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Teachers' engagement with language practices through a geometry lesson study

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ABSTRACT

Our study focuses on teachers' engagement with language practices through their participation in an adapted lesson study (LS) focusing on introduction to the concept of similarity and the meaning of 'similar' in a lesson on similar triangles. We use data from textbook analysis, lesson planning, lessons and lesson reflection sessions to explore teachers' engagement with language practices and how these evolved over a LS cycle. Working with the notion of dilemmas in teaching as 'sources of praxis' (Adler, 1998), we identified inter-connected teaching dilemmas, engagement with which led to a wider use of language registers and representations in the second lesson and with these, opening opportunities for elaborating the meaning of similar triangles. We describe the dilemmas that emerged and their related language practices, evidence teachers' engagement with these in their talk and their teaching, and argue that LS can create conditions for teachers' learning about language practices in teaching.

1. Introduction

The teaching and learning of secondary level geometry is recognised as challenging in Malawi, with calls for professional development (PD) to improve teaching quality (Ministry of Education, Science and Technology, 2020) (MoEST). In response to this challenge and call, in late 2021, we initiated an exploratory lesson study (LS) research project focused on lower secondary geometry with ten secondary mathematics teachers from two schools. We were inspired by the recent, and the relatively new take-up of, and success with, LS in Malawi by colleagues working at the elementary mathematics level (Fauskanger, Jakobsen, & Kazima, 2019). Research has shown that when LS is introduced in a new context, structuring frameworks, or more broadly what has been referred to as theory-informed approaches, can scaffold teachers' entry and enable systematic research of teacher learning through LS (Bahn & Winsløw, 2019; González & Deal, 2019; Huang, Gong, & Han, 2019). We introduced the teachers to a Mathematics Teaching Framework (MTF), a theoretically informed mathematics lesson planning, observation and reflection tool previously used in a LS project in a similar teaching and learning context to Malawi (Adler & Alshwaikh, 2019). The MTF (elaborated below) brings into focus two key interacting mathematics teaching practices, exemplification and explanatory communication, with the latter including the specific language practices of word use (and so attention to the mathematics register[2] in relation to other language registers in use in classroom communication) and justifying (and so attention to building mathematical explanations).

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[2] Our use of 'register' and 'mathematics register' is elaborated below

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Our exploration has involved two LS cycles, with the teaching practices foregrounded in the MTF shaping our research. We have reported on our study of the teachers' participation in the first cycle, where we confirmed the importance of theory-informed LS when adapting it in a new context (Adler, Mwadzaangati, & Takker, 2022), and showed how LS can support teachers' learning of exemplification practices (Mwadzaangaati, Takker, & Adler, 2022). Throughout cycle 1 we also observed that the teachers' interpretations of explanatory communication remained largely within their well-honed routines, with little opportunity for extended mathematical talk. While the teachers themselves attended to more meaningful use of definitions (Adler et al., 2022), student activity and related talk was limited to empirical angle measure tasks and reporting these measures. Students need to speak, read and write mathematically, and so appreciate the specificity of mathematical language (Pimm, 1987), especially in multilingual countries like Malawi where the language of instruction is not the learners' main language (e.g. Barwell, 2018; Wilkinson, 2018). We thus set out in cycle 2, where the content focus selected by the teachers was similar triangles, to work more deliberately with the teachers to plan, enact and reflect on language practices, with specific attention to word use and justifying as valued in the MTF. We were interested in the forms these language practices would take in the Malawian instructional context, and what we, as a field, could learn from the teachers' praxis.

There has been growing attention to the need for studies on mathematics teachers' learning of language practices, particularly in multilingual contexts, and for these to be content (Planas, 2021; Prediger, 2022; Zahner, Calleros, & Pelaez, 2021) and context specific (Adler, 2021). Concomitantly, while LS has been increasingly adopted, adapted and researched in mathematics teacher PD across contexts, including in resource-constrained contexts, research linked with LS in such contexts remains under-developed, with few studies reported in international research collections and surveys (Huang et al., 2019; 2023). A key affordance of LS is through teachers' collaborative planning, teaching and reflection, creating conditions for teaching/learning issues and/or dilemmas to emerge for discussion and action. As has been argued, again from research in multilingual classrooms, teaching dilemmas can be "sources of praxis", for teachers to work on their practices as well as for researchers to advance the field (Adler, 1998; Barwell & Pimm, 2016). The study reported in this paper sits in the confluence of, and contributes to, these lines of research: it investigates the language-related dilemmas a group of Malawian secondary teachers engaged with as they participated in an adapted LS where the content focus was the concept of similarity, and similar triangles. We pursue the following research questions:

RQ1. What, if any, language-related dilemmas emerged, and how did teachers talk about these as they planned and reflected their lessons on similarity through the LS cycle?

RQ2. What did the teachers enact and how did these language practices unfold through the LS and so from Teaching 1 to Teaching 2?

2. Malawian educational context, approach to mathematics, and location of 'similarity' in the curriculum

As in most low income countries, secondary school teachers in Malawian public schools work with large class sizes (40 students is the desired or official maximum, though typically classes are much larger than that) and limited material and curriculum resources (MoEST, 2020). Printed textbooks remain the readily available and used resource for the teachers (Mwadzaangati, 2016: Fauskanger, Helgevold, Kazima, & Jakobsen, 2022). Many languages are spoken in Malawi, with Chichewa being the national language, and spoken by the majority. Nevertheless, and as in some other postcolonial contexts, English is the official language, and consequently, the language of instruction in secondary school classrooms and in teacher education. This constellation of languages in use is different from countries like Germany or the United Kingdom, where the language of instruction is the language of the majority, and teachers are predominantly first language speakers of the language of instruction while some learners are not (e.g. Uribe & Prediger, 2021). In Malawi, teacher educators, teachers and learners speak diverse languages in their homes and outside the classroom, but work in English during instruction. Curriculum materials for secondary level mathematics are in English, including the syllabus and all textbooks. As we will see in the data below, the selected teacher discussions and their classroom teaching are in English, and we return to reflect on this language practice for work going forward.

The approach to secondary geometry is relatively formal and static in the Malawian secondary mathematics syllabus. Geometry (including transformations) is typically taught deductively, i.e. from definitions to illustrations, and then examples, their rehearsal and applications. Teaching is typically through 'direct' (whole class teacher-led) instruction. Teachers embrace values related to learner centred education and are concerned to connect with and involve learners through the lesson (Mwadzaangati, Adler, & Kazima, 2022). It is unfortunately beyond the scope of this paper to illuminate teachers' interpretations of learner centeredness in their materially constrained classrooms. What is important here is to point out that this rarely manifests in extended discussion and exploration of meanings learners bring and construct. Ultimately, the deductive approach to geometry functions to obscure well known student difficulties in geometry, and particularly working simultaneously and in connected ways with visual and verbal representations of geometric objects (e.g. Bansilal & Ubah, 2019).[3]

Congruent and similar triangle topics are both covered, though separately, in form 2 (grade 10). Congruent triangles come first, usually in the first term. Similar triangles are taught in term three in most schools, including the two schools that participated in the PD and LS. The syllabus presents the topics of transformations (taught in form 1, Grade 9), congruent triangles and similar triangles as independent topics, as do three out of the four textbooks that were analysed by the teachers and used for planning the lessons. Only one textbook linked congruent and similar triangles in terms of their geometric properties such as proportionality and using enlargement.

[3] Duval (2006) has been influential in bringing the difficulties of working both within and across what he calls semiotic registers – particularly the visual and verbal in geometry. We acknowledge this important work, used in studies referred, but have not drawn on his constructs or tools of analysis in this study

The other three textbooks move quickly into defining similar triangles as 'triangles with same shape', or 'triangles with same shape but different sizes'. As we will see, during the PD, the teachers decided to link properties of congruent (the known) and similar triangles (the new). We set out to work with this content selection, available resources and an appreciation of the teachers' situated expertise, interested to see how their teaching of similarity and similar triangles and the language practices entailed would unfold through a LS cycle, the dilemmas they faced, and what could be learned from this content and context specific investigation.

3. Literature Review

3.1. Lesson study as professional development context

LS teacher PD has its roots in Japan. Teachers identify a teaching and learning research problem and then work collaboratively to plan, observe, and reflect on their practice with guidance from a knowledgeable other (Ding et al., 2024; Fujii, 2014; 2016). Benefits of LS for teachers have included opportunities for teachers to learn to improve their lessons, learn new teaching pedagogies, deepen their subject matter knowledge, work collaboratively, and become self-reflective (Huang et al., 2019). LS also has improved students' learning and enhanced building of connections between research and practice (Lewis et al., 2009; Huang et al. 2019). These learning opportunities are provided through the lesson planning and post-lesson reflection discussion stages of the LS cycle, where collaborative comprehensive planning and in-depth analyses about the quality of teaching and learning is done (Lewis et al., 2009).

LS has been adopted and adapted in many countries for improving teaching quality and for connecting research and practice (Lewis et al., 2009; Huang & Shimizu, 2016). A recent systematic research review found, however, that "success" was "indecisive" in contexts outside of Japan (Ding et al., 2024, p. 95), notwithstanding ranging theoretical frameworks informing this research (Huang et al., 2019), and called for more research across cultural contexts. Recent research has also called for structuring LS with learning theories and specific mathematics topics, arguing that when teachers plan and reflect on mathematics lessons using theory, there has been improvement in their ability to analyse critical features of the learning object, and their sensitivity to students learning (Huang, et al., 2019), Moreover, topic-specific interventions have been argued as more effective for improving mathematics learning than topic-independent interventions (Erath et al., 2021). These findings have informed the design of our LS in geometry with selected secondary school teachers in Malawi and the value of our investigation into whether and how through attention to language practices in a LS, dilemmas emerge and then function as sources of praxis.

