(3.48) $$\lim_{n\to\infty}(\varphi_{n,n+k}g_{n,n+k})(w) = (w)^{-1},\ |w| > 1;$$

(3.49) $$\lim_{n\to\infty}(g_{n,n+k+1}(g_{n,n+k})^{-1})(w) = (w)^{-1},\ |w| > 1.$$

Proof:

Note that the degree of

$$\frac{\varphi_{n,n+k}(w) - \varphi_{n,n+k}(z)}{w-z} \le n+k-1,$$

so by orthonormality,

$$\int_0^{2\pi} \overline{\varphi_{n,n+k}(z)}\left[\frac{\varphi_{n,n+k}(w) - \varphi_{n,n+k}(z)}{w-z}\right] d\mu_n(\theta) = 0.$$

Thus,

(3.50) $$\int_0^{2\pi} \frac{|\varphi_{n,n+k}(z)|^2}{w-z} d\mu_n(\theta) = \int_0^{2\pi} \frac{\varphi_{n,n+k}(w)\overline{\varphi_{n,n+k}(z)}d\mu_n(\theta)}{w-z}.$$

So from (3.50) we get,

(3.51) $$(\varphi_{n,n+k}g_{n,n+k})(w) = \int_0^{2\pi} \frac{|\varphi_{n,n+k}(z)|^2}{w-z} d\mu_n(\theta).$$

Now we use (3.14) with $\ell = 0$ and $f = (w-z)^{-1}$ together with (3.51) and Lemma 3.12 to get (3.47) and (3.48). Notice that

$$\frac{g_{n,n+k+1}}{g_{n,n+k}}(z) = (g_{n,n+k+1}\varphi_{n,n+k+1})\cdot(\varphi_{n,n+k}(\varphi_{n,n+k+1})^{-1})\cdot(g_{n,n+k}\varphi_{n,n+k})^{-1}.$$

Thus using (3.47), (3.48) with (3.8) yields (3.49).

We now show uniform convergence.

$$(\varphi_{n,n+k}g_{n,n+k})(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{|\varphi_{n,n+k}(z)|^2}{w-z} d\mu_n(\theta)$$

by (3.51). Let $\epsilon > 0$ and define

$$K = \{w : d(w, \tau) \geq \epsilon\}.$$

Then K is compact and $\varphi_{n,n+k}g_{n,n+k}$ is uniformly bounded in K as

$$\begin{aligned}|\varphi_{n,n+k}g_{n,n+k}(w)| &\leq \frac{1}{2\pi}(d(w,\tau))^{-1}\int_0^{2\pi}|\varphi_{n,n+k}(z)|^2 d\mu_n(\theta)\\ &\leq \frac{1}{2\pi}(d(w,\tau))^{-1} = C_9.\end{aligned}$$

Let

$$P(w) = \begin{cases} 0, & |w| < 1, \\ (w)^{-1}, & |w| > 1. \end{cases}$$

By the Vitali–Montel theorem choose a subsequence of $\varphi_{n,n+k}g_{n,n+k}$ converging uniformly on K with limit $p(w)$. If the full sequence did not converge to $P(w)$ uniformly in K there exists $\Omega \subseteq \mathbb{N}$ and $z_n \in K$, $n \in \Omega$ such that

$$|g_{n,n+k}\varphi_{n,n+k}(z_n) - P(z)| \geq \epsilon,\ n \in \Omega. \tag{3.52}$$

But by Vitali choose $\Omega' \subseteq \Omega \subseteq \mathbb{N}$ such that for $n \in \Omega'$,

$$\max_{z\in K}|\varphi_{n,n+k}g_{n,n+k}(z) - P(z)| < \epsilon/2$$

as $n \to \infty$, $n \in \Omega'$. But this contradicts (3.52) so we have uniform convergence. ■

Remark

In this chapter and in the subsequent chapters ratio and mean analogues are given to theorems 2.17, 2.18 and 2.19. What form the strong asymptotic analogues for these will take is still open to much research.

Comparative Asymptotics

When deriving Szegő asymptotics on and off τ, we used the Szegő function. However, when this was no longer defined we saw in Chapter 3 that we derived asymptotics with "μ" acting as a "substitute" for D. Another option open to us is to consider the so called comparative asymptotic. The study of comparative asymptotics was initiated by P. Nevai in 1979 and continued with his collaborators Máté and Totik. The idea is the following. Pick two Borel measures μ and σ say with weight functions f_1 and f_2 say. That is $\frac{d\mu}{d\theta} = f_1$, $\frac{d\sigma}{d\theta} = f_2$. We let $\rho = f_1 \cdot (f_2)^{-1}$ and then prove asymptotics of the form $\lim_{n\to\infty} \frac{\varphi_n(d\mu, z)}{\varphi_n(d\sigma, z)} = D(\rho, z)$, where $D(\rho, z)$ is defined. It is this type of asymptotic which will occupy our attention in the next two chapters.

Chapter 4

Comparative Asymptotics off τ

Notation

Throughout, $d\mu$ will denote a finite, positive Borel measure on $\tau \subseteq \mathbb{C}$ with $supp\, d(\mu)$ infinite. We associate with $d\mu$ a non decreasing distribution μ on $[0, 2\pi]$ with the corresponding $(\varphi_n)_{n=0}^{\infty}$ uniquely determined by

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(z)\overline{\varphi_m(z)}d\mu(z) = \delta_{mn},\ z = e^{i\theta},\ \gamma_n > 0,\ \varphi_n(d\mu, z) = \varphi_n(z).$$

Also g will always denote a non negative $d\mu$ integrable function and Q a suitable polynomial such that either $\hat{Q}g^{\pm 1} \in L^{\infty}(d\mu)$ or $\hat{Q}g^{\pm 1}$ is Riemann integrable (see Definition 4.1). In both cases $D(g, z)$ is well defined. We also define a new measure $\mu_g(E)$ as follows. $\mu_g(E) = \int_E g\,d\mu$ for all $d\mu$ measurable sets E. Write $d\mu_g = g\,d\mu$ and note that g will always be chosen such that $supp(g\,d\mu)$ is infinite. $C_1, C_2 \ldots$ will denote constants independent of n, z, ξ, w.

Definition 4.1

Let G be a function defined on τ and define $\hat{G}(t) = G(e^{it})$. Let

Statement 4.2

We now state the main result to be proved in this chapter.

Theorem 4.8

Assume $d\mu$ is such that

$$\lim_{n\to\infty} \Phi_n(d\mu, 0) = 0.$$

Let $g \geq 0$, $g \in L^1(d\mu)(\tau)$ and Q a polynomial such that $\hat{Q}g^{\pm 1}$ is Riemann integrable. Then,

$$\lim_{n\to\infty} \Phi_n(gd\mu, 0) = 0; \tag{4.1}$$

$$\lim_{n\to\infty} w_n(gd\mu, z)\big(w_n(d\mu, z)\big)^{-1} = \left|D(g, z)\right|^2 \tag{4.2}$$

uniformly for $|z| \leq r < 1$;

$$\lim_{n\to\infty} \frac{\varphi_n^*(gd\mu, z)}{\varphi_n^*(z)} = D(g^{-1}, z) \tag{4.3}$$

uniformly for $|z| \leq r < 1$;

$$\lim_{n\to\infty} \frac{K_n(gd\mu, z, \xi)}{K_n(d\mu, z, \xi)} = \overline{D(g^{-1}, z)}D(g^{-1}, \xi) \tag{4.4}$$

uniformly for $|z| \leq r < 1$ and $|\xi| \leq r < 1$;

$$\lim_{n\to\infty} \gamma_n(gd\mu)\big(\gamma_n(d\mu)\big)^{-1} = D(g^{-1}, 0); \tag{4.5}$$

$$\lim_{n\to\infty} \frac{\varphi_n(gd\mu, z)}{\varphi_n(z)} = \overline{D(g^{-1}, z^{-1})} \tag{4.6}$$

uniformly for $|z| \geq S > 1$.

