A STUDY OF THE CONSTITUTIVE CRITERIA FOR ALGEBRAIC EXPLANATIONS AND ACTS OF EXPLAINING FROM A PROFESSIONAL DEVELOPMENT COURSE AND IN TEACHERS' PRACTICES

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Declaration

I declare that this thesis is my own unaided work. It is being submitted for the degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

(Signature of candidate)

15 July 2019

Abstract

Pedagogically, it is important to know what an explanation in itself can be understood to be. It is also useful for the mathematics teacher to have explicit guiding principles that are constitutive criteria of an explanation and acts of explaining. I set out on this study to first find theoretically the constitutive criteria of an explanation, and acts of explaining and secondly to empirically apply the constitutive criteria using data from the course, Wits Maths Connect Secondary Professional Development (WMCS PD), and teachers' practices. A conceptual separation between explanation and acts of explaining was adopted from the philosophical literature, where I used Ruben's (1992) interpretation of Aristotle's four criteria of explanation. I then mobilised the PD and mathematics education literature in order to particularise and re-describe the criteria for an algebraic expressions mathematics education focus. The separation between explanation and acts of explaining served as an organisational structure through which I then read and engaged with the literature. In this methodology, the four criteria of explanation were then operationalised by translating them from four criteria into 8 codes for explanation. These were matter (M1 and M2), form (F1 and F2), process (P1 and P2) and goal (G1 and G2). I found that the course distinguished between *what* and *how* explanations with a possibility of *why* and *when* attachments for both types of explanations. Criteria transmitted by the course for acts of explaining were examples and their representations, as well as language which I coded as Rx and Rn respectively. I found from the classroom data that there were communicative techniques which I classified as acts of explaining such as re-voicing Rv, finishing teachers' sentence Rf, gestures Rt, chorusing Rc, and evaluating Re. There was regulative talk in the classroom which I coded as Rg. The dominant and most privileged criteria of explanation in teachers' practices were the reading of constitutive elements (M1) and process (P1). For acts of explaining the dominant criteria were regulation (Rg) and re-voicing (Re). I concluded that there is merit in having the analytic separation between criteria for explanation, and acts of explaining. The implication the findings had for PD was that more attention ought to be focused on criteria for explanation and acts of explaining as well as effective ways of communicating and transmitting these to teachers. The interpretation I made about the findings was that poor learner performances are linked to the most dominant criteria of explanation transmitted by teachers.

Key words: Explanation, Acts of explaining, Criteria, Algebraic expressions

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Shortenings

BODMAS	Brackets of Division Multiplication Addition Subtraction
CCK	Common Content Knowledge
CK	Curricular Knowledge
CHC	Confucian Heritage Culture
DIPIP	Data Informed Practice Improvement Project
FB	Formally Based explanation
FET	Further Education and Training
FOIL	First Outer Inner Last
GET	General Education and Training
HOD	Head of Department
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
MB	Mathematically Based Explanation
MDI	Mathematics Discourse in Instruction
MKT	Mathematical Knowledge for Teaching
MTF	Maths Teaching Framework
MWT	Mathematical Work of Teaching
PB	Practically Based Explanation
PCK	Pedagogical Content Knowledge
PD	Professional Development
PLC	Professional Learning Communities
PPT	PowerPoint
SCK	Specialised Content Knowledge
SMK	Subject Matter Knowledge
TIMSS	Trends in International Mathematics and Science Study

TM1	Transition Maths 1
TM2	Transition Maths 2
WMCS	Wits Maths Connect Secondary

CHAPTER 1 INTRODUCTION TO THE STUDY

1.1 Introduction

Explanation and acts of explaining are ubiquitous in school mathematics classroom practice. Whether this is teacher to learner, or learner to teacher, or other learners, mathematics classrooms, constituted as they are by communication (interaction), involve ongoing explanation of mathematical objects, processes and practices.

In the professional development (PD) work of the Wits Maths Connect Secondary project (WMCS), attention is focused on the 'what' and 'how' of explanations as a way for teachers to engage with what they are teaching and to whom. In this study, I explore the explanations offered in the WMCS PD, and in teachers' practices. The underlying assumption is that school mathematics practice and mathematics teacher education practice are two different practices in which mathematics is learned. It is not possible then to see a smooth, linear and unfettered move between the two. The explanations presented in school mathematics will be different from those presented in mathematics teacher education, because teachers' take up from a PD program could be according to their needs and understanding.

I begin this chapter by briefly presenting the background literature on teacher take up as a rationale for the study. The literature that has been reviewed in this study was mainly done for the purposes of locating the study and identifying a gap in the literature. There are three concepts in the literature from which the study is developed and develops that is 'PD', 'school algebra', and 'explanation'. Firstly, the study is located within a PD programme offered by WMCS, and thus it becomes necessary to review literature on PD to determine how the notion of explanation is dealt with in PD programmes, if at all. Secondly, the content focus for the study is algebra and so a literature survey on what aspects of school algebra current research is focussed on is conducted. Lastly the pedagogic focus of the study is the idea of explanations, and so an examination of literature concerned with the theorisation of the notion of explanation is presented. The literature on explanation makes a distinction between 'explanation' and 'acts of explaining' (Ruben, 1992). I adopted this distinction for the study, and it had methodological implications for how I developed the codes and undertook the analysis of the data. The methodological tool I used to chunk and segment the data into units of analysis was derived from the literature on 'explanation', specifically the notion 'explanandum', viz. that which needs to be explained. The data collected and analysed for this study came from two sources: the WMCS PD course and teachers' videotaped lessons. I briefly elaborate in what follows on the notion of teacher take up from a PD programme.

1.2 Take up from PD

I explored the process of how teachers took up and adopted ideas of explanation and acts of explaining presented in the WMCS PD course offered to them at WITS University into their own practice in the classroom. The notion of teacher learning has been thought of using different words and expressions such as 'teacher take up', recontextualisation, appropriation, or transformation of knowledge from a PD programme into classroom practice (Graven, 2004; Xu, 2011). In one context, the teacher is a learner who has to learn, and in another context the teacher has to transform the knowledge acquired into her own practice. Adler and Reed (2002) named this process 'teacher take up'. For my study I adopted the expression 'take up', because it captures the idea that teachers selectively learn according to their needs from a PD programme. I do not provide a theoretical elaboration of the notion of take up because it is no longer central, but backgrounded in the study to form the setting for the main object of contemplation. The main focus of the study is to investigate constitutive criteria for explanation and acts of explaining algebraic expressions.[1] Even though the notion of take up has now taken a less prominent position in the study, it is still something that undergoes scrutiny because in Chapter Nine I do compare between criteria obtained from teachers' practices and criteria transmitted by the course.

In the study, I followed teachers after their participation in the WMCS professional development course to see their practices in class when providing explanations and when engaged in the acts of explaining algebraic concepts. There was a focus on the practice of teaching in the WMCS PD course, where criteria for both explanations and acts of explaining were defined in particular ways, and with specific foci, words, and language use. The

[1] There was a slight shift in the focus of the study after I realised that there was not enough in the field concerning criteria for explanation and acts of explaining to constitute an analytic tool. I then backgrounded teacher take up as the original aim of the study, so as to pursue a conceptualisation of constitutive criteria for explanation and acts of explaining as the main aim of the study. Thus, the important background to the study, had this been detected early enough, would have been constitutive criteria for explanation in algebra instead of teacher take up.

fundamental idea is that the notion of explanation and acts of explaining will not travel unproblematically to the classroom. Thus, the idea of explanation and acts of explaining in the classroom will not be the same as that presented in the WMCS. Firstly the teacher's practice in the classroom has its own logic, and this will shape what teachers come to privilege there. Secondly, the teachers' own knowledge for teaching mathematics will be another factor at work. The last factor is similar for all teachers, in the sense that all teachers have gone through the same course; but of course, they will not 'learn' in precisely the same way.

The PD programme considered is the WMCS Course, composed of university sessions and school-based sessions. I provide a description of the design of the WMCS PD programme in the next chapter. I also discuss teacher knowledge in relation to explanation and acts of explaining in the next chapter. Teacher's context of work and specific details of the PD programme are defined in the methodology chapter. In what follows, I present the literature, which looked at teacher learning as the rationale and background for having conducted this study.

1.3 Rationale and Background

There is a need to investigate take up from a PD programme. Studies about teacher learning within the context of teacher education in South Africa suggest that there is no direct transfer from PD into the classroom. As such, the important background and rationale to this study is the take up of PD. My review of the literature in this chapter focuses on studies which use Bernstein's notion of recontextualisation as a theoretical orientation to refer to the notion of take up. Literature on teacher take up from various mathematics teacher education programmes in South Africa that have emerged post 1994, is highlighted by three studies, viz. Ensor (1999), Graven (2004) and Adler & Reed (2002). All these studies were moves to elaborate and problematise notions of take up, and to challenge assumptions in studies where learning from the site of teacher education was focused solely on the teacher. In these studies, emphasis is particular to the influence of context and teacher agency in what was taken up (Graven, 2002; Adler & Reed, 2002); and context and teacher subjectivity (Ensor, 1999) in what was taken up. Of significance for my study is that in all this work, the role of knowledge is backgrounded. I elaborate on each of these studies below.

Drawing substantially from Bernstein (1996), Ensor followed seven pre-service mathematics teachers in their first year of teaching after observing them through a methodology course in their final year of training to become mathematics teachers in South Africa. She was concerned with how teachers recruit from the privileged repertoire in their methodology course to constitute their own discourse and practice. She carefully examined the privileged discourse presented in the methodology course and the teacher's recruitment, or not, from the privileged discourse. The results of her study show that teachers recruit from other reservoirs of knowledge, outside of the privileged methodology course discourse. She argues that the context within which the teachers were working was a key factor influencing the teachers' recontextualisation practices. The connection with my study is the important assumption that teachers' appropriation in relation to privileged discourse (here, algebraic explanations and acts of explaining from WMCS PD Course), depend on reservoirs of knowledge available in the wider context in which the teachers work.

Graven (2002) conducted a study of mathematics teacher learning through participation in a curriculum-based PD programme, which she designed in South Africa. The results of her study point to the importance of Subject Matter Knowledge (SMK) (Shulman, 1986). She found that the progress of teachers with weak SMK was constrained (Graven, 2002). SMK was a specific outcome in Graven's study, and not the object of study. Similarly, SMK was not the object of study in Ensor's work. I am thus offering a focus that introduces SMK, a component of teacher knowledge, to be considered in the explanation and acts of explaining of algebraic expressions.

Adler & Reed (2000),also working in South Africa, have found that take up (a notion they offer to amplify what is at stake in recontextualisation) is largely dependent upon the availability of resources; where the notion of resources is elaborated to include social, cultural and knowledge resources (Adler, 2000). The study investigated teachers' take up from an in-service course developed and implemented at the University of the Witwatersrand. The main finding was that teacher take up is mediated by resources available in a context (Adler & Reed, 2002). Once again, context, as for Ensor, is foregrounded as a factor, and while all teachers were teaching mathematics, the mathematics per se was not a specific focus of the study, nor was the explanation and acts of explaining algebra.

These constitute the major extant studies in South Africa in mathematics education with a focus on teacher learning of a privileged discourse of mathematics (for) teaching into the classroom context. Internationally, studies on appropriation of PD or of curriculum documents have been conducted. The two I discuss here both foreground teachers' knowledge of mathematics for teaching as a key factor, and have orientation to take up as recontextualisation.

Xu (2011) probed teachers' understanding of the new reform curriculum (official discourse) in China and carefully examined twelve teachers' implementation practices. These teachers had not been through any course or substantial PD work concerning the new curriculum, except for a 2-3 day training offered by the department of education in China. She explored the role played by different discourses in the take up process. She mentions that other discourses, that is: 1) official discourses at the level of examinations; 2) conventional discourse which is the mathematics content; and 3) local discourse – this includes the organisational structure of the school, collegiality and departmental practice, parents, teachers and learners, all influence how the take up process occurs. Xu points out after studying the interviews and classroom observations she conducted with teachers, that the most influential and hence dominant discourse for the teachers in China was the conventional discourse, the mathematics. She then concludes that strong content knowledge structured teachers' take up practices of the reformed curriculum. The context is different to that of South Africa, where strong mathematical content in most teachers is lacking, due to previous training, which was shaped by apartheid policy of segregated and unequal educational policies.

Huillet, Adler and Berger (2011) draw from Huillet's (Huillet, 2007) study that focuses on the evolution of teachers' knowledge of the limit concept, through their participation in a research project in Mozambique. They show by way of literature review that in previous PD studies, teacher knowledge is assumed, rather than interrogated, even for studies where specific topics were in focus (Huillet, Adler, & Berger, 2011). In Huillet's study, mathematical issues were overt. Huillet et al. (2011) argue the importance of foregrounding content in PD, and note that teachers were uncomfortable when their mathematical knowledge was challenged.

Foregrounding the explanation and acts of explaining algebraic notions as I put forward in this study is one way of bringing mathematical knowledge into focus: the teacher needs to know the explanation of a mathematical notion that she is explaining to her learners, taking context and learner needs into account. Put another way, this study makes a contribution to the literature on PD in mathematics by focusing on teachers' knowledge of algebraic explanations for teaching, and how this translates into acts of explaining in the classroom. By focusing on explicit criteria for my investigation of teachers' explanations and acts of explaining, as will be shown later, I intend to contribute to the above research in the mathematics education literature.

The work of Wilson and Berne (1999) is now 20 years old, where more has been done in PD, however, they were the first in mathematics education to talk about the importance of content-focused PD. The issue has remained, where effective PD must be content-focused. However, I am studying the course and the teachers not in themselves, but in relation to building the theoretical clarification of explanation.

In their literature review of PD, Wilson and Berne (1999) were concerned with whether and how teachers acquire professional knowledge. They segmented professional knowledge into three categories according to studies that provided opportunity to discuss: 1) SMK; 2) learners and learning; and 3) teaching (Wilson & Berne, 1999). They show from the studies they reviewed that teacher learning was by chance and a 'patchwork of opportunities':

> Opportunities for teacher learning (while they may be more authentic) are happenstance, random, and unpredictable. In sum, teacher learning has traditionally been a patchwork of opportunities—formal and informal, mandatory and voluntary, serendipitous and planned—stitched together into a fragmented and incoherent "curriculum"... of what exactly it is that teachers learn and by what mechanisms that learning takes place? What knowledge do teachers acquire across these experiences? How does that knowledge improve their practice? These questions are left unanswered (Wilson & Berne, 1999, pg. 174).

Wilson and Berne argue that in many PD studies the object of study is unclear, which is why the kinds of knowledge and skills acquired by teachers are not known. They also stress that in order to understand teaching; the actual practice of teaching must itself be studied; and note that to be successful at studying teaching, teachers need more knowledge of the mathematics so that they are able to talk about learners and learning of the content.

The studies discussed above together suggest the importance of teachers' knowledge in offering 'good' explanations and acts of explaining. Acts of explaining offer a way of talking about and studying the enactment of teachers' knowledge into pedagogic practice.

What is emerging from all these studies is that a specific content area and a deliberate pedagogic focus, such as that of explanation, as well as the acts of explaining algebra, remain open for further research. The current study differs from these in that it is focusing on a specific programme (WMCS PD), which foregrounds mathematical content, specifically mathematical explanations, in mathematics in general, and in algebra specifically. Researchers have agreed about the importance and need to improve teachers' knowledge (Even, 1993; Rowland, 2012). What we do not yet know and still need to determine is the notion of explanation and acts of explaining as components of teacher knowledge. Teachers who participate in the WMCS PD course come with different levels of qualifications and knowledge. Inequities from apartheid education perpetuate in South Africa, and the quality of teachers' mathematical explanation for teaching thus has traces in the apartheid past. By the end of the course teachers have gone through the same programme and they work in different and similar environments. The reported study, therefore, adds to the research on teachers' practices in mathematics education in South Africa by examining that which has become privileged as algebraic explanation and acts of explaining, first in the PD, and then in teachers' work. The study also adds to the existing descriptions of what constitutes an explanation in algebra teaching, and what elements of explanation are privileged as well as separating out and illuminating acts of explaining.

The study is located in the wider field of PD elaborated on in Chapter Two on the one hand, and research on teaching and learning of algebra and explanation on the other; which I now introduce here as part of the study rationale.

1.3.1 Learning and teaching of algebra

I have noted already that in the take-up of PD studies conducted in South Africa, there is a lack of a specific content focus. A content focus is important for a study that seeks to theorise the notion of explanation and acts of explaining. I have chosen to focus on algebra for a content focus because algebra constitutes the unifying thread of all mathematics. This suggests that skills learnt in algebra are necessary for advancement into other parts of mathematics. The history of algebra as will be elaborated in Chapter Four teaches us that algebra is a way of thinking, and that algebraic thinking transcends any mathematical notation and expression. The historical development of algebraic notions illustrates that algebraic expressions have progressed from pictographic symbols to rhetoric, to syncopated expressions, and on to the modern symbolic expressions we now use. The debate is that different expressions afford different aspects of the explanation and different approaches to the acts of explaining. In

Chapter Four I engage with this and other important debates in the field concerning the teaching and learning of school algebra such as the relationship between concepts and procedures (Kieran, 2013). Thus, algebra projected itself as an important part of mathematics to focus on and the WMCS PD course also allocated a considerable amount of attention to algebra.

1.3.2 Explanation and acts of explaining

As noted before, and further defined and clarified in Chapter 3, this study adopts a distinction provided in the philosophical literature between 'explanation' and 'acts of explaining' (Ruben, 1992). This is a deliberate move to mark out the absence of this distinction in the current literature in mathematics education, and thus a potential contribution of this study. From an epistemological perspective, Ruben (1992) provides a theory of explanation by drawing on various theorists; I specifically adopt the discussion on Aristotle because Ruben argues that Aristotle provides a general theory of what constitutes an explanation. I operationalise Ruben's theory by looking at school algebra. Both 'explanation' and 'acts of explaining' have epistemic entailments. Both are about knowledge, in pedagogical terms. Explanation is about 'what' knowledge needs to be transmitted, while explaining is about 'how' the knowledge needs to be unpacked for a particular audience. Criteria for what constitutes an explanation are obtained from Aristotle as interpreted by Ruben (1992). Aristotle argues that for one to have a fuller understanding of the explanandum (that which needs to be explained) one needs to know about its material composition, and so its constitutive elements; the form or structure the elements take; the motion originator, in this case, the procedural activity; and goal or purpose for which the entire activity is performed. The distinction of explanation from acts of explaining methodologically informed the ways in which I took on the analysis and development of codes for coding data. Acts of explaining are both subject matter specific, and contain broad 'pedagogical moves'. These were the features I considered in the literature, which structured the study. The research question addresses the problem stated below.

1.4 Problem statement

In South Africa it is evident that inequalities from apartheid continue to reproduce themselves in the education system for various reasons, and that the vast majority of learners continue to underperform in National Senior Examinations and international comparative tests (DoE, 2017). The Trends in International Mathematics and Science Study (TIMSS) is ordinarily tested in Grade Eight, however, in South Africa, for the years 2011 and 2015 it was tested at Grade Nine level, because in previous studies it was observed that most learners did not provide responses (Reddy, Visser, Winnaar, Arends, Juan, Prinsloo, Isdale, 2016). South Africa still performed poorly compared to other participating countries (Reddy et al., 2016).

Morrow (1994) has problematised the fixation for most educational activists that they have mainly focused on physical access by examining school infrastructure and facilities, without paying attention to epistemological access once learners are in the school building. Scholars have argued that access to powerful, theoretical, scientific, forms of knowledge is one way of disrupting the inequality reproducing nature of schooling (Young, 2011b). In the words of my study, access to powerful forms of knowledge is made possible and available through the explanation and acts of explaining. In acknowledging this, focusing attention on what is offered as an explanation, and the actions that are taken to provide this explanation in PD and in teachers' classrooms, becomes important.

The objectives of the reported study were to determine what is constituted as explanation and acts of explaining in the WMCS PD course and in participating teachers' classroom practices. Below is the key research question accompanied by critical research questions:

1.4.1 Research Question:

How do the WMCS PD course and participating teachers present criteria for explanation and acts of explaining algebraic expressions in Grade Nine?

1.4.1.1 Critical research questions:

1. What are the constitutive criteria of explanation and acts of explaining?
2. What is legitimated as an explanation, and acts of explaining in algebra in the WMCS course?

2.1 What constitutive criteria of explanation and acts of explaining are privileged?

2.2 What messages about 'quality' of both explanation and acts of explaining are transmitted?

3. What is legitimated as explanation and acts of explaining in algebra in the classroom?

3.1 What constitutive criteria of explanation and acts of explaining are privileged in teachers' practices?

3.2 In what ways are teachers privileging practices similar to and or different from what was privileged as explanation and acts of explaining in the WMCS course?

4. What and how do these empirical findings and their inter-relation further illuminate the criteria for explanation and acts of explaining?

1.5 The usefulness of the study

Fulfilling the purpose of the study was important to me and the wider field of education, particularly mathematics education. The purpose of the study was important specifically to researchers who are also interested in the problem and are alongside seeking solutions to the same problem explored in this study.

The study also adds to the knowledge, as noted before, the distinction between the explanation and acts of explaining is missing in the field. There is a blurring of boundaries between explanation and acts of explaining criteria as presented in the field. The study will be valuable to teachers who are seeking clear criteria for an explanation distinct from criteria for acts of explaining to utilise in the planning and implementation of their lessons.

Furthermore, in teacher education, the study will be especially meaningful to undergraduate courses for preservice teachers that prepare teachers by developing their mathematical knowledge for teaching. Explanation and acts of explaining are important constituents of mathematics knowledge for teaching.

The learner dropout rates and underachievement in mathematics are partly owing to learners not understanding the content taught in schools. The separated constitutive criteria of explanation and acts of explaining might intervene in bringing about better understanding and thus lessen learner frustration. In this way, the criteria have the potential to bring into focus for the learner the subject matter in more meaningful ways if taken into consideration in the planning and delivery of the lesson.

1.6 Scope of the study

This study focuses on the notion of explanation and acts of explaining in the course and in the teachers' practices in their own right, where the impact of the PD course on teacher learning has received focus elsewhere. As such, in this study, the aim was not to evaluate the course but to see what it presented as explanation and acts of explaining and how participating teachers also presented explanation and acts of explaining. Reporting and studying a PD programme that focuses on explanation, and which equally has a theory of explanation, was important to highlight. I compared what the course presented as criteria for explanation and acts of explaining with that presented by teachers, while the effects and impact of the PD course was not necessarily the focus of the study. I conduct the comparison so as to develop a language for explanation and acts of explaining criteria.

The original empirical scope for the study was wider than that reported here. I originally collected data from five high school teachers. I specifically videotaped three consecutive lessons focused on a topic in algebra and conducted a follow up reflective interview with each teacher. As the study progressed, it became clearer that a strong focus and amplification of the notion of explanation itself was needed, since the lack of explicit criteria for explanation in the literature was identified. This led to a reduction in the original empirical space for the study, which caused me to focus on two teachers' teaching the same grade and topic. I return to reflect on strengths and limitations of this choice for the study in the concluding chapter.

The study was conducted for the fulfilment of a doctoral degree in mathematics education. Due to the boundaries set up in the requirements of the degree, a longitudinal large-scale study was not possible, where the nature of the problem I was interested in influenced the design of the study.

1.7 Thesis outline

Following the introduction of the study in this chapter, in Chapter Two, I offer a presentation of the literature concerned with PD, as PD is the empirical space in which the study is located.

Chapter Three is devoted to the theorisation of the notion of explanation and acts of explaining. Algebra is the content focus of the study and so a review of literature pertaining to algebra is discussed in Chapter Four. Chapter Five deals with the methodological aspects of the study from data collection to coding and analysis. In Chapter Six I report on the WMCS PD course and what it presented as criteria for explanation and acts of explaining. In Chapter Seven I report on the analysis of the teachers' practice and what they presented as explanation and acts of explaining algebraic expressions. Chapter Eight presents a discussion of the findings from Chapter Seven to answer the question of what was privileged as explanation and acts of explaining in teachers' practices. In Chapter Nine, I compare what was privileged as explanation and acts of explaining in the WMCS PD course and in teachers' practices. To end, I then discuss the conclusions and recommendations of the study in Chapter Ten.

1.8 Conclusion

In this chapter I outlined the research problem and how it is important to understand it within the wider field of PD, teaching and learning of algebra and research on explanations in education and in philosophy. In the presentation of the concepts developed in the study, I have discussed the contributions made by the study to teacher learning literature, and explanations literature. In the next chapter, I provide a detailed discussion of PD literature, because it is within this that the study is empirically situated, and so it is important to see what the field says is PD and how PD should be structured.

CHAPTER 2 PROFESSIONAL DEVELOPMENT AND TEACHER KNOWLEDGE

2.1 Introduction

The literature of mathematics teacher education forms the context in which the study is developed and develops, specifically PD, that has a content focus with deliberate attention to explanation. In this chapter, I discuss the nature of PD, as well as the relevant studies that have been done both generally and specifically in mathematics education. This information is obtained from literature surveys and reviews covering different periods of times conducted by different researchers. After discussing literature concerning PD, I then provide a brief description of the WMCS PD project. Within mathematics education, I am interested in the literature concerned with teacher knowledge. Ball and colleagues in their refinement of Shulman's domains of teacher knowledge, have generated subject-specific categories of teacher knowledge. They derived the categories from observing and studying the actual teaching practice in mathematics classrooms. Of particular interest in my study are the actual teacher knowledge categories from Shulman (1986), and the observed actions which Ball, Thames and Phelps (2008) call 'the works of teaching'. Apart from the list of 16 'works of teaching', in their 2005 publication, Ball and colleagues enumerate three 'works of teaching' which can be broadly described as: 1) working with learner error; 2) representing the content using pictures and diagrams; and 3) explaining concepts and terminology. Working with learner error, representations, and explanations have all been researched within the context of teacher development in mathematics education, however, I present a different approach and conceptualisation of the notion of explanation by separating it from acts of explaining. The purpose of this chapter is to highlight this approach and conceptualisation of the notion of explanation and acts of explaining within teacher knowledge domains using Shulman's SMK and PCK categories of teacher knowledge and categories from Ball's 'work of teaching'.

Before embarking on the literature concerned with teacher knowledge, I first provide a general discussion of what is meant by PD. This discussion highlights that the literature has thought of PD as structured by mainly four features. I now turn to this general discussion.

2.2 General

Research focused on PD tends to be concerned with answering questions of whether and how teachers learn, and thus finding models of high quality effective PD programmes on teacher learning. Furthermore, concern is also given to investigating how teacher learning impacts learner achievement. The discussions focused on models of high quality PD programmes provide an enumeration of features/characteristics of a PD programme. This study reports on what the WMCS PD programme presented, elaborated and exemplified as criteria for explanation, as well as acts of explaining to teachers, where the features are important. However, the initial focus in my study was on the content of what was made available and possible to learn as explanation and acts of explaining in the PD programme. As already noted in the previous chapter, the focus on teacher take up has now been backgrounded in order to foreground the constitutive criteria of explanation and acts of explaining. Nevertheless, PD is the empirical space in which I conducted the study of constitutive criteria for explanation and acts of explaining. In what follows I discuss the four features of a PD programme.

There is general agreement among researchers that features of a PD programme include facilitator; the PD programme and what it makes available to learn; the context; and the teacher as a learner (Adler, Ball, Krainer, Lin, & Novotna, 2005; Avalos, 2011; Borko, 2004; Borko, Liston, & Whitcomb, 2007; Desimone, 2009; Goldsmith, Doerr, & Lewis, 2014; Opfer & Pedder, 2011; Sztajn, Borko, & Smith, 2017; Webster-Wright, 2009; Wilson & Berne, 1999). My research interest is to study what the WMCS PD presents as explanation and acts of explaining to teachers. Most of the reviews of PD research reported here are not in mathematics education. Reviews concerned with mathematics education are provided by Adler et al. (2005), Goldsmith et al. (2014) and Sztajn et al. (2017). The rest are concerned to know about designs of PD programmes and its features, the subject specific reviews that are included in this general section also highlight the same features for a PD programme, which are: facilitator, teachers as learners, PD programme and context. I proceed into a discussion of each in what follows.

