h(t, x, y)$ is a solution of the partial differential equation (5.4.2), then the left-hand-side of (5.4.28) is zero and

$$D_t\left[c_1 h + c_2 y h\right] + D_x\left[c_1 h^3 + c_2 y h^3\right] + D_y\left[c_1\left(-h^3 h_y\right) + c_2\left(-y h^3 h_y + \frac{1}{4}h^4\right)\right] = 0. \tag{5.4.29}$$

Any conserved vector for the partial differential equation (5.4.1) with multiplier $\Lambda(t, x, y, h, h_t, h_x, h_y)$ is therefore a linear combination of the two conserved vectors

$$T^1 = h, \quad T^2 = h^3, \quad T^3 = -h^3 h_y, \tag{5.4.30}$$

$$T^1 = yh, \quad T^2 = yh^3, \quad T^3 = -y h^3 h_y + \frac{1}{4}h^4. \tag{5.4.31}$$

The conserved vectors (5.4.30) and (5.4.31) are the same as the conserved vectors obtained using the direct method. In order to investigate other conserved vectors we would need to consider multipliers that depend on second and higher derivatives of h. When we considered a multiplier which depended only on first order derivatives the determining equation, (5.4.10), contained many terms. When considering multipliers with higher derivatives computer algebra could be used to simplify the calculations.

5.5 Partial Lagrangian

Once again we start from the partial differential equation (5.2.1) in the expanded form (5.2.2):

$$E = h_t - 3h^2h_y^2 - h^3h_{yy} + 3h^2h_x = 0. \tag{5.5.1}$$

In order to introduce the idea of a partial Lagrangian consider the k^{th}-order partial differential equation

$$E\left(x, h, h_{(1)}, h_{(2)}, ..., h_{(k)}\right) = 0 \tag{5.5.2}$$

where $x = (x^1, x^2, ..., x^n)$ denotes the n independent variables and $h_{(i)}$ denotes the collection of all i-order partial derivatives of h. Then if there exists a function $L\left(x, h, h_{(1)}, h_{(2)}, ..., h_{(l)}\right)$ where $l \leq k$ such that

$$E_h\left(L\right)|_{E=0} = 0 \tag{5.5.3}$$

where E_h is the Euler operator, then L is called a Lagrangian of equation (5.5.2). Not all partial differential equations have a Lagrangian. Suppose that equation (5.5.2) can be written as

$$E = E^0 + E^1 = 0. \tag{5.5.4}$$

Then a partial Lagrangian of equation (5.5.3) is defined as follows [10].

Definition. If there exists a function $L = L(x, h, h_{(1)}, \ldots, h_{(l)})$, $l \leq k$ and a non-zero function f such that (5.5.4) can be written as

$$E_hL|_{E^0+E^1=0} = fE^1 \tag{5.5.5}$$

then, provided $E^1 \neq 0$, L is called a *partial Lagrangian* of equation (5.5.4). Otherwise it is the standard Lagrangian.

As a partial Lagrangian of the partial differential equation equation (5.5.1) consider

$$L = \frac{1}{2}h^3 h_y^2 + h^3 h_x. \tag{5.5.6}$$

The Euler operator is defined by equation (5.4.5):

$$E_h = \frac{\partial}{\partial h} - D_t \frac{\partial}{\partial h_t} - D_x \frac{\partial}{\partial h_x} - D_y \frac{\partial}{\partial h_y} + \dots \tag{5.5.7}$$

Therefore

$$E_h \left[h^3 h_x + \frac{1}{2} h^3 h_y^2 \right] = -\frac{3}{2} h^2 h_y^2 - h^3 h_{yy}. \tag{5.5.8}$$

Now from the partial differential equation (5.5.1)

$$h^3 h_{yy} = h_t + 3h^2 h_x - 3h^2 h_y^2. \tag{5.5.9}$$

Substituting (5.5.9) into (5.5.8) and simplifying gives

$$E_h \left[h^3 h_x + \frac{1}{2} h^3 h_y^2 \right] |_{E=0} = -\left(h_t + 3h^2 h_x - \frac{3}{2} h^2 h_y^2 \right). \tag{5.5.10}$$

Define

$$E^1 = h_t + 3h^2 h_x - \frac{3}{2} h^2 h_y^2 \tag{5.5.11}$$

and

$$E^0 = E - E^1 = -\frac{3}{2} h^2 h_y^2 - h^3 h_{yy}. \tag{5.5.12}$$

Then (5.5.10) becomes

$$E_h \left[h^3 h_x + \frac{1}{2} h^3 h_y^2 \right] |_{E=0} = f E^1, \tag{5.5.13}$$

where $f = -1$ and

$$E = E^0 + E^1 \tag{5.5.14}$$

where E^0 and E^1 are defined by (5.5.11) and (5.5.12). Hence L defined by (5.5.6) is a partial Lagrangian of the partial differential equation (5.5.1).

In order to obtain a conserved vector we first derive the partial Noether symmetry [10],

$$X = \xi^1 (t, x, y, h) \frac{\partial}{\partial t} + \xi^2 (t, x, y, h) \frac{\partial}{\partial x} + \xi^3 (t, x, y, h) \frac{\partial}{\partial y} + \eta (t, x, y, h) \frac{\partial}{\partial h}. \tag{5.5.15}$$