Several challenges of adapting LS in new contexts have been reported (e.g. Ebaeguin & Stephens, 2014), including in African countries (Fujii, 2014; 2016). Significant systemic constraints, particularly on teachers' time and thus challenges in enacting a typical LS cycle have been reported in South Africa (Jita, Maree, & Ndlalane, 2008; Sekao & Engelbrecht, 2021). At the lesson level, challenging cultural practices include teachers' unfamiliarity with planning lessons in relation to an identified research problem, and with mathematical problem solving. Suggestions for addressing the challenges include paying more attention in lesson planning to analysing tasks in curriculum materials and anticipating students' reasoning (Fujii, 2014; 2016), a challenging practice in contexts where problem-solving tasks are not common in the available curriculum materials, and teachers are new to LS (Adler et al., 2022). Closer to this study, drawing insights from the above, Adler & Alshwaikh (2019) working in South Africa, adapted to time constraints, familiarity and the desirability of structure in a few ways. The LS cycle sessions were held after school hours, once a week over a four-week period, and the cycle was extended to include a second revised version of the same lesson taught to a different class, both using a teaching framework as a (theoretical) guide. Our use of the MTF as a tool for lesson planning, observation and reflection has been informed by these studies and our accumulating experience.

Research reporting on adapted LS with geometry brings the issues of tasks and time simultaneously into focus. Moss, Hawes, Naqvi, & Caswell, (2015) adapted the Japanese LS model adding in four steps to the typical cycle of planning, teaching and reflecting: (1) PD workshops, (2) teachers designing and conducting task-based clinical interviews, (3) teachers and researchers co-designing and carrying out exploratory lessons and activities, and (4) the creation of resources for other educators. Each of these added considerable time through focused PD work that was integrated into an extended cycle. Moss et al. (2015) evidenced several areas of teacher learning. These included teachers' moving beyond static views and enactments towards dynamic orientations, reflecting these approaches to geometry and related tasks were a focus of the PD, with these forming the implicit theoretical framing for teaching geometry. The study thus exemplifies adaptations that expand the LS cycle to include theoretically-informed attention to mathematics content for teaching, and the design and use of related tasks with teachers (Huang, Barlow, & Prince, 2016; Prediger, 2022b). Another recent study and adaptation of LS with a geometry content focus further illuminates that bringing mathematical content into focus in LS research and practice where theoretical frameworks for mathematics teaching are at work is not trivial. Bahn and Winslow (2019) found challenges in how teachers used the language of their framework in their discussions during LS, with geometry in this case the context for the study rather than its theory-informed focus. Our own experience in cycle 1 attested to teachers' selective attention to aspects of the guiding framework in use (Adler et al., 2022).

Our review of LS above is relatively brief and selective, with attention to adaptations and geometry, and a function of LS being the context for our study, and not our direct research object. Nevertheless, the above insights from LS research are all significant in that they informed the design of our LS which is described in the methodology below, creating the opportunities for teachers to engage with language practices. We thus now move on to literature relevant to the teaching and learning of similarity, and language practices, the research foci of our study.

3.2. Teaching and learning similarity and similar triangles

Geometric similarity has long and widely been recognised as an important yet challenging concept to teach and to learn (e.g. Clark-Wilson & Hoyles, 2017; Cox, 2013; Friedlander, Lappan & Kazima, 1985; Moss et al., 2015; Seago et al., 2014). It is thus not surprising that Malawian teachers experienced difficulties in their teaching and their students' learning of similarity. Much of this cited research has explored and argued for the benefits of dynamic software for teaching and learning congruence and similarity through transformations. Seago et al. (2014) argue that when learners are equipped with a definition of similarity which is based on geometric transformations and helped to understand congruence as a special case of similarity, they are likely to develop a more robust conceptual understanding. A dynamic resource-based approach was not possible in our resource constrained context. Nevertheless, the underlying issues of definition, and the connections between multiple representations of geometric objects discussed in the literature are pertinent whatever the approach used. We focus on these as they illuminate initial conceptualizations of similarity and similar triangles and so the lesson parts we explore in this paper. We acknowledge there is a great deal more to teaching and learning similarity, and an expansive literature on this.

Similar and congruent figures have commonly been defined in textbooks (including in Malawi) as 'same shape, different size' and 'same shape, same size' respectively. These words, typically illustrated with diagrams, focus on appearance and are imprecise (Seago et al., 2013). Expressed as they are in everyday and school mathematics language, they could lead to faulty conceptions. Without variation in orientations, supported by knowledge of rotation or general transformations, learners may not know that figures can be similar or congruent when they are oriented differently, and they might face challenges in identifying their corresponding sides (Seago et al., 2013). Recent research in South Africa revealed this difficulty for pre-service secondary mathematics teachers (Bansilal & Ubah, 2019).

A more mathematically precise conceptualisation of congruency and similarity is as a numeric relationship between angles and length measures in two figures (Seago et al., 2013; Seago et al., 2014). Congruent figures are figures having *equal corresponding* angles and *equal corresponding sides*, while similar figures are *figures having equal corresponding angles with proportional corresponding length of sides*. Signalled in the highlighting of 'corresponding', is that the way similarity is talked about – the way words are used – matters. A less precise way of talking could assist with meaning-making, by relating to the appearance of the objects being referred to. Once orientation varies, however, it becomes necessary to identify which sides and angles are being referred to and so the importance now of the word 'corresponding' – and its meaning in mathematics is needed to progress in relation to both similarity and congruency.

The use of multiple representations, and we would add, their varying orientations, has been recommended across studies as a means for learners to understand geometric figures and their properties. Of course, in school geometry, visual representations are mainly plane figures (Moss et al., 2015; Seago et al., 2014). Consequently, teachers need to work with their learners to understand both visual representations and abstract properties of the figures, using both diagrammatic and real objects. Through bridging of a real object and diagramatic visual representations, learners' geometric mental image of a figure is enabled. But research shows that learners' misconceptions of geometric properties sometimes occur because they are not able to meaningfully interpret geometric figures (Seago et al., 2013; Bansilal & Ubah, 2019). This was also noticed by the Malawian LS teachers during cycle 1.

Relating the focused review above to the Malawian educational context, and our study, teachers were working within their curriculum and textbooks, and imprecise introductory definitions of, and a static approach to, similarity and similar triangles. This, together with their material conditions, opened up questions about the dilemmas that the teachers might confront as they worked to introduce their learners to this concept, and so as they (might) work across language use, and representational forms through participation in a LS cycle.

3.3. Language practices and representations in multilingual mathematics classrooms

Over two decades ago, Adler (1998) elaborated how mediation in multilingual classrooms created dilemmas for teachers. S/he drew on the notion of dilemmas of schooling in Berlak & Berlak (1981) and in mathematics teaching in Lampert (1985) to articulate situations in mathematics teaching in multilingual classrooms where there were competing demands on the teacher, creating tensions for in-the-moment actions where choices would need to be made between equally compelling needs or goals. Dilemmas arose from the complexity of teaching where there was simultaneous need for learners to access the language of instruction (in Malawi this would be English), the language of the classroom (ways of talking and engaging in school) and the specialised language of mathematics, or in Halliday's terms, the English mathematics register, where register is defined as:

> "a set of meanings that is appropriate to a particular function of language, together with the words and structures, which express these meanings. […] We can refer to a 'mathematics register', in the sense of the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself) and that a language must express if it is used for mathematical purposes. […]". (Halliday, 1975, p. 65)

The '*dilemma of code-switching' i.e. 'to switch or not to switch'* (Adler, 1998, p.26) captured the tension teachers expressed and enacted between the need to develop and so use and have students use spoken mathematical English in class vs. enabling meaning-making through use of the learners' main language(s). The *'dilemma of mediation'* captured the tension between 'validating pupil meanings vs. developing mathematical communicative competence' (p. 30) and so drawing on learners' meanings and their everyday experiences vs. providing explicit instruction towards scientific concepts. The third '*dilemma of transparency'* captured the tensions between language as object and language as means, and so the importance of the teacher modelling mathematical English in her talk, making mathematical language visible vs. talking too much. Drawing on Lave & Wenger (1991), Adler (1998) described this as the dilemma of "visibility vs. invisibility of language as a resource for learning" (pp. 31). The significance of the dilemmas for our study goes beyond their identification and naming. As Adler (1998) argued, these tensions in teaching, while expressed as binary opposites, are not either-ors in the complexity of classroom teaching. They are managed by teachers in their day-to-day practice, sometimes, but not always, with awareness. Consequently, they could be "sources of praxis, of working with, and possibly transcending, tensions in the dialectical teaching-learning process" (p. 32).

We have turned our attention to these language-related dilemmas, firstly, as recent research reveals that they persist (e.g. Barwell, 2010; Turner et al., 2019): they are not simply an outcome of context-specific research in a particular time and place. Turner et al. (2019) theorised orientations to language practices in mathematics teaching from their review of the field as ranging from lexical (focus on vocabulary), and register (focus on word *use* and ways of arguing) to sociocultural (involving communication). They used this framing with Adler's (1998); (2001) language of dilemmas to explore language practices of early career teachers (ECTs). They found that ECTs too faced dilemmas, naming these as between content and language (related to Adler's transparency), between meaning-making and participation in discussion (related to Adler's, mediation), and between different languages (related to Adler's code switching). The ECTs, in different ways, struggled with how to simultaneously *explore* specific content *and use* content specific vocabulary; and with whether and how to *model* mathematical ways of talking while engaging students in *inquiry* practices. When the teachers attended to vocabulary and precise ways of using mathematical language, they experienced dilemmas e.g. do you formally name an idea and then elaborate it, or introduce it more informally/experientially and build the lexicon through activity, only naming the idea after there has been some engagement with it. Turner et al. (2019) found "varying degrees" of evidence of attention to word meaning, multiple meanings and multiple languages. They called for research in teacher education and PD to go beyond "precision with language …" to include supporting teacher learning and practices "related to the mathematics register and situated sociocultural perspectives" (p. 22), and so to word use and communication in class.