2.2.1 Facilitator

The facilitator has been counted as one of the most important features of a PD programme in Borko (2004), while in other reviews, the facilitator is not counted as one of the factors that influence teacher learning process (Opfer and Padder, 2011; Avalos, 2011). In other reviews, the role of the facilitator has simply been stated as that of managing the challenge of designing the PD programme, teaching it, as well as researching the PD programme (Adler et al., 2005). The nature of facilitation and preparation of facilitators is seldom reported or studied in the literature. To place emphasis on this, Sztajn et al. (2017) report that the role of facilitator or PD leader is understudied. They argue that it is important to study the facilitation practices of PD leaders so as to know what it takes to lead PD, and what knowledge is required to enact the practices. Cereseto (2015) studied the role of the facilitator in the professional learning community (PLC). She found that facilitators need to have sound SMK, and at a personal level, that facilitator need to have an encouraging and supportive temperament (Cereseto, 2015). There is also a lack of literature reporting on what it takes to prepare facilitators. In this regard, Cereseto (2015) argues that facilitors are likely to be more effective when training in facilitation is provided. In the current study, I interviewed the WMCS PD course presenter to elaborate and clarify on what was presented as explanation and acts of explaining in the WMCS PD course documents.

2.2.2 Teacher as learner in PD

The teacher is seen as a learner and a professional in most reviews concerned with PD. As part of what she calls the core features of a PD programme, Desimone (2009) describes the kinds of learning processes that should be made available in a PD programme, and those are collective participation and active learning. When it comes to the notion of development, Webster-Wright (2009) argue that teacher learning is dependent on the interaction between the teacher, context, and what is learnt. Teacher knowledge has been thought of as change in beliefs and identity, in studies concerned with teacher development, as well as teacher knowledge (Graven, 2004). The focus in Desimone (2009) and Webster-Wright (2009) is to enumerate features of a PD programme, and so the notion of teacher knowledge is not elaborated. I pick up on the discussion of teacher knowledge shortly in this chapter, because explanations and acts of explaining are tantamount to SMK and PCK teacher knowledge categories, respectively.

2.2.3 PD programme

The programme and what it makes available to learn has been the subject of many PD studies and is enumerated as one of the features of a PD programme. Studies in this regard have examined the duration of the programme, where the argument is that teacher learning occurs over time, where short PD courses are unlikely to be effective in the learning process (Lydon & King, 2009). Studies concerned with effectiveness of PD programme have been dominated by what is called process-product studies. There have been suggestions by researchers for a content focus in PD programmes and coherence in the presentation of the content (Desimone, 2009). Researchers echo that in a PD programme, there ought to be opportunities for teachers to talk about and work on content while at the same time noting that content on its own does not constitute the full substance of teaching. The literature recommends that content should be presented in interrelation with teaching, and how learners learn (Wilson and Berne, 1999). In this study, the WMCS PD course that teachers attended had sessions focused on content and sessions focused on the teaching practice, particularly the notion of explanation and acts of explaining.

2.2.4 Context of PD and teachers' work

In the reviews about PD, context is noted as a feature of PD. Context is used in two ways; to refer to the context where the PD programme is delivered, mostly the university, and the context of work for the teacher which is the school. In the first instance, where context refers to place where PD is offered, context is viewed as a factor that influences the learning that will occur. Borko (2004) and Webster-Wright (2009) are using context as a feature of PD in the first sense to refer to context where PD is offered. The WMCS PD, as the name suggests, was conducted in the University of the Witwatersrand. In the second sense, context is seen as a place that either enables or impedes the implementation process of what was learnt in PD. Specifically, researchers have highlighted the organisation of the school, its working conditions, its culture, and the resources both material and human (Adler, 2000) as factors that regulate how the teacher will learn and take-up from the PD. In my study, the context of teachers' work is the school. The school is the institutional setting, which has its policies of teaching and its own culture of administration. At the level of human resource, as conceptualised by Adler (2000), the school context can have a lack or a surplus of mathematical leadership in terms of more experienced and knowledgeable teachers and HOD. There is also the political aspect of the school context, where the district has certain regulations in the running of the affairs of the school. There is a recommendation for PD programmes to take into account the local context of teachers' work as it is a factor that influences implementation of what is learnt in PD. Specifically, PD programmes are required to seek strategies of how to support teachers to implement what they have learnt from the PD in their contexts of work, where school-based PD programmes are encouraged (Sztajn et al, 2017). The WMCS PD programme started off as school-based, and over time included the university based courses to support the ongoing school-based PD work. I collected video data by following teachers into their classrooms.

The four are listed as features of a PD programme, while at the same time, some are listed as factors that influence the learning process. Here the aim is to study constitutive criteria for explanation and acts of explaining in the WMCS PD project. The mathematical content and teacher knowledge are not sufficiently elaborated in the general literature, because the focus is on features that constitute a PD programme. In Chapter Six and Nine, I offer an elaboration of the mathematical content used in the WMCS PD to teach the notion of explanation and acts of explaining. I now move on to a discussion of teacher knowledge as the key focus of the chapter, as well as the object of PD.

2.3 Teacher knowledge

The question of what is it that teachers in general and specifically mathematics teachers need to know, and know how to do, to bring about quality instruction has been with the field for a long time. Shulman (1986) presented what he called *the missing paradigm* in literature concerned with teacher knowledge and teacher learning 30+ years ago. Shulman (1986) narrates that the nature of examinations that were used in the United States to measure teacher knowledge focused on content, such that a 13-year-old who took the elementary teachers' examination was mistakenly certified for practice. Shulman addresses this misrepresentation of teacher knowledge, by referring to the medieval university, and showing how pedagogy and content were one body of knowledge. Shulman shows that available literature concerned with teacher education at that time was predominantly focused on general pedagogical matters, such as classroom management, and that there was an absence of studies focusing on content of the lesson and explanations offered. He then suggested a way of thinking about teachers' knowledge as content knowledge with three categories namely: subject matter knowledge (SMK); pedagogical content knowledge (PCK); and curricular knowledge (CK). I elaborate on SMK and PCK below, and show how explanation and acts of explaining respectively relate to these knowledge domains.

2.3.1 Subject matter knowledge (SMK)

Shulman, when referring to SMK, is referring to the organisation of knowledge in the teachers' mind that it involves understanding the structures of the subject matter. He explains what he means by structures of the subject matter by drawing on Schwab (cited in Shulman, 1986), who provides a definition that is substantive and syntactic structures. By substantive structures, he is referring to the organisation of the basic facts and principles of the discipline. By syntactic structures he is referring to "… a set of rules for determining what is legitimate to say in a disciplinary domain and what breaks the rules" (p.9).

There is academic content to be taught to learners and so teachers' expertise in SMK is of utmost importance. SMK refers to the knowledge the teacher possesses. With reference to the amount, there is a minimum requirement that Shulman argues is expected of teachers. Shulman argues that teachers' content understanding should be "at least equal to that of their colleagues, the mere subject major" (p.9).

What can be surmised is that SMK is the knowledge required to produce quality explanations that are coherently organised according to the principles of the discipline, which take into consideration the rules of what to say and what not to say.

2.3.2 Pedagogical Content Knowledge (PCK)

With regards to PCK, Shulman says: "I still speak of content knowledge here, but of the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (p.9). This is a teacher's transformation of content knowledge, viz. the mathematical concepts, into pedagogical actions. These are ways of talking, representing ideas, and demonstrating them so that the learner can come to know mathematics. In the language of the WMCS PD, as will be shown later in Chapter Six, these are ways of making the mathematics available to learners. PCK involves being able to transform one's understanding of a particular concept into instruction that learners can comprehend. Included in PCK is the understanding of the difficulties learners encounter in the form of misconceptions and errors. In the language of the current study, PCK, borrowing from Ruben (1992), is understood as acts of explaining.

The relationship between SMK and PCK is that of interdependence. PCK relies on SMK and PCK is compromised when SMK is weak. Shulman points out that there are pedagogical prices that are paid when SMK is weak. This is the case because PCK is a combination of content with pedagogy. PCK is the transformation of one's own understanding of mathematical concepts into pedagogical actions. The talk, the representation of ideas, and demonstrations are actions that a teacher performs in order for learners to learn the subject matter, SMK. When Shulman's first two categories of content knowledge (SMK and PCK) are appropriated into the language of my study, there is a relation between SMK and the explanation, more specifically the knowledge the teacher is expected to have about a specific mathematical problem or concept; while PCK relates better with acts of explaining, because it is the methodology of enacting the explanation. Of course, not all aspects of acts of explaining can be related to PCK. Some relate to general pedagogic knowledge.

Ever since this seminal work that Shulman presented, there has been volumes of studies in mathematics education focused on content. However, the notion of teacher knowledge as the source of mathematically sound explanations has not received a lot of attention except for the work of Gael Leinhardt,[2] which Shulman acknowledges. Shulman argues that explanations are generated from the content expertise of the teacher, and so the explanation is what a teacher

[2] I discuss Leinhardt's work in Chapter Three, where the focus is to find criteria for explanation and acts of explaining in mathematics. The focus in this chapter is on teacher knowledge, specifically PD.

knows about mathematics in general, and specific knowledge of school algebra. While questions about the content of the lesson might have been addressed in the literature, questions about the nature of explanations offered, the constitutive criteria of an explanation and acts of explaining remain unanswered several decades later, ever since Shulman highlighted the missing paradigm.

Researchers in mathematics education have done some work specifically focused on Shulman's categories of content knowledge. In the following discussion, I report on how Ball and colleagues have refined Shulman's categories for mathematics education.

2.4 Mathematical knowledge for teaching

The concern in this section is to continue the discussion on knowledge, particularly mathematics teachers' knowledge, because the explanation is knowledge-centred and established upon knowledge. The phrase mathematical knowledge for teaching (MKT) is used by Ball and colleagues to contextualise Shulman's knowledge domains for mathematics. They redefine Shulman's categories of teacher knowledge by looking at what they call 'works of teaching'. They first start with three then extend to 16 'works of teaching' in a later publication in 2008. Before I elaborate on the works of teaching I first discuss MKT.

MKT has been thought of as the kind of knowledge required to do the work of teaching (Ball, Hill, & Bass, 2005). The authors define MKT in this way:

> From this we derived a practice-based portrait of what we call "mathematical knowledge for teaching"—a kind of professional knowledge of mathematics different from that demanded by other mathematically intensive occupations, such as engineering, physics, accounting, or carpentry (Ball, Hill and Bass, 2005, p.17).

To categorise the professional knowledge they refer to, they redefine the categories of knowledge from Shulman. They provide subject (mathematics) specific sub-categories for PCK and SMK shown in Figure 1 below. PCK is segmented into three categories, viz.: knowledge of content and students (KCS); knowledge of content and teaching (KCT); and knowledge of content and curriculum. Similarly, SMK is also segmented into three categories viz. common content knowledge (CCK); horizon content knowledge; and specialised content knowledge (SCK). The actual activities they embarked on to generate the sub-categories of mathematics knowledge for teaching are observing classrooms, and documenting what actually occurs in class. They included the activities which occur in class in their works of teaching. I move on into a discussion of the works of teaching from which the knowledge categories shown in Figure 2:1 were derived.

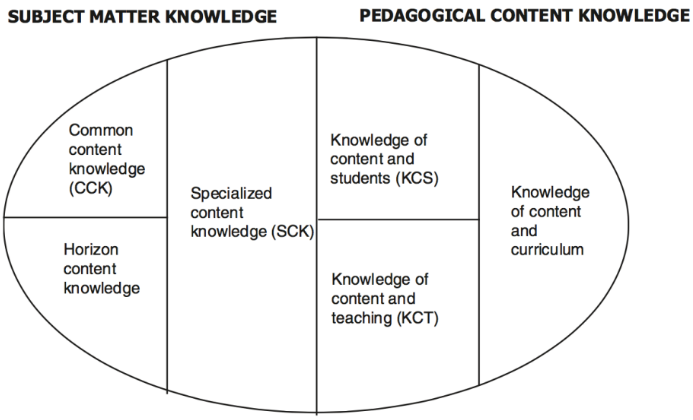


Figure 2:1 – Categories of MKT (Ball et al., 2008, p.403)

2.4.1 Works of teaching

The categories listed under the ‘works of teaching’ were obtained by observing what mathematics teachers do in class as they teach and outside of class. Ball, Thames and Phelps (2008) define the word teaching and at the same time enumerate the works of teaching:

> By “teaching,” we mean everything that teachers must do to support the learning of their students. Clearly we mean the interactive work of teaching lessons in classrooms and all the tasks that arise in the course of that work. But we also mean planning for those lessons, evaluating students’ work, writing and grading assessments, explaining the classwork to parents, making and managing homework, attending to concerns for equity, and dealing with the building principal who has strong views about the math curriculum (Ball, Thames & Phelps, 2008, p. 396).

The definition of teaching for Ball, Thames and Phelps (2008) is not limited to what goes on in class, but includes out of class activities such as planning and talking to parents (Ball, Thames, & Phelps, 2008). The description of what they mean by works of teaching in a subsequent publication highlights this fact by stating that activities included in the work of teaching are found inside and outside of the classroom (Ball & Forzani, 2009).

> By "work of teaching," we mean the core tasks that teachers must execute to help pupils learn. These include activities carried on both inside and beyond the classroom, such as leading a discussion of solutions to a mathematics problem, probing students' answers, reviewing material for a science test, listening to and assessing students' oral reading, explaining an interpretation of a poem, talking with parents, evaluating students' papers, planning, and creating and maintaining an orderly and supportive environment for learning. The work of teaching includes broad cultural competence and relational sensitivity, communication skills, and the combination of rigor and imagination fundamental to effective practice (Ball and Forzani, 2009, p. 497).

The descriptions given for both teaching and works of teaching is in essence the same. However, Ball, Thames and Phelps (2008) minimise confusion by providing a figure with a list of what they refer to as mathematical tasks of teaching – they seem to be using the words 'task' and 'work' interchangeably. The numbering on the left hand side of Figure 2:2 below is my own insertion for organisational purposes later in Table 2-1.

1	Presenting mathematical ideas
2	Responding to students' "why" questions
3	Finding an example to make a specific mathematical point
4	Recognising what is involved in using a particular representation
5	Linking representations to underlying ideas and to other representations
6	Connecting a topic being taught to topics from prior or future years
7	Explaining mathematical goals and purposes to parents
8	Appraising and adapting the mathematical content of textbooks
9	Modifying tasks to be either easier or harder
10	Evaluating the plausibility of students' claims (often quickly)
11	Giving or evaluating mathematical explanations
12	Choosing and developing useable definitions
13	Using mathematical notation and language and critiquing its use
14	Asking productive mathematical questions
15	Selecting representations for particular purposes
16	Inspecting equivalencies

Figure 2:2 – Mathematical tasks of teaching (Ball, Thames and Phelps, 2008, p. 400)

The mathematical tasks of teaching list has been criticised for being an unstructured list in the sense that there is no theoretical organising principle (Mosvold, 2016). Mosvold then suggests and uses a theoretical frame from a discourse perspective to organise the list. He concludes that most items in the list are not necessarily observable discourse practices and those include planning, choosing examples, and their representation. As already noted in Ball, Thames and Phelps's (2008) definition of teaching the list is comprised of activities which occur inside and outside of the classroom. Mosvold (2016) drawing from Sfard's (2008) discursive theory of thinking as commognition, and with the help of Venkat and Adler (2012) as well as Adler and Ronda's (2015) Mathematical Discourse in Instruction (MDI) framework, reduces the list of 16 into three 'discourse practices', that is, examples, explanations, and responding to learners. In the classroom discourse I am particularly interested in the notion of explanations for algebraic expressions, however, for the explanation to come into existence, it has to be exemplified and represented. Furthermore the explanation is cooperatively constituted in the interaction between learners and teacher. Thus from the MDI I work with examples and explanations.

Selling, Garcia and Ball (2016) agreed to the fact that the list needed structure: “As the list of teachers’ mathematical work grew, we needed to create an organizational [sic] structure that would make the list more orderly, systematic, and useful for item writers” (p. 40). Selling et al. (2016) argue that the need to structure the list came as their work continued to add and expand the list. Their way of structuring the list was to develop what they call mathematical work of teaching framework (MWT). They articulate that the framework provides guidelines for item writers who are interested in assessing teachers’ SCK. The individual items in the list of 16 works of teaching are now called *mathematical instructional object*s, for example explanations, definitions, representations, and notation (Selling, Garcia, & Ball, 2016). In the quote below, they specify the ways in which they use the notion of object, and how it differs from the traditional way in which it has been used in the field. The notion of explanation is seen as one such mathematical instructional object as shown in the quote below:

> ...teachers regularly give, use, and encounter mathematical explanations; in this case, the mathematical explanation is the “object”. Teachers give mathematical explanations themselves, but they also make sense of student explanations, compare different explanations in textbooks, determine if a student’s explanation is valid, or critique written explanations for the purpose of improving them. We recognize that we define “mathematical objects” here in a way that is different from the way “objects” is typically used in mathematics to refer to objects such as numbers, functions, and polygons (p.40).

Before the explanation can become the mathematical instructional object which teachers provide, compare and critique, its constitutive elements are necessary to know first. My own interest, on the other hand, in this study, is not to develop tasks per se, but to obtain constitutive criteria for explanation and acts of explaining that I can use in the analysis of the data.

I now go back to the classification of the three broad categories once used by Ball and her colleagues in earlier publications in 2005. I go back to these because I find them more useful in terms of limiting the number of works of teaching to focus on and also because they include the category explanation. My main aim in using the three broad categories is to structure the discussion of PD research conducted in mathematics education. The three broad categories in what they called works of teaching which were: 1) using pictures and diagrams as representations for mathematical ideas, concepts and procedures; 2) providing mathematically careful explanations is one of the tasks of teaching; 3) analyse learner contributions (Ball, Hill & Bass, 2005; and Hill, Rowan & Ball, 2005). In the PD research focused on mathematics education it is clear that there is a presence of studies focused on teachers learning about the use of representations, and teachers' learning about learner errors. There is also research focused on teachers' explanation of leaners' errors. However, there is far less research focused on elaborating teachers' learning about explanations and acts of explaining even though it is identified in the list shown in Figure 2:2. I elaborate on this in the discussion below by referring to four texts focused on mathematics education PD research.

2.4.2 Research focused on the three broad works/tasks of teaching in mathematics education PD

There are four papers I discuss below in relation to the work of teaching. I discuss these four papers because all of them examine teacher knowledge of varius concepts, such as multiple representations, learner error, and explanations. The first is Dreher and Kuntze (2015), who examine teachers' understanding of multiple representations. The second and third are Brodie (2014) and Shalem, Sapire, & Sorto (2014), who examine how teachers' knowledge is increased by analysing learner errors. They developed explanation criteria to judge teachers' analysis of learner error. The last is a study focused on explaining a "lesson", by Peng (2007). The representation carries the knowledge structure for the particular mathematical object in focus, also the act of selecting the representation is a pedagogic decision. Thus, I argue in this study that representations is an element of explanation and acts of explaining. Analysing learner error is an act of explaining. Both explanation and acts of explaining are components of teacher knowledge, SMK and PCK, respectively.

In Germany, Dreher and Kuntze (2015) compared expert (in-service) and novice (pre-service) teachers' understanding or 'noticing' of the role that multiple representations play in mathematics instruction. The interest in their study was to explore the extent to which teachers knew about the role of multiple representations. That multiple representations have both negative and positive consequences, which can aid learning by enhancing understanding and/ or impede learning by shifting focus. They gave teachers a questionnaire and a vignette based questionnaire. They found that both in-service and pre-service teachers did not fully understand/notice the role played by multiple representations and how to use them appropriately (Dreher & Kuntze, 2015). For this study, representations are a way of making visible the mathematics requiring explanation. Mathematics and its objects are abstract and intangible, and so need to be represented. Since representations are not the thing in itself, the combination and linking of multiple representations constitute the actions which a mathematics teacher performs to create an explanation.

In South Africa, Brodie (2014) reports on a PD project called Data Informed Practice Improvement Project (DIPIP) that supported teachers to learn and understand learners' errors as 'reasoned' and 'reasonable'; and ways to engage them in the classroom. Professional Learning Communities (PLC) is the model of PD that was used where a group of teachers with a team leader would work together. In the PLC, teachers learnt to identify and engage the reasoning behind the error. Brodie identifies three important shifts that teachers made: i) from identifying to interpreting errors; ii) from interpreting to engaging; and iii) from focusing on learner errors to focusing on their own knowledge as teachers (Brodie, 2014). In the language of the current study, the process of identifying and engaging learner error requires that constitutive criteria of explanation be known to the teacher so that in turn the teacher uses the criteria to assist the learner.

In the same project, the concept of explanation was used to look at the quality of teachers' analysis of learners' error (Shalem, Sapire, & Sorto, 2014). Shalem et al. provide criteria for explanation which they use in the analysis of leaners' errors produced in the assessment texts. In a similar fashion, they segment the criteria of an explanation into SMK (conceptual and procedural) and PCK (everyday, diagnostic and multiple). The difference is that I aligned the SMK and PCK categories to explanation and acts of explaining respectively, while for Shalem et al. (2014), both SMK and PCK constitute two types of explanation with subcategories. The authors draw a distinction between partial, full, incomplete, absent explanations to judge the quality of teachers' learner error analysis, but they do not provide an epistemic analysis of explanation; specifically of what are the main criteria, which constitutes explanation and how explanation differs from explaining.

Peng (2007) writes in the context of China a paper based on Confucian Heritage Culture (CHC) that explored something they call *shuoke,* but for the benefit of international readership, choose to call it *explaining a lesson*. The aim was not focused on what is the explanation and its components a teacher offers to learners, but rather, on the explanation of the lesson by a teacher to fellow teachers. Peng reports that the process of explaining a lesson has three components to it, viz.: 1) explaining the mathematical content; 2) explaining the teaching style chosen; and

3) explaining the teaching procedure of the lesson referring to sequencing of ideas in the lesson (Peng, 2007). The word explaining is a verb, and so the paper was not concerned with the components of an explanation, but with the actions taken to explain the lesson, and how teachers' PCK and SMK develops in the process of engaging in lesson analysis. Following these four papers, there is still a need for the exploration of what constitutes explanation and acts of explaining and how teachers learn the explanation and acts of explaining from a PD.

Representations are a means for expressing the mathematics which requries explanation. The act of explaining occurs in a classroom environment, where learners are the audience, and so learner contributions, in whatever form, may have misconceptions and errors. Misconceptions and errors are again clarified by providing an explanation, they have to be explained. The last paper was not necessarily focused on explanation per se, but provides a discussion of a lesson delivered.

As Mosvold (2016) has already noted, the weakness in the listing of works of teaching is that it is unstructured, as there is no theoretical organising principle. I suggest a way of organising the list differently, in line with the distinction between explanation and acts of explaining. The categories listed in Figure 2.2 above have words describing a process, highlighted in grey, that are continuous present participles telling of an action which is in process. I briefly mentioned in the introductory chapter that the explanation is the product of the acts of explaining. The words highlighted in Figure 2.2 are descriptions of what is done to produce the explanation. Table 2-1 below shows a reorganisation of the descriptions into four categories, that is, acts of explaining, examples, explanation and audience. In separating the categories I also clustered them into four groups 1) goals/aims/purpose; 2) examples and their representations; 3) maths content and 4) questions. The four groups reflect general classroom practice where the goal/ purpose/topic is declared first before anything else. The goals, purposes, and topics are taught through examples and tasks, which are represented in a particular way and ordered according to levels of difficulty. Then, in the actual teaching process, mathematical content is achieved through the explanation of notation and language used. In the interaction, there are questions asked by the teacher or learners to justify the work in which they are engaged. In grouping the list, I have kept the words the same as they appear in Figure 2:2, no rephrasing has been made, the only words that are excluded are the conjunctions 'and' and the preposition 'to'. For example, the seventh mathematical task of teaching '*explaining mathematical goals and* *purposes to parents*', the word *explanining* was categorised as a continuous verb, *mathematical goals/purposes* were indentified under explanation, and *parents* were indentified as audience.

Table 2-1: Works of teaching are organised to align with explanation and acts of explaining

No	Acts of explaining		Explanation	Audience
	Continuous verbs	**Examples, representations & language**		
	Goals/purpose			
7	Explaining		Mathematical goals/purposes	Parents
6	Connecting, being		Topics prior, future	
	Examples and their representation			
3	Finding	Examples of a concept or process		
15	Selecting	Representations		
4	Recognising	Representations		
5	Linking	Representations		
9	Modifying	Easier/ harder tasks		
	Maths content			
8	Appraising, adapting		Mathematical content of textbooks	
12	Choosing, developing		Useable definitions	
13	Using, critiquing	Language	Mathematical notation	
1	Presenting		Mathematical ideas	
11	Giving, Evaluating		Mathematical explanations	
10	Evaluating		Claims	Students (learners)
16	Inspecting		Equivalences	
	Questions/ justification			
14	Asking		Productive mathematical questions	
2	Responding		Why questions	Students (learners)

The organisation of the works of teaching in Table 2-1 above was useful in the sense that it enabled me to see that constitutively the exaplanation is composed of goals, definitions, notation, ideas, and 'why questions', to name a few. Table 2-1 also enabled me to see the constitutive criteria for acts of explaining such examples and their representations and language use. The elaboration of explanation and acts of explaining that follow in Chapter Three confirms this. In other words, the organisation of the works of teaching in this fashion brought into the spotlight some of the constitutive criteria for explanation and acts of explaining.

In summary. I first started with PD literature in general, whereafter from this literature, features of a PD programme were emphasised, and there was less emphasis placed on teacher knowledge. The explanation and acts of explaining are epistemic concepts, and so have to do with teacher knowledge. I then moved on to teacher knowledge, and I noted the relationship between Shulmans' SMK as explanation and PCK as acts of explaining.

I then focused on mathematics teacher education, and how it has conceptualised and researched the subject of teacher knowledge. From this I found that there were what researchers call 'works of teaching', viz. actions researchers observed inside and outside of mathematics classrooms. I argued that all the actions listed had to do with explanation or acts of explaining. I then separated the list into three broad categories, that is, acts of explaining, explanation, and audience. I also grouped the list according to general pedagogic practice activities I have observed over the years that in fact there is first, not always, a statement of the goal. In classroom practice, the purpose and goals for the day are mentioned via a topic which must be justified in terms of how it connects to previous and future topics. The statement of the goal is frequently followed by a series of examples, representations and tasks that have been mostly planned and selected prior to the lesson. The talk that accompanies the examples and their representation often highlights the notation and the mathematical language, and there is a posing and answering of questions. In fact, Adler and Ronda's (2015) MDI framework gives this kind of organisation, even though their aims when constructing the MDI framework were not to give an organisational structure to the 'works of teaching'. Mosvold (2016) noted that the MDI framework does assist in terms of organising the 'works of teaching' into observable discourse practices. My own input is not divorced from the arguments presented by Mosvold (2016). I bring a dimension that views the works of teaching in the language of my study as acts of explaining, explanation, and audience, as shown in Table 1 above.

There is specific mention of the act of explaining in the list. Explaining is mentioned in relation to parents, and it is an act done out of the classroom. My interest is in how the activity of explaining occurs in the classroom to or with learners as the audience to produce the explanation. The explanation is seen as something a teacher provides to learners, where I am interested in what the criteria are for an explanation that a teacher provides to learners. What I obtained from the literature is that the 'works of teaching' list has merely listed these two (explanation and explaining), where there are no elaborations of what might be the criteria for both.

In the next section, I discuss the WMCS project from which the study is derived. What I describe are the initial goals of the course, and the activities the project has implemented and how the design of activities evolved over time, as the nature of the problem and need became clearer. One of the needs that became clear as the project progressed over the years is the notion of explanation in mathematics classrooms.

2.5 WMCS PD course description

The Wits Maths Connect Secondary Professional Development (WMCS PD) project was a five-year (2010 – 2014) research and development project.[3] I have been with the project from 2010-2016, two years after its extension in 2014. The project supported teachers from ten secondary schools in one district in the Johannesburg area Gauteng Province in South Africa. The PD work supports Mathematics teachers to deepen and extend their mathematical content knowledge, the quality of their teaching, their knowledge and awareness of key transition points in the Curriculum and in mathematics (Adler, 2014).

In the first two years of operation, the project identified the transition from Grade Nine to 10 Mathematics as critical to the overall goals of the project. In some schools, too few learners were proceeding to Grade 10 Mathematics; and in others, far too many learners moved into Grade 10 Mathematics seriously underprepared (Adler, 2017). In addition to a concern with quantity, qualitatively there are critical transitions that learners need to make in their understanding of mathematics as they move from General Education and Training (GET) to Further Education and Training (FET) Mathematics. Work that the project team did with the schools was mainly in the afternoon, so as to not disturb the running of schools during the day. However, there were constraints, the after school phase did not provide enough time to engage

[3] The second phase of the project started in 2015 and goes to 2020. This study was conducted in the first phase of the project which started in 2010-2014.

with the mathematical content and its teaching, and family commitments which made teachers unable to attend consistently. As a result, in the third year, the PD was structured into 20 day courses,[4] so as to give teachers sufficient time to work not only on their mathematics, but also on their explanations and acts of explaining.