Lemma 4.3 (Gronwall type inequalities)

Let $f, g \geq 0$ be functions defined on integers and let $C \geq 0$. Suppose

$$f(x) \leq C + \sum_{t=1}^{x-1} f(t)g(t), \; x \geq 1,$$

then

$$f(x) \leq C \exp\left(\sum_{t=1}^{x-1} g(t)\right), \; x \geq 1. \tag{4.7}$$

Let $f, g \geq 0$ be functions defined on integers. Let $C \geq 0$ and suppose $g(x) \leq \epsilon$, $x \geq 1$ and

$$f(x) \leq C + \sum_{t=1}^{x-1} f(t)g(t), \; x \geq 1.$$

Then,

$$f(x) \leq \frac{C}{1-\epsilon} \exp\left(\frac{1}{1-\epsilon} \sum_{t=1}^{x-1} g(t)\right), \; x \geq 1. \tag{4.8}$$

Remark

We note that we will require the weaker (4.8) for what follows.

Proof:

Write

$$F(x) = C + \sum_{t=1}^{x-1} f(t)g(t).$$

Then $F(x+1) - F(x) = g(x)f(x)$ and by the assumption, $f(x) \leq F(x)$. Thus $F(x+1) - F(x) \leq F(x)g(x)$, thus $F(x+1) \leq F(x)\big(1 + g(x)\big) \leq F(x)\exp\big(g(x)\big)$, $x \geq 1$. So by induction for some $K \geq 1$,

$$f(K+1) \leq F(K+1) \leq F(K)\exp\big(g(K)\big) \leq \cdots F(1)\exp\left(\sum_{t=1}^{x-1} g(t)\right)$$

$$= C\exp\left(\sum_{t=1}^{x-1} g(t)\right), \; x \geq 1.$$

So (4.7) is proved. Now as

$$f(x) \leq C + \sum_{t=1}^{x-1} f(t)g(t), \; x \geq 1$$

we have that

$$f(x)\big(1 - g(x)\big) \leq C + \sum_{n=1}^{x-1} f(t)g(t).$$

Also as $g(x) \leq \epsilon$ for all $x \geq 1$ we have

$$f(x) \leq C(1-\epsilon)^{-1} + \sum_{t=1}^{x-1} \frac{f(t)g(t)}{1-\epsilon}.$$

Thus

$$f(x) \leq \frac{C}{1-\epsilon} \exp\left(\frac{1}{1-\epsilon} \sum_{t=1}^{x-1} g(t)\right)$$

by (4.7) for $x \geq 1$. ■

<u>Lemma 4.4</u>

Let $d\mu$ be an arbitrary distribution on τ and define functions f and h as follows:

$$f(z) = \begin{cases} 0, & |z| \leq 1, \\ |z|^2 - 1, & |z| > 1; \end{cases} \qquad h(z) = \begin{cases} 1 - |z|^2, & |z| \leq 1, \\ 0, & |z| > 1. \end{cases}$$

Then the following are equivalent:

(a)

(4.9) $$\lim_{n \to \infty} \Phi_n(0) = 0.$$

(b) There exists $z \in \mathbf{C}$, $|z| \neq 1$ such that

(4.10) $$\lim_{n \to \infty} \left|\varphi_n(z)\right|^2 w_n(d\mu, z) = f(z).$$

(c) There exists $z \in \mathbf{C}$, $|z| \neq 1$ such that

(4.11) $$\lim_{n \to \infty} \left|\varphi_n^*(z)\right|^2 w_n(d\mu, z) = h(z);$$

(d)

$$(4.12)\qquad \lim_{n\to\infty}\sup_{z\in\mathbf{C}}\left||\varphi_n(z)|^2 w_n(d\mu,z)-f(z)\right|(1+|z|^2)^{-1}=0;$$

(e)

$$(4.13)\qquad \lim_{n\to\infty}\sup_{z\in\mathbf{C}}\left||\varphi_n^*(z)|^2 w_n(d\mu,z)-h(z)\right|(1-|z|^2)^{-1}=0.$$

Also for every $z\in\mathbf{C}$, $|z|=1$, there exists $d\mu=d\mu_{(z)}$ such that

$$\lim_{n\to\infty}|\varphi_n(z)|^2 w_n(d\mu,z)=0$$

and none of (4.9) — (4.13) hold.

Proof:

This is Theorem 4 in ([13]) together with Lemma 4.3 ■

Lemma 4.5

Let $d\mu$ satisfy $\lim_{n\to\infty}\Phi_n(0)=0$, then for all $d\mu$ measurable and Riemann integrable functions F,

$$(4.14)\qquad \lim_{n\to\infty}\int_0^{2\pi}|\varphi_n(z)|^2 F(\theta)d\mu(\theta)=\int_0^{2\pi}F(\theta)d\theta,\ (z=e^{i\theta});$$

$$(4.15)\qquad \lim_{n\to\infty}\int_0^{2\pi}\varphi_n(z)\overline{\varphi_n^*(z)}F(\theta)d\mu(\theta)=0,\ (z=e^{i\theta}).$$

Remark

We note that this is stronger than Theorem 3.10 as $\mu'>0$ *a.e.* implies $\lim_{n\to\infty}\Phi_n(0)=0$.

Proof:

We have by Theorem 3.8 that

$$\lim_{n\to\infty} \frac{\varphi^*_{n+1}(z)}{\varphi^*_n(z)} = 1$$

and

$$\lim_{n\to\infty} \frac{\varphi_{n+1}(z)}{\varphi_n(z)} = z$$

uniformly for $|z| = 1$. Now suppose $F(z) = z^K$, $(z = e^{i\theta})$ then for example the left-hand side of (4.15) becomes after an application of Theorem 3.8

$$\int_0^{2\pi} \varphi_{n+k}(z)\overline{\varphi^*_{n+k-1}(z)}[1 + o(1)]\,d\mu(\theta) \to 0$$

by orthonormality as $n \to \infty$. Now given F continuous and 2π periodic, we can always find a trigonometric polynomial T such that $\|F - T\|_{L^\infty([0,2\pi])}$ and $\|F - T\|_{L^\infty(d\mu)([0,2\pi])}$ are arbitrarily small. Hence,

$$\begin{aligned}
&\left|\int_0^{2\pi} |\varphi_n|^2 F(\theta)\,d\mu(\theta) - \int_0^{2\pi} F(\theta)\,d\theta\right| \\
&= \left|\int_0^{2\pi} |\varphi_n|^2 T(\theta)\,d\mu(\theta) - \int_0^{2\pi} T(\theta)\,d\theta + \int_0^{2\pi} |\varphi_n|^2 (F(\theta) - T(\theta))\,d\mu(\theta) + \right. \\
&\qquad \left. + \int_0^{2\pi} (T(\theta) - F(\theta))\,d\theta\right| \\
&\le \left|\int_0^{2\pi} |\varphi_n|^2 T(\theta)\,d\mu\theta - \int_0^{2\pi} T(\theta)\,d\theta\right| + \\
&\qquad + \|F - T\|_{L^\infty(d\mu)([0,2\pi])} \cdot 1 + 2\pi\|F - T\|_{L^\infty([0,2\pi])} \\
&= o(1) \text{ as } n \to \infty.
\end{aligned}$$