In our field, it has long been recognized that language practices in mathematics teaching need to include both the communicative and epistemic function of language (e.g. Pimm, 1987). By epistemic function we mean both vocabulary work (word use) and explaining words and phrases, thus requiring teaching to create opportunities for students to talk about and express their developing mathematical thinking (e.g. Barwell, 2012; Moschkovich, 2015; Prediger, 2019). Such language practices enable learners' access to the mathematics register (Pimm, 1987), and ways of speaking, reading and writing mathematics, with increasing emphasis on mathematical reasoning and explanations (Moschkovich, 2010; Prediger, 2019). Accessing the mathematics register requires understanding the structures of everyday language and their relationship to the structures of specialised mathematical language (Wilkinson, 2018), raising the dilemma of mediation, and then transparency, given the role of the teacher and her mathematical talk (see also Khisty & Chval, 2002; Planas, 2020).

The issues of context and content emerge from this 'dilemma' focused review. It is useful to consider the teachers in focus in Adler's and Turner et al.'s studies. Both sets of teachers were working in 'new' settings: recently deracialised, and so multilingual classrooms in South Africa; and as beginning teachers in the USA. That dilemmas emerge is thus not surprising, and their identification and description important for the field. How teachers might engage with dilemmas in their teaching was not in focus in either study. In our study, the Malawian teachers, while experienced, were introduced to a new framework for working on their teaching, a framework that now asked them to think about and work on their language practices. It is this context of 'newness' as well as the specificity of the Malawian educational context described above, and that the teachers would then engage with these over a LS cycle, that suggested to us that dilemmas might arise for the teachers. Their engagement with these would be of interest, not only for their own praxis, but for the field more widely.

The content issue that emerges but is out of focus in research on language dilemmas, and in much of the research on language practices discussed above, is the representations that are being talked about (see also Poo & Venkat, 2021). Prediger's (e.g. 2019) design principles for language responsive mathematics teaching deliberately and explicitly connects these. The principles emerged from content-specific design research studies with a strong focus on explicit mediation of practice-based identification of categories of language responsive teaching (Prediger, 2019). The relevant principle here of *relating multiple languages, registers, and representations* combines three approaches: (a) connecting different languages (state languages such as German and Spanish or languages within countries); (b) connecting the everyday register, school academic register, and technical/formal register; and (c) connecting multiple

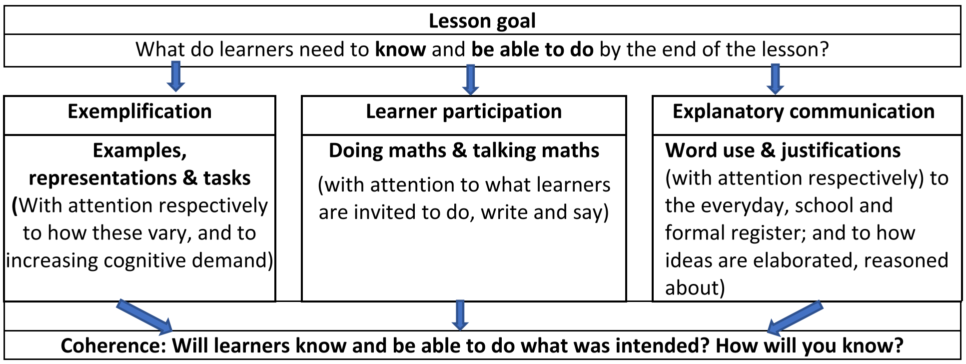


Fig. 1. The WMCS Mathematics Teaching Framework (Adapted from Adler & Alshwaikh, 2019).

representations (Prediger, 2022). We thus draw from Prediger's framing going forward that the issue on language is not only about different languages and registers, but also about how these are occurring in relation to representations. We now turn to our theoretical framing and the MTF framework that we adapted and used in our LS, to bring into sharper focus the key language practices of word use and justifications and their connections with representations for conceptualising our study.

4. Overarching theory and the Mathematics Teaching Framework (MTF)

The MTF in Fig. 1 below is underpinned by sociocultural theory and thus a view of teaching/learning as goal-directed, with mediation towards increasing sophistication in disciplinary knowledge (Vygotsky, 1978). This is an orientation that informs much of the language literature reviewed above. As those elaborating Vygotskian theory have argued, the subject-object relationship is mediated by both cultural tools (Kozulin, 2003) and human communication (Wertsch, 2007). In school mathematics with its abstract objects, there is extensive research on how and why curriculum materials and artifacts are infused with cultural meaning (e.g. Remillard & Kim, 2020). They do not speak for themselves - and are mediated in classroom communication and so through language in use. Moreover, mediation towards the more specialised ways of thinking that define disciplines requires movement towards scientific, rather than spontaneous, concepts (Kozulin, 2003). Of course, learners may generalise from their everyday experiences, but the spontaneous concepts formed could remain at the level of the "unsystematic, empirical and unconscious" (Karpov & Bransford, 1995, p. 61), and disconnected from organised and interconnected networks that constitute disciplinary knowledge. It is in this context that we understand why Seago et al. (2014) were concerned with a conceptualisation of similarity as "same shape, different size", as described and illustrated in Malawian textbooks. In addition, one cannot take for granted that learners will detect symbolic relations, nor diagrammatic representations of geometric ideas and concepts, no matter how obvious they might seem to the teacher, they require mediation. As Kozulin (2003) explains: "Symbolic tools (e.g. letters, codes, mathematical signs) have no meaning whatsoever outside the cultural convention that infuses them with meaning and purpose." (p. 26)

Critically, then, the way ideas are exemplified and represented, and the way words are used to communicate mathematical meaning, by both teachers or learners, requires deliberate planning, use and reflection so as to strengthen how these might function as mediational means.

The starting point of the MTF is that teaching is purposeful and always about something – an object or objects of learning which, in school mathematics lessons, is akin to the lesson goal. From a sociocultural perspective, this object requires mediation (Vygotsky, 1978), and two key teaching mediational means are exemplification and explanatory communication, each with respective elements (examples, tasks and representation; word use and justifications) (Adler & Alshwaikh, 2019). For 'something' to come to be known, it will need to be experienced and so exemplified in some form, and then elaborated. Exemplification will entail selection of examples, their representational form and associated tasks that can be used to bring the object into focus with learners. Their elaboration unfolds through explanatory communication in the classroom, where language use includes both how concepts and procedures are named (the words used) and how they are explained, and so justified. Learner participation with the mediational means through doing, writing and speaking is critical, and all these elements need to connect towards and cohere with the object of learning – the lesson goal. Our study focused on language practices in geometry teaching and learning, where representational forms are understood as interconnected with working between language registers.

In our PD it was important that we worked with the language practices we had observed to support a practice and a discourse that positioned the teacher's role as crucial in offering learners opportunities to hear and use the formal mathematics register, and in producing with their learners, mathematically robust explanations (answers to the question "why") for both concepts and procedures. Uribe and Prediger (2021) associate the technical or formal mathematics register with 'reporting' mathematics (the presentation of a solution to a problem), the school mathematics register with 'explaining meanings' and the everyday register with 'telling stories', all of which matter in the mediation process. In the MTF, we distinguish between the everyday register (or colloquial language) used in the home and in the street, the school mathematics register (the words we use in school to describe talk about mathematics) and the formal or technical register where mathematical ideas and relationships are symbolised and described, without association to communicational forms. In elaborating justifications we worked with teachers to move from stating what it is they want learners to know, to elaborating meanings by asking and emphasising 'why', invoking the dilemma of mediation – of moving between everyday meanings and the scientific. How we establish whether a solution or procedural step or reason given in mathematics is "valid or true" is critical to learning what it means to do mathematics.

Two points are important to make here. First is to emphasise 'moving between' the everyday and scientific. While not evident in the figure, in the PD our mediation of the MTF and its elements is that movement between registers and between everyday and scientific concepts is not unidirectional in mathematics teaching and learning, reflecting research findings of others (e.g. Zahner & Aquino-Sterling, 2022). There is likely a forward and back movement between these in classroom talk. The second is to note the recent work of Planas & Pimm (2024) where they interrogate the relationship between language and communication and introduce the notion of a mathematics communication register. The exploration of this, we suspect, will enable a fuller description of the school mathematics register as we have used it, but this lies beyond our scope here.

In the MTF, representations are distinguished by type (verbal, symbolic, diagramatic or graphic) and as we have moved into geometry, we have, through our interaction with the data distinguished them further as illustrated in the emerging analytic framework

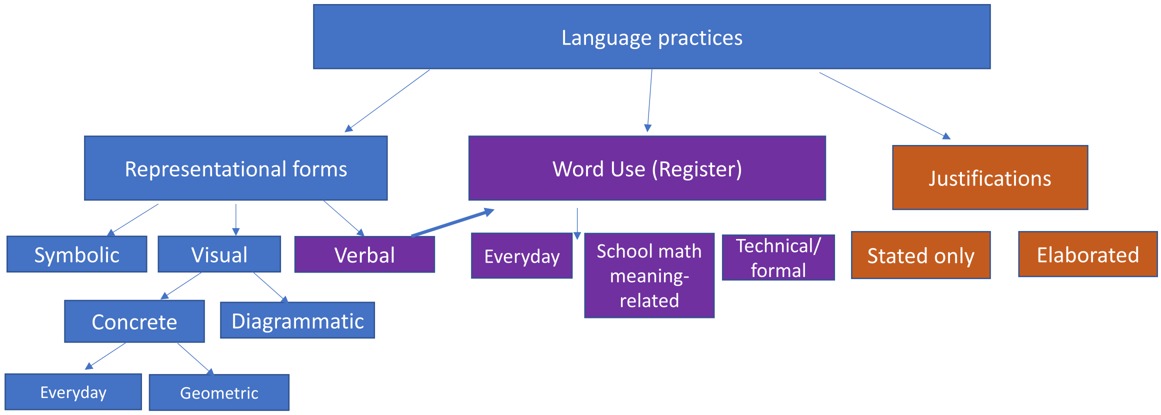


Fig. 2. Analytic framework for studying language practices in geometry.

for our language practices in Fig. 2 below.[4] Of course, how the elements in the framework cohere and connect in classroom practice matters, and this informs our analysis.