Two courses were developed to suit the range of teacher mathematical abilities within and across schools. Both courses were called Transition Mathematics courses, and named TM1 and TM2 in line with the identified critical transition points. TM1 course is a year-long course for Grades 8, 9 and 10 teachers focused on core mathematics content for this transition, and related pedagogy. TM2 is also a year-long course, focused on mathematical content critical to progress into Higher Education, and pedagogic practices that increase mathematical and cognitive demands on learners (Adler, 2017).

In the current study, I focused on the TM1 course. In the TM1 course I looked for what was stated, elaborated and exemplified as explanation, as well as acts of explaining in the teaching sessions focused on explanation. I used the course handouts presented to teachers, as well as the course presenter's PowerPoint slides. The documentary analysis of the course was accompanied by a follow up interview with the course presenter. I provide more methodological elaborations and analysis in Chapter Five and Six, respectively.

The WMCS PD project acts in accordance with the listed features of PD. In terms of teacher knowledge, there are content-based sessions focusing on teachers' SMK, and there are teaching sessions focusing on teachers' SMK and PCK.

2.6 Conclusion

The main aim of this chapter has been to show that the notion of explanation and acts of explaining are components of teacher knowledge not explicitly raised in the extant PD literature. I have shown in the section that deals with general PD literature that the emphasis in the literature is on the features of a PD programme, and teachers' knowledge of explanation is backgrounded. Where there is a focus on mathematics teachers' knowledge, producing and elaborating on what counts as explanations is backgrounded. The knowledge of what

[4] Course documentation refers varyingly to 16 and 20-day courses. The 16 days refers to contact sessions at the university. Related support work in schools combines to constitute the remaining four days – hence 20 days.

constitutes an explanation and how to enact the explanation – that is the acts of explaining – is an important aspect of teachers' knowledge, which requires some detailed study and research. PD literature in mathematics reported on representations, which is a key act of teaching, learner error, and explanation from a different perspective to the one I use in this study. I have shown by reworking the list of the works of teaching that explanation and acts of explaining are very important works of teaching. In Chapter Three, I enter into the detail of literature focused on explanation and acts of explaining.

CHAPTER 3 EXPLANATION AND ACTS OF EXPLAINING

3.1 Introduction

Explanation is the heart of teaching, and teaching proceeds through uttering of statements, which are aimed at providing the explanation of the content learners are supposed to learn. It is hard to imagine mathematics teaching without considering explanations; an unquestionably central part of algebra teaching. In the classroom, therefore, explanations are ubiquitous. However, the question remains as to what the explanation itself is? What are the constitutive criteria of an explanation? It is important to have the constitutive elements that compose an explanation established, because they can be used to judge whether the explanation offered has met the criteria of an explanation.

The notion of explanation is an epistemic concept and thus philosophically located within the theory of knowledge. In this chapter I present literature on explanation from the field of philosophy, followed by literature from the field of education. I do this because beyond the field of education, the philosophical literature offers necessary conceptual clarity and analysis of the word explanation by delineating what it is as a noun, as distinguishable from its common relative verb.

In the mathematics education literature I observed the notions of explanation and explaining to have been used interchangeably, where the distinction between the words is not drawn, and hence the criteria provided can be characterised as both explanation, as well as acts of explaining. The weakness of not making the distinction is that what is classifiable as criteria for explanation becomes ambiguous.

The notion of explanation is of great interest in this chapter and the aim is to find constitutive criteria of explanation and acts of explaining by reviewing literature in philosophy and education. After the literature survey, I conclude the chapter by outlining what I obtain as criteria for explanation, and acts of explaining, respectively.

3.2 Explanation from philosophy

Ruben (1992) in his book *Explaining Explanation* is interested to formulate a theory of explanation. Ruben distinguishes explanation from acts of explaining, focusing on the former to the exclusion of the latter. I take up this distinction as an organisational structure to separate between criteria for explanation and acts of explaining here. The educational context in which I am working of classroom practice allowed me to think about the explanation within classroom practice as a product of the act of explaining. In the planning of the lesson the explanation is the object of the acts of explaining. In essence, the explanation is both the object and product of the acts of explaining.

I briefly discuss below the notions of verbs and nouns, the epistemic nature of the concept of explanation and acts of explaining; the relationship between the verb and the noun; the audience as recipients of an explanation, and the terms describing the object which needs to be explained. All these are points associated with the notion of explanation, which require some clarification.

Verb vs noun

From a linguistic point of view, there is a difference between the words *explanation* and *explaining,* and the philosophical literature highlights this difference by distinguishing explanation from explaining. Explanation is a noun, while explaining is a verb; explaining is the methodology of enacting an explanation. To highlight the verb and verbal aspect of the word explaining, Achinstein (1983) argues that it is an illocutionary act; a speech act; something uttered.

Epistemic concepts

The notions of knowledge conception and knowledge applied further emphasise the distinction between explanation and acts of explaining. Explanation is the knowledge conception, while on the other hand, acts of explaining are a practical and concrete demonstration of the knowledge conception. Explanation and acts of explaining subsists within the parameters of two epistemic perceptions of the cognitive and the experiential, in the sense that explanation is a cognitive perception of algebraic concepts, while the acts of explaining refer to the algebraic knowledge applied to the practice of teaching for a particular audience. Explanation can be used to judge whether the activity of explaining has captured the essence of the knowledge conception. From this it can be concluded that one provides an explanation of what one knows, or otherwise put, an explanation is a reflection of the teachers' knowledge. I have already associated explanation with SMK and the acts of explaining with PCK in the previous chapter concerned with teacher knowledge. The PCK or acts of explaining are a manifestation in practical form of the cognitive perception that is the explanation or SMK. Providing insights on what it takes to produce an explanation, Wilson and Keil (1998) argue that offering an explanation is a cognitive demand, and the only way to avoid providing a shallow explanation about X is to learn a theory of X. This already links up with teacher knowledge discussed in the previous chapter, playing a central role in the production of an explanation. The claim Wilson and Keil (1998) are making is that to learn a theory about X is to acquire a web of knowledge about X, and therefore, having a theory about X entails being able to offer an explanation of X with greater depth.

Relationship between E and AoE

Explanation and the acts of explaining are closely related, and together they create the capacity which enables learners to fully grasp the concepts of mathematics. It is necessary not to create a slippage between explanation and the acts of explaining, rather, to note that together as distinct concepts, they bring about the understanding of algebraic concepts. The notion of explanation in conjunction with acts of explaining is of vital importance in the learning of mathematical concepts, because teachers' explanations are to be composed of demonstrative acts of explaining, in order to provide the means by which learners can gain access to the mathematical concepts and processes.

The separation between explanation and acts of explaining is conceptual, and is a necessary methodological strategy for the current study, which sought to understand the notion of explanation. In real time, while observing the teacher's practice in the classroom, what is actually collected are acts of explaining from which the explanation can be derived. In this sense, the explanation is the full view of what went on in the act of explaining, it is the product and object of the acts of explaining. The idea of an explanatory act and product are mutually determined and we cannot speak of one without the other; in the sense that the actions are informed by the knowledge the teacher possesses, and the same knowledge is observable in the acts of explaining.

Audience

When engaged in the act of explaining there is an audience to whom the explanation is offered, and hence the criteria in the act of explaining might be unpacked in different ways, and may not necessarily resemble the criteria in the explanation.

Explanandum

Before there can be an explanation, there has to be something that needs to be explained. The act of explaining is performed on something which needs to be explained called the explicandum or the explanandum, or explananda in the plural. In the language of education research, the explicandum is what needs to be learned and therefore would be understood more broadly as the object of learning (Marton, Runesson, & Tsui, 2004). That which is done to explain the explanandum is called explanans, the plural of which is explanan. This discussion informed the ways in which I went about the analysis, specifically in the methodology, where I defined the unit of analysis as that which needs to be explained, which was, in broad terms, the goal (explanandum) of the lesson accompanied by or nested within a series of examples (explanan). The explanation is the object and product of the explanan used.

Historical section

Apart from Chapter One, Ruben segmented his book into two sections, the historical section, chapters 2, 3 and 4; and the last three chapters Ruben reserves for the elaboration of his own views on explanation 5, 6 and 7. I have taken up one aspect from the discussion from his Chapter One as already demonstrated in the discussion above, namely the separation between the respective terms explanation, and acts of explaining. The historical section provides an assessment of philosophical texts as presented by Plato, Aristotle, John Steward Mill and Carl Hempel[5] for the purposes of finding what these theorists established as criteria for explanation, adequately providing insight for the purposes of this study into the question of constitutive criteria for explanation.

Theorists (Plato, Aristotle, Mill and Hempel) as discussed by Ruben (1992) were interested in what information has to be conveyed in order to explain X[6] and, therefore, provide criteria for what constitutes an explanation. My interpretation of Plato and Aristotle's criteria was that they are related in the sense that they build on each other, and so I discussed them jointly. Mill and Hempel's criteria were also similar as shown by Ruben and I also discussed them jointly. I begin with a discussion from Plato and Aristotle.

[5] I settled for a secondary reading of these theorist because of the sharp focus Ruben provides for explanation which is very useful for this study.
[6] In all the talk X refers to the explanandum i.e., that which needs to be explained.

3.2.1 Explanation from Plato and Aristotle

Ruben traces the history of explanation as far back as Plato (423BCE-347BCE) and Aristotle (384BCE-322BCE). Ruben tells us that Plato's writings (which were mainly composed of dialogues and letters which were used to teach a range of subjects including philosophy, mathematics, ethics, rhetoric and logic) were concerned with knowledge. Plato did not discuss explanation explicitly as a concept but Ruben argues that since knowledge and explanation are closely related, it can be concluded that Plato was discussing explanation. Two salient points about explanation emerge from Plato's writings as interpreted by Ruben; which are enumerating the elements and stating the mark:

> First, 'giving an account of X' might mean 'enumerating X's elements' (207a). Second, 'giving an account of X' might mean 'being able to name some mark by which the thing one is asked about differs from everything else' (208c). (p.75)

From what Plato is arguing, one can conclude that enumerating the elements of X is necessary in the explanation of X, even though not sufficient. This is one point which I think is useful to take forward in the conceptualisation of what might constitute criteria for an algebraic explanation - enumerating the elements of X amounts to stating that of which X is composed. Marking that which is distinctive about X differentiates it from something else. In Plato's view then, an explanation is giving an answer to what X is, specifically enumerating its elements and pointing out the distinctive mark of X. Further emphasis on the epistemic nature of the concept of explanation is argued by Aristotle.

Ruben moves on to discuss Aristotle's view on explanation. Aristotle argues that one does not have a full type of knowledge about X unless one possesses an explanation for X. Therefore, knowing something is being able to give an explanation of it.

The translation of the Greek word *aitai* and its interpretation in the work of Aristotle is debated in the field of philosophy. Ruben highlights two ways in which the Greek word *aitai* has been interpreted in the field of philosophy or by readers of Aristotle. First, that the translation of the word was understood as referring to *cause,* and thus, most readers concluded that Aristotle was providing a causal theory of explanation. On the contrary, both Ruben (1992) and Moravcsik (1974) agree that Aristotle was providing a theory of explanation. Second, Aristotle is quoted to have said that the *aitai* is whatever answers the question of why. Ruben argues that not all questions in the form of what and how are translatable to why-questions without losing their meaning. Ruben and Moravcsik argue on the contrary that in fact Aristotle was giving a theory of explanation, where the *aitai* cannot be limited to a theory of *causes* or *why questions,* but is a theory of explanation.

Ruben refers to the interrogative particle *why* when used as part of an explanation as answering a question, and should be used to address an area of interest and not necessarily act as an obligatory, ample criterion of explanation. In other words, the why question in the explanation can be invoked as the need arises. I quote below from Ruben the supporting statement to this argument.

> Aristotle's remarks on why questions should be taken merely as a heuristic device for picking out the area in which he is interested, not as an adequate philosophical criterion offering necessary and sufficient conditions for when an answer to a question is an explanation (Ruben, 1992; p.80).

In the above, Ruben argues that explanations are not limited to the why question. He continues to argue that there are explanations in response to "what" and "how" questions. *What, how* and *why* questions are interrogative particles within the context of English as a language, and the argument Ruben presents highlights that the explanation derives from them the drive to provide answers to these interrogative particles. The WMCS PD from which I collected data used this approach in offering criteria for explanation. This is relevant in a mathematics classroom context, where explanations of definitions (what) and process (how) are presented.

Aristotle argues that there are four criteria that can be employed in the provision of an explanation. The first explanation is *matter,* described as the elements that compose X, out of which X emanates. The second: *form* refers to the structuring of the matter/elements, the pattern the matter takes to make X. It refers to the general principles or laws that underlie the arrangement of the matter. The third is *motion originator,* which refers to that which initiates change. The last is the *goal or end,* which refers to the purpose or that for the sake of which a thing is done. Ruben brings the following quote in which Aristotle states his four criteria to be included in an explanation:

> Knowledge is the object of our inquiry, and men do not think that they know a thing till they have grasped the 'why' of it… In one sense, then, (1) that out of which a thing

> comes to be and which persists, is called 'explanation'… In another sense (2) the form or the archetype…and its genera are called 'explanations'… Again (3) the primary source of the change or coming to rest… Again (4) in the sense of end or 'that for the sake of which' a thing is done… This then perhaps exhausts the number of ways in which the term 'explanation' is used…. As the word has several senses, it follows that there are several explanations of the same thing… Further, the same thing is the explanation of contrary results. For that which by its presence brings about one result is sometimes blamed for bringing about the contrary by its absence. Thus, we ascribe the wreck of a ship to the absence of a pilot whose presence was the cause of its safety. (Physics II, Chapter 3). (In Ruben, 1992; p. 78)

Ruben believes that Aristotle's four criteria – the matter, form, goal or end, and motion-originator – is a theory of explanation: "the doctrine of the four causes is about four explanatory principles" (p.78). Ruben shows how Aristotle's four criteria together constitute criteria for an explanation. In other words, the explanation for a particular thing is done in terms its constitutive elements: form, motion, originator, and goal.

> I therefore stick to the less semantic-involving formulations: that 'explanation' can be used in four ways or with regard to four different elements of things or that there are four different types of explanation (p.81).

In the above, Ruben finally describes these as four different "types" of explanation. However, this suggests they each constitute an explanation on their own. This is not appropriate, hence I redescribe these as criteria that come together to constitute an explanation. At the same time, Ruben notes that Aristotle was not committed to the idea that every explananda will have an explanation in each of the four criteria. As will be evident through the study these do combine and are all important for an explanation in mathematics education, and hence are referred from hereon as constitutive criteria.

When looking across the two antiquity philosophers it is apparent that both agree on the first two as constitutive criteria. For the first criterion, Plato discusses enumerating the elements, while Aristotle refers to this as material composition; this is the same criterion, because the material composition is made up of elements. Again, Plato and Aristotle agree on the second criterion of explanation, namely form. A mark that distinguishes X from the rest serves as a special property of X. This is to say, properties of X are possible to notice from the structure or arrangement the elements take. There is no way of observing the distinguishing mark outside of the particular form the elements have taken. It is only possible to classify according to the distinguishing mark when there is an already existing arrangement/form of the elements.

Aristotle develops Plato's criteria further to include explanations of the motion originator, which is the process and explanations of purpose or the goal.

Aristotle's theory of explanation was developed through the study of nature, but Ruben as well as Moravcsik (1974) argue that it can be applied to other studies not necessarily concerned with physical nature directly (Ruben, 1992, p.95). Specifically, in this study I am interested in what constitutes an explanation for algebraic expressions, which are abstract, and do not possess any material composition. In the discussion that follows, I briefly describe how I am re-describing each of the four criteria for use in my study of algebraic expressions. Following this, I then exemplify the criteria using a school mathematics algebraic expression.

3.2.1.1 Matter

The first criterion is *matter,* which is described as the elements that compose X, that out of which X derives. Algebraic expressions are not material phenomena. Aristotle's concept of explanation reflects beliefs about reality that are incompatible with the kinds of objects dealt with in mathematics. The nature of algebraic expressions is abstract, and so the term *matter* needs to be re-described to be usable for algebraic expressions context. Within the mathematical context, that which needs to be explained is referred to as the mathematical object in a metaphorical sense because the word object implies a tangible, concrete object that one can hold, while algebraic expressions are represented in spoken or written words, symbolic, or graphical forms. For the reason that there are ways of representing algebraic expressions, it is then favourable to think about constitutive elements of the algebraic expression, instead of matter. Aristotle talks about matter within the context of physics and biology, where the objects are tangible and perceivable through the senses viz., you can see, touch and taste solids and liquids, smell solids and gases, and hear sound. Speaking to this very notion, Moravcsik (1974) argues that the esoteric might not be able to be understood as matter, but is nonetheless a product of constituents, where the constituents of algebraic expressions are the signs and symbols put together in a particular fashion to create form/structure.

3.2.1.2 Form

The *form* refers to the general principles or laws that underlie the binding together and arrangement of the elements. The form is the essence of the algebraic expression. It is the structure or pattern the elements take. The principles and laws that govern the patterning or structuring of the matter are part of the form and need to be unpacked in the explanation. Principles and laws are responsible for the properties the algebraic expression ultimately possesses. The form contains the qualities and character of the algebraic expression. One is required to know the relations between elements. The elements are the entities from which the structure is made.

Pertaining to both matter and form there is a close relation in the sense that the deconstructed form/structure are the constitutive elements. In other words, knowing the form requires knowing the parts and the manner of putting them together. The notion of form, as will be shown in the next chapter concerned with algebra literature, can be re-described to refer to algebraic structure.

3.2.1.3 Motion originator

The third criterion is *motion originator*. Ruben, in his interpretation of Aristotle, uses the expressions 'efficient cause' and 'change initiator' interchangeably with 'motion originator'. Moravscik (1974), when it comes to the notion of motion originator, argues that *kinesis* is the closest word in terms of translation which can be better understood as *process*. Motion originator refers to that which initiates change, for example, the human being that acts on the physical matter/elements is the one that initiates change. Within the context of mathematics, the intangible matter which has now been put together into a particular structure has inherent rules of how to perform procedural activity on the structure to change into equivalent forms.

Processes refer to the careful steps of a method that are performed on the algebraic expressions. Process is the word I have used to refer to the third criterion of explanation. In the exemplification, this idea is illustrated further. I now discuss the last criterion of explanation, viz. goal.

3.2.1.4 Goal

The last criteria of explanation is *goal or end*. This refers to the purpose or the sake for which a thing is done. This is a requirement to put forth a justification. In the classroom the broader aim is to achieve understanding of how to factorise algebraic expressions. For a specific lesson the goal might be the statement of the topic to be learnt.

To exemplify, I use the following school mathematics algebraic expression: factorise $(q-4)^2+3(q-4)+2$.

$(q-4)^2+3(q-4)+2$ is an algebraic expression. The variable q; the specific integers 2, 3 and 4; the specific operation signs $+,-$; and a specific mathematical word *factorise* are the elements that compose the expression.

The *form* refers to the arrangement of the elements, and for the specific example it is a quadratic expression. This specific quadratic expression can be classified as a polynomial with three terms, and because of the number of terms in the expression, it is called a trinomial. The terms are separated by the $+$ sign, with the first two terms having a common expression inside the brackets $(q-4)$, and the last term being the only term in the expression without the common $(q-4)$. The exponent 2 in the first term, is responsible for the expression being classified as a quadratic expression, and is the distinguishing mark, to use Plato's words.

The *process* to be performed on the expression is determined by the word "factorise", which requires a transformation of the expression without changing the expression. The process of factorising the expression $(q-4)^2+3(q-4)+2$ can be approached in two ways; first by expanding or multiplying out, simplifying, and finally finding the factors of the quadratic trinomial.

$$q^2-8q+16+3q-12+2$$

$$q^2-5q+6$$

$$(q-2)(q-3)$$

The second process highlights the relationship between factorisation and the distributive property, and requires an understanding of the structure of the expression, where the middle term $3(q-4)$ is broken into parts $(q-4)^2+(q-4)+2(q-4)+2;$ a factorisation by grouping is performed $(q-4)(q-4+1)+2(q-4+1);$ a common factor taken out $(q-4+1)(q-4+2)$, and when simplified gives $(q-3)(q-2)$. The reverse process of factorisation is the multiplication of the two factors which is the application of the distributive property $(q-4)(q-4)+(q-4)(1)+(2)(q-4)+(2)(1)$.

A quick remark follows on the structure of the two equivalent expressions, the original expression and the transformed expression. The original expression $(q-4)^2+3(q-4)+2$ is clearly a quadratic expression, while the transformed expression $(q-2)(q-3)$ is the product of two linear factors. Each linear factor is a polynomial with two terms and thus called a binomial. The structural appearance of these two expressions looks different, and their descriptions seem different one from the other, however, the structure they possess is the same, because the transformational activity performed on the quadratic expression does not change the expression, but maintains equivalence. I pick up on the relationship between structure and process in more detail in the next chapter focused on school algebra.

The purpose or the *goal* of performing the factorisation is to be able to find linear factors and the ability to find factors is important for future topics and doing mathematics in general. The expression on its own without the accompanying instruction is meaningless; instructions that accompany the expression are also important. Achinstein (1983) says "instructions are rules imposing conditions on answers to a question" (p.64). In the algebraic expression I used to exemplify the four criteria of explanation, the word factorise is an instruction demanding a particular understanding of what procedure to perform in order to produce the answer. A set of instructions or an instruction that helps to explain X is/are necessary. Thus, one can say that the explanation has been provided correctly if the instructions accompanying X have been satisfied.

To this end, the criteria obtained are the four constitutive criteria for explanation, while the constitutive criteria for acts of explaining still need to be found. I now discuss the last two theorists in order to establish the constitutive criteria for explanation.

3.2.2 Explanation from Mill and Hempel

Explanation is conceived of as something we do individually or collectively, and so is a product of human action. To place emphasis on the notion of human action, Achinstein (1983) defines explanation as "uttering something with the intention of rendering X understandable (in a certain way)" (p.23). The concern is with what causes human actions and the conclusion that Doyal and Harris (1986) come to is that *thought* and *action* go hand in hand. Because human actions are subject to so many unknown and unknowable circumstances, Mill and Hempel believe that the science of humans can never be as exact as the science of the physical world, but is a possible science.

The last two theorists Ruben discusses in his historical summation are Mill and Hempel. I discuss Mill and Hempel's views of explanation to the extent and scope necessary for my study, where the rest of the view on statistical probabilistic types of explanation argued for by Hempel lie outside the scope of the current study, where the aim is to take from the historical section of the book the relevant theorisation of the constitutive criteria of explanation. Contrary to the experience transcendent ideas of explanation, Mill is an empiricist, and for Mill, the explanation is an empirically accepted idea. Ruben states that "for Mill, all explanations are a subset of the set of deductively valid arguments" (p.110). To this Hempel agrees, however, he adds that *some* and *not all* explanations are subsets of the sets of deductively valid arguments. For Mill and Hempel, an object is said to be explained by pointing out laws which are responsible for its construction.

> An individual fact is said to be explained by pointing out its cause, that is, by stating the law or laws of causation of which its production is an instance. Thus, a conflagration is explained when it is proved to have arisen from a spark falling into the midst of a heap of combustibles; and in a similar manner, a law of uniformity of nature is said to be explained when another law or laws are pointed out, of which that law itself is but a case, and from which it could be deduced (Mill 1970:305 in Ruben, 1992, p.116).

In this quote, Mill is arguing that the explanandum comes out of a deductive process where other laws responsible for the explanandum's existence need to be ascertained. From Mill's argument, I am particularly interested in the notion of laws behind the deduction or the underlying principles from which the deduction is extracted. In essence, the argument posits that the history and background of the explanandum's existence is an important part of the explanation.

The notion of laws refers to the fact that the truth behind the law needs to be accounted for so that it is clear why the law works. The process of accounting for the truth of the law is a process of derivation by stating the fundamental principles. The algebraic procedure has fundamental laws which are responsible for its existence. It does not independently surface. What needs to be unpacked in the explanation is the gathering up of mathematical laws, which constitute the general law. Within the context of algebraic expressions, the distributive property $a(a + b)$ which is $ab + ac$ example of a principle, which needs to be derived so that the fundamental principles behind it are unpacked. In terms of Aristotle's four criteria discussed earlier, Mill's argument has relevance for the third criterion of explanation, process, because it is in procedural activity that mathematical laws and principles are applied and performed. Because of the close relationship between form and process, Mill's argument also has relevance for form. What I have taken forward from Mill is that a derivation of the procedure is necessary in order to account for the underlying principles behind the mathematical procedure.

To summarise, the philosophical literature draws a distinction between explanation and acts of explaining. From the discussion, constitutive criteria for an explanation came from the work of theorists Plato and Aristotle, Mill, and Hempel. It is also important to know the constitutive criteria for acts of explaining – that is, the specific actions taken to produce the explanation. I continue this discussion on explanation and acts of explaining in the education and mathematics education literature.

3.3 Explanation from Mathematics education literature

I reviewed literature on explanation with the aim of answering the question: what is legitimated as constitutive criteria for explanation in education literature and mathematics education literature? The results of this search revealed that education and mathematics education literature provided criteria that were conflated, containing both explanation and acts of explaining criteria.

Research in mathematics education concerning explanations can be separated into three branches. The first describes aspects of a good or effective explanation, the second views the notion of explanation as a discourse practice, and the third is focused on learner preferences of a certain type of explanation. I discuss the last two under one section, because they all focus on explanations offered by learners. All these branches intend to provide criteria for explanation, but in the main, presented descriptions of actions taken when explaining, or the desired result after an explanation has been offered. This is the case because the distinction is not made in the field, and thus the criteria provided are mingled. Together the three approaches are important for the current study as the aim is to develop clarity on the constitutive criteria for explanation, and acts of explaining. Alongside this discussion, the criteria for explanation are particularised and redescribed for a mathematics education context. Subsequent to the three approaches, I then discuss the constitutive criteria for acts of explaining according to what has emerged in the discussion.

3.3.1 Effective or good explanation

The main person who has been studying the notion of effective instructional explanations in mathematics classrooms is Gaea Leinhardt. Her work separates four types of locations where explanations and acts of explaining occur, viz.: (1) common explanations are part of everyday life and rely on sharing of local context; (2) disciplinary explanations are shared rules that can exist across time, distances and space; (3) self-explanations, where the self can be group or intra-individual; and (4) instructional explanations, that are designed for the communication of particular subject matter knowledge in this study mathematics and specifically algebraic expressions (Leinhardt, 1987, 1989, 1997, 2009, 2010). From the four, Leinhardt locates her work within instructional explanation.

On the point of instructional explanations, Leinhardt posits that there are goals and actions associated with provision of instructional explanation. The goals are the ends; they are what the teacher wants the class to know. Actions include selection of examples, problems, and tasks. Leinhardt presents a model of instructional explanation with the following four points:

1) an instructional explanation should identify the nature of the problem; 2) know the representational base being used; 3) the explanation should be complete and connect back to the nature of the problem and its representation; and 4) the principles of the discipline should be accessed and addressed.

The word goal for Leinhardt seemingly refers to the broader goals and objectives of the lesson. The identification of the nature of the problem seems to be separated from the broader goal. The goal and the ways in which it has been defined is similar to how Aristotle as interpreted by Ruben has conceptualised it as the end, or the purpose.

The actions that Leinhardt enumerates are acts of explaining, such as: selection of examples, representation, problems and tasks, even though actions taken in the act of explaining are not limited to these. In fact, those enumerated by Leinhardt are not directly observable in the act of explaining, as already discussed in the previous chapter. These (selection of examples, problems and tasks) are actions performed in the planning of the lesson, and then implemented in the lesson depending on how closely the teacher follows the plan. Except for following the plan, there can be examples generated in the lesson depending on how the lesson proceeds. However, a collective set of examples used in the lesson can be compiled after observing the implementation of the lesson.

The first point that Leinhardt highlights is that an instructional explanation should *identify the nature of the problem*. This is basically establishing that which needs to be explained and so can be taken as a criterion of explanation related to the goal. The second refers to *knowledge of the representational base*; this is related to the different structures in which the elements can be arranged. This is again a point related to the pedagogic act of selecting examples and their representations, that is, the particular form/structure the act should take. In terms of the criteria for explanation, the knowledge of the representational base is related to knowledge of form/structure.

The third point is about *coherence.* Coherence is understood as an expected result at the end of the act of explaining. However an elaboration of details on how to achieve the desired coherence is missing.