The determining equation for the partial Noether symmetry is

$$X(L)+L \left(D_t\xi^1 + D_x\xi^2 + D_y\xi^3\right) = D_tB^1+D_xB^2+D_yB^3+\left(\eta - \xi^1u_t - \xi^2u_x - \xi^3u_y\right) E_h(L)|_{E=0} \tag{5.5.16}$$

where L is the partial Lagrangian and B^1, B^2 and B^3 are the components of the gauge vector. We consider gauge functions of the form

$$B^i = B^i (t, x, y, h) \,, \; i = 1, 2, 3. \tag{5.5.17}$$

The gauge functions are obtained from (5.5.16) in the process of deriving the partial Noether symmetries. Once the partial Noether symmetry and gauge functions have been derived the components of the conserved vector are given by

$$T^i = B^i - \xi^i L - \left(\eta - \xi^k h_k\right) E_{h_i}(L) + \sum_{s\geq 1} D_{l_1l_2\ldots l_s} \left(\eta - \xi^k h_k\right) E_{h_il_1l_2\ldots l_s}(L). \tag{5.5.18}$$

The partial Lagrangian (5.5.6) does not depend on second order and higher order partial derivatives of h. The summation term im (5.5.18) is then identically zero and

$$E_{h_i}(L) = \frac{\partial L}{\partial h_i}. \tag{5.5.19}$$

We start by evaluating the terms in the determining equation (5.5.16). Now with the partial Lagrangian L given by (5.5.6),

$$X(L) = \eta \left[\frac{3}{2}h^2h_y^2 + 3h^2h_x\right] + \zeta_x h^3 + \zeta_y h^3 h_y \tag{5.5.20}$$

where ζ_x and ζ_y are given by (3.2.11) and (3.2.12). Hence

$$\begin{aligned}X(L) = \eta\left[\frac{3}{2}h^2h_y^2 + 3h^2h_x\right] + h^3\left[\frac{\partial\eta}{\partial x} + h_x\frac{\partial\eta}{\partial h} - h_t\frac{\partial\xi^1}{\partial x} - h_th_x\frac{\partial\xi^1}{\partial h}\right.\\ -h_x\frac{\partial\xi^2}{\partial x} - h_x^2\frac{\partial\xi^2}{\partial h} - h_y\frac{\partial\xi^3}{\partial x} - h_yh_x\left.\frac{\partial\xi^3}{\partial h}\right] + h^3h_y\left[\frac{\partial\eta}{\partial y} + h_y\frac{\partial\eta}{\partial h}\right.\\ -h_t\frac{\partial\xi^1}{\partial y} - h_th_y\frac{\partial\xi^1}{\partial h} - h_x\frac{\partial\xi^2}{\partial y} - h_xh_y\frac{\partial\xi^2}{\partial h} - h_y\frac{\partial\xi^3}{\partial y} - h_y^2\left.\frac{\partial\xi^3}{\partial h}\right],\end{aligned} \tag{5.5.21}$$

where $\xi^i = \xi^i(t,x,y,h)$ and $\eta = \eta(t,x,y,h)$. Next we determine

$$L\left(D_t\xi^1 + D_x\xi^2 + D_y\xi^3\right)$$

$$= \left(\frac{1}{2}h^3h_y^2 + h^3h_x\right)\left[\left(\frac{\partial}{\partial t} + h_t\frac{\partial}{\partial h}\right)\xi^1 + \left(\frac{\partial}{\partial x} + h_x\frac{\partial}{\partial h}\right)\xi^2 + \left(\frac{\partial}{\partial y} + h_y\frac{\partial}{\partial h}\right)\xi^3\right]$$

$$\begin{aligned}&= \frac{1}{2}h^3\left[h_y^2\frac{\partial\xi^1}{\partial t} + h_th_y^2\frac{\partial\xi^1}{\partial h} + h_y^2\frac{\partial\xi^2}{\partial x} + h_xh_y^2\frac{\partial\xi^2}{\partial h} + h_y^2\frac{\partial\xi^3}{\partial h} + h_y^3\frac{\partial\xi^3}{\partial h}\right]\\ &\quad + h^3\left[h_x\frac{\partial\xi^1}{\partial t} + h_th_x\frac{\partial\xi^1}{\partial h} + h_x\frac{\partial\xi^2}{\partial x} + h_x^2\frac{\partial\xi^2}{\partial h} + h_x\frac{\partial\xi^3}{\partial y} + h_xh_y\frac{\partial\xi^3}{\partial h}\right].\end{aligned} \tag{5.5.22}$$

Finally using (5.5.13) we have

$$\begin{aligned}&\left(\eta - \xi^1h_t - \xi^2h_x - \xi^3h_y\right)E_h(L)|_{E=0}\\ &= \left(\eta - \xi^1h_t - \xi^2h_x - \xi^3h_y\right)\left(\frac{3}{2}h^2h_y^2 - h_t - 3h^2h_x\right)\\ &= \frac{3}{2}h^2\eta h_y^2 - \eta h_t - 3h^2\eta h_x - \frac{3}{2}h^2\xi^1h_th_y^2 + \xi^1h_t^2 + 3h^2\xi^1h_th_x - \frac{3}{2}h^2\xi^2h_xh_y^2\\ &\qquad + \xi^2h_th_x + 3h^2\xi^2h_x^2 - \frac{3}{2}h^2\xi^3h_y^3 + \xi^3h_th_y + 3h^2\xi^3h_xh_y.\end{aligned} \tag{5.5.23}$$

