# Emergent Spacetime

Kagiso Mathaba

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supervised by

Professor Robert De Mello Koch

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## Abstract

In this dissertation we explore the connection between entanglement and geometry. Recent work in the AdS/CFT correspondence has uncovered fascinating connections between quantum information and geometry, suggesting that entanglement in the CFT results in the emergence of spacetime in the bulk . We work in the $1/2$ BPS sector of the duality between $\mathcal{N} = 4$ super Yang Mills on $R \times S^3$ and IIB string theory on $AdS_5 \times S^5$. We aim to test this connection by calculating the Renyi entropies in the presence of $1/2$ BPS operators heavy enough to deform the background geometry. This allows us to calculate the entanglement of these operators via the *replica trick*. The Ryu-Takayanagi formula relates this calculation to a minimal surface in the dual supergravity geometry, thus allowing us to observe how the boundary entanglement affects the bulk spacetime. We build a formula to calculate correlation functions of $1/2$ BPS operators on the Riemann sheet that arises from the replica trick. This is a recursive formula based on group theory techniques. We demonstrate how the formula works for light operators and discuss how it can be generalised to include heavy operators by considering symmetric groups of higher order.

# Declaration

I declare that the work contained in this thesis is my own work, any work done previously by others or by myself has been acknowledged and put to references. This thesis is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg and it has not been submitted before in any other tertiary institution.

K.Mathaba

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Kagiso Mathaba.

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# Chapter 1

# Introduction

The goal of quantising gravity remains one of the greatest challenges in contemporary theoretical physics. The 20th century has seen the emergence of two frameworks that explain most of the observed universe. One of these is Einstein's theory of General Relativity (GR). GR provides a framework that explains the force of gravity and the structure of spacetime. The theory is very successful at large scales, however the theory breaks down for very strong gravitational fields arising from a large amount of energy amassed at a very small scales. This is the domain of Quantum Mechanics (QM). QM is the foundation of Quantum Field Theory (QFT) , a theory that successfully describes the electromagnetic, strong and weak interactions through the Standard Model of particle physics.

Applying the principles of QM to gravity results in a theory that is non-renormalizable. Consider a gravitational theory defined by the Einstein-Hilbert action in $D$-dimensions

$$S_{EH} = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} R. \tag{1.1}$$

The action is dimensionless, hence the coupling has dimension $[G_N] = 2 - D$. Therefore for $D > 2$, the theory has an irrelevant coupling. Thus while the theory is well behaved in the IR, it becomes ill-defined as we flow to the UV. The coupling can be written in terms of the Planck mass as $8\pi G_N = 1/M_{pl}$, where $\hbar = c = 1$. Thus a perturbative expansion of the theory is in terms of $M_{pl}^{-1}$, and hence only valid for

$$E^2 << M_{pl}. \tag{1.2}$$

This can be taken as an indication that some new physics comes into play as the energies approach the $M_{pl}$. This is reminiscent of the electroweak theory which was originally described by four fermion interactions. This theory was not renormalised and perturbation theory broke down at sufficiently high energies, or small length scales. At high energy scales the theory is not described by a quartic interaction amongst fermions but by a local gauge theory mediated by the $W^+$, $W^-$ and $Z$ bosons. Thus the four fermion interaction is an approximation of an interaction of fermions and $W^+$, $W^-$ and $Z$ particles.

A consistent theory of quantum gravity probably needs to properly account for new physics that occurs as we approach the Planck scale. String theory is one such framework that provides a consistent quantisation of gravity. Although much is still unknown about the theory, it has managed to produce some fascinating and insightful results. The most profound of which is the AdS/CFT correspondence proposed by Maldacena [1].

The AdS/CFT correspondence, is a concrete realisation of the holographic principle, the idea that the degrees of freedom of a given region of spacetime are not proportional to the volume of the region but the boundary of the region. The AdS/CFT correspondence states an equivalence between a String theory on a certain spacetime and a Conformal Field Theory (CFT) on a boundary of that spacetime. The most extensively studied instance of the correspondence is that between IIB string theory on a 10 dimensional $AdS_5 \times S^5$ spacetime and $\mathcal{N} = 4$ supersymmetric Yang Mills theory on a $3 + 1$ dimensional boundary of the spacetime.

The correspondence has been the subject of intense research in the past two decades. Despite this, many of the aspects of the correspondence remain unclear. The precise mechanism behind the correspondence remains a mystery. In particular we do not know how the 6 extra dimensions of the string theory are holographically reconstructed in the boundary CFT. The goal of the dissertation is to study how these extra dimensions and the local physics on these dimensions emerge from the strongly coupled limit of the CFT. Our motivation comes from the significant amount of evidence in recent years that the organisation of the quantum information in the boundary CFT is directly related to the geometric structure of the dual spacetime [2].

One of the biggest questions about the correspondence is what information on the boundary CFT encodes the geometry of a region in the bulk spacetime. The Bekenstein-Hawking formula [3, 4]

$$S = \frac{Area(\Sigma)}{4G_N^d}$$

relates the entropy of a black hole and the area of its horizon. It is one of the earliest connections between geometry and microscopic information. Extending this to holography in the context of the AdS/CFT correspondence, we expect to gain insight into the emergence of space and gravity in the bulk in terms of the information in the boundary CFT.

Rather than probe this connection with observable that are specific to a particular CFT, we choose the Entanglement Entropy (EE), a universal observable that can be defined for any Quantum system. The EE defines a recent entry in the AdS/CFT dictionary and will be the focus of our investigation as we exploit its holographic properties.

Our approach is to calculate the Entanglement Entropy on both sides of the correspondence. To calculate the entanglement entropy of a CFT in a given state $|\psi\rangle$, we divide the system into two part A and B. The state of A is defined by the reduced density matrix

$$\rho_A = Tr_B(|\psi\rangle \langle\psi|)$$

by tracing out the degrees of freedom of system B from the total density matrix $\rho$. The entanglement entropy of system A is defined as the von Neumann entropy of the reduced density matrix

$$S_A = -Tr\rho_A \log \rho_A.$$

The entanglement entropy gives a measure of how entangled system A is with system B. On the gravity side, the entanglement entropy of system A is given by the formula proposed by Ryu-Takayanagi [5, 6]

$$S_A = \frac{Area(\gamma_A)}{4G_N^{d+1}}$$

where $\gamma_A$ is a codimensional-2 minimal surface that extends into the bulk. The surface has the same boundary a system A.

We will be working in the 1/2 BPS sector of the $\mathcal{N} = 4$ super Yang Mills on $R \times S^3$ and its dual gravity description. The dynamics of this sector are reduced on the $S^3$ to a $0 + 1$ dimensional matrix quantum mechanics. The matrix model is known to be completely solvable [5], due to the fact that in the eigenvalue basis, the eigenvalues behave as fermions in a harmonic oscillator potential. The states in this sector can be characterised by the distribution of the fermions on a two dimensional phase space.

In the dual gravity description, each 1/2 BPS state on the boundary CFT corresponds to a 1/2 BPS IIB supergravity solution discovered by Lin, Lunin and Maldacena [7]. These so-called 1/2 BPS geometries have the same symmetry and supersymmetry as the boundary states. The geometries are characterised by a 2 dimensional plane, embedded in the 10 dimensional spacetime, that can be identified with the two dimensional phase space of the fermions. The whole family of these geometries can be constructed in terms of an auxiliary function $z$ that is determined by the fermion distribution.

The AdS/CFT correspondence is a strong/weak duality. This means when the CFT is weakly coupled allowing for calculation to be performed, the dual gravity description has a highly curved geometry that is difficult to treat. When the gauge theory is strongly coupled and cannot be treated pertubatively, in the gravity dual, the spacetime is nearly flat.

The $1/2$ BPS states we consider are useful since their conformal dimension is protected from quantum corrections. This means we can calculate these states at weak coupling and extrapolate these results to strong coupling with confidence. It is known that the energies in the gravity side are related to the conformal dimension in the gauge side [1]. For our purposes, we need states that have high enough energy in the bulk to backreact to new spacetime backgrounds. These states correspond to operators of large conformal dimension $\Delta$.

We calculate the correlators of the $1/2$ BPS states in order to determine the entanglement entropy in the CFT. We then determine the metric of the LLM geometry. With this, we will be able to gain insight into how the Entanglement Entropy of the boundary CFT is related to the geometry of the bulk.

The dissertation is structured as follows: In chapter 2 we focus on the $1/2$ BPS sector of the $\mathcal{N} = 4$ super Yang Mills and its matrix model description. We also introduce the group theory technology used in calculating the correlators in this sector. In chapter 3 we we look the LLM construction of $1/2$ BPS IIB supergravity backgrounds. In chapter 4, we look at Holographic Entanglement entropy. We study how the entanglement entropy of the CFT can be calculated using the replica trick. We also look at how the dual calculation can be done with the Ryu-Takayanagi formula.

# Chapter 2

# $\mathcal{N} = 4$ super Yang-Mills theory and Matrix Models

In this chapter we study the $1/2$ BPS sector of the $\mathcal{N} = 4$ super Yang-Mills theory. In this sector we find observables whose dimensions are protected by supersymmetry and are annihilated by half of the supercharges. By considering a reduction of the theory to a class of $1/2$ BPS states, we demonstrate how the dynamics of this sector can be described by a single matrix model. Motivated by this we will study a simplified matrix model that has the same basic mathematical structure as more complicated matrix field theories. We see a manifestation of the duality in the matrix model as we take the large N limit, since the correlators of the gauge theory have a dual description as string worldsheets. The matrix model correlators we consider are treated with group theory techniques. The rest of the chapter focuses on group representation theory.

## 2.1 $\mathcal{N} = 4$ SYM theory

The field content of the maximally supersymmetric field theory in 4d contains gauge bosons $A_\mu$, six massless real scalar fields $\phi^I$, $I = 1 \ldots 6$, four chiral fermions $\psi^a_\alpha$ and four anti-chiral fermions $\bar{\psi}_{\dot{\alpha} a}$, with $a = 1 \ldots 4$. The indices $\alpha, \dot{\alpha} = 1, 2$ are the spinor indices of the two independent $SU(2)$ algebras that make up the 4 dimensional Lorentz algebra. All the fields transform in the adjoint representation of the $U(N)$ gauge group. There is also a global $SU(4) \simeq SO(6)$ R-symmetry under which the scalars transform in the $\mathbf{6}$, $\psi^a_\alpha$ in the $\mathbf{4}$ (raised a index) and $\bar{\psi}_{\dot{\alpha} a}$ in the $\bar{\mathbf{4}}$ (lowered a index) representation of the R-symmetry algebra. The action in the Euclidean signature reads [8],

$$S = \frac{1}{g_{YM}^2}\int d^4x\, Tr\Big(\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{g_{YM}^2\theta}{8\pi^2}F^{\mu\nu}\tilde{F}_{\mu\nu} + D_\mu\phi^I D^\mu\phi^I + i\bar{\Psi}\Gamma^\mu D_\mu\Psi \\ - \frac{1}{2}[\Phi^I,\Phi^J][\Phi^I,\Phi^J] + i\bar{\Psi}\Gamma^I[\Phi^I,\Psi]\Big), \tag{2.1}$$

where $g_{YM}$ is the Yang-Mills coupling, $\Gamma^\mu$ and $\Gamma^I$ are the ten dimensional $16\times16$ Dirac matricies, $\theta$ is the instanton angle and the four fermions have been written as a single Majorana-Weyl fermions in ten dimensions.

The Lagrangian is scale invariant at the classical level since all terms are of dimension 4. This together with Poincare invariance forms full conformal symmetry with $SO(4,2) \simeq SU(2,2)$ group. Additionally, the combination of the conformal symmetry with the $\mathcal{N}=4$ Poincare supersymmetry results into an even larger superconformal symmetry given by the $PSU(2,2|4)$ group. This superconformal symmetry continues to hold at the quantum level. The theory is found to be UV finite, and the $\beta$-function is believed to vanish to all orders of pertubation theory [8, 9]. As a consequence the coupling constant $g_{YM}$ is a non-running parameter and can be fixed to a desired value. The theory is therefore specified by the value of the coupling $g_{YM}$ and the rank of the gauge group $N$.

## 2.2 The Superconformal Group $PSU(2,2|4)$

Special conformal transformation and Poincare supersymmetry generators do not commute. Their commutators give rise to new symmetry generators. The complete symmetry group is known as the $\mathcal{N}=4$ superconformal group and is given by the supergroup $PSU(2,2|4)$. In this section we give a short review of the group $PSU(2,2|4)$.

The superconformal group $PSU(2,2|4)$ contains the bosonic subgroup $SU(2,2)\times SU(4)$. The full global symmetry group of the $\mathcal{N}=4$ SYM is given by the conformal symmetry, R-symmetry, Poincare symmetry and the superconformal symmetry. The conformal algebra has 15 generators, 10 of which are in the Poincare algebra that contains 4 spacetime translations $P_\mu$ and 6 Lorentz transformations $M_{\mu\nu}$ which form the group $SO(1,3)\simeq SU(2)\times SU(2)$. We also have one generator of dilatations $D$ and four generators of special conformal transformations $K_\mu$. The generators obey the following algebra:

$$\begin{aligned}
&[D,P_\mu] = -iP_\mu, \qquad [D,M_{\mu\nu}] = 0, \qquad [D,K_\mu] = +iK_\mu,\\
&[M_{\mu\nu},P_\lambda] = -i(\eta_{\mu\lambda}P_\nu - \eta_{\lambda\nu}P_\mu), \qquad [M_{\mu\nu},K_\lambda] = -i(\eta_{\mu\lambda}K_\nu - \eta_{\lambda\nu}K_\mu),\\
&[P_\mu,K_\nu] = 2i(M_{\mu\nu} - \eta_{\mu\nu}D).
\end{aligned}$$

The R-symmetry is generated by $T^A$, where $A = 1, \ldots, 15$. This symmetry forms the $SO(6) \simeq SU(4)$ group. This is an internal symmetry that rotates the supercharges into one another. The R-symmetry is important since it corresponds to the $S^5$ isometry group in the dual gravitational theory, which will be important in the derivation of the LLM geometries. The R-symmetry commutes with the conformal symmetry.

The generators of the supersymmetry are the supercharges. There are 16 supercharges $Q_{\alpha\, a}$ and their conjugates $\tilde{Q}^a_{\dot\alpha}$, where $\alpha, \dot\alpha = 1, 2$ are the spinor indices and $a = 1, \ldots, 4$. Here the $a$ labels the Weyl fields. Subscript $a$ labels the components of spinors in the $\mathbf{4}$ and superscript $a$ labels the components of the spinors in the $\bar{\mathbf{4}}$ of the $R$-symmetry. The supersymmetry group does not commute with the entire conformal group. This leads to another symmetry called the superconformal symmetry. This is a graded Lie algebra that contains commuting and anti-commuting commutation relations of the supercharges and Poincare generators given by

$$\begin{aligned} \{Q_{\alpha\, a}, \tilde{Q}^a_{\dot\alpha}\} =& \gamma^{\mu}_{\alpha\,\dot\alpha}\delta^b_a P_\mu, \qquad \{Q_{\alpha\, a}, Q_{\alpha\, a},\} = \{\tilde{Q}^a_{\dot\alpha}, \tilde{Q}^a_{\dot\alpha}\} = 0, \\ & [P_\mu, Q_{\alpha\, a}] = [P_\mu, \tilde{Q}^a_{\dot\alpha}] = 0, \\ & [M^{\mu\nu}, Q_{\alpha\, a}] = i\gamma^{\mu\nu}_{\alpha\beta}\epsilon^{\beta\gamma}Q_{\gamma\, a}, \qquad [M^{\mu\nu}, \tilde{Q}^a_{\dot\alpha}] = i\gamma^{\mu\nu}_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\tilde{Q}^a_{\dot\gamma}, \end{aligned} \tag{2.2}$$

where $\gamma^{\mu\nu}_{\alpha\beta} = \gamma^{[\mu}_{\alpha\beta}\gamma^{\nu]}_{\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta}$. The dimension of the supercharges is $\Delta = 1/2$ and their commutator with the dilatation operator gives

$$[D, Q_{\alpha\, a}] = -\frac{i}{2}Q_{\alpha\, a}, \qquad\qquad [D, \tilde{Q}^a_{\dot\alpha}] - \frac{i}{2}\tilde{Q}^a_{\dot\alpha}. \tag{2.3}$$

The supercharges do not commute with the special conformal generator. As a result, a new set of supercharges arise from the commutators

$$[K^\mu, Q_{\alpha\, a}] = \gamma^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\tilde{S}_{\dot\beta a}, \qquad [K^\mu, \tilde{Q}^a_{\dot\alpha}] = \gamma^\mu_{\alpha\dot\alpha}\epsilon^{\alpha\beta}S^a_\beta.$$

These are the superconformal charges. The superconformal charges transform differently under the R-symmetry representation from the supercharges, and thus combine for a total of 32 supersymmetry generators. The superconformal charges have dimension $\Delta = -1/2$ and have the following anti-commuting relations

$$\begin{aligned} &\{S^a_\alpha, \tilde{S}_{\dot\alpha b}\} = \gamma^\mu_{\alpha\dot\alpha}\delta^a_b K_\mu, \qquad\qquad \{S^a_\alpha, S^a_b\} = \{\tilde{S}_{\dot\beta a}, \tilde{S}_{\dot\beta b}\} = 0, \\ &[K_\mu, S^a_\alpha] = [K_\mu, \tilde{S}_{\dot\beta a}] = 0. \end{aligned} \tag{2.4}$$

The algebra is completed by the anti-commuting relations between the superconformal charges and the supercharges

$$\begin{aligned}
&\{Q_{\alpha\, a}, S^b_\beta\} = -i\epsilon_{\alpha\beta}\sigma^{IJb}{}_a R_{IJ} + \gamma^{\mu\nu}_{\alpha\beta}\delta^b_a M_{\mu\nu} - \frac{1}{2}\epsilon_{\alpha\beta}\delta^b_a D,\\
&\{\tilde{Q}^a_{\dot\alpha}, \tilde{S}_{\dot\beta b}\} = -i\epsilon_{\dot\alpha\dot\beta}\sigma^{IJb}{}_a R_{IJ} + \gamma^{\mu\nu}_{\dot\alpha\dot\beta}\delta^b_a M_{\mu\nu} - \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\delta^b_a D,\\
&\{Q_{\alpha\, a}, \tilde{S}_{\dot\beta b}\} = \{\tilde{Q}^a_{\dot\alpha}, S^b_\beta\} = 0,
\end{aligned} \tag{2.5}$$

where $R_{IJ}$ is the $SU(4) \simeq SO(6)$ R-symmetry generators with $I, J = 1, \ldots, 6$. For more details see [4].

Let $\mathcal{O}(x)$ be a local operator in the field theory. The operators transform under scale transformations generated by the dilatation operator D. The action of the dilatation operator yields

$$[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0) \tag{2.6}$$

where $\Delta$ is the eigenvalue of the dilatation operator, called the conformal dimension. The conformal dimension can also be obtained from the two-point function

$$\langle\mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{x^{2\Delta}} \tag{2.7}$$

The special conformal generator $K_\mu$ lowers the conformal dimension of the operator $\mathcal{O}(0)$ by 1, which can be seen by acting on $[K_\mu, \mathcal{O}(0)]$ with the dilatation operator D

$$\begin{aligned}
&[D, [K_\mu, \mathcal{O}(0)]]\\
&= [[D, K_\mu], \mathcal{O}(0)] + [K_\mu, [D, \mathcal{O}(0)]]\\
&= i[K_\mu, \mathcal{O}(0)] - i\Delta[K_\mu, \mathcal{O}(0)]\\
&= -i(\Delta - 1)[K_\mu, \mathcal{O}(0)].
\end{aligned} \tag{2.8}$$

A unitary representation requires that the conformal dimension be positive. Thus by repeatedly acting with $K_\mu$ on the operator $\mathcal{O}(0)$, we must find an operator of lowest dimension $\tilde{\mathcal{O}}(0)$. This operator satisfies the condition

$$[K_\mu, \tilde{\mathcal{O}}(0)] = 0. \tag{2.9}$$

This defines a conformal primary operator[1]. By acting on the primary with the translation operator $P_\mu$, we create operators of higher dimension called descendants of the primary. The primary operator together with all of its descendants determine an irreducible representation of the conformal group.

[1]This is defined at $x = 0$.

Since the superconformal charges have dimension $-1/2$, analogous to the special conformal generator $K_\mu$, acting with these charges creates operators with a lower dimension $\Delta$. An operator of lowest dimension must still exist and satisfy

$$[S^a_\alpha, \tilde{\mathcal{O}}(0)] = [\tilde{S}_{\dot{\alpha}a}, \tilde{\mathcal{O}}(0)] \tag{2.10}$$

This defines a superconformal primary operator (SPO). It is clear from the anti-commuting relations (2.5), that this operator is also a primary. The SPO and it's descendant make up an infinite dimensional irreducible represention of the $PSU(2,2|4)$ superalgebra. The representation is necessarily infinite dimensional as a consequence of the fact that $PSU(2,2|4)$ is a non-compact group. This is demonstrated by the fact that we can act on the SPO an infinite number times with the $P_\mu$ and $Q^a_\alpha$, to create operators of higher dimension. The SPO has the highest weight in this representation.

The representations are labelled by the quantum numbers of the bosonic subgroup $SO(3,1) \times SO(1,1) \times SU(4)$ of the superconformal group. The labels for the Lorentz group are the pair of the integers $(s_+, s_-)$, the conformal dimension labels the $SO(1,1)$ and the Dynkin numbers $[r_1, r_2, r_3]$ determine the R-charge.

We are particularly interested in a shortened representation of $PSU(2,2|4)$ referred to as the 1/2 BPS multiplet. This reduced representation is only possible if the primary operator $\tilde{\mathcal{O}}(0)$ commutes with some of the supercharges $Q^a_\alpha$

$$[Q^a_\alpha, \tilde{\mathcal{O}}(0)] = 0 \qquad \text{for some } \alpha, a. \tag{2.11}$$

The operators are called chiral primaries.

The conformal dimension of the chiral primaries is uniquely determined by their R-symmetry charge. This can be shown as described in [10], by considering the anti-commuting relations (2.5). Indeed

$$[\{Q_{\alpha\, a}, S^b_\alpha\}, \tilde{\mathcal{O}}(0)] = [-i\epsilon_{\alpha\beta}\sigma^{IJ\,a}{}_b R_{IJ} - \epsilon_{\alpha\beta}\delta^a_b D + \sigma^{\mu\nu}_{\alpha\beta}\delta^b_a M_{\mu\nu}, \tilde{\mathcal{O}}(0)] = 0.$$

We assume the chiral primaries are scalars and hence have vanishing Lorentz numbers. We are left with the simple relation between the conformal dimension and the R-charge

$$\sigma^{IJ\,a}{}_b[R_{IJ}, \tilde{\mathcal{O}}(0)] = \Delta\delta^b_a\tilde{\mathcal{O}}(0). \tag{2.12}$$

The generators belonging to the Cartan subalgebra of the $SU(4)$ Lie algebra $\sigma^{IJ\,a}{}_b$ can be chosen as [10],

$$\sigma^{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \sigma^{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \sigma^{56} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.13}$$

SO(6) is a rank 3 group and thus has 3 commuting generators in its Cartan subalgebra. The generators can be choosen to be $R_{12}, R_{34}, R_{56}$ and the charges a $(J_1, J_2, J_3)$. Thus for a conformal primary with R-charge $(J_1, 0, 0)$ is annihilated by the charges $Q_{\alpha 1}$ and $Q_{\alpha 2}$ for $\Delta = J_2$. It is clear from (1.3) that the operator will also be annihilated by $\tilde{Q}^3_{\dot{\alpha}}$ and $\tilde{Q}^4_{\dot{\alpha}}$. Therefore this operator will be annihilated by half of the supercharges. These arguments are valid for a chiral primary with $(0, J_2, 0)$ and $\Delta = J_2$. But this operator would be in the same $SO(6)$ representation as $(J_1, 0, 0)$, and hence the $PSU(2,2|4)$ representation. Thus we see that the 1/2 BPS multiplets are specified by the operators with R-charge $(J, 0, 0)$ and conformal dimension $\Delta = J$.

In general the conformal dimension depends on the coupling $\lambda$ , $\Delta = \Delta_0 + \gamma(\lambda)$, where $\Delta_0$ is the classical (or engineering) dimension and $\gamma$ is the anomalous dimension. Chiral primaries commute with half the supercharges independent of the coupling $\lambda$. The anomalous dimension is a continuous parameter that cannot be changed continuously by the charges of the $PSU(2,2|4)$ group. These can only produce discrete changes in the dimension $\Delta$. Since the $R$-charges are discrete and by (2.12) the $R$-charge equals the conformal dimension, these operators are protected from quantum corrections [8, 10].

These protected operators are very useful in testing the AdS/CFT correspondence, since when we consider the 1/2 BPS geometries in the supergravity regime in the string theory, we have a strongly coupled regime in the gauge theory, for which quantum corrections cannot be calculated using perturbation theory.

A general chiral primary is given by the single trace operator

$$\mathcal{O}_\Delta(x) = C_{I_1 \ldots I_\Delta} Tr(\phi^{I_1} \ldots \phi^{I_\Delta}) \tag{2.14}$$

where $C_{I_1 \ldots I_\Delta}$ is an SO(6) symmetric tensor. Using the complex basis $X = \phi^1 + i\phi^2, Y = \phi^3 + i\phi^4$ and $Z = \phi^5 + i\phi^6$, a general 1/2 BPS chiral primary can be written as the multi-trace operator

$$\mathcal{O} \sim \prod_{a=1}^{k} Tr Z^{n_a} \tag{2.15}$$

where $n_a$ is bounded by the rank of the Hermitian matrices N.

## 2.3 Matrix Model description of the 1/2 BPS sector

In this section we review how the dynamics of the 1/2 BPS sector of $\mathcal{N} = 4$ SYM can be described by a gauged matrix quantum mechanics as described in [11, 12]. The action of the

$\mathcal{N}=4$ SYM compactified on $R \times S^3$, reads as:

$$S = \frac{2}{g_{YM}^2}\int d^4x\sqrt{|g|}Tr\bigg( -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - D^\mu\phi_i D_\mu\phi_i - \frac{R}{12}\phi_i^2 + \frac{1}{4}[\phi_i,\phi_j]^2 - 2i\lambda_A^\dagger\sigma^\mu D_\mu\lambda^A + i\rho_i^{AB}\lambda_A^\dagger\sigma^2[\phi_i,\lambda_B^*] - i(\rho_i^\dagger)_{AB}(\lambda^A)^T\sigma^2[\phi_i,\lambda^B]\bigg) \tag{2.16}$$

where $x^\mu = (t,\theta,\psi,\chi)$ with $\mu,\nu = 0,\ldots,3$ and $i,j=4,\ldots,9$.

To obtain the $0+1$ dimensional matrix model, we decompose the field $\phi^i$ into it's Kaluza-Klein (KK) modes on the $S^3$

$$\phi_i(x) = \sum_{\vec{l}=0}^{\infty}\sum_{\vec{m}=1}^{(l+1)^1}\phi_i^{\vec{l}\vec{m}}(t)Y_{(0)}^{\vec{l}\vec{m}}(x) \tag{2.17}$$

where $Y_0^{\vec{l}\vec{m}}$ is the scalar spherical harmonic on $S^3$. Since fields on a circle recieve a mass contribution from the momentum modes on that circle, the KK modes of the fields $\phi_i$ recieve a mass from this decomposition. The mass is given by

$$\Delta = l+1 \qquad \text{with} \qquad l=0,1,\ldots$$

Only the lowest mode in the expansion satisfies the 1/2 BPS condition $\Delta = J = 1$ [2]. Therefore we consider only the zero modes of the field

$$\phi_i(x) \to \phi_i(t) \tag{2.18}$$

with the action reducing to just the scalar part

$$S = \frac{1}{2g_{YM}^2}\int dt\, \mathsf{Tr}[(D_t\phi_i)^2 - \phi_i^2] \tag{2.19}$$

We build the local operators from the single complex scalar $Z = \phi^5 + i\phi^6$. We then find the action of a single complex matrix model [11, 13]. The action of the matrix quantum mechanics reduces to

$$\begin{aligned} S &= \frac{1}{2}\int dt\, \mathsf{Tr}\Big((\dot\phi_5)^2 + (\dot\phi_6)^2 - (\phi_5)^2 - (\phi_6)^2\Big) \\ &= \frac{1}{2}\int dt\, \mathsf{Tr}\Big(|\dot Z|^2 - |Z|^2\Big). \end{aligned} \tag{2.20}$$

[2]The fields $\phi_i$ have dimension $\Delta = 1$ in $d=4$ dimensions

The Gauss law amounts to projecting onto the U(N) singlet states. Choosing the gauge $A_t = 0$ and have a Gauss law constraint on the gauge singlet states $\frac{\delta \mathcal{L}}{\delta A_0}|\psi\rangle = 0$.