The last point in the model of instructional explanations is about *access to the principles of the discipline*. This point refers to the integrity of the explanation. However, there is no specification of what exactly should be done to achieve access to the principles of the discipline.

The notion of coherence as a criterion of explanation is a judgement of quality in the explanation offered, that is, whether or not the explanation is mathematically sound and consistent. For coherence to work as a criterion of explanation, there needs to be specification stating that it is coherence in relation to matter, form or goal. When coherence is merely stated without the things it refers to, it is too vague, and is broadly applicable to just about anything. In the 2010 publication the word *connections* is used instead in a different sense by providing specific areas where the connections are supposed to occur; viz. between ideas, examples, connection to prior knowledge, and connections between error and misapplication. The connections aspect is not stated on its own as a criteria of explanation, but is accompanied by specific points where it is supposed to transpire. The idea behind these criteria is that they suggest pedagogic strategies instead of the constitutive criteria of explanation. From the pedagogic strategies they suggest, the notions of error and misapplication are issues that occur in the teaching, in relation to matter, form, process or goal it is still important to identify what the error and misapplication is. The discussion of instructional explanations offered by Leinhardt needs to be specific and articulate just what exactly the principles of the discipline are and what the coherence/connections relate to.

The studies of effective instructional explanations in education and mathematics education were predominantly focused on comparison of novice and expert teacher's explanations. I move into the detail of these studies. Early on, Leinhardt (1987) studied a series of eight consecutive lessons from an expert teacher to see the development of an expert explanation. In the next publication, she is concerned to see how expert and novice teachers' explanations and acts of explaining differ and how this knowledge can be used to enhance novice teachers' acts of explaining (Leinhardt, 1989). Duffy, Roehler, Meloth and Vavrus (1986) also studied explanations of grades Two, Three and Five expert and novice literacy teachers for effectiveness. They present what they call 'properties of effective verbal explanations' which are: 1) responsive information giving; 2) developing awareness; 3) nature of information given; and 4) devices for assisting learners (Duffy, Roehler, Meloth, & Vavrus, 1986). Responsive information giving can include the act of re-voicing, which is a term used to describe teacher echoing of learner contributions; it is not the constitutive criteria of explanation, but an act of explaining. The notion of developing awareness is broad, and requires specific mention of that which the awareness is in relation to. The assumption one can make in relation to the criterion:

devices for assisting learners is that it refers to a variety of pedagogic strategies the teacher chooses. The point here is that the criteria provided need to be interpreted and the process of interpreting the criteria is open to a variety of assumptions the interpreter makes. I am working with mathematics teachers' verbal explanations, while Duffy et al. were looking at literacy teachers. However, the notion of responsive information giving seen as re-voicing is relevant[7] for my study as a criterion for acts of explaining.

A similar study comparing expert and novice teachers' acts of explaining was conducted by Borko and Livingston (1989). In their study, novices were assigned to expert teachers. The common finding across these studies is that effective explanations came from expert teachers with years of experience. This, of course, is necessarily a context-based conclusion, which cannot be extended to other contexts such as ours where years of experiences do not necessarily correlate with production of effective explanations. In the South African context, teachers with years of experience obtained their training during apartheid. Disrupting the inequalities from our past has been a slow process where the inequalities, in the main, reproduce themselves. Thus, drawing parallels between years of experience and the notion of effective teaching within the South African context, has to be done with caution. The poor results we continuously obtain in the international comparative tests are a testimony to this fact.

Except for literature focused on comparing expert and novice teachers' explanations and acts of explaining, some researchers have focused on pre-service teachers demonstrating lessons and conclude that the act of explaining is a learnable practice (Charalambous, Hill, & Ball, 2011). Four pre-service teachers who differed in SMK were studied imitating how they would provide explanations to learners about the invert and multiply algorithm for division of fractions. They outline the following seven points as criteria for *good explanations*: 1) establish central question; 2) build on learners' existing knowledge; 3) consider learner difficulties and misconceptions; 4) clarify what is to be learnt; 5) chunk and sequence information; 6) hinge on judicious use and choice of examples; and 7) provide precise, accurate and meaningful information.

The notion of a *central question* and *clarifying what is to be learnt* are related, and so can be thought of as the goal. The act of *building on learners' existing knowledge* is a regulative pedagogic strategy, suggesting that the explanation for the current lesson ought to be connected

[7] The selection of responsive information giving as re-voicing and also as relevant to my study is a function of data analysis.

to knowledge acquired and constructed in previous lessons. The knowledge can be the knowledge of matter, form, process, and goal. The act of *chunking and sequencing information* is a pedagogic decision, which speaks to the notion of regulating the rate at which the knowledge is transmitted. The notion of examples has already been highlighted as a criterion for acts of explaining. The notion of accurate, precise, and meaningful information can be taken to refer to matter, form, process, and goal. Similarly, the notion of learner misconceptions and difficulties can be identified in relation to matter, form, process and goal. Thus, the list provided by Charalambous et al. (2011) mainly consists of acts of explaining.

The act of explaining is one out of ten observable didactic strategies (Andrews, 2009). Andrews (2009) reports on a project that observed teachers' practices in European countries, namely: England, Spain, Hungary, and Flanders. The definition of explaining was constituted of the following list of actions the teacher performs:

> The teacher explains (presents) instructions. The teacher explains (demonstrates) how to complete routine tasks. The teacher explains (describes) new concepts. The teacher explains (models) how to solve non-routine or unfamiliar problems. The teacher explains (derives) links between the current topic and others covered earlier (Andrews, 2009, p. 565).

These are actions that occur in the classroom. The author discusses how explaining could be construed as having several subcategories. The subcategories of explaining include: questioning, coaching, motivating, assessing, activate, and sharing.

A similar study in terms of comparing explanations offered in mathematics classrooms in different countries was conducted by Perry (2000). Perry compared explanations offered in US, Japanese and Chinese mathematics classrooms. Perry coded the data according to two types of explanation determined by their duration. The first was what the author called an *extended explanation* lasting five minutes or more, and the second was the brief explanation lasting a minute or less (Perry, 2000). They show through their results that there was more of the extended explanations in Asian classrooms than in US classrooms. Perry's description of an extended explanation included explanations of *how* to do something and/or *why*. The *how* and *why* are interrogative questions seeking to know the procedural steps and the reasons behind them. Viewed in this sense, the explanation as observed by Perry in Asian classrooms must interrogate the how to do as well as the why. To connect this to Aristotle's four criteria, the 'how' goes hand in hand with process, and the why corresponds to goal. The goal was defined as purpose for which a thing is done, however, within purpose the justification, specifically as an answer to the why interrogative question is fitting. The how and why are part of the explanation criteria the WMCS PD course transmits to teachers as will be shown in Chapter Six.

A literature review focused on instructional acts of explaining across different contexts such as: classroom instruction, tutoring, cooperative learning, and cognitive skill acquisition, learning from texts, computer-supported learning, and multimedia learning was conducted by Witter and Renkl (2008). From this review, the authors present what they call general guidelines for designing instructional explanations. The guidelines are as follows: 1) explanation should be adapted to learners' specific knowledge level; 2) should focus on concepts and principles; 3) should be integrated to learners' ongoing cognitive skills; and 4) should not replace learners' knowledge construction activities (Wittwer & Renkl, 2008). What is highlighted is the importance of the audience (learners) as recipients of the explanation offered, and their role in the activity of constructing the explanation. The criteria they provide have already been covered in the above discussion.

On the point of effective explanations, van de Sande and Greeno (2010) have argued that instructional explanations are likely to be more effective when both teacher and learners have shared expectations in relation to patterns of interaction and organisation of information. They argue that the explainer and the recipient must have framings that are aligned for the explanation to be meaningful (van de Sande & Greeno, 2010).

To summarise, the discussion on effective explanations has highlighted explanation and acts of explaining criteria. For explanation the notion of goal at two levels has been highlighted, the first is the broad clarification of the topic as the central question and what is to be learnt. The second is the justification as a response to the *why* interrogative question. Also, for explanation, the notion of process has been noted as a response to the *how* interrogative question. For acts of explaining, the importance of examples and their representations, responsive information giving, which I interpreted as a re-voicing communicative technique, as well as the notion of regulating information have also been highlighted. In the next section, I present a discussion of literature that looks at the notion of explanation and acts of explaining by studying learners' contributions.

3.3.2 Discourse practice and learners' explanations

In this section, I first discuss studies which view explanation as a discourse practice, and then move on to discuss studies which focused on types of explanations learners prefer.

There are two studies that focus on explanations offered by learners from a discourse perspective from Esmonde (2009) and Forman, and McCormick and Donato (1998). Both studies emphasise the importance of precision and accuracy in the language used to offer the explanation. For instance, Esmonde noticed that learners' speeches were filled with hedges when they were unsure of what they were talking about, and the use of pronouns to show ownership and confidence in what they say. On the other hand, Forman et al. (1998) noticed that the teacher endorsed decontextualised specialised speech compared to contextualised non-specialised language. This is important, because the speech used to utter the explanation is in full sentences.

The notion of explanation has also been studied from the perspective of the learner in terms of the types a learner affiliates themselves with and prefer (Levenson, 2010, 2013) .

There are three types of explanations Levenson (2010) used to code learner data and that is practically based (PB), mathematically based (MB), and formally based (FB). Levinson found that there are two types of explanation learners affiliate themselves to; that is, MB, and PB explanations. PB explanations use daily contexts, visual aids and manipulatives to give meaning to mathematical expressions, while MB explanations are based on mathematical definitions or previously learnt mathematical properties, and often use mathematical reasoning. MB explanations are based on mathematical ideas, but are not formal or rigorous. Levinson (2013) demonstrates that leaner preferences progress and shift as the learner develop from Grades 5 to Grade 10 from PB to FB.

Even though the studies reported here were focused on learners' discourses, while explaining there is relevance to my study, because the explanation in the classroom is co-constructed in the interaction between teacher and learners. This means learner contributions may reflect contextualised non-specialised language as well as their practically based preferences. For my study which is focused on teachers' explanations and acts of explaining the implication this work has is that while it may be acceptable for learners' discourses to use hedges, everyday contextualised language, and practically based explanations, the teachers' explanation must be precise, coherent, and mathematically based progressing towards formally based explanations in terms of how they talk about matter, form, process, and goal.

3.3.3 Criteria for acts of explaining

The effective explanations literature has emphasised examples and their representations as part of the acts of explaining; in addition, the notion of regulative acts has been highlighted from this literature. From the literature which viewed the explanation as a discursive practice, language has been emphasised as an important aspect used in the act of explaining. Achinstein (1983) defined the notion of explaining as an illocutionary act, a speech act; from the word act it can be concluded that the speech act is accompanied by gestures as part of the communication. Radford (2010a) noted that words and gestures are signs in their own right and thus semiotic means of explanation. The discussion which follows is very selective in the sense that it is not encompassing of the whole breadth of the literature that is available in the field in relation to each of the criteria in the acts of explaining. However, the selected papers discussed below do highlight the depth of the issues pertaining to examples, language, gestures, re-voicing, and regulation as acts of explaining. I discuss below, in more detail, the five criteria in the acts of explaining obtained from the literature thus far,[8] which are: 1) examples and their representations; 2) language; 3) gestures; 4) re-voicing; and 5) regulation, in that order.

3.3.3.1 Examples and their representations

The foregoing literature has emphasised the notion of examples and their representations as an important aspect of the acts of explaining. Bills, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky (2006) argue that examples are at the core of learners' mathematical experiences. The authors continue to argue that concepts as well as procedures are illustrated through examples. The meaning of mathematical terminology is elaborated through examples. Optimal support is made possible when attention is given to the careful selection, design, structure and sequencing of examples (Bills et al., 2006).

[8] The criteria in the acts of explaining are not limited to these, they extend in the methodology chapter.

The act of selecting good examples is a pedagogic decision informed by goals of the lesson and the teachers' knowledge. Because the example carries the knowledge, what Zaslavsky (2014) refers to as 'explanatory power' in and of itself is not the explanation without the matching explanatory talk (Adler & Venkat, 2014). The notion of representations presents another dimension, where they are not only part of the acts of explaining, because the representation also portrays the structure of the expression which is the form. In this way, the notion of representations fit in explanation as both matter and form, and in the acts of explaining as a representation of the example.

In other words, there is matter that makes up the example and form, which is the representation of the example whether spoken about or not in the discussion offering the explanation, as much as each example may require transformational activity to be performed on the example for the accomplishment of a particular goal. The process of unpacking the example may exhaust or may not do justice to what the example offers, this makes the constitutive criteria for explanation essential. The choosing of examples and their representation makes available what is possible to say in the explanation in terms of matter, form, process, and goal.

Zaslavsky (2014) enumerates three types of example use and those are: 1) spontaneous examples; 2) evoked example production; and 3) example provisioning. Spontaneous examples are randomly generated, try out examples. Evoked example production is the act of explicitly requiring learners to generate examples. Finally, example provisioning refers to worked out examples that are related to the learning goals. Zaslavsky goes on to clarify that example provisioning is where considerations are made in advance in the planning of the lesson and that their selection depends on a special kind of knowledge defined by Ball, Thames and Phelps (2008). Beyond enumerating types of examples used in teaching and learning, the author goes on to argue the principles that govern the selection and compilation of examples (Zaslavsky, 2010, 2014). On the act of selecting examples, Zaslavsky says the following:

> What is this an example of? In other words, the person who generates or selects it is able to articulate what property, principle, concept, or idea the specific example is a case of (Zaslavsky, 2014, p.2).

The extent and limits of what the example makes possible to learn – it's explanatory power – can be judged through the principles of selecting an example Zaslavsky offers in the above quote. These are indeed plausible principles to guide the selection of examples with ample explanatory power. On the structuring of examples, the author argues thus:

Example spaces are not just lists, but have internal structure in terms of how the elements in the space are interrelated. In my work, I consider an example space as the collection of examples one associates with a particular concept at a particular time or context. (Zaslavsky, 2014, p2).

The notion of examples space mentioned in the quote above emphasises a point earlier noted, namely that the collective set of examples used in the explanation can be compiled after lesson observation. In addition, the *internal structure* of the example space used in the lesson can then be examined. Mathematical examples have to be represented somehow, and the selected representation has structure. Leinhardt noted the importance of knowing the representational base in order to deliver an effective explanation.

Research has emphasised the importance of using and moving between different representations when teaching mathematical concepts (Wilkie, 2016). The strategy of using different or multiple representations and making connections among them assists learners to better understand mathematical ideas and explanations. The types of representations that are often reported and commonly used in the literature concerned with school algebra are: written/spoken words; symbols; tabular representation; geometric representation; and graphical representation. Zazkis (2016) in particular highlights that research examining contextualised problems is disjoint from research examining multiple representations in mathematics, and argues for a bridging of the two areas of research based on the commonality they have, namely that of understanding how one representation of a problem informs another. The diagramme below summarises the common types of representations used in mathematics education literature.

Verbal Representation *Spoken*	Tabular representation *Numerical*	Graphical representation *Visual*
Real life / imagined scenario *Contextual*	Algebraic representation *Analytical*	Geometric representation *Visual*

Figure 3:1 – multiple representations

The main skill emphasised in the literature as important to develop is that learners must be able to interpret, connect, and develop fluency in working with multiple representations (Zazkis, 2016). However, the main issue that I wish to highlight in connection with the explanation criteria is that multiple representations, which learners are expected to develop the agility for, have form/structure. In other words it is multiple forms/structures that learners must make sense of in order for them to interpret multiple representations correctly. The different forms/structures used to represent mathematical ideas provide diverse opportunities for learning and reasoning about the mathematics. I have already noted in the previous chapter that representation as a mathematical idea is the explanation and that representation in the selection stage is a pedagogic decision and thus the act of explaining.

3.3.3.2 Language

In the explanation, language is the main ingredient, in the sense that there is no way of conveying the explanation outside of language. The provision of a definition for a mathematical word/term is a commonly used method in the teaching process. Riccomini, Smith, Hughes, and Fries (2015) argued that "without the instructor first teaching basic understanding and facilitating fluency with vocabulary words, the purposeful and effective use of the language of mathematics will likely not occur" (p.238). Mathematical words such as variable, constant, coefficient, monomial, binomial, trinomial, polynomial, exponent, term represent abstract concepts, which require the use of language to render them accessible. By providing definitions to the mathematical vocabulary access to understanding the mathematical concept is made possible (Riccomini, Smith, Hughes, & Fries, 2015). Boulet (2007) in particular presented examples of how language is used to read and interpret notation, describe mathematical processes and define mathematical terms. The author argues that the language used to talk about the mathematics needs careful attention; for instance reading a number – 0,42 – as zero comma fourty-two, instead of fourty-two hundredths, does not assist learners to make sense of the mathematics (Boulet, 2007).

3.3.3.3 Gestures

I noted earlier in the chapter that Achinstein argued that explaining is a speech act, an illocution, which is the art of drawing a picture in a person's mind through speaking. Thus, gesticulation goes along with speaking, and particularly explaining. A taxonomy of gestures is presented and used by Roth (2001) in the review of educational research as an organising framework. There is a list of four types of gesture, that is, beat, deictic, iconic, and metaphoric. Beat gestures are interactive in nature, and their purpose, amongst others, is to regulate and coordinate speaking turns. Deictic gestures are pointing gestures, which are context-dependent, and prevalent in developmental stages for kids who are still in the process of acquiring language. Deictic terms include *here, there, that, this, I, you,* which require a pointing motion to indicate the appropriate referent (Roth, 2001). Iconic gestures, are concrete demonstrations which are visually similar to the objects they refer to e.g., a teacher drawing a circle in the air with her finger is an iconic gesture, because there is some resemblance. Metaphoric gestures also refer to images, as in the iconic, but the difference is that the images in the metaphoric gestures are abstract without a physical form. Gestures are a communicative technique used in the classroom to convey the explanation. In this sense, gestures could be considered an act of explaining.

3.3.3.4 Re-voicing

Re-voicing is at best the act of reuttering learners' inputs. In the interaction between teacher and learners, where the explanation is constructed, conversation is mainly coordinated through the act of re-voicing. Franke, Kazemi and Battey (2007) have enumerated a number of functions the act of re-voicing often serves and those include, rephrasing, repeating, expanding, and reporting what a learner said. The act of re-voicing provides the teacher with opportunities to clarify the learners' contribution and substitute the learner's contextualised, informal voice by rephrasing and putting it in the voice of the discipline. Thus, the most important role re-voicing plays according to Franke, Kazemi and Battey (2007) is "...an encouragement of students' developing mathematical voice" (p.233). The "mathematical voice" is the explanation that is ultimately produced as a result of having engaged in the act of re-voicing. Besides the supportive roles that re-voicing can play in the classroom and in the constitution of the explanation, there is a limitation about the act of re-voicing that literature has highlighted. Re-voicing can lead to learner disengagement in the conversation, if it is merely an uncritical repetition of learners' inputs without the necessary probing 'why' question that seeks a reason and justification for the input.

3.3.3.5 Regulation

The notion of regulating the nature of information provided in the explanation has been emphasised in the criteria from the education and mathematics education literature. The manner in which information is regulated is context specific and depends on the culture established in the classroom. Regulation is an important criterion of the acts of explaining because the quality of explanation depends on the structure of the classroom culture and how it facilitates effective communication of the explanation. Güven and Dede (2017), drawing substantially on Paul Cobb's work, distinguish between general classroom culture (governed by social norms) and mathematics classroom culture (governed by socio mathematical norms). Norms refer to shared expectations for behaviour. They argue that there are values and beliefs that are established in order to regulate mathematical content and behavior (Güven & Dede, 2017). Social norms are the kind of talk, which involves negotiating with leaners the context for learning e.g., one person talks at a time. These are rules for how teacher and learners engage each other and in this way the classroom culture is created. Social norms regulate who participates e.g., I want someone who has not been to the board or said anything. Socio-mathematical norms refer to the kinds of solutions that are acceptable and when a mathematical process is finished. Learners' beliefs and values about how to act as a member of a mathematics classroom are developed through socio-mathematical norms. Classroom interaction and communication is governed by these norms and the explanation is crafted in the interaction.

3.4 Conclusion

The philosophical literature has been useful in sense that it provides the distinction between explanation and explaining, and offers an analysis of the concept of explanation. The philosophical literature has also offered criteria for an explanation, which I took from Ruben's interpretation of Aristotle's four criteria of explanation: matter, form, motion originator, and goal. In this Chapter, I have re-described the four criteria of explanation as offered by Aristotle for use in the lessons concerned with school algebra. The philosophical literature emphasised the idea of learning the theory about that which you will explain. This was in the language of the previous chapter highlighted as the importance of teacher knowledge specifically SMK.

The notion of examples and their representations, language, speech and accompanying gestures was emphasised as actions humans perform to produce an explanation. The entire classroom interaction is regulated by some norms, both social and socio-mathematical. The discussion in this chapter and in the previous chapter illuminated that the acts of explaining are mainly pedagogic decisions and rely on the teacher's PCK.

Table 3-1: Tools obtained for explanation and acts of explaining

Explanation (SMK)	Acts of explaining	
	(PCK)	**General pedagogy**
• Matter • Form • Motion originator • Goal	• Examples and their representations • Language • Gestures • Re-voicing	• Regulation (not PCK)

From the philosophical literature, we see that there is no specification of the audience and that which needs to be explained because the criteria of explanation provided are decontextualised and applicable in any context, as Moravscik and Ruben argued. In mathematics education research, the audience is at the centre of the explanation. From the philosophical literature in the social sciences and literature in mathematics education, I have shown how the acts of explaining are context dependent. The audience determines the kinds of actions and decisions the teacher will make in terms of establishing what is to be learnt, goals, and how this is to be known.

Fuller elaborations of how I have adopted and developed a coding system for explanation and acts of explaining to use in the analysis of the data is provided in the methodology chapter. Methodologically, the philosophical literature has not only contributed by separating the noun *explanation* from the continuous verb *explaining,* but also some terminology has been outlined whose definition helps in segmenting the data into manageable units of analysis described as *that which needs to be explained*. More elaboration is found in the methodology chapter. In the next Chapter, I go into a discussion of algebra, since it is the content focus for the study.

CHAPTER 4 HISTORY AND DEVELOPMENT OF ALGEBRAIC EXPRESSION

4.1 Introduction

Providing an explanation in the teaching and learning of beginning algebra poses serious challenges for teachers and learners. In this chapter, I pay attention to 'recent' developments of algebraic expressions. A discussion on algebraic expressions is useful in this study, in the sense that they are the specific content which both teachers and the project focused. The topic itself has a long history, which began in antiquity. In this long history, I merely present a selective review of algebraic expressions literature by looking at three contexts, the first being their historical development, the second is the field of mathematics education, and lastly the textbook. The general purpose is to understand the ways in which algebraic notions are spoken about in different communities. My interest with historical account is the development of algebraic notions. Specifically, I am interested in how algebra came to be expressed in its modern symbolic form, and in the implications this has for explanation. Within mathematics education research, my attraction is bent towards certain aspects of the main debates concerning the teaching and learning of school algebra, specifically, the relationship between concepts and procedures as components of explanation. The purpose of looking at the textbook is to see how the algebraic concepts and procedures are defined. Within the broad area of algebraic expression, the distributive property was in focus in the WMCS PD, and in teachers' lessons. In addition, the four criteria of explanation obtained in the philosophical literature need to be contextualised for school algebra. Thus, my aim in the current chapter, is to particularise the descriptions obtained in philosophical literature to the context of algebraic expressions using three settings. I find the strategy of using Ruben's criteria of explanation as an organisational structure for this chapter useful to the extent that it adds to the understanding of Aristotle's criteria for an algebraic expressions mathematics education context.

4.2 History of algebra

It is befitting to reiterate that the problem I am investigating in this chapter is the notion of explanation. As such, my burden is to trace what occurred long ago with the view to orient and entrench the current study on its traditions. What we have come to know as explanations in algebra are necessarily an inheritance from our past. Mathematics, in particular, was not created in the spur of the moment, it took shape over a significant period. As time progressed, it was refashioned into its current form which we are most familiar. In my view, history is essential not only for its own sake, but to inform a composite approach to gain a good understanding of algebra broadly and in particular the notion of explanation. It is indeed a truism that it would be almost impossible making meaning of the present without the past. Cajori (1993), puts it that there are great lessons to learn from the past that will guide us for the current time:

> history constitutes a mirror of past and present conditions in mathematics, which can be made to bear on the notational problems now confronting mathematics. The successes and failures of the past will contribute to a more speedy solution of the notational problems of the present time (Cajori, 1993, p.1).

Indeed, a historical analysis provides us with referential understanding of the problems we face today. Granted that the present is an inheritance from the past, we are then connected to the past through our learning about mathematics and interaction with it in the present. What is important for me as I discuss the historical account is to understand the nature of these connections. By way of metaphor, trees have roots, stem and branches; in this application roots are important because that is where mathematics and its branches is anchored, by extension, explanation too. Above all, history brings us into contact with bigger questions of origin. Since it is difficult to pin down actual beginnings due to rigorous contestation of novelty, the narratives provided in the texts are filled with uncertainties. As it is difficult to decipher pictographic information of extinct and dead forms of communication. Nevertheless, it is a worthy expedition.

The word algebra comes from the title of a book that was written in Arabic language by Al-Khworizmi (780-850 AD) entitled: *al-kitab al-mukhtasar fi hisab al-jabr wa'l-muqabala* (the compendious book on calculation by completion and balancing). The word *al-jabr* in Arabic is understood to mean restoration. It is said that the book was later translated into Latin. If the above claim is true, then the word algebra as we have come to know it is simply a Latinised corruption of the Arabic word *al-jabr,* which means restoration or completion. Confrontation and restoration is the actual meaning of algebra, because the mathematical objects that early inventors dealt with were equations. And so, the branch of mathematics concerned with balancing and solving equations came to be known as algebra (Groza, 1968).

In terms of what is recorded, algebra has its roots in ancient Egypt and Babylon. For instance, there is a school of thought contesting that what we know as Greek mathematics was originally Egyptian mathematics. Proponents of this view posit that Greek was the language in which Egyptian mathematics was recorded (Groza, 1968). However, the Greeks did not only take the mathematical knowledge of the Egyptians and Babylonians, but also made their own original contribution informed by their own process of thinking and culture. For instance, where the demonstrative 'how' mathematics of the Egyptians was no longer enough, the Greeks wanted to develop propositional knowledge in terms of 'why'. And so, mathematics developed into a deductive system. Kvasz (2006) argues that Greek ways of knowing were centred on the close link between thought and sight, where the aim was to envision the result. Such that the problems we now consider as algebraic were geometrically expressed. This preoccupation with envisioning algebraic problems confined the growth of algebra and limited the structure of mathematical objects that were possible to engage with to squares and cubes. The foregoing discussion gives us a cursory understanding that is often left unacknowledged, that there is wider contribution to the earliest forms of algebra. Its inventive intellectual property is not entirely of hellenist originality. In the next section, I present an account of the development of the modern symbolic expression of algebra.

4.2.1 The history of algebra – the development of symbolic expression of algebra

Prior to modern symbolic expressions, earlier expression of algebra employed pictographic symbols which were exotic imaginaries and extremely metaphoric. These, the pictographic ways of writing, are sometimes referred to as rhetoric expression. It is recorded that the rhetoric expressions of algebraic notions were written in prose form. The prose was clustered into syncopated expressions of algebraic notions, which refers to the mixture of words and abbreviations. The latter ordering gave rise to what we may be most familiar with as modern mathematicians. What would develop from this are algebraic expressions without words, as words were completely abandoned in the early versions of modern symbolic expression. The significance of this historical account lies with potency to highlight omissions that recur in the reading and authoring of literature. In particular, there are three stages recorded in the development of algebra literature, but the pictographic stage tended to be omitted by many writers. On the other hand, pictographs are not a grammatical language, but pictorial and ideographic, and so cannot be considered to be a rhetorical expression of algebra.

Frutiger (1989) provides a historical account of ancient writing systems in the early Babylon and early Egyptian times as having been the origin of writing.

> Pictorial signs were certainly at the origin of all scripts that have come into existence through a natural course of development (p.113).

The Sumerian script first expressed ideas pictographically then progressed to the Cuneiform or wedge shaped writing (Frutiger, 1989). Egyptians used Hieroglyphic (written in monumental walls, wood or metal), Hieratic (cursive form of writing which sprang out of the hieroglyphic recorded on papyrus) and Demotic forms of writing. The Hieroglyphic and Hieratic writings did not necessarily evolve to Demotic writing. Instead, both Hieroglyphic and Hieratic forms of writing were in constant use. The Demotic writing was merely colloquial speech, which was later recorded as a more abbreviated form of the cursive Hieratic writing. I bring in Figure 4:1 taken from Frutiger (1989) as an illustration.