5. Methodology

5.1. Doing the professional development and lesson study cycle – the study context

Like others (e.g. Moss et al., 2015) we adapted LS by adding in PD sessions and a second teaching (Adler & Alshwaikh, 2019). Cycle 2 began with a two-day PD workshop with the two groups of teachers (five teachers from each school) at the university, with a focus on language practices. The two schools are national secondary schools which are characterised by high performing students, a relatively low teacher to student ratio of 1: 40 as recommended by the MoEST, with qualified and experienced teachers. All ten teachers were qualified to teach secondary mathematics and had some years of experience. Initial PD activities included attention to word use and justifying (explaining why). Being mindful that curriculum material analysis is important for linking the lesson study to specific learning goals in the curriculum (Moss et al., 2015), the language sessions were followed by analysing mathematics textbooks for key words, definitions, explanations and justifications used under a chosen topic. In the final session teachers began planning the first lesson (LP1). Each school identified a problem with one school selecting learners' challenges in defining and applying conditions of similar triangles (as identified in the literature) as the focus for their LS. The full first lesson plan (LP1) was completed by the teachers in their school, before Teaching 1. Reflection 1 and planning for the revised Teaching 2 (LP2) followed, and took place in the second PD session. Then came teaching 2, followed by reflection 2. One knowledgeable other (KO) (first author here) participated in all stages of each LS cycle and video recorded the sessions, except the completion of the planning of L1. We deliberately planned that teachers finalise their LP1 on their own without further input from the KO, so that we could learn from the teachers' situated knowledge and interpretations from the PD, including what was available in textbooks.

5.2. Researching languages practices - focus on RQs 1 and 2

In line with our sociocultural framing, and following Adler (2001), data included teachers' talking about and within their practice i.e. the lesson planning and reflection sessions and the two enacted lessons, all of which were video recorded and transcribed.

To identify and then explore the dilemmas that emerged during lesson planning and reflection discussions we conducted content analysis of the transcribed data. We read the data several times to identify firstly, instances if, and then where the teachers appeared to have different ideas about, or debated amongst themselves, an issue of teaching involving language. We then analysed the utterances that communicated these discussions and/or disagreements inferring and naming dilemmas related to word use, justifications and representations.[5] In the analysis of episodes in the lessons, we make connections between the dilemmas we identified, and apparent challenges in enactment.

To examine the language practices the teachers enacted and how these unfolded during the LS cycle, we focused on lesson episodes related to the specific dilemmas discussed, with these turning out to be concentrated in the first two episodes. The teachers' plans for lessons 1 and 2 both followed the MoEST lesson plan template; starting with introduction, followed by development and conclusion. We used this sequence to divide the lesson transcripts into episodes, and sub-episodes within episodes identified by a change in content

[4] This elaboration of the MTF does not do justice to extensive work in multiple representations in our field, and we discuss this in the limitations of our work, and signal it for further work going forward.

[5] There were dilemmas evident in the data about, for example, how time was used, a significant an ever-present dilemma of teaching (Adler, 2001). We have not included such in and of themselves, but when they interacted with language dilemmas.

focus, typically by the teacher bringing a new/different mathematical object/process into focus. In both lessons, though with differences within each episode as we will show, Episode 1 focused on recalling knowledge of congruent triangles. Episode 2 introduced the concept of similarity. This was followed in Episode 3 by measuring activities for establishing the relationship between sides and angles of similar triangles, with examples for applying the relationships in Episode 4. Episode 5 concluded the lesson. Notwithstanding the richness across the episodes in each lesson, we zoom in on episodes 1 and 2 where inter-related language dilemmas emerged as sources of action and reflection, through the LS cycle. Each (sub) episode was analysed to see what was said and what was done. To pursue our research questions (RQ), we were interested in whether and how word use, justifications and connections with representations unfolded, and so in the teachers' praxis in relation to the dilemmas we had inferred in their planning and/or reflective talk. Coding thus captured language registers (word use), justifications, and representations, and so the analytic categories indicated in Fig. 2.

We analysed whether words used were in everyday language, for example referring to the colour of objects and coded these **LE**; whether they were school mathematics language, for example, 'same shape, same size', and coded these **LSM**; or formal mathematical language, for example 'corresponding angles are equal' and use of mathematical vocabulary and coded these **LF**. For justifications, we examined if dialogic turns (typically student responses to teacher questions and teacher follow-on) were statements only and whether by student or the teacher (**JSs or JSt**), or these included some elaboration and so extended talk and coded these as (**JEs of JEt).** In relation to what was done, we identified and then coded different representations that were used. Here we distinguished first between whether these were symbolic **(RS)** or visual; and further whether the visual representations were concrete **(RC)** or diagrammatic **(RD),** and further still whether concrete objects were everyday **(RCE)** or geometric **(RCG).**

To distinguish the different mathematical registers in the text, *we italicised everyday and school mathematics language register*, and underlined the formal mathematics register. We further **bolded the justifications that were elaborated**. We illustrate the coding of the lesson episodes using part of lesson episode 2, from each of Teaching 1 and Teaching 2, where the notion of similar triangles is introduced, as shown in Table 1. Note that T stands for teacher, S for student, MS for many students.

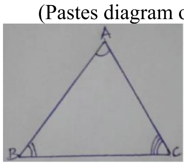
In Teaching 1, we coded lines 70, 71, 74 and 75 as LE and italicised them because the word similar and the definitions such as similar, same appearance and same colour are used in the context of everyday language. For lines 72 and 73, we coded LSM and italicised as the words same shape and same size are in common school mathematics register. Since the learners and teachers explanations are only words and statements, we coded these as JSs and JSt. In Teaching 2, we coded line 49 as LF and LSM because there is use of formal mathematical language 'two triangles where the corresponding angles are equal' as well as use of common school mathematical language 'but the sizes of the triangles are different'. We also underlined the formal mathematical register and italicised the school mathematics register. We coded the diagram RD because it is a Visual diagrammatic representation.

To ensure trustworthiness of the findings, both authors started by conducting independent analysis of the teachers' discussions to identify instances where the teachers debated/disagreed on a language practice issue and compared these. We did the same with the lesson episodes, each author coded the lesson episodes independently then compared the codes. Where we disagreed we discussed our codes, and refined these as appropriate to reach consistent agreement across the data.

6. Findings

In both lessons, the focused objects of learning were the meaning of similar triangles and the relationship between congruent triangles and similar triangles in terms of proportionality of the corresponding sides and equal corresponding angles. As part of planning for teaching 1, the teachers analysed the topic of similarity in the textbooks they used in their schools and that were made

Table 1
Part of each lesson episode 2 on introducing similar triangles.

Teaching 1	Teaching 2
70. T: *Okay, anyone who can define or explain what you know about similar? yes, thank you.* **LE** **71. S1:** *Same appearance.* **LE, JSs** **72. S2:** *Same shapes.* **LSM, JSs** **73. S3:** *Same size.* **LSM, JSs** **74. S4:** *Same colour.* **LE, JSs** **75. T:** *Same size, so things are similar if they have the same appearance, the same shape and the same size.* Do we agree with those three points? **LE, LSM, JSt**	**49. T:** Can we have two triangles where the corresponding angles are equal but *the sizes of the triangles are different*? Can that be possible? **LF, LSM** **50. MS:** Yes/No! **JSs** **51. T:** it cannot be possible? **52. MS**: Yes! **JSs** **53. T:** *What if we do something like this* (Pastes diagram on board).  **RD**

available in the PD. While only one textbook linked congruency with similarity, the teachers agreed that this connection was important for their learners and that they would start their lesson by checking learners' understanding of congruent triangles. They also noticed and agreed that focusing only on the abbreviated conditions for congruency, for example side side side (SSS) and side angle side (SAS), and not their meaning could become confusing, as similar abbreviations were used for similarity, indicating appreciation of the need for elaborating meanings as discussed in the PD. As they moved on to discuss introducing similarity, they identified key geometric words, and how they were defined, explained or justified in their textbooks. While most of these words were in the formal mathematics register (e.g. similar, triangle, proportional, corresponding and enlargement), they spent time discussing which school mathematics definition of similar triangles to use. One of the textbooks had 'triangles with same shape', while another had 'triangles with same shape but different size'. They decided the latter, saying it would be more helpful for learners. They also considered the visual representations used in the textbooks seeing these as supporting understanding of the definition of similar objects and as a lead to similar triangles which included pictures of similar houses and match boxes. The teachers did not raise any issues in their discussion of the everyday objects in the textbooks, but they agreed to bring their own everyday objects into the lesson.

In summary, on a first pass through the data, we noted that the main planning points included recalling congruent triangles and meaning of conditions of congruency, emphasising both shape and size when defining similarity and bringing everyday representations into the lesson. Our deeper analysis revealed the emergence of and engagement with inter-related dilemmas. We therefore begin by presenting findings on RQ1 which focuses on the language-related dilemmas that emerged in the teachers' planning and reflection discussions.