We define the conjugate momenta of the adjoint scalars as $P_5$ and $P_6$. The conjugate momentum $\Pi$ of the field $Z$ is defined as

$$\Pi = -i\frac{\partial}{\partial Z^\dagger} = \frac{1}{\sqrt{2}}(P_5 + iP_6) \tag{2.21}$$

We can define two sets of creation and annihilation operators $A, A^\dagger$ and $B, B^\dagger$ as

$$A^\dagger = \frac{1}{2}(Z^\dagger - i\Pi^\dagger) \qquad B^\dagger = \frac{1}{2}(Z^\dagger + i\Pi^\dagger) \tag{2.22}$$

With these operator we can express the Hamiltonian of the reduced Hilbert space and the U(1) charge J as

$$H = Tr(A^\dagger A + B^\dagger B) \qquad J = Tr(A^\dagger A - B^\dagger B) \tag{2.23}$$

The 1/2 BPS sector is defined by $H = J$. It is therefore clear that we only keep one set of operators $A, A^\dagger$. In this way we get a one dimensional harmonic oscilator Hilbert space, with a two dimensional phase space. We look for states of the form [11],

$$|\psi\rangle = \prod_{a=1}^{k} \mathrm{Tr}(A^\dagger)^{n_a} |0\rangle \tag{2.24}$$

which are invariant under unitary transformations of Z. We can write the field as $Z = U^\dagger(z+T)U$, where $z$ is the diagonal part and $T$ is an upper triangle matrix. Integrating out $U$, the wave function take the form

$$|\psi\rangle \sim \Delta(z)\prod_{a=1}^{k}\left(\sum_i z_i^{n_a}\right) e^{-1/2\sum_i z_i\bar{z}_i - 1/2\mathrm{Tr}(TT^\dagger)} \tag{2.25}$$

where $z_i$ are the eigenvalues and

$$\Delta[z] = \prod_{i<j}(z_i - z_j) \tag{2.26}$$

is the Vandermonde determinant. For more details see [14, 15].

The determinant has total anti-antisymmetry in the eigenvalues $z_i, z_j$. Therefore the eigenvalues behave as fermions. We see then that the 1/2 BPS sector can be described by a state of N fermions in a harmonic oscillator potential. The state of the system can be specified by the configuration of the fermions in the two dimensional phase space. This phase space configuration is of critical importance when we look for the supergravity solutions in the string theory side.

## 2.4 Matrix Models

In this section we motivate the idea that the string theory can have dual description as a non-abelian gauge theory with a large number of colours (large N). This idea can be understood from 't Hooft's insight [16], that Feynman diagrams of the $U(N)$ gauge theory can be classified according to their topology. This insight gives an intuitive idea of the correspondence, as the perturbative expansion on the gauge theory can be interpreted as the genus expansion in the string theory.

We consider a $0$-dimensional $U(N)$ gauge theory, a so-called matrix model of hermitian $N \times N$ matrices. The matrices $M$ transform in the adjoint representation $M \to UMU^{\dagger}$. The gauge invariant action of the theory is given by

$$S = \omega Tr(M^2), \tag{2.27}$$

The $n$-point correlation function of operators $\mathcal{O}_i$ is defined by

$$\langle \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \rangle = \frac{\int [DM]\, e^{-S} \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n}{\int [DM]\, e^{-S}} = \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_1} \dots \frac{\delta}{\delta J_1} Z[J] \tag{2.28}$$

with the partition function is given by

$$Z = \mathcal{N} \int [dM] e^{-S + \mathrm{Tr}(JM)}. \tag{2.29}$$

The measure $[dM]$ is defined as

$$[dM] = \prod_{i=1}^{N} \prod_{i<j}^{N} dM_{ii} dReM_{ij} dImM_{ij}. \tag{2.30}$$

Correlation functions for the matrix theory are given by

$$\langle M_{ij} M_{kl} \dots M_{rs} \rangle = \mathcal{N} \int [dM] e^{-S} M_{ij} M_{kl} \dots M_{rs}. \tag{2.31}$$

To calculate the correlation functions we define the generating function

$$Z[J] = \mathcal{N} \int [dM] e^{-S + TrJM}. \tag{2.32}$$

The above correlation functions can then be calculated by taking the appropriate derivatives of the generating function

$$\langle M_{ij} M_{kl} \dots M_{rs} \rangle = \frac{d}{dJ_{ji}} \frac{d}{dJ_{lk}} \dots \frac{d}{dJ_{sr}} Z[J] \bigg|_{J=0}. \tag{2.33}$$

The normalization constant $\mathcal{N}$ is determined by the setting $Z[J = 0] = 1$, which implies

$$1 = \mathcal{N} \int [dM] e^{-\omega Tr(M^2)}. \tag{2.34}$$

The normalization depends on the type of matrix we consider. For a Hermitian matrix M, the trace term in the interaction is given by

$$\begin{aligned} Tr(M^2) &= \sum_i^N M_{ii}^2 + \sum_{i<j}^N (ReM_{ij})^2 + (ImM_{ij})^2 \\ &= \sum_i^N M_{ii}^2 + 2\sum_{i<j}^N (ReM_{ij})^2. \end{aligned} \tag{2.35}$$

We now find

$$\begin{aligned} 1 &= \mathcal{N} \int \prod_{i=1}^N \prod_{i<j}^N dM_{ii} dReM_{ij} dImM_{ij} e^{-\omega \sum_i^N M_{ii}^2 - 2\omega \sum_{i<j}^N (M_{ij})^2} \\ &= \mathcal{N} \int \prod_i^N dM_{ii} e^{\sum_i^N M_{ii}^2} \int \prod_{i<j}^N dReM_{ij} dImM_{ij} e^{-2\omega \sum_{i<j}^N (M_{ij})^2} \\ &= \mathcal{N} \left(\frac{\pi}{\omega}\right)^{N/2} \left(\frac{\pi}{2\omega}\right)^{N(N-1)/2} \\ &= \mathcal{N} \left(\frac{\pi}{\omega}\right)^{N^2} \left(\frac{1}{2}\right)^{N(N-1)/2}. \end{aligned} \tag{2.36}$$

Hence,

$$\mathcal{N} = \left(\frac{\pi}{\omega}\right)^{-N^2} 2^{N(N-1)/2}. \tag{2.37}$$

To explicitly evaluate the correlation functions, the generating function is evaluated by completing the square in the exponential:

$$\begin{aligned} (TrM^2 - TrJM) &= TrM^2 - TrJM + Tr(J^2/4) - Tr(J^2/4) \\ &= Tr((M - J/2)^2) - \frac{1}{4} TrJ^2. \end{aligned} \tag{2.38}$$

Then

$$\begin{aligned} Z[J] &= \mathcal{N} \int [dM]\, e^{-\omega TrM^2 + TrJM} \\ &= \mathcal{N} \int [dM]\, e^{-\omega(Tr((M-J/2\omega)^2)) + \frac{1}{4\omega} TrJ^2} \\ &= \mathcal{N} e^{1/4\omega TrJ^2} \int [dM]\, e^{-Tr((M-J/2\omega)^2)}. \end{aligned} \tag{2.39}$$

Introducing a new integration variable $M' = M - J/2\omega$, and noting that J is not a function of M, we have $dM'_{ij} = dM_{ij}$. Consequently

$$\begin{aligned} Z[J] &= \mathcal{N} e^{1/4\omega Tr J^2} \int [dM']\, e^{-Tr(M'^2)} \\ &= e^{1/4\omega Tr J^2}, \end{aligned} \tag{2.40}$$

where in the first line the normalization condition has been used. We use this result to illustrate the calculation of a few correlation functions:

$$\langle M_{ij} M_{kl} \rangle = \frac{\partial}{\partial J_{ji}} \frac{\partial}{\partial J_{lk}} e^{Tr(J^2/4\omega)} \Big|_{J=0} = \frac{1}{4\omega}(\delta_{la}\delta_{bk}\delta_{jb}\delta_{ia} + \delta_{aj}\delta_{ib}\delta_{bl}\delta_{ak}) = \frac{1}{2\omega}\delta_{jk}\delta_{il} \tag{2.41}$$

$$\langle M_{ij} M_{kl} M_{pq} M_{mn} \rangle = \frac{1}{4\omega^2}(\delta_{jk}\delta_{il}\delta_{qm}\delta_{pn} + \delta_{jp}\delta_{iq}\delta_{lm}\delta_{kn} + \delta_{jm}\delta_{in}\delta_{lp}\delta_{kq}). \tag{2.42}$$

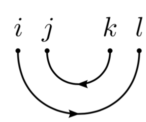
$$\begin{aligned} \langle Tr(M^2) Tr(M^2) \rangle &= \frac{1}{4\omega^2}[\delta_{ii}\delta_{jj}\delta_{ll}\delta_{kk} + \delta_{kj}\delta_{li}\delta_{il}\delta_{kj} + \delta_{ki}\delta_{lj}\delta_{jl}\delta_{ki}] \\ &= \frac{1}{4\omega^2}(N^4 + 2N^2) \end{aligned} \tag{2.43}$$

## 2.5 Ribbon diagrams

The correlations functions above can be evaluated using diagrammatic notation called ribbon diagrams. The following rules are used to construct the diagrams:

1. For each matrix $M_{ij}$, two dots are drawn labeled with the matrix indices.

$$\langle M_{ij} M_{kl} \rangle_0 =$$

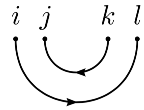



.

2. Each Kronecker delta function of two indices is represented as a line joining the two dots representing the indices.

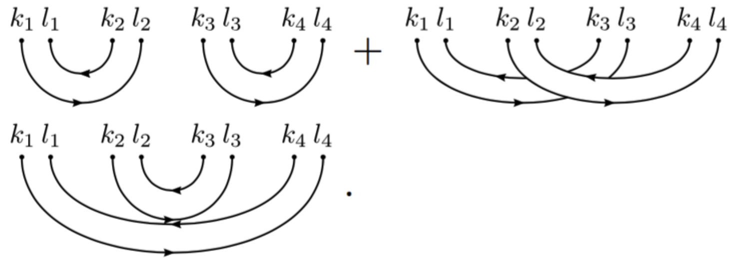
3. Each pair of adjacent edges is a ribbon. For each ribbon, a factor of $1/\omega$ is included. The procedure is illustrated with a few examples below.

$$\langle M_{ij} M_{kl} \rangle_0 =$$




$$= \frac{1}{\omega} \left( \delta_{il} \delta_{jk} \right)$$

$$\langle M_{k_1 l_1} M_{k_2 l_2} M_{k_3 l_3} M_{k_4 l_4} \rangle_0 =$$

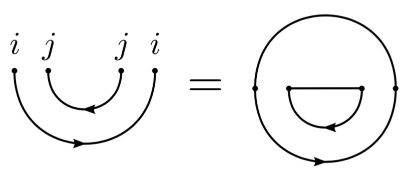



$$= \frac{1}{4\omega^2} ( \delta_{k_3 l_4} \delta_{k_4 l_3} \delta_{k_1 l_2} \delta_{k_2 l_1} + \delta_{k_2 l_3} \delta_{k_3 l_2} \delta_{k_4 l_1} \delta_{k_1 l_4} + \delta_{k_2 l_4} \delta_{k_4 l_2} \delta_{k_1 l_3} \delta_{k_3 l_1} ) .$$

For correlation functions involving traces of powers of M, all Kronecker delta functions in the final result contain repeated indices. These indices are summed over. Diagrammatically the procedure is represented by joining another edge to dots that share the same index, allowing for the omission of such indices. With this comes an additional rule to evaluating the diagrams.

4. For k closed loops in the diagram, include a factor of $N^k$. This process is demonstrated below:

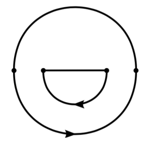
$$\delta_{ii} \delta_{jj} =$$




$$= N^2$$

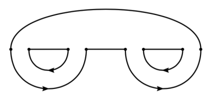
For an $N \times N$ matrix M, each kronecker delta function with repeated index leads to a factor $\delta_{ii} = N$. A few examples are demonstrated below:

$$\langle Tr(M^2)\rangle = \langle M_{ij}M_{ji}\rangle =$$



$$= \frac{1}{\omega^2}\, N^2$$

$$\langle Tr(M^4)\rangle =$$



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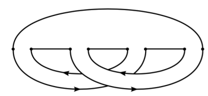

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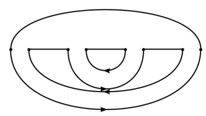


$$= \frac{1}{(2\omega)^2}\left(2N^3 + N\right).$$

For traces with an odd power of M, the result includes terms with uncontracted indices multiplied by a source term. Such terms vanish under the condition $J = 0$.

## 2.6 Planar Limit and Factorisation

The diagrams obtained from calculating the correlators $\langle \prod_k Tr(M^{2n_k})\rangle$ include planar and non-planar diagrams. Planar diagrams are diagrams that do not cross when drawn on a plane, while non-planar diagrams do. We now introduce the planar limit of the matrix model. This is achieved by taking the limit $N \to \infty$, as these two limits coincide. This results in the suppression of non-planar diagrams and we find that only planar diagrams contribute to the calculations of correlation functions. Thus the planar limit results in a significant simplification of the theory. We begin the process by setting the normalization such that:

$$\lim_{N\to\infty} \mathcal{N}\langle Tr(M^{2n})\rangle = \mathcal{O}(1). \tag{2.44}$$

This is done by setting

$$\mathcal{N}Tr(M^{2n}) = \frac{Tr(M^{2n})}{N^{(2n+2)/2}}. \tag{2.45}$$

With this normalization, implementing the large N limit results in:

$$\lim_{N\to\infty}\langle Tr(M^2)\rangle = \frac{1}{2\omega}, \tag{2.46}$$

$$\begin{aligned}\lim_{N\to\infty}\langle Tr(M^2)Tr(M^2)\rangle &= \lim_{N\to\infty}\frac{1}{4\omega^2}\left(\frac{N^4+2N^2}{N^4}\right)\\ &= \lim_{N\to\infty}\frac{1}{4\omega^2}\left(1+2\frac{1}{N^2}\right)\\ &= \frac{1}{4\omega^2}\\ &= \langle Tr(M^2)\rangle^2,\end{aligned} \tag{2.47}$$

$$\begin{aligned}\lim_{N\to\infty}\langle Tr(M^2)Tr(M^2)Tr(M^2)\rangle &= \lim_{N\to\infty}\frac{1}{8\omega^3}\frac{N^6+6N^4+8N^2}{N^6}\\ &= \lim_{N\to\infty}\frac{1}{8\omega^3}\left(1+6\frac{1}{N^2}+8\frac{1}{N^4}\right)\\ &= \frac{1}{8\omega^3}\\ &= \langle Tr(M^2)\rangle^3.\end{aligned} \tag{2.48}$$

Hence from the ribbon graph technology introduced above, it is clear that the calculation of correlations functions in the large $N$ limit reduces to the summation of planar ribbon diagrams, while the contribution from the non-planar diagrams is suppressed by a factor of $\frac{1}{N^2}$. We also notice that the general form of the correlators is

$$\left\langle \prod_k Tr(M^{2n_k})\right\rangle = \prod_k \langle Tr(M^{2n_k})\rangle\left[1+\mathcal{O}\frac{1}{N^2}\right]. \tag{2.49}$$

Thus in the large N limit, the expectation value of a product of observables is equal to the product of the expectation values of the observables. This process is called *factorisation*. Factorisation in the matrix model implies that the theory can be described by classical dynamics, i.e in the large $N$ limit, the matrix model can be formulated as a classical theory. To demonstrate this, suppose our system has the following set of basis states $\{i\}$ and the probability to be in the $i$-th state to be $\mu_i$, with $\sum_i \mu_i = 1$. Labelling our operators as $O_A$, the expectation value is given by

$$\langle O_A\rangle = \sum_i \mu_i O_A(i) \tag{2.50}$$

where $O_A(i)$ is the value of $O_A$ in state $i$. For a number of operators $O_{A_j}$ where $j=1,\ldots,n$, factorisation implies

$$\begin{aligned}\sum_i \mu_i O_{A_1}(i)O_{A_2}(i)\ldots O_{A_n}(i) &= \sum_{i_1}\mu_{i_1}O_{A_1}(i_1)\sum_{i_2}\mu_{i_2}O_{A_2}(i_2)\cdots\sum_{i_n}\mu_{i_n}O_{A_n}(i_n)\\ &= \sum_i\prod_j^n \mu_{i_j}O_{A_j}(i_j) + \text{mixed terms},\end{aligned} \tag{2.51}$$

where the *mixed terms* are terms that mix the states $\{i\}$. For (2.51) to be satisfied, we require that the coefficients of the mixed terms to be zero and

$$\mu_i = \prod_j^n \mu_{i_j}. \tag{2.52}$$

These conditions are satisfied by $\mu_{i^*}=1$ for some state $i=i^*$, while $\mu_i=0$ for any other state $i\neq i^*$, which implies $\mu_{i^*{}_j}=1$ for all $j$. Thus in the large $N$ limit, the system is in definite state with no uncertainty and the correlation functions reduce to being evaluated at this single configuration. In this way, the model behaves as a classical theory.

It is important to note that the planar limit only holds when the number of fields in our operators is kept constant and is of $\mathcal{O}(1)$. As the number of fields increased beyond $\mathcal{O}(1)$, the planar limit is no longer valid. In this project we are interested in heavy operators, i.e operators that are dual to new spacetime geometries. The number of fields in these operators is of $\mathcal{O}(N^2)$ and is therefore dependant on $N$. Thus as we take the large $N$ limit, the number of fields increase and there is a significant increase in the number of wick contractions that need to be performed. This results in large combinatorial factors that counter the $1/N^2$ suppression of non-planar diagrams. As a result, when considering heavy operators, the planar approximation fails as we take the large $N$ limit. In later sections, we consider other methods of evaluating these correlation functions.

## 2.7 Interacting Matrix Theory

To include interactions in the matrix theory, we modify the action by adding an interaction term:

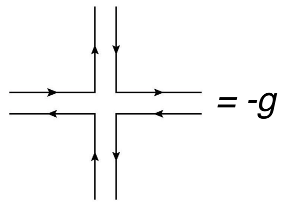
$$S = \omega Tr(M^2) + gTr(M^4), \tag{2.53}$$

where $g$ is a weak coupling ($g << 1$). The generating function then takes the form

$$Z[J,g] = \int [dM] e^{-\omega Tr(M^2) - gTr(M^4) + Tr(JM)}. \tag{2.54}$$

With the interaction, another rule is introduced.

5. For each vertex, include a factor of $-g$ :




To first order in the coupling $g$, the $0$-point function is given by expanding the exponential and only keeping terms to first order in $g$

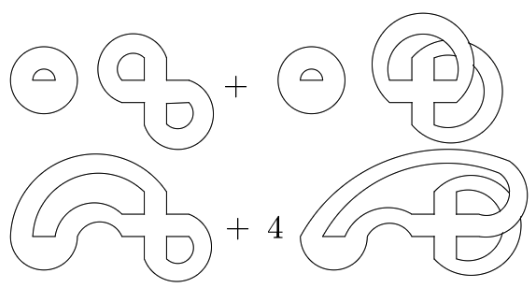
$$\begin{aligned}\langle 1\rangle &= \int [dM]e^{-\omega \mathsf{Tr}(M^2)}(1-g\mathsf{Tr}(M^4)+\mathcal{O}(g^2)) \\ &= 1-g\langle \mathsf{Tr}(M^4)\rangle \\ &= 1-\frac{gN^3}{(2\omega)^3}(2+\frac{1}{N^2}).\end{aligned} \tag{2.55}$$

Similarly for the $2$-point function, we find

$$\begin{aligned}\langle Tr(M^2)\rangle &= \int [dM]e^{-\omega TrM^2}Tr(M^2)(1-gTr(M^4)+\mathcal{O}(g^2)) \\ &= \langle Tr(M^2)\rangle - g\langle Tr(M^2)Tr(M^4)\rangle + \mathcal{O}(g^2) \\ &= \frac{N^2}{2\omega}-\frac{gN^5}{8\omega^3}(2+\frac{9}{N^2}+\frac{4}{N^4})+\mathcal{O}(g^2).\end{aligned} \tag{2.56}$$

In terms of ribbons graph, the first order in $g$ part of the $2$-point function is calculated as

$\langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^4)\ \rangle =$ 2 + 8 + 4



In the interacting theory, the normalisation is chosen as $Z[J,g]\big|_{J=0,g=0}$. This is equivalent to summing over Feynman diagrams that do not include vacuum diagrams. Dropping these diagrams, the normalised $2$-point function to first order in $g$ is

$$\langle Tr(M^2)\rangle = \frac{N^2}{2\omega} - \frac{gN^3}{8\omega^3}(8 + \frac{4}{N^2}) + \mathcal{O}(g^2). \tag{2.57}$$

These correlators demonstrate that in the large $N$ limit, the first order corrections are no longer small and thus perturbation theory is not valid. To overcome this, the 't Hooft parameter $\lambda$ is introduced:

$$\lambda = gN. \tag{2.58}$$

In the large N limit, the 't Hooft parameter is kept constant by requiring that $g \to 0$ as $N \to \infty$. Using the parameter to rewrite the correlators just calculated, we find:

$$\begin{aligned}\langle Tr(M^2)\rangle_{con} &= \frac{N^2}{2\omega} - \frac{1}{(2\omega)^3}[8\lambda N^2 + 4\lambda] \\ &= N^2\left(\frac{1}{2\omega} - \frac{9\lambda}{8\omega^3} + \mathcal{O}(\lambda^2)\right) + N^0\left(\frac{-4\lambda}{8\omega^3} + \mathcal{O}(\lambda^2)\right) + \mathcal{O}\left(\frac{1}{N^2}\right).\end{aligned} \tag{2.59}$$

Thus the 't Hooft parameter makes the pertubative approach consistent by tuning the coupling to zero in such a way that we get non-trivial contributions for each order in $\lambda$.

## 2.8 Matrix model correspondence with string theory

The ribbon graphs can be drawn on different surfaces with different topologies in such a way that the ribbons do not twist or intersect. In this way, we say the ribbon diagrams triangulate surfaces of different topologies. The different topological surfaces can be classified by their genus $g$, which is related to the number of handles the surface has. In the large N limit, the powers of N for a ribbon diagram can be directly identified with the topological surface it triangulates. Planar diagrams are ribbon graphs that can be drawn on the plane without any of the ribbons crossing. These diagrams triangulate a sphere, which has a genus $g = 0$. Each planar ribbon graph has factor $N^2$. We can therefore identify a ribbon diagram with factor $N^2$ with a surface of genus $g = 0$. The next ribbon graphs in the correlation functions have factor $N^0$. These ribbon graphs triangulate a torus, a surface of genus $g = 1$. We then match a factor $N^0$ with genus $g = 1$. The next ribbon has factor $N^{-2}$ and triangulates a pretzel with genus $g = 2$. Therefore $N^{-2}$ is identified with genus $2$. Continuing in this way, we see that each ribbon diagram in the calculation of the correlation function at large N, can be mapped to a surface with a genus $g$.

Each of these diagrams can be identified with a worldsheet of a closed string. Closed string worldsheets are two dimensional surfaces. These surface have no boundaries, and only differ by their genus $g$. These surfaces are used in the genus expansion of the string theory, which is a way of calculating string scattering amplitudes. A propagating string has a worldsheet with no handles. This worldsheet has genus $g = 0$. For a single closed string that splits into two closed strings and then recombines into a single string, we have a world worldsheet of genus $g = 1$. This corresponds to an $\hbar$ correction to the string propagator. Continuing with this analysis, we can see that each $\hbar$ correction, corresponds to an increase by 1 in the genus of the corresponding worldsheet. In this way, we see that not only can the correlators on the matrix model be given in terms of string worldsheets but further, the $\hbar$ in the string theory can be identified with the $1/N^2$ in the matrix model.

We can state this matching of parameters more precisely. Setting $\omega = 1/(2g_{YM}^2)$, we see that each correlator in the matrix models comes with a factor

$$(g_{YM}^2)^{E-V} N^F = \lambda^{E-F} N^{F-E+V} \tag{2.60}$$

where E is the number of propagators, V is the number if vertices and F is the number of closed loops. From Euler's theorem, we know that a polyhedron with a number of edges E, a number of vertices V, a number of face F and genus g, obeys

$$V - E + F = 2 - 2g \tag{2.61}$$

so we can rewrite (2.60) as $\lambda^F \left(\frac{\lambda}{N}\right)^{2g-2}$ and consequently express any correlator in the form

$$\langle \dots \rangle = \sum_{g=0}^{\infty} \lambda^F \left(\frac{\lambda}{N}\right)^{2g-2} \sum_{n=0}^{\infty} c_{g,n} \lambda^n \tag{2.62}$$

In this way we can see that the genus expansion is controlled by the parameter $\lambda/N$ with the coupling $\lambda$ enumerating the quantum loops. We deduce that the string coupling $g_s$ must be

$$g_s \sim \frac{\lambda}{N} = g_{YM}^2. \tag{2.63}$$

For more details see [17].

## 2.9 Group Represention Theory

We consider the Lie algebra of $u(p)$ instead of the group $U(p)$ itself, since it is easier. Most of the results we obtain for representations in $u(p)$ still hold on $U(p)$. This also has the benefit that the Clebsch-Gordon coefficients that are used in the construction of the representations of $u(p)$ and $U(p)$ are identical. The structure of $u(p)$ is explored using its fundamental representation, of which there is only one. This representation gives the elements of the Lie algebra as $N \times N$ matrices. Let $E_{ij}$ with $1 \leq i, j \leq p$ be the matrix

$$(E_{ij})_{rs} = \delta_{ir}\delta_{js}, \tag{2.64}$$

which has one non-zero matrix element. The defining representation of the Lie algebra is given by the matrices

$$\begin{array}{lll} iE_{kk}, & i \leq k \leq p & \\ i(E_{k,k-1}+E_{k-1,k}), & E_{k,k-1}-E_{k-1,k}, & 1 < k \leq p. \end{array} \tag{2.65}$$

$u(p)$ is spanned by real linear combinations of these matrices. The restriction of any irreducible representation of $GL(p,\mathbb{C})$ onto its subgroup $U(p)$ is also irreducible. We have that the carrier space of the irreducible representations of $U(p)$ share the same basis as the irreducible representation of $GL(p,\mathbb{C})$. As a consequence of this, labelling for $gl(p,\mathbb{C})$ irreducible representations is also a labelling for $u(p)$ irreducible representations. We therefore work in the basis for $gl(p,\mathbb{C})$ given by complex linear combinations of matrices

$$J_Z^{(l)} = E_{ll}, \qquad 1 \leq l \leq N, \tag{2.66}$$

$$J_+^{(l)} = E_{l,l+1}, \qquad J_-^{(l)} = E_{l+1,l}, \qquad 1 \leq l \leq N-1. \tag{2.67}$$

For each $l$, they satisfy the Lie algebra

$$\left[J_z^{(l)}, J_\pm^{(l)}\right] = \pm J_\pm^{(l)}, \qquad \left[J_+^{(l)}, J_-^{(l)}\right] = 2J_z^{(l)}. \tag{2.68}$$

The set of matrices $J_z^{(l)}$ form a maximal set of mutually commuting matrices.

The Gelfand-Tsetlin labelling for the irreducible representations of $u(p)$ and the basis states of their carrier spaces will now be introduced. The basis states chosen by this labelling are simultaneous eigenstates of all matrices $J_z^{(l)}$. Further, the explicit formulas of the matrix elements of $J_z^{(l)}$ with respect to this basis are known. Once the representations for all $J_\pm^{(l)}$ on a given carrier space are know, all other elements of the algebra $gl(p,\mathbb{C})$ and $u(p)$ are also known. An inequivalent irreducible representation for $GL(p,\mathbb{C})$ is uniquely given by specifying its weight. The weight is given by the sequence of $p$ integers

$$\mathbf{m} = (m_{1p}, m_{2p}, \ldots, m_{pp}), \tag{2.69}$$

that satisfy $m_{kp} \geq m_{(k+1)p}$. Such a sequence can be identified with the row lengths of a Young diagram R. The restriction of this irreducible representation onto the subgroup $GL(p-1,\mathbb{C})$ is reducible. It is decomposed into a direct sum of $GL(p-1,\mathbb{C})$ irreducible representations with highest weights

$$\mathbf{m}' = (m_{1,p-1}, m_{2,p-1}, \ldots, m_{p-1,p-1}), \tag{2.70}$$

for which the "betweenness" conditions

$$m_{kp} \geq m_{k,p-1}, \geq, m_{k+1,p} \qquad for \qquad 1 \leq k \leq p-1 \tag{2.71}$$

hold. After restriction to the $GL(p-1,\mathbb{C})$ subgroup, the carrier spaces of the $GL(p-1,\mathbb{C})$ irreducible representations give rise to the $GL(p-2,\mathbb{C})$ irreducible representations. This procedure can be repeated until we get to $GL(1,\mathbb{C})$ which has a one-dimensional carrier space. This sequence of subgroups is used to label the basis state in what is called the Gelfand-Tsetlin patterns. These are triangular arrangements of integers, denoted by $M$, with the structure

$$M = \begin{bmatrix} m_{1p} & & m_{2p} & & \ldots & & m_{p-1,p} & & m_{pp} \\ & m_{1,p-1} & & m_{2,p-1} & & \ldots & & m_{p-1,p-1} & \\ & & \ldots & \ldots & \ldots & & & & \\ & & m_{12} & & & m_{22} & & & \\ & & & m_{11} & & & & & \end{bmatrix} \tag{2.72}$$

The irreducible representation of the state is contained in the top row, with the lower rows subject to the betweenness condition. In this way the lower rows give the irreducible representations the state belongs to as we pass successively through the restrictions from $GL(p,\mathbb{C})$ to $GL(p-1,\mathbb{C})$ to $\ldots$ to $GL(1,\mathbb{C})$. For an irreducible representation with weight $\mathbf{m}$, its dimensions are given by the number of valid Gelfand-Tsetlin patterns with $\mathbf{m}$ as their top row.