We now substitute equations (5.5.21), (5.5.22) and (5.5.23) into the determining equation for the partial Noether symmetry, (5.5.16), which becomes

$$\begin{aligned}
&\eta\left[\frac{3}{2}h^2h_y^2+3h^2h_x\right]+h^3\left[\frac{\partial\eta}{\partial x}+h_x\frac{\partial\eta}{\partial h}-h_t\frac{\partial\xi^1}{\partial x}-h_th_x\frac{\partial\xi^1}{\partial h}-h_x\frac{\partial\xi^2}{\partial x}-h_x^2\frac{\partial\xi^2}{\partial h}-h_y\frac{\partial\xi^3}{\partial x}\right.\\
&\quad\left.-h_yh_x\frac{\partial\xi^3}{\partial h}\right]+h^3\left[h_y\frac{\partial\eta}{\partial y}+h_y^2\frac{\partial\eta}{\partial h}-h_th_y\frac{\partial\xi^1}{\partial y}-h_th_y^2\frac{\partial\xi^1}{\partial h}-h_xh_y\frac{\partial\xi^2}{\partial y}-h_xh_y^2\frac{\partial\xi^2}{\partial h}\right.\\
&\quad\left.-h_y^2\frac{\partial\xi^3}{\partial y}-h_y^3\frac{\partial\xi^3}{\partial h}\right]+\frac{1}{2}h^3\left[h_y^2\frac{\partial\xi^1}{\partial t}+h_th_y^2\frac{\partial\xi^1}{\partial h}+h_y^2\frac{\partial\xi^2}{\partial x}+h_xh_y^2\frac{\partial\xi^2}{\partial h}+h_y^2\frac{\partial\xi^3}{\partial h}+h_y^3\frac{\partial\xi^3}{\partial h}\right]\\
&\quad+h^3\left[h_x\frac{\partial\xi^1}{\partial t}+h_th_x\frac{\partial\xi^1}{\partial h}+h_x\frac{\partial\xi^2}{\partial x}+h_x^2\frac{\partial\xi^2}{\partial h}+h_x\frac{\partial\xi^3}{\partial y}+h_xh_y\frac{\partial\xi^3}{\partial h}\right]=\frac{\partial B^1}{\partial t}+h_t\frac{\partial B^1}{\partial h}+\frac{\partial B^2}{\partial t}\\
&\quad+h_x\frac{\partial B^2}{\partial h}+\frac{\partial B^3}{\partial y}+h_y\frac{\partial B^3}{\partial h}+\frac{3}{2}h^2\eta h_y^2-\eta h_t-3h^2\eta h_x-\frac{3}{2}h^2\xi^1h_th_y^2+\xi^1h_t^2\\
&\quad+3h^2\xi^1h_th_x-\frac{3}{2}h^2\xi^2h_xh_y^2+\xi^2h_th_x+3h^2\xi^2h_x^2-\frac{3}{2}h^2\xi^3h_y^3+\xi^3h_th_y+3h^2\xi^3h_xh_y.
\end{aligned} \tag{5.5.24}$$

Since ξ^i, η and B^i are independent of the derivatives of h we separate (5.5.24) by powers and products of the derivatives of h. Consider first

$$h_t^2: \qquad \xi^1=0, \tag{5.5.25}$$

$$h_xh_t: \qquad -\frac{\partial\xi^1}{\partial h}h^3+\frac{\partial\xi^1}{\partial h}h^3=\xi^2+3h^2\xi^1, \tag{5.5.26}$$

$$h_yh_t: \qquad -h^3\frac{\partial\xi^1}{\partial y}=\xi^3. \tag{5.5.27}$$

Hence

$$\xi^1=0,\quad \xi^2=0,\quad \xi^3=0 \tag{5.5.28}$$

and the determining equation (5.5.24) reduces to

$$\begin{aligned}
&\eta\left[\frac{3}{2}h^2h_y^2+3h^2h_x\right]+h^3\left[\frac{\partial\eta}{\partial x}+h_x\frac{\partial\eta}{\partial h}\right]+h^3\left[h_y\frac{\partial\eta}{\partial y}+h_y^2\frac{\partial\eta}{\partial h}\right]=\frac{\partial B^1}{\partial t}+h_t\frac{\partial B^1}{\partial h}\\
&\quad+\frac{\partial B^2}{\partial x}+h_x\frac{\partial B^2}{\partial h}+\frac{\partial B^3}{\partial y}+h_y\frac{\partial B^3}{\partial h}+\frac{3}{2}h^2\eta h_y^2-\eta h_t-3h^2\eta h_x.
\end{aligned} \tag{5.5.29}$$

We now separate equation (5.5.29) by the remaining derivatives of h:

$$h_y^2: \qquad \frac{\partial \eta}{\partial h} = 0, \tag{5.5.30}$$

$$h_t: \qquad \frac{\partial B^1}{\partial h} = \eta, \tag{5.5.31}$$

$$h_x: \qquad \frac{\partial B^2}{\partial h} = 6h^2\eta + h^3\frac{\partial \eta}{\partial h}, \tag{5.5.32}$$

$$h_y: \qquad \frac{\partial B^3}{\partial h} = h^3\frac{\partial \eta}{\partial y}, \tag{5.5.33}$$

$$R: \qquad \frac{\partial B^1}{\partial t} + \frac{\partial B^2}{\partial x} + \frac{\partial B^3}{\partial y} = h^3\frac{\partial \eta}{\partial x}. \tag{5.5.34}$$

From (5.5.30),

$$\eta = \eta(t, x, y). \tag{5.5.35}$$

Since η does not depend on h we can integrate equations (5.5.31), (5.5.32) and (5.5.33) to obtain the following results:

$$B^1(t, x, y, h) = h\eta + A(t, x, y), \tag{5.5.36}$$

$$B^2(t, x, y, h) = 2h^3\eta + C(t, x, y), \tag{5.5.37}$$

$$B^3(t, x, y, h) = \frac{1}{4}h^4\frac{\partial \eta}{\partial y} + D(t, x, y). \tag{5.5.38}$$

We then substitute equations (5.5.36) to (5.5.38) into (5.5.34) which gives

$$h\frac{\partial \eta}{\partial t} + \frac{\partial A}{\partial t} + 2h^3\frac{\partial \eta}{\partial x} + \frac{\partial C}{\partial x} + \frac{1}{4}h^4\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial D}{\partial y} = h^3\frac{\partial \eta}{\partial x}. \tag{5.5.39}$$