So (4.14) is true for F continuous and 2π periodic. Similarly (4.15) is true under the same conditions. Next if F is Riemann integrable we can choose G continuous and 2π periodic such that $\|F - G\|_{L^2}$, is arbitrarily small. By a slightly more complicated argument [27, Theorem 1.5.4, p.11], (4.14) is true for F. Now as (4.14) is true for F, we have using (4.14) with $F(\theta) = (F(\theta))^{1/2}(F(\theta))^{1/2}$ and the Schwarz inequality that

$$\text{(4.16)} \qquad \limsup_{n\to\infty} \left|\int_0^{2\pi} \varphi_n(z)\overline{\varphi^*_n(z)} F(\theta)\,d\mu(\theta)\right| \le \int_0^{2\pi} |F(\theta)|\,d\theta, \; (z = e^{i\theta}).$$

Proof:

We have by Theorem 3.8 that

$$\lim_{n\to\infty}\frac{\varphi^*_{n+1}(z)}{\varphi^*_n(z)}=1$$

and

$$\lim_{n\to\infty}\frac{\varphi_{n+1}(z)}{\varphi_n(z)}=z$$

uniformly for $|z|=1$. Now suppose $F(z)=z^K$, $(z=e^{i\theta})$ then for example the left–hand side of (4.15) becomes after an application of Theorem 3.8

$$\int_0^{2\pi}\varphi_{n+k}(z)\overline{\varphi^*_{n+k-1}(z)}\,[1+o(1)]\,d\mu(\theta)\to 0$$

by orthonormality as $n\to\infty$. Now given F continuous and 2π periodic, we can always find a trigonometric polynomial T such that $\|F-T\|_{L^\infty([0,2\pi])}$ and $\|F-T\|_{L^\infty(d\mu)([0,2\pi])}$ are arbitrarily small. Hence,

$$\begin{aligned}
&\left|\int_0^{2\pi}|\varphi_n|^2F(\theta)d\mu(\theta)-\int_0^{2\pi}F(\theta)d\theta\right|\\
&\quad=\left|\int_0^{2\pi}|\varphi_n|^2T(\theta)d\mu(\theta)-\int_0^{2\pi}T(\theta)d\theta+\int_0^{2\pi}|\varphi_n|^2(F(\theta)-T(\theta))d\mu(\theta)+\right.\\
&\qquad\left.+\int_0^{2\pi}(T(\theta)-F(\theta))d\theta\right|\\
&\quad\le\left|\int_0^{2\pi}|\varphi_n|^2T(\theta)d\mu\theta-\int_0^{2\pi}T(\theta)d\theta\right|+\\
&\qquad+\|F-T\|_{L^\infty(d\mu)([0,2\pi])}\cdot 1+2\pi\|F-T\|_{L^\infty([0,2\pi])}\\
&\quad=o(1)\quad\text{as}\quad n\to\infty.
\end{aligned}$$

So (4.14) is true for F continuous and 2π periodic. Similarly (4.15) is true under the same conditions. Next if F is Riemann integrable we can choose G continuous and 2π periodic such that $\|F-G\|_{L^2}$, is arbitrarily small. By a slightly more complicated argument [27, Theorem 1.5.4, p.11], (4.14) is true for F. Now as (4.14) is true for F, we have using (4.14) with $F(\theta)=(F(\theta))^{1/2}(F(\theta))^{1/2}$ and the Schwarz inequality that

(4.16) $$\limsup_{n\to\infty}\left|\int_0^{2\pi}\varphi_n(z)\overline{\varphi^*_n(z)}F(\theta)d\mu(\theta)\right|\le\int_0^{2\pi}|F(\theta)|d\theta,\ (z=e^{i\theta}).$$

So then we have, using (4.16)

$$
\begin{aligned}
&\limsup_{n\to\infty}\left|\int_0^{2\pi} \varphi_n\overline{\varphi_n^*}(z)F(\theta)d\mu(\theta)\right| \\
&\quad\le \limsup_{n\to\infty}\left|\int_0^{2\pi} \varphi_n\overline{\varphi_n^*(z)}(F(\theta)-G(\theta))d\mu(\theta)\right| + \\
&\quad + \lim_{n\to\infty}\left|\int_0^{2\pi} G(\theta)\varphi_n\overline{\varphi_n^*(z)}d\mu(\theta)\right| \\
&\quad\le \int_0^{2\pi} |F(\theta)-G(\theta)|d\theta + \lim_{n\to\infty}\left|\int_0^{2\pi} G(\theta)\varphi_n\overline{\varphi_n^*(z)}d\mu(\theta)\right| \\
&\quad\text{(by (4.16) as } F-G \text{ is Riemann integrable)} \\
&\quad\to 0
\end{aligned}
$$

using the fact that $\|F-G\|_{L^2}$ is arbitrarily small and that (4.15) holds for continuous G. ■

Lemma 4.6

Let $d\mu$ satisfy $\lim_{n\to\infty}\Phi_n(0)=0$. Let F be a $d\mu$ measurable and Riemann integrable function. Then for all $0<r<1$,

(a) $$\lim_{n\to\infty} G_n(d\mu,F,z)=P(F,z)$$

holds uniformly for $|z|\le r<1$. Further, let G be a bounded $d\mu$ measurable function and let $z=e^{i\theta}$. If G is continuous at θ then

$$\lim_{n\to\infty} G_n(d\mu,G,z)=G(\theta)$$

and if G is uniformly continuous on an open set $\Omega\subseteq[0,2\pi]$ then given any compact $\Omega'\subseteq\Omega$,

(4.17) $$\lim_{n\to\infty} G_n(d\mu,G,z)=G(\theta)$$

uniformly for $\theta\in\Omega'$.

Proof:

The proof of (a) follows by the definition of G_n, K_n, Lemma 4.5, (4.12) and (4.13). Next we prove (4.17). First note that if $u = e^{it}$, using (4.12), (4.13) and the definition of K_n we have,

$$\lim_{n\to\infty} \frac{w_n(z)}{2\pi} \int_0^{2\pi} |z-u|^2 |K_n(z,u)|^2 d\mu(\theta) = 0 \tag{4.18}$$

uniformly for $|z| = 1$.