When applying group theory to Quantum Field Theory, two weights are used extensively. These are the $\Sigma$-weights and the $\Delta$-weights. Define the row sum

$$\sigma_l(M) = \sum_{k=1}^{l} m_{k,l}, \tag{2.73}$$

then the sequence of row sums defines the sigma weight

$$\Sigma(M) = (\sigma_p(M), \sigma_{p-1}(M), \ldots, \sigma_l(M)). \tag{2.74}$$

The sigma weights do not provide a unique label for states in the carrier space. It is possible that $\Sigma(M) = \Sigma(M')$ with $M \neq M'$. The number of states $\vec{v}(M)$ in the carrier space with the same sigma weight $\Sigma = \Sigma(M)$ is called the inner multiplicity $I(\Sigma)$ of the state. In a later section we will introduce the restricted Schur polynomials. The inner multiplicity plays an important role in determining the number of possible restricted Schur polynomials. The delta weights are defined in terms of the differences between the row sums

$$\begin{aligned}\Delta(M) &= (\sigma_p(M)-\sigma_{p-1}(M), \sigma_{p-1}(M)-\sigma_{p-2}(M), \ldots, \sigma_1(M)-\sigma_0(M)) \\ &\equiv (\delta_p(M), \delta_{p-1}(M), \ldots, \delta_1(M)),\end{aligned} \tag{2.75}$$

where $\sigma_0(M)=0$. From this it is clear that $I(\Sigma)=I(\Delta)$, where $I(\Delta)$ is the number of states in the carrier space with the same $\Delta$. The delta weights are important in determining how the Young diagram labels $R,(r,s)$ of the restricted Schur polynomial $\chi_{R,(r,s)\alpha\beta}$ are translated into a set of $U(p)$ labels. From them, we can find how boxes are removed from R to obtain r. Furthermore, the multiplicity labels $\alpha\beta$ of $\chi_{R,(r,s)\alpha\beta}$ run over the inner multiplicity.

## 2.10 Relation between Young Diagrams and Gelfand-Tsetlin Patterns

There is a bijection between Young diagrams and $\Sigma$-weights and a bijection between semi-standard Young tableaux (Young diagrams with labeled boxes) and Gelfand-Tsetlin patterns. For a $u(p)$ irreducible representation there are at most p rows and every Young diagram uniquely labels an irreducible representation of $u(p)$. For a semi-standard Young tableau, the rule for labelling is that each row of boxes weakly increase from left to right, that is each number is equal to or greater than the one to its left, with the numbers in each column strictly increasing from top to bottom. The set of all semi-standard Young tableaux identified by the Young diagram R can then be used to uniquely identify the basis states of a $u(p)$ representation. The dimensions of a carrier space labeled by a Young diagram is equal to the number of valid Young tableaux with the same shape as the Young diagram. As stated before, there is a correspondence between Gelfand-Tsetlin patterns and Young tableau. We show the explicit construction of a Young tableau for a given Gelfand-Tsetlin pattern $M$. The Gelfand-Tsetlin pattern considered in this example is given by

$$M = \begin{bmatrix} 4 & & 3 & & 1 & & 1 \\ & 3 & & 2 & & 1 & \\ & & 3 & & 2 & & \\ & & & 2 & & & \end{bmatrix}. \tag{2.76}$$

We begin with an empty Young diagram R. The first line of M tells us about the shape of R. There are four number in the first row of M, therefore the corresponding Young diagram R has 4 rows. We have that R is given by a top row of 4 boxes, with the row below it given by 3 boxes, the one following this by 1 box and the final row by 1 box. R is therefore given by

$$R = \ydiagram{4,3,1,1} \quad . \tag{2.77}$$

To label R, we read the bottom row of M, that tells us which boxes of R are to be labeled with a 1. To do this imagine superposing the smaller Young diagram defined by the last row of M, in this case $2 \equiv \ydiagram{2}$, onto the topmost and leftmost boxes of R. In this the following boxes are labeled with a 1,

$$\begin{ytableau} 1 & 1 & \none[] & \none[] \\ \none[] & \none[] & \none[] \\ \none[] \\ \none[] \end{ytableau} \quad . \tag{2.78}$$

The row before last of M tells us which boxes are labeled with a 2. We again superpose the Young diagram defined by this row to the larger diagram R. In this case $3, 2 \equiv \ydiagram{3,2}$. This Young diagram is superposed to the topmost and leftmost boxes. All boxes superposed, except those already labeled, are labeled with a 2, as shown below.

$$\begin{ytableau} 1 & 1 & 2 & \\ 2 & 2 & \\ \\ \end{ytableau} \quad . \tag{2.79}$$

This procedure is repeated until the rows of M are used to label the boxes of R with integers up to p. For our example, the semi-standard Young tableau obtained from M is given by

$$\begin{ytableau} 1 & 1 & 2 & 4 \\ 2 & 2 & 4 \\ 3 \\ 4 \end{ytableau} \quad . \tag{2.80}$$

We therefore have demonstrated how to obtain a semi-standard Young tableau from a Gelfand-Tsetlin pattern. A simple way of obtaining a Gelfand-Tsetlin pattern from semi-standard Young tableau is briefly detailed. We know that the shape of the Young diagram gives the first row of the Gelfand-Tsetlin pattern, in the case of (56), this is $[4 \quad 3 \quad 1 \quad 1]$. The second row is given by the sequence of numbers, one for each row, that give the number of boxes that do not contain the largest integer in the Gelfand-Tsetlin pattern, in this case $4$. The second row will therefore be given by the sequence $[3 \quad 2 \quad 1]$. Similarly the third row will be given by the sequence of numbers giving the number of boxes not containing the second largest number in $M$, in this case $3$ (Note that in counting the number of boxes that do not include the integer in question, boxes including larger integers are not counted. In the case of the third row, boxes with $4$ are not counted). Progressing this way, we obtain the original Gelfand-Tsetlin pattern.

The number of boxes containing the number $l$ in tableau row $k$ is given by $m_{k,l} - m_{k,l-1}$ and we set $m_{kl} \equiv 0$ if $k > l$. The converse process of transcribing a semi-standard Young tableau to a Gelfand-Tsetlin pattern is now obvious. The components $\delta_l(M)$ of the $\Delta$ weight of a Gelfand-Tsetlin pattern $M$, is the number of boxes containing $l$ in the tableau corresponding to $M$. Thus, the tableau corresponding to two patterns with the same $\Delta$ weight contain the same set of entries (i.e. the same number of $l$-boxes) but arranged in different ways. One interpretation for the inner multiplicity is that it simply counts the number of ways to arrange the relevant fixed set of entries in the tableau.

## 2.11 Schur Polynomials

In studying the planar limit, we find that there is no mixing between multi-trace structures. As a result, this allows us to focus on single trace operators. When we consider a large N but non-planar limit, mixing between different trace structures occurs in an uncontrollable way. To demonstrate this consider the $1/2$ BPS operator

$$\mathcal{O}_J = \frac{Tr(Z^J)}{\sqrt{JN^J}}. \tag{2.81}$$

In the dual gravity theory J gives the angular momentum of the giant graviton. The operator is normalised in such a way that the two point correlation function in the large N limit is given by

$$\langle \mathcal{O}_J \mathcal{O}^{\dagger}{}_J \rangle = 1 + O\left(\frac{1}{N^2}\right). \tag{2.82}$$

This result is obtained by summing only planar diagrams. The result is only accurate for $J^2 << N$. Consider now a multi-trace two point function composed of a double trace $O_{J_1} O_{J_2}$ and a single trace $O_{J_1+J_2}$

$$\langle O_{J_1} O_{J_2} O^{\dagger}_{J_1+J_2} \rangle = \frac{\sqrt{J_1 J_2 (J_1+J_2)}}{N}. \tag{2.83}$$

We see that, for $J_1, J_2$ fixed, the two point function vanishes if we take the limit $N \to \infty$. We see then that the single and double trace structures do not mix in the large N limit. Although not demonstrated, matrix models have the general property that different trace structures do not mix in the large N limit. This property allows us to calculate the anomalous dimensions of the theory only using single trace operators. This can be done since single trace operators containing m fields can be mapped to a spin chain on a lattice with m sites, with the fields in the trace determining the spins on each site. The Hamiltonian of the spin chain is integrable and can be put in correspondence with the planar dilatation operator. In this way we have a bijection between spin chain states and single trace operators.

Another explanation of why the three point function (2.83) vanishes is that in the dual gravity, the number of traces can be identified with the number of particles. In this case the two point function is an overlap between orthogonal states given by a two particle state of two gravitons with $\mathcal{R}$-charge $J_1$ and $J_2$ and a single particle state of a graviton with an $\mathcal{R}$-charge of $J_1+J_2$.

If the $J_i's$ are allowed to depend on N, then the $RHS$ of (2.83) shows that the three point function need not vanish. A careful analysis shows that when the $J_i's$ grows faster than $\mathcal{O}(\sqrt{N})$ there is a mixing of the different trace structures and we must sum non-planar diagrams. To properly deal with this, techniques based on Schur polynomials are introduced. This technique is useful since, two point correlation functions of Schur polynomials are known exactly and higher point correlation functions can be calculated from the two point correlators.

### 2.11.1 Projectors

We first introduce projection operators that play an important role in the Schur polynomial basis. In the matrix model, the field $Z$, is a complex matrix operator on the $N$ dimensional vector space $V_N$. We can combine n fields in a tensor product $Z^{\otimes n} = Z \otimes Z \otimes \cdots \otimes Z$ to produce an operator in the larger vector space $V_N^{\otimes n}$. The vector space $V_N^{\otimes n}$ is obtained by taking a tensor product of n copies of the vector space $V_N$. For $n=3$ the possible trace structures on $V_N$ are $Tr(Z^3), Tr(Z^2)Tr(Z)$ and $Tr(Z)^3$. If we consider an element $\sigma \in S_3$, then the multi-trace fields can be written as a single trace $Tr(\sigma Z^{\otimes 3})$ in the larger space $V_N^{\otimes 3}$, for example:

$$
\begin{aligned}
Tr((123)Z^{\otimes 3}) &= (123)^{i_1 i_2 i_3}_{j_{\sigma(1)} j_{\sigma(2)} j_{\sigma(3)}} Z^{j_1}_{i_1} Z^{j_2}_{i_2} Z^{j_3}_{i_3} \\
&= \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \delta^{i_3}_{j_{\sigma(3)}} Z^{j_1}_{i_1} Z^{j_2}_{i_2} Z^{j_3}_{i_3} \\
&= Z^{i_3}_{i_1} Z^{i_1}_{i_2} Z^{i_2}_{i_3} \\
&= Tr(Z^3).
\end{aligned} \tag{2.84}
$$

Similarly $Tr((23)Z^{\otimes 3}) = Tr(Z)Tr(Z^2)$ and $Tr((1)Z^{\otimes 3}) = Tr(Z)^3$. In general, for n fields, each multi-trace structure in the vector space $V_N$ can be written as a single trace $Tr(\sigma Z^{\otimes n})$ in the larger space $V_N^{\otimes n}$.

### 2.11.2 Conjugacy classes

Consider elements of a group $g_1, g_2 \in G$, if two of the elements are related to each other as

$$g_1 = g^{-1} g_2\, g \quad \text{for some}\, g \in G, \tag{2.85}$$

then the two elements $g_1, g_2$ are conjugate to each other. The conjugacy relation enjoys various properties discussed below. Reflexivity, i.e. every element is conjugate to itself

$$g = e^{-1} g e \tag{2.86}$$

where $e$ is the identity element of the group. If

$$g_1 = g^{-1} g_2 g, \tag{2.87}$$

this implies,

$$g_2 = (g^{-1})^{-1} g_1 g^{-1}, \tag{2.88}$$

so that conjugacy is symmetric. Conjugacy is also transitive, since if,

$$g_2 = g^{-1} g_1 g, \qquad \text{and} \qquad g_3 = \tilde{g}^{-1} g_2 \tilde{g}, \tag{2.89}$$

then,

$$g_3 = (g\tilde{g})^{-1} g_1 (g\tilde{g}) \tag{2.90}$$

This shows that conjugacy is an equivalence relation on the group $G$. This relation partitions the set $G$ into equivalence classes. For two elements $g_1, g_2$ in the same equivalence class, we have

$$
\begin{aligned}
\chi_R(g_2) &= Tr(\Gamma_R(g_2)) \\
&= Tr(\Gamma_R(g^{-1} g_1 g)) \\
&= Tr(\Gamma_R(g^{-1}) \Gamma_R(g_1) \Gamma_R(g)) \\
&= Tr(\Gamma_R(g)^{-1} \Gamma_R(g_1) \Gamma_R(g)) \\
&= Tr(\Gamma_R(g_1)) \\
&= \chi(g_1).
\end{aligned} \tag{2.91}
$$

This shows that elements in the same conjugacy class have the same character. The conjugacy classes of the symmetric group collect elements with the same cycle structure. In particular, let $g_1, g_2, g \in S_n$ with $g_2 = g^{-1}g_1g$ and $g_1(i) = j$, then $g_2(g(i)) = g(j)$ and the cycle structures of $g_1$ and $g_2$ are the same.

**Proof** : Using the rule that multiplication corresponds to function composition. We have,

$$\begin{aligned} g_2(g(i)) &= (g^{-1}g_1g)(g(i)) \\ &= g(g_1(g^{-1}(g(i)))) \\ &= g(g_1(i)) \\ &= g(j). \end{aligned} \tag{2.92}$$

Let $g_1$ have cycle structure,

$$\begin{aligned} g_1 &= (a_1, a_2, \ldots a_{k_1})(b_1, b_2, \ldots b_{k_2}) \ldots (c_1, c_2, \ldots c_{k_m}) \\ &= \sigma_1\sigma_2 \ldots \sigma_m, \end{aligned} \tag{2.93}$$

where $\sigma_i$ is a disjoint $k_i$-cycle. Then

$$\begin{aligned} g_2 &= g^{-1}(\sigma_1\sigma_2 \ldots \sigma_m)g \\ &= (g^{-1}\sigma_1 g)(g^{-1}\sigma_2 g) \ldots (g^{-1}\sigma_m g). \end{aligned} \tag{2.94}$$

Considering a single cycle $\sigma = (a_1, a_2, \ldots a_k)$, let $d_i = g(a_i)$, then from the first part of the proof, $g_2$ must have cycle $(d_1, d_2, \ldots d_k)$. Moreover if $x$ is an element that is shifted by $g_2$, then $g_2(x) = (g^{-1}g_1g)(x) \neq x \implies g_1(g(x)) \neq g^{-1}(x)$, hence $g^{-1}(x) = a_i$ for some $i$. This means $x = d_i$ for some $i$. It follows that

$$g_2 = (d_1, d_2, \ldots d_k). \tag{2.95}$$

We see then that if $g_1$ is a $k$-cycle permutation, then for any cycle permutation $g$, $g_2$ is also a $k$-cycle.

We therefore have that each cycle structure in the $Tr(\sigma Z^{\otimes n})$ gives a physical observable, and each cycle structure belongs to a conjugacy class. Therefore the conjugacy classes of the symmetric group $S_n$ correspond to physical observables.

### 2.11.3 Young diagrams

Young diagrams are rows of boxes which describe partitions. They have a one-to-one correspondence with the conjugacy classes and irreducible representations of the symmetric group. A partition n is represented by a Young diagram with n boxes. To avoid over counting a row must be as long or shorter than the row above it and all the rows are left aligned. The Young diagrams are best understood by studying examples. Consider the $S_4$ group. This group has five conjugacy classes. $(1, 1, 1, 1)$ which represents four 1-cycles, the identity is the only element in this group. $(2, 1, 1)$, this class has one 2-cycle and two 1-cycles. The following classes $(2, 2)$ and $(3, 1)$ with two 2-cycles and a single 3-cycle with a 1-cycle respectively. Then finally $(4)$ which is a single 4-cycle. Each of these conjugacy classes is represented by a Young diagram:

$$(1,1,1,1) \rightarrow \ydiagram{1,1,1,1} \qquad (2,1,1) \rightarrow \ydiagram{2,1,1} \qquad (2,2) \rightarrow \ydiagram{2,2} \tag{2.96}$$

$$(3,1) \rightarrow \ydiagram{3,1} \qquad (4) \rightarrow \ydiagram{4} \tag{2.97}$$

The Young diagrams provide convenient labels for the irreducible representations of the symmetric group $S_n$. They can can also be used to determine the dimensions of an irreducible representation and provide a way of constructing an orthonormal basis for the carrier space of the irreducible representations of $S_n$.

### 2.11.4 Schur Polynomials

As stated above, the Schur polynomials provide a particularly useful basis for half BPS operators [13]. There are mathematical and physical reasons to motivate this. Mathematically, the two point correlators of Schur polynomials are known to all orders in 1/N and are diagonal in the Young diagrams that label the the Schur polynomials. Schur polynomials also satisfy a product rule that allows higher point correlators to be evaluated using results from two point correlators.

The general form of the Schur polynomial is

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \dots Z^{i_n}_{i_{\sigma(n)}}, \tag{2.98}$$

where $\chi_R(\sigma)$ is the character of the element $\sigma \in S_n$ in irreducible representation R, which is given by a Young diagram of n boxes.

We consider the simplest case of $n = 2$. In this case the symmetric groups has two elements $S_2 = \{(1), (12)\}$. The possible irreps for the group are $R = \substack{\square\\\square}$ or $\square\square$, known as the anti-symmetric and symmetric representations respectively. The complete set of characters are given by $\chi_{\substack{\square\\\square}}((1)) = 1, \chi_{\substack{\square\\\square}}((12)) = -1$ and $\chi_{\square\square}((1)) = 1, \chi_{\square\square}((12)) = 1$. The Schur polynomials for $S_2$ are given by

$$\begin{aligned}\chi_{\square\square}(Z) &= \frac{1}{2}\sum_{\sigma\in S_2} \chi_{\square\square}(\sigma) Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \\ &= \frac{1}{2}[\chi_{\square\square}((1)) Z^{i_1}_{i_1} Z^{i_2}_{i_2} + \chi_{\square\square}((12)) Z^{i_1}_{i_2} Z^{i_2}_{i_1}] \\ &= \frac{1}{2}[Tr(Z)^2 + Tr(Z^2)],\end{aligned} \tag{2.99}$$

and

$$\begin{aligned}\chi_{\substack{\square\\\square}}(Z) &= \frac{1}{2}\sum_{\sigma\in S_2} \chi_{\substack{\square\\\square}}(\sigma) Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \\ &= \frac{1}{2}[\chi_{\substack{\square\\\square}}((1)) Z^{i_1}_{i_1} Z^{i_2}_{i_2} + \chi_{\substack{\square\\\square}}((12)) Z^{i_1}_{i_2} Z^{i_2}_{i_1}] \\ &= \frac{1}{2}[Tr(Z)^2 - Tr(Z^2)].\end{aligned} \tag{2.100}$$

Similarly, the Schur polynomials for the $S_3$ symmetric group are given by

$$\begin{aligned}\chi_{\square\square\square}(Z) &= \frac{1}{6}\big[Tr(Z)^3 + 3Tr(Z^2)Tr(Z) + 2Tr(Z^3)\big], \\ \chi_{\substack{\square\\\square\\\square}}(Z) &= \frac{1}{6}\big[Tr(Z)^3 - 3Tr(Z^2)Tr(Z) + 2Tr(Z^3)\big], \\ \chi_{\substack{\square\square\\\square}}(Z) &= \frac{1}{3}\big[Tr(Z)^3 - Tr(Z^3)\big].\end{aligned} \tag{2.101}$$

The two point function for the Schur polynomials is given by

$$\langle \chi_R(Z)\chi^\dagger_S(Z)\rangle = \delta_{RS}\frac{n!}{d_R} Dim_R. \tag{2.102}$$

Here $d_R$ is the dimension of the irreducible representation labeled by the Young diagram R, given by,

$$d_r = \frac{n!}{hooks_R}, \tag{2.103}$$

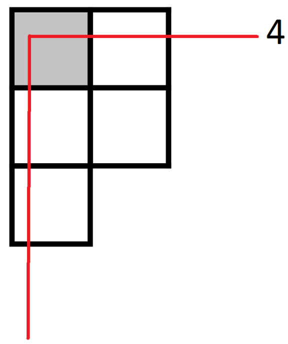



Figure 2.1: The hook length of the shaded box is 4

where n is the number of boxes in the Young diagram R and $hooks_R$ is the product of the hook lengths of each box in the Young diagram R. The hook length of a box is obtained by drawing a vertical line from the bottom of the Young diagram to the box, then drawing a horizontal line from the box that joins the original. The horizontal line moves from the box to the outside on the Young diagram in the right direction as shown in figure 2.1.

We calculate the three possible two point functions for $S_2$:

$$\begin{aligned}\langle\chi_{\square\square}(Z)\chi^{\dagger}_{\substack{\square\\\square}}(Z)\rangle &= \frac{1}{4}\big\langle[Tr(Z)^2+Tr(Z^2)][Tr(Z^{\dagger})^2-Tr(Z^{\dagger 2})]\big\rangle\\ &= \frac{1}{4}\big\langle[Tr(Z)^2Tr(Z^{\dagger})^2-Tr(Z)^2Tr(Z^{\dagger 2})+Tr(Z^2)Tr(Z^{\dagger})^2-Tr(Z^2)Tr(Z^{\dagger 2})]\big\rangle\\ &= \frac{1}{4}[2N^2-2N+2N-2N^2]\\ &= 0,\end{aligned} \tag{2.104}$$

$$\begin{aligned}\langle\chi_{\square\square}(Z)\chi_{\square\square}{}^{\dagger}(Z)\rangle &= \frac{1}{4}\big\langle[Tr(Z)^2+Tr(Z^2)][Tr(Z^{\dagger})^2+Tr(Z^{\dagger 2})]\big\rangle\\ &= \frac{1}{4}[2N^2+2N+2N+2N^2]\\ &= N(N+1).\end{aligned} \tag{2.105}$$

We see that the answer $N(N+1)$ is the product of the weights of the Young diagram that gives the irreducible representation $R=\square\square$.The final two point function is

$$\begin{aligned}
\langle \chi_{\substack{\square\\\square}}(Z)\chi^{\dagger}_{\substack{\square\\\square}}(Z)\rangle &= \frac{1}{4}\big\langle[Tr(Z)^2 - Tr(Z^2)][Tr(Z^{\dagger})^2 - Tr(Z^{\dagger 2})]\big\rangle \\
&= \frac{1}{4}[2N^2 - 2N - 2N + 2N^2] \\
&= N(N-1).
\end{aligned} \tag{2.106}$$

Again, the solution is given by the product of the weights of the irrep $R = \substack{\square\\\square}$.

From these calculations we notice that, the two point functions of Schur polynomials of different irreps are orthogonal and that the two point functions of the Schur polynomials of the same irrep R is given by the product of the weights of the Young diagram of R. To see if these properties apply to the general case, the general form of the Schur polynomial is written as

$$\begin{aligned}
\chi_R(Z) &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)Z^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}\dots Z^{i_n}_{i_{\sigma(n)}} \\
&= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)Tr(\sigma Z^{\otimes n}) \\
&= Tr\left(\frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma Z^{\otimes n}\right) \\
&= \frac{1}{d_R}Tr(P_R Z^{\otimes n}),
\end{aligned} \tag{2.107}$$

where

$$P_R = \frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma \tag{2.108}$$

is a projection operator. The projection operator projects to the subspace of the irrep R. To see how this operator works, we consider the case of $n=2$, with $R = \square\square$ or $\substack{\square\\\square}$.

$$\begin{aligned}
P_{\square\square}Z^{\otimes n} &= \frac{1}{2}[\chi_{\square\square}((1))^{i_1 i_2}_{j_1 j_2} + \chi_{\square\square}((12))^{i_1 i_2}_{j_1 j_2}]Z^{j_1}_{k_1}Z^{j_2}_{k_2} \\
&= \frac{1}{2}[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}]Z^{j_1}_{k_1}Z^{j_2}_{k_2} \\
&= \frac{1}{2}[Z^{i_1}_{k_1}Z^{i_2}_{k_2} + Z^{i_1}_{k_2}Z^{i_2}_{k_1}],
\end{aligned} \tag{2.109}$$

and

$$\begin{aligned}
P_{\substack{\square\\\square}} Z^{\otimes n} &= \frac{1}{2}[\chi_{\substack{\square\\\square}}((1))^{i_1 i_2}_{j_1 j_2} + \chi_{\substack{\square\\\square}}((12))^{i_1 i_2}_{j_1 j_2}] Z^{j_1}_{k_1} Z^{j_2}_{k_2} \\
&= \frac{1}{2}[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}] Z^{j_1}_{k_1} Z^{j_2}_{k_2} \\
&= \frac{1}{2}[Z^{i_1}_{k_1} Z^{i_2}_{k_2} - Z^{i_1}_{k_2} Z^{i_2}_{k_1}].
\end{aligned} \tag{2.110}$$

We see therefore that the projector $P_{\square\square}$ projects to the symmetric subspace of R, while $P_{\substack{\square\\\square}}$ projects to the anti-symmetric subspace of R. The projection operator also satisfies the following properties:

$$\begin{aligned}
(P_{\square\square})^{i_1 i_2}_{j_1 j_2} (P_{\square\square})^{j_1 j_2}_{k_1 k_2} &= \frac{1}{4}[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}][\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}] \\
&= \frac{1}{4}[\delta^{i_1}_{k_1}\delta^{i_2}_{k_2} + \delta^{i_1}_{k_2}\delta^{i_2}_{k_1} + \delta^{i_1}_{k_2}\delta^{i_2}_{k_1} + \delta^{i_1}_{k_1}\delta^{i_2}_{k_2}] \\
&= \frac{1}{2}[\delta^{i_1}_{k_1}\delta^{i_2}_{k_2} + \delta^{i_1}_{k_2}\delta^{i_2}_{k_1}] \\
&= (P_{\square\square})^{i_1 i_2}_{k_1 k_2}
\end{aligned} \tag{2.111}$$

and

$$\begin{aligned}
(P_{\square\square})^{i_1 i_2}_{j_1 j_2} + (P_{\substack{\square\\\square}})^{i_1 i_2}_{j_1 j_2} &= \frac{1}{2}[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}] + \frac{1}{2}[\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}] \\
&= \delta^{i_1}_{j_1}\delta^{i_2}_{j_2} \\
&= (1)^{i_1 i_2}_{j_1 j_2}
\end{aligned} \tag{2.112}$$

where $(1)$ is the identity of $V_N^{\otimes 2}$.