Since η is independent of h we can separate equation (5.5.39) by powers of h:

$$h^4: \qquad \frac{\partial^2 \eta}{\partial y^2} = 0, \tag{5.5.40}$$

$$h^3: \qquad \frac{\partial \eta}{\partial x} = 0, \tag{5.5.41}$$

$$h: \qquad \frac{\partial \eta}{\partial t} = 0, \tag{5.5.42}$$

$$R: \qquad \frac{\partial A}{\partial t} + \frac{\partial C}{\partial x} + \frac{\partial D}{\partial y} = 0. \tag{5.5.43}$$

From (5.5.41) and (5.5.42) it follows that

$$\eta = \eta(y) \tag{5.5.44}$$

and therefore from (5.5.40),

$$\eta(y) = b_1 y + b_2, \tag{5.5.45}$$

where b_1 and b_2 are constants. From (5.5.28) and (5.5.45) the partial Noether symmetry is

$$X = (b_1 y + b_2)\frac{\partial}{\partial h}. \tag{5.5.46}$$

Substituting (5.5.45) into equations (5.5.36), (5.5.37) and (5.5.38) we obtain the gauge functions for B^1, B^2 and B^3,

$$B^1 = b_1 yh + b_2 h + A(t, x, y), \tag{5.5.47}$$

$$B^2 = 2b_1 yh^3 + 2b_2 h^3 + C(t, x, y), \tag{5.5.48}$$

$$B^3 = \frac{1}{4} b_1 h^4 + D(t, x, y), \tag{5.5.49}$$

where A, C and D satisfy the condition (5.5.43).

Since the partial Lagrangian L, given by (5.5.6), does not depend on the second partial derivatives of h, the conserved vector (5.5.18) reduces to

$$T^i = B^i - \xi^i L - \left(\eta - \xi^1 h_t - \xi^2 h_x - \xi^3 h_y\right)\frac{\partial L}{\partial h_i} \tag{5.5.50}$$

and further since $\xi^1 = \xi^2 = \xi^3 = 0$ by (5.5.28),

$$T^i = B^i - \eta\frac{\partial L}{\partial h_i}, \quad i = 1, 2, 3. \tag{5.5.51}$$

Hence

$$\begin{aligned} T^1 &= B^1 - \eta\frac{\partial L}{\partial h_t} \\ &= b_1 yh + b_2 h + A(t, x, y), \end{aligned} \tag{5.5.52}$$

$$
\begin{aligned}
T^2 &= B^2 - \eta \frac{\partial L}{\partial h_x} \\
&= b_1 y h^3 + b_2 h^3 + C(t,x,y),
\end{aligned} \tag{5.5.53}
$$

$$
\begin{aligned}
T^3 &= B^3 - \eta \frac{\partial L}{\partial h_y} \\
&= \left(\frac{1}{4} h^4 - y h^3 h_y \right) b_1 - b_2 h^3 h_y + D(t,x,y).
\end{aligned} \tag{5.5.54}
$$

When $b_1 = b_2 = 0$ we obtain the conserved vector

$$
T^1 = A(t,x,y),\ T^2 = C(t,x,y),\ T^3 = D(t,x,y) \tag{5.5.55}
$$

where A, C and D satisfy (5.5.43). Since (5.5.43) is satisfied independently of the partial differential equation, (5.5.55) is a trivial conserved vector. We therefore set $A = 0$, $C = 0$ and $D = 0$.

Setting $b_1 = 0$ and $b_2 = 1$ gives

$$
T^1 = h, \quad T^2 = h^3, \quad T^3 = -h^3 h_y \tag{5.5.56}
$$

which is the elementary conserved vector. Setting $b_1 = 1$ and $b_2 = 0$ gives

$$
T^1 = yh, \quad T^2 = yh^3, \quad T^3 = -y h^3 h_y + \tfrac{1}{4} h^4. \tag{5.5.57}
$$

The conserved vectors (5.5.56) and (5.5.57) are the same conserved vectors as found using the direct method and the multiplier method. In order to find other conservation laws, if they exist, we would need to include partial derivatives of h in the gauge functions, B^1, B^2 and B^3. The derivation will be longer but computer algebra may be able to assist with the calculations.

5.6 Generating new conserved vectors from known conserved vectors

Conservation laws for a partial differential equation can be generated from known conserved vectors for the partial differential equations and known Lie point symmetries of the partial differential equation.

Consider the partial differential equation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h^3\right) - \frac{\partial}{\partial y}\left(h^3 \frac{\partial h}{\partial y}\right) = 0 \tag{5.6.1}$$

and let $T = (T^1, T^2, T^3)$ be a conserved vector for (5.6.1) and X a Lie point symmetry generator of (5.6.1). Then

$$T_*^i = X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i), \quad i = 1, 2, 3 \tag{5.6.2}$$

are the components of a conserved vector for (5.6.2), that is

$$D_k T_*^k = 0 \tag{5.6.3}$$

for all solutions of the partial differential equation (5.6.1).