6.1. The emergence of the dilemma of transparency

As a reminder, this is a dilemma of the visibility/invisibility of language, including when and how to name or define a concept and so whether and how the meaning of an idea or concept could be explored through discussion before it was formally defined. In their planning, the teachers aimed to introduce the concept of similarity and to have learners involved in describing and giving examples of similar objects. As they discussed how to do this, a dilemma emerged of whether it was necessary to first define similarity as they would routinely do – for how do you talk about something if it hasn't yet been named and defined? Extract 1 from the LP1 planning session is part of the debate around this issue. Note that Tn stands for teacher number

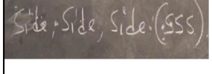
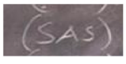
Extract 1: from Lesson planning 1

223 T4: … first [we] will be discussing examples of similar objects.

224 T5: Examples?

225 T4: Yes, examples of similar objects.

Table 2
Episode 1 - Recalling congruent triangles.

What was said by the teacher and learners (students)	**What was done – i.e. shown on the board**
Episode 1.1: Defining congruent triangles **9. T:** What do you know about <u>congruent triangles</u>? **10. S1:** These are *angles of the same shapes and size*. **11. T:** <u>Congruent triangles</u> are *angles of the same shapes and size*? **12. S2**: Are *triangles of the same shape and size*. **13. T**: What is wrong in the first definition? **14. MS**: Angles! **15. T**: <u>Angles, because when we are talking about congruent triangles,</u> we cannot say they are *angles of the same shape and size*. <u>**We talk about angles as amount of turn which is in degrees. So, triangles are said to be congruen**</u>**t if they have *same shape and size.*** **LSM, LF; JSs, JSt, JEt.**	T Wrote <u>"Similar Triangles"</u> on the board and then began the lesson with the first question.
Episode 1.2 Naming conditions of congruent triangles **16. T:** What are the <u>tests for congruency</u>? **17. S3**: *Side, Side, Side.* **18. T**: *Side, Side, Side,* yes what else? **19. S4**: *Side, Angle, Side* **20. T**: *Side, Angle, Side* **LF, LSM, JSs, JSt, RS**	He wrote the following on the board as he spoke:  

226 T3: How can we start discussing examples of similar triangles before defining similar objects?

227 T4: They should define using examples of similar objects that they give. We can say, these are examples of similar objects, so what do you think is the definition of similar objects? Students will answer after giving examples of similar objects and seeing some of their characteristics.

228 T5: For me I also think after seeing and identifying examples of similar objects then we will go to defining similar objects …

As extract 1 shows, the teachers began by debating how to discuss 'similar' objects, if these had not yet been defined (Line [L] 226). Following T3's question, T4 (L227) attempted to clarify suggesting that the learners 'give' objects, and the teacher names these as similar, and then asks for characteristics. T5's response was in agreement that after seeing examples of similar objects, they could be 'defined', though it was not clear here who would show the objects. The discussion continued and the dilemma of when to name and define 'similar' persisted …

275 T1: Ok, since we are saying if we start with defining similar objects then probably, we are a bit rushing … when we ask these kids to say give us examples of similar objects, how are they going to know that these objects are similar before we tell them the definition of similar objects?

278 T2: You just want to notice if they know the word similar and if they can define it mathematically. From their knowledge of things that are similar you want to [ask] them to define and use the word similarity mathematically.

T1 articulated the dilemma differently, suggesting that defining first as they typically do is 'a bit rushing', but questioned how learners could give examples of 'similar' objects without a definition. And T2's response focused in on finding out if learners knew what the word 'similar' meant (i.e. in everyday terms) and then moving to define it mathematically. There were no final resolutions in the discussion, suggesting these issues might emerge through the enactment, and so how the word 'similar' was introduced and then defined, and when and whether similar 'objects' were brought in for discussion and by whom. What was reflected in the discussion, nevertheless, was the teachers' expressed aim to get learners involved in naming and defining similar triangles and showing representations of similar triangles - different language practices from their routine deductive practice. So, what was enacted?

6.2. Language practices in Teaching 1 lesson enactment – the dilemma in practice?

Teaching 1 was taught by teacher 3. Tables 2 and 3 below present Episode 1 on recalling congruency and Episode 2 on introducing similar triangles respectively. We discuss each in turn. Again, the abbreviations used in the extracts were T: teacher, S: Student, MS: many students; *Italics for everyday and/or school mathematics word use;* <u>underline for formal word use</u> **and bold for elaborations.**

We have included Episode 1 even though it does not relate to the dilemma of whether and when to name or define. We have included it as it illustrates the language practices of meaning elaboration that were discussed during textbook analysis. The language registers in use were mainly the school mathematics register (LSM) with elaboration done by the teacher (JEt). There was some vocabulary in the formal mathematics register (LF) - for the words congruent, angles and triangles (L11, L14). The talk was accompanied by spoken words written on the board, with accompanying symbols for representing conditions of congruent triangles (RS) in sub-Episode 1.2. It is also useful to note that although the teacher asked questions in the formal register (LF), for example 'what do you know about congruent triangles' (L9), in sub-Episode 1.1 a student responded in LSM by defining them as 'angles with same shapes and size' (L10). A correction was needed and it was T who stated that describing congruency in terms of angles was not correct, and explained why by elaborating the meaning of an angle (L15). In sub-Episode 1.2, the talk is mainly in the formal and school mathematics registers (LF, LSM), naming the conditions for congruency using abbreviations, which T captured on the board in strings of symbols (L15-L20). We point to this as in the planning the teachers had agreed that referring to symbols only for congruency could become confusing, and that they would ask students to explain meanings for the abbreviations of the conditions, perhaps reflecting the challenge of shifting routine practices to elicit learner talk.

As T moved from eliciting learners' meanings for the word similar (2.1) to discussion of the identical and similar objects (2.2 and 2.3) and then defining similar objects, difficulties were apparent. These resonate with the dilemma that emerged as the teachers planned when and how to name and define similar objects. Just as he asked them what the word 'similar' meant to them (L35) so, after illustration with pairs of objects, he asked what he thought was the "same question" of the two plates: "are these similar?" (L77), and students could not agree. Of course we cannot claim that this was experienced in-the-moment as a dilemma for T, and we hold this question over to the reflection amongst the teachers. We focus first on the language practices enacted through this episode.

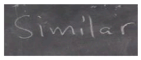
In episode 2.1, T started in everyday language (LE), by asking learners to 'define' or 'explain' the word 'similar' (35). Learners responded by stating associated words using both the school mathematics register (LSM), 'same shape and size', and the everyday register 'same appearance' (36−38). These were the same words associated with congruent triangles in Episode 1. T pointed this out and posed the question whether 'congruent' and 'similar' were 'connected'. In episode 2.2, T used two pairs of everyday objects (RCE) to represent and differentiate congruent and similar objects. After seeing the juice boxes, the learners stated the everyday properties that they noticed using the same school mathematics and everyday registers they used for the word 'similar' i.e. 'same shape, same size, same appearance', now adding other everyday properties: 'same name', 'same colours' and 'appearance' (39−42). T repeated and noted these and moved directly on to showing the circular plates, and learners stated these were different sizes, same shape and appearance (72, 76). In both sub-episodes 2.1 and 2.2, the justifications were predominantly statements only, without elaboration.

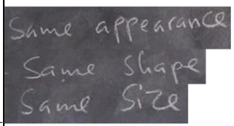
It was in sub-episode 2.3 that there was opportunity for some elaboration before T stated and wrote a definition for similar. In this episode, justifications for being similar or not similar were in the everyday language register and in school mathematics register 'they are similar because they have the same flowers', 'because they have the same shape and appearance', 'they do not have the same size' (80−86). The definition of similar objects given by the teacher 'objects with same shape' remained in the everyday and school mathematics registers (96). As explained earlier, during textbook analysis, the teachers agreed that this definition was insufficient and

Table 3
Episode 2 - Defining and showing similar objects.

<table>
<tr><th>What was said by the teacher and students</th><th>What was done</th></tr>
<tr><td>Episode 2.1 Introducing ‘similar’

(After summing up the definition of congruent triangles as having same shape and size and the conditions for congruency, T continued)
33. T: Ok let’s see now the topic of the day, <u>Similar Triangles</u>. … First we should dwell much on the word similar. So, have you ever heard about that word similar?
34. MS: Yes!
35. T: Okay, anyone who can define or explain what you know about similar, yes, thank you.
36. S1: Same appearance.
37. S2: Same shapes.
38. S3: Same size.
39. S4: Same colour.
40. T: Same size, so things are similar if they have the same appearance, the same shape and the same size. Do we agree with those three points?
41. MS: Yes
(T pointed out that these exact points were made for congruent triangles and then asked:)
42. T: Is there a connection between similar <u>and congruent</u> objects?
43. MS: Yes/no

LE, LSM, JSs</td><td>T wrote ‘similar’ on the board


T nominated students who raised their hands to respond, one at a time and wrote the words as they spoke

</td></tr>
<tr><td>Episode 2.2 Representing similar objects

44. T: Some are saying yes, some are saying no. I don’t have enough materials so I will just be demonstrating. I have these two objects, alright. What do we notice about these objects?
45. S1: Same shapes and size.
46. S2: Same colour.
47. S3: Same name.
48. S4: Same appearance.

…
71. T. Sorry it is not time for lunch [some laughter], but I have got these two objects which I believe you know them, what can you say about these two? Yes!</td><td>T showed two identical ‘enjoy’ box juices

MS offered other ideas and T drew their attention to the words on the board that they had offered, specifically appearance, shape and size

T put down the boxes and picked up two plates, and</td></tr>
</table>

(*continued on next page*)

that including size would be helpful for learners. Although the teacher did not include size in the definition, he elaborated the difference between congruent and similar objects by emphasising size as a differentiating property when clarifying students’ confusions (87).