This shows that for $n=2$, $P_R$ has all the properties of a projection operator. We now consider the general case. To prove that $P_R P_S = \delta_{RS} P_R$ holds for the general case, the Fundamental Orthogonal Relation is used. The relation states:

$$\sum_{\alpha \in G} [\Gamma_a(\alpha)]_{ij} [\Gamma_b(\alpha^{-1})]_{kl} = \delta_{il}\delta_{jk} \frac{|G|}{d_a} \delta_{ab}, \tag{2.113}$$

where $|G|$ is the order of the group and $d_a$ is the dimension of the representation $a$. We therefore have

$$P_R P_S = \frac{d_R d_S}{(n!)^2} \sum_{\phi \in S_n} \sum_{\psi \in S_n} \chi_R(\phi)\chi_S(\psi)\phi\psi. \tag{2.114}$$

Making the substitution $\tau = \psi\phi$, where $\tau \in S_n$, we find

$$\begin{aligned} P_R P_S &= \frac{d_R d_S}{(n!)^2} \sum_{\phi \in S_n} \sum_{\tau \in S_n} \chi_R(\phi)\chi_S(\tau\phi^{-1})\phi\tau\phi^{-1} \\ &= Tr\left(\frac{d_R d_S}{(n!)^2} \sum_{\tau \in S_n} \tau[\Gamma_S(\tau)]_{jk} \sum_{\phi \in S_n} \tau[\Gamma_R(\phi)]_{ii}[\Gamma_R(\phi^{-1})]_{kj}\right) \\ &= Tr\left(\frac{d_R d_S}{(n!)^2} \sum_{\tau \in S_n} \tau[\Gamma_S(\tau)]_{jk}\delta_{kj}\frac{n!}{d_R}\delta_{RS}\right) \\ &= \frac{d_S}{n!} \sum_{\tau \in S_n} \tau\chi_S(\tau)\delta_{RS} \\ &= \delta_{RS} P_S. \end{aligned} \tag{2.115}$$

Therefore the projection property holds for the general case. We also have that the projector $P_R$ commutes with any element of the symmetric group $S_n$. Let $\psi \in S_n$, then

$$\begin{aligned} P_R\psi &= \left(\frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\sigma\psi\right) \\ &= \left(\frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\psi\tau\psi^{-1})\psi\tau\right) \\ &= \left(\psi\frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\tau)\tau\right) \\ &= \psi P_R. \end{aligned} \tag{2.116}$$

Where in the second line the substitution $\sigma = \psi\tau\psi^{-1}$ is used. One last result we need is

$$\mathrm{Tr}(P_R) = d_R Dim_R, \tag{2.117}$$

which we will not prove. It is now rather simple to derive the two point function of Schur polynomials

$$\begin{aligned} \langle \chi_R(Z)\chi_S^\dagger(Z)\rangle &= \frac{1}{d_R d_S} \sum_{\sigma \in S_n} \mathrm{Tr}(P_R\sigma^{-1}P_S\sigma) \\ &= \frac{1}{d_R d_S}\mathrm{Tr}(P_R P_S) \sum_{\sigma \in S_n} \\ &= \frac{n!}{d_R d_S}\mathrm{Tr}(P_R)\delta_{RS} \\ &= \frac{n! Dim_R}{d_R}\delta_{RS} \end{aligned} \tag{2.118}$$

## 2.12 Restricted Schur polynomials

In order to study more complicated systems with more than one field, the restricted Schur polynomials are introduced. These are a multi-matrix generalization to the Schur polynomials considered above. The $\mathcal{N} = 4$ super Yang-Mills Lagrangian in 4 dimensions is given by

$$\mathcal{L} = \frac{1}{g^2}\Big\{D_\mu\phi_i D^\mu\phi_i + \frac{1}{2}[\phi_i, \phi_j]^2 + \dots\}. \tag{2.119}$$

Terms in the Lagrangain involving fermions and gauge boson have been suppressed. The Lagrangian contains six scalar Higgs fields $\phi_i(i = 1, ..., 6)$. These fields transform in the adjoint of the $U(N)$ gauge group. The fields can therefore be considered as $N \times N$ matrices. The six hermittian scalar fields are grouped into three complex matrix fields:

$$Z = \phi_1 + i\phi_2, \quad Y = \phi_3 + i\phi_4, \quad X = \phi_5 + i\phi_6, \tag{2.120}$$

The restricted Schur polynomials we consider are built from these matrix fields. The two point functions of the fields are as follows:

$$\langle Z_{ij}Z^\dagger_{kl}\rangle = \delta_{il}\delta_{jk}, \quad \langle Y_{ij}Y^\dagger_{kl}\rangle = \delta_{il}\delta_{jk}, \quad \langle X_{ij}X^\dagger_{kl}\rangle = \delta_{il}\delta_{jk} \tag{2.121}$$

The spacetime dependence of the above correlators is concealed since it plays no role in the techniques we consider below. Conformal invariance is used to determine the spacetime dependent factors and these terms can then be reinstated in the final result if so desired. For two matrices, the restricted Schur polynomials are defined as

$$\chi_{R,(r,s)\alpha\beta}(Z, Y) = \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)Tr(\sigma Z^{\otimes n}Y^{\otimes m}). \tag{2.122}$$

Here R is an irreducible representation of $S_{n+m}$, labeled by a Young diagram with n+m boxes. The r and s are irreducible representation of $S_n$ and $S_m$ respectively, while $(r, s)$ is an irreducible representation of $S_n \times S_m$. $\chi_{R,(r,s)\alpha\beta}(\sigma)$ is the restricted trace. The trace is restricted in that it is not carried over the carrier space of R, but over the subspace of R that is identified by the carrier space of $(r, s)$. This is possible since the carrier space of $(r, s)$ is a subspace of the carrier space of R. This can be determined by subducing the Young tableaux of $(r, s)$ from the Young tableaux of R. The indices $\alpha$ and $\beta$ are multiplicity indices. They account for the case when a particular irreducible representation is subduced more than once when restricting from $S_{n+m}$ to the subgroup $S_n \times S_m$. Each of these copies is carried by a different subspace of R.

Consider an example with $n=2$ and $m=2$. We have $S_{n+m}=S_4$, $R=\yng(2,2)$ and $(r,s)=(\yng(1,1),\yng(1,1))$. The basis for R is given by the Young tableaux as

$$|\young(43,21)\rangle=\begin{bmatrix}1\\0\end{bmatrix}\equiv|1\rangle \qquad |\young(42,31)\rangle=\begin{bmatrix}0\\1\end{bmatrix}\equiv|2\rangle. \tag{2.123}$$

To obtain $(r,s)$ from R, we remove the relevant number of boxes from R, such that we have two valid Young tableaux. In this case, 2 boxes have to be removed. We can see that this is only possible if we subduce $(r,s)$ from $|2\rangle$. There is only one way of doing this so there is no multiplicity. We therefore have

$$\begin{aligned}\chi_{R,(r,s)\alpha\beta}(\sigma)&=Tr_{(r,s)}(\Gamma_R(\sigma))\\&=\langle 1|\,\Gamma_R(\sigma)\,|1\rangle+\langle 2|\,\Gamma_R(\sigma)\,|2\rangle\\&=\langle 2|\,\Gamma_R(\sigma)\,|2\rangle\,.\end{aligned}\tag{2.124}$$

For the representation $R=\yng(2,2)$, the elements of the group $S_4$ are written as:

$$\Gamma_{\yng(2,2)}((1))=\Gamma_{\yng(2,2)}((13)(24))=\Gamma_{\yng(2,2)}((13)(24))=\Gamma_{\yng(2,2)}((14)(23))=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$\Gamma_{\yng(2,2)}((12))=\Gamma_{\yng(2,2)}((34))=\Gamma_{\yng(2,2)}((1324))=\Gamma_{\yng(2,2)}((1423))=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

$$\Gamma_{\yng(2,2)}((13))=\Gamma_{\yng(2,2)}((24))=\Gamma_{\yng(2,2)}((1234))=\Gamma_{\yng(2,2)}((1432))=-\frac{1}{2}\begin{pmatrix}1&\sqrt{3}\\\sqrt{3}&-1\end{pmatrix},$$

$$\Gamma_{\yng(2,2)}((14))=\Gamma_{\yng(2,2)}((23))=\Gamma_{\yng(2,2)}((1243))=\Gamma_{\yng(2,2)}((1324))=\frac{1}{2}\begin{pmatrix}-1&\sqrt{3}\\\sqrt{3}&1\end{pmatrix},$$

$$\Gamma_{\yng(2,2)}((123))=\Gamma_{\yng(2,2)}((134))=\Gamma_{\yng(2,2)}((142))=\Gamma_{\yng(2,2)}((243))=-\frac{1}{2}\begin{pmatrix}1&-\sqrt{3}\\\sqrt{3}&1\end{pmatrix},$$

$$\Gamma_{\yng(2,2)}((132))=\Gamma_{\yng(2,2)}((143))=\Gamma_{\yng(2,2)}((124))=\Gamma_{\yng(2,2)}((234))=-\frac{1}{2}\begin{pmatrix}1&\sqrt{3}\\-\sqrt{3}&1\end{pmatrix}.\tag{2.125}$$

The $\sigma=(1)$ and $\sigma=(12)$ terms in the restricted Schur polynomial are given by,

$$\begin{aligned}
&\frac{1}{n!m!}\chi_{R,(r,s)\alpha\beta}((1))Tr((1)Z^{\otimes n}Y^{\otimes m})\\
&=\frac{1}{2!2!}\left\langle 2\middle|\Gamma_{\yng(2,2)}((1))\middle|2\right\rangle Tr((1)Z^{\otimes n}Y^{\otimes m})\\
&=\frac{1}{4}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\left(Z^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}Y^{i_3}_{i_{\sigma(3)}}Y^{i_4}_{i_{\sigma(4)}}\right)\\
&=\frac{1}{4}(1)\left(Z^{i_1}_{i_1}Z^{i_2}_{i_2}Y^{i_3}_{i_3}Y^{i_4}_{i_4}\right)\\
&=\frac{1}{4}Tr(Z)^2Tr(Y)^2,
\end{aligned}\tag{2.126}$$

and

$$\begin{aligned}
&\frac{1}{n!m!}\chi_{R,(r,s)\alpha\beta}((12))Tr((12)Z^{\otimes n}Y^{\otimes m})\\
&=\frac{1}{2!2!}\left\langle 2\middle|\Gamma_{\yng(2,2)}((12))\middle|2\right\rangle Tr((1)Z^{\otimes n}Y^{\otimes m})\\
&=\frac{1}{4}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\left(Z^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}Y^{i_3}_{i_{\sigma(3)}}Y^{i_4}_{i_{\sigma(4)}}\right)\\
&=\frac{1}{4}(-1)\left(Z^{i_1}_{i_2}Z^{i_2}_{i_1}Y^{i_3}_{i_3}Y^{i_4}_{i_4}\right)\\
&=-\frac{1}{4}Tr(Z^2)Tr(Y)^2.
\end{aligned}\tag{2.127}$$

Following the same procedure for the remaining 22 terms $\sigma\in S_4$, the complete restricted Schur polynomial is given by,

$$\begin{aligned}
\chi_{\yng(2,2),(\yng(1,1),\yng(1,1))}(Z,Y)=\frac{1}{2}\big[&Tr(Z)^2Tr(Y)^2+Tr(ZY)^2+Tr(YZYZ)+Tr(Z)Tr(Y)Tr(ZY)\\
&+Tr(Z^2Y^2)-Tr(Z)Tr(ZY^2)-Tr(Y)Tr(YZ^2)\\
&-\frac{1}{2}Tr(Z^2)Tr(Y)^2+\frac{1}{2}Tr(Z)^2Tr(Y^2)\big].
\end{aligned}\tag{2.128}$$

The two point function of the restricted Schur polynomial is given by [18],

$$\langle\chi_{R,(r,s)}(Z,Y)\chi^{\dagger}_{T,(t,u)}(Z,Y)\rangle = \delta_{RT}\delta_{rt}\delta_{su}\frac{(hooks)_R}{(hooks)_r(hooks)_s}f_R \tag{2.129}$$

In general the restricted Schur polynomials can be comprised of more than two types of matrices. For instance [19]:

$$\chi_{R,(r,s,t)\alpha\beta}(Z,Y,X) = \frac{1}{n!m!l!}\sum_{\sigma\in S_{n+m+l}}\chi_{R,(r,s,t)\alpha\beta}(\sigma)Tr(\sigma Z^{\otimes n}Y^{\otimes m}X^{\otimes l}), \tag{2.130}$$

defines a Schur polynomial consisting of three types of matrices. The restricted trace $\chi_{R,(r,s,t)\alpha\beta}(\sigma)$ is now taken over the carrier space of R which can be identified as the carrier space of $(r,s,t)$, where $(r,s,t)$ is an irreducible representation of $S_n\times S_m\times S_l$.

# Chapter 3

# Geometry of 1/2 BPS states

## 3.1 Introduction

In the previous section we studied the $1/2$ BPS sector of the $\mathcal{N} = 4$ super Yang Mills theory. We found the operators of this sector that are protected by supersymmetry and annihilated by $1/2$ the supercharges. The states of this sector can be described by a single matrix quantum mechanics with a harmonic oscillator potential. In the semi-classical limit, these states can be characterised by free fermions in a 2 dimensional phase space.

In this chapter we study the dual gravity description of the $1/2$ BPS sector. These are the $1/2$ BPS IIB supergravity solutions discovered by Lin, Lunin and Maldacena [7]. These so called LLM geometries have the same symmetry and supersymmetry as the $1/2$ BPS states of the dual gauge theory. Moreover the fermion phase space can be identified with a 2 dimensional plane embedded in the 10 dimensional geometry. The whole family of LLM solutions are given in terms of a single auxiliary function $z$, that also determines the fermion distribution on the phase space. There are strong constraints on the supergravity solutions due to the symmetry requirements. This together with the condition for regularity, result in a considerable simplification on this class of solutions. The regularity of the solutions is imposed by suitable boundary conditions on the auxiliary function $z$.

In the next section we review the explicit construction of the LLM geometries as in [7]. We begin by building an ansatz metric for the geometries based on the bosonic symmetries of the gauge theory operators. We then study the Killing spinor equation, which imposes maximal supersymmetry on the solutions. In section 3.3 we check if the geometries are indeed regular. We then analyse the solutions in section 1.4 and provide an example of a supergravity solution in section 1.5. In the last section we discuss the $1/2$ BPS dictionary and the specify the operators relevant for our purposes.

## 3.2 Review of the LLM construction

In this section we review the construction of the 1/2 BPS supergravity backgrounds, discovered by Lin, Lunin and Maldacena [7]. These are the so called LLM geometries.

We begin by matching the symmetries of the gauge theory with the isometries of the supergravity metric. The relevant symmetry group of the 1/2 BPS states is the bosonic subgroup $SO(4) \times SO(4) \times \mathcal{R}$, where the first $SO(4)$ corresponds to the R-charge and is due to the invariance of the operators under the rotation of the other four scalar fields $\phi^i$. The second $SO(4)$ is from the isometry of the $S^3$ from the conformal reduction to $R \times S^3$. The $\mathcal{R}$ comes from the Hamiltonian $\Delta - J$, which arises from the 1/2 BPS condition. Additionally the states preserve 16 of the supersymmetries.

We therefore look for supergravity solutions whose metric has the $SO(4) \times SO(4) \times \mathcal{R}$ isometry. This class of solutions asymptote to $AdS_5 \times S^5$, while preserving 1/2 the supersymmetries of the $AdS_5 \times S^5$. An ansatz for the metric is thus given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + e^{H+G} d\Omega_3^2 + e^{H-G} d\tilde{\Omega}_3^2, \tag{3.1}$$

where $\mu, \nu = 0, 1, 2, 3$. The two 3-spheres correspond to the two $SO(4)$ groups, while the killing vector corresponds to the time translation of the Hamiltonian $\Delta - J$. From the gauge operators chosen, we expect only D3 branes to be included in the geometry. Thus we assume only the five-form flux $F_{(5)}$ is excited. The five-form field is expected to admit the same symmetry as that of the geometry, thus the following ansatz

$$F_{(5)} = F_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3. \tag{3.2}$$

The Hodge dual of the five form field, must itself be a five form field in the coordinates transverse to those of the original field. This implies the two 3-spheres are dual to each other and that $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ are dual to each other in the four dimensional base space

$$F = e^{3G} *_4 \tilde{F}, \qquad F = dB, \qquad \tilde{F} = d\tilde{B}. \tag{3.3}$$

In addition to satisfying the bosonic symmetries, the LLM geometries are required to be maximally supersymmetric. This is imposed by requiring that all supersymmetry variations of the fields vanish. Additionally we assume the dilaton and axion are constant and the 3-form field strength from string excitations is zero. The variation of the gravitino field $\psi^I_{\mu\alpha}$ are also set to zero. This leads to the Killing spinor equation

$$\nabla_M \eta + \frac{i}{480}\Gamma^{M_1M_2M_3M_4M_5}F^{(5)}_{M_1M_2M_3M_4M_5}\Gamma_M\eta = 0. \tag{3.4}$$

The solution to this equation is the Killing spinor on the geometry given by the metric ansatz (3.1). The Killing spinor allows us to construct vector and tensors that preserve maximal supersymmetry under isometries of the metric. By solving the Killing spinor equation, we are able to build supersymmetric invariant fields and relate the various quantities in the ansatz. To analyse (3.4), we follow closely the methods described in [20–22].

We begin by selecting a suitable basis for the 10 dimensional gamma matrices in terms of 4 dimensional gamma and Pauli matrices

$$\Gamma_\mu = \gamma_\mu \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \qquad \Gamma_a = \gamma_5 \otimes \sigma_a \otimes \mathbb{1} \otimes \hat{\sigma}_1, \qquad \Gamma_{\hat{a}} = \gamma_5 \otimes \mathbb{1} \otimes \tilde{\sigma}_a \otimes \hat{\sigma}_2, \tag{3.5}$$

where the $\mu$ refers to the 4 dimensional space-time and $a, \tilde{a}$ refer to the two 3-spheres. The spinor $\eta$ satisfies the chirality condition

$$\Gamma_{11}\eta = \gamma^5\hat{\sigma}_3\eta = \eta \tag{3.6}$$

and the 10 dimensional gamma matrices satisfy relations

$$\Gamma_{11}\eta = \prod \Gamma_\mu \prod \Gamma_a \prod \Gamma_{\tilde{a}} = \gamma^5\hat{\sigma}_3, \qquad \gamma^5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3. \tag{3.7}$$

We can consider a spinor $\chi$ on an arbitrary unit 3-sphere. From the isometry of the 3-sphere, the spinor representation that transforms under the rotation group associated with the isometries is

$$\nabla_c\chi = a\frac{i}{2}\gamma_c\chi, \qquad a = \pm 1 \tag{3.8}$$

where $a$ is correlated with the chirality and $\nabla_c$ is the covariant derivative on the unit 3-sphere. Taking into account the warp factors appearing in the metric ansatz, the covariant derivatives on the two 3-spheres are given by

$$\nabla_a = \nabla'_a - \frac{1}{4}\Gamma^\mu{}_a\partial_\mu(H+G), \qquad \nabla_{\tilde{a}} = \nabla'_{\tilde{a}} - \frac{1}{4}\Gamma^\mu{}_{\tilde{a}}\partial_\mu(H-G) \tag{3.9}$$

where $\nabla'$ contains the spin connection on the unit 3-spheres. The 10 dimensional Killing spinor can now be decomposed into a product of two spinors $\chi_a$ and $\tilde{\chi}_{\tilde{a}}$ on the 3-spheres and a spinor $\epsilon_{a,b}$ on the remaining 4 dimensions

$$\eta = \epsilon_{a,b} \otimes \chi_a \otimes \tilde{\chi}_{\tilde{a}} \tag{3.10}$$

where $\chi_a, \tilde{\chi}_{\tilde{a}}$ obey equation (3.8). We define

$$M = \frac{i}{480}\Gamma^{M_1M_2M_3M_4M_5}F^{(5)}_{M_1M_2M_3M_4M_5}. \tag{3.11}$$

Applying the change in basis,

$$M = -\frac{1}{4}e^{-\frac{3}{2}(H+G)}\gamma^{\mu\nu}F_{\mu\nu}\gamma^5\hat{\sigma}^1. \tag{3.12}$$

With this, the Killing spinor equation reduces to a system of equations involving only the 4 dimensional spinor $\eta$

$$(iae^{-\frac{1}{2}(H+G)}\gamma_5\hat{\sigma}_1 + \frac{1}{2}\gamma^\mu\partial_\mu(H+G))\epsilon + 2M\epsilon = 0, \tag{3.13}$$

$$(ibe^{-\frac{1}{2}(H-G)}\gamma_5\hat{\sigma}_2 + \frac{1}{2}\gamma^\mu\partial_\mu(H-G))\epsilon - 2M\epsilon = 0, \tag{3.14}$$

$$\nabla_\mu\epsilon + M\gamma_\mu\epsilon = 0. \tag{3.15}$$

We now define a set of spinor bilinears, whose variations are used to construct the supergravity solution, these are

$$\begin{aligned} f_1 = i\,\bar{\epsilon}\hat{\sigma}_1\epsilon, \qquad f_2 = i\bar{\epsilon}\hat{\sigma}_2\epsilon, \qquad K_\mu = -\bar{\epsilon}\gamma_\mu\epsilon, \\ L_\mu = -\,\bar{\epsilon}\gamma^5\gamma_\mu\epsilon, \qquad Y_{\mu\nu} = \bar{\epsilon}\gamma_{\mu\nu}\hat{\sigma}_1\epsilon, \qquad \omega_\mu = \epsilon^t\Gamma^2\gamma_\mu\epsilon, \end{aligned} \tag{3.16}$$

where $\bar{\epsilon} = \epsilon^\dagger\Gamma^0$. The variations are given by

$$\begin{aligned} \nabla_\mu f_1 &= -e^{-\frac{3}{2}(H-G)}\tilde{F}_{\mu\sigma}K^\sigma, \qquad \nabla_\mu f_2 = -e^{-\frac{3}{2}(H+G)}\tilde{F}_{\mu\nu}K^\nu, \\ \nabla_\mu K_\nu &= -e^{-\frac{3}{2}(H+G)}\left(F_{\mu\nu}f_2 - \frac{1}{2}\epsilon_{\nu\mu\alpha\beta}F^{\alpha\beta}f_1\right), \\ \nabla_\mu L_\nu &= -e^{-\frac{3}{2}(H+G)}\left(\frac{1}{2}g_{\mu\nu}F_{\alpha\beta}Y^{\alpha\beta} + F_\mu{}^\rho Y_{\rho\nu} + F_\nu{}^\rho Y_{\rho\mu}\right). \end{aligned} \tag{3.17}$$

We also note that $K_\mu$ can be shown to be a Killing vector

$$\nabla_\nu K_\mu + \nabla_\mu K_\nu = 0, \tag{3.18}$$

and that $L_\mu dx^\mu$ is a locally exact form. This implies the space is contractable, and thus a new coordinate y can be defined as

$$L_\mu dx^\mu = \gamma dy, \qquad \gamma = \pm 1. \tag{3.19}$$

We are now able to build the 4 dimensional metric using the coordinate y as

$$ds^2 = h^2 dy^2 + g_{mn} dx^m dx^n, \qquad m, n = 1, 2, 3 \tag{3.20}$$

where the other 3 coordinates are orthogonal to y. We can show using the Fierz identities, that

$$K \cdot L = 0, \qquad L^2 = -K^2 = f_1^2 + f_2^2. \tag{3.21}$$

This implies the Killing vector $K_\mu$ has no $y$-component $K_\mu = K_m$.

We require a single time-like Killing vector to correspond to the Hamiltonian. This implies that $K_m$ must have a single component $K_m = K_t$. This component can be normalized as $K_t = 1$, such that the time-like Killing vector is expressed as $K_t = \partial_t$. $K_t$ imposes certain restrictions on the metric (3.20), such that it becomes

$$ds^2 = h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + h_{ij} dx^i dy^j), \qquad i, j = 1, 2 \tag{3.22}$$

where the second Fierz identity $K^2 = -L^2$, is used to relate the two metric components $g_{tt}$ and $g_{yy}$. We now use the spinor billinears $f_1$ and $f_2$ to derive the expressions of the forms in terms of the 4 dimensional scalar fields $H, G$ and the gauge potentials $B$ and $\tilde{B}$.

Since $K_t = 1$, the variation of $f_2$, becomes

$$\partial_\mu f_2 = -e^{\frac{3}{2}(H+G)} F_{\mu t} = -e^{\frac{3}{2}(H+G)} \partial_\mu B_t \tag{3.23}$$

where the fact that $B_i$ is independant of $t$ is used. We can express $\partial_\mu B_t$ as

$$\partial_\mu B_t = F_{\mu\nu} K^\nu = -F_{\mu\nu} \bar{\epsilon} \gamma^\mu \epsilon = \frac{1}{4} \bar{\epsilon} [\gamma_\mu, \not{F}] \epsilon. \tag{3.24}$$

Substituting (3.12) into (3.13), and multiplying with $\gamma^5 \hat{\sigma}_1$, we find the following expression together with it's adjoint

$$\begin{aligned} \frac{1}{2} e^{-\frac{3}{2}(H+G)} \not{F} \epsilon &= (iae^{-\frac{1}{2}(H+G)} + \frac{1}{2} \gamma_5 \not{\partial}(H+G) \hat{\sigma}_1) \epsilon \\ \frac{1}{2} e^{-\frac{3}{2}(H+G)} \bar{\epsilon} \not{F} &= \bar{\epsilon} (iae^{-\frac{1}{2}(H+G)} + \frac{1}{2} \gamma_5 \not{\partial}(H+G) \hat{\sigma}_1), \end{aligned} \tag{3.25}$$

where $\not{F} = \gamma_{\lambda\sigma} F^{\lambda\sigma}$. Using the last two equations, the expressions for $\partial_\mu B_t$ and $\partial_\mu f_2$ become

$$\partial_\mu B_t = -e^{\frac{3}{2}(H+G)} \frac{1}{2} \partial_\mu (H+G) f_2, \tag{3.26}$$

$$\partial_\mu f_2 = \frac{1}{2} f_2 \partial_\mu (H+G). \tag{3.27}$$

The solutions of these equation are given by

$$f_2 = 4\alpha e^{\frac{1}{2}(H+G)}, \qquad \tilde{B}_t = -\alpha e^{2(H+G)}. \tag{3.28}$$

We follow a similar procedure to find the expressions of $f_1$ and $B_t$ as

$$f_1 = 4\beta e^{\frac{1}{2}(H-G)}, \qquad B_t = -\beta e^{2(H-G)}. \tag{3.29}$$

We can find an expression for $H$ by adding (3.14) and (3.15), multiplying with $\hat{\sigma}_1$ and using the chirality condition (3.6) to find

$$\hat{\sigma}_1 \partial\!\!\!/ H\epsilon = (-iae^{-\frac{1}{2}(H+G)}\gamma_5 + be^{-\frac{1}{2}(H-G)})\epsilon, \tag{3.30}$$

$$\bar{\epsilon}\hat{\sigma}_1 \partial\!\!\!/ H = -\bar{\epsilon}(-iae^{-\frac{1}{2}(H+G)}\gamma_5 + be^{-\frac{1}{2}(H-G)}). \tag{3.31}$$

By multiplying (3.30) with $\frac{1}{2}i\bar{\epsilon}\gamma_\mu$ and (3.31) with $\frac{1}{2}i\gamma_\mu\epsilon$, the coordinate dependance of the scalar $H$ is found to be $y = e^H$, where the constants $\alpha, \beta$ are chosen to be $\alpha = \beta = 1/4$. This imposes $a = b$. This choice leads to the expression for $h$

$$h^{-2} = f_1^2 + f_2^2. \tag{3.32}$$

Using the fact that $H$ only depends on $y$, (3.30) can be written as

$$\left(\frac{1}{hy}\sigma\Gamma^3 + iae^{-\frac{1}{2}(H+G)}\gamma_5 - be^{-\frac{1}{2}(H-G)}\right)\epsilon = 0. \tag{3.33}$$

The equation can be further simplified by using (3.28), (3.29),(3.32) and factoring out $e^{-\frac{1}{2}(H-G)}$, to find

$$\left(\sqrt{1+e^{-2G}}\hat{\sigma}_1\Gamma^3 + aie^{-G}\gamma_5 - a\right)\epsilon = 0. \tag{3.34}$$

Using the time-like Killing vector $K_t$, together with $\bar{\epsilon} = \epsilon^\dagger\gamma^0 = \epsilon^\dagger\Gamma^0$ and the fact that $L_y = -a$ we find the following relation

$$\epsilon^\dagger\Gamma^0\Gamma^5\Gamma^3\epsilon = -a. \tag{3.35}$$

The operator $\Gamma^0\Gamma^5\Gamma^3$ is unitary and $|a| = 1$, thus we find the projection conditions

$$\left(1 + a\Gamma^0\Gamma^5\Gamma^3\right)\epsilon = 0, \qquad (1 + ia\Gamma_1\Gamma_2)\,\epsilon = 0 \tag{3.36}$$

using $\Gamma^5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$.