Also, by orthonormality and the definition of K_n we have

$$\frac{w_n(z)}{2\pi} \int_0^{2\pi} |K_n(z,u)|^2 d\mu(t) = 1, \ (u = e^{it}). \tag{4.19}$$

Now fix $\delta \in (0,1)$. Then $G_n(d\mu, G, z) = I_1 + I_2$ where

$$I_1 = \frac{w_n(d\mu, z)}{2\pi} \int_{|t-\delta|>\delta} G(t) |K_n(d\mu, z, u)|^2 d\mu(\theta),$$

$$I_2 = \frac{w_n(d\mu, z)}{2\pi} \int_{|t-\theta|\le\delta} G(t) |K_n(d\mu, z, u)|^2 d\mu(\theta).$$

Now if $|t-\theta| > \delta$ then

$$|z-u| = \left|2\sin\left(\frac{t-\theta}{2}\right)\right| \ge C_1\delta.$$

Thus

$$\lim_{n\to\infty} \frac{w_n(z)}{2\pi} \int_{|t-\theta|>\delta} |K_n(z,u)|^2 d\mu(t)$$

$$\le \lim_{n\to\infty} \frac{w_n(z)}{2\pi} \int_{|t-\theta|>\delta} \frac{|K_n(z,u)|^2 |z-u|^2}{C^2\delta^2} = 0$$

(by (4.18)) uniformly for $|z| = 1$. Thus by (4.19) and the fact that G is bounded independent of n gives that

$$I_1 \to 0 \text{ as } n \to \infty \tag{4.20}$$

uniformly for $|z| = 1$.

Next

$$\left|w_n(z)\frac{1}{2\pi}\int_0^{2\pi} G_n(t)\left|K_n(z,u)\right|^2 d\mu(t) - G(\theta)\right|$$

$$\leq \left|\frac{w_n(z)}{2\pi}\int_{|t-\theta|>\delta}(G(t)-G(\theta))\left|K_n(z,u)\right|^2 d\mu(t)\right| +$$

$$+\left|\frac{w_n(z)}{2\pi}\int_{|t-\theta|\leq\delta}(G(t)-G(\theta))\left|K_n(z,u)\right|^2 d\mu(t)\right|$$

$$\leq o(1) + C_2 \max_{|t-\theta|\leq\delta}\left|G(t)-G(\theta)\right|.$$

Thus we have,

$$\left|w_n\frac{1}{2\pi}\int_0^{2\pi} G_n(t)\left|K_n(z,u)\right|^2 d\mu(t) - G(\theta)\right|$$

$$\leq o(1) + C_2 \max_{|t-\theta|\leq\delta}\left|G(t)-G(\theta)\right|. \tag{4.21}$$

So if G is continuous, let $\delta \to 0$. Then by definition of G_n, (4.18) and (4.19) the result holds. If G is uniformly continuous, the modulus of continuity $w(G,\delta) \to 0$ as $\delta \to 0$. So use (4.21) to yield result. ■

Lemma 4.7

Let $d\mu$ be a distribution on τ. Let g be a non negative $d\mu$ integrable function, such that $gd\mu$ is also a distribution. Let m be a non negative integer. Then for all $n \geq m$ and for all $z \in \mathbf{C}$ we have,

$$\frac{w_n(gd\mu,z)}{w_{n-m}(d\mu,z)} \leq \left|\pi_m(z)\right|^{-2} G_{n-m}\left(d\mu, \left|\widehat{\pi_m}\right|^2 g, z\right), \tag{4.22}$$

$$\frac{w_{n+m}(d\mu,z)}{w_n(gd\mu,z)} \leq \left|\pi_m(z)\right|^{-2} G_{n+m}\left(d\mu, \left|\widehat{\pi_m}\right|^2 g^{-1}, z\right), \tag{4.23}$$

(where $\widehat{\pi_m}(t) = \pi_m(e^{it})$).

Proof:

By the definition of W_n and K_n and (1.28) we have that

$$w_n(gd\mu,z) \leq \left|\pi_{n-1}(z)\right|^{-2}.\frac{1}{2\pi}\int_0^{2\pi}\left|\pi_{n-1}(u)\right|^2 g(t)d\mu(t), \ (u=e^{it})$$

for every π_{n-1} of degree $n-1$. Now set

$$\pi_{n-1}(u) = \pi_m(u)K_{n-m}(d\mu, z, u)$$

of degree $n-1$. Then,

$$\begin{aligned}
&\frac{w_n(gd\mu, z)}{w_n(d\mu, z)} \\
&\quad \leq \left(\frac{|\pi_m(z)|^{-2}|K_{n-m}(d\mu, z, z)|^{-2}}{w_{n-m}(d\mu, z)} \right) \times \\
&\qquad \times \left(\frac{1}{2\pi} \int_0^{2\pi} |\pi_m(u)|^2 |K_{n-m}(z, u)|^2 g(t)d\mu(t) \right) \\
&\quad = |\pi_m(z)|^{-2} G_{n-m}\left(d\mu, |\widehat{\pi_m}|^2 g, z\right)
\end{aligned}$$

which gives (4.22).

Next, by (1.26) we have

$$\begin{aligned}
&\pi_m(z)K_n(gd\mu, z, z) \\
&\quad = \frac{1}{2\pi} \int_0^{2\pi} \pi_m(u)K_n(gd\mu, z, u)K_{n+m}(d\mu, u, z)d\mu(t), \ (u = e^{it}).
\end{aligned}$$

Thus by the Schwarz inequality,

$$\begin{aligned}
&|\pi_m(z)|^2 K_n^2(gd\mu, z, z) \\
&\quad \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |K_n(gd\mu, z, u)|^2 g(t)d\mu(t) \right) \times \\
&\qquad \times \left(\frac{1}{2\pi} \int_0^{2\pi} |\pi_m(u)|^2 |K_{n+m}(d\mu, z, u)|^2 g^{-1}(t)d\mu(t) \right) \\
&\quad = K_n(gd\mu, z, z)G_{n+m}\left(d\mu, |\widehat{\pi_m}|^2 g^{-1}, z\right)
\end{aligned}$$

which gives (4.23) ∎

We can now give the **proof of Theorem 4.8.**

<u>Proof:</u>

We first show that (4.2) is true, so choose a polynomial Q such that Q has no zeros in $|z| < 1$ and such that $\hat{Q}g^{\pm 1}$ is Riemann integrable. Let m_1 be the degree of Q. Choose m_2 a positive integer and let π_{m_2} be a polynomial of degree m_2. Put $m = m_1 + m_2$ so that $\pi_m = Q\pi_{m_2}$. Now with Lemma 4.7, $n \geq m$ and for every $z \in \mathbb{C}$, we have

$$\frac{w_n(gd\mu, z)}{w_n(d\mu, z)} = \frac{w_{n-m}(d\mu, z)}{w_n(d\mu, z)} \cdot \frac{w_n(gd\mu, z)}{w_{n-m}(d\mu, z)}$$
$$\leq \frac{w_{n-m}(d\mu, z)}{w_n(d\mu, z)} \cdot \left|\pi_m(z)\right|^2 G_{n-m}\left(d\mu, |\widehat{\pi_m}|^2 g, z\right).$$

Hence we get,

$$\frac{w_n(gd\mu, z)}{w_n(d\mu, z)}$$
$$\leq \frac{w_{n-m}(d\mu, z)}{w_n(d\mu, z)} \left|\pi_m(z)\right|^{-2} G_{n-m}\left(d\mu, |\widehat{\pi_m}|^2 g, z\right) \tag{4.24}$$

and similarly

$$\frac{w_n(d\mu, z)}{w_n(gd\mu, z)}$$
$$\leq \frac{w_n(d\mu, z)}{w_{n+m}(d\mu, z)} \cdot \left|\pi_m(z)\right|^{-2} G_{n+m}\left(d\mu, |\widehat{\pi_m}|^2 g^{-1}, z\right). \tag{4.25}$$