A proof of the result for the general case is given by Kara and Mahomed [9, 12]. The generated conservation law (5.6.3) may be trivial. The conserved vector T_* may be a linear combination of known conserved vectors or zero. In (5.6.2), X is prolonged to as many partial derivatives as required and the total derivatives D_i are defined by

$$D_1 = D_t = \frac{\partial}{\partial t} + h_t \frac{\partial}{\partial h} + h_{tt} \frac{\partial}{\partial h_t} + h_{xt} \frac{\partial}{\partial h_x} + h_{yt} \frac{\partial}{\partial h_y} \ldots, \tag{5.6.4}$$

$$D_2 = D_x = \frac{\partial}{\partial x} + h_x \frac{\partial}{\partial h} + h_{tx} \frac{\partial}{\partial h_t} + h_{xx} \frac{\partial}{\partial h_x} + h_{yx} \frac{\partial}{\partial h_y} \ldots, \tag{5.6.5}$$

$$D_3 = D_y = \frac{\partial}{\partial y} + h_y \frac{\partial}{\partial h} + h_{ty} \frac{\partial}{\partial h_t} + h_{xy} \frac{\partial}{\partial h_x} + h_{yy} \frac{\partial}{\partial h_y} \ldots. \tag{5.6.6}$$

The linear combination of the Lie point symmetries of the partial differential equation (5.6.1) is given by (3.2.30). Since there is no constant c_3 in (3.2.30) we rename c_4 as c_3, c_5 as c_4 and c_6 as c_5. The Lie point symmetry (3.2.30) becomes

$$X = (c_1 t + c_3)\frac{\partial}{\partial t} + (c_2 x + c_4)\frac{\partial}{\partial x} + \left(\frac{1}{4}(3c_2 - c_1)\, y + c_5\right)\frac{\partial}{\partial y} + \frac{1}{2}(c_2 - c_1)\, h\frac{\partial}{\partial h}. \quad (5.6.7)$$

We now investigate if new conserved vectors can be generated from the known conserved vectors (5.5.56) and (5.5.57) using (5.6.2) with X given by (5.6.7). In expanded form the components of (5.6.2) are

$$T^1_* = X(T^1) + T^1 D_2(\xi^2) + T^1 D_3(\xi^3) - T^2 D_2(\xi^1) - T^3 D_3(\xi^1), \quad (5.6.8)$$

$$T^2_* = X(T^2) + T^2 D_1(\xi^1) + T^2 D_3(\xi^3) - T^1 D_1(\xi^2) - T^3 D_3(\xi^2), \quad (5.6.9)$$

$$T^3_* = X(T^3) + T^3 D_2(\xi^2) + T^3 D_2(\xi^2) - T^1 D_1(\xi^3) - T^2 D_2(\xi^3). \quad (5.6.10)$$

We consider first the elementary conserved vector T with components

$$T^1 = h, \quad T^2 = h^3, \quad T^3 = -h^3\frac{\partial h}{\partial y} \quad (5.6.11)$$

and start by calculating T^1_* from equation (5.6.8). In order to find T^1_* we need to determine each term of equation (5.6.8):

$$X(T^1) = \frac{1}{2}(c_2 - c_1)h, \quad (5.6.12)$$

$$T^1 D_2(\xi^2) = c_2 h, \quad (5.6.13)$$

$$T^1 D_3(\xi^3) = \frac{1}{4}(3c_2 - c_1)\, h, \quad (5.6.14)$$

$$T^2 D_2(\xi^1) = 0, \quad (5.6.15)$$

$$T^3 D_3(\xi^1) = 0. \quad (5.6.16)$$

Substituting equations (5.6.12) to (5.6.16) into (5.6.8) we obtain

$$T^1_* = \frac{3}{4}(3c_2 - c_1)\, h = \frac{3}{4}(3c_2 - c_1)\, T^1. \quad (5.6.17)$$

We now calculate T_*^2 starting from (5.6.9):

$$X(T^2) = \frac{3}{2}(c_2 - c_1)h^3, \tag{5.6.18}$$

$$T^2 D_1(\xi^1) = c_1 h^3, \tag{5.6.19}$$

$$T^2 D_3(\xi^3) = \frac{1}{4}\left(3c_2 - c_1\right)h^3, \tag{5.6.20}$$

$$T^1 D_1(\xi^2) = 0, \tag{5.6.21}$$

$$T^3 D_3(\xi^2) == 0. \tag{5.6.22}$$

Substituting equations (5.6.18) to (5.6.22) into (5.6.9) we obtain

$$T_*^2 = \frac{3}{4}\left(3c_2 - c_1\right)h^3 = \frac{3}{4}\left(3c_2 - c_1\right)T^2. \tag{5.6.23}$$

Finally we calculate T_*^3 from equation (5.6.10) once again determining each term in the equation starting from the first:

$$\begin{aligned} X(T^3) &= \left[\xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \xi^3\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial h} + \zeta_y\frac{\partial}{\partial h_y}\right]\left[-h^3 h_y\right] \\ &= -\frac{3}{2}(c_2 - c_1)h^3 h_y - h^3\zeta_y. \end{aligned} \tag{5.6.24}$$

Now from equation (3.2.12)

$$\zeta_y = \frac{1}{4}(-c_2 - c_1)h_y. \tag{5.6.25}$$

Hence equation (5.6.24) becomes,

$$X(T^3) = -\frac{1}{4}(5c_2 - 7c_1)h^3 h_y. \tag{5.6.26}$$

Also

$$T^3 D_1(\xi^1) = -h^3 h_y c_1, \tag{5.6.27}$$

$$T^3 D_2(\xi^2) = -h^3 h_y c_2, \tag{5.6.28}$$

$$T^1 D_1(\xi^3) = 0, \tag{5.6.29}$$

$$T^2 D_2(\xi^3) = 0. \tag{5.6.30}$$

Substituting equations (5.6.26) to (5.6.30) into (5.6.10) we obtain

$$T_*^3 = -\frac{3}{4}(3c_2 - c_1)h^3 h_y = \frac{3}{4}(3c_2 - c_1)T^3. \tag{5.6.31}$$

Thus the generated conserved vector satisfies

$$T_* = (T_*^1, T_*^2, T_*^3) = \frac{3}{4}(3c_2 - c_1)(T^1, T^2, T^3) \tag{5.6.32}$$

and therefore

$$T_* = \frac{3}{4}\left(3c_2 - c_1\right) T. \tag{5.6.33}$$

The generated conserved vector T_* is a constant multiple of the elementary conserved vector T and is therefore not a new conserved vector.