With the projector conditions, we can determine the explicit form of the Killing spinor $\epsilon$ in terms of the gamma matrices, a parameter $\delta$ related to the scalar $G$, and a second spinor $\epsilon_1$

$$\epsilon = e^{i\delta\gamma^5\Gamma^3\hat{\sigma}^1}\epsilon_1, \qquad \Gamma^3\hat{\sigma}^1\epsilon = a\epsilon_1, \qquad \sinh 2\delta = ae^{-G}. \tag{3.37}$$

By inserting the last expression into the definition of $f_1$, we find constraints on the scale of $\epsilon_1$ in terms of another spinor $\epsilon_0$:

$$\epsilon_1 = e^{\frac{1}{4}(H-G)}\epsilon_0, \qquad \epsilon_0^\dagger\epsilon_0 = 1. \tag{3.38}$$

We can set the value of $\epsilon_0$ to be a constant on the space-time by setting it's phase to zero with an appropriate Lorentz rotation. In this way the coordinate dependance of the Killing spinor comes only from it's dependance on the fields. Using this form of the Killing spinor, we can now calculate the closed form of the bilinear $\omega_\mu$. The only non-vanishing components $\omega_1$ and $\omega_2$ are

$$\omega_1 = \epsilon^t\Gamma^2\Gamma^1\epsilon = -iah^{-1}\epsilon_0^t\epsilon_0, \qquad \omega_2 = \epsilon^t\Gamma^2\Gamma^2\epsilon = h^{-1}\epsilon_0^t\epsilon_0. \tag{3.39}$$

Using vielbeins $e^a_\mu$, the one-form $\omega_\mu$ can be expressed as

$$\omega = \omega_{\hat{c}}e^{\hat{c}}_\mu dx^\mu = h^{-1}(-iae^{\hat{1}}_\mu + e^{\hat{2}}_\mu)dx^i. \tag{3.40}$$

The relation $h^{-2} = y(e^G + e^{-G})$ and the fact that $\omega$ is a closed one-form $d\omega = 0$, implies the vielbeins have no $y$ dependance and that the two dimensional metric $h_{ij} = \delta_{ij}$ is flat. Using the self duality condition of the field (3.3), the gauge potentials $B$ and $\tilde{B}$ can be separated into temporal and spatial components and be expressed in terms of the vectors $V_i$ as

$$\begin{aligned} B &= B_t(dt+V) + \hat{B}, \qquad & d\hat{B} + B_t dV &= -h^2e^{3G}*_3 d\tilde{B}_t, \\ \tilde{B} &= \tilde{B}_t(dt+V) + \hat{\tilde{B}}, \qquad & d\hat{\tilde{B}} + \hat{\tilde{B}}_t dV &= h^2e^{-3G}*_3 dB_t, \end{aligned} \tag{3.41}$$

where $*_3$ is the flat 3 dimensional Hodge dual operator in the $y, x_1, x_2$ directions. Using he expression $K = -h^{-2}(dt+V)$ for the Killing vector, we can write the anti-symmetric part of the variation of $K_\nu$ in (3.17) as

$$\frac{1}{2}dK = -\frac{1}{2}d[h^{-2}(dt+V)] = e^{-(H+G)}F + e^{-(H-G)}\tilde{F}. \tag{3.42}$$

We are now able to find an expression for $V$ as

$$\begin{aligned} dV &= 2h^2[e^{-(H+G)}(d\tilde{B} + B_t dV) - e^{-(H-G)}(d\hat{\tilde{B}} + \tilde{B}_t dV)] \\ &= h^2(e^{-(H-2G)}*_3 d\tilde{B}_t - e^{-(H+2G)}*_3 dB_t) \\ &= 2h^4 y *_3 dG = \frac{1}{y}*_3 dz. \end{aligned} \tag{3.43}$$

Where we have used $h^{-2} = y(e^G + e^{-G})$, together with the following expression of the gauge fields:

$$\begin{aligned} \tilde{B}_t &= -\frac{1}{4}e^{2(H-G)}, \qquad d\tilde{B}_t = -\frac{1}{2}e^{2(H-G)}(dH - dG), \\ B_t &= -\frac{1}{4}e^{2(H+G)}, \qquad dB_t = -\frac{1}{2}e^{2(H+G)}(dH + dG), \end{aligned} \tag{3.44}$$

and have defined a new function $z$ as

$$z = \frac{1}{2}\tanh G. \tag{3.45}$$

From (3.41) and (3.45), we find the gauge fields in terms of $z$ as

$$d\hat{B} = -\frac{1}{4}y^3 *_3 d\left(\frac{z + 1/2}{y^2}\right), \qquad d\hat{\tilde{B}} = -\frac{1}{4}y^3 *_3 d\left(\frac{z - 1/2}{y^2}\right). \tag{3.46}$$

The fact that $dV$ is closed $d(dV) = 0$, leads to the following equation for $z$:

$$\begin{aligned} \partial_\alpha\left(\frac{1}{y}*_3 \partial_\alpha z\right) &= 0, \\ \frac{1}{y}\partial_i^2 z + \partial_y\left(\frac{1}{y}\partial_y z\right) &= 0, \end{aligned} \tag{3.47}$$

where $\alpha = x_1, x_2, y$ and $i = x_1, x_2$. Thus we have the solutions to the general 1/2 BPS IIB supergravity background as:

$$ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2 \tag{3.48}$$

$$h^{-2} = 2y\cosh G, \qquad z = \frac{1}{2}\tanh G, \tag{3.49}$$

$$y\partial_y V_i = \epsilon_{ij}\partial_j z, \qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z, \tag{3.50}$$

$$F = dB_t \wedge (dt + V) + B_t dV + d\hat{B}, \tag{3.51}$$

$$\tilde{F} = d\tilde{B}_t \wedge (dt + V) + \tilde{B}_t dV + d\hat{\tilde{B}}, \tag{3.52}$$

$$B_t = -\frac{1}{4}y^2 e^{2G}, \qquad \tilde{B}_t = -\frac{1}{4}y^2 e^{-2G}, \tag{3.53}$$

$$d\hat{B} = -\frac{1}{4}y^3 *_3 d\left(\frac{z + \frac{1}{2}}{y^2}\right), \qquad d\hat{\tilde{B}} = -\frac{1}{4}y^3 *_3 d\left(\frac{z - \frac{1}{2}}{y^2}\right). \tag{3.54}$$

We have then the remarkable result that the entire family of solutions to the 10 dimensional 1/2 BPS supergravity backgrounds are given in terms of a single function $z(x_1, x_2, y)$. The function obeys a simple linear differential equation (3.47). It is important to note that the coordinate $y$ is the product of the two radii of the two 3-spheres.

## 3.3 Regularity of the Supergravity solutions

With the 1/2 BPS supergravity solutions specified by the function $z$, there is an additional constraint that needs to be imposed on the backgrounds. We need to check if the geometries have any singularities, that is, verify if the solutions are regular.

The 1/2 BPS backgrounds have a metric with two 3-spheres with radii squared $r^2 = ye^{-G}$ and $\tilde{r}^2 = ye^{G}$, thus, $r^2\tilde{r}^2 = y^2$. It is clear that on the coordinate $y = 0$, where the product of the radii vanishes, there is a potential singularity in the background solution, as it would seem six of the dimensions from the 3-spheres have shrunk to a point at this coordinate. The singularity can be avoided by realising that one of the 3-spheres combine with $y$ to form a 4 dimensional flat space-time with spherical coordinates, while the other radius remains finite. For this to occur, the following constraints are imposed at $y = 0$:

$$G \to \pm\infty, \implies z = \pm\frac{1}{2}. \tag{3.55}$$

The two cases are analysed. For $z = \frac{1}{2}$, the function $z(x_1, x_2, y)$ is expanded for $y$ close to zero:

$$z(x_1, x_2, 0) = g_0(x_1, x_2) + yg_1(x_1, x_2) + y^2 g_2(x_1, x_2) + \dots \tag{3.56}$$

From (3.49), it can be shown that $z = \frac{1}{2}\sqrt{1 - 4h^4y^2}$, thus

$$g_0 = z(x_1, x_2, 0) = \frac{1}{2} \qquad \text{and} \qquad g_1 = \left.\frac{\partial^2 z}{\partial y^2}\right|_{y=0} = 0. \tag{3.57}$$

On the other hand

$$z = \frac{1}{2}\frac{e^G - e^{-G}}{e^G + e^{-G}} \approx \frac{1}{2}(1 - 2e^{-2G} + \dots). \tag{3.58}$$

Thus we have $-e^{-2G} = y^2 g_2$. We can set $g_2 = -b^2$, then $e^{-G} = by$ and $h^{-2} = \frac{y}{yb} + y^2 b \to \frac{1}{b}|_{y=0}$. The metric becomes

$$
\begin{aligned}
ds^2 &= \frac{1}{b}(dt + V_i dx^i)^2 + b(dy^2 + dx^i dx^i) + \frac{1}{b} d\Omega_3^2 + b y^2 d\tilde{\Omega}_3^2, \\
ds^2 &= \frac{1}{b}(dt + V_i dx^i)^2 + b(dy^2 + y^2 d\tilde{\Omega}_3^2) + b dx^i dx^i + \frac{1}{b} d\Omega_3^2.
\end{aligned}
\tag{3.59}
$$

We can see the 3-sphere $\tilde{\Omega}_3$ has combined with the coordinate y to form a 4 dimensional flat spacetime with a spherical coordinate metric, while the second 3-sphere $\Omega_3$ has a finite radius $r = \frac{1}{b}$. In this way the singularity is avoided at $y = 0$. Particularly, in the $y$ and $\tilde{\Omega}_3$ directions, the metric becomes

$$ds^2 \sim b(dy^2 + y^2 d\tilde{\Omega}_3^2). \tag{3.60}$$

A similar argument applies for $z = -\frac{1}{2}$, with the metric becoming

$$ds^2 = \frac{1}{b}(dt + V_i dx^i)^2 + b(dy^2 + y^2 d\Omega_3^2) + b dx^i dx^i + \frac{1}{b} d\tilde{\Omega}_3^2 \tag{3.61}$$

and the metric at $y = 0$ is

$$ds^2 \sim b(dy^2 + y^2 d\Omega_3^2). \tag{3.62}$$

The transformation $z \to -z$ and the exchange of the two 3-spheres in the metric, corresponds to the particle-hole transformation in the fermion phase space.

## 3.4 Analysis of the Supergravity Solutions

The constraints $z = \pm\frac{1}{2}$ required for the supergravity solutions to be regular, impose boundary conditions on (3.47). Therefore a solution to the 1/2 BPS background is fully specified by the data on the $(x_1, x_2)$ plane on which $z(x_1, x_2, 0) = \pm\frac{1}{2}$. It is convention to assign black and white colours respectively to points on the $(x_1, x_2)$ plane where $z = -\frac{1}{2}$ and $z = \frac{1}{2}$. In this way, the $(x_1, x_2)$ plane can be identified with the fermion phase space, where a point coloured black denotes a fermion and a point coloured white denotes a hole. This connection can be further demonstrated by considering a black circular region on the $(x_1, x_2)$ plane with center on the origin, denoted by $\mathcal{D}$. With this boundary condition, the supergravity solution has energy

$$\Delta = J = \int_{\mathcal{D}} \frac{d^2x}{2\pi\hbar} \frac{(x_1^2 + x_2^2)}{2\hbar} - \frac{1}{2}\bigg(\int_{\mathcal{D}} \frac{d^2x}{2\pi\hbar}\bigg)^2. \tag{3.63}$$

This coincides with the energy of N fermions in a harmonic oscillator potential minus the energy of the ground state of the N fermions in the same potential. In this way, the $(x_1, x_2)$ plane has a natural interpretation as a 2 dimensional fermion phase space. The fermion phase space is the same as that of the eigenvalue description of the matrix model description in the dual CFT. For a total area $\mathcal{A}$, the quantisation condition of the droplet is related to the five from flux as

$$N = \frac{\mathcal{A}}{2\pi\hbar} \tag{3.64}$$

where $\hbar = 2\pi l_p$ and $N$ is the number of fermions in the phase space.

## 3.5 Example of a Supergravity solution: The ground state configuration

In this section we consider an example of a 1/2 BPS supergravity solution. With the boundary conditions for the function $z$ specified by the regularity of the supergravity solutions, a new function $\phi = \frac{z}{y^2}$ can be defined. With this, (3.47) becomes Laplace's equation in a 6 dimensional space-time, where $(x_1, x_2)$ make up the flat 2 dimensional plane, and the remaining 4 dimensions have spherical symmetry. The coordinate $y$ corresponds to the radial direction in the 4 dimensional subspace. For boundary conditions specified on the $(x_1, x_2)$ plane, the general solution is given by

$$z(x_1, x_2, y) = \frac{y^2}{\pi} \int_{\mathcal{D}} \frac{z(x_1', x_2', 0) dx_1' dx_2'}{[(\mathbf{x} - \mathbf{x}')^2 + y^2]^2}, \tag{3.65}$$

$$V_i(x_1, x_2, y) = \frac{\epsilon_{ij}}{\pi} \int_{\mathcal{D}} \frac{z(x_1', x_2', 0)(x_j - x_j') dx_1' dx_2'}{[(\mathbf{x} - \mathbf{x}')^2 + y^2]^2}. \tag{3.66}$$

The simplest configuration in the fermion phase space is a circular droplets radius $R_0 = \sqrt{2\hbar N}$. This configuration corresponds to the $AdS_5 \times S^5$ geometry with a five form flux of N units.

With the radial symmetry of the configuration it is convenient to move to spherical coordinates $(x_1, x_2) \to (R, \phi)$. Then the radial component of $V_i$ vanishes, $V_R = V_1 \cos\phi + V_2 \sin\phi = 0$, thus $V$ becomes $V \equiv V_\phi = -R(V_1 \sin\phi - V_2 \cos\phi)$. The relation between $z$ and $V$ (3.50) becomes

$$y\partial_y V = -R\partial_R z, \qquad \frac{1}{R}\partial_R V = \frac{1}{y}\partial_y z, \tag{3.67}$$

and the solutions (3.65) and (3.66) are

$$\begin{aligned} z(R,y) &= -\int z(R',0)\frac{\partial}{\partial R'} z_0(R,y;R')dR' \\ z_0(R,y;R') &= \frac{R^2 - R'^2 + y^2}{2[(R^2+R'^2+y^2)^2 - 4R^2R'^2]^{1/2}} \end{aligned} \tag{3.68}$$

$$\begin{aligned} V(R,y) &= \int z(R',0)V_0(R,y;R')dR' \\ V_0(R,y;R') &= \frac{-2R^2R'^2(R^2 - R'^2 + y^2)}{2[(R^2+R'^2+y^2)^2 - 4R^2R'^2]^{3/2}} \end{aligned} \tag{3.69}$$

This can be inserted into the metric ansatz (3.1), together with the change in coordinates[1]

$$y = R_0 \sinh\rho \sin\theta, \quad R = R_0 \cosh\rho\cos\theta, \quad \phi = \tilde{\phi} + t, \tag{3.70}$$

to recover the $AdS_5 \times S^5$ metric

$$ds^2 = R_0(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\theta^2 + \cos^2\theta d\tilde{\phi}^2 + \sin^2\theta d\tilde{\Omega}_3^2). \tag{3.71}$$

## 3.6 Discussion

We have reviewed the explicit construction of the full set of 1/2 BPS supergravity solutions discovered by Lin, Lunin and Maldacena [7]. These geometries are dual to the 1/2 BPS states in the gauge theory. The supergravity solutions are found by considering the symmetry and supersymmetry of the gauge theory states. The solutions are also required to be regular. The geometries are found to have an $SO(4) \times SO(4) \times \mathcal{R}$ isometry groups and are asymptotically $AdS_5 \times S^5$. The whole set of solutions is constructed from a single function $z(x_1, x_2, y)$. The requirement of regularity imposes boundary conditions $z = \pm\frac{1}{2}$ on the $(x_1, x_2)$-plane. The $(x_1, x_2)$-plane has a natural interpretation as a fermion phase space with harmonic oscillator potential, that matches exactly that of the matrix model description of the gauge theory.

The energy of the supergravity solutions $J$ matches exactly that of the fermion distribution in the phase space. The $AdS_5 \times S^5$ space-time can be viewed as the ground state solution, while other solutions are excitations of the $AdS_5 \times S^5$ space-times. The solutions can be described in terms of the excitation energy $J$. The BPS condition $\Delta = J$ relates the conformal dimension[1]of the gauge theory to the excitation energy $J$. A dictionary of the 1/2 BPS sector between $\mathcal{N} = 4$ SYM and IIB string theory can be organised according to $J$.

For small excitations $J << N$ [2], we study operators of small dimension that are dual to gravitons in the bulk. Increasing the energy $J \sim \sqrt{N}$, the gauge theory operators describe string propagating in the bulk. The operators are better described in the trace basis due to orthogonality. Increasing the energy further $J \sim N$, the trace basis is no longer orthogonal. The operator are now better described by the Schur polynomials [11, 13]. These operators are dual to $D3$-branes wrapping on either the $S^3 \subset S^5$ or the $S^3 \subset AdS_5$ 3-spheres of the $AdS_5 \times S^5$ background. With an excitation energy $J \sim N^2$, we have operators of large dimension. These operators are dual to order N $D3$-branes, whose backreaction in the bulk leads to new classical geometries that are asymptotically $AdS_5 \times S^5$.

In our investigations of how the entanglement in the gauge theory is related to the space-time in the bulk, we are particularly interested in these operators of large dimension. The states dual to these operators are heavy enough that they affect the bulk space-time in non-trivial ways. In the next chapter we will study the entanglement entropy of such states in the gauge theory. The entanglement entropy has a dual description in term of the bulk geometry, thus these states provide a platform on which we are able to compare the entanglement entropy on both sides of the duality.

[1]In the 1/2 BPS sector, the conformal dimension of a operator is equal to the number of field in the operator.
[2]Here, N is the rank of the matrix field $Z$

# Chapter 4

# Entanglement Entropy and Holography

## 4.1 Introduction

It is still not known how holography works. In recent times, entanglement has played a key role in understanding the nature of holography. This stems from the fact that entanglement connects the quantum information of a system to its geometry. This connection is provided by the Ryu-Takayanagi formula, which proposes that the entropy of a subsystem corresponds to a particular surface in the corresponding bulk spacetime. Entanglement is a property of any quantum system and has no classical analog. Thus by studying its role in holography, we hope to learn how gravity can emerge from some fundamental description.

In this chapter we begin by studying the entanglement entropy as a measure of entanglement in quantum mechanics and its properties. We then look at entanglement entropy in the AdS/CFT correspondence. We study how entanglement entropy is calculated in the CFT with the *replica trick*. We then proceed to study the Ryu-Takayanagi formula as a calculation of entanglement entropy in terms of the bulk spacetime. With the background theory presented, we provide calculations based on group theory techniques for relevant correlators in the CFT that result in a recursive formula. In the final section we review the idea of emergent spacetime from entanglement.

## 4.2 Definition and properties of Entanglement Entropy

### 4.2.1 Quantum states and Von Neumann Entropy

The physical state $\psi$ of a quantum system can be often described by a vector $|\psi\rangle$ living on a Hilbert space $\mathcal{H}$. The state can more generally be given by the density matrix $\rho$. The density matrix is a self-adjoint, positive, linear operator with unit trace $\text{Tr}(\rho) = 1$. The state of the quantum system is said to be a pure state if it is known exactly and can be represented by a single vector $|\psi\rangle$ in the Hilbert space $\mathcal{H}$. In this case the density matrix reduces to $\rho = |\psi\rangle \langle\psi|$, with a single eigenvalue $\lambda = 1$. Otherwise, the system is in an ensemble of pure states $\{p_i, \psi_i\}$ referred to as a mixed state, where $p_i$ is the probability of the system to be in pure state $|\psi\rangle_i$. The associated density matrix is given by

$$\rho = \sum_i p_i \, |\psi\rangle_i \, \langle\psi|_i \, , \tag{4.1}$$

where the coefficients $p_i$ are non-negative and $\sum_i p_i = 1$. The density matrix provides a convenient quantity to describe not only pure and mixed states of closed quantum systems, but also open subsystems of a composite system. With the density matrix formalism, a state can be determined to be pure or mixed with the simple criterion

$$\text{Tr}(\rho^2) \leq 1, \tag{4.2}$$

where the equality only holds for a pure state.

Suppose we perform a quantum measurement of an operator $\mathcal{O}$ on a quantum system in a state described by density matrix $\rho$. Assume the operator $\mathcal{O}$ and the density operator $\rho$ have a common set of eigenstates, and that the measurement has $N$ possible outcomes, where $N = dim\{\mathcal{H}\}$. Prior to the measurement, the von Neumann entropy provides a measure of our ignorance about the outcome. Alternatively, the von Neumann entropy can be seen as a measure of the amount of information that can be gained from the measurement. For a system in a state $\rho$, the von Neumann entropy of the system is defined as

$$S(\rho) = -\text{Tr}[\rho \log \rho] \tag{4.3}$$

Choosing a basis $|\psi_i\rangle$ that diagonalises the density matrix $\rho$, the von Neumann entropy can be written as

$$S(\rho) = -\sum_{i=1}^{N} [\lambda_i \log \lambda_i] \tag{4.4}$$

where the $\lambda_i$'s are the eigenvalues of $\rho$. From this definition, it is clear that the von Neumann entropy of a pure state $\rho = |\psi\rangle \langle\psi|$ is $S(\rho) = 0$, since $\lambda = 1$. This is in agreement with the interpretation above since there is no uncertainty in the measurement of such a state. Moreover for an operator $\mathcal{O}$ such that

$$\mathcal{O}\left|\psi\right\rangle = \lambda\left|\psi\right\rangle, \tag{4.5}$$

no information is gained when the measurement is performed, which is reflected in zero entropy $S(\rho) = 0$. For a maximal mixed state $\rho = \sum_i^N \frac{1}{N}\left|\psi_i\right\rangle\left\langle\psi_i\right|$, where a measurement reveals the most information about the state, the entropy attains its maximum value

$$S(\rho) = -\sum_i^N \frac{1}{N}\log\frac{1}{N} = \log N \tag{4.6}$$

For a general non-maximally mixed state, the von Neumann entropy has range $0 \leq S(\rho) \leq \log N$.

### 4.2.2 Quantum subsystems and Entanglement

Suppose we have a quantum system in a pure state $\rho = \left|\psi\right\rangle\left\langle\psi\right|$, with a corresponding Hilbert space $\mathcal{H}$. We can divide the system into two parts $A$ and its compliment $B$. We then have a composite system and the corresponding Hilbert space factorises into a tensor product of the Hilbert spaces of the two subsystems as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The pure state $\left|\Psi\right\rangle_{AB}$ of the composite system can be written in the terms of the Schmidt decomposition as

$$\left|\Psi\right\rangle_{AB} = \sum_{k=1}^D \lambda_k \left|\phi_k^A\right\rangle \otimes \left|\chi_k^B\right\rangle, \tag{4.7}$$

where $\left|\phi_k^A\right\rangle$ and $\left|\chi_k^B\right\rangle$ are an orthonormal basis of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Suppose we only have access to a subsystem $A$. How can we describe the physics of this subsystem? To do this we would need to neglect all information contained in $B$, thereby defining a reduced density matrix $\rho_A$ of the subsystem $A$. This is achieved by performing a partial trace on the density matrix of the composite state

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}), \tag{4.8}$$

where $\mathrm{Tr}_B$ is the partial trace over the subsystem $B$. The partial trace is the unique operator that maps from the density operator $\rho_{AB}$ on the composite space $\mathcal{H}_A \otimes \mathcal{H}_B$, to the reduced operator $\rho_A$ of the subsystem A on $\mathcal{H}_A$ [23]. The density matrix of the composite system can be written as $\rho_{AB} = \sum_{ijkl} c_{ijkl}\left|\psi_i^A\right\rangle\left\langle\psi_j^A\right| \otimes \left|\phi_k^B\right\rangle\left\langle\phi_l^B\right|$, where $\{\psi_i\}$ is a basis for $\mathcal{H}_A$ and $\{\phi_k\}$ is a basis for $\mathcal{H}_B$. The partial trace acts as follows

$$\mathrm{Tr}_B\left[\left|\psi_i^A\right\rangle\left\langle\psi_j^A\right| \otimes \left|\phi_k^B\right\rangle\left\langle\phi_l^B\right|\right] = \left|\psi_i^A\right\rangle\left\langle\psi_j^A\right| \mathrm{Tr}_B\left(\left|\phi_k^B\right\rangle\left\langle\phi_l^B\right|\right). \tag{4.9}$$

Thus, we find the reduced density matrix of the subsystem $A$ as

$$\mathrm{Tr}_B \rho_{AB} = \sum_{ijkl} c_{ijkl} \left\langle \phi_k^B \middle| \phi_l^B \right\rangle \left|\psi_i^A\right\rangle \left\langle \psi_j^A\right| \equiv \rho_A \tag{4.10}$$

where $\left\langle \phi_k^B \middle| \phi_l^B \right\rangle \in \mathbb{C}$. From (4.2), we can see that the state of the subsystem can be either pure or mixed. For $\mathrm{Tr}(\rho_A^2) = 1$, the subsystem $A$ is in a pure state and the composite state (4.7) can be written as a

$$|\Psi_{AB}\rangle = |\psi_A\rangle \otimes |\phi_B\rangle \tag{4.11}$$

and the von Neumann entropy of subsystem $A$, $S(\rho_A) = -1\log 1 = 0$. However for $\mathrm{Tr}(\rho_A^2) \leq 1$, the state $\rho_A$ is mixed. This means the state is entangled with $B$, and thus the composite state cannot be written as a product state. The state of $A$ cannot be represented as a single vector in the Hilbert space $\mathcal{H}_A$, but as an ensemble of pure states $\{\psi_i\}$. The entanglement on $A$ is due to the fact that the degrees of freedom of the subsystem are mixed with that of subsystem $B$. The entanglement of subsystem $A$ is measured by the entanglement entropy which is defined as

$$S(\rho_A) = -\mathrm{Tr}[\rho_A \log \rho_A]. \tag{4.12}$$

The entanglement entropy of $A$ can be interpreted as a measure of the amount of information we can gain about the rest of the system from a measurement on $A$. Thus if $S(\rho_A) = 0$, the composite state $|\Psi_{AB}\rangle$ is a product state and we gain no information about system $B$ from a measurement of system $A$. However if the subsystems are entangled $S(\rho_A) \neq 0$, we gain some information about $B$ from a measurement on $A$. This implies that there are correlations between the two subsystems. To illustrate this, we consider an example.

Suppose we have a spin-less scalar particle decaying into two spin $1/2$ half particles $A$ and $B$. The conservation of angular momenta requires that the two particles have opposite spin about the $z$-axis, without specifying the spin of either particle. Such a state can be expressed as

$$|\Psi_{AB}\rangle = \sqrt{p}\,|{\uparrow_z}\rangle_A\,|{\downarrow_z}\rangle_B + \sqrt{1-p}\,|{\downarrow_z}\rangle_A\,|{\uparrow_z}\rangle_B\,, \tag{4.13}$$

where $p \in [0,1]$. Thus for $p = 0$ or $1$, the composite state is a product state and both $A$ and $B$ have pure states. However in general where $p \in (0,1)$, $A$ and $B$ are entangled, and the state of $A$ cannot be expressed as a pure state, but as a statistical ensemble

$$\rho_A = \mathrm{Tr}_B\left(|\Psi_{AB}\rangle\langle\Psi_{AB}|\right) = p\,|{\uparrow_z}\rangle_A\,\langle{\uparrow_z}|_A + (1-p)\,|{\downarrow_z}\rangle_A\,\langle{\downarrow_z}|_A\,. \tag{4.14}$$

It is clear that a measurement on $A$ reveals information about $B$ since the systems must have opposite spin to each other. The entanglement entropy of $A$ is given by

$$S(\rho_A) = -\mathrm{Tr}\left(\rho_A \log \rho_A\right) = -[p\log p + (1-p)\log(1-p)]. \tag{4.15}$$

Thus the entanglement entropy is zero only for $p = 0$ or $1$ as expected. The entanglement entropy has a maximum value for $p = 1/2$, where either particle has equal probability to be spin up or spin down.

## 4.3 Properties of Entanglement Entropy

The entanglement entropy has several properties that are useful and emphasise why it is a measure of entanglement. The first of these is that for a composite system in a pure state $\rho_{AB}$, the entanglement entropy of $A$ is equal to that of its compliment $B$

$$S(\rho_A) = S(\rho_B). \tag{4.16}$$

This follows from the Schmidt decomposition (4.7). Since the eigenvalues $\lambda_i$ are real, positive and sum to unity, the density matrix of the two subsystems $A$ and $B$ are

$$\rho_A = \sum_i^n \lambda_i \left|\psi_i^A\right\rangle \left\langle\psi_i^A\right| \qquad \rho_B = \sum_i^n \lambda_i \left|\phi_i^B\right\rangle \left\langle\phi_i^B\right|, \tag{4.17}$$

where $n \leq \min\{\dim\{\mathcal{H}_A\}, \dim\{\mathcal{H}_B\}\}$. Hence we have (4.16). It is clear that the entanglement entropy is an intensive quantity i.e. independent of the system size, thus we are able to discuss the entanglement entropy between subsystems without considering the asymmetry of the Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$. Another consequence of $(4.16)$ is that the entanglement entropy can be at most the log of the dimension of the smaller subsystem. This also highlights the fact that the entanglement entropy gives a measure of the degrees of freedom of the system. This becomes an important property when we consider entanglement entropy in AdS/CFT [24].The equality $(4.16)$ does not hold however when the composite system is in a mixed state. In this case the entanglement entropy not only encodes quantum entanglement of the system, but also thermodynamic correlations.

The entanglement entropy also obeys several important inequalities. The most important of these is the strong subadditivity of entanglement entropy. Consider dividing the system into three subsystems $A, B$ and $C$. Then the entanglement entropy of the subsystems satisfy the following inequalities

$$S(\rho_{AB}) + S(\rho_{BC}) \geq S(\rho_C) + S(\rho_A), \tag{4.18}$$

$$S(\rho_{AB}) + S(\rho_{BC}) \geq S(\rho_{ABC}) + S(\rho_B). \tag{4.19}$$

From these inequalities we can derive the triangle inequality

$$|S(\rho_A) - S(\rho_B)| \leq S(\rho_{AB}) \tag{4.20}$$

and the subadditivity property

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B). \tag{4.21}$$

The subadditivity inequality saturates if the composite state factorises as $\rho_{AB} = \rho_A \otimes \rho_B$. The inequalities $(4.18), (4.19), (4.20)$ and $(4.21)$ hold even if the composite state $\rho_{ABC}$ is mixed. This can be seen from the fact that for a mixed state $\rho_{ABC}$, an auxiliary system can be added such that the combination is a pure state of a larger system. This process is called a purification of $\rho_{ABC}$ [25]. Thus the state of the total system is a mixed state found by tracing out the auxiliary system. This together with (4.18), (4.19) and (4.16) leads to (4.20) and (4.21) [26].

From the subadditivity property (4.21), we can naturally define another measure of entanglement between $A$ and $B$ called *mutual information*, defined as

$$I(A : B) = S(\rho_A) + S(\rho_A) - S(\rho_{AB}) \geq 0. \tag{4.22}$$

The mutual information measures the amount of correlation between the two subsystems. It can be used as a bound on the correlation functions $\mathcal{O}_A$ and $\mathcal{O}_B$ acting only within $A$ and $B$ respectively as [27],

$$I(A : B) \geq \frac{(\langle \mathcal{O}_A \mathcal{O}_A \rangle - \langle \mathcal{O}_A \rangle \langle \mathcal{O}_A \rangle)^2}{2 \left| \mathcal{O}_A \right|^2 \left| \mathcal{O}_A \right|^2}. \tag{4.23}$$

The mutual information vanishes if the subsystems $A$ and $B$ are not entangled

$$I(A : B) = 0 \iff \rho_{AB} = \rho_A \otimes \rho_B. \tag{4.24}$$

## 4.4 Entanglement Entropy in Holography

So far we have only discussed entanglement entropy in quantum mechanics. However, entanglement entropy has played a key role in recent development in high energy physics, following attempts to understand how gravity emerges from quantum field theory in the context of holography. One of the most important proposals in the framework of the AdS/CFT correspondence, is the *Holographic Entanglement Entropy* (HEE), which conjectures that the entanglement entropy of a conformal field theory (CFT) defined on a $(d+1)$ dimensional boundary of an AdS spacetime, can be found by a minimal surface in the $(d+2)$ dimensional AdS spacetime [5, 6, 28].