Next, using (1.20) we obtain that

$$\left(w_{n+1}(d\mu, z)\right)^{-1} = \left(w_n(d\mu, z)\right)^{-1} + \left|\varphi_n(d\mu, z)\right|^2. \tag{4.26}$$

So using (4.12) we may deduce that

$$\lim_{n\to\infty} \frac{w_n(d\mu, z)}{w_{n+1}(d\mu, z)} = 1$$

uniformly for $|z| \leq 1$. Thus for every fixed m,

$$\lim_{n\to\infty} \frac{w_n(d\mu, z)}{w_{n+m}(d\mu, z)} = \lim_{n\to\infty} \frac{w_{n-m}(d\mu, z)}{w_n(d\mu, z)} = 1 \tag{4.27}$$

uniformly for $|z| \leq 1$.

Now using Lemma 4.6, (4.24) and (4.25) become

$$\limsup_{n\to\infty} \frac{w_n(gd\mu, z)}{w_n(d\mu, z)} \leq \left|\pi_m(z)\right|^{-2} P\left(|\widehat{\pi_m}|^2 g, z\right), \tag{4.28}$$

$$(4.29) \qquad \limsup_{n\to\infty} \frac{w_n(d\mu,z)}{w_n(gd\mu,z)} \le |\pi_m(z)|^{-2} P(|\widehat{\pi_m}|^2 g^{-1}, z)$$

uniformly for $|z| \le r < 1$.

Now for a given $z \in \mathbb{C}$, let π_1 be defined by $\pi_1(u) = 1 - z\bar{u}$ and $\pi_m = Q\pi_1\pi_{m_3}$ where $m_3 = m_2 - 1$, $u = e^{it}$. Then using Schwarz's inequality, (4.28), (4.29) and the definition of π_1 we obtain,

$$\limsup_{n\to\infty} \frac{w_n(gd\mu,z)}{w_n(d\mu,z)}$$

$$(4.30) \qquad \le |Q(z)|^{-2}(1-|z|^2)^{-1}|\pi_{m_3}(z)|^{-2} \cdot \frac{1}{2\pi}\int_0^{2\pi} |\pi_{m_3}(t)|^2 |Q(t)|^2 g(t)\,dt$$

and

$$\limsup_{n\to\infty} \frac{w_n(d\mu,z)}{w_n(gd\mu,z)} \le |Q(z)|^{-2}(1-|z|^2)^{-1}|\pi_{m_3}(z)|^{-2} \times$$

$$(4.31) \qquad \times \frac{1}{2\pi}\int_0^{2\pi} |\pi_{m_3}(t)|^2 |Q(t)|^2 (g(t))^{-1}\,dt.$$

Now use (1.28) and take the minimum of the righthand side of (4.30) and (4.31) over all polynomials π_{m_3} of degree $m_2 - 1$ to obtain

$$\limsup_{n\to\infty} \frac{w_n(gd\mu,z)}{w_n(d\mu,z)}$$

$$(4.32) \qquad \le |Q(z)|^{-2}(1-|z|^2)^{-1} w_{m_2}(|\hat{Q}|^2 g\,dt, z)$$

and

$$\limsup_{n\to\infty} \frac{w_n(d\mu,z)}{w_n(gd\mu,z)}$$

$$(4.33) \qquad \le |Q(z)|^{-2}(1-|z|^2)^{-1} w_{m_2}(|\hat{Q}|^2 g^{-1}\,dt, z).$$

Since Q has no zeros in the open unit disc by (2.46) we obtain

$$\lim_{m_2\to\infty} |Q(z)|^{-2}(1-|z|^2)^{-1} w_{m_2}(|\hat{Q}|^2 g\,dt, z)$$
$$= |Q(z)|^{-2}|D(|\hat{Q}|^2 g, z)|^2 = |D(g,z)|^2$$

and similarly

$$\lim_{m_2\to\infty}\left|Q(z)\right|^{-2}\left(1-|z|^2\right)^{-1}w_{m_2}\left(|\hat{Q}|^2(g)^{-1}dt,z\right)$$
$$=\left|D(g,z)\right|^2$$

uniformly for $|z|\le r<1$.

Thus letting $m_2\to\infty$ in (4.32) and (4.33) yields (4.2). Now we show (4.1) and (4.3). Firstly, as

$$\varphi_n^*(0)=\gamma_n=\left(w_{n+1}(0)\right)^{-1/2}$$

and as $\log g\in L^1[0,2\pi]$ we have $D(g,0)>0$. So putting $z=0$ and using (4.2) we have,

$$\lim_{n\to\infty}\frac{w_n^{-1/2}(gd\mu,0)}{w_n^{-1/2}(d\mu,0)}=D(g,0)^{-1}=D(g^{-1},0).$$

So (4.3) holds for $z=0$. But as (4.3) holds for $z=0$ together with $\varphi_n^*(0)=\gamma_n(d\mu)$, (1.18) and (3.6) we have (4.1). We show (4.3) in more generality uniformly for $|z|\le r<1$. Now by (4.13) and (4.2) we have that,

$$\lim_{n\to\infty}\left|\frac{\varphi_n^*(gd\mu,z)}{\varphi_n^*(z)}\right|=\left|D(g,z)\right|^{-1}=\left|D(g^{-1},z)\right| \tag{4.34}$$

uniformly for $|z|\le r<1$.

Put

$$f_n(z)=\frac{\varphi_n^*(gd\mu,z)}{\varphi_n^*(z)}D(g,z).$$

We must show that,

$$\lim_{n\to\infty}f_n(z)=1 \tag{4.35}$$

uniformly for $|z|\le r<1$. Now $f_n(z)$ is analytic in $|z|<1$ and φ_n^* has all its zeros in $|z|>1$. Assume that (4.35) is not true. Fix $0<r<1$, $\epsilon>0$. Then there exists $\Omega\subseteq\mathbb{N}$, and (z_n), $n\in\Omega$ such that as $n\to\infty$, $n\in\Omega$, $|z_n|\le r$,

$$\left|f_n(z_n)-1\right|>\epsilon. \tag{4.36}$$

Now as (4.3) holds for $z = 0$ we have $\lim_{n\to\infty} f_n(0) = 1$, $\lim_{n\to\infty}|f_n(z)| = 1$ uniformly for $|z| \leq \frac{r+1}{2}$ say and in particular for $n \in \Omega$. Thus $(f_n(z))n \in \Omega$ is uniformly bounded in $|z| \leq \frac{r+1}{2}$. So by the Vitali–Montel theorem select $\Omega' \subseteq \Omega \subseteq \mathbb{N}$ such that $\lim_{n\in\Omega'} f_n(z) = f(z)$ uniformly for $|z| \leq r$, f analytic in $|z| \leq r$. Then we must have $f(0) = 1$ and $|f(z)| = 1$, $|z| \leq 1$. But then this contradicts (4.36) so (4.3) is shown. Next we show (4.5). Using the definition of K_n (Lemma 1.7), (4.12) and (4.13) we have

$$\frac{K_n(gd\mu, z, \xi)}{K_n(d\mu, z, \xi)} = \frac{\overline{\varphi_n^*(gd\mu, z)}\varphi_n^*(gd\mu, \xi')}{\overline{\varphi_n^*(z)}\varphi_n^*(\xi)}(1 + o(1)_{z,\xi}) \tag{4.37}$$

uniformly for $|\xi|$, $|z| \leq r$ as $n \to \infty$. Thus using (4.3) with (4.37) yields (4.5). Now (4.5) is (4.3) with $z = 0$ as $\varphi_n^*(0) = \gamma_n$ and finally (4.6) is the same as (4.3) taking the star transform. ■

We end this chapter by proving a variation of Theorem 4.8.