In overview, we see that in Episode 2 everyday concrete objects were used to represent first same shape, appearance and size and then same shape, appearance and different size. Accompanying talk was predominantly in the everyday and school mathematics registers with only one extended justification by a student and a few by the teacher.

The summary in Table 4 shows that the language register shifted between school mathematics and formal mathematics during Episode 1 recalling of congruent triangles and these were with only some abbreviated mathematical symbols written on the board. Elaboration of meanings was done by the teacher. In episode 2, concrete everyday representations were used, and the accompanying register for defining similar objects remained in the everyday and school mathematics. There was no smooth path from revising congruence to asking for meanings for the word ‘similar’ and defining similar objects. T moved directly from his explanation at the end of Episode 2 onto Episode 3 and an activity that involved measurement of geometric diagrams – pairs of similar and congruent triangles - and so from opportunities opened for learners to use everyday and school mathematics language registers for talking about similar

Table 3 (*continued*)

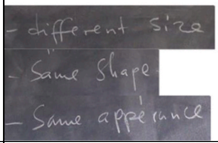
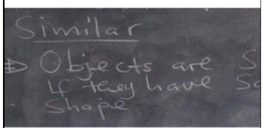
72. S: They are *different in size.* **73. T:** They are *different in size.* Anything else? **74. S:** They have the *same shape but different sizes.* **75. T:** *Same shape different sizes*. We are agreeing? Yes? Is that all? **76. S:** They have the *same appearance.* **LE, LSM, JSs, RCE**	wrote the words offered by students  
Episode 2.3 Elaboration of similar objects **77. T:** Now I am going to ask the same question: *Are these plates similar*? **78. MS:** Yes/No **79. T:** Can one stand and say *why these plates are similar*? **80. S1:** They are similar because they have the *same flowers*. **81. T:** They have the *same flowers*. **82. S2:** Because they have *the same shape and appearance*. **83. T:** They have *the same shape and appearance*. **84. S3:** They are *not similar* **85. T:** ***Why are you saying they are not similar*?** **86. S3: Because we are talking about *similar things, they have the same shape and size*. So, if you can look at those plates, *they do not have the same size.*** **87. T:** so S3 is saying ***these are not similar because we have defined though we haven't yet finally defined but because of the statements same size … these cannot be similar unless if they have same size. But many are saying these are similar.*** … **so, let's now define similar objects. Who can come up with a definition of similar objects basing on what we have been discussing?** [There is a lengthy discussion between T and various students some of whom continue to bring in same size, until T offers a definition himself] **96. T:** *Let's just say objects are similar if they have the same shape* [T then discussed the 'issue of size', and referring to the juice boxes and then holding the plates, he attempted to explain why congruent objects were also similar, finally asserting] **97. T: now when we are looking at similar objects *we should not only be limited to the size but the way the objects are looking. …So if they exactly look the same in everything, then you can say that the figures are <u>congruent</u>. But if they might appear the same despite having different sizes they are also similar.*** **LE, LSM, JSs, JEs, JEt, RCE**	Still holding the two plates   Holding the two plates? 

Table 4
Summary analysis of language practices Episode1 and 2.

Episode 1.1	**LSM, LF**	**JSs, JSt, JEt**	
Episode 1.2	**LSM, LF**	**JSs, JSt**	**RS**
Episode 2.1	**LE, LSM**	**JSs**	
Episode 2.2	**LE, LSM**	**JSs**	**RCE**
Episode 2.3	**LE, LSM**	**JSs, JEs, JEt**	**RCE**

concrete everyday objects, to working on geometric figures.

6.3. The re-emergence of language related dilemmas in reflection and lesson planning for Teaching 2

PD2 began with the teachers watching the video-recording of the lesson. They expressed appreciation of the overall lesson. With respect to Episodes 1 and 2 on recalling knowledge and introducing similar triangles, the teachers agreed that learners were able to identify and define concrete and similar objects. However, they viewed the move from defining similar objects as 'same shape' (Episode 2) to similar triangles (Episode 3) as 'sudden'. T3 shared the difficulty he faced of connecting congruent and similar objects, explaining that it was because this was "new to me". At the same time, he appreciated this approach learnt during the LS, but that it all took "too long". In the reflection, the KO re-emphasised issues from PD1 and the initial planning, that simply naming concepts and ideas was insufficient as learners needed opportunities to elaborate meanings and to link congruent and similar objects more explicitly. She also suggested that it might help in teaching 2 to use objects of the same kind to contrast congruency and similarity – for example, two small 'enjoy' boxes and one larger 'enjoy' box; showing the three together, asking what was the same and different about them. That dilemmas were lurking in the reflection became apparent as the teachers moved from their reflective comments to plan re-teaching the lesson.

As the teachers began planning teaching 2 (LP2), they discussed the KO's suggestion of working with three objects to contrast congruence and similarity but agreed quite quickly that they would start with recall of congruent triangles. They would, however, make sure conditions were elaborated. Only after that would they introduce similarity, and thus stayed within their routine practice of working with one idea at a time, and not simultaneously. At this point a similar discussion of whether learners could explore objects before defining 'similar' re-emerged. Now, however, discussion included how to represent the similar objects to be talked about, whether to start by showing everyday objects (as was illustrated in all the textbooks) or to start immediately with triangles, and thus an intertwining of the dilemmas of transparency (when to name; language as object vs language as means) and of mediation (moving between representations and specifically the everyday towards the geometric).

Extract 2: The dilemma of transparency intertwined with the dilemma of mediation.

53. T3: We show those that are congruent and debate on congruency … But the moment we want to start similarity we should bring what we are talking about.

54. T1: Ok so what should we ask them when we show the boxes that are different in terms of size?

55. T3: … When that debate on congruency is over now you go to similarity. So you can now say when we started [with two of the same boxes] you were saying this. But now what do you say about these two objects the bigger and the smaller one?

56. T5: Are the learners going to say that those two objects are congruent?

57. T3: That's what I said because when we are talking about congruency we just refer to triangles. Instead of bringing the objects why can't we try improvising triangles? Because there [i.e. objects] you can go and talk and it will come back to …

58. T5: … that they have the same shapes, same size, same name

59. T2: same height.

60. T4: … time consuming also, it will be time consuming.

61. T3: But still if they will say they have same shape, same size you can now say mathematically when objects have the same shape but different size, what do you call them? Maybe because the most important thing is for them to mention and understand these properties.

62. T2: So in answering that question, yes any objects can be brought but it will be convenient if we show them triangles. Remember we are also required to improve on time management.

63. T5: You are going to use similar and congruent triangles?

64. T2: Yes.

65. T1: Not objects?

66. T5: Objects?

67. T2: No problem even if they are triangular shapes or squares and the like. The objects can then come as a conclusion right?

68. T4: We can take the set square – the instrument of the chalkboard.

69. T3: We can use set squares on the board, but they should be similar, yes, its ok… we should look at them as triangles.

And a little later after further discussion of using the board instruments.

78. T1: OK but now which will come first? Bringing in the triangles or asking them if they have ever heard of the word similar, what does it mean and how do they understand it?

79. T2: They are going to mention similarity on their own after they have seen the materials.

80. T5. So we are starting with the triangles?

81. T2. Yes … 'enjoy' [juice boxes] will not be there. With time they can come in at the end, to show similarity is not only triangles. Similarity can be in bottles, enjoy, whatever.

The discussion continued, as not all the teachers were yet convinced with the move to using triangular chalkboard instruments instead of everyday objects to discuss similarity.

93. T3: I think we have now completely changed the approach … I started showing materials and now that step is completely cut off as we are now using the triangles … we are eliminating objects.

94. T2: Not necessarily … we have said knowing the concept of similarity through the triangles … those materials can also be shown in the end.

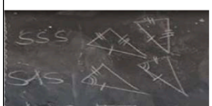
95. T4: …but I have already said the workload will increase.

96. T1: We will test it.

The three-parts of extract 2 show that the teachers were concerned with several inter-related issues. First is the concern with what to ask the learners when they are shown objects with different sizes (L54). Following T3’s suggestion that learners say what they notice, about the same and then the different sized objects, the dilemma re-emerges as T5 asks (again) whether the learners will say what is expected i.e. ‘congruent’ (L56). The dilemma of transparency, whether to define first or ask students for meanings first, is ever present. At this point, T3 initiates a significant move, given his own difficulties as he taught the lesson, asking “why can’t we try improvise triangles?” (L57). The discussion thus turns to what representations to use (L57), and so an intertwining of the dilemma of mediation – of moving between the everyday and the scientific with the added concerns that using everyday objects could also waste time as learners would again refer to appearances without mentioning ‘congruent’ (L58-61).

T2 (L62) picks up the suggestion ‘show them triangles’, and T4 extends this offering to taking ‘the set square – the instrument of the chalkboard’ and so a concrete object, albeit geometric (L68). T1 returns with a question of what ‘comes first’ (L78), attention to the meaning of ‘similar’ or showing objects, with others reinforcing starting with the triangles (L81). However, this ‘total’ change of ‘approach’ away from starting with everyday objects and learner notions remained a concern for T3, with some resolution brought by T1’s comment that they ‘will test it’ (L96) - a possibility precisely because of the LS process, and an appreciation that their teaching in the LS is for their learning.

Table 5
Episode 1. Recalling congruent triangles.