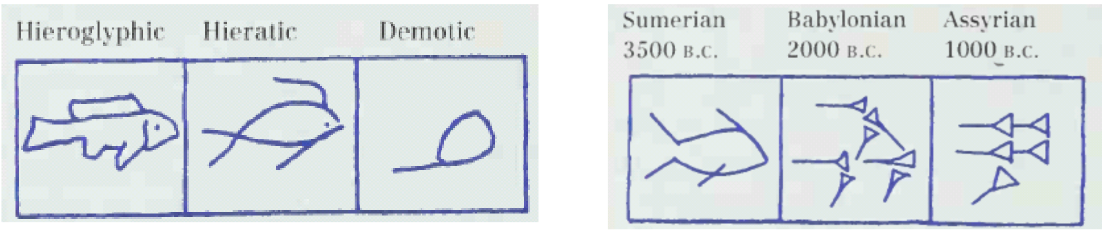


Figure 4:1 – Inscriptions taken from Frutiger (1989, p. 124)

The notion of symbolisation as a recent development that occurred roughly around 16th and 17th centuries is somewhat misleading. Forms of ancient writing used peculiar marks, which are pictographic symbols with extended meanings. In essence, there were four consecutive stages in the development of algebraic expressions. The pictographic way of writing is often referred to as the prehistoric era. This exclusion of pictographic symbols from modern history may be due to political conquer, but has debilitating effects on the epistemology of algebraic expression. While it may be true, that the Sumerian Cuneiform and Egyptian Hieroglyphics, hieratic and Demotic forms of writing are extinct and dead symbolic languages, it does not follow that contributions these languages made should be rendered extinct. The omission of instances of historical significance is an injustice that risks perpetuating epistemological fallacies. The inscriptions of both Cuneiform and hieroglyphics are not well understood, because there was no conservation of the past, and thus, current archaeological discoveries and the monumental inscriptions are not easily decipherable. These are necessarily given new meanings according to the understandings of the people. In my view, the activity of providing new meanings for the pictographic writings is better that rendering the contribution these languages made as extinct. I call the modern day attempts at reading the ancient pictographic ways of writing an act of giving them new meaning, because we may not know what supplementary description or customary way of communication was used to render the meaning of the pictures in so-called prehistoric drawings. This ‘not knowing’ is a reason good enough to search for more ways of making sense of the writings. Frutiger highlights that the difficulty we face today is that “the drawings have come down to us, but the speech, and with it the meaning of the signs, has not been directly handed down” (p. 112). This being the case, we are nonetheless still obligated, at the very least, to acknowledge early developments[9].

Trying to track changes that occurred over time may appear as a trivial endeavour, until we come to an appreciation that losing pictographs to words, subsequently words into modern symbols, has implications for retaining the very notion of explanation on which the current study hinges. The words that were used to provide rules and procedures for explanation of pictures are indeed lost, such that it seems whatever we use today as modern symbolic expressions have no real connection and continuity with the past, at least from the point of view of recognition. As such, in modern historical accounts of algebraic expression, the pictographic stage of algebra is omitted. It may well be that the omission is owed to a poor archiving system, but this does not justify complete negation, especially in the context of present day post-independence Africa. They may well signal something powerful about equal opportunity to make contributions into universal knowledge that we would be ill-advised to continue to overlook. I now turn to the rhetoric, syncopated and symbolic expressions of algebra.

It is said that Greek and Arabic algebra used the rhetoric expressions for some time until Diophantus began to introduce abbreviations to shorten the lengthy texts of writing. Accordingly, Diophantus (circa 250 CE) and Brahmagupta (circa 598-668 CE) are recorded as

[9] There are other ideographic languages that notably extend continuity from ancient to contemporary times which are not in focus in this aspect of historical pespectives e.g. Chinese writing.

the first to use symbols in the form of abbreviations in the syncopated stage of algebra (Cajori, 1993). Until then, algebra was written in words in the rhetorical stage. Thereafter, from Diophantus' days up until the 16th century, algebra was written both in words and symbols. With the progress of time, this would change significantly, towards total symbolisation.

Around the 1500CE, the need for more compact and efficient notation arose. This era signals the beginning of the development of symbolic algebra. Often two factors are singled out as major contributions to the standardisation of mathematical symbols, one being the travelling of scholars, which allowed for the transpollination of ideas, as well as the invention of the printing press. This begins to show how dissemination of ideas occurred, first through contact with other persons who themselves had their own ideas, and latter through print.

Cajori highlights, for instance, that the European renaissance of the 16th century pioneered the translation of texts from rhetoric to syncopated. This is the record we have on how symbolic forms of expressing algebraic notions came about. Translation from words to symbols became a preoccupation of each mathematician going forward to use symbols to denote quantities, relations, and operations. The process of standardisation was fraught with personal struggles for influence, recognition, and ascendency into power. In some ways, contestations of novelty eventuated dissimilarities. To date, some of the mathematical concepts such as similarity and congruency have different symbols in different countries. In this light, it can be argued that the standardisation process was not a comprehensively successful project. Cajori further reveals that, prior to total displacement of words in the 18th century, Thomas Hobbes, the philosopher, is said to have protested against the symbolisation of mathematics as shown in the quote below:

> Symbols though they shorten the writing, yet they do not make the reader understand it sooner than if it were written in words. For the conception of the lines and figures [...] must proceed from words either spoken or thought upon. So that there is a double labour of the mind, one to reduce your symbols to words, which are also symbols, another to attend to the ideas which they signify. Besides, if you but consider how none of the ancients never used any of them in their published demonstrations of geometry, nor in their books of arithmetic [...] you will not, I think, for the future be so much in love with them... (Cajori, Vol. 1, p.427)

The advantage of symbols, as Hobbes points out here, is that writing is shortened, however the subsequent challenge is that words must be used to decode the symbol, because the symbol is a concrete representation of abstract notions. The power of words is that the double labour of interpretation Hobbes points to here is removed. Hobbes foresaw the difficulty we are now facing of an unending search for the correct words for explanations of algebraic symbols. In this case it would appear that apparent advancement has been retrogressive, obfuscating mathematical concepts. From what is recorded by Cajori, we note that Hobbes seemed to be arguing against proposals for knowledge advancements that may hinder the ease of comprehension, but for a preservation of words while developing the symbols. Certainly, narrative and explanatory forms of eloquence, which were useful for talking about abstract notions, have been lost and replaced with signs representing those abstract ideas. For learners in the mathematics classroom, the symbol is seen externally through the eye, and assumptions are made to decode the symbol.

The process that led us to reach the modern symbolisation that algebra is known for today was a gradual slow, process, although Hobbes and his reader point to some of the negative aspects that emerged with modern symbolisation. The development of symbolic algebra came with many advantages. One being that mathematical objects began to exist through the symbolic representation, albeit in their abstract form. The quest to recognise pictography, should not be seen as an attempt to do away with symbolic objects. Instead, it should be acknowledged that symbolic form of algebraic objects could feasibly take up expressions other than words.

The symbols are numerals which stand for number (0123456789); letters which stand for unknowns, variables, coefficients and constants; operation signs of addition, subtraction, multiplication, and exponentiation; signs to denote relations like equality and inequality, to mention just a few. There is a particular way of putting together the notation such that it forms structures which can be operated on using specific rules. Figure 4:2 below provides an example of an algebraic equation expressed in rhetoric, syncopated, and symbolic forms.

Rhetoric expression	Syncopated expression	Symbolic expression
Three multiplied by the cube of the unknown number from which six multiplied by the square of the unknown number is subtracted is equal to the sum of five and the product of four and the unknown number.	3 Cu m 6 Sq ae 4 no p 5 (3 cubes minus 6 squares equals 4 numbers plus 5)	$3x^3 - 6x^2 = 4x + 5$

Figure 4:2 – An example of the three algebraic expressions (Groza, 1968, p.251)

In the progression from the rhetoric expression to the symbolic expression, what has been lost is the way in which algebra is spoken. Words were lost and this had repercussions for explanation and acts of explaining algebra in the classroom. It would have been ideal for teachers to know the relationship between rhetoric and modern symbolic expressions of algebra simply because explanation relies on talk and the rhetoric expression in terms of offering quality explanations would have been more advantageous.

In the following section, I structure the discussion of the development of algebra using the four constitutive criteria of an explanation as outlined in the previous chapter by Ruben (1992).

4.2.2 The history of algebra – development of the constitutive elements of algebraic expressions

The story of algebra is not only about a progression from words to symbols, it is in fact a story of number to algebra. The history of algebra has its roots in the development of number or algebra is a result of the development of number. The four criteria of explanation that I use to organise the discussion in this section are: 1) matter; 2) form; 3) process; and 4) goal. The notions that I discuss under matter are number, letters, and relational signs. Under form I discuss the structure of algebraic expressions. Within the criterion of process I discuss operation signs (symbols for operating on number like addition, subtraction, multiplication, division and properties of number) and basic real number properties. I then mentioned briefly what the goal of algebra used to be. To end the historical section, I summarise before proceeding to the discussion of literature focused on mathematics education.

4.2.2.1 Matter: Number and its symbolisation, letters and relational sign

Algebra began to have its independence from geometry through the development of the numerals that we now use. The numerals that we now use (0123456789) have their origin in India, where Brahmagupta is the mathematician recorded to have used these before the Arabic mathematicians. Consequently, the numerals then, come to be to be called Hindu-Arabic numerals, because it is through Arabic mathematicians' work that they became published and known (Cajori, 1993).

Each ancient civilisation and kingdom had their own cultural artefacts and tools that they used to denote number. The Babylonians had their own notation for number, which used wedge shaped characters to denote the notion of number. It is recorded that two scales of number were used by the Sumerians, namely the decimal (base 10) and sexagesimal (base 60) scales. Numeral symbols in the Sumerian clay tablets were denoted using circles and half lunar signs, and then progressed to use the wedge shaped characters as shown in the third row of Table 4-1 below. There are three forms of Egyptian numerals, that is, the hieroglyphic, hieratic, and demotic. There was evidence of counting based on scales 5, 10, 12 and 60 in Egypt. The Greek alphabets were used as numerals.

Table 4-1: Development of the numerals

Egyptian	𓏺	𓎆	𓍢	𓆼	𓂭	𓆐	𓁨
Cuneiform	𒁹	𒌋	𒁹𒈠				
Greek	I	Δ	H	X	M		
Hindu	१	१०	१००	१०००			
Hindu-Arabic	1	10	100	1000	10 000	100 000	1000 000

The notion of negative number was still rejected by ancient mathematicians, where they only accepted positive solutions, because the solution was a geometric picture, and so only positive values were true. Also, the notion of zero was pivotal to the development of algebra.

The sets that make up the real numbers are natural numbers (used for counting); whole numbers (natural numbers plus zero); integers (natural numbers, zero, and the negative of natural numbers); rational numbers (all numbers that can be expresses as the quotient of integers, terminating or repeating decimals); and irrational numbers (non-terminating decimals and non-repeating decimal digits, have to be approximated). Therefore, the set of real numbers is the set of numbers that are either rational or irrational. On the number line, real numbers increase from left to right. Diophantus introduced the use of alphabets to denote unknown numbers. Viete (1540-1603 CE) introduced the signs for numbers that are constants and variables. Viete again is credited with the introduction of coefficients as letters and was the one who used capital vowels A, E, I, O, U; for unknowns. Descartes (1596-1650 CE) used the first five letters (a, b, c, d, e) instead of vowels for coefficients, and the letters x, y, z for variables/unknowns (Cajori, 1993).

I decided from the symbols used to represent relations to discuss the equal sign, because of the interesting journey it went through, and also because it still poses problems for learners beginning algebra. The symbol that we now use for equality, =, once had different meanings among different mathematicians and denoted other concepts either than equality. The symbol = was once used to signify arithmetic difference, addition and subtraction as in $\pm$, separatrix in decimal fractions, once used to designate parallel lines. Apart from the different meaning the symbol once signified for the notion of equality, competing symbols were once used to denote equality to mention a few the square bracket [, vertical parallel lines ||, and a single vertical line |.

The history of the standardisation of the equality sign is a story of how it survived competition with other symbols and then fully established itself. Before the final symbols suggested by Recorde took hold in Europe, the rhetoric expressions of equality were: aequales, aequantur, esgale, faciunt, ghelijck, gleich, that is, European languages. And the syncopated expression of equality was: aeq. Descartes used the mark ∝ as a symbol of equality, which was suggested by æ in aequales, a union of the two letters reversed. Recorde used the mark = as a symbol for equality. Cajori (1993) argues that the use of Recorde's symbol of equality by great names, Newton and Leibniz, in the 18th century supported its general adoption. The period of standardisation demonstrates that the symbol has no original historical connection, our learning and understanding of it is through acquaintance and familiarisation.

The signs are put together under certain mathematical principles to form different structures of mathematical objects. I now turn to this discussion.

4.2.2.2 Form: Structure

Algebra begins with the attempt to find the unknown on which a certain operation was performed and a given result obtained. The actual objects of algebra that were dealt with over the 4000 year period were equations until the end of the 19th century. The forms of objects ancient people dealt with were equations to the third degree, only because of the geometric conceptualisation of algebra. Kvasz (2006) provides an account of how Al-Khowarzmi offers a systematic study of methods for solving equations in the book Al-Khowarizmi wrote. Kvasz (2006) mentions that in the first part of the book, which Al-Khowarzmi wrote, is a presentation of procedures for solving six types of equations; $ax^2 = bx$, $ax^2 = c$, $bx = c$, $ax^2 + c = bx$, $ax^2 + c = bx$, and $bx + c = ax^2$.

Groza (1968) records that in the second part of the book, Al-Khowarizmi presented geometric justifications for the equations. This is because geometry was the foundation of Babylonian, Egyptian and Greek mathematics. Furthermore, Kvasz (2006) mentions that Al-Khowarismi was a student of Greek, and so might have included the second part of the book in order to appeal to the Greek way of thinking. The following quote from Kvasz highlights this fact.

> For Euclid the problem that we today understand as a quadratic equation $x^2 + bx = C$, and solve using the well-known formula, was a geometric problem. The task was to find a segment x such that a square having it as a side, together with a rectangle having this segment as one side and b as the other, together have an area equal to the area of a given figure C. Euclid (VI, problem 28) shows us how to construct the segment x. But even though Euclid was able to provide nice constructions for quadratic equations, the geometric approach to such problems has a fundamental limitation. When we interpret x^2 as the area of a square, then x^3 is the volume of a cube, and we must stop there, because the higher powers have no counterparts in three-dimensional space. This was why the Greeks, confined to geometry, were able to capture only a small fragment of the world of algebraic equations, a fragment that may have been too limited to give rise to an independent mathematical discipline. This may be why the world of algebra remained hidden from the Greeks (Kvasz, 2006, p.290).

There is operational activity performed on the different structures of mathematical object. The next discussion is focused on operational signs and certain properties of number.

4.2.2.3 Process: Operations on number

I present the history for the notation of the four operation signs (modern$+$, $-$, $\div$, $\times$) as recorded by Cajori, and also as a progression from the pictographic symbols for addition and subtraction, rhetoric, syncopated to the modern symbol. Cajori records that the addition and subtraction were denoted as feet walking forward, and feet walking away, respectively, in the hieratic writing of Ahmes, later translated into hieroglyphic.

Fruitiger noted earlier argued that the pictures have come down to us but the accompanying speech and explanation has been lost. Nonetheless, the notion of feet to depict the concepts of addition and subtraction is an interesting one, and worth looking at. It demonstrates that the pictorial and rhetorical forms of communication that have been lost could have served a complementary role in the explanation of mathematical concepts.

The illustrative intention of feet is that they take steps, and thus cover distance in a particular direction. In the forward direction the result obtained from the movement or steps taken is addition and in the reverse direction the operation is subtraction. However, the backward direction for the Egyptians is the forward direction for us, and the forward direction is the backward direction, because the Hieroglyphics was written and read from right to left, while Greek is written and read from left to right.

The notion of number line addition and subtraction, shown in Figure 4:3 below, seems to be derived exactly from the ideas expressed by feet in the Hieroglyphics. The number line is a teaching aid for the notions of addition and subtraction, between two points there is a distance of unit length. The drawing below marks the positive integers, however, all real numbers are included in the number line.

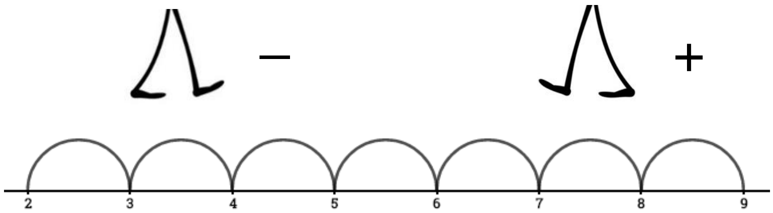


Figure 4:3 Pictographic depictions for addition and subtraction

The modern addition or subtraction signs are meaningless symbols compared to the ideogram from which they were derived. The power of the ideogram (feet) is that, the ideas it conveys are self-contained in the symbol, in the sense that its explanatory speech and gestures are not divorced from its function (taking steps in forward or backward direction). Nevertheless, the ideas expressed by the picture of feet are still in use in the teaching and learning of the operations of addition and subtraction of numeric expressions as shown by the semicircle curves, representing steps taken, on the number line.

In the syncopated stage of algebra, the letters *p* from the word plus and *m* from the word minus were used to denote the notions of addition and subtraction. The notions of addition were symbolised in the translation of Arabic mathematics to the Latin language. The Latin word meaning *and* is *et* and when written rapidly it looks like $+$. Cajori notes that the German mathematician Widman used the signs $+$ and $-$ to denote addition and subtraction. In the 17^{th} century, the signs received general use and adoption in Europe and America. Unlike addition and subtraction, multiplication and division both have a number of different ways of representing them in modern symbolic algebra.

In the rhetoric stage, words were used to talk about the notion of multiplication. Cajori narrates that factors were placed next to each other in a notation we use in modern symbolic algebra. In the syncopated stage, the letter M or m was used to denote multiplication, and the letter D or d was used to denote the notion of division. The symbol which survived competition and was later adopted to denote the concept of multiplication is called the St. Andrew's cross $\times$. Leibniz argued that the mark for multiplication was not fit for use as it resembled x, a letter used to label the unknown, and instead proposed the dot, $9 \cdot 5$ to denote multiplication. Both notations are now used in modern symbolic algebra. Important to note in this process is that the act of finding symbols no longer had significance and relevance to people's bodies and objects of familiarity. The process was not about the symbol carrying its meaning with it as it were in the hieroglyphic pictographs, the aim was now to have a symbol to function as an abstract entity.

Diophantus used the word *μορὶον* to denote the notion of division. Cajori reports that in the Bakhali arithmetic the word *bhâ,* which is an abbreviation of the *bhâga* meaning *part*, was used in operations that involved division. Indeed, when you divide you are partitioning. The fractional notation where the divisor is written beneath the dividend came from Al-Hassar, an Arabic author in the 12^{th} century. Al-Hassar is recorded by Cajori to have given the following instruction regarding the writing of division as fractions.

> Write the denominators below a [horizontal] line and over each of them the parts belonging to it; for example, if you are told to write three-fifths ..., write thus, $\frac{3}{5}$ (Cajori, 1993, p.269).

The waxing lunar sign was used by Michael Stifel to perform short or long division e.g.: 8)24. Rahn in 1659 introduced the sign ÷ of division which was later adopted by English writers. There is no comment in Cajori about the division sign, but one can infer from Al-Hassar's instruction above that the dots stand for the dividend and divisor and thus the division sign is a derivation from the instruction Al-Hassar provided. All three signs, the fractional, waxing lunar and the ÷ are symbols used for division to date.

What is becoming clear in the symbolisation stage is that some of the inventors of the modern symbols provided explanations to justify the symbol. The explanations provided conceptual clarification for some, whereas for others, the symbols were merely symbols for their own sake meant to be taken and adopted as mathematical convention with no conceptual explanations attached to them. On the other hand, symbols in the scripts of the Sumerians and Egyptians had pictographic and functional meanings, which were important for understanding and providing explanations for the concepts.

The order of operations rules existed in the words the people used to talk about mathematics before the symbolic notation. Multiplication and division have precedence over addition and subtraction. The BODMAS (Brackets of division multiplication addition and subtraction) mnemonic was formalised in the 19^{th} century, with the growth of the textbook industry. Operation with numbers leads mathematicians to observe certain properties of number namely the commutative and associative properties, the distributive property of multiplication over addition, the identity property for both addition and multiplication, and the inverse property for both addition and multiplication. I am interested in the distributive property. The discussion continues in this direction below.

4.2.2.3.1 Basic real number properties

In the investigation of the ways of formalising number, mathematicians would notice things that will always hold and those things would be given the name "property". These properties, once noticed and formalised, were critical and foundational in the growth and extension of the discipline. Properties of number that beginning algebra learners are introduced to are the associative property, the commutative property, and the distributive property. The associative and commutative properties are principles which apply only in the case of addition and multiplication. The associative law states that $a + b + c$ is the same as $a + (b + c)$ or as $(a + b) + c$. For multiplication, the associative property of number states that abc is the same as $(ab)c$ or $a(bc)$. The commutative property of number states that $a + b$ can be written as $b + a$; similarly for multiplication ab as ba. In other words, the order of operation does not alter the numerical value. As already noted, the distributive property was in focus in the PD and in the teachers' practices. In this section, I discuss briefly the distributive property.

The distributive property and other properties are the foundation of algebra. The formal name for the property is "distribution of multiplication over addition". The distributive property which is the focus of this study is sitting in the discussion on arithmetic, because it is a generalised property of number, which we express algebraically. The distributive property works for any set of real numbers. In order for one to know how to transform, rewrite and factorise an algebraic expression, one has to know the distributive property. The distributive property is one of the properties that enable us to do algebra. This is the form in which the distributive property is stated: $a(b + c) = ab + ac$. It states that the value obtained by multiplying a number, a, by the sum of two other numbers $(b + c)$ is the same as that obtained by multiplying the number by each addend $(b + c)$. The constituents: a, multiplicand, and the multiplier b, and c are any real numbers.

FOIL (First, Outside, Inside, Last) is a short-cut method for the Distributive Property. The FOIL method only works for a product of two binomials, and so it is a mnemonic used to remember the process of multiplying two binomials. FOIL method is simply a trick, when learners understand the distributive property then they will understand why FOIL method works. Below, I mention a few goals that algebra was used for according to the needs of people in antiquity which are similar our needs today.

4.2.2.4 Goal

Historically, it is recorded that the objectives for engaging in algebraic activity were for application in, and response to, everyday life. Algebra developed as a response to practical needs in agriculture, business and industry. In ancient Egypt, algebra had everything to do with the building of pyramids, land surveying, volume measurement, transportation, and astronomical calculations (Groza, 1968). The relevance of this is that the activities people in antiquity used algebra for are still the same activities we engage in today. Architecture, astronomy, business etc. are all studies in higher education today, which require the knowledge of algebra. The difference is in the fact that algebra was used as a means to the end, while today, it is both a means and an end in itself in studies concerned with learning mathematics for its own sake.

4.2.3 The history of algebra – Summary

In this section of the literature review, I have presented an historic account for the development of algebra as a branch of mathematics. What emerged is that algebra has progressed from pictographic symbols, rhetoric, syncopated to the modern symbolic expression. I have argued in this section that the pictographs and words that were used is what has been lost in the explanation of algebraic notions. A preservation of the correct interpretation of the pictographic meanings as well as the words could have assisted in the production of quality explanations and acts of explaining. What also appeared in the discussion above is that algebra is a study of number, relations between numbers and ways of operating with number using specific properties of number. The importance of understanding the properties of number has been emphasised as necessary to do algebra. The understanding of symbols and their meaning is of utmost importance for learners to be successful in learning algebra in its modern symbolic form. Mathematics education, which I now turn to, takes up and deepens an exploration into the subject of teaching and learning of algebra in its modern symbolic form.

4.3 Mathematics education literature

Mathematics education researchers are concerned to define school algebra, and are interested in the historical development of algebraic notions and the implications it has for teaching and learning school algebra. I begin this section with a short discussion of what is school algebra. I then discuss the historical account of algebraic notions from the perspective of mathematics education researchers, moving on to discuss mathematics education research according to the four criteria of explanation.

4.3.1 What is school algebra?

Algebra is an investigation of number systems and their operations. Vermeulen (2007) presents four perspectives for thinking about school algebra. The first is *algebra as generalised arithmetic*. Problems in the transition from arithmetic to algebra have been noted, namely that arithmetic deals with known number, while algebra deals with unknown letters and variables; letters in arithmetic if used at all are usually abbreviations or units, while in algebra letters are usually stand-ins for variables; and the interpretation of the equality sign in arithmetic is simply announcing a result, while in algebra, it is stating equivalence (Herscovics & Linchevski, 1994; Kieran, 1981; Watson, 2009). The second is *algebra as a problem solving tool* used to model situations, represent real life situations, and translate word problems into algebraic expressions, and graphs. The third is *algebra as the study of relations*. Algebra is used as a language to study and talk about relations to list but a few: growing exponentially or logarithmically, direct or inverse proportion, relations approaching asymptotically, etc. (Vermeulen, 2007). The last is *algebra as the study of structure*. Recognition of mathematical structures in algebraic expressions is a sign of reification, for example seeing 8+g as an object instead of a computational process (Sfard, 1991). From these ways of thinking about school algebra, letters in symbolic algebra are the constitutive elements of expression. The study of relations among constitutive elements is the study of structure and its properties.

Kieran (2007) proposed algebra as an *activity*, and synthesises the activities of school algebra into three elements, namely generational, transformational and global/meta-level, all three discussed below.

The generational activity involves "the forming of the expressions and equations which are the objects of algebra" (Kieran, 2007, p.713). This includes the equations containing unknowns, which are expressions of generality arising from geometric and numeric patterns. This includes reading and understanding words so as to generate the equation or expression. The focus of generational activities is the representation of situations. In other words, the generational activities provide structure and form for situations. The author argues that most of the meaning making in algebra is considered to be situated in generational activity (Kieran, 2007).

Transformational activity involves “collecting like terms, factoring, expanding, substituting, adding and multiplying polynomial expressions, and exponentiation with polynomials, solving equations, simplifying expressions, working with equivalent expression and equations and so on” (Kieran, 2007, p.714). A great deal in this activity involves changing the expression to maintain equivalence. Within the South African mathematics curriculum for high schools, both transformational and generational activities are privileged. In fact, most of the work done by the teachers participating in this study is within the transformational activity, and so is an important consideration here.

In global/meta-level activities algebra is used as a tool. These include problem-solving and modelling. This is the weaker component as this activity is not exclusive to algebra and so both the generational and global/meta-level are backgrounded, and the focus in this study is on transformational activities. What Kieran highlights in this discussion is that the explanation of algebraic notions in school is an explanation of mainly three types of activities. The relevance for this study is that the three types of activities Kieran outlines describe some of the criteria for explanation especially form and process as will be shown in this section.

4.3.2 Historical account in mathematics education

Researchers in mathematics education are concerned to know the order in which the mathematical content emerged during its development. The sequence in the historical development of mathematical ideas is then compared to the sequence of acquisition of concepts in modern day educational settings. Some mathematics educators have argued that there are parallels in the historical path and the learning process of mathematical ideas (Karts & Barton, 2007; Sfard and Linchevski, 1994). The conclusion has serious recommendations for teaching and learning such that mathematics education researchers endorse that learners should be made to rediscover and reinvent mathematical ideas. The recommendations have serious implications for teacher knowledge about and of the historical development of the discipline, that the teacher knows how mathematical ideas evolved and will be able to unpack and design learning activities in accordance with the historical development (Caspi & Sfard, 2012; Harper, 1987).

On the contrary, Fried (2001, 2018) argues that the treatment of the historical development of mathematical ideas in empirical mathematics education research is problematic. He argues that the history of mathematics is something to be studied, instead of being used as a tool. Fried outlines a few justifications that are used to motivate for the inclusion/ use of history of mathematics in mathematics education: 1. The history of maths humanises mathematics because it tells us about how diverse cultures contributed to the development of the discipline; 2. the history of maths makes mathematics more interesting, understandable and approachable; and 3. the history of mathematics provides insights into the concepts, problems and problem-solving (Fried, 2001, 2018).