In order to distinguish between the two conserved vectors we will denote the second conserved vector by S. Thus

$$S^1 = yh, \quad S^2 = yh^3, \quad S^3 = -yh^3 h_y + \frac{1}{4}. \tag{5.6.34}$$

Equations (5.6.8) to (5.6.10) for the generated conserved vector apply for the second conserved vector with T replaced by S.

Consider first S_*^1 given by

$$S_*^1 = X\left(S^1\right) + S^1 D_2\left(\xi^2\right) + S^1 D_3\left(\xi^3\right) - S^2 D_2\left(\xi^1\right) - S^3 D_3\left(\xi^1\right). \tag{5.6.35}$$

Then

$$X(S^1) = \frac{1}{4}(5c_2 - 3c_1)yh + c_5 h, \tag{5.6.36}$$

$$S^1 D_2(\xi^2) = c_2 yh, \tag{5.6.37}$$

$$S^1 D_3(\xi^3) = \frac{1}{4}\left(3c_2 - c_1\right) yh, \tag{5.6.38}$$

$$S^2 D_2(\xi^1) = 0, \tag{5.6.39}$$

$$S^3 D_3(\xi^1) = 0 \tag{5.6.40}$$

Substituting equations (5.6.36) to (5.6.40) into (5.6.35) we obtain

$$S_*^1 = (3c_2 - c_1)\, yh + c_5 h = (3c_2 - c_1)\, S_{(2)}^1 + c_5 T^1. \tag{5.6.41}$$

We now calculate S_*^2 given by

$$S_*^2 = X\left(S^2\right) + S^2 D_1\left(\xi^1\right) + S^2 D_3\left(\xi^3\right) - S^1 D_1\left(\xi^2\right) - S^3 D_3\left(\xi^2\right) \tag{5.6.42}$$

and again determining each term in the equation starting with the first term:

$$X(S^2) = \frac{1}{4}(9c_2 - 7c_1)yh^3 + c_5 h^3, \tag{5.6.43}$$

$$S^2 D_1(\xi^1) = yh^3 c_1, \tag{5.6.44}$$

$$S^2 D_3(\xi^3) = \frac{1}{4}\, (3c_2 - c_1)\, yh^3, \tag{5.6.45}$$

$$S^1 D_1(\xi^2) = 0, \tag{5.6.46}$$

$$S^3 D_3(\xi^2) = 0. \tag{5.6.47}$$

Substituting equations (5.6.43) to (5.6.47) into (5.6.42) we obtain

$$\begin{aligned} S_*^2 &= (3c_2 - c_1)\, yh^3 + c_5 h^3 \\ &= (3c_2 - c_1)\, S^2 + c_5 T^2. \end{aligned} \tag{5.6.48}$$

Finally we calculate S_*^3 from the equation

$$S_*^3 = X\left(S^3\right) + S^3 D_1\left(\xi^1\right) + S^3 D_2\left(\xi^2\right) - S^1 D_1\left(\xi^3\right) - S^2 D_2\left(\xi^3\right). \tag{5.6.49}$$

once again determining each term in the equation starting from the first:

$$\begin{aligned} X(S^3) = &\left[(c_1 t + c_3)\frac{\partial}{\partial t} + (c_2 x + c_4)\frac{\partial}{\partial x} + \left(\frac{1}{4}(3c_2 - c_1)y + c_5\right)\frac{\partial}{\partial y} \right. \\ &\left. + \left(\frac{1}{2}h(c_2 - c_1)\right)\frac{\partial}{\partial h} + \zeta_y \frac{\partial}{\partial h_y}\right]\left(-yh^3 h_y + \frac{1}{4}h^4\right) \\ = &\left[\frac{1}{4}(3c_2 - c_1)y + c_5\right](-h^3 h_y) + \frac{3}{2}(c_2 - c_1)(-yh^3 h_y) + \frac{1}{2}(c_2 - c_1)h^4 - yh^3\zeta_y. \end{aligned} \tag{5.6.50}$$

Now ζ_y has already been calculated and is given in equation (5.6.25) as

$$\zeta_y = \frac{1}{4}(-c_2 - c_1)h_y. \tag{5.6.51}$$

Therefore equation (5.6.50) becomes,

$$X(S^3) = 2(c_2 - c_1)(-yh^3h_y + \frac{1}{4}h^4) + c_5(-h^3h_y). \tag{5.6.52}$$

Also

$$S^3D_1(\xi^1) = c_1(-yh^3h_y + \frac{1}{4}h^4), \tag{5.6.53}$$

$$S^3D_2(\xi^2) = c_2(-yh^3h_y + \frac{1}{4}h^4), \tag{5.6.54}$$

$$S^1D_1(\xi^3) = 0, \tag{5.6.55}$$

$$S^2D_2(\xi^3) = 0. \tag{5.6.56}$$

Substituting equations (5.6.52) to (5.6.56) into (5.6.49) we obtain

$$\begin{aligned} S^3_* &= (3c_2 - c_1)(-yh^3h_y + \frac{1}{4}h^4) + c_5(-h^3h_y) \\ &= (3c_2 - c_1)S^3 + c_5T^3. \end{aligned} \tag{5.6.57}$$

The generated conserved vector therefore is

$$S_* = \left(S^1_*, S^2_*, S^3_*\right) = (3c_2 - c_1)\left(S^1, S^2, S^3\right) + c_5\left(T^1, T^2, T^3\right) \tag{5.6.58}$$

which in vector form is

$$S_* = (3c_2 - c_1)\, S + c_5T. \tag{5.6.59}$$

The generated conserved vector is a linear combination of the two conserved vectors, S and T, we have already found. It is therefore not a new conserved vector.

In both cases the generated conserved vector does not lead to a new conserved vector. However we can use the results derived in this section to determine the Lie point symmetry associated with each conserved vector.