What was said	**What was done**
Episode 1.1:defining congruent triangles **T2** wrote ‘Triangles’ on the board, stating that they were continuing learning about triangles, and then asked: **6. T**: What geometric shapes are these? **7. MS:** Triangles. **8. T:** Triangles, these are triangles. Okay these triangles I will *re-orient them so that I put them something like fitting together like this* so that it looks like something one, *what do you call triangles that can fit each other like this*? **9. MS:** Congruent triangles. **10. T:** How do you define congruent triangles? **11. S1:** ***Congruent triangles are triangles which have exactly the same shape and size***. **12. T:** ***Congruent triangles are triangles with same shape and size.*** **LF, LSM, JSs, JSt, RCG**	He held up two same sized chalkboard set squares  
Episode 1.2 Naming and elaborating conditions of congruent triangles **13. T:** What are the conditions for congruency, anybody who remembers? **14. S2:** By using *Side Side Side*. **15. T:** By using *Side Side Side, what does Side Side Side mean?* **16. S3: All corresponding sides are equal to each another.** **17. T:** Yes, ***Side Side Side means that we have two triangles in which this side equal to that side…*** **18. T:** Another condition? **19. S4**: *Side, Angle, Side* **20. T**: ***We can have two triangles where this side is equal to that side, this angle equal to that angle and this side equal to that side.*** **LF, LSM, JSs, JEs, JEt, RS, RD**	He drew pairs of triangles and marked equal sides as these were discussed, with abbreviations alongside 

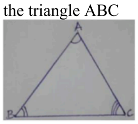
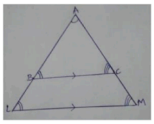
While we have extracted parts of a lengthy discussion amongst the teachers, Extract 2 illuminates that and how initial joint planning, followed by teaching 1 and lesson observation, and then reflection and replanning in the phases of LS provided the teachers with opportunities for engaging with language dilemmas, illustrating these as resources for praxis – for practicing, evaluating and improving on the language responsive teaching practices of naming, justifying and representing that they were introduced to during the PD.

Having illuminated the interconnected dilemmas of transparency and mediation in play, we now move to how the teachers' decisions to change representations for introducing similarity were enacted in teaching 2. We turn our attention again to the language practices in use, and whether and how, in relation to teaching 1, these consequently changed.

6.4. Language practices in Teaching 2 lesson enactment – intertwined dilemmas in practice

Teaching 2 was done by T2. Tables 5 and 6 present Episode 1 on recalling congruency and Episode 2 on introducing similar triangles respectively. Again we discuss each episode in turn, using the same abbreviations: T: teacher, S: student, MS: many students;

Table 6
Episode 2. Introducing similar triangles.

Episode 2.1 Illustrating equiangular triangles with geometric diagrams	
Having recalled congruent triangles and their conditions, T asked: **49. T:** Can we have two triangles where the corresponding angles are equal but *the sizes of the triangles are different*? Can that be possible? **50. MS:** Yes/No! **51. T:** It cannot be possible? **52. MS**: Yes! **53. T:** *what if we do something like this?* After drawing triangle ABC, he continued: **54. T:** So this triangle ABC, what will happen if we extend the sides AB with a continuous line and also extend side AC for them to have continuous straight lines and draw a line which is parallel to BC **55. T**: How many triangles are you able to see in that diagram? **56. S1:** One triangle **57. T:** One triangle, what is its name? **58. S1:** Triangle ABC. **59. T:** He sees one triangle which is triangle ABC, is there anyone else with a different idea? **60. S2:** Two triangles. **61. T:** Can you name them? **62. SS:** triangle ABC and triangle ALM. As we discuss below, seeing both triangles was not easy for some, and T spent some time on this, separating the triangles visually first, and then continuing returning to his focus.	After students' disagreement, as he spoke , T pasted a chart on the board and drew and labelled the triangle ABC  He extended the sides to form an enlarged triangle as shown in figure below and used it to do the proof. 
63. T: Alright, let's compare angles in these two triangles ABC and ALM, do we have an angle in ALM which is equal to an angle in triangle ALM? **64. MS:** Yes! **65. T:** Which angles? **66. S3:** Angle ABC is equal to angle ALM. **67. T:** Angle ABC is equal to angle ALM, why are these angles equal? **68. S4:** They are corresponding angles. … **LF, JSs, JEs, JEt, RD**	
Episode 2.2: Showing visual concrete geometric manipulatives to introduce similar triangles **139. T:** For *triangles like these which have correspondingly equal angles but the sizes are the same, how do we describe these triangles*? **140. MS:** Congruent triangles **141.T:** Congruent triangles not so? **142. MS:** Yes **143. T:** Okay, now how do we describe these two triangles, *are there any words that you can use to describe two triangles with same shape but different size?* **144. S4:** Similar triangles **145. T:** These are similar triangles now today we are going to look at similar triangles, triangles which have correspondingly equal angles *but the sizes are different.* **LSM, LF, JSs, JEt, RCG**	He showed the following geometric objects  

Italics for everyday and/or school mathematics word use; underlining formal word use **and bold for elaborations**

In episode 1 of recalling congruent triangles, while both the teacher and the students also talked in the school mathematics register (LSM e.g. L6-7; L12-15), there was additional formal register use particularly in stating that it was corresponding angles and sides that were equal (LF, L13). The definition of congruent triangles, as in teaching 1 remained in school mathematics register, here with the inclusion of 'exactly the same shape and size' (8). Also, the conditions of congruency were elaborated by both S3 and T (L13-L14). Students were not only naming the conditions for congruent triangles, they also elaborated their meanings in the formal mathematics register 'all corresponding sides are equal to each another' (17). The representations included concrete geometric objects (RCG) and geometric diagrams (RD). The questions T asked and the language registers elicited worked together with the visual representations in use to open up opportunities for discursive elaboration and possible meaning making. We note here, nevertheless, that as in L15, as T continued with the conditions for congruency it was him rather than the students that elaborated meanings –he explained meanings of the remaining conditions of congruent triangles, indicating, perhaps, that enabling students' elaborated talk would need more deliberate attention. Note that SS means same student.

In episode 2.1 of introducing of 'similar triangles' the register also moved between school mathematics (LSM) and formal mathematics register (LF). T asked whether it was possible to have two triangles where the corresponding angles are equal (LF) but the sizes of the triangles are different (LSM) register (61). He then introduced a diagram of a triangle (RVD) to support illustrating how it is possible to have two equiangular triangles with different sizes (71). He started with ABC then extended sides AB and AC to produce a second triangle ALM. Initially, the students could not see the two triangles and so he had to separate the complex diagram by drawing two different triangles, a small triangle labelled ABC and bigger triangle labelled ALM to help illustrate to the learners the two triangles (67−74). During illustration of the two triangles, he struggled to relate the sides of the smaller triangle ABC and those of the bigger triangle ALM to help the students understand the idea of corresponding sides of similar triangles. This is further evidence of dilemmas of how to move between representations (diagrams) and the registers (one, two triangles and similarity). After illustrating the two triangles, he moved back to the initial complex triangle containing triangles ABC and ALM to prove similarity (75−80). He asked learners to justify why some pairs of angles are equal (80) by stating reasons (JEt). He thus engaged learners in elaborated talk in school mathematics register and the formal words like corresponding angles come from a student and were then used by the teacher and other students.

After the illustration of equiangular diagrammatic objects, he showed the learners visual concrete geometric representations (RCG) of two congruent and two similar triangles for more elaboration on how it is possible for two triangles to have equal corresponding angles but have different sizes (139−143) and so enable a smoother link to introduce similar triangles. This again indicates the difficulty of the moving between representations and registers. He then proceeded to elaborating the meaning of similar triangles (JEt) in formal and school mathematics register as 'triangles which have correspondingly equal angles but the sizes are different' (145). Although the definition was drawn from the formal register in terms of shape as meaning equal angles, it remained with an implication that similar triangles cannot have equal corresponding sizes. We see that in episode 2, T introduced diagrams not objects to introduce similarity indicating again the difficulty of the move between representations and registers. However, as we will see later during teaching 2 reflection, the teachers agreed that this approach enabled a strong link between representations and introducing of the words 'similar triangle'.

We point out here, given the concern in the planning about removing everyday objects, that in then final episodes for teaching 2, and so after discussing how to apply the similar triangle properties to calculate sizes of angles and lengths of sides of triangles, T2 showed learners visual concrete representations (the three box juices) and they easily distinguished the different sized ones as similar. In this way T reinforced the meaning attached to similar by linking the scientific back to the everyday. Interestingly, at this point the language register also changed. Here, similar objects were not explained in terms of angles, but only in terms of the everyday and school mathematics aspects that were visible to the learners like shape and size. As the representations changed, the type of language register used also changed, indicating again how representations and language register used are intertwined. The summary of language registers enacted in both episodes 1 and 2 during Teaching 1 and 2 are presented in Table 4.

Table 7 shows that the language register moved between school mathematics and formal mathematics during both recalling of congruent triangles and introducing similar triangles in the second lesson. There were more opportunities for learners to elaborate meanings of conditions for congruent triangles during the discussion of the condition of AAA for similar triangles. The verbal and visual representations were connected in elaborating both congruent and similar triangles, suggesting that the representations used supported the learners' and the teacher's talk to move between the school mathematics and formal registers.

In the teaching 2 reflection, the teachers agreed that Teaching 2 was very successful as learners were involved in more language practices. They highlight learner involvement in talking mathematics as the main success of the lesson as well as the teacher's own ability to clarify the definition of similar triangles through the use of the triangular shapes and demanding for more explanation from

Table 7
Summary analysis of language practices enacted in both episodes during Teaching 1 and 2.