Fried highlights that the teacher faces the *dilemma of relevance,* where a selective approach to history of mathematics is used to pick what aligns with the modern mathematics curriculum and throw away what does not. Fried strongly argues that the history of mathematics education cannot coexist with mathematics education without being ahistorical. In this way, the history of mathematics is used as a tool instead of studied in the sense that we measure and provide direction to the past through the present (Fried, 2001). To solve the dilemma, Fried argues that there are three choices that teachers can associate themselves with 1) to remain committed to modern mathematics and modern techniques; 2) risk being unhistorical in approach or trivialise history; and 3) take a genuinely historical approach and risk spending time on irrelevant things to the mathematics curriculum the teacher has to teach. (Fried, 2001)

What Fried is highlighting here captures the dilemma I faced as I was engaged in the narrative of historical development in the previous section. Nevertheless, my interest in the historical development is the conceptual insight it provides for the explanation of some of the modern mathematical notions. I have engaged in a selective reading, where my interest, Ruben's four criteria of explanation was used to structure my approach to the historical account, instead of studying history for its own sake.

I discuss briefly two papers reporting on the historical development of algebraic notions within the context of mathematics education.

It is generally agreed that there are three stages in the development of algebraic expressions, the rhetoric, syncopated and symbolic stage (Katz & Barton, 2007; Sfard & Linchevski, 1994). Sfard and Linchevski (1994) provide a strong criticism of rhetoric algebra by arguing that words are not manipulable as symbols are. They argue that algebraic objects began to possess object like quality because of the symbols. Alongside these stages of development, Katz and Barton (2007) record that there were four conceptual stages, which are: geometric; static equation solving; function; and abstract. I discuss Capsi and Sfard's (2012) account in more detail, because it focuses on this idea.

It is worth noting that the earlier work Sfard and Linchevski (1994) conducted pertaining to the learning of school algebra had a different reseach focus compared to the most recent work of Caspi and Sfard (2012). The earlier work had a cognitive theoretical orientation, while the recent work has shifted to focus on the discourse practices learners engage in as they learn school algebra.

Caspi and Sfard (2012) are interested in the development of algebraic thinking in schools. To investigate algebraic thinking, they first analyse the history and the development of algebraic thinking; they then study how Grade 5 and Grade 7 learners progress from informal to formal talk about numerical processes as they encounter formal school algebraic discourse. From their analysis of the history, they mention that school algebra was shaped through a process called formalisation, which occurs in three stages. The first stage is disambiguation – a process of preventing multiple meanings for the same expression. The second is standardisation, which is a process of making sure that the speakers follow the same rules of communication. Lastly compression is a process of turning lengthy narratives into short, concise, easily manipulable symbolic statements. The previous section has already highlighted for us that the formalisation processes in the enlightenment era were not smooth, but were an activity filled with power struggles for influence. Radford (2012) on a different but related matter, lamenting the separation of the individual from the socio-cultural world, highlights that the enlightenment era was marked by an epistemological shift, where universal values were formulated within the power struggles of capitalist entrepreneurs. Most of the focus when it comes to school algebra is on the symbolic expression, but the symbolic expression is only one form of expression, and the last phase in what Sfard and Caspi term the formalisation process.

Caspi and Sfard (2012) argue that anything expressed symbolically has its counter-symbolic expression, especially that symbolic expression is as a recent development of the modern era. In the account they provide for the historic development of algebraic expression they point out that the mathematical objects dealt with were fixed value algebra and variable value algebra. Fixed-value algebra deals with constant numerical values either as unknown, known, given or sought. With variable algebra, the concern is to describe and represent a process of change or movement. Their main concern is to understand the process of algebraic thinking. The authors argue that the historic path mirrors the present. I return to this paper in the discussion focused on process.

Hitherto, I have discussed what school algebra is and the approach mathematics education researchers take towards the history of mathematics.

I now turn to discuss studies that are concerned with algebraic elements that constitute algebraic structure, processes, and the distributive property. I have specifically organised the discussion in mathematics education research according to the four criteria of explanation obtained in the previous chapter in order to develop the detail of the criteria according to their descriptions as formulated by mathematics education researchers for school algebra.

4.3.3 Constitutive elements

Mathematics education researchers have identified some problematic areas in the teaching and learning of algebra. These include the constitutive elements, such as variables and signs. I discuss these below.

4.3.3.1 Matter: Letters as variable, placeholders, unknowns, and generalized number

Algebraic letters have several uses, which are important to distinguish. The ways in which the word variable is used in mathematics classrooms is such that it refers to any letter appearing in an algebraic expression. Researchers argue that the role of the variable presents a critical point in the transition from arithmetic to algebra. In particular, Radford (2000) argues that variables expressed in the form of boxes and empty spaces in numeric expressions are a ‘transitional language’ from arithemetic to algebra. McNeil, Weinberg, Hattikudur, Stephens, Asquith, Knuth, and Alibali (2010) studied learners’ misinterpretation of literal symbols in algebraic expressions. They argue that learners’ misconception in interpreting literal symbols might be activated and reinforced by the exposure they receive from interacting with textbook material and classroom examples (McNeil et al., 2010).

Solving an arithmetic problem means to carry out one or more operations with specific data for reaching a solution (almost always unique), while this is not always the case in algebraic expressions (Malisani & Spagnolo, 2009). For example, the arithmetic expression: $4 \times 3 + 12$ can be simplified to the answer 24, whereas in algebra the expression $4x + 12$ can be simplified to $4(x + 3)$, which is not a specific numeric answer.

Likewise, Stacey and McGregor (2009) agree that learners struggle with the concept of variable. They highlight that learners are unable to see and interpret algebraic letters as generalised number or unknown. Instead, learners ignore the letter (e.g. $4x + 12$ will be treated as $4 + 12$ equals to 16), treat it as shorthand name or substitute a numerical value for it (e.g. $4e + 12$ will be treated as $4 \times 5 + 12$ because of position five in the alphabet system) (Stacey & Vincent, 2009). They present evidence for the origins of learner misinterpretations of the algebraic notation. The message they promote is for teachers to know how learners interpret and use letters in order to provide explanations that will assist learners.

A more comprehensive discussion of a variable is offered by Ely and Adams (2012) and Usiskin (1999). Ely and Adams point out that the idea of variable evolved from the early use of letters as unknowns which marked the beginning of symbolic algebra. I have already shown briefly in the historical section above the evolution path for the notion of variable from Diophantus, Viete and ultimately Descartes. They argue that the historical shifts in the conception and use of letters have important lesson for the teaching and learning of symbolic algebra as we have it today. The conceptions of the notion of variable and its different uses is first provided by Usiskin in the chapter conceptions of school algebra, and these correspond with the ways in which Ely and Adam, who have shown that letters in algebra have multiple uses which include placeholder, variable, unknown, and generalised number (Ely & Adams, 2012). Usiskin (1999) acknowledges that the concept of variable is manifold and illuminates the point with specific examples.

The first case to which the idea of a variable is applied given by Usiskin is: $A = lb$ a formula for calculating area where A, l and b stand for quantities area, length and breadth, respectively and are all unknown (Usiskin, 1999), while $80 = 8x$ is a linear equation with an unknown number, a determinate quantity. The value of the letter x is unique and is equal to 10.

The equation $y = mx$ is a linear function, where x is an independent variable of a function, while the letter y is a dependent variable, and m a parameter. The letter x in this expression represents a range of values and so the value is not particular like the unknown. Earlier algebra was thought of as a study of relations. In the expression $y = mx$, the two sets of values, x and y, stand in a particular relation to each other. The letter m is an indeterminate coefficient that represents all cases of a straight line, once a numerical value is substituted for it then it becomes a specific case of a straight line. And so m is a placeholder rather than a variable.
In the trigonometric identity $sinx = cosx \cdot tanx, x$ is an independent variable of the function. While in the following scenario x is a generalised number for the arithmetic pattern: $T_x = 5x - 7$. Therefore, from this discussion we can see how the concept of variable has different meanings in different contexts and demands that the concept itself is explained according to the context in which it is expressed.

Symbol sense has been identified as problematic in the teaching and learning of algebra (Arcavi, 1994). Signs serve to record mathematical knowledge and concepts. The work of education, within the symbolic expression of algebra, is concerned not only with teaching algebraic thinking, but also with the meaning of algebraic symbols. In a nutshell, the elements that compose the algebraic expression are identifiable in the expression by breaking it into individual elements, which are coefficients, variables, unknowns, constants, and operation signs.

Research reveals that a great number of important difficulties encountered in the learning of algebra are related to learners' understanding of the meaning of signs and the syntax of the algebraic language (Radford & Puig, 2007). It is important that signs and symbols become transparent, hence the need to provide explanations. Radford (2010b) suggests a conceptualisation of learning algebraic symbols as a process of objectification, which is a sophisticated process of educating the eye to perceive mathematical entities intellectually. Radford argues that there are many semiotic ways of expressing algebraic thinking other than alphanumeric symbols. Radford (2010) argues that gestures, words, as well as alphanumeric symbolism are all part of the semiotic system of algebra.

The discussion on symbol sense is inseparable from a discussion on structure sense, because symbols are the constituent elements that make up the structure of the algebraic object. I now turn to a discussion on form or structure taken by the symbols.

4.3.3.2 Structure sense

The term *structure sense* was first used by Linchevski and Livneh (1999). They observed three dominant errors that Grade Six in Israel committed when presented with similar structures in arithmetic and algebraic representations. They observed that there was a tendency to detach a term form its designated operation; a misunderstanding of the order of operations and the jumping off with the posterior operation (Linchevski & Livneh, 1999). The authors argue that learners must be exposed to activities and instruction that will enable the development of structure sense through activities that require creative and flexible use of equivalent structure of algebraic expressions. They go on to state that:

> Instruction should promote the search for decomposition and recomposition of expressions and guarantee that the mental gymnastics needed in manipulating expressions makes sense (Linchevski & Livneh, 1999, p. 191).

In the above quote, the importance of the constitutive elements, which make up the structure has been highlighted through the words ‘decomposition’ and ‘recomposition’. Hoch and Dreyfus (2004) specifically studied how young learners struggle with structure of arithmetic expressions in their learning of early algebra. The authors provide definitions of structure sense, which are derived from interviews they held with mathematics education researchers (Hoch & Dreyfus, 2004). They define structure sense in this way:

> Structure in mathematics can be seen as a broad view analysis of the way in which an entity is made up of its parts. This analysis describes the systems of connections or relationships between the component parts (Hoch and Dreyfus, 2004, p.50).

This definition highlights the fact that structure sense is an analysis of the algebraic expression, specifically how the algebraic expression is made up of its parts. The definition also highlights that structure is the study of the manner in which the component parts are arranged as systems of connections. The last part of the definition highlights the notion of relationship between the elements that make up the structure. There is a correlation between the definition of structure offered by Hoch and Dreyfus, and that which was offered by Ruben for Aristotle’s notion of form. In the previous chapter, Ruben presented that Aristotle's definition of form stated that it is the pattern the elements take, within this context, the pattern or structure that the component parts take. The authors continue to provide definitions of structure as:

> Any algebraic expression or sentence represents an algebraic structure. The external appearance or shape reveals, or if necessary can be transformed to reveal, an internal order. The internal order is determined by the relationships between the quantities and operations that are the component parts of the structure (Hoch and Dreyfus, 2004, p.50).

Here, the authors highlight the importance of transformational activity which when performed reveals the internal order of the expression. This is to say there are structural features of the expression not immediately apparent from the external appearance unless some transformations are performed. We have seen from the example used in the previous chapter how transforming the expression $(q-4)^2+3(q-4)+2$ to an equivalent expression $(q-2)(q-3)$ revealed the 'internal order' of the expression. By making reference to the external appearance and internal orders, the authors seem to be alluding to two levels of structure for algebraic expressions. From the example used thus far, the trinomial which is a quadratic expression $(q-4)^2+3(q-4)+2$ is the external structure the *broad view* of the expression or its *shape*, the *sentence*. The internal order are the individual terms: first term $(q-4)^2$; second term $3(q-4)$ and last term 2 of the sentence. From this the authors go on to include more detail to the definition of structure by relating it to the transformational activity:

> Structure sense, as it applies to high school algebra, can be described as a collection of abilities. These abilities include the ability to: see an algebraic expression or sentence as an entity, recognise an algebraic expression or sentence as a previously met structure, divide an entity into sub-structures, recognise mutual connections between structures, recognise which manipulations it is possible to perform, and recognise which manipulations it is useful to perform (Hoch and Dreyfus, 2004, p.51).

Apart from seeing the algebraic expression as an object and dividing the expression into sub-structures there are three ways in which the word 'recognise' has been used even though appearing four times in the definition. The word recognise refers to the activity of seeing something with the eyes and then connecting and confirming it in your cognisance. First, the word refers to algebraic expression that has been encountered before. Second, the word is used to refer to identification of commonalities between structures. Last, the word is used to refer to the procedural activity, the learner has to recognise from the procedural activities that are possible to perform which one is useful to perform. This points out that the notion of structure sense is in fact a perceptual activity which then informs the nature of procedural activity in terms of what transformations are useful to perform.

Going back to the notion of dividing the expression into sub-structures, researchers argue that articulating the structure of an algebraic expression involves identifying and describing its basic constructions (Hoch & Dreyfus, 2004; Banerjee & Subramaniam, 2012; Ruede, 2013) Identification of structure means recognition of individual terms that make up the expression terms. Banerjee and Subramaniam (2012) have delineated ways of perceiving sub-structures for symbolic algebraic expressions. Barnerjee and Subramaniam (2012) defined terms as units of the algebraic expression, parted by the + or – sign. They present two types of terms, which are what they call simple term and complex term. The definition they provide for a simple term is that it is a constant term, in this example $x^2 + 3x + 2$, the last term 2 is a simple term. For a complex term, there were two descriptions, namely a product term and a bracket term. In the example, a product term is $3x$. A bracket term in the expression $(x + 3)^2 \ + 7(x + 3) + 6$ would be the first two terms $(x + 3)^2$ and $7(x + 3)$.

When the viewing of the expression shifts from its component parts to consider it as a whole, then we see that algebraic expressions take different structures. Rüede (2013) mentions that in the 2010 paper Hoch and Dreyfus give suggestions of algebraic structures that are central to secondary education, namely:$a^2 – b^2$; $a^2 + 2ab + b^2$; $ab + ac + ad$; $ax + b = 0$ and $ax^2 + bx + c = 0$. For them, the structure of an algebraic expression refers to a class of expressions to which it belongs. Structure understood in this way is in fact a mathematical property of the expression, according to Plato’s description, which as interpreted by Ruben, is the distinguishing mark.

An emphasis on the notion of recognition and perception was cited earlier from Sfard and Linchevski (1994) as they argued that the meaning of algebraic expression also depends on what the person “…is prepared to notice and able to perceive” (p. 88). They say that symbols do not speak for themselves, which indicates that it is not immediately obvious what the algebraic expressions mean. When you look at an algebraic expression: $5(x + 3) + 2$, what do you see? Your response can be in computational terms (add three to the number, multiply the sum by 5 and add 2 to the product), where you see a sequence of instructions, or in structural terms, where you perceive the product of the computation than the computation itself, for instance $(5(x+3)+2)$ represents a number even though not specified at the moment; or alternatively, a string of symbols, which represent nothing but an algebraic object.

In their work they are interested in finding out to what extent learners are capable of seeing and using the variety of possible interpretations of algebraic expressions. They mention that the computational process and the structural approach are processes that go on in one's head, rather than what actually gets communicated by means of written records. From this they conclude that the computational and structural processes are thinking models. They argue that the computational process should not be demonised as inefficient because this does give meaning to the symbols and so to give meaning to the symbols, one has to first perceive them computationally (Sfard & Linchevski, 1994). In Caspi and Sfard's (2012) most recent work, mentioned earlier, a way of dealing with the computational process for purposes of explanation has been theorised and suggested in discursive terms.

On the other hand, Novotna and Hoch (2008) talk about learners in high school who come from primary school, having high scores in mathematics, but are poor at algebraic manipulation. They explain this problem as a lack of *structure sense*, which they explain to be a built up from *number sense,* which extends to *symbolic sense,* which in turn extends to *structure sense* (Novotná & Hoch, 2008). Number sense is having a perception for numbers and choosing the correct operation. While symbolic sense is described to be:

> a complex feel for symbols, ability to manipulate and interpret symbolic expressions, and a sense of different roles symbols play in different context (Novotná & Hoch, 2008). p.97).

They say the act of unpacking the symbolism reveals meaning and structure, therefore, symbols display structure. The significance of this is that it highlights the connectedness of structure to its constitutive elements. I pick up on the notion of process of these authors in the next discussion focused on process.

4.3.3.3 Process

The field of mathematics education research has had the debates about the nature of the relationship between procedures and concepts for a long time. Star (2005) refers to these debates as “maths wars”. In the maths wars, conceptual understanding has been the most preferred, due to the view that it is necessary for learners to perform better in mathematics. However, the mathematics learning study committee of the Kilpatrick, Swafford and Findell (Kilpatrick, Swafford, & Findell, 2001) demonstrated that conceptual and procedural understanding are interwoven. The interwoven nature of concepts and procedures is something I also observed as I was interacting with the data in chapters Six, Seven and Eight. There are three arguments that present pertinent views on conceptual procedural debates, which extend beyond defining conceptual and procedural understanding, but look more closely into the nature of the relationship between conceptual and procedural understanding. The first argument comes from Sfard and Linchevski (1994), the second from Rittle-Johnson and Alibali (1999), and the third from Kieran (2013).

To begin with, Sfard and Linchevski (1994) do not use the terms conceptual and procedural, and instead use the terms operational and structural. Their argument against the separation was that the conceptual and procedural constitute different aspects of the same algebraic expression. As they argue their theory of reification, the authors state that operational activity comes first, where structural thinking is developed and reified later. In this way, a precedence and sequential relationship between conceptual and procedural is established. The key term proposed by Sfard and Linchevski is *duality,* instead of *dichotomy*.

On the other hand, Rittle-Johnson and Alibali (1999) provide a description of the nature of the relationship between conceptual and procedural, by stating that conceptual understanding is the principles that govern a domain and interrelationships between pieces of knowledge, while procedural knowledge is defined as action sequences for solving problems. The reseachers claim that these two kinds of knowledge lie on two separate ends of a continuum (Rittle-Johnson & Alibali, 1999). As far as these authors are concerned, the two ends of the continuum represent two different kinds of knowledge, which do not develop independently. They argue that the generation of a procedure relies on conceptual understanding and acknowledge a causal relationship between the conceptual and procedural.

Kieran (2013) argues the false dichotomy between concepts and procedures. She traces the debates in the field back to the NCTM volume publications. She highlights the dichotomous manner in which the relationship between concepts and procedures has been presented. For some writers, a precedence relationship where mastery of procedures is said to occur first before conceptual understanding takes place is advanced. For other writers, conceptual understanding is accorded status and respect above that of procedural activity. She argues that the contrast conferred between concepts and procedures has caused more harm than good in the field of mathematics education. Furthermore, Kieran claims that this has been the case simply because the nature of the relationship between concepts and procedures has not been well investigated. Kieran posits that procedures in their stage of *elaboration* are conceptual in nature. The elaboration is the process of establishing a procedure. In other words, concepts and procedures are indistinguishable in the stage of elaboration. In fact, the reverse is true, even though Kieran does not mention that a concept is elaborated through procedural activity. What makes the reverse true is the fact that procedural activity is the application of a concept. The stage of elaboration is followed by a process Kieran terms *automitisation*, where the elaborated procedures now serve a more 'pragmatic role'. Kieran argues that the stage of automatisation is not a mindless drill activity of procedural work, as it has been spoken about in the field, but rather, a conscious activity. Following the automitisation stage is the *updating* stage, where automitised skills need to undergo a revision and thus an updating of the procedural skill to be functional with expanding mathematical objects. For example, the notion of factorising a linear expression is different from factorising a quadratic or cubic expression, and thus requires that the skill of factorisation be updated and elaborated according to the 'widening' or 'broadening' mathematical objects, or what Kieran refers to as the 'conceptual domain' (p.223). To sum up the nature of the relationship Kieran suggests between concepts and procedures, Kieran highlights that it is an ongoing recursive process:

> The procedural, which in its initial elaboration constitutes new conceptual knowledge related to the given technique, acquires pragmatic, skill-related characteristics, which are in turn updated and revised by means of further technical activity of an epistemic nature" (Kieran, 2013, p223).

The following diagramme devised for this study illustrates the recursive relationship Kieran refers to between concepts and procedures. The relationship between procedures and concepts suggests a discontinuous broken path where procedures function independently of the concepts in the automitisation stage.

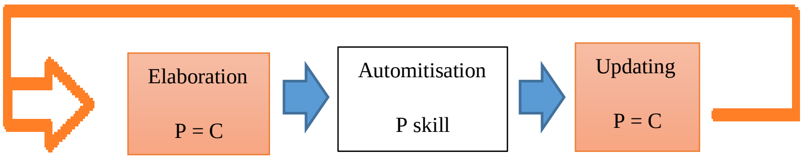


Figure 4:4 – Relationship between concepts and procedures

My own argument is that there is no way of offering an explanation of a concept without using a process to illustrate it. In the same way, there is no way of offering an explanation of process without a mention of the underlying concepts. Viewed in this way, the relationship between concepts and procedures is that they are interrelated. The real issue is that procedural activity is application of concepts, whether or not elaborated at the beginning and in their updating stages.

The field has been lamenting the fact that often times, the aim in the teaching of mathematics has been to jump to drill exercises – the more pragmatic role – which are meant to automatise a skill learnt without elaboration and conceptual derivation of the procedure. The key issue is elaboration, and in practice, we seem to have two cases, one of teaching which includes elaboration, and another of teaching, which excludes elaboration of a process. In the latter case, the procedural activity may seem to be devoid of conceptual understanding, because the procedure is merely stated, and applied as steps of a method without any conceptual derivation and elaboration for learners to understand where the procedure comes from and why it works in the expression under examination. Take the case of the distributive property: the procedure can be elaborated through the ideas of a rectangular array and repeated addition; or the procedure can merely be stated and demonstrated by drawing the distributive curved arrows without any establishment of the procedure, viz. elaboration.

On the contrary, Kieran (2013) erases the precedence, causal relationship, duality argument and continuum suggestions advocated for above, by aguing that the dichotomy between concepts and procedures is false, citing the co-emergence of the procedural and conceptual entities in the teaching and learning of algebra. Processes are informed by mathematical principles and properties, which are in themselves conceptual, and so concepts are embedded in the procedures.

What emerges from this discussion is that there seems to be two distinct arguments presented here because the earlier argument in the 90's acknowledges the dichotomy, while emphasising the need for both, and hence the use of the word duality and continuum, while the most recent argument nullifies the separation by stating that it is a false separation. However, the separation seems to occur at specific 'moments' according to Kieran, namely the stage of elaboration and the stage of updating. When procedures serve a more pragmatic role – technical skill – then they are not conceptual. My own thinking around this is that there are no moments when procedures act independently of concepts, because procedures are applications of concepts from which they were derived. In other words, to speak of a separation at any moment doesn't make sense. The implication this has for the current study is that it highlights the blurred nature of boundaries between structure and process as constitutive criteria for explanation.

I discussed Capsi and Sfard's (2012) historical account for stages in the formalisation of algebraic notions earlier. I now revisit this discussion, because the account continues from unknown and variable algebra to include descriptions of a computational process, which is the focus of this criterion of explanation. Their focus was on constant value algebra, and within this they identify three levels of description for a computational process, that is processual, granular, and objectified. A computational process is processual when the description features one-to-one operational steps to be taken. The granular description is still formulated as a description of a computational process, with some of the verbs such as subtract replaced with nouns, such as difference. In the objectified level complex algebraic expressions are used in a depersonalised way to describe the relations between objects. Both the processual and granular descriptions feature the human actor, while the objectified description is depersonalised. Consider the example in figure 4:2 below taken from Caspi and Sfard's (2012) paper for the example: $3 + 2(n - 1)$, to illustrate these computational descriptions.

Processual	Subtract 1 from n, multiply by two, and add 3
Granular	Multiply the difference between n and 1 by 2 and add 3
Objectified	The sum of 3 and the product of 2 by the difference between n and 1

Figure 4:5 – Computational processes (Caspi and Sfard, 2012, p.51)

Although Caspi and Sfard (2012) have contributed a language of description for learner talk in algebra, their contribution is useful in this study, which is focused on teachers, and seeks to find criteria for explanation. Specifically, their study provides ways of understanding talk about process and the specific language used to discuss mathematical explanations.

The following discussion focuses on the distributive property, as a form of content used to teach the notion of explanations and acts of explaining in the WMCS PD project. The teachers I followed also taught the distributive property. I have included this discussion under process because the distributive property is both process and structure i.e., it is a generalised property, and used to multiply out or expand expressions.

4.3.3.3.1 Distributive property

The concept of the distributive property is difficult to learn because of its abstract nature and lack of relevance to people's lives (Ding & Li, 2014). Abstract refers to decontextualised, depersonalised representation of mathematical ideas.

Distributive property refers to the distributive nature of multiplication over addition. To understand the distributive property is an essential knowledge and skill in algebra. In the solving of equations, the distributive property is one of the rules of simplifying algebraic expressions that one might need to apply. Ding and Li (2014) have noted that the distributive property is necessary for learning linear functions, and is an underlying idea in partial products that lead to understanding of binomials, for example the product of 15 and 24 can be decomposed to tens and units to form a product of two binomials, where the distributive property can be applied as shown below: $15 \times 24 = (10 + 5) \times (20 + 4)$ distributive property: $10(20 + 4) + 5(20 + 4)$, and then multiply out: $10 \times 20 + 10 \times 4 + 5 \times 20 + 5 \times$

4 to obtain the partial products: $200 + 40 + 100 + 20$, which is a sum of 360. We see that the distributive property is used to remove brackets and this is the manner in which it is often spoken about and referred to in operational activity. This is a specific case of the application of the distributive property, where the general case is expressed algebraically as the product of two binomials $(a + b)(c + d) = a(c + d) + b(c + d);\ ac + ad + bc + ad.$ The notion of the distributive property is discussed in the course, and teachers taught it to their learners. Specifically, in the course, emphasis is placed on being able to communicate in the explanation, the reasons why the algorithm works. I pick up this discussion in Chapters Six and Nine, where the important point mentioned here is that the notion of distributive property is both process and structure. The last criterion of explanation is the notion of goal to which I now turn.

4.3.3.4 Goals of algebra

The goals of teaching algebra are articulated as outcomes that must be achieved. In particular, Ernest, speaking more generally on the question of why teach mathematics outlines that there are visionary intended and unintended aims, as well as opportunities its teaching affords. The author argues that there are functional, practical, and advanced reasons for teaching mathematics (Ernest, 2000, 2005). The very basic goal of teaching mathematics is for individuals to be able to function as responsible, critical citizens in society. At school, for learners, the goal of learning mathematics seems not to be immediately appreciated. The practical goals are employment related, and advanced goals are specialised study of STEM-related academic careers.

4.3.4 Summary

The studies presented thus far have mainly been focusing on operational activity and fluency in carrying out the procedure. The different kinds of algebraic activities have been discussed. The need for precision in the talk about computational processes and the need for recognition of structural understanding to accompany procedural knowledge have been emphasised.

In this section, the nature of school algebra from the perspective of mathematics education literature has received discussion. Also presented are views on what is problematic in the teaching and learning of algebra. What has appeared is that school algebra is an activity; ways of thinking about school algebra have also been noted. The steps in the process of formalisation as occurred in the historical development of algebra has been suggested as a teaching approach for school algebra. What also came to light from the discussion of mathematics education literature is that the concepts or construction elements for algebraic expression, such as variables and signs, are problem areas for learners. The discussion has also highlighted the importance of understanding and recognising structure. The next discussion focuses on what is constituted as explanation for algebra in a college textbook.

4.4 Algebraic expressions from the textbooks

The title of the book I used as a source from which this discussion is taken is: *College algebra essentials*. The chapter, which deliberates on how an algebraic expression is constructed is entitled: *Algebraic expressions, mathematical models and real numbers* (Blitzer, 2013). It has been chosen because it was available to me as a precalculus text and had the description of the topic “algebraic expressions”. The specific elements that constitute an algebraic expression are outlined. The forms that the elements, once combined, take are also discussed. The operations and the agreed order of operation are also specified in the book. I organised the discussion in the chapter according to the four criteria of explanation.

4.4.1 Construction elements

The idea of algebra is defined in the book as:

> Algebra uses letters, such as x and y to represent numbers. If a letter is used to represent various numbers, it is called a variable (p.2).

The constitutive elements of algebraic expressions that are discussed in the chapter are variable, coefficient, exponent, where the notion of variable is defined in the textbook as “a letter is used to represent various numbers, it is called a variable” (p.2).