Lemma 4.9

Lemma 4.6 (a) holds if $\mu' > 0$ a.e. $(meas)$ and $F \in L^\infty(d\mu)$.

Proof:

This follows from (4.12), (4.13), (3.6) and Theorem 3.10.

Theorem 4.10

Let $\mu' > 0$ $a.e.$ $(meas)$ in $[0, 2\pi]$. Choose g and Q as in Theorem 4.8 except that $\hat{Q}g^{\pm 1} \in L^\infty(d\mu)$. Then (4.1) — (4.6) are true.

Proof:

The proof is identical to Theorem 4.8 except that Lemma 4.9 and Theorem 3.10 are used instead of Lemma 4.6 and Lemma 4.5. ■

Remark

We note here that in order to lessen the condition on $\hat{Q}g^{\pm 1}$ we must strengthen that on μ.

Chapter 5

Comparative Asymptotics on the unit circle

In Chapter 4, we considered comparative asymptotics off τ. Now we continue our study of comparative asymptotics except we consider the case of asymptotics on τ.

Notation

Throughout μ and σ will denote positive finite Borel measures on τ. $supp(d\mu)$ and $supp(d\sigma)$ will be infinite, $\varphi_n(z) = \varphi_n(d\mu, z)$ will always be the nth orthonormal polynomial with respect to $d\mu$, uniquely determined by

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(z)\overline{\varphi_m(z)}d\mu(z) = \delta_{mn}, \ \gamma_n(d\mu) > 0.$$

g will be a non negative $d\mu$ integrable function and Q a suitable polynomial such that $\hat{Q}g^{\pm 1} \in L^\infty(d\mu)$. g will be choeen such that $supp(gd\mu)$ is infinite. $D(\rho, z)$ and $D(\rho, z, ex)$ will denote the interior and exterior Szegõ functions with respect to ρ. If ρ is a continuous function

$$\mathcal{L}(\rho, z) = \max_{\xi\in\tau} \frac{|\rho(z) - \rho(\xi)|}{|z - \xi|}$$

will be the Lipschitz constant with respect to ρ. $C_1, C_2, \ldots$ will denote constants indepedent of n, z, ξ, w. We denote by $\tau_\epsilon(z) = \{e^{i\alpha}: |\theta - \alpha| \le \epsilon\}$, $z = e^{i\theta}$ the open arc neighbourhood of $z \in \tau$ and $\overline{\tau_\epsilon(z)} = \{e^{i\alpha}: |\theta - \alpha| \le \epsilon\}$, $z = e^{i\theta}$ the corresponding closed arc.

Statement 5.1

The main results to be proved in this chapter are the following.

Theorem 5.2

Let $d\mu$ be such that $\lim_{n\to\infty} \Phi_n(d\mu, 0) = 0$. Let $g \geq 0$ and $g \in L^1(d\mu)(\tau)$. Choose Q such that $\hat{Q}g^{\pm 1} \in L^\infty(d\mu)(\tau)$. Let $|z| = 1$ and let g be continuous for $\theta \in [0, 2\pi]$. Then we have,

$$\lim_{n\to\infty} \frac{w_n(g d\mu, z)}{w_n(d\mu, z)} = g(\theta), \ \theta \in [0, 2\pi]. \tag{5.1}$$

Further, if g is uniformly continuous on $\Omega \subseteq [0, 2\pi]$, Ω open then for every compact $\Omega' \subseteq \Omega$ (5.1) holds uniformly for $\theta \in \Omega'$.

Theorem 5.11

Let μ and σ be two finite, positive Borel measures on τ. Let μ and σ have weight functions w_1, w_2 respectively, w_1, $w_2 > 0$ a.e. (meas). Let $\rho = w_1(w_2)^{-1} = \frac{d\mu}{d\sigma}$ be positive, continuous and 2π periodic on τ. Further let $\mathcal{L}_\rho(z) \leq C$, $z \in \gamma \subseteq \tau$ where γ is an arbitrary subset of τ. Then, $\lim_{n\to\infty} \varphi_n(z)\left(\varphi_n(d\sigma, z)\right)^{-1} = D(\rho, z, ex)$ uniformly for $z \in \gamma$.

Theorem 5.13

Let $\mu' > 0$ a.e. (meas) in $[0, 2\pi]$ and let g and Q be as in Theorem 5.2. Then we have,

$$\lim_{n\to\infty} \int_0^{2\pi} \left|\varphi_n(g d\mu, u)\overline{D(g, u)} - \varphi_n(u)\right|^2 \mu'(t)\,dt = 0, \quad (u = e^{it}).$$

Remarks:

(a) We mention that Nevai et al in [15] have achieved the following two comparative results.

(1) Let g and $d\mu$ be as in Theorem 5.13 and that for $\theta \in [0, 2\pi]$, $g(\theta) > 0$ and $\left|g(t) - g(\theta)\right| \leq k|t - \theta|$ when $|t - \theta| < \delta$ for some k and $\delta > 0$.

Then

$$(5.2) \qquad \lim_{n\to\infty} \frac{\varphi_n(gd\mu, e^{i\theta})}{\varphi_n(e^{i\theta})} = \overline{D(g^{-1}, e^{i\theta})}.$$

(2) Let g and $d\mu$ be as in (1) and $g(\theta) > 0$ for $\theta \in \Omega \subseteq [0, 2\pi]$, Ω closed and $|g(t) - g(\theta)| \leq k|(t - \theta)|$ when $|t - \theta| < \delta$ some k and $\delta > 0$. Then (5.2) holds uniformly for $\theta \in \Omega$.

(b) [15] has also given a comparative analogue to Theorem 2.19 (b) which is stronger than (5.2) and (5.3) in the sense that weaker conditions are imposed on g, however uniform boundedness of φ_n is required, in a neighbourhood of θ. We note that this boundedness was not required in Theorem 2.19 (b).

Let g and $d\mu$ satisfy the conditions of Theorem 5.1. Let $\delta > 0$, $\eta > 0$, $k > 0$ and suppose for $\theta \in [0, 2\pi]$, $|\varphi_n(e^{it})| \leq k$, $t \in (\theta - \delta, \theta + \delta)$, $n = 0, 1, 2, \ldots$, $\eta < g(t) < k$ and $\int_{\theta-\delta}^{\theta+\delta} \left|\frac{g(t)-g(\theta)}{t-\theta}\right|^2 d\mu(t) < \infty$. Then (5.2) is valid.