5.7 Associated Lie point symmetries

A Lie point symmetry X of a partial differential equation is said to be **associated** with a conserved vector T for the partial differential equation if the generated conserved vector T_* is zero [9, 12]. Thus from (5.6.2), X is associated with T provided

$$X\left(T^i\right) + T^i D_k\left(\xi^k\right) - T^k D_k\left(\xi^i\right) = 0, \quad i = 1, 2, \ldots, n. \tag{5.7.1}$$

Sjoberg [17] has shown that if a partial differential equation of order q admits a Lie point symmetry X which is associated with a conserved vector T for the partial differential equation, then the partial differential equation can be reduced to an ordinary differential equation of order $q-1$. A double reduction can therefore be performed on a partial differential equation (PDE) of order q by a Lie point symmetry X of the PDE associated with a conserved vector for the PDE. The first reduction is to reduce the PDE to an ordinary differential equation (ODE) of order q and the second reduction is to reduce the ODE of order q to an ODE of order $q-1$.

We now investigate when the Lie point symmetry

$$X = (c_1 t + c_3)\frac{\partial}{\partial t} + (c_2 x + c_4)\frac{\partial}{\partial x} + \left(\frac{1}{4}(3c_2 - c_1)\, y + c_5\right)\frac{\partial}{\partial y} + \frac{1}{2}(c_2 - c_1)\, h\frac{\partial}{\partial h} \tag{5.7.2}$$

of the partial differential equation (5.2.1),

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h^3\right) - \frac{\partial}{\partial y}\left(h^3 \frac{\partial h}{\partial y}\right) = 0 \tag{5.7.3}$$

is associated with conserved vectors

$$T: \qquad T^1 = h, \quad T^2 = h^3, \quad T^3 = -h^3 h_y, \tag{5.7.4}$$

$$S: \qquad S^1 = yh, \quad S^2 = yh^3, \quad S^3 = -yh^3 h_y + \frac{1}{4}h^4, \tag{5.7.5}$$

for the PDE (5.7.3). We will also investigate the double reduction of the partial differential equation (5.7.3) by associated Lie point symmetries.

Consider first the elementary conserved vector given by (5.7.4). Then from (5.6.33),

$$T_* = \frac{3}{4}\left(3c_2 - c_1\right)T \tag{5.7.6}$$

and therefore the Lie point symmetry (5.7.2) is associated with the elementary conserved vector provided

$$\alpha = \frac{c_2}{c_1} = \frac{1}{3}. \tag{5.7.7}$$

The associated Lie point symmetry is

$$X = \left(c_1 t + c_3\right)\frac{\partial}{\partial t} + \left(\frac{1}{3}c_2 x + c_4\right)\frac{\partial}{\partial x} + c_5\frac{\partial}{\partial y} - \frac{1}{3}c_1 h\frac{\partial}{\partial h}. \tag{5.7.8}$$

Consider next the conserved vector (5.7.5). Since from (5.6.59),

$$S^* = \left(3c_2 - c_1\right)S + c_5 T \tag{5.7.9}$$

the Lie point symmetry (5.7.2) is associated with the conserved vector S provided

$$\alpha = \frac{c_2}{c_1} = \frac{1}{3} \;\; \textit{and} \;\; c_5 = 0. \tag{5.7.10}$$

The associated Lie point symmetry is

$$X = \left(c_1 t + c_3\right)\frac{\partial}{\partial t} + \left(\frac{1}{3}c_2 x + c_4\right)\frac{\partial}{\partial x} - \frac{1}{3}c_1 h\frac{\partial}{\partial h}. \tag{5.7.11}$$

For the general case the partial differential equation (5.7.3) is reduced by the Lie point symmetry (5.7.2) to the ordinary differential equation (3.3.31) with $V = 0$:

$$\frac{9}{16}c_1\left(\alpha - \frac{1}{3}\right)^2\frac{d}{d\eta}\left(H^3\frac{dH}{d\eta}\right) + \frac{3}{4}\alpha\frac{d}{d\eta}\left(\eta H^3\right) - \frac{1}{4}\frac{d}{d\eta}\left(\eta H\right) + \frac{3}{4}H\left(1 - 3\alpha H^2\right) - \frac{1}{c_1}V = 0. \tag{5.7.12}$$

Equation 5.7.12 is independent of c_5. When $\alpha = \frac{1}{3}$ the ordinary differential equation (5.7.12) reduces to

$$\frac{d}{d\eta}\left(\eta H^3\right) - \frac{d}{d\eta}\left(\eta H\right) + 3H\left(1 - H^2\right) = 0 \tag{5.7.13}$$

This result clearly illustrates double reduction of a partial differential equation by an associated Lie point symmetry. The partial differential equation (5.7.3) was order $q = 2$ while the ordinary differential equation (5.7.13) is of order $q - 1 = 1$. Equation (5.7.13) can be expressed as

$$\eta \frac{d}{d\eta}\left(H^3 - H\right) = 2\left(H^3 - H\right) \tag{5.7.14}$$

which is variables separable. The solution of (5.7.14) is

$$H^3 - H = C\eta^2 \tag{5.7.15}$$

where C is a constant. Equation (5.7.15) is an implicit cubic equation for H.