The idea of a coefficient is defined in the chapter as: "Placing a number and a letter next to one another indicates multiplication" (p.2), where for example 7x7 is a coefficient. This statement is often found in mathematics classrooms, particularly the two teachers practices reported in chapter 7, as a definition of a coefficient. "The terms of an algebraic expression are those parts that are separated by the addition" (p.13). A numeral term without a variable is called a constant.

$$7x - 9y + z - 3$$

For example, -9 is a coefficient of y and 1 is a coefficient of z while -3 is a constant term.

The notion of factors and multiplication are important aspect to highlight in the definition of a coefficient, a concept which both teachers make an attempt to define. As the book notes, "The parts of each term that are multiplied are called the factors of the term. The factors of the term $7x$ are 7 and x" (p.13).

Many algebraic expressions involve exponential notation. The expression x^2 is read x squared or x to the second power and it means $x \cdot x$, the exponent indicates that the base appears as a factor two times. To refer back to the historical section, the dot as proposed by Leibniz to denote multiplication is used in the textbook.

Highlighted in this discussion from the textbook thus far is the terminology: exponent, power, roots, variable, constant, coefficient, factors, term, and like terms.

4.4.2 Form – arrangement of the construction elements

The term is a component part of the expression and it has its own constitutive elements, which the textbook has already defined in the previous discussion on construction elements of an algebraic expression. The chapter notes that "The terms of an algebraic expression are those parts that are separated by the addition" (p. 13), and that "Like terms are terms that have exactly the same variable factors." (p. 13), which means the same variable raised to the same power.

The definitions states that a polynomial is "a single term or the sum of two or more terms containing variables with whole-number exponents" (p.46). The textbook definition of a polynomial goes against the meaning of the word poly – which means many – since one would normally consider "many" from two things instead of one. Simplified polynomials with one, two, three terms are called monomial, binomial and trinomial, respectively (Blitzer, 2010). There are no special names for polynomials with four or more terms. The degree of a polynomial is the highest degree of all the terms of the polynomial in one variable. Blitzer (2010) provides the following form of an algebraic expression to define a polynomial:

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0 \text{ (p. 47)}$$

Where a_n is the leading coefficient and a_0 the constant term $a_n \neq 0$ and n is a non-negative integer.

An algebraic expression is described in the book as

> A combination of variables and numbers using the operations of addition, subtraction, multiplication, or division, as well as powers or roots, is called an algebraic expression. Here are some examples of algebraic expressions:
>
> $x + 6;\ x - 6;\ 6x;\ \frac{x}{6};\ 3x + 5;\ x^2 - 3;\ \sqrt{x} + 7$ (Blitzer, 2010, p.2).

Even though the mathematical focus is not equations, it is noteworthy to state that an equation is formed when an equal sign, =, is placed between two algebraic expressions to express an equivalence relation between the two expressions.

4.4.3 Process – Operations

Adding and subtracting polynomials involves grouping and adding like terms. Multiplying polynomials involves using the laws of exponents, and the distributive property. There is a specific order of operations that have been agreed when operating on algebraic expressions, famously known as BODMAS in most school textbooks. BODMAS stands for brackets of division multiplication addition and subtraction. The basic principle behind it is that you first perform operations within the bracket, then perform multiplication and division working from left to right and lastly perform additions and subtractions working from left to right. Simplification of algebraic expressions involves the removal of brackets and the addition of like terms.

There are specific words the textbook uses to talk about process involved when working with algebraic expressions, such as simplification, factoring, adding, subtracting, grouping, distributive, BODMAS.

4.4.4 Goal – the purpose of algebra?

It is simply stated in the book that learning algebra is important, because it has a special language that can be used to describe our world. From how one's lifestyle changes when the price of a litre of petrol increases, to the cost of university education.

4.5 Summary – for all three sections

To this end, I have presented literature on algebraic expressions from three different perspectives; first, the historical account; second, mathematics education; and third, a college textbook. The literature in the historical section has highlighted the journey from rhetoric to symbolic expressions of algebra, where the main argument presented is that it is unfortunate that the rhetoric way of expressing algebraic notions has been lost because the provision of an explanation relies on words. One fact that I argue is that the notion of symbolic algebra as having been a project of the enlightenment era is misleading because the hieroglyphic, hieratic writing of the Egyptians and the Cuneiform scripts of the Babylonians were written in symbolic languages. Table 4-1 below summarises how the literature on algebra helped interpret Aristotle's criteria for school algebra.

Table 4-2: Summary table for the chapter

Matter	Form	Process	Goal
History			
Pictographs, Words Symbols	Equations Geometry	Solving, adding, subtracting,	Building of pyramids, commerce, land surveying, astronomy, etc.
Maths Education			
Variable, signs and symbols	Terms, simple terms, product terms, complex terms, Structure sense	Procedural Different kinds of talk from processual, granular to objectified talk	Teaching and learning for future career pursuits and responsible citizenry
Textbook			
Numerals, letters, unknowns, variables, equal sign, signs, coefficient, constant, terms, operation signs, powers/roots Brackets /parenthesis	Terms Polynomial, monomial, binomial, trinomial Linear/ quadratic equations inequalities	Add, subtract, multiply, divide, Factor, Simplify Solve Squaring, modelling, grouping and adding like terms BODMAS	To understand and describe our world

4.6 Conclusion

The discussion above has managed to illuminate at the level of algebraic expressions that the basic elements that construct the expression are important to understand in order to manipulate and operate on the expression successfully. When it comes to manipulation, it has been noted in the above discussion that, much as computational processes are vital in the formation of an abstract perception of algebraic objects, they must be meaningful and conceptual. The literature survey presented above has highlighted that once the symbols have been combined together through the operations of addition and subtraction, they take specific forms in the manner of terms and structures such as polynomials and equations. It has also been stressed in the discussion of the literature that the goal of algebra is not only to solve, and carry out procedures correctly, but to develop algebraic thinking, which was defined as the ability to generalise number and model situations. With all this in consideration, I now proceed to discuss the methodology for the study.

CHAPTER 5 METHODOLOGY

5.1 Introduction

In this chapter, my aim is to discuss the methodology of the study. The aim of the research was to study what comes to be constituted as criteria for explanation and acts of explaining in the WMCS PD first and secondly in the teachers' practices. I discuss the methodology in the order in which the research activities were completed. The analysis of the WMCS chapter was completed first, and from this criteria transmitted by the course for explanation and acts of explaining were matched up with Ruben's criteria, which were used to analyse the teachers' practices. I provide an elaboration of how the coding system used to analyse the data was developed and used in the analysis. A detailed explanation of the analysis process follows in Chapters 6, 7 and 8.

5.2 Methodology

The type of research methodology used here is informed by certain ontological and epistemological considerations (Hatch, 2002; Maxwell, 2012; Opie, 2004), where the research instruments used to collect data were guided by the methodology. The ontological consideration enlightened the type of methodology chosen when it came to the nature of reality, and the epistemological considerations informed the methodology about the nature of knowledge and where and how to seek for it.

When it comes to the nature of reality, it is assumed that there is a social reality, which is socially constructed by its participants. In this sense, reality is a product of social constructions, and thus subjective, and context bound. The very process of meaning making within this study is reality (Maxwell, 2012). The small aspect of reality with which I was interested in this study was the notion of constitutive criteria for explanation and acts of explaining. I thus understand this as socially constructed reality.

Knowledge of reality for this study is observable in social settings by taking records of actions and what the participants say about them (Maxwell, 2012). Knowledge and what counts as knowledge is obtained from what the participants say. The participants' subjective experiences is what counts as knowledge. Access to the knowledge is through the language of the participants. Participants include the course presenter and teachers who participated in the course. I interviewed the course presenter and observed teachers' practices.

The research methodology implied by the abovementioned ontological and epistemological considerations is qualitative, and used here as means of conducting the study and generating knowledge (Opie, 2004). As already noted in the ontological and epistemological considerations the qualitative methodology assumes that knowledge is found in the participants' practices and experiences. The knowledge is then facilitated, organised, and reported by myself, the researcher, whose own perceptions and biases are minimised because I was guided by the criteria I obtained from the literature to code and analyse the participants' utterances.

Research methods undertaken in the study follow from the qualitative methodological considerations. In this study, I observed teachers' practice and interviewed teachers and the course presenter. The aim was to find out what constitutes explanation and acts of explaining in the course, and in the teacher's practices after attending the PD course. The criteria for explanation, as mentioned, come from the literature and were expanded into different levels through interaction with the data. Some of the criteria for acts of explaining were obtained from the literature and others emerged through interaction with the data. The discussion on development of codes for both explanation and acts of explaining elaborates this process.

According to the ontological and epistemological assumptions I align with in this study, the research paradigm is interpretivist. However, I also used some characteristics associated with the quantitative paradigm in the sense that I quantified qualitative data. This was for descriptive purposes and thus the overall methodology of the study is largely qualitative in approach because only qualitative methods have been used to collect the data.

5.2.1 Data collection

Research design and data collection obtains its shape and direction from the underlying philosophical considerations I discussed earlier. I had initially articulated a set of questions based on how the notion of explanation as presented in the course was taken up by some of the participating teachers. I then realised that the notion of explanation was in itself not sufficiently conceptualised to enable the study of take up, because I needed to know what it was that teachers were supposed to have taken up. The focus shifted to a study now concerned with the constitutive criteria of explanation and acts of explaining. Since the empirical settings remained, regardless of the change in focus (from take up to constitutive criteria), the PD course and the teachers' practices became the empirical contexts, through which I could then apply and develop the criteria of explanation and acts of explaining obtained from the literature. I provide here a description of the original research design and the data collection path I followed.

I started off by collecting document data from the project in 2012 while alongside developing the research proposal. In the year 2013, I collected video recorded data from five teachers who had successfully completed the 2012 course. One of the teachers who I named Teacher A was on maternity leave in 2013, and so I could only collect video recorded data from her classroom in 2014. In 2014, I conducted follow up, video stimulated interviews with the four teachers I had collected video recorded data in 2013. I then conducted the video stimulated follow up interview with the Teacher A in the same year I collected video recorded data, 2014.

At the time when I was collecting data from the course and the teachers' classrooms, I was collecting data for a study, which sought to understand teachers' take up as the main focus. As a result, when the focus on take up shifted, I was obliged to limit what was useful to focus insofar as the data was able to answer the question of constitutive criteria. I was able to do this within the framework of a qualitative study, because of the flexibility it allows that the research design may change during the study (Opie, 2004) as it occurred in my study. I had to use the available data as empirical resources on which I could apply and develop the conceptually obtained criteria of explanation and acts of explaining. Because of this, the amount of data I then needed to use could be reduced. I discuss the two focal teachers in detail in the sampling section.

Qualitative data were collected for both the course and the teachers. For the WMCS course, all available course materials in the form of handouts and PowerPoint slides were collected. From the data, it was possible to describe the structure of the course, and then localise myself to those sessions with a specific focus on explanation and acts of explaining. These were called teaching sessions. For discursive data, only a follow up interview with the course presenter was conducted. There was no video records of the sessions. For the classroom data three consecutive lessons were video recorded from each teacher. There were follow up interviews with teachers and the course presenter. Figure 5:1 is a schematic illustration of the research design and data collection.

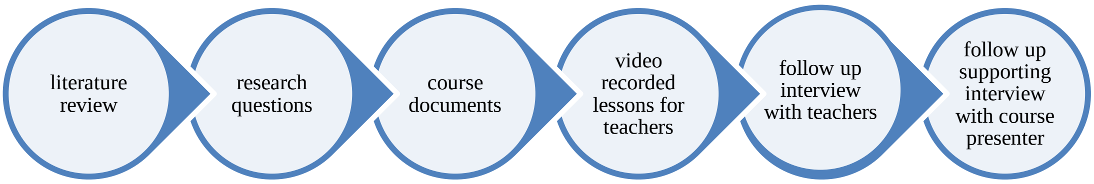


Figure 5:1 – Flow chart for data collection

A discussion of the background information to the empirical fields I worked in order to collect data is important.

5.2.2 Setting, school contexts and sampling process

The first data collection site was the WMCS PD project. From this I collected document data in the form of handouts and PowerPoint presentations. I then constructed a follow up interview with the course presenter once the document analysis of the course material was completed.

To obtain teacher participants for this study, I used purposeful sampling. I made a conscious decision about which teachers from the course were to participate in the study. Firstly, they had to successfully complete the TM1 course; and secondly, they had to be high achievers in the course. I initially followed five teachers who participated in the WMSC PD course into their classroom and video recorded data for three consecutive lessons from each teacher based on the teaching of algebraic expressions in Grade Nine and algebraic equations in Grade 10. I also audio recorded a follow up video stimulated interview with the teachers. However, once the initial focus of the study shifted, and I needed to reduce and selected to focus only on the video recorded data of the teachers' practices. From the five teachers, three teachers taught the same topic in Grade Nine, that is, operations on algebraic expressions. The remaining two teachers taught equations to Grade 10. The story reported in this thesis is of two teachers in Grade Nine. The reason for this was because the third teacher's lessons did not add anything new to the development of the codes, both for explanation and acts of explaining, than what the first two teachers' lessons had already contributed. As discussed, the initial purpose for collecting video data from teachers' classrooms was to see teachers' take-up of the notion of explanations and acts of explaining algebra from the course into their classrooms. When the focus shifted to focus on constitutive criteria of explanation and acts of explaining, take-up was backgrounded and the original sample, video recorded data, and follow up interviews with teachers was no longer needed, because the aim was now to see the application of the conceptual criteria for explanation. For this purpose, two teachers were deemed sufficient and I proceeded to focus in on them, their context and their practices. I refer to the two teachers as Teacher A and Teacher B.

Teacher B's school context – township,[10] no-fee school serving learners from mainly informal settlements. There is a feeding scheme at the school. Learners are stationed in a classroom, while different teachers for different subject areas move around to their classes. Homework is written on the chalkboard, because learners at the time when data was collated were not in possession of individual textbooks. This impacts writing time, especially for learners who are slow to copy from the board, which has to be erased by the next subject teacher. Teacher B is a non-specialised Mathematics teacher teaching up to Grade Nine before participating in the course, now teaching up to Grade 10 Mathematics. Teacher B has more years of teaching experience than Teacher A.

Teacher A's school was an inner-city, previously "white" school, and better resourced. Each learner had a textbook. The textbook reference stipulating homework practice exercises was written on the board. Learners walk to teachers' classrooms and so classroom spaces are specialised according to subject area, viz. mathematics. The Mathematics classroom for Teacher A has mathematics charts on the walls. There were extra Mathematics textbooks on the teacher's shelves. Teacher A is a specialised mathematics teacher teaching all the way up

[10] In South Africa, a township refers to a residential area formerly designated, under apartheid legislation, for occupation by black (African, Indian and coloured) people.

to Grade 12 Mathematics. Teacher A had been teaching for less than 10 years at the time when I collected data.

5.2.3 Ethical consideration

The teachers who participated in this study voluntarily agreed to their practice being observed by me. The necessary consent documents were signed by the teachers. In the documents, I stated what the video recordings of their lessons were going to be used for, and how they were going to be used. I assured the teachers that their real names and the names of the students present in their lessons were going to be changed to pseudonyms to ensure confidentiality. For anonymity, I also ensured the teachers that the videos were only going to be viewed by me alone and where necessary with my supervisors and no one else. I further emphasised and guaranteed the research participants that there was no fourth person outside of these who would view the videos and I have kept to the promises of confidentiality and anonymity I made to the participants. The data is kept safely, according to the policy specifications of the University of the Witwatersrand.

5.2.4 Data analysis

As indicated, there were two different sets of data. The first came from the WMCS PD course and the other from classroom data.

5.2.4.1 The document data – WMCS PD

In the documents, I collected from the course, I examined what was stated, elaborated and exemplified as explanation and acts of explaining criteria. The process is illustrated in table 5-1 below.

Table 5-1: The stated, elaborated, and exemplified criteria

Stated	Elaboration	Exemplification in relation to distributive law
	Explanation	
Two types of explanation: 1. What Why and When 2. How Why and When	1. What Definition Concept Notation 2. How Executing a procedure Application of formulae	1. What Big idea Repeated addition Area model 2. How Repeated addition Distributive arrows
	Acts of explaining	
Ways of making mathematics available to learners: Examples and their representations Language, terminology and notation	Producing a written explanation given a list of acts of explaining to employ	Critiquing the teacher's choice of examples and thinking of other examples to suggest

I then matched up what was stated as criteria for explanation with Aristotle's four criteria of explanation. The analysis of the criteria transmitted by the course could not be completed from the course documents alone. There were issues that needed confirmation and further elaboration, and for these I constructed a follow-up supporting interview with the course presenter (see Appendix B2 for a full transcript). Because this was a supporting interview, relevant utterances from the course presenter were drawn on to accompany my analysis of the documents.

In the videotaped data of teachers' practices, I looked for what was constituted as explanation and acts of explaining in the transcript using a coding system. I describe below how the codes developed from literature and the data.

5.2.4.2 The transcripts – teachers' practices

I videoed and transcribed the three lessons for each teacher. The transcript is the actual utterances from both teacher and learners. From the teacher, where applicable, I described the gestures in brackets. Also in brackets is the description of what the teacher was doing, for instance, if she was walking around, marking learners' books, or writing on the board. For all writing on the board, I took a snapshot from the video, and included it in the transcript. The transcript is basically a conversation between teacher and whole class, mostly initiated by the teacher, where the teacher had more turns to speak and elaborate.

The segmentation/explanandum

The unit of analysis for this study is called the explanandum, namely that which needs to be explained. Attention would be focused on one explanandum from beginning to end, and as such, I segmented the transcript according to what the attention was focused on. The explananda were mainly made up of a discussion of an example or a concept, statement of the goal for the lesson, or the homework given to learners. The segments across the three lessons for each teacher, came together to constitute a larger unit of analysis, which was the goal of the three lessons together – operating on algebraic expressions. In each segment, I used a coding system that separates explanation from acts of explaining. I only coded the teacher's utterances and excluded learner talk in the coding because my interest in this study is on the teachers' explanation. However, these explanations occur and are co-constituted in the interaction between teacher and learners. I begin the discussion of the codes with explanation then followed by acts of explaining.

5.2.4.3 Explanation codes

I started with the four criteria of explanation as discussed in Chapter Three. Each of the four criteria developed as I worked with data. In total for explanation I had eight codes, namely: M1 and M2 for matter, F1 and F2 for form; P1 and P2 for process and goal, G1 and G2. I describe each code below with examples from the data.

1. **Matter**

There were two ways of describing and coding utterances related to matter, and that is M1, and M2. All utterances which were a reading of a string of symbols, I coded as M1. Also when there was a calling out of names for the elements without defining them, I coded these as M1. M2 refers to utterances that seeks to provide a definition for an individual element.

Table 5-2 below shows examples of utterances I coded as constitutive elements. In the first column is a description of the code; the second column provides the data source from where the example in the third column was taken (for instance, in Row 1 the source from which I took the example is Teacher A/L1/Sg4/T1 where Teacher A is a code for teacher, L1 refers to Lesson

Number 1, Sg refers to Segment/Explanandum Number 4, and T1 refers to Turn 1. All the codes for both explanation and acts of explaining are described in this manner.

Table 5-2: Codes for Matter

Description	Source	Example
Reading a string of symbols M1	Teacher A/L1/Sg4/T1	*T: Okay, we want to add five x plus five y. minus three x plus two y.*
Calling out names of the elements M1	Teacher A /L1/Sg1/T12, 14, 16, 19, 22, 26	*T: terms, alphabets, equations, variable, coefficient, constant, exponents,*
Defining the elements M2	Teacher A /L1/Sg2/T24-33	*T: Okay maybe we should define what is a coefficient?… So coefficient is a number next to a variable*

2. **Form**

There are two ways of describing the ways in which structure of the elements is spoken about. When the utterance focuses on the structure of the individual term within the expression, viz. a substructure, it is allocated the code F1. When the utterance focuses on the structure of the entire explanandum, it is allocated the code F2.

Table 5-3: Codes for Form

Description	Source	Example
Talk about structure of a term F1	Teacher A /L1/Sg4/T6, 30	*T: How do you see that these are like terms? ... So the term goes together with its sign…*
	Teacher A /L1/Sg1/T10	*T: Algebraic terms are separated by a plus or a minus.*
Talk about structure of entire expression F2	Teacher A /L1/Sg4/T44	*T: So they are not like terms so this is our answer?*
	Teacher A /L1/Sg/T13	*T: Okay now if I say to you subtract 4 from 6 how are you going to write this one* [written on the board].
	Teacher B /L3/ Sg2.5/ T42	*T: So now we take now this whole thing. So what is it* [writing a bracket along the length and along the breadth] *going to be called? So this is called what? Length and this breadth* [writing the labels on length and breadth, respectively].

3. **Process**

When the utterance was about steps to be followed these were coded P1. When the steps were derived and justified they were coded P2.

Table 5-4: Codes for Process

Description	Source	Example
Steps of the method P1	Teacher A/L1/Sg5/ T9, 17, 39, 41, 54	*T: Okay first let's write this instruction the way they are telling us*"*… So the first thing that you do is to remove the brackets… Okay we've grouped the like terms then what do we do next?... Then we simplify … Because there are no like terms. We accept this as the final answer. So we accept that as the final answer*.
Justified steps P2	Teacher B/L3/Sg2. 5/T42	*T: That's my area, right? But now when we, let's say we go back now. We want to multiply. But we don't want to multiply because they are, they might be long when we using area. So, we want to multiply a monomial and a binomial.*

4. **Goal**

Goal has two aspects to it, and that is, statement of the goal, and the why question. The statement of the goal refers to the purpose of the entire lesson, which I coded as G1. The occurrence of the G1 code was in relation to the announcement of the topic for the day's lesson.

I decided to have a separate code for the why question, so as to highlight when it appears and notice its usage in the teachers' practices. In cases where the why question was not explicitly asked as long as there was the word "because" in the sentence I coded as G2. Scholars in Chapter Three have argued that the word because is a response to why question. The G code is the only code in the explanation where G1 and G2 are not related in a hierarchical manner.

Table 5-5: Codes for Goal

Description	Source	Example
Stated goal of the lesson G1	Teacher A /L3/Sgt3/T3	*T:Multiplying and dividing expressions. So that is our next topic. So write today's date and write multiplying and diving expressions* [writing on the board multiplying and dividing expressions].
A why question G2	Teacher A /L3/Sg6/T30	*T: So my question was why are we writing it like that?*
	Teacher A /L3/Sg10/T18	*T: Why do you agree?*
	Teacher A /L3/Sg5/T20	*T: Okay why do you say no?*
	Teacher A /L2/Sg1/T9	*T: Trinomial, why trinomial?*
	Teacher A /L1/Sg4/T22	*T: Minus, why not plus three x?*

5.2.4.4 Acts of explaining codes

Acts of explaining as earlier shown in the PD literature are a way of transforming SMK into PCK in pedagogic practice. Aside from SMK and PCK, there are actions taken in class that have nothing to do with knowledge of the content and how to teach it. These refer to general pedagogic actions that have to do with manners, and establishing of order in the classroom. The codes for acts of explaining are the R codes and they can be separated into three parts.

The first is purely control of behaviour related utterances. These utterances have no explanation in themselves except for social order which is important for the learning environment, in which an explanation is to be produced. I coded these as regulative (Rg). These include utterances like controlling who should speak, who should go to the board, regulating the noise, when to write, etc. These are rules about how to behave in class.

Table 5-6: Codes for Regulative talk

Description	Source	Example
Regulative (Rg)	Teacher A /L1/Sg1/T17	*T: Okay I said no choir so put up your hand. Mm, so that thought that comes to your mind that is what I'm looking for.*

The second part are pedagogic communicative techniques, and those are: re-voicing (Rv), requirement to finish teachers sentence (Rf); chorusing (Rc); gesturing (Rt); and evaluation; (Re) these are what drives the communication of the explanation forward.

For re-voicing, the teacher basically repeats learner responses to highlight errors and acknowledge ideas that come from learners. I coded the act of repeating learners' utterances as Rv. From the famous classroom interaction/communication of the explanation pattern known as IRE pattern: teacher initiation, learner response, and teacher evaluation (Mehan 1979, cited in Franke, Kazemi and Battey, 2007) or teacher feedback IRF (Wells, 1993, cited in Franke, Kazemi and Battey, 2007). The response aspect is only noted from the learners' according to the famous initiation response pattern, however, I would like to think about it from a teacher's perspective as re-voicing, because the response from the teacher, as I have observed the interaction pattern, was often in the form of re-voicing. I observed that the teachers' re-voicing utterance was in the form of evaluation, as shown in the second and last row of Table 5-8 below. The evaluation were utterances where the teacher provided opportunity for learners to apply their thinking on another learners' contribution by asking if they agree, "*is it correct?*", "*Let's look at what is written on the board*" – all these I coded 'Re'. Gestures are signs that the teacher gives to learners and for these signs to make sense the accompanying words are important. Utterances where the teacher makes use of arms, hands, fingers, I coded 'Rt'.

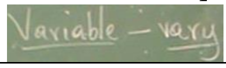
Chorusing is where the whole class says the same thing at the same time, this happens more for Teacher B where she explicitly requires a unison response by repeating her question when the volume of learners responses in low. I coded these as 'Rc'. Completing the teachers' utterance served as an indication of whether or not learners have the understanding the teacher expects or whether or not they see ideas from the teacher's perspective. I coded all utternaces which were a requirement to finish the sentence as 'Rf'.

Table 5-7: Codes for modes of communication

Description	Source	Example
Re-voicing (Rv)	Teacher A /L1/Sg2/T20	*T: He says x to the exponent of two, do you agree?*
Finishing sentences (Rf)	Teacher A /L1/Sg2/T47	*T: The highest exponent is* [putting up two fingers].
Chorusing (Rc)	Teacher B /L1/Sg4/T47	*T: What do we call an alphabet in maths?*
Gesturing (Rt)	Teacher A /L1/Sg2/T47	*T: The highest exponent is* [putting up two fingers].
Evaluating (Re)	Teacher A /L1/Sg2/T20	*T: He says x to the exponent of two, do you agree?*

The last part are the acts of exemplifying (Rx) and use of mathematical language (Rn). What an explanation comes to be does not only depend on the nature of examples selected and made available, but also on how the examples are dealt with. All the examples, whether they were examples of a property, concept or procedure, and whether they were examples countering an assertion or non-examples of a concept, I coded Rx. Mathematical processes are demonstrated but their comprehension depends on the formal mathematical language used to describe the underlying principles. Essentially, what comes to be constituted as explanation is equally dependent on language. All utterances concerned with naming, definition of words, meaning of words I coded 'Rn'.

Table 5-8: Codes for Examples and Language

Description	Source	Example
Exemplifying (Rx)	Teacher A /L1/Sg2/T1	*T: Okay now let's use one example of an algebraic expression* [walking to the desk, then to the board to write] *minus six y squared plus four* ($3x^2 + 3xy - 6y^2 + 4$). *So that's… Is that an expression?*
Language (Rn)	Teacher B /L1/Sg4/T35	*T: We call it a variable. Okay, but there is a reason why it's called a variable. So this word variable comes from an English word* [writing the word vary next to variable] 

General Comments on coding

An utterance can have more than one code in the explanation (E) and in the acts of explaining (AoE). For instance, in one utterance in the AoE, there can be re-voicing, requirement to finish sentence, evaluation, and general regulative talk going on simultaneously. With the explanation, since the explanation itself is about a specific object, then the elements are always present whether the talk is predominantly focused on form or process.

5.2.4.5 Analysis

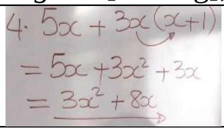
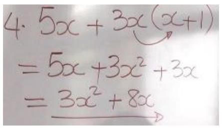
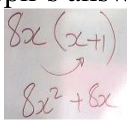
The format that I used to present the transcript is as follows:

- Turn numbered: T
- Explanation code: E
- Acts of explaining code: AoE
- Transcript: Classroom dialogue. In brackets are the researcher's annotations.

There were four main questions I asked for each explananda: 1: why is this a sub-unit? 2. What components of explanation are present and how? 3. What is the achieved elaboration? What is the nature of pedagogy and learner participation? These are exemplified in the discussion of the transcript exerpt from Teacher A's lesson three Segment 3.4 below.