(c) The following theorem is found in [15] and we note that the conditions on μ are weakend but those on $\hat{Q}g^{\pm 1}$ are strengthened. Let $d\mu$ satisfy $\lim_{n\to\infty} \Phi_n(d\mu, 0) = 0$, let $g \geq 0$, $g \in L^1(d\mu)(\tau)$ and choose Q such that $\hat{Q}g^{\pm 1}$ are Riemann integrable on $[0, 2\pi]$. Then under the assumptions of (a)(1) and (b), (a)(1) and (b) remain valid. If $d\mu$ is absolutely continuous then Theorem 5.13 holds in this new setting.

(d) We note that Theorem 5.13 is the natural analogue of (2.58) and (2.59).

(e) Some open questions concerning these remarks will be raised at the end of this chapter.

We can already give **The proof of Theorem 5.2:**

Proof:

We assume first that $g(\theta) > 0$. By continuity of g, we may also assume without loss of generality that $Q(e^{i\theta}) \neq 0$. Now we use the inequalities (4.22) and (4.23) with $\pi_m = Q$ together with (4.17) and (4.27) to deduce the result for this case. Next if $g(\theta) = 0$ and $\epsilon > 0$ then using (1.28) we have that

$$(5.3) \qquad 0 \leq \frac{w_n(g d\mu, z)}{w_n(d\mu, z)} \leq \frac{w_n((g+\epsilon)d\mu, z)}{w_n(d\mu, z)} \to g(\theta) + \epsilon \text{ as } n \to \infty$$

by the case above. Letting $\epsilon \to 0$ in (5.3) gives the result pointwise. Next we show uniform convergence on Ω' under the given assumptions. Suppose as before that $g(\theta) > 0$, $\theta \in \Omega'$ and assume as before that $Q(e^{i\theta}) \neq 0$ for $\theta \in \Omega'$. Thus uniform convergence follows as in Lemma 4.6, since g^{-1} and Q^{-1} are bounded away from zero and so uniformly continuous on some open neighbourhood of Ω'. Now without the above assumption let $\epsilon > 0$ and write $\Omega_1'(\epsilon) = \{\theta \in \Omega' : g(\theta) \geq \epsilon\}$ and $\Omega_2'(\epsilon) = \{\theta \in \Omega' : g(\theta) \leq \epsilon\}$. Then we have uniform convergence in (5.1) on $\Omega_1'(\epsilon)$ by the result just proved. As $\epsilon > 0$ is arbitrary it is enough to show that for large enough n independent of θ,

$$\frac{w_n(g d\mu, z)}{w_n(d\mu, z)} \leq 3\epsilon, \ (z = e^{i\theta})$$

holds for $\theta \in \Omega_2'(\epsilon)$. But this is true by observing that we have uniform convergence on $\Omega_2'(\epsilon)$ in the limit in (5.3) according to the result just established. So we are done. ■

Remark:

We note that the above result holds if $\mu' > 0$ a.e. (meas) by (3.6).

We can now proceed to prove Theorem 5.11. We need first some auxillary results.

Lemma 5.3

Let μ and σ be two finite Borel measures on τ. Suppose $d\mu = \rho d\sigma$ and $\rho(e^{i\theta}) = t_1(\theta)\left(t_2(\theta)\right)^{-1}$ where t_1 and t_2 are non negative trigonometric polynomials. Then,

$$(5.4) \qquad \lim_{n\to\infty} \frac{\varphi_n(z)}{\varphi_n(d\sigma, z)} = D(\rho, z, ex)$$

uniformly on compact subsets of $\left\{z: |z| \geq 1,\, z \neq \exp(i\beta_k),\, k = 1, \ldots, m\right\}$ where $\{\beta_k\}_1^{\infty}$ is the set of zeros of t_1 and t_2.

Proof:

By Lemma 2.3 there exist algebraic polynomials P_1 and P_2 such that

$$t_i(\theta) = \left|P_i(e^{i\theta})\right|^2,\ i = 1,2$$

with zeros in $|z| \leq 1$. We make the important observation that it is sufficient to prove the result for

$$t_1 = \left|P_1(e^{i\theta})\right|^2,\ t_2 = 1$$

or

$$t_2 = \left|P_2(e^{i\theta})\right|^2,\ t_1 = 1$$

by (2.32) and the fact that Borel measures are closed under multiplication of non negative trigonometric polynomials. Thus it is enough to prove the result for

$$d\mu(z) = |z - z_0|^2 d\sigma(z),\ z = e^{i\theta},$$

z_0 fixed such that $|z_0| \leq 1$. Now let $\hat{z}_0 = (\bar{z}_0)^{-1}$. If $z_0 = 0$ then $\rho = \dfrac{d\mu}{d\sigma} = 1$ which implies

$$\frac{\varphi_n(z)}{\varphi_n(d\sigma, z)} = 1$$

and

$$D(\rho, z, ex) = \exp(0) = 1.$$

So the result holds trivially. Suppose then without loss of generality that $z_0 \neq 0$, that is $\hat{z}_0 \neq \infty$. Let

$$Q_{n+2}(z) = (z - z_0)(z - \hat{z}_0)\Phi_n(z) - \alpha_n \Phi_n^*(d\sigma, z)$$

where

$$\alpha_n = -\hat{z}_0 \overline{a_n}(d\mu) \frac{\gamma_n^2(d\sigma)}{\gamma_n^2(d\mu)},$$

a_n as in Lemma 1.5.

We must show that

$$\int_0^{2\pi} Q_{n+2}(z)(z)^{-m} d\sigma(z) = 0, \; m = 0, \ldots, n \tag{5.5}$$

well,

$$\int_0^{2\pi} \Phi_n^*(d\sigma, z)(z)^{-m} d\sigma(z)$$

$$= \overline{\int_0^{2\pi} \Phi_n(d\sigma, z) z^{-(n-m)} d\sigma(z)} = \begin{cases} 0, & m = 1, 2, \ldots, n, \\ 2\pi\gamma_n^{-2}(d\sigma), & m = 0, \end{cases}$$

by Lemma 1.5. So we have shown

$$\int_0^{2\pi} \Phi_n^*(d\sigma, z)(z)^{-m} d\sigma(z) = \begin{cases} 0, & m = 1, 2, \ldots, n, \\ 2\pi\gamma_n^{-2}(d\sigma), & m = 0. \end{cases} \tag{5.6}$$

Also for $z \in \Gamma$, $(z - z_0)(z - \hat{z}_0) = -z\hat{z}_0 |z - z_0|^2$, so

$$\int_0^{2\pi} (z - z_0)(z - \hat{z}_0)\Phi_n(z)(z)^{-m} d\sigma(z)$$

$$= -\hat{z}_0 \int_0^{2\pi} \Phi_n(z) z^{-(m-1)} d\mu(z)$$

$$= \begin{cases} 0, & m = 1, 2, \ldots, n, \\ 2\pi\gamma_n^{-2}(d\sigma)\alpha_n, & m = 0 \end{cases}$$

by Lemma 1.5. So we have,

$$\int_0^{2\pi} (z - z_0)(z - \hat{z}_0)\Phi_n(z)(z)^{-m} d\sigma(z)$$

$$= \begin{cases} 0, & m = 1, 2, \ldots, n, \\ 2\pi\gamma_n^{-2}(d\sigma)\alpha_n, & m = 0. \end{cases} \tag{5.7}$$