Thus the HEE proposal turns entropy into a geometric quantity by connecting quantum information and gravitational physics. Therefore HEE allows us to tackle the question of how the extra dimensions of the bulk spacetime emerge from the quantum information of the boundary CFT.

We begin by considering entanglement entropy in the boundary CFT and its calculation *via* the *Replica Trick*. Then we look at the formula proposed by Ryu and Takayanagi to calculate entanglement entropy in the bulk spacetime [5].

## 4.5 Entanglement Entropy in the CFT

### 4.5.1 Area law in the CFT

We consider a CFT in a $(d+1)$ dimensional spacetime. The definition and properties of entanglement entropy in the previous section apply to any general CFT. The degrees of freedom (DOF) of a CFT are local. Thus the theory can be formulated such that the Hilbert space is a tensor product of Hilbert spaces at each point. This is best demonstrated when the theory is formulated on a lattice, where the DOF are associated with the lattice points, for more see [29, 30] . Thus separating the Hilbert space of the CFT into a tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ corresponds to separating the spacetime on which the theory is defined into two spatial regions $A$ and $B$. The boundary $\partial A$ that separates the two regions is called the entangling surface. The entangling surface can be either a physical horizon or an imaginary surface.

With the entangling surface $\partial A$ defined, the DOF of fields in region $B$ can be traced out, to find the reduced density matrix $\rho_A$ and the entanglement entropy $S(\rho_A)$ of system $A$. We can consider the CFT in a vacuum state $\rho_{AB} = |0\rangle \langle 0|$. The vacuum of the CFT is a highly entangled state and as a result the entanglement entropy of the subsystem is divergent. This can be seen by noting that the field modes can have arbitrarily short wavelengths. Hence the subsystems have an infinite number of DOF. Thus an ultra-violet cut-off $\epsilon$ must be introduced. Then we find that the entanglement entropy of subsystem A has, up to a constant, the following expression

$$S(\rho_A) = \frac{\text{Area}(\partial A)}{\epsilon} + \dots \tag{4.25}$$

where the leading term is proportional to the area of the entangling surface $\partial A$. This is called the area law of entanglement entropy. This can be understood intuitively by considering that in the CFT, the entanglement between the two subsystems occurs mostly on the entangling surface $\partial A$. In this way, the entanglement entropy depends on the geometry of the boundary $\partial A$

### 4.5.2 The Replica Trick

In general, the calculation of the entanglement entropy in a CFT is a difficult procedure. The standard technique used is called the *Replica Trick* [31–33]. To implement the procedure, we begin by introducing a general set of quantities useful in characterising entanglement. These are the Rényi entropies of the reduced density matrix $\rho_A$ defined as

$$S_n(\rho_A) = \frac{1}{1-n}\log \mathsf{Tr}(\rho_A^n), \tag{4.26}$$

where $n \in \mathbb{R}$, $n \geq 0$ and $n \neq 1$. The entanglement entropy $S(\rho_A)$ is obtained by taking the limit $n \to 1$,

$$S(\rho_A) = \lim_{n\to 1}\frac{\log \mathsf{Tr}(\rho_A^n)}{1-n} = -\lim_{n\to 1}\frac{\mathsf{Tr}(\rho_A^n \log \rho_A)}{\mathsf{Tr}(\rho_A^n)} = -\mathsf{Tr}(\rho_A \log \rho_A). \tag{4.27}$$

Where in the first equality, l'Hospital's rule is used, along with $\mathsf{Tr}(\rho_A) = 1$ on the last line. An equivalent and more convenient way of writing this is

$$S(\rho_A) = \lim_{n\to 1}\frac{\partial}{\partial n}\mathsf{Tr}(\rho_A^n). \tag{4.28}$$

Thus the problem of calculating the entanglement entropy in a given CFT reduces to finding the quantity $\mathsf{Tr}(\rho_A^n)$ for $n \geq 1$. The usefulness of writing the problem in this way comes from the fact the reduced density matrix $\rho_A$, and hence $\mathsf{Tr}(\rho_A^n)$, has a simple representation in the path integral formalism. The problem is then left to the existence and uniqueness of a proper analytic continuation of $n$, to extract the entanglement entropy from (4.28).

### 4.5.3 Path Integral Formalism

For simplicity we consider a $(1+1)$-dimensonal CFT in a flat Euclidian spacetime $(t_E, x) \in \mathbb{R}^2$, where $t_E$ is the Euclidean time. The dynamical fields of the theory are denoted by $\phi(t_E, x)$. We take the spatial subsystem $A$ to be an interval $x \in [u, v]$ at a time $t_E = 0$. The ground state of the CFT is found by a Euclidean path integral from $t_E = -\infty$ to $t_E = 0$

$$\Psi[\phi_0(x)] = \int_{\phi(t_E=-\infty,x)}^{\phi(t_E=0,x)=\phi_0(x)} D\phi e^{-S_E[\phi]} \tag{4.29}$$

where $S_E[\phi]$ is the Euclidean action. The value of the fields at the boundary $t_E = 0$ depend on the spatial coordinate $x$. The conjugate ground state is found by path integrating from $t_E = \infty$ to $t_E = 0$. We can construct the total density matrix as $[\rho]_{\phi_0\phi_0'} = \Psi(\phi_0)\bar{\Psi}(\phi_0')$. To define the reduced density matrix $\rho_A$, the fields can be demarcated in two sets belonging to the two subsystems as $\phi_0(x) = \{\phi_0^A(x), \phi_0^B(x)\}$ and the boundary conditions

$$\phi^A|_{t=0^-} = \phi_-, \qquad \phi^A|_{t=0^+} = \phi_+ \tag{4.30}$$

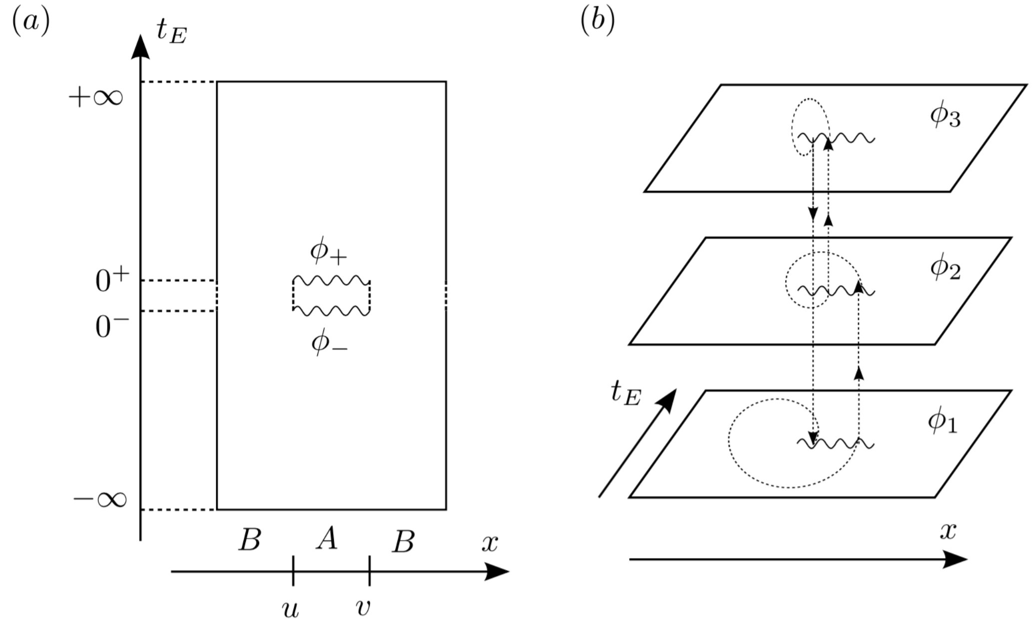



Figure 4.1: (a) The path integral representation of the reduced density matrix $[\rho_A]_{\phi_+\phi_-}$ (b) The $n$-sheeted Riemann surface with $n = 3$. This figure is taken from [28].

can be imposed for the fields in $A$ and $\phi_- = \phi_+$ for all $x \in B$. These boundary conditions have the effect of introducing a branch cut for the path integral on the interval $A$. Imposing these boundary conditions, the elements of the reduced density matrix are found to be

$$[\rho]_{\phi_+\phi_-} = (Z_1)^{-1} \int_{\phi(t_E=-\infty,x)}^{\phi(t_E=-\infty,x)} D\phi e^{-S_E[\phi]} \prod_{x\in A} \delta(\phi(0^+,x) - \phi_+(x))\, \delta(\phi(0^-,x) - \phi_-(x)) \quad (4.31)$$

where $Z_1$ is the vacuum partition function on $\mathbb{R}^2$ and its inverse is a normalization factor that ensures $tr_A\rho_A = 1$. This computation is depicted in figure 4.1 (a).

The interpretation of this formula is that the partial trace $\mathrm{Tr}_B[\rho]$ is achieved by allowing the periodicity of the fields on $B$, while imposing boundary conditions (4.30) on $A$. The functions $\phi_-$ and $\phi_+$ can be seen as indices of the reduced density matrix $[\rho_A]_{\phi_-\phi_+}$. To calculate the quantity $\mathrm{Tr}[\rho_A^n]$, we introduce $n$-copies, also called *replicas*, of the two dimensional spacetime on which the CFT is defined. To find the trace, the replicas are glued to each other along the branch cuts as

$$[\rho]_{\phi_{1+}\phi_{1-}}[\rho]_{\phi_{2+}\phi_{2-}} \dots [\rho]_{\phi_{n+}\phi_{n-}}$$

by imposing the boundary condition on $\{\phi_{i\pm}(x)\}$ as $\phi_{i-}(x) = \phi_{(i+1)+}(x)$ and $\phi_{n-}(x) = \phi_{1+}(x)$, where $1 \le i \le n$ for all $x \in A$. The result of this process is a path integral on a spacetime structure that is an $n$-sheeted Riemann surface $\mathcal{R}_n$. This is shown in figure 4.1 (b), for $n = 3$. Thus the quantity Tr$[\rho_A^n]$ is given by

$$\text{Tr}\rho_A^n = (Z_1)^{-n} \int_{(t,x)\in\mathcal{R}_n} D\phi e^{-S[\phi]} = \frac{Z_{\mathcal{R}_n}}{(Z_1)^n}$$

where $Z_n$ is the partition function on the n-sheeted Riemann surface $\mathcal{R}_n$ and $Z_1$ is the partition function on $\mathbb{R}^2$. Depending on the existence of a unique analytical continuation to Re$(n) > 1$, the entanglement entropy can be extracted as

$$S(\rho_A) = -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_{\mathcal{R}_n}}{(Z_1)^n}. \tag{4.32}$$

Thus the entanglement entropy of $A$ reduces to a path integral on Riemann surface $\mathcal{R}_n$. It is important to note the theory has a symmetry under the cyclic permutations of the fields $\phi_i$ i.e. $i \to i+1$ or $i \leftarrow i+1$ for $i = 1, \ldots, n$ and $n+1 \equiv 1$.

## 4.6 Rényi Entanglement entropy of Local Operator Excited states

Here we study the Rényi entanglement entropy of local operator excited states. Consider a CFT on a d-dimensional flat spacetime $\mathbb{R}^{1,d-1}$, whose time and $d-1$ dimensional space is denoted by $t$ and $x^i$ $(i = 1, 2, \ldots, d-1)$. The excited state $|\Psi_O\rangle$ is defined by acting the local operator $O(t, x^i)$ on the vacuum state $|0\rangle$ at a time $t = 0$ and position $x^i$

$$|\Psi_O\rangle = O(0, x^i)\,|0\rangle$$

with the time evolution under the Hamiltonian described by

$$|\Psi_O(t)\rangle = \sqrt{\mathcal{N}} e^{-iHt} e^{\epsilon H} O(0, x^i)\,|0\rangle\,,$$

where $\mathcal{N}$ is the normalization and the parameter $\epsilon$ is the ultraviolet regulator of the local operator $O(0, x^i)$.

We aim to calculate the Rényi entanglement entropy $S_A^{(n)}$. To do this, A is chosen to be the half space $x_1 > 0$ and we insert the local operator $O$ at position

$$x^1 = -l(< 0) \qquad \mathbf{x} = (x^2, x^3, \ldots, x^{d-1})$$

at a time $t$. Perform the Euclidian continuation $\tau = it$ and introduce the complex coordinates

$$\omega = x^1 + i\tau, \qquad \bar{\omega} = x^1 - i\tau.$$

The density matrix for the locally excited state is:

$$\begin{aligned}\rho(t) =&\mathcal{N}\, e^{-iHt}e^{-\epsilon H}O(0,x^i)\left|0\right\rangle \left\langle 0\right| O^\dagger(0,x^i)e^{-\epsilon H}e^{iHt}\\ =&\mathcal{N}O(\omega_2,\bar{\omega_2},\mathbf{x})\left|0\right\rangle \left\langle 0\right| O^\dagger(\omega_1,\bar{\omega_1},\mathbf{x})\end{aligned}$$

where $\mathcal{N}$ is determined by the condition Tr$[\rho(t)]=1$ and

$$\begin{aligned}\omega_1 =&i(\epsilon-it)-l, \qquad \omega_2=-i(\epsilon+it)-l,\\ \bar{\omega}_1=&-i(\epsilon-it)-l, \qquad \bar{\omega}_2=i(\epsilon+it)-l.\end{aligned}$$

We are interested in studying the excess of the Rényi entanglement entropies $\Delta S_A^{(n)}$ obtained by subtracting the ground state entropy from $S_A^{(n)}$:

$$\Delta S_A^{(n)}=\frac{1}{1-n}\log\left(\frac{tr\rho_A^n}{tr(\rho_A^0)^n}\right)$$

where $\rho_A^{(0)}$ is the ground state reduced density matrix. We will compute $\Delta S_A^{(n)}$ by the replica method. Introducing polar coordinates, the wave functional $(4.29)$ becomes

$$\Psi[\phi_0^i(x)]=\int_{\phi(\tau=-\infty,x^i)}^{\phi(\tau=0,x^i)=\phi_0(x^i)} D\phi\,\mathcal{O}(r_1,\theta_1)e^{-S[\phi]} \tag{4.33}$$

and the components of the reduced matrix are now given by

$$[\rho]_{\phi_1(x^i)\phi_2(x^i)}=\frac{1}{Z_1^{EX}}\int_{\phi(-\infty,x^i)}^{\phi(\infty,x^i)} D\phi\mathcal{O}^\dagger(r_2,\theta_2)\mathcal{O}(r_1,\theta_1)e^{-S[\phi]}\delta(\phi(-\delta,x^i)-\phi_1(x^i))\delta(\phi(\delta,x^i)-\phi_2(x^i))$$

where $Z_1^{EX}$ is defined as

$$Z_1^{EX}=\int_{\phi(-\infty,x^i)}^{\phi(\infty,x^i)} D\phi\mathcal{O}^\dagger(r_2,\theta_2)\mathcal{O}(r_1,\theta_1)e^{-S[\phi]}.$$

As before, the $n$-sheeted Riemann surface $\Sigma_n$ is constructed by gluing subsystem A on a sheet to subsystem A on the next sheet. The local operators are periodically inserted on $\Sigma_n$, with $\theta_{i,k}=\theta_i+2\pi(k-1)(i=1,2.\quad k=1,\dots,n)$, where $k$ denotes the $k$-th sheet. Then

$$\begin{aligned}tr_A\hat{\rho}_A^n=\frac{Z_n^{EX}}{(Z_1^{EX})^n}=\frac{1}{(Z_1^{EX})^n}\int_{\phi(-\infty,x^i)}^{\phi(\infty,x^i)} &D\phi\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\dots\\ &\mathcal{O}^\dagger(r_2,\theta_{2,k})\mathcal{O}(r_1,\theta_{1,k})\dots\mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})e^{-S[\phi]}.\end{aligned} \tag{4.34}$$

The ground state reduced density matrix is given by

$$tr_A\hat{\rho}_0^n = \frac{Z_n}{Z_1}. \tag{4.35}$$

Here $Z_n$ is the partition function on $\Sigma_n$ and $Z_1$ is the partition function on $\mathcal{R}^{d+1}$. We find the path integral representation of the excess to be

$$\begin{aligned}
\Delta S_A^{(n)} =& \frac{1}{1-n}\left(\log\frac{tr_A\rho_A^n}{\rho_{0A}} - n\log\frac{tr\rho}{tr\rho_0}\right)\\
=& \frac{1}{1-n}\left(\log\frac{Z_n^{EX}}{Z_n} - n\log\frac{Z_1^{EX}}{Z_1}\right)\\
=& \frac{1}{1-n}\big(\log\langle\mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\ldots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\rangle_{\Sigma_n} -\\
& n\log\langle\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\rangle\big).
\end{aligned} \tag{4.36}$$

## 4.7 The Ryu-Takayanagi Formula

Having reviewed the replica trick as a way of calculating the entanglement entropy in the CFT, we will now consider the dual calculation of the entropy in the bulk spacetime. We review the Ryu-Takayanagi formula that proposes the entanglement entropy of a spatial subsystem $A$ on the $d$-dimensional boundary CFT is proportional to a $d$-dimensional surface in the bulk $AdS_{d+1}$ spacetime [5, 6, 28].

To state the proposal, consider a holographic CFT defined on a $d$-dimensional spacetime, in a state $|\Psi\rangle$. We choose a time-slice and then divide the spacetime into two spatial subsystems $A$ and $B$. The entangling surface $\partial A$ is a $(d-1)$-dimensional surface separating $A$ and $B$. We know from the area law that the entanglement entropy of $A$ is proportional to the entangling surface $\partial A$.

To find the entanglement entropy of $A$ in terms of the bulk spacetime, the entangling surface $\partial A$ is extended into the bulk geometry to produce a $d$-dimensional surface $\gamma_A$. A restriction on this surface $\gamma_A$, is that its boundary must coincide with that of the entangling surface $\partial\gamma_A = \partial A$. There is in fact, an infinite number of bulk surfaces that satisfy this condition. The Ryu-Takayanagi conjecture claims that the correct surface that is proportional to the entanglement entropy of $A$, is the one whose surface takes a minimum value. Thus the first order variation of the area functional of the surface $\gamma_A$ must vanish. Thus the Ryu-Takayanagi formula gives the entanglement entropy of the subsystem $A$ as

$$S(\rho_A) = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+1)}} \tag{4.37}$$

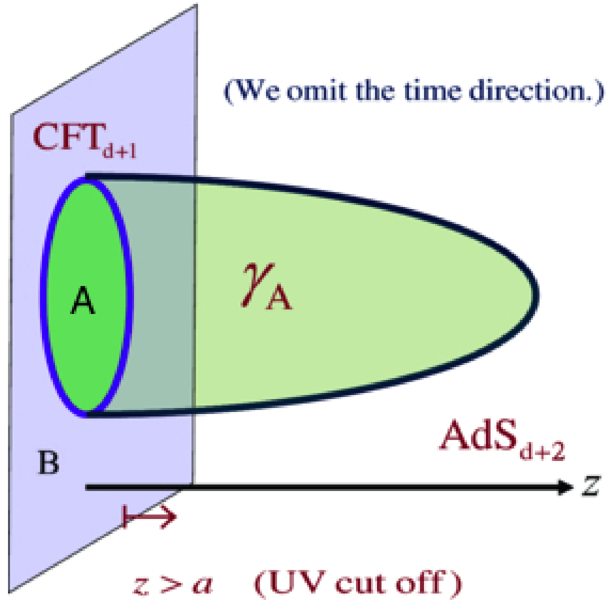



Figure 4.2: Minimal Surface used to compute $S(\rho_A)$ by the Ryu-Takayanagi formula.

where $G_N^{(d+1)}$ is Newton's constant in a $(d+1)$-dimensional $AdS_{(d+1)}$ gravitational theory. The minimal surface is shown in Figure 4.2.

## 4.8 Properties of the Ryu-Takayanagi Formula.

The Ryu-Takayanagi formula (4.37) is divergent. This is due to the fact that the surface $\gamma_A$ extends all the way to the boundary of the $AdS$ spacetime. This is consistent with the earlier observation that the entanglement entropy in the CFT is a divergent quantity. We must regulate by introducing a UV cut-off $\epsilon$. This is dual to an IR cut-off in the bulk which corresponds to keeping only the $z > \epsilon$ part of the bulk geometry. With this, the entanglement entropy on both sides of the correspondence can be shown to match up to terms that vanish as the cut-off is removed.

If the CFT is in a pure state $|\Psi\rangle$, then the equality (4.16) is satisfied by the Ryu-Takayanagi formula, as both $A$ and $B$ will have the same minimal surface in the bulk i.e. $\gamma_A = \gamma_B$.

We saw in the CFT that the strong subadditivity (4.18) and (4.19) are important properties of the entanglement entropy. Using the Ryu-Takayanagi formula, we can now prove this formula in terms of the bulk geometry. Consider dividing the boundary CFT into three spatial subsystems $A, B$ and $C$. The subsystems are chosen such that they do not overlap. We consider the entanglement

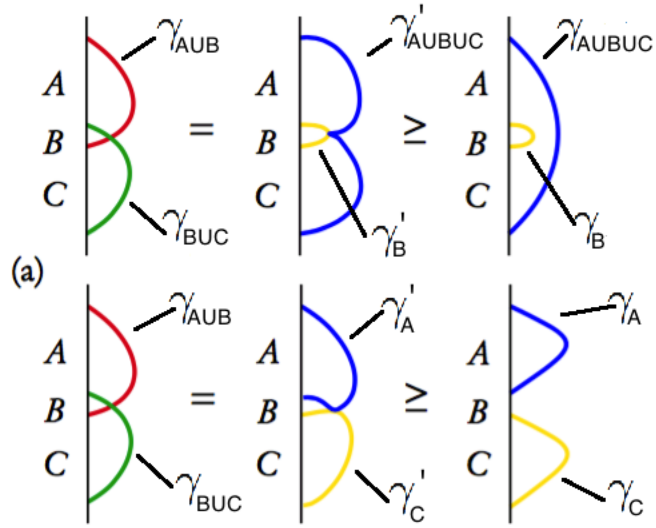



Figure 4.3: Strong subadditivity proof in terms of the minimal surface.

entropies $S_{A\cup B}$ and $S_{B\cup C}$ given by the minimal surfaces $\gamma_{A\cup B}$ and $\gamma_{B\cup C}$ respectively. The surfaces satisfy $\partial\gamma_{A\cup B} = \partial(A\cup B)$ and $\partial\gamma_{B\cup C} = \partial(B\cup C)$. As shown in Figure 4.3, the minimal surfaces $\gamma_{A\cup B}$ and $\gamma_{B\cup C}$ can be divided and then modified into the intermediate surfaces $\gamma'_B$ and $\gamma'_{A\cup B\cup C}$, as well as $\gamma'_A$ and $\gamma'_C$, such that they satisfy the strong subadditivity inequalities

$$S(\rho_{AB}) + S(\rho_{BC}) \geq S(\rho_C) + S(\rho_A), \tag{4.38}$$

$$S(\rho_{AB}) + S(\rho_{BC}) \geq S(\rho_{ABC}) + S(\rho_B). \tag{4.39}$$

Note the inequality arises from the fact that the intermediate surfaces are not minimal. By satisfying the strong subadditivity inequality, the Ryu-Takayanagi formula subsequently satisfies the triangle inequality as well as the subaddivity inequality.

The Ryu-Takayanagi formula can be seen as a generalisation of the Bekenstein-Hawking formula [3, 4],

$$S = \frac{\text{Area}(\Sigma)}{4G_N} \tag{4.40}$$

where $\Sigma$ is the area of the event horizon of the black hole. Consider the boundary CFT at a finite temperature. This is dual to an $AdS$ spacetime with a black hole. In this set up, the CFT is not in a pure state, but a thermal state and thus as stated before $S_A \neq S_B$. This is due to the fact that minimal surfaces $\gamma_A$ and $\gamma_B$ have to wrap around the black hole horizon $\Sigma$, and in doing so become different and unequal.

In the limit where $B$ is shrunk to zero and $A$ covers the entire boundary, the minimal surface $\gamma_A$ is expected to completely wrap around the black hole horizon $\Sigma$. In this way, the Bekenstein-Hawking formula (4.40) is recovered. In this limit, the entropy of the boundary is not due quantum, but rather thermal correlations.

## 4.9 Greens functions on the $n$-sheeted Riemann surface.

For the n-sheeted Riemann surface we are considering, the Green's function is given by (see Appendix A for derivation) [33],

$$G_n(r,s,\theta,\theta',\mathbf{x},\mathbf{x}') = \frac{1}{2\pi n}\sum_{l=0}^{\infty} d_l \int_0^\infty dk \int \frac{dk_\perp^{d-1}}{(2\pi)^{d-1}} \frac{k\cdot J_{l/n}(kr)J_{l/n}(ks)}{k^2+k_\perp^2} e^{i\mathbf{k}_\perp\cdot(\mathbf{x}-\mathbf{x}')}\cos\left(\frac{\theta-\theta'}{n}l\right), \tag{4.41}$$

where $d_0=1$, $d_{l>0}=2$, $k$ is the momentum on the $(\tau,x^1)$ and $k_\perp$ is the momentum on the $(\tau,\mathbf{x})$ coordinates. To begin the integration, the denominator $k^2+k_\perp^2$ is rewritten using Schwinger's trick

$$\frac{1}{k^2+k_\perp^2} = \int_0^\infty du e^{-(k^2+k_\perp^2)u}. \tag{4.42}$$

We can then perform the integration over $k_\perp$, as follows

$$\begin{aligned}
&\frac{1}{(2\pi)^{d-1}}\int dk_\perp^{d-1} e^{-uk_\perp^2+i\mathbf{k}_\perp\cdot(\mathbf{x}'-\mathbf{x})}\\
&= \frac{1}{(2\pi)^{d-1}}\int dk_2\dots dk_d e^{-u(k_2^2+\dots+k_d^2)+i[k_2(x_2'-x_2)+\dots+k_d(x_d'-x_d)]}\\
&= \frac{1}{(2\pi)^{d-1}}\int dk_2 e^{-uk_2^2+k_2[i(x_2'-x_2)]}\dots\int dk_d e^{-uk_d^2+k_d[i(x_d'-x_d)]}\\
&= \frac{1}{(2\pi)^{d-1}}\left(\sqrt{\frac{\pi}{u}}e^{-\frac{(x_2'-x_2)^2}{4u}}\right)\dots\left(\sqrt{\frac{\pi}{u}}e^{-\frac{(x_d'-x_d)^2}{4u}}\right)\\
&= \frac{1}{(2\pi)^{d-1}}\left(\sqrt{\frac{\pi}{u}}\right)^{d-1} e^{-\frac{1}{4u}[(x_2'-x_2)^2+\dots+(x_d'-x_d)^2]}\\
&= \frac{1}{(2\sqrt{\pi u})^{d-1}}e^{-\frac{1}{4u}(\mathbf{x}'-\mathbf{x})^2}
\end{aligned} \tag{4.43}$$

For the integration over $k$, we use an identity for Bessel functions [34], to find

$$\int_0^\infty dkke_{-uk^2}J_{l/n}(kr)J_{l/n}(ks) = \frac{1}{2u}e^{-\frac{r^2+s^2}{4u}}I_{l/n}\Big(\frac{rs}{2u}\Big), \tag{4.44}$$

where $I_{l/n}(x)$ is the modified Bessel function, which is given in terms of the Schlafli integral

$$I_{l/n}\left(\frac{rs}{2u}\right) = \frac{1}{2\pi i}\int_{\infty-i\pi}^{\infty+i\pi} dt e^{\frac{rs}{2u}\cosh t - \frac{l}{n}t}. \tag{4.45}$$

The $u$ integral is given by

$$\int_0^\infty du u^{-\frac{d+1}{2}} e^{-\frac{\Delta}{4u}}, \tag{4.46}$$

where $\Delta = (\mathbf{x}' - \mathbf{x})^2 + r^2 + s^2 - 2rs\cosh t$. Making the substitution $v = \Delta/4u$, we find

$$\begin{aligned}
\int_0^\infty du u^{-\frac{d+1}{2}} e^{-\frac{\Delta}{4u}} &= \int_\infty^0 \left(-\frac{\Delta}{4}v^{-2}dv\right)\left[\left(\frac{4}{\Delta}\right)^{\frac{d+1}{2}} v^{\frac{d+1}{2}}\right]e^{-v} \\
&= \left(\frac{4}{\Delta}\right)^{\frac{d-1}{2}} \int_0^\infty v^{\frac{d+1}{2}} e^{-v} dv \\
&= \left(\frac{4}{\Delta}\right)^{\frac{d-1}{2}} \Gamma\left(\frac{d-1}{2}\right).
\end{aligned} \tag{4.47}$$

After the $u$ integral the Green's function is given by

$$G(r,s,\theta,\theta',\mathbf{x},\mathbf{x}') = \frac{\Gamma(\frac{d-1}{2})}{4\pi n(2\pi^{1/2})^{d-1}}\frac{1}{2\pi i}\sum_{l=0}^{\infty} d_l \int_{\infty-i\pi}^{\infty+i\pi} dz \frac{4^{\frac{d-1}{2}} e^{-\frac{l}{n}z}\cos(\frac{l(\theta-\theta')}{n})}{[(\mathbf{x}-\mathbf{x})^2 + r^2 + s^2 - 2rs\cosh z]^{\frac{d-1}{2}}}. \tag{4.48}$$