To investigate the boundary conditions consider η and h given by (3.3.32) and (3.3.33) with the constants c_4, c_5 and c_6 renamed as c_3 c_4 and c_5:

$$\eta\left(t, x, y\right) = \frac{\left[\frac{1}{4}\left(3c_2 - c_1\right) y + c_5\right]\left(c_1 t + c_3\right)^{\frac{1}{4}}}{\left(c_2 x + c_4\right)^{\frac{3}{4}}}, \tag{5.7.16}$$

$$h(t, x, y) = \left(\frac{c_2 x + c_4}{c_1 t + c_3}\right)^{\frac{1}{2}} H(\eta). \tag{5.7.17}$$

Consider first the Lie point symmetry (5.7.8) associated with the elementary conserved vector. Then $3c_2 - c_1 = 0$ and $c_5 \neq 0$. Thus

$$\eta = \frac{c_5\left(c_1 t + c_3\right)^{\frac{1}{4}}}{\left(c_2 x + c_4\right)^{\frac{3}{4}}}. \tag{5.7.18}$$

The similarity variable η is independent of y and therefore $h = h(t, x)$. The symmetry boundary condition

$$y = 0: \qquad \frac{\partial h}{\partial y} = 0 \tag{5.7.19}$$

is identically satisfied. The condition at the boundary of the rivulet,

$$y = \pm a(t, x): \qquad h = 0 \tag{5.7.20}$$

cannot be imposed for $H \neq 0$ because h is independent of y. The solution (5.7.15) is a trivial solution of the partial differential equation but it does illustrate the double reduction of a partial differential equation by an associated Lie point symmetry.

For the Lie point symmetry (5.7.11) associated with the second conserved vector, S, we have $c_5 = 0$ and (5.7.18) gives $\eta = 0$. To analyse this special case we would need to return to the derivation of the invariant solution and set $3c_2 = c_1$ and $c_5 = 0$ at the start of the derivation. We will not do that in this dissertation.

5.8 Concluding remarks

We were able to derive two conservation laws and therefore two conserved vectors for the partial differential equation for the height of the rivulet with zero leak-off velocity. One conserved vector was the elementary conserved vector while the second conserved vector was new.

Three methods were used to derive the conserved vectors, the direct method, the multiplier method and the partial Lagrangian method. Each method required an assumption which restricted the range of conserved vectors that could be derived. The direct method required an assumption on the variables on which the components of the conserved vector depend, the multiplier method an assumption on the variables on which the multiplier depends and the partial Lagrangian method an assumption on the variables on which the gauge function depend. To search for more conservation laws these quantities would have to depend on higher derivatives. This would make the calculations much longer but they could be considered with the aid of interactive computer packages.

We were not able to derive new conserved vectors from the known conserved vectors and the Lie point symmetries of the partial differentials. The calculation however gave the Lie point symmetry of the partial differential equation associated with each conserved vector. The Lie point symmetry associated with second conserved vector was a special case ($c_5 = 0$)

of the Lie point symmetry associated with the elementary conserved vector. The results clearly illustrated the double reduction of a partial differential equation to an ordinary differential equation by an associated Lie point symmetry although the solution of the ordinary differential equation lead to a solution of the problem that was not physically significant.

Chapter 6

CONCLUSION

By not specifying the form of the leak-off velocity in the model, considerable analytical progress could be made. The partial differential equation for the height of the rivulet had an invariant solution and could be reduced to an ordinary differential equation in two steps provided the leak-off velocity satisfied a linear first order partial differential equation in three variables. The solution of this partial differential equation for the leak-off velocity was quite general and permitted physically significant models for the leak-off velocity to be considered and compared.

Most of the known solutions for rivulet flow down an inclined plane are numerical solutions. The analytical solution we derived may be useful for checking the accuracy of numerical methods for rivulet flow down an inclined plane. The leak-off velocity in the analytical solution is a linear combination of the leak-off velocities we considered in the two numerical solutions. The analytical solution contained a dry patch in the central region of the rivulet and confirmed that dry patches can occur in rivulet flow. For known solutions of rivulet flow with a dry patch, the dry patch is caused by thermal and surface tension effects [8]. In our analytical solution the dry patch is due to fluid leak-off into the porous substrate.

We found that the shooting method was an effective method to derive the numerical solution for rivulet flow down an inclined plane. It transformed the boundary value problem into an initial value problem. The boundary of the rivulet was determined by the shooting method as well as the initial value of the unknown functions. The two numerical solutions clearly demonstrated how the fluid leak-off at the porous based determined the width and height of the rivulet.

Two conserved vectors for the partial differential equation for the height of the rivulet with no leak-off at the base were found. One conserved vector was the elementary conserved vector while the second was a new conserved vector. The three methods of deriving the conserved vectors gave the same results. The multiplier method gave first a multiplier and further elementary operations had to be performed to derived the conserved vector. The direct method gave the conserved vector explicitly at the end of the calculation. In the partial Lagrangian method the conserved vector was calculated from a formula once the gauge functions and Noether symmetry had been obtained. New conserved vectors were not derived from the Lie point symmetry of the partial differential equation and the known conserved vectors, but the calculation gave the Lie point symmetry associated with each conserved vector. The double reduction of a partial differential equation by an associated Lie point symmetry was clearly illustrated but the solution obtained by the double reduction was not physically significant.

In all calculations of Lie point symmetries and invariant solutions and in the derivation of conservations laws only the general case was considered. Special cases in which the constants in the linear combination of the Lie point symmetries take special values were not considered. These special cases may lead to non-trivial solutions and should be investigated as further work.

Extensions of the research undertaken in this dissertation could be considered. Rivulets consisting of non-Newtonian fluids such as power-law fluids could be investigated by the analytical and numerical methods developed in this dissertation. Other leak-off velocities at the porous based could be studied. In the two models we considered the leak-off velocity depended explicitly on t and x as well as on the height h. A model in which the leak-off velocity depends explicitly on y as well as on h could be investigated. Rivulet flow with leak-off at the base together with thermal and surface tension effects could be investigated. Higher order conservation laws in which the conserved vectors depend on higher derivatives could be considered as well as conservation laws for the partial differential equation for the rivulet height containing the leak-off velocity. An investigation of higher order conservation laws may require the assistance of interactive computer packages.

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