Episode	Teaching 1			Teaching 2		
Episode 1.1	**LSM, LF**	**JSs, JSt, JEt**		**LF, LSM**	**JSs, JSt**	**RCG**
Episode 1.2	**LSM, LF**	**JSs, JSt**	**RS**	**LF, LSM**	**JSs, JEs, JEt,**	**RS, RD**
Episode 2.1	**LE, LSM**	**JSs**		**LF**	**JSs, JEs, JEt**	**RD**
Episode 2.2	**LE, LSM**	**JSs**	**RCE**	**LF**	**JSs, JEs, JEt,**	**RD**
Episode 2.3	**LE, LSM**	**JSs, JEs, JEt**	**RCE**	**LSM, LF**	**JSs, JEt**	**RCG**

learners.

7. Discussion

We have analysed extracts from lesson planning, lessons and lesson reflections on recalling knowledge for teaching similarity and introducing and defining similar triangles to engage with two aims. Firstly, we explored language-related dilemmas that emerged when the teachers were planning and reflecting on lessons and consequently how they intended to act on these through the LS cycle. Of course, these dilemmas were interpreted by the researchers, not stated as such by the teachers. Secondly, we examined the language related practices that the teachers enacted and how these unfolded through the LS cycle from teaching 1to teaching 2. The findings showed that the teachers engaged with two dilemmas during the lesson planning and reflection sessions, confirming earlier research (Adler, 2001; Turner et al., 2019) and going further to illustrate how deeply inter-related these were. From the teachers' discussions during planning for Teaching 1, we identified and described the dilemma that emerged of whether to define 'similar' first (and so begin with the formal register) then elaborate its meaning through showing of objects (and so elaborate through the everyday and school mathematics registers), or to first ask learners to explore objects before they name and define 'similar' later. This is a dilemma of transparency as it relates to the visibility/invisibility of language, and of language as means and/or ends in mathematics education. This dilemma emerged in the context of 'new' practices for teachers and so a disturbance to their routine practice, which is to name and define similar triangles, then show how to prove and apply its properties, and finally give an exercise. These results are similar to Moss et al. (2015) who also reported that teachers were initially concerned about introducing activities and ideas without first naming and defining, for example, various figures in the tasks, a well-known language-related dilemma in mathematics teaching (Adler, 2001; Barwell, 2010; Planas, Adler & Mwadzaangati, 2023; Turner et al., 2019). Our findings confirm previous research that the teaching and learning of mathematics is challenging because among other reasons, it involves mediating complex relationships among linguistic, symbolic and visual forms of representing mathematical knowledge (Wilkinson, 2018). However, the teachers' challenges with these language practices through the stages of the LS cycle show that this particular language related dilemma is not easy to deal with and suggests the potential benefit of continuous PD in language responsive teaching practices (Moss et al., 2015; Prediger, 2019).

The second dilemma was the dilemma of mediating between everyday and the scientific meanings, and so between everyday and formal language registers in connection with multiple representations. We have shown that as the representations changed, so too did the language register in use. And conversely, as the language register in use changed, the representations also changed. In teaching 1 when only everyday representations were used, the discussion remained in the everyday and school mathematics language registers as these were the visual properties of the objects in focus. In contrast, in Teaching 2, when geometric concrete representations and geometric diagrams were used, the discussion moved between school mathematics and formal language registers. The corresponding sizes of angles being equal and the sizes of triangles being different were in focus. Ultimately connections were also made with the everyday through representations and language used, with these not being unidirectional i.e. from the everyday (Vygotsky & Cole, 1978; Zahner & Aquino-Sterling, 2022).

And so we illustrated that and how the nature of representations and the language register that was engaged were intertwined, and so too the dilemmas of transparency and mediation. The study thus confirms findings elsewhere that language responsive teaching entails practices of connecting representations with mathematics language registers (eg. Prediger, 2022; Wilkinson, 2018). What is added here is a different content and context of study – with its focus on registers and representations in relation to similarity on the one hand, and in the context of Malawian education on the other. Specifically, the study has made visible teachers' praxis through their LS and shown the forms the dilemmas take in a materially constrained context, and with a more traditional approach to geometry teaching on the other.

Notwithstanding opportunities for praxis, teachers learning language responsive teaching practices is no trivial affair, whatever the context. Our analysis of teachers' enactments from teaching 1 to teaching 2 brought to the fore evidence of different language practices unfolding in teaching 2 in terms of both the registers used, meaning elaboration and representations used. In teaching 1, the register only reached to formal mathematical naming in lesson episode 1 during recalling of congruent triangles. However, it remained in the everyday and school mathematics in episode 2 when introducing similar triangles and as we observed, the teachers attributed this challenge to their use of everyday objects to introduce similar triangles. In teaching 2, the teacher worked hard to provide an experience of moving from congruency to similarity in the school mathematics register and in formal mathematics language registers by using AAA as a counter example for conditions for congruency with the support of visual concrete geometric shapes, a confirmation about their learning of exemplification in cycle 1 of the LS (Mwadzaangaati, Takker, & Adler, 2022). As we observed during the lesson reflection, the teachers agreed that their use of concrete geometric objects and diagrammatic representation of similar triangles promoted shifting from everyday meaning of similar triangles to school mathematics and formal mathematics registers through meaning elaboration using the angles and the sides. This helped the learners to understand in the later stages of the lesson that the difference between similar and congruent triangles lies in proportionality of sides. However, the moves between diagrams and objects, notwithstanding the use of concrete triangles, to illustrate and elaborate similarity were not smooth. This illustrated and confirms the complexity of linking representations and language registers (Wilkinson, 2018). Furthermore, although the teacher pushed for more explanations and justifications in teaching 2, he still did much of the explaining of meanings – he did not create conditions/questions for more learner talk about the geometric concepts, confirming that designing tasks that can engage learners in rich mathematical talk is challenging for teachers (Moss et al., 2015; Prediger, 2019). This suggests that not only is further research needed in relation to the findings here, but more opportunities for supportive teacher learning in creating such tasks is needed.

It is important, before concluding, to return to the dilemma of code-switching and to confront how in our description and representation of this particular setting in Malawian teaching, code-switching was not visible, and the dilemma of code-switching unsurprisingly then not visible either. There was no visible use of Chichewa in either of the lessons, though it is possible that when learners talked amongst themselves, they used all their language resources. The teachers did on occasion 'chat' in Chichewa while planning and reflecting on their lessons, but not so in the extracts selected. We thus became aware that, in further studies, and given the power of English and its place in the curriculum, it will be important to work explicitly and visibly with the teachers and what might well be, but was not observed here, their translanguaging practices the related dilemmas that might arise, and how these are engaged. This is important as recent research in South Africa, albeit in elementary level contexts, points to the significance of the interaction across language registers and representations simultaneously with moving across different languages (Poo & Venkat, 2021; Robertson & Graven, 2020).

8. Limitations

As with any small qualitative study, there are limitations to the work reported here. First, as we have noted, while this was the second LS cycle for the teachers, this was the only one conducted with a focus on language practices. Empirically, moreover, we have selected episodes from lessons from this cycle, and focused only on the first two episodes where disagreement and debate was visible. Consequently, we are offering what is possible in these episodes but not beyond these. In addition, we have not attended to translanguaging even though the teachers did at times speak in the local language during the discussions. Thirdly, we deliberately did not attempt to introduce teachers to more open tasks, nor dynamic approaches, working instead within their curriculum constraints. The PDs were not long enough to work with them on new resource intensive approaches to linking similarity and transformation. We believe that it is important to expand teachers' knowledge and practice by introducing them to more tasks and opportunities for working with dynamic approaches to geometry, and figuring our what is enabled and constrained when moving them beyond their curriculum requirements, and significantly the forms and functions such moves might take and serve. This, together with translanguaging will inform further work up ahead, with the time and resource investment needed for LS that is expansive, and inclusive, and works from where teachers are and with their responsibilities.

9. Conclusion

We have shown through our analysis that language-related dilemmas did indeed arise as the teachers planned, taught, reflected on their teaching, and retaught a lesson on similarity and similar triangles. As they embraced new practices related to language use in their teaching, they discussed over time and through their lessons, how and when to introduce the notion of similarity, and what representations might support this process. There were tensions involved in when and how to name, in mathematical terms, what was intended to be learned; and in how to navigate between everyday and scientific meanings of similarity. We have thus illustrated that and how the lesson study cycle provided opportunities for teachers' praxis through engaging with the dilemmas of transparency and mediation, and so opportunities for their learning language responsive teaching practices. Of course, one LS cycle can only be illustrative of possibilities, and further research, through engagement with LS or other PD contexts is needed.

As the findings show, the teachers did engage with dilemmas, and these dilemmas were sources for them to work on their language practices which included naming, elaborating meaning, and work on representations. Through comparing the two lessons and analysing the plans and reflections we were able to see how LS as a professional practice opened up learning opportunities for the teachers to debate and enact language practices valued and emphasised in the PD. This cycle thus further confirms the productive role of theory driven LS. In the PD and through the LS cycle we attempted to provide a structure for language practice through attention to word use and justifications which involved probing different language registers and linking these to different representations.

We draw two conclusions. First, studying teaching through lesson study, even in a new context and where language practices are complex opens learning opportunities for teachers as they learn in situ. Second, the teachers had multiple things to do at same time as they learnt the language practices and doing LS. As such they still faced challenges in elaborating meanings and engaging leaners in extended mathematical talk, despite their appreciation of the importance of elaborating meanings as discussed in the PD. We recommend that this new practice is expected to extend and become part of what they do over time. Thus LS is not something like a short intervention to be expected to have an immediate impact, but a PD approach where teachers begin to reflect as professionals on what they do and improve gradually. LS is promising and thus a form of PD to develop over time.

CRediT authorship contribution statement

Lisnet Elizabeth Mwadzaangati: Writing – review & editing, Writing – original draft, Conceptualization. **Jill Adler:** Writing – original draft, Supervision, Conceptualization, Writing – review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

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Data Availability

The data are not publicly available due to restrictions.

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