So (5.6) and (5.7) yield, (5.5). Next it follows from (5.5) that if we expand Q_{n+2} in

$$\left\{\Phi_m(d\sigma)\right\}_{m=0}^{n+2}$$

we must have two terms higher order than n and since Q_{n+2} is monic, we must have by definition of Q_{n+2},

(5.8) $(z-z_0)(z-\hat{z}_0)\Phi_n(z) = \Phi_{n+2}(d\sigma,z)+\beta_{n+1}\Phi_{n+1}(d\sigma,z)+\alpha_n\Phi_n^*(d\sigma,z).$

Also by definition of $d\mu = |z-z_0|^2 d\sigma$ and (5.6) with $m=0$ we have $d\mu \leq 4d\sigma$ and

(5.9) $$\gamma_n^2(d\sigma) \leq 4\gamma_n^2(d\mu).$$

Thus (5.9) implies that

(5.10) $$|\alpha_n| \leq 4\left|\hat{z}_0 a_n(d\mu)\right| \to 0 \text{ as } n \to \infty$$

by (3.6). Now divide (5.8) by $\Phi_{n+1}(d\sigma,z)$, set $z=\hat{z}_0$ and we obtain

(5.11) $$\beta_n = \frac{-\Phi_{n+2}(d\sigma,\hat{z}_0)}{\Phi_{n+1}(d\sigma,\hat{z}_0)} - \frac{\alpha_n\Phi_n^*(d\sigma,\hat{z}_0)}{\Phi_{n+1}(d\sigma,\hat{z}_0)} \to -\hat{z}_0 \text{ as } n\to\infty$$

by (5.10) and Theorem 3.8. Next divide (5.8) by $\Phi_n(d\sigma,z)$. Use (5.10), (5.11) and Theorem 3.8 to yield

(5.12) $$\lim_{n\to\infty}\left[(z-z_0)(z-\hat{z}_0)\frac{\Phi_n(z)}{\Phi_n(d\sigma,z)}\right] = z(z-\hat{z}_0),\ |z|\geq 1.$$

Now using (2.39) we have $z(z-z_0)^{-1} = D(\rho,z,ex)$. Thus we have that

(5.13) $$\lim_{n\to\infty}\frac{\Phi_n(z)}{\Phi_n(d\sigma,z)} = D(\rho,z,ex) = z(z-z_0)^{-1},$$

uniformly for $|z|\geq R>1$, $z\neq z_0$, $\rho(z)=|z-z_0|^2$ where the uniformity follows by (5.10) and Theorem 3.8. Now recalling the definition of Φ_n and $\Phi_n(d\sigma)$ we must show that

$$\lim_{n\to\infty}\frac{\gamma_n(d\mu)}{\gamma_n(d\sigma)} = 1.$$

If $|z_0| < 1$ (5.13) together with (1.20) gives the result. Next if $|z_0| = 1$, then we notice that the following is true

$$\lim_{n\to\infty} \int_\tau |\varphi_n(z)|^2 d\mu(\theta) = meas\ \tau. \tag{5.14}$$

This is because

$$\begin{aligned}
&\left|\int_\tau |\varphi_n(z)|^2 d\mu(\theta) - meas\ \tau\right| \\
&= \left|\int_\tau |\varphi_n(z)|^2 \mu'(\theta) d\theta + \int_\tau |\varphi_n(z)|^2 d\mu_s(\theta) - \int_\tau d\theta\right| \\
&\le \int_\tau \left|\left(|\varphi_n(z)|^2 \mu'(\theta) - 1\right)\right| d\theta + \int_\tau |\varphi_n(z)|^2 d\mu_s(\theta) \\
&= I_1 + I_2.
\end{aligned}$$

$I_1 \to 0$ by (3.11) and $I_2 \to 0$ by (3.45). Thus given $\epsilon > 0$ we have for $n \ge N$ by (5.13) and (5.14)

$$\begin{aligned}
1 - 3\epsilon &\le \frac{1}{2\pi} \int_{\tau\setminus\tau_\epsilon(z_0)} |\varphi_n(z)|^2 d\mu(z) \\
&= \frac{\gamma_n^2(d\mu)}{2\pi} \int_{\tau\setminus\tau_\epsilon(z_0)} |\Phi_n(z)|^2 d\mu(z) \\
&\le (1+\epsilon)\gamma_n^2(d\mu) \cdot \frac{1}{2\pi} \int_{\tau\setminus\tau_\epsilon(z_0)} |\Phi_n(d\sigma, z)|^2 d\sigma \\
&\le (1+\epsilon) \frac{\gamma_n^2(d\mu)}{\gamma_n^2(d\sigma)}
\end{aligned}$$

which implies

$$1 - 3\epsilon \le (1+\epsilon) \frac{\gamma_n^2(d\mu)}{\gamma_n^2(d\sigma)} \tag{5.15}$$

and

$$1 - 3\epsilon \le (1+\epsilon) \frac{\gamma_n^2(d\sigma)}{\gamma_n^2(d\mu)} \tag{5.16}$$

(interchanging $d\mu$ and $d\sigma$). So letting $\epsilon \to 0$ by (5.15) and (5.16) we get the result. ■

Lemma 5.4

Let $d\mu(z) = \rho(z)d\sigma(z)$ be a positive Borel measure on τ. Then for $z \in \tau$,

$$\left|\frac{\varphi_n(z)}{\varphi_n(d\sigma, z)} - \frac{\gamma_n(d\mu)}{\gamma_n(d\sigma)}\right| \le \left(\pi\rho(z)\right)^{-1} \int_0^{2\pi} \left|\frac{\rho(z) - \rho(\xi)}{z - \xi}\right| |\varphi_n(d\sigma, \xi)| |\varphi_n(\xi)| d\sigma(\xi).$$

Proof:

We first prove a Korous identity,

$$\varphi_n(z) - \frac{\gamma_n(d\mu)}{\gamma_n(d\sigma)}\varphi_n(d\sigma, z) = \frac{1}{2\pi}\int_0^{2\pi} \varphi_n(\xi)K_n(d\sigma, \xi, z)\Big(d\sigma(\xi) - \left(\rho(z)\right)^{-1}d\mu(\xi)\Big). \tag{5.17}$$

Well

$$\varphi_n(z) - \frac{\gamma_n(d\mu)}{\gamma_n(d\sigma)}\varphi_n(d\sigma, z)$$

is a polynomial of degree $n - 1$. So by Lemma 1.7,

$$\begin{aligned}\varphi_n(z) &- \frac{\gamma_n(d\mu)}{\gamma_n(d\sigma)}\varphi_n(d\sigma, z) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left[\varphi_n(\xi) - \frac{\gamma_n(d\mu)}{\gamma_n(d\sigma)}\varphi_n(d\sigma, \xi)\right] K_{n-1}(d\sigma, \xi, z)d\sigma(\xi) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \varphi_n(\xi)K_{n-1}(d\sigma, \xi, z)d\sigma(\xi)\end{aligned}$$

by orthonormality.

Also

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(\xi)K_{n-1}(d\sigma, \xi, z)d\mu(\xi) = 0$$

which implies

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(\xi)K_{n-1}(d\sigma, \xi, z)\left(\rho(z)\right)^{-1}d\mu(\xi) = 0.$$