We are interested in the case of $3+1$ dimensions. This sets $d = 3$. The contour integral is evaluated using the Cauchy residue theorem. To do this, the integral is simplified by setting $A = (\mathbf{x} - \mathbf{x})^2 + r^2 + s^2$ and $B = rs$. Then

$$\oint_C dz f(z) = \int_{\infty-i\pi}^{\infty+i\pi} dz \frac{e^{-\frac{l}{n}z}}{[A - 2B\cosh z]}. \tag{4.49}$$

Further setting

$$f(z) = \frac{g(z)}{h(z)}, \tag{4.50}$$

with $g(z) = e^{-\frac{l}{n}z}$ and $h(z) = A - 2B\cosh z$, we see that $h(z)$ is periodic in the imaginary axis since $\cosh z = \cosh(z + 2n\pi i)$. The integral has a single pole at

$$h(z_0) = 0 \implies z_0 = \cosh^{-1}(A/2B). \tag{4.51}$$

By the residue theorem, the integral evaluates to

$$\begin{aligned}
\oint_C dz f(z) &= 2\pi i Res[f(z), z_0] \\
&= 2\pi i \left[\frac{g(z_0)}{h'(z_0)}\right] \\
&= 2\pi i \frac{e^{-\frac{l}{n} z_0}}{-2B \sinh z_0} \\
&= -2\pi i \frac{e^{-\frac{l}{n} \cosh^{-1}(A/2B)}}{2B \sinh[\cosh^{-1}(A/2B)]} \\
&= -2\pi i \frac{e^{-\frac{l}{n} \log[A/2B + \sqrt{(A/2B)^2 - 1}]}}{2B \sinh[\cosh^{-1}(A/2B)]} \\
&= 2\pi i \frac{a^{-\frac{l}{n}}}{B(a - a^{-1})}
\end{aligned} \tag{4.52}$$

where $a = \frac{A}{2B} + \sqrt{\left(\frac{A}{2B}\right)^2 - 1}$. The Green's function becomes

$$\begin{aligned}
G(r, s, \theta, \theta', \mathbf{x}, \mathbf{x}') &= \frac{1}{2\pi^2 nrs} \sum_{l=0}^{\infty} d_l \frac{a^{-\frac{l}{n}}}{(a - a^{-1})} \cos\left(\frac{l(\theta - \theta')}{n}\right) \\
&= \frac{1}{2\pi^2 nrs} \frac{1}{(a - a^{-1})} \left[1 + \frac{2[-1 + a^{\frac{1}{n}} \cos((\theta - \theta')/n)]}{1 + a^{\frac{2}{n}} + 2a^{\frac{1}{n}} \cos((\theta - \theta')/n)}\right] \\
&= \frac{1}{2\pi^2 nrs(a - a^{-1})} \frac{a^{\frac{1}{n}} - a^{-\frac{1}{n}}}{a^{\frac{1}{n}} + a^{-\frac{1}{n}} + \cos((\theta - \theta')/n)}.
\end{aligned} \tag{4.53}$$

## 4.10 Preliminary Calculations

In previous sections we have explained that the path integral computation of the Rényi entropy entails the computation of correlation functions on an $n$ sheeted Riemann sphere. Since we are ultimately interested in the computation of Rényi entropies in the presence of operators that are heavy enough to deform the background geometry, we need to consider the computation of many point correlators, in a large $N$ but non-planar limit. This is a challenging problem and we will only take some first steps towards solving it. The simplest correlator we might consider is the trace of a product of two fields. The backreaction from this operator is negligible, but it is a

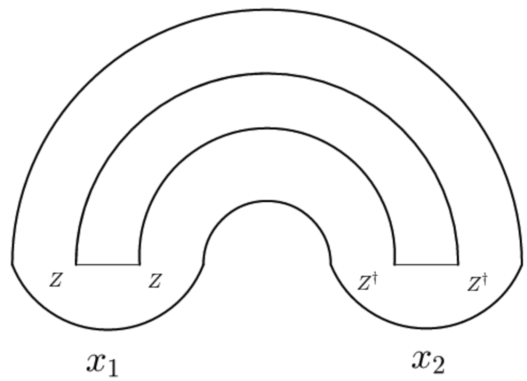



Figure 4.4: A two point function of the basic gauge invariant operator we consider. This corresponds to setting $n = 1$ in (4.54).

useful simple first example, by raising this operator to a large power we do get an operator heavy enough to deform the geometry and, this “simple toy problem” is already highly non-trivial.

We attempt to evaluate correlators of the form

$$\langle \mathsf{Tr}(ZZ)(x_1)\ldots \mathsf{Tr}(ZZ)(x_n)\mathsf{Tr}(Z^\dagger Z^\dagger)(x_{n+1})\ldots \mathsf{Tr}(Z^\dagger Z^\dagger)(x_{2n})\rangle. \tag{4.54}$$

This will be done by making the substitution

$$Z \to Z_i \qquad\qquad Z^\dagger \to Z_i^\dagger \qquad\qquad i = 1,\ldots,n, \tag{4.55}$$

where $Z_i$ is one chosen element of the set $\{Z_1,\ldots,Z_n\}$. This replacement is motivated by the need to keep track of spacetime dependences and we expect that this will be a useful step even in the computation of more complicated correlators. We begin by calculating a few cases of (4.54). This will be done by employing ribbon diagrams.

For $n = 1$, we have an expectation value that corresponds to the ribbon diagram on Figure 4.4. This diagram evaluates to:

$$\langle \mathsf{Tr}(ZZ)(x_1)\mathsf{Tr}(Z^\dagger Z^\dagger)(x_2)\rangle = 2N^2(12)^2 \tag{4.56}$$

where $(ij) \equiv G(x_i, x_j)$. The factor of 2 comes from the fact that there are two ways of contracting the 1st field $Z$ and only one way of contracting the remaining field. The $N^2$ comes from the number of faces on the diagram. In this case there are two.

For n=2, there are two types of diagrams, the first of which is shown on Figure 4.5. These are two disconnected diagrams, each of which is the same as Figure 4.4. The contribution is therefore

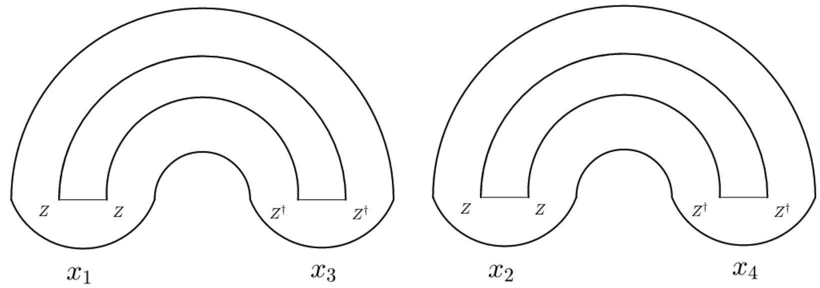



Figure 4.5: The disconnected contribution to (4.54) when n=2.

$4N^4$. We also need to consider the position permutations. We can pair the fields on $x_1$ with those on $x_3$ or $x_4$, with the fields on $x_2$ contracting with one remaining position. We therefore have $4N^4[(13)^2(24)^2 + (14)^2(23)^2]$. The second diagram for $n = 2$ is shown on Figure 4.6. This is a connected diagram with no position permutations. We only consider the number of ways the fields contract. The 1st field $Z$ on position $x_1$ has 4 possible ways of contracting with the four $Z^\dagger$ fields in position $x_3$ and $x_4$. The second field on position $x_1$ cannot contract in the same position as the first field as this will give us the same disconnected diagrams as those on Figure 4.5. Therefore the are only 2 possible ways of contracting the second field. For the first field at $x_2$, the are two possible ways contracting with the fields left over at the different positions. The second field at position $x_2$ has only one way of contracting. We then have $4 \times 2 \times 2 \times 1$ ways in which the field can contract, with each such configuration giving the Green's functions $(13)(14)(23)(24)$ .Figure 4.6 has 2 faces, an so has factor $N^2$. We therefore get for $n = 2$:

$$\begin{aligned}&\langle \mathrm{Tr}(ZZ)(x_1)\mathrm{Tr}(ZZ)(x_2)\mathrm{Tr}(Z^\dagger Z^\dagger)(x_3)\mathrm{Tr}(Z^\dagger Z^\dagger)(x_4)\rangle \\ &\quad = 4N^4[(13)^2(24)^2 + (14)^2(23)^2] \ + 16N^2[(13)(14)(23)(24)]\end{aligned} \tag{4.57}$$

We consider the case $n = 3$. Here we get 3 types of diagrams, the first of which is shown on figure 4.7. Each diagram can be paired in two ways, hence $2 \times 2 \times 2 = 8$ possible diagrams of this sort with this position configuration. Each diagrams has two faces, giving a factor $N^2$ for each. For the position permutations, we can hold $x_4, x_5, x_6$ fixed, while permuting $x_1, x_2, x_3$ .This gives the set of position permutations

$$S(^{123}P, 1) = \{(14)(25)(36), (14)(26)(35), (15)(24)(36), (15)(26)(34), (16)(24)(35), (16)(25)(34)\}. \tag{4.58}$$

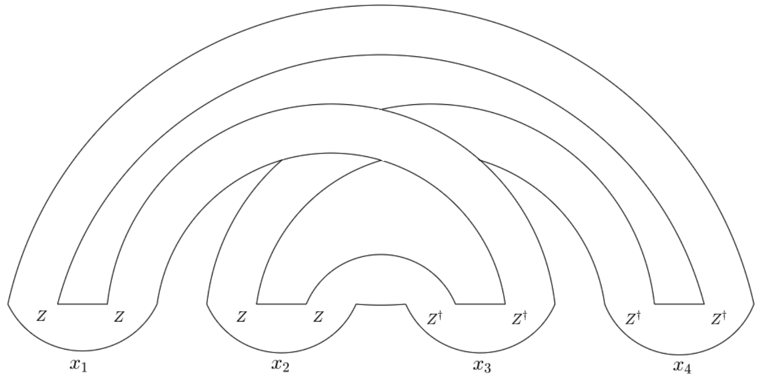



Figure 4.6: The connected contribution to (4.54) when n=2.

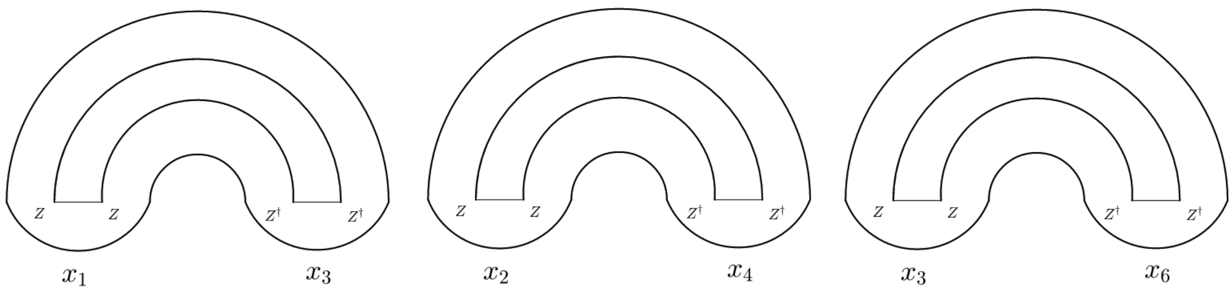



Figure 4.7: A disconnected contribution to (4.54) when n=3.

From this set, we get the Green's function of this diagram

$$\begin{aligned} G({}^{123}P,1) = (14)^2(25)^2(36)^2 + (14)^2(26)^2(35)^2 + (15)^2(24)^2(36)^2 + \\ (15)^2(26)^2(34)^2 + (16)^2(24)^2(35)^2 + (16)^2(25)^2(34)^2. \end{aligned} \tag{4.59}$$

The contribution is therefore

$$8N^6 G({}^{123}P,1). \tag{4.60}$$

The second diagram for $n = 3$ is shown on figure 4.8.

We have two disconnected graphs. From the previous analysis we know there are $2$ ways to pair the first and $16$ to pair the second. For the position permutations, we combine $2$ out of the $3$ positions $\{x_1, x_2, x_3\}$, this can be denote as $C(x_1, x_2, x_3, 2) \equiv {}^{123}C_2$. We do the same for

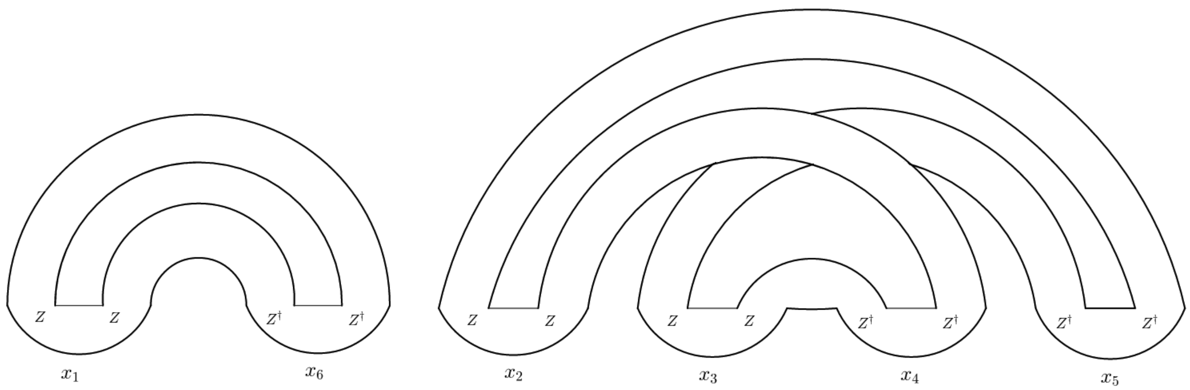



Figure 4.8: A second disconnected contribution to (4.54) when n=3.

$\{x_4, x_5, x_6\}$ to get ${}^{456}C_2$. We get the set of position permutations

$$S({}^{123}C_2, {}^{456}C_2) = \{(1245)(36), (1246)(35), (1256)(34), (1345)(26), (1346)(25), (1356)(24), (2345)(16), (2346)(15), (2356)(14)\}, \tag{4.61}$$

From this set, we are able to directly get the Green's functions. These are given by

$$G({}^{123}C_2, {}^{456}C_2) = \{(1245)(36)^2, (1246)(35)^2, (1256)(34)^2, (1345)(26)^2, (1346)(25)^2, (1356)(24)^2, (2345)(16)^2, (2346)(15)^2, (2356)(14)^2\}, \tag{4.62}$$

where, $(1245)(36)^2 \equiv G(x_1, x_4)G(x_1, x_5)G(x_2, x_4)G(x_2, x_6)G(x_3, x_6)G(x_3, x_6)$. The contribution from this diagram is therefore

$$32N^4 G({}^{123}C_2, {}^{456}C_2). \tag{4.63}$$

The third diagram for $n = 3$ is shown on Figure 4.9. The number of pairings for this diagram, such that we don't get the previous graphs, is $6 \times 4 \times 4 \times 2 \times 2 \times 1$. The diagram has only two faces and there are no position permutations, the contribution is

$$384N^2 G({}^{123}C_3). \tag{4.64}$$

Therefore the result for $n = 3$ is

$$\begin{aligned} &\langle \mathrm{Tr}(ZZ)(x_1)\mathrm{Tr}(ZZ)(x_2)\mathrm{Tr}(ZZ)(x_3)\mathrm{Tr}(Z^\dagger Z^\dagger)(x_4)\mathrm{Tr}(Z^\dagger Z^\dagger)(x_5)\mathrm{Tr}(Z^\dagger Z^\dagger)(x_6)\rangle \\ &= 8N^6 G({}^{123}P) + 32N^4 G({}^{123}C_2, {}^{456}C_2) + 384N^2 G({}^{123}C_3). \end{aligned} \tag{4.65}$$

Following the same procedure we calculate cases $n = 4$ and $n = 5$. For $n = 4$ there are 5 diagrams shown on figure 4.10. In the first row, each diagram gives contribution $2N^2$. We get

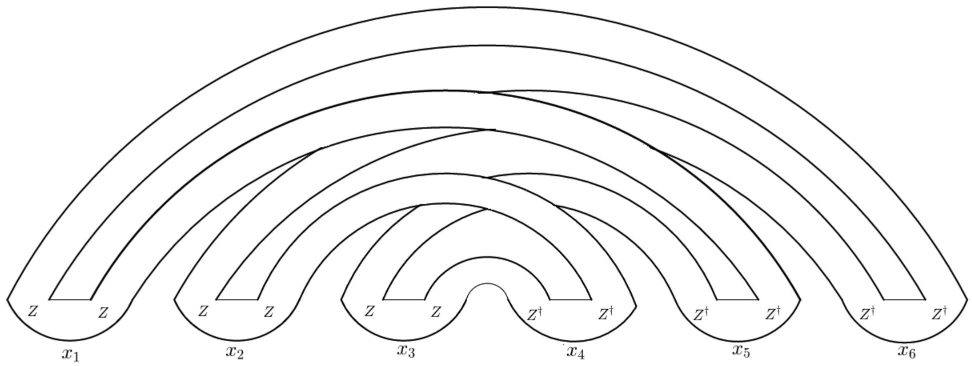



Figure 4.9: The connected contribution to (4.54) when n=3.

the position permutations by permuting $\{x_1, x_2, x_3, x_4\}$ while holding $\{x_5, x_6, x_7, x_8\}$ fixed, this gives ${}^{12345}P$. The contribution is therefore $16N^8 G({}^{12345}P)$

For the diagrams on the second row we get $(2N^2)^2 \times 16N^2$. For the position permutations, we can combine two positions from the $\{x_1, x_2, x_3, x_4\}$, we do the same for $\{x_5, x_6, x_7, x_8\}$, this gives ${}^{1234}C_2 \times {}^{5678}C_2$, for each such configuration, we swap the remaining two positions from either set, in this case we pick $(1234)$, we can denote this as ${}^{\overline{1234}}P_2$. We get

$$64N^6 G({}^{1234}C_2, {}^{5678}C_2, {}^{\overline{1234}}P_2). \tag{4.66}$$

The two diagrams on the third row give $16N^2 \times 16N^2$. To get the position permutation, we notice that ${}^4C_2 \times {}^4C_2$ over counts, since the same position configurations on the two different graphs should count only once. We get the correct permutations by holding 1 position fixed, in this case $x_1$ and permuting every other position. We get ${}^{1,234}C_1 \times {}^{5678}C_2$. The contribution is

$${}^3C_1 \times {}^4C_2 \times 16N^2 \times 16N^2 = 256N^4 G({}^{1,234}C_1, {}^{5678}C_2). \tag{4.67}$$

For the two diagrams on the fourth row, we get $2N^2 \times 384N^2$, with position permutations ${}^{1234}C_3 \times {}^{5678}C_3$, then

$$768N^4 G({}^{1234}C_3, {}^{5678}C_3). \tag{4.68}$$

For the final diagram, the number of pairing is $8 \times 6 \times 6 \times 4 \times 4 \times 2 \times 2 \times 1$. The diagram has two faces, so that we get

$$18432N^2G({}^{1234}C_4). \tag{4.69}$$

Therefore:

$$\begin{aligned}&\langle \mathrm{Tr}(ZZ)(x_1)\mathrm{Tr}(ZZ)(x_2)\mathrm{Tr}(ZZ)(x_3)\mathrm{Tr}(ZZ)(x_4)\\ &\times \mathrm{Tr}(Z^\dagger Z^\dagger)(x_5)\mathrm{Tr}(Z^\dagger Z^\dagger)(x_6)\mathrm{Tr}(Z^\dagger Z^\dagger)(x_7)\mathrm{Tr}(Z^\dagger Z^\dagger)(x_8)\rangle\\ &= 16N^8G({}^{12345}P) + 64N^6G({}^{1234}C_2, {}^{5678}C_2, {}^{\overline{1234}}P_2) + 256N^4G({}^{1,234}C_1, {}^{5678}C_2)\\ &+ 768N^4G({}^{1234}C_3, {}^{5678}C_3) + 18432N^2G({}^{1234}C_4).\end{aligned} \tag{4.70}$$

For the case $n = 5$, we find:

$$\begin{aligned}&\langle \mathrm{Tr}(ZZ)(x_1)\mathrm{Tr}(ZZ)(x_2)\mathrm{Tr}(ZZ)(x_3)\mathrm{Tr}(ZZ)(x_4)\mathrm{Tr}(ZZ)(x_5)\\ &\times \mathrm{Tr}(Z^\dagger Z^\dagger)(x_6)\mathrm{Tr}(Z^\dagger Z^\dagger)(x_7)\mathrm{Tr}(Z^\dagger Z^\dagger)(x_8)\mathrm{Tr}(Z^\dagger Z^\dagger)(x_9)\mathrm{Tr}(Z^\dagger Z^\dagger)(x_{10})\rangle\\ &= 32N^{10}G({}^{12345}P) + 128N^8G({}^{12345}C_2, {}^{678910}C_2, {}^{\overline{12345}}P_3) + 1536N^6G({}^{12345}C_1, {}^{678910}C_1, {}^{\overline{12345}}P_2)\\ &+ 512N^6G({}^{12345}C_2, {}^{678910}C_2, {}^{\overline{12345}}C_2, {}^{\overline{678910}}C_2) + 6144N^4G({}^{12345}C_3, {}^{678910}C_3)\\ &+ 136864N^4G({}^{12345}C_4, {}^{678910}C_4) + 1474560N^2G({}^{12345}C_5).\end{aligned} \tag{4.71}$$

The proposed formula to simplify these calculations is shown below.

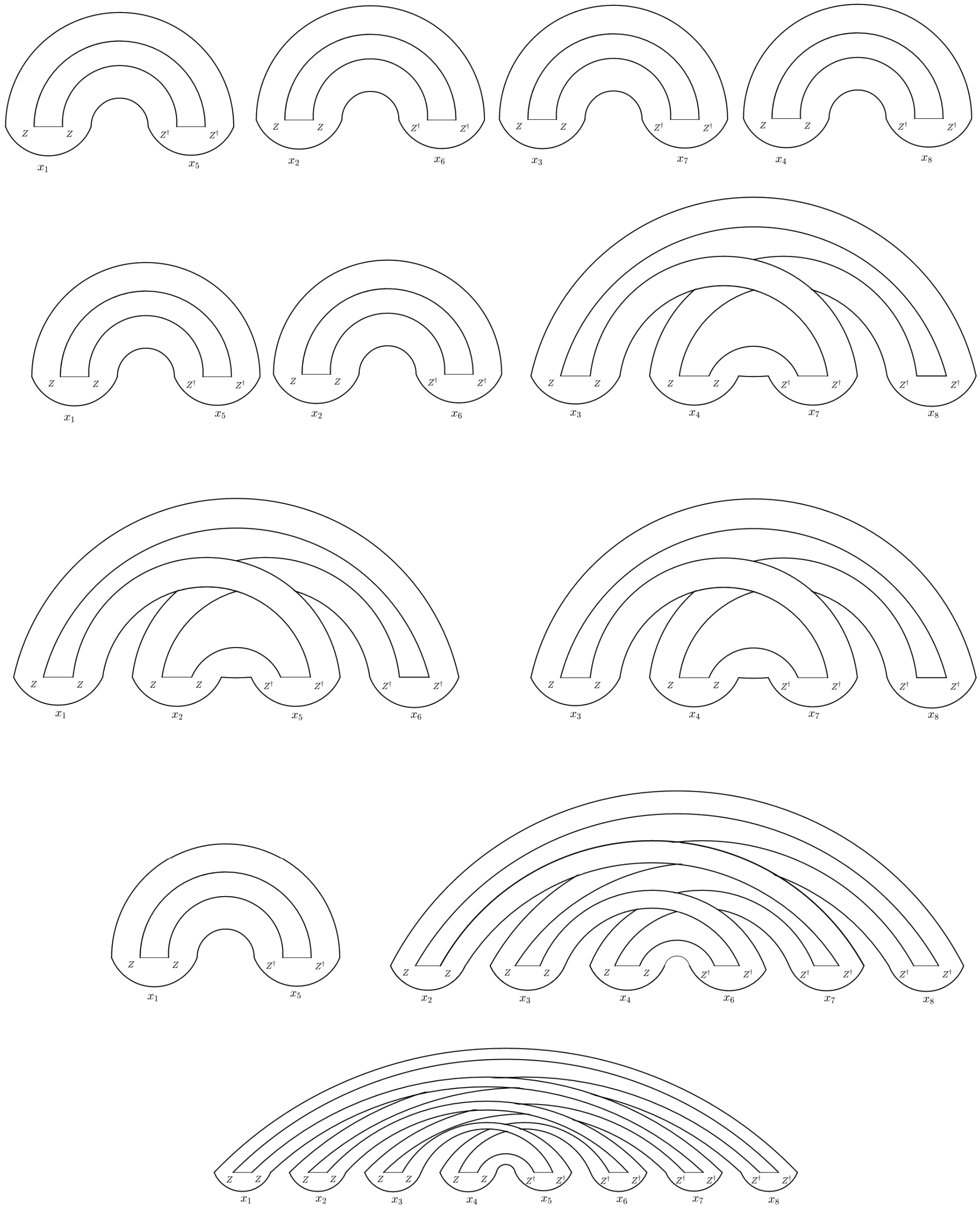



Figure 4.10: The diagrams contributing to (4.54) when n=4.

$$
\begin{aligned}
&\langle \mathrm{Tr}(ZZ)(x_1)\mathrm{Tr}(ZZ)(x_2)\ldots\mathrm{Tr}(ZZ)(x_n)\mathrm{Tr}(Z^\dagger Z^\dagger)(x_{n+1})\ldots\mathrm{Tr}(Z^\dagger Z^\dagger(x_{2n})\rangle \\
&= a_{N^{2n}}\langle \mathrm{Tr}(Z_1Z_2)(x_1)\mathrm{Tr}(Z_3Z_4)(x_2)\ldots\mathrm{Tr}(Z_{2n-1}Z_{2n})(x_n)\mathrm{Tr}(Z_1^\dagger Z_2^\dagger)(x_{n+1})\ldots\mathrm{Tr}(Z_{2n-1}^\dagger Z_{2n}^\dagger)(x_{2n})\rangle(1,n+1)\ldots(n,2n) \\
&+ a_{N^{2n-2}}\langle \mathrm{Tr}(Z_1Z_2)(x_1)\mathrm{Tr}(Z_3Z_4)(x_2)\ldots\mathrm{Tr}(Z_{2n-1}Z_{2n})(x_n)\mathrm{Tr}(Z_1^\dagger Z_3^\dagger)(x_{n+1})\mathrm{Tr}(Z_2^\dagger Z_4^\dagger)(x_{n+2})\mathrm{Tr}(Z_5^\dagger Z_6^\dagger)(x_{n+3})\ldots\mathrm{Tr}(Z_{2n-1}^\dagger Z_{2n}^\dagger)(x_{2n})\rangle \\
&(1,2;n+1,n+2)(3,n+3)\ldots(n,2n) \\
&+ a_{N^{2n-2}}\langle \mathrm{Tr}(Z_1Z_2)(x_1)\ldots\mathrm{Tr}(Z_{2n-1}Z_{2n})(x_n)\mathrm{Tr}(Z_1^\dagger Z_3^\dagger)(x_{n+1})\mathrm{Tr}(Z_2^\dagger Z_4^\dagger)(x_{n+2})\mathrm{Tr}(Z_5^\dagger Z_7^\dagger)(x_{n+3})\mathrm{Tr}(Z_6^\dagger Z_8^\dagger)(x_{n+4})\mathrm{Tr}(Z_9^\dagger Z_{10}^\dagger)(x_{n+5})\ldots \\
&\mathrm{Tr}(Z_{2n-1}^\dagger Z_{2n}^\dagger)(x_{2n})\rangle(1,2;n+1,n+2)(3,4;n+3,n+4)\ldots(n-1,n;2n-1,2n) \\
&\vdots \\
&+ a_{N^{n}}\langle \mathrm{Tr}(Z_1Z_2)(x_1)\mathrm{Tr}(Z_3Z_4)(x_2)\ldots\mathrm{Tr}(Z_{2n-1}Z_{2n}) \\
&\times (x_n)\mathrm{Tr}(Z_1^\dagger Z_3^\dagger)(x_{n+1})\mathrm{Tr}(Z_2^\dagger Z_4^\dagger)(x_{n+2})\mathrm{Tr}(Z_5^\dagger Z_7^\dagger)(x_{n+3})\mathrm{Tr}(Z_6^\dagger Z_8^\dagger)(x_{n+4})\ldots\mathrm{Tr}(Z_{2n-1}^\dagger Z_{2n-3}^\dagger)(x_{2n-1})\mathrm{Tr}(Z_{2n-2}^\dagger Z_{2n}^\dagger)(x_{2n})\rangle \\
&(1,2,3;n+1,n+2,n+3)(4,n+4)\ldots(n,2n) \\
&+ a_{N^{n-1}}\langle \mathrm{Tr}(Z_1Z_2)(x_1)\ldots\mathrm{Tr}(Z_{2n-1}Z_{2n})(x_n)\mathrm{Tr}(Z_1^\dagger Z_3^\dagger)(x_{n+1})\mathrm{Tr}(Z_2^\dagger Z_5^\dagger)(x_{n+2})\mathrm{Tr}(Z_4^\dagger Z_6^\dagger)(x_{n+3})\mathrm{Tr}(Z_7^\dagger Z_8^\dagger)\ldots\mathrm{Tr}(Z_{2n-1}^\dagger Z_{2n}^\dagger)(x_{2n})\rangle \\
&(1,2,3;n+1,n+2,n+3)(4,5;n+4,n+5)(6,n+6)\ldots(n,2n) \\
&+ a_{N^{n-2}}\langle \mathrm{Tr}(Z_1Z_2)(x_1)\ldots\mathrm{Tr}(Z_{2n-1}Z_{2n})(x_n)\mathrm{Tr}(Z_1^\dagger Z_3^\dagger)(x_{n+1})\mathrm{Tr}(Z_2^\dagger Z_5^\dagger)(x_{n+2})\mathrm{Tr}(Z_4^\dagger Z_6^\dagger)(x_{n+3})\mathrm{Tr}(Z_7^\dagger Z_9^\dagger)x_{n+4}\mathrm{Tr}(Z_8^\dagger Z_{10}^\dagger)x_{n+5}\mathrm{Tr}(Z_{11}^\dagger Z_{12}^\dagger)x_{n+6}\ldots \\
&\mathrm{Tr}(Z_{2n-1}^\dagger Z_{2n}^\dagger)(x_{2n})\rangle(1,2,3;n+1,n+2,n+3)(4,5;n+4,n+5)(6,7;n+6,n+7)\ldots(n-1,n;2n-1,2n) \\
&\vdots \\
&+ a_{N^{n/3}}\langle \mathrm{Tr}(Z_1Z_2)(x_1)\ldots\mathrm{Tr}(Z_{2n-1}Z_{2n})(x_n) \\
&\times \mathrm{Tr}(Z_1^\dagger Z_3^\dagger)(x_{n+1})\mathrm{Tr}(Z_2^\dagger Z_5^\dagger)(x_{n+2})\mathrm{Tr}(Z_4^\dagger Z_6^\dagger)(x_{n+3})\ldots\mathrm{Tr}(Z_{2n-5}^\dagger Z_{2n-3}^\dagger)(x_{2n-2})\mathrm{Tr}(Z_{2n-4}^\dagger Z_{2n-1}^\dagger)(x_{2n-1})\mathrm{Tr}(Z_{2n-2}^\dagger Z_{2n}^\dagger)(x_{2n})\rangle \\
&(1,2,3;n+1,n+2,n+3)\ldots(n-2,n-1,n;2n-2,2n-1,2n) \\
&\vdots \\
&+ a_{N^{2}}\langle \mathrm{Tr}(Z_1Z_2)(x_1)\ldots\mathrm{Tr}(Z_{2n-1}Z_{2n})(x_n) \\
&\times \mathrm{Tr}(Z_1^\dagger Z_3^\dagger)(x_{n+1})\mathrm{Tr}(Z_2^\dagger Z_5^\dagger)(x_{n+2})\mathrm{Tr}(Z_4^\dagger Z_7^\dagger)(x_{n+3})\mathrm{Tr}(Z_6^\dagger Z_9^\dagger)(x_{n+4})\ldots\mathrm{Tr}(Z_{2n-6}^\dagger Z_{2n-3}^\dagger)(x_{2n-2})\mathrm{Tr}(Z_{2n-4}^\dagger Z_{2n-1}^\dagger)(x_{2n-1})\mathrm{Tr}(Z_{2n-2}^\dagger Z_{2n}^\dagger)(x_{2n})\rangle \\
&(1,\ldots,n;n+1,\ldots,2n)
\end{aligned}
$$

Where $(r,s) \equiv [G(r,s)]^2$ and $(r, r+1, \dots, r+n; s, s+1, \dots, s+n) \equiv G(r,s)G(r,s+1)\dots G(r,s+n)\dots G(r+n,s)\dots G(r+n,s+n)$. The coefficients $a_N$ are determined by mixing of the conjugate fields. This is shown in the example below.

Checking the formula. For $n=3$:

$$
\begin{aligned}
&\langle \mathsf{Tr}(ZZ)(x_1)\mathsf{Tr}(ZZ)(x_2)\mathsf{Tr}(ZZ)(x_3)\mathsf{Tr}(Z^\dagger Z^\dagger)(x_4)\mathsf{Tr}(Z^\dagger Z^\dagger)(x_5)\mathsf{Tr}(Z^\dagger Z^\dagger)(x_6)\rangle \\
&= (2\times 2\times 2)\langle \mathsf{Tr}(Z_1Z_2)(x_1)\mathsf{Tr}(Z_3Z_4)(x_2)\mathsf{Tr}(Z_5Z_6)(x_3)\underbrace{\mathsf{Tr}(Z_1^\dagger Z_2^\dagger)}_{2}(x_4)\underbrace{\mathsf{Tr}(Z_3^\dagger Z_4^\dagger)}_{2}(x_5)\underbrace{\mathsf{Tr}(Z_5^\dagger Z_6^\dagger)}_{2}(x_6)\rangle \\
&+ [(4\times 2\times 2)\times 2]\langle \mathsf{Tr}(Z_1Z_2)(x_1)\mathsf{Tr}(Z_3Z_4)(x_2)\mathsf{Tr}(Z_5Z_6)(x_3)\underbrace{\mathsf{Tr}(Z_1^\dagger Z_3^\dagger)(x_4)\mathsf{Tr}(Z_2^\dagger Z_4^\dagger)(x_5)}_{4\times 2\times 2}\underbrace{\mathsf{Tr}(Z_5^\dagger Z_6^\dagger)(x_6)}_{2}\rangle \\
&+ (6\times 4\times 4\times 2\times 2)\langle \mathsf{Tr}(Z_1Z_2)(x_1)\mathsf{Tr}(Z_3Z_4)(x_2)\mathsf{Tr}(Z_5Z_6)(x_3)\underbrace{\mathsf{Tr}(Z_1^\dagger Z_3^\dagger)(x_4)\mathsf{Tr}(Z_2^\dagger Z_5^\dagger)(x_5)\mathsf{Tr}(Z_4^\dagger Z_6^\dagger)(x_6)}_{6\times 4\times 4\times 2\times 2}\rangle \\
&= 8\langle \mathsf{Tr}(Z_1Z_2)(x_1)\mathsf{Tr}(Z_3Z_4)(x_2)\mathsf{Tr}(Z_5Z_6)(x_3)\mathsf{Tr}(Z_1^\dagger Z_2^\dagger)(x_4)\mathsf{Tr}(Z_3^\dagger Z_4^\dagger)(x_5)\mathsf{Tr}(Z_5^\dagger Z_6^\dagger)(x_6)\rangle \\
&+ 32\langle \mathsf{Tr}(Z_1Z_2)(x_1)\mathsf{Tr}(Z_3Z_4)(x_2)\mathsf{Tr}(Z_5Z_6)(x_3)\mathsf{Tr}(Z_1^\dagger Z_3^\dagger)(x_4)\mathsf{Tr}(Z_2^\dagger Z_4^\dagger)(x_5)\mathsf{Tr}(Z_5^\dagger Z_6^\dagger)(x_6)\rangle \\
&+ 384\langle \mathsf{Tr}(Z_1Z_2)(x_1)\mathsf{Tr}(Z_3Z_4)(x_2)\mathsf{Tr}(Z_5Z_6)(x_3)\mathsf{Tr}(Z_1^\dagger Z_3^\dagger)(x_4)\mathsf{Tr}(Z_2^\dagger Z_5^\dagger)(x_5)\mathsf{Tr}(Z_4^\dagger Z_6^\dagger)(x_6)\rangle.
\end{aligned}
$$

This demonstrates how the coefficients are determined. For $n=4$:

$$
\begin{aligned}
&\langle \mathsf{Tr}(ZZ)(x_1)\mathsf{Tr}(ZZ)(x_2)\mathsf{Tr}(ZZ)(x_3)\mathsf{Tr}(ZZ)(x_4)\mathsf{Tr}(Z^\dagger Z^\dagger)(x_5)\mathsf{Tr}(Z^\dagger Z^\dagger)(x_6)\mathsf{Tr}(Z^\dagger Z^\dagger)(x_7)\mathsf{Tr}(Z^\dagger Z^\dagger)(x_8)\rangle(1,4)(2,5)(3,6)(4,8) \\
&= 16\langle \mathsf{Tr}(Z_1Z_2)(x_1)\mathsf{Tr}(Z_3Z_4)(x_2)\mathsf{Tr}(Z_5Z_6)(x_3)\mathsf{Tr}(Z_7Z_8)(x_4)\mathsf{Tr}(Z_1^\dagger Z_2^\dagger)(x_5)\mathsf{Tr}(Z_3^\dagger Z_4^\dagger)(x_6)\mathsf{Tr}(Z_5^\dagger Z_6^\dagger)(x_7)\mathsf{Tr}(Z_7^\dagger Z_8^\dagger)(x_8)\rangle(1,2;5,6)(3,7)(4,8) \\
&+ 64\langle \mathsf{Tr}(Z_1Z_2)(x_1)\mathsf{Tr}(Z_3Z_4)(x_2)\mathsf{Tr}(Z_5Z_6)(x_3)\mathsf{Tr}(Z_7Z_8)(x_4)\mathsf{Tr}(Z_1^\dagger Z_3^\dagger)(x_5)\mathsf{Tr}(Z_2^\dagger Z_4^\dagger)(x_6)\mathsf{Tr}(Z_5^\dagger Z_6^\dagger)(x_7)\mathsf{Tr}(Z_7^\dagger Z_8^\dagger)(x_8)\rangle(1,2;5,6)(3,6)(4,8) \\
&+ 256\langle \mathsf{Tr}(Z_1Z_2)(x_1)\mathsf{Tr}(Z_3Z_4)(x_2)\mathsf{Tr}(Z_5Z_6)(x_3)\mathsf{Tr}(Z_7Z_8)(x_4)\mathsf{Tr}(Z_1^\dagger Z_3^\dagger)(x_5)\mathsf{Tr}(Z_2^\dagger Z_4^\dagger)(x_6)\mathsf{Tr}(Z_5^\dagger Z_7^\dagger)(x_7)\mathsf{Tr}(Z_6^\dagger Z_8^\dagger)(x_8)\rangle(1,2;5,6)(4,5;7,8) \\
&+ 768\langle \mathsf{Tr}(Z_1Z_2)(x_1)\mathsf{Tr}(Z_3Z_4)(x_2)\mathsf{Tr}(Z_5Z_6)(x_3)\mathsf{Tr}(Z_7Z_8)(x_4)\mathsf{Tr}(Z_1^\dagger Z_3^\dagger)(x_5)\mathsf{Tr}(Z_2^\dagger Z_5^\dagger)(x_6)\mathsf{Tr}(Z_4^\dagger Z_6^\dagger)(x_7)\mathsf{Tr}(Z_7^\dagger Z_8^\dagger)(x_8)\rangle(1,2,3;5,6,7)(4,8) \\
&+ 18432\langle \mathsf{Tr}(Z_1Z_2)(x_1)\mathsf{Tr}(Z_3Z_4)(x_2)\mathsf{Tr}(Z_5Z_6)(x_3)\mathsf{Tr}(Z_7Z_8)(x_4)\mathsf{Tr}(Z_1^\dagger Z_3^\dagger)(x_5)\mathsf{Tr}(Z_2^\dagger Z_5^\dagger)(x_6)\mathsf{Tr}(Z_4^\dagger Z_7^\dagger)(x_7)\mathsf{Tr}(Z_6^\dagger Z_8^\dagger)(x_8)\rangle(1,2,3,4;5,6,7,8).
\end{aligned}
$$

## 4.11 Emergence of spacetime from Entanglement.

In this section we discuss the role that entanglement has in the "*connectedness*" of spacetime. We review a thought experiment by van Raamsdonk on the emergence of spacetime from entanglement [2, 35].

We begin by considering the boundary CFT in a pure state $|\Psi\rangle$, that is dual to pure $AdS$ spacetime. We consider the entanglement entropy of two spatial subsystems $A$ and $B$ on the boundary CFT. From the Ryu-Takayanagi formula, we know that the entanglement entropy of $A$ is given by a minimal surface $\gamma_A$ in the bulk spacetime. This minimal surface can be seen as a boundary separating the bulk spacetime regions dual to the subsystems $A$ and $B$. Now we consider what happens when the entanglement entropy between $A$ and $B$ is decreased. We expect the minimal surface $\gamma_A$ to decrease proportionally. However since $\gamma_A$ is a boundary separating the two spacetime regions, this implies that as the entanglement decreases, the two regions become further and further apart. In the extreme case where the entanglement decreases to zero, the two regions must therefore become disconnected. Thus by removing the entanglement between $A$ and $B$, we are able to pull apart the spacetime into two disjoint regions. This process can be taken further by dividing the boundary into more subsystems and removing the entanglement between them. This corresponds to splitting the bulk spacetime into smaller disjoint regions. The limiting case is therefore that if the boundary entanglement is completely removed, the bulk spacetime would become completely disconnected and thus vanish. Thus we can deduce the entanglement is crucial to the emergence of spacetime in holography.

The thought experiment implies the CFT is more fundamental than the gravitational theory. This argument can be made more quantitative by considering the mutual information $I(A:B)$. Recall that $I(A:B)$ is a pure measure of the entanglement between $A$ and $B$ and provides an upper bound on the correlation functions in the two subsystems as (4.23). Thus if the entanglement is decreased to zero, the mutual information $I(A:B)$ vanishes, as well as any correlation functions of local operators between $A$ and $B$. We can choose a boundary operator that is dual to massive particle in the bulk. In this case, the two point function in the bulk is given by [36],

$$\langle \mathcal{O}_A(x_A)\mathcal{O}_B(x_B)\rangle \sim e^{m\delta x} \tag{4.72}$$

where $m$ is the mass of the particle and $\delta x$ is the length of the shortest geodesic between $x_A$ and $x_B$. Combining (4.23) and (4.72), we see that no entanglement between $A$ and $B$, requires the proper distance $\delta x$ between the two subsystems, to be infinite[1].

[1] Note the CFT has an explicit cut-off such that the correlators considered are finite.

# Chapter 5

# Conclusion and Outlook

In this dissertation we have tried to understand the mechanism behind holography. In particular, how the 6 extra dimensions of the string theory are holographically reconstructed in the CFT. The approach we followed, is an investigation of the interplay between quantum information and geometry via the entanglement entropy. Our considerations took place in the 1/2 BPS sector of the duality between the $\mathcal{N} = 4$ super Yang-Mills on $R \times S^3$ and IIB string theory on $AdS_5 \times S^5$.

In chapter 2, we began by reviewing $\mathcal{N} = 4$ super Yang-Mills theory and the 1/2 BPS sector. Here we found the operators of interest that are protected from quantum corrections. We studied how the dynamics of this sector can be described by a single matrix model. We then discussed the large $N$ limit of the theory as the classical limit. The operators we considered are of large dimension such that they are dual to heavy objects that deform the spacetime. In the large $N$ limit, the evaluation of correlation functions of such operators entailed the summation of planar and non-planar diagrams. We tackled this difficult task by reviewing group theory techniques in the remainder of the chapter.

In chapter 3 we reviewed the 1/2 BPS supergravity solutions by Lin, Lunin and Maldacena. These solutions allow us to build the spacetime geometries dual to the operators we considered.

In chapter 4, we studied the role entanglement entropy plays in holography and the conjecture that spacetime is built from entanglement. We began by defining entanglement in quantum mechanics and defining how it can be measured with the entanglement entropy. We then looked at properties of entanglement entropy. Moving to the AdS/CFT correspondence, we extended the definition and properties of entanglement entropy to the CFT. We then discussed how the area law demonstrates the dependance of entanglement entropy on geometry. We proceeded to discuss the explicit calculation of entanglement entropy via the replica trick. This method entails the calculation of Renyi entropies of the reduced density matrix and Green's functions on an $n$ sheeted Riemann surface. We then derived the Green's function in section 4.9.

We moved to the gravity side of the duality where we studied the Ryu-Takayanagi formula for calculating entanglement entropy in terms of a minimal surface in the bulk. In terms of the minimal surface, we saw how the entanglement entropy has the same properties as those in the CFT.

With the necessary ingredients introduced, we began to build a formula to calculate correlation functions in section 4.10. We began with the substitution $Z \to Z_i$, that results in a recursive formula built from some of the group theory techniques we reviewed in chapter 2. We demonstrated a few calculations using the formula. Those examples demonstrate how the formula works by transpositions on the field $Z_i$. This process can be mapped to the symmetric group $S_2$. The result was due to the fact that we included 2 fields at each position. The heavy operators that we were interested in, require the inclusion of more fields at each position. Thus the extension of this formula would require us to consider symmetric groups of higher degree. This will be part of future work planned for the project. We ended chapter 4, by discussing the idea that entanglement is the "glue" that holds together spacetime.

Future work will focus on building a formula for correlation functions with an arbitrary number of fields at each position. This will entail studying symmetric groups of higher order. In particular we plan to look at the subgroup structures and how they are partitioned by transpositions. The formula is of critical importance to our investigation. If we are successful in building such a formula, we will be able to perform explicit calculations of the entanglement entropy of heavy operators in the CFT via the replica trick. Using the supergravity solutions and the Ryu-Takayanagi formula, we will be able to perform the dual calculation of entanglement entropy. In this way, we will be able to concretely test the conjecture that entanglement entropy plays a role in the emergence of spacetime and gain insight into the inner workings of the correspondence.

# Appendix A

# Sturm-Liouville Problem

## A.1 Sturm Liouville Problem

Consider the following two-point boundary value problem

$$\frac{d}{dx}\left[p(x)\frac{d\phi_n(x)}{dx}\right] + q(x)\phi_n(x) + \lambda r(x)\phi_n(x) = 0 \qquad a < x < b \tag{A.1}$$

$$\alpha_1\phi_n(a) + \alpha_2\phi_n'(a) = 0 \qquad \beta_1\phi_n(b) + \beta_2\phi_n'(b) = 0 \tag{A.2}$$

where $p, p', q$ and $r$ are continuous functions on $a \leq x \leq b$ and $p(x) \geq 0$ and $r(x) > 0$ on $a \leq x \leq b$. We define the Sturm-Liouville eigenvalue problem as:

$$\mathcal{L}\phi_n(x) = \lambda r(x)\phi_n(x) \quad \text{where} \quad \mathcal{L}\phi_n(x) = \frac{d}{dx}\left[p(x)\frac{d\phi_n(x)}{dx}\right] + q(x)\phi_n(x) \tag{A.3}$$

$$\alpha_1\phi_n(a) + \alpha_2\phi_n'(a) = 0 \qquad \beta_1\phi_n(b) + \beta_2\phi_n'(b) = 0 \tag{A.4}$$

$$p(x) > 0 \quad \text{and} \quad r(x) > 0. \tag{A.5}$$

The operator $\mathcal{L}(x)$ is hermitian.

*Proof.*

$$\begin{aligned}
&\int_a^b dx\phi_m(x)^*\mathcal{L}(x)\phi_n(x) - \Big[\int_a^b dx\phi_n(x)^*\mathcal{L}(x)\phi_m(x)\Big]^* \\
&= \int_a^b dx\phi_m(x)^*\frac{d}{dx}[p(x)\phi_n'(x)] + \phi_m(x)^*q(x)\phi_n(x) \\
&- \int_a^b dx\phi_n(x)\frac{d}{dx}[p(x)(\phi_m^*(x))'] + \phi_n(x)q(x)\phi_m^*(x)
\end{aligned} \tag{A.6}$$

Applying the boundary conditions and integrating by parts,

$$\begin{aligned}
&= -\int_a^b dx(\phi_m^*(x))'p(x)\phi_n'(x) + \int_a^b dx\phi_n'(x)p(x)(\phi_m^*(x))' \\
&= 0
\end{aligned} \tag{A.7}$$

Therefore

$$\int_a^b dx\phi_m(x)^*\mathcal{L}(x)\phi_n(x) = \Big[\int_a^b dx\phi_m(x)^*\mathcal{L}(x)\phi_m(x)\Big]^*. \tag{A.8}$$

□

As a result of this property, the eigenvalues of the operator $\mathcal{L}(x)$ can be shown to be real and its eigenfunctions to be orthogonal. If

$$\mathcal{L}(x)\phi_m(x) = \lambda_m w(x)\phi_m(x) \implies \int_a^b dx\phi_m(x)^*\mathcal{L}(x)\phi_n(x) = \lambda_m \int_a^b dx\phi_m(x)^*w(x)\phi_n(x)$$

and

$$\mathcal{L}(x)\phi_n(x) = \lambda_n w(x)\phi_n(x) \implies \int_a^b dx\phi_n(x)^*\mathcal{L}(x)\phi_m(x) = \lambda_n \int_a^b dx\phi_n(x)^*w(x)\phi_m(x)$$

Using (4),

$$(\lambda_m - \lambda_n^*)\int_a^b dx\phi_m(x)^*w(x)\phi_n(x) = 0. \tag{A.9}$$

Therefore the eigenvalues $\lambda_n$ are real and the eigenfunctions $\phi_n(x)$ are orthogonal. Additionally the eigenfunctions of $\mathcal{L}(x)$ have the following properties:

$$\sum_n \phi_n(x)w(x')\phi_n(x') = \delta(x-x'), \tag{A.10}$$

and any arbitrary function $f(x)$ on $[a,b]$ can be expressed as a generalised Fourier expansion in terms of the eigenfunctions $\phi_n(x)$

$$f(x) = \sum_n C_n\phi_n(x) \quad \text{where} \quad C_m = \int dxw(x)\phi_m(x)f(x). \tag{A.11}$$

Consider the Green's function $G(x,x')$ that satisfies

$$\mathcal{L}(x)G(x,x') = \delta(x-x'), \tag{A.12}$$

with boundary conditions $G(x,x') = 0$ for $x = a$ and $x = b$. The Green's function can be expanded as

$$G(x,x') = \sum_m C_m(x')\phi_m(x). \tag{A.13}$$

Then

$$\sum_m \lambda_m w(x)\phi_m(x)C_m(x') = \delta(x-x'). \tag{A.14}$$

Multiplying by $\phi_n^*(x)$ and integrating

$$\lambda_m C_m(x')\delta_{mn} = \phi_n^*(x'). \tag{A.15}$$

The Green's function can therefore be expressed in terms of the eigenfunctions

$$G(x,x') = \sum_n \frac{\phi_n^*(x')\phi_m(x)}{\lambda_n}. \tag{A.16}$$

## A.2 Green's function on a Riemann surface

We consider again the eigenvalue problem

$$\mathcal{L}y = \lambda y, \tag{A.17}$$

where $y$ is a function on $r \in [0, a]$ with boundary condition $y(a) = 0$. The function is regular at $r = 0$. Here the operator $\mathcal{L}$ is

$$\mathcal{L} = \Big[ -\frac{1}{r}\frac{d}{dr}\Big(r\frac{d}{dr}\Big) + \frac{m^2}{r^2} \Big]. \tag{A.18}$$

The eigenvalue $\lambda$ is written as $k^2$ below. We again have that the eigenfunctions $y$ and $z$ of $(13)$ with distinct eigenvalues $\lambda$ and $\mu$ are orthogonal

$$(\lambda - \mu)\int_0^a y(r)z(r)rdr = 0, \tag{A.19}$$

if $y$ and $z$ are regular at $r = 0$ and $y(a) = z(a) = 0$. The solutions to the eigenvalue problem $\mathcal{L}y = k^2 y$ are the Bessel functions $J_m(kr)$ defined as

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(m+n+1)}\Big(\frac{x}{2}\Big)^{2n+m}. \tag{A.20}$$

For the boundary condition $y(a) = 0$, the parameter k must satisfy the condition

$$J_m(ka) = 0. \tag{A.21}$$

The Bessel function $J_m(x)$ has an infinite sequence of zeros, denoted by $\{\alpha_{mn}\}_{n=1}^{\infty}$, it follows that there is an infinite sequence of eigenvalues $\{k_{mn}\}_{n=1}^{\infty}$ given by

$$k_{mn} = \frac{\alpha_{mn}}{a}. \tag{A.22}$$

The functions

$$\phi_{mn}(r) = J_m(k_{mn}r) \tag{A.23}$$

therefore form an infinite sequence of orthogonal functions since

$$\mathcal{L}\phi_{mn} = k_{mn}^2\phi_{mn} \tag{A.24}$$

and eigenfunctions of $\mathcal{L}$ with distinct eigenvalues are orthogonal. The Bessel functions obey the orthogonality relation

$$\int_0^a J_m(k_{mn}r)J_m(k_{mn'}r)rdr = \frac{a^2}{2}[J_m'(k_{mn}r)]^2\delta_{nn'} \tag{A.25}$$

For an arbitrary function $f(r, \theta)$ defined on a region $0 \le r < a, 0 \le \theta < 2\pi$, the functions

$$\phi_{mn}^a = J_m(k_{mn}r)\cos(m\theta), \phi_{mn}^b = J_m(k_{mn}r)\sin(m\theta) \tag{A.26}$$

form a complete orthonormal set of eigenfunctions, where $m$ ranges through the integers 0,1, ... and for each each $m$, $n$ range through the positive integers. As before the function $f(r,\theta)$ can be expanded, now in the Fourier-Bessel series:

$$f(r,\theta) = \sum_{n=1}^{\infty} A_{0n} J_0(k_{0n}r) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}r)[A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)] \tag{A.27}$$

where

$$A_{0n} = \frac{1}{\pi a^2 J_0'(k_{0n}a)^2}\int_0^{2\pi}\int_0^a f(r,\theta)J_0(k_{0n}r)rdrd\theta \tag{A.28}$$

$$A_{mn} = \frac{1}{\pi a^2 J_m'(k_{mn}a)^2}\int_0^{2\pi}\int_0^a f(r,\theta)J_m(k_{mn}r)\cos(m\theta)rdrd\theta \tag{A.29}$$

$$B_{mn} = \frac{2}{\pi a^2 J_m'(k_{mn}a)^2}\int_0^{2\pi}\int_0^a f(r,\theta)J_m(k_{mn}r)\sin(m\theta)rdrd\theta. \tag{A.30}$$

As before the Greens functions $G(r,r')$ on the region $0 \le r < a, 0 \le \theta < 2\pi$ with appropriate boundary conditions is the solution to

$$\mathcal{L}G(r,r') = \delta(r-r') \tag{A.31}$$

and admits an eigenfunction expansion

$$G(r,r') = \sum_n N_n \phi^a_{mn}(r)\phi^b_{mn}(r') \tag{A.32}$$

where $N_n$ is the normalisation constant determined by the orthogonality condition.

We are interested in a Green's function on an n-sheeted Riemann surface. The eigenvalue problem $\mathcal{L}y = k^2 y$ on this surface is solved in polar coordinates $r = (x,y) = (r\cos\theta, r\sin\theta)$. Consider a finite region on this surface given by $0 \le L, 0 \le \theta \le 2\pi n$. Then the complete set of orthogonal eigenfunctions is given by

$$\phi^a_{\nu,i} = J_\nu(\lambda_i r)\cos(i\nu\theta), \qquad \phi^b_{\nu,i} = J_\nu(\lambda_i r)\sin(i\nu\theta). \tag{A.33}$$

Imposing the $2\pi n$ periodicity boundary condition, $\nu$ becomes $\nu = k/n$ and setting $\phi^a_{\nu,i}(L) = \phi^a_{\nu,i}(L) = 0$, the eigenvalues are then $\lambda_i L = \alpha_{\nu,i}$, where $\alpha_{\nu,i}$ is the $i$-th zero of the Bessel function. The orthogonality relation is now given by

$$\int_0^L rdrJ_\nu(\alpha_{\nu,i}r/L)J_{\nu'}(\alpha_{\nu',i}r/L) = \frac{L^2}{2}[J_{\nu+1}(\alpha_{\nu,i}r/L)]^2\delta_{\nu\nu'}. \tag{A.34}$$

From this, the normalisation $N_{i,k}$ is determined to be

$$N_{i,k} = \frac{d_k}{2\pi n} \frac{2/L^2}{[J_{\nu+1}(\alpha_{\nu,i} r/L)]^2}. \tag{A.35}$$

The Green's functions in the eigenfunction expansion then becomes

$$G(r,\theta,r',\theta') = \frac{1}{2\pi n} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} d_k \frac{2/L^2}{[J_{\nu+1}(\alpha_{\nu,i} r/L)]^2} \frac{J_\nu(\alpha_{\nu,i} r/L) J_{\nu'}(\alpha_{\nu',i} r/L)}{\alpha_{k/n,i}^2/L^2 + m^2} \cos\Big(\frac{\theta-\theta'}{n} l\Big). \tag{A.36}$$

Taking the limit $L \to \infty$, the index $i$ becomes continuous. Then $\alpha_{\nu,i}/L \to \lambda$ and $\delta_{\nu\nu'} \to \delta(\lambda - \lambda')$, with the normalisation becoming $N_k(\lambda) = d_k \lambda/(2\pi n)$. The Green's function becomes

$$G(r,\theta,r',\theta') = \frac{1}{2\pi n} \sum_{k=0}^{\infty} d_k \int_0^\infty \lambda d\lambda \frac{J_{k/n}(\lambda r) J_{k/n}(\lambda r')}{\lambda^2 + m^2} \cos\Big(\frac{\theta-\theta'}{n} l\Big). \tag{A.37}$$

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