Table 5-9: Segment 3.4: Example 4

Turn	E	AoE	Transcript	
1		Rx	T:	*Okay number four [writing on the board]. Okay, let's use plus one.*
2				$4.\ 5x + 3x(x-1)$
3	M1		T:	*Okay. Five x plus three x into x plus one.*
4			J:	*Ma'am…*
5			T:	*…yes, Joseph?*
6			J:	*You will first add the numbers outside the bracket you will say five plus three its gonna give eight x.*
7				
8	P1 M1	Rv Rf	T:	*Okay, so Joseph is saying we must add five x and three x to get.*
9			J:	*Eight x open bracket x plus one* [written], *eight x plus eight.*
10				
11		Rv	T:	*The answer is eight x plus eight?*
12				$8x(x+1)$
13			Lrns:	[telling Joseph he's wrong]
14		Rg Re	T:	*Give him a chance, okay before we move on Joseph do you agree with him?*
15			Lrns:	*no*
16	G2		T:	*Why?*
17			Lrns:	*BODMAS*
18			J:	*You gonna get the same answer.*
19		Rv	T:	*You gonna get the same answer?*
20			Lrns:	*Yes, no.*
21	P1	Rg Re	T:	*Okay, let's write Joseph's answer this side and we'll double check if we get the same answer. Now those who say we are starting with brackets, let's start with brackets, so we'll leave Joseph's answer that side.*
22			Lrns:	*Ma'am write five x plus*
23		Rg	T:	*I write*
24			L:	*Five x*
25	M1		T:	*Plus*
26			L:	*Then you multiply three x times x*
27	P1		T:	*So you multiply three x by x*
28			L:	*Three x squared*
29		Rv	T:	*So that is three x squared*
30			L:	*Plus three x*
31		Rv	T:	*Plus three x*
32				$= 5x + 3x^2 + 3x$
33			L:	*Group the like terms.*
34	P1	Rv	T:	*Right, you group the like terms.*
35			Lrns:	*Yes.*

36		Rf Rt	T:	*And, if we group the like terms so five x will go that side [gesturing with hand] then we'll be having three x squared and…?*
37			L:	*…three x squared plus eight x.*
38	M1	Rv	T:	*Eight x* [writing],
39				
40			Lrns:	*Final answer.*
41	P1	Rv	T:	*Is that the final answer?*
42			Lrns:	*Yes.*
43	F2	Re Rt	T:	*Okay, now is it the same as this one?* [pointing at Joseph's answer written on opposite side]
44			Lrns:	*No.*
45		Re	T:	*You said it would be the same thing.*
46			L:	*He said it's the same thing.*
47	F2 P1	Re	T:	*Would it be the same answer if we open the bracket?*
48			Lrns:	*No.*
49	P1 F2	Re Rg	T:	*If we open the bracket here, we will be having eight x squared plus eight x* [written], *which is not the same as what we have there.* Teacher and learners Joseph's answer   *So we must use BODMAS, so we have to use BODMAS, so that is how important BODMAS is.* [writing example five on the board]

1. **Why is this a sub-unit?**

I have classified this as a sub-unit based on a discussion of one example from beginning to the end. The discussion is based on an example: $5x + 3x(x - 1)$, which is immediately changed to: $5x + 3x(x + 1)$. A learner, whom I have chosen to call Joseph, was given a chance to share his own explanation on how to simplify the expression. Joseph conjoins $5x + 3x$ and his answer at the end of his simplification method is incorrect. The BODMAS approach is suggested by the rest of the learners as the approach that will produce the correct answer. Joseph protests that the BODMAS method will produce the same answer as his. The two answers are compared so that Joseph can see why his method is incorrect. When the teacher shifts learner attention from the current example to focus on the next example, then that marks the end of the current sub-unit and a beginning of a new sub-unit.

2. **What components of explanation are present and how?**

   a. **Are all the elements of matter present?**

Teacher writes the example and then reads through it, T3: "*Okay. Five x plus three x into x plus one.*" This is coded as M1 because it is a reading of a string of symbols.

   b. **Are all types of form present?**

The teacher requires learners to study the structure of the expressions in Joseph's answer and the answer generated by following BODMAS rule in T43: *Okay now is it the same as this one?* [Pointing at Joseph's answer written on opposite side]. Even though the word structure or form is not explicitly mentioned in the teacher's utterance, for learners to do the comparison, they necessarily have to study the structure of both answers. Joseph's answer was left as $8x(x+1)$ and the teacher asks the following question in T47: *Would it be the same answer if we open the bracket?* The study of structure is still implied by this utterance. In turn 49, the teacher says: *If we open the bracket here we will be having eight x squared plus eight x (written), which is not the same as what we have there*. This I coded as F2 because the utterance still requires learners to study structure. The study of structure in this example is required through the comparisons that the teacher wants the learners to make between Josephs' answer and the answer generated by applying BODMAS rule.

   c. **Are all kinds of process present?**

The teacher writes an incorrect process offered by the learner and then compares it with the BODMAS process to highlight the importance of using BODMAS. T21: "*Okay, let's write Joseph's answer this side and we'll double check if we get the same answer, now those who say we are starting with brackets, let's start with brackets, so we'll leave Joseph's answer that side.*" Not all kinds of process are present, because the underlying principle, distributive law, that Joseph is currently struggling with is not derived.

   d. **Is the goal stated and justified?**

The goal is correctly simplify the algebraic expression dealt with in this example.

   e. **Is anything absent and why?**

What is absent in the example is an explicit mention of the nature of Joseph's error and why it is not correct to do what he did. Also missing is the explicit study of structure to highlight why $5x + 3x$ must not be added because the structure of the expression is such that it is composed of two unlike terms; $5x$ as term number one and $3x(x+1)$ as the second term. These cannot be added together because they are unlike terms. This kind of approach by breaking down the expression into its individual terms was going to further convince Joseph and help him understand why he must not add $5x$ with $3x$ in the expression: $5x + 3x(x + 1)$.

3. What is the achieved explanation?

In turn 49, the teacher emphasises the importance of using BODMAS, T49 – "*so we must use BODMAS, so we have to use BODMAS so that is how important BODMAS is*". This is the achieved elaboration.

4. What is the nature of interaction and learner participation?

Learners provide solution steps, correct or incorrect, the teacher writes these on the board and seeks whole class approval. Learners respond to teacher questions, and finish the teacher's sentences.

The foregoing discussion illustrated how I conducted the analysis for each segment across the two teachers' lessons. From the coded utterances in each segment, I then isolated the codes for explanation and acts of explaining from the transcript. The following table is a demonstration of the codes when separated from the utterances. The purpose was to establish a count for each code for both explanation and acts of explaining so as to construct an argument for what was constituted and privileged as explanation and acts of explaining for across the two teachers' practices. A record of the three lessons for Teacher A and Teacher B is found in Appendix C1 and C2, respectively.

Table 5-10: Codes for Teacher B lesson 1

Sg																						
1.1	Calling out Mental Maths activity																					
	Turn	2	4	6	8	9	11	13														
	Exp			M1	M1 P1	P1		P1 M1														
	AoE	Rg	Rg	Rg			Rv	Rg Rt														
1.2	Teacher walking around talking to individual learners and groups																					
	Turn	1	2	3	4	7	12	14	16	18	20	21	23	25	26	27	29	31	32			
	Exp		M1 P1			M1	M1 P1					M1	P1		M1	P1	P1					
	AoE	Rg		Rg	Re	Rg	Rf	Rv Rf	Rf	Rg Rf Rt	Rg		Rv	Rv Re		Re Rf Rt	Rv	Rg	Rg			
1.3	Whole class discussion of Mental Maths																					
	Turn	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	33	35	37			
	Exp	M1	P1		P2	P1	P2 G2	P2 G2		P1			F1		P1	P2	P2	P1				
	AoE	Rg Rf	Rv	Rf	Rv Rf		Rv Rg	Rv Rf Rt	Rv	Rv Rf	Rv	Rg	Rv Rg Rt	Rv Rf	Rv	Rv Rf	Rv Rg	Rv	Rv Rg			
2	Variable (V)																					
	Turn	1	5	7	8	11	15	20	21	23	24	26	27	29	31	33	35	36	38	40	42	44
	Exp	M1	G2	M1	P1	M1	M1		M1 P1		M1 P1	P1	F1	M1	P1 M1	M2	G2 M2		M2			M2

	AoE	Rg		Rx Rv		Rv		Rx		Rv	Rx	Rv	Rt	Rt	Rv Rn	Rn	Rn	Rn	Rn Rv Rt	Rt	Rv	Rn Rt Rc
	Turn	46	47	49	52																	
	Exp			M2 G2	G2																	
	AoE	Rv	Rn Rc	Rv Rn	Rg																	
3	Monomial (Mo)																					
	Turn	1	2	3	4	5	6	7	9	10	11	13	15	16	19	21	22	23	25	27	28	
	Exp	M1	M2 M1		M1			M1	M2 G2	M1		M1	M1	M2 F1			M1 F1	F1				
	AoE	Rg	Rt	Rg	Rt	Rg	Rg		Rv		Rv		Rv	Rg Rt Rc Rn	Rn Rf	Rv	Rx		Rg	Rv	Rt	
	Turn	30	32	34	36	39																
	Exp		F1 P1	G2 F2	F2 P1	P1 F2																
	AoE	Re	Rv	Rf Rt	Rf	Rv Rt Rn																
4	Number next to variable (Cf)																					
	Turn	2	3	4	6	8	10															
	Exp	M1 P1	M1	P1 M1	P1 G2	P1	P1															
	AoE		Rt	Rt	Rv Rf																	
5	Binomial (Bi)																					

	Turn	1	3	5	7	11	13	15	17	19	21	23	25	27	28	29	31	33	36			
	Exp	M1 F1			G2	M1	P1	P1 F2	F2	M1 G2	P1	F2	F2	F2	M1		F1	F1				
	AoE		Rg	Rg		Rt		Rf Rt	Rf	Rg		Rf		Rv Rn Rf	Rn	Rf	Rn Rf Rt	Rn Rf Rt	Rv Rg			
6.1	Trinomial (Tr)																					
	Turn	1	2	4	6																	
	Exp	F1 M1	P1	G2	F2																	
	AoE	Rx			Rg																	
6.2	Side																					
	Turn	1	3	5	7	9	11	13	15	17	19	21	23	25	27	28	32	34	35	37	40	42
	Exp	M1	M1	M1	M1 P1	M1	P1 M1		F1		F1				M1	P1	P1	P1	F1			P1
	AoE	Rx	Re	Rv Re	Rv Re Rt	Rv Re Rt	Rv Re	Rg Rt	Rt	Rt	Rg	Rt	Rg Re Rt	Re	Rv	Re Rt	Rv	Rv	Rt	Re Rg Rt	Rt	Rg
	Turn	46	48	50	52	54	56	58	60	62	64	66										
	Exp	P1	P1	P1	P1		P1	P1	P1		F1											
	AoE		Rv		Rg Rt	Rv	Rg	Rv	Rg	Rc	Re Rt	Rv										
6.1	Trinomial (Tr)																					
	Turn	9	11	13	15	16	18	20	23	24	26	29										
	Exp	F2	F2	F2		F2	F2			F2												

	AoE	Rg Rn	Rg	Rn Rf Rt	Rn Rv Rg	Rn	Rv	Rn	Rn	Rn	Rv Rn	Rg	
7	Goal: algebraic expressions (tpc)												
	Turn	1	3	5	7	9	11						
	Exp	G1 G2	G2			G2 G1 F2	G1						
	AoE	Rg	Rg Rn	Rn Rt	Rn	Rn Rg							
8.1	Example 1: Ascending and descending powers (Eg1)												
	Turn	1	2	4									
	Exp	M1	F2	F2									
	AoE	Rx	Rn	Rv Rt									
8.2	Side												
	Turn	1	3	5	7	9	10	11	12	14	16		
	Exp	F2	F2	F2	M1 F2	G2 P1 F1	F2 G2			F2	P1 F2		
	AoE	Rg Rt	Rg	Rt Rf	Rg	Rv	Rf	Rv	Rg	Rg	Rv		
8.1	Example 1: Ascending and descending powers (Eg1)												
	Turn	5	7	9	10	12	13	14	16	18	20	22	24
	Exp	M1 F1	F1	F1	F1		M1 G2	M1 F1		F1	F1	F1 M1	M1 G2

	AoE	Rg Rt	Rv Rf Rt	Rv Rn	Rn	Rv Rn	Rg	Rg	Rv Rn	Rf	Rv	Rv Rf	Rv Rg
9	Classwork/ Homework (HW)												
	Turn	1	3	5	6								
	Exp				F2 F1								
	AoE	Rg	Rv	Rg	Rg								

For each lesson transcript, I constructed the same table shown above for explanation and acts of explaining codes. The table enabled me to do frequency counts of the codes for explanation and acts of explaining. For Teacher B's Lesson 1 the frequency counts were as shown in the table below.

Table 5-11: Frequency counts table for

ACTS OF EXPLAINING	**Rg**	4	7	6	2	6		4	1	8	4	3		5	4	3	58
	Re		3			1				10							14
	Rc				2	1											3
	Rt	1	2	2	5	6	2	4		13	1	1	1	2	2		42
	Rf		5	7		3	1	7			1			2	3		29
	Rv	1	4	14	9	7	1	2		11	3		1	3	7	1	64
	Rn				8	3		4			8	4	1		4		32
	Rx				3	1			1	1			1				7
EXPLANATION	**G_2**			3	4	2	1	2	1			3		2	2		20
	G_1											3					3
	P_2																0
	P_1	3	5	5	5	3	5	3	1	13				2			45
	F_2					3		5	1		6	1	2	7		1	26
	F_1				1	4		3	1	4				1	8	1	23
	M_2				5	3											8
	M_1	3	5	1	8	8	3	4	1	7			1	1	5		47
	Act	MM	MM	MM	V	Mo	Cf	Bi	Tr	Sd	Tr	tpc	DP	Sd	DP	HW	
	Sg	1.1	1.2	1.3	2	3	4	5	6.1	6.2	6.1	7	8.1	8.2	8.1	9	
	Time	00:00	03:35	11:15	13:54	19:20	22:28	24:18	26:01	26:35	31:44	33:18	35:22	35:50	37:58	41:07	Total
	L	L1: ALGEBRAIC EXPRESSIONS **(48:05minutes)**															

The frequency counts table highlighted gaps, P2, in explanation codes and dominances in both explanation and acts of explaining criteria. The table shows that there was a dominance of M1 which is talk about elements as a string of symbols. There was also a dominance of talk about process at P1 level, which refers to procedural activity focused on steps of a method. Acts of explaining codes are present from highest in terms of frequency to lowest: re-voicing Rv, regulative Rg, gestures Rt and use of mathematical language Rn. This is how I carried out the coding and analysis of the data in chapters 7 and 8.

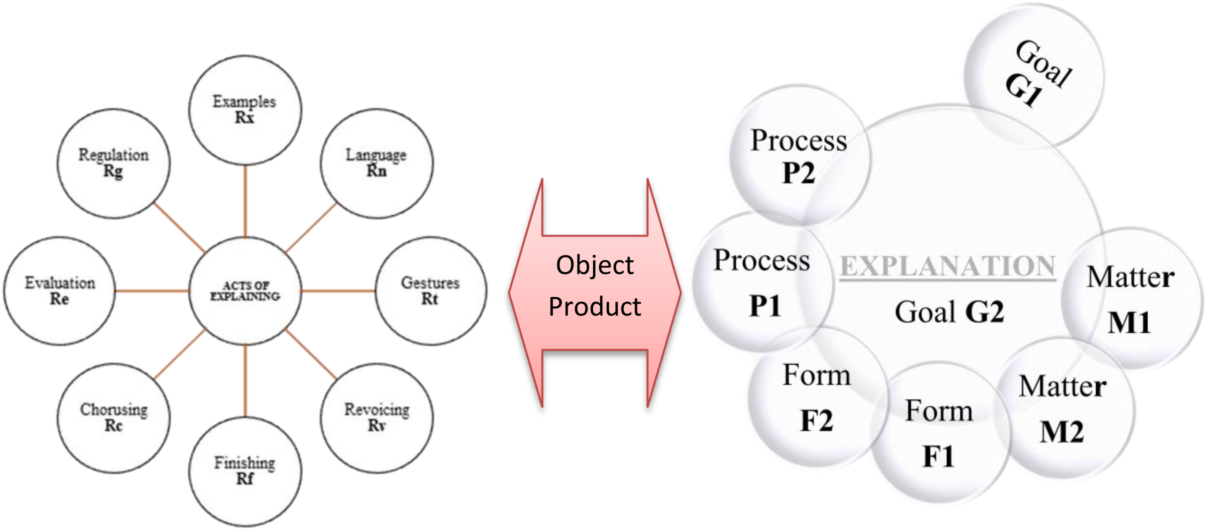


Figure 5:2 – Tools obtained from the literature and data for explanation and acts of explaining

The meaning of the double arrow in Figure 5:2 above illustrates that the acts of explaining have the explanation criteria as their object while the explanation itself is a product of the acts of explaining. As already noted, the acts of explaining are criteria I obtained from the literature and from observations I made on the data. With regards to explanation criteria, I have already discussed that the four criteria presented by Ruben were expanded and further developed in the interaction with the data.

5.2.5 Reliability and validity

The instruments used to collect the data required no reliability test; the video recorder produces data that is specific to the context in the same way that the semi-structured interview with course presenter produces data that is specific to the interview. The reliability question is focused on the processes I used to analyse the data from both course and teachers, especially the interpretations I then made as a result of the analysis.

The contexts in which I used the qualitative approach was teacher professional development, where I collected document data from the course, and classroom practice, where I video recorded lessons, in order to find constitutive criteria for explanation and acts of explaining. To be objective in my interpretation of the notions of explanation and acts of explaining displayed by the course and teachers, I recruited from the philosophical literature the distinction between explanation and acts of explaining. Specifically, I used Ruben's interpretation of Aristotle's criteria of explanation. My reading of the literature was guarded from turning into my own fallible constructions to more objective interpretations through the use of Ruben's criteria as an organisational tool to structure the ways in which I engage with the literature on the historical development section in Chapter 4 and the learning and teaching of school algebra section in Chapter 4. To make sure of a fair representation of the different contexts in which I collected data, I implemented extensive display of the primary text to evidence the analysis in analysis chapters.

I engaged in the analysis process using the criteria of explanation as codes. The credibility of the codes was authorised by the rigorous process of particularising the criteria of explanation and reinterpreting them using three different contexts where algebraic expressions are spoken about in the previous chapter. Matter was the elements and it occurred in the data that matter as a code had a continuous presence, because of the mode of talk employed in the lessons. I coded each utterance according to the descriptions of the criteria as elaborated in Chapter 3 and 4. It emerged that each criterion had two levels, with the first three, (M, F and P), related hierarchically, except for the last one, G.

The inferences I made about the results of the analysis are valid, because I was consistent in the application of the codes throughout the analysis of the data. The interpretations and conclusions I drew about explanation and acts of explaining were based firstly on my reading and synthesis of the literature, and secondly on the results of the analysis. The interpretations and conclusions I drew about teachers' practices, about the course were based on the results I obtained from analysing the data. For instance, I found that the most privileged criteria of explanation for both teachers was the reading of constitutive elements, with the code M1, and carrying out steps of a method, with the code P1. In the acts of explaining the dominant and most privileged criteria was regulative talk, Rg, and re-voicing, Rv.

The theoretical engagement I went through in the process of finding the criteria for explanation and acts of explaining assisted in making sense of the particular topic I was focused on, that is algebraic expressions, however, application of the same criteria in a different topic within the discipline of mathematics would probably result in further extension and elaboration of the criteria. The sample size and the focus on one topic limits the transferability of the findings obtained from this study, however, the criteria of explanation I obtained from the philosophical literature are generalisable. Also, the notion of separating between explanation and acts of explaining is transferable and not limited to a study of algebraic expressions. I revisit the discussion of limitations in the conclusions chapter.

5.3 Conclusion

In this chapter, I have described the methodology as well as the coding and analysis of the data. I provided a description showing and supporting my choice of sampling, data collection and analysis. It emerged that each code in the explanation had two levels and the acts of explaining codes were expanded by the data with the inclusion of communicative techniques. I have described how the data was transcribed, and exemplified the organisation and coding of data as well as the analysis. I was then able to see gaps and dominances from the utterance by utterance coding. I then argued the reliability and validity of the study and stated the ethical considerations taken. I provide an in-depth analysis of the course and the classroom data in chapters 6, 7 and 8, respectively.

CHAPTER 6 WITS MATHS CONNECT SECONDARY PROFESSIONAL DEVELOPMENT PROJECT

6.1 Introduction

The aim in this chapter is to describe what was legitimated as explanation and acts of explaining in the Wits Maths Connect Secondary (WMCS) project. The purpose is to determine the criteria of what counts as a 'good' explanation, and as an act of explaining were privileged and transmitted in the course. From the sessions presented to teachers, I specifically paid attention to those that had a focus on explanation and acts of explaining. I describe how explanation and acts of explaining are themselves explained to teachers in the course. My focus of analysis in the PD project is what mathematically was explained, and how a mathematical explanation was set out and elaborated. The evidence I provide for the course description in this chapter are the overhead clips from PowerPoint presentations used in the sessions, and the handouts that accompanied these in the sessions as well as the follow up supporting interview with the course presenter. As discussed in Chapter 5, there are no video records of the sessions, so discursive evidence of the course is based on the follow up interview I conducted with the course presenter for selected work on explanation (see appendix B1.2 for the interview schedule). The full PowerPoint presentations and handouts are found in Appendices B1.1. The specific research question I am seeking to answer in this chapter is as follows:

2. What is legitimated as an explanation, and acts of explaining in the WMCS course?

2.1 What constitutive criteria of 'explanation' and 'acts of explaining' are privileged?

2.2 What messages about 'quality' of both explanation and acts of explaining are transmitted?

In Chapter Three, a distinction between explanation and acts of explaining was made. Explanation was perceived of as the knowledge conception while acts of explaining were the methodology of enacting the explanation. Explanation is focused on the content, while acts of explaining are pedagogically oriented. This kind of separation between explanation and acts of explaining is only possible for analytic purposes and is used in this chapter to analyse the criteria transmited by the course for explanation and acts of explaining.

After looking through the course documents and the supporting interview it became apparent that in the course two types of explanation were presented and discussed with teachers; that is a “what” and a “how” explanation. However, for an explanation to be full it has to be justified or legitimated in some way and so both the “what” and “how” explanations could have a “why” and a “when” legitimation component. Thus the two types of explanation transmitted by the course were:

- What – could be accompanied by why and or when
- How – could be accompanied by why and or when

In Chapter 4, I observed that a dominant distinction between conceptual and procedural understanding is typically framed as a distinction between “why” and “how”, repectively. However, recent literature in the same field of mathematics education has shown and argued that procedures have a conceptual base that contributes to the understanding of the mathematical object being handled. Thus, it is misleading to separate the two. This discussion can be extended to the distinction between “what” and “how” whether it is also not as misleading as the conceptual/procedural distinction. This reasoning and debate appears to have informed the course which does not separate between “why” and “how”, but posits that there is a why component for both “what” and “how” types of explanations. Added to the “why” justification or legitimation for an explanation is the “when”, so both the “what” and “how” explanations may have a “when” component.

In the PowerPoint slides and handouts, there is a typology for what the philosophical literature referred to as acts of explaining that had been explicitly transmitted by the course. These are what the PD literature refers to as PCK, which is a way of transforming content knowledge (SMK) into pedagogical actions. The course, in this regard, has transmitted that careful selection and sequencing of examples, and their representations are important as well as the language used to talk about these.

I related the criteria transmitted by the course for explanation with Ruben’s explanations framework. In doing the analysis of the course, I code some of the wording from the PowerPoint slides and handouts using the criteria transmitted by the course for explanation, namely: “what”, “how”, “when” and “why”. I also coded wording in the powerpoint slides and handouts related to acts of explaining using the criteria transmitted by the course for acts of explainng such as examples and their representations as well as language. I further associated explanation with SMK and acts of explaining with PCK. At the end of this chapter it was possible for me to read from the PowerPoint slides, handouts and supporting interview, criteria transmitted by the course for explanation and acts of explaining.

To arrive at what has been constructed as explanation and acts of explaining in the course, I begin with a description of the wider project from which this study arises and in which it is located. This is then followed by a description of the content of the course the teachers participating in my study attended. In the description of the content of the course, special attention is directed to sessions that focused on explanation and acts of explaining. These sessions were structured by a Maths Teaching Framework (MTF), from which criteria for explanation and acts of explaining were made explicit and so the MTF and its elaborations are discussed. Exemplifications of the criteria for both explanation and acts of explaining are discussed. The entire discussion is substantiated by excerpts from the follow up interview I conducted with the course presenter. I then conclude the chapter by presenting a synthesis of what the course transmitted as criteria for a 'good' explanation, and acts of explaining in mathematics.

6.2 Overview of TM1 course

In Chapter 2, I briefly described the courses offered by the WMCS PD project and those are transition maths 1 and 2 (TM1 and TM2). I noted that my focus is on TM1 and thus briefly described in this section is the overview of TM1 course.

Even though there were no video records of the presentations in the course, I am able to recollect the content of what was done, and provide a description of the structure of how explanation and acts of explaining were dealt with. This is a result of my partly monitoring the course on the one hand, and more so due to the documentation that exists and the supporting follow up interview I conducted with the course presenter.

I provide an account of the TM 1 course in 2012, because the teachers participating in my study all attended and completed TM 1 successfully in that year.[11] Criteria for 'success' included: attendance at 14 of the 16 days, submission of mathematics and mathematics teaching tasks prepared before sessions, results in mid and post mathematics tests (there was no formal assessment of the mathematics teaching in which the work on explanation and explaining took place, and so no further criteria transmitted).

[11] It is worthy to note that one of the teachers (Teacher A) has since continued to do TM 2 course and also completed it successfully.

The 16 days in TM1 were composed of 8×2 full day university sessions spread over the academic year. Each day session started at 08h30 and ended at 16h30. From the 16 days conducted at the University, four of the sessions, specifically Sessions 1.5 to 1.8, had a focus on explanation and explaining as highlighted in Table 6-1 below.

Table 6-1: TM1 course dates and topics for the year 2012

COURSE	DATE	DESCRIPTION
TM 1.1	22 Feb – Thursday	(pre-test) Connecting number, algebra and proof, equality and variables,
	23 Feb – Friday	Algebraic procedures
TM 1.2	16 March – Friday	Equations, inequalities, balancing syntax and meaning
	17 March – Saturday	Quadratic and cubic inequalities
TM 1.3	11 May – Friday	Key features of functions
	12 May – Saturday	Functions as models
TM 1.4	14 June – Thursday	Geometry – Euclidean geometry
	15 June – Friday	Geometry – ratio and proportion
TM 1.5	12 July – Thursday	(mid-test) Functions and their inverses, mathematical explanations
	13 July – Friday	Transformation of functions
TM1.6	24 August – Friday	Explaining key ideas in trigonometry, Trigonometric basics, trigonometric functions
	25 August – Saturday	Trigonometric equations
TM1.7	19 September – Wednesday	Pattern generalization, functions, exponents and explaining
	20 September – Thursday	Sequences and series, exponents
TM1.8	26 October – Friday	(post-test) Algebraic expressions – working on explanations *
	27 October – Saturday	Functions

Even though not all days followed exactly the same pattern, in each day there were four sessions, two before lunch and two after lunch. In the four days highlighted above one session from the four was a teaching focused session. The remaining three sessions were called mathematics focused sessions. Over and above the mathematics focus and teaching focus sessions there were tutorial session where teachers received feedback on their submitted work and there was a support time session for help with the mathematics. As we can see from the table above and what will follow, three quarters of the day's session had a mathematics focus and the TM1 course was referred to as a mathematics for teaching course. A quarter of the course was on teaching and some of those focused on explanation and acts of explaining, to which I now turn.

6.2.1 The sessions with a specific focus on explanation and acts of explaining

From the TM1 course overview there are four days highlighted that had a focus on explanations that is session 1.5, 1.6, 1.7, and 1.8. In spite of the title for the teaching session for 1.5 indicating that the session is focused on a mathematical explanation, the attention for the session was on selecting and sequencing examples, while the remaining sessions were focused on both explanation and acts of explaining. Below is a detailed programme for each day.

Table 6-2: Detailed programme for each of the four two day sessions

Time	TM1.5	TM1.6	TM1.7	TM1.8
	Day 1			
08h30	Midcourse test	Maths focus 1: revisiting the trig basics 1	Maths focus (patterns1: generalisation)	Things that disappear in algebra how do we explain this?
	Tea	Tea	Tea	Tea
	Maths focus 1: function notation, domain and range	Maths focus 2: revisiting the trig basics 2	Maths focus 2 (patterns 2: links to functions)	Working on explanations,
	Lunch	Lunch	Lunch	Lunch
	Teaching focus 1: mathematical explanation	Maths focus 3: trigonometric functions	Maths focus 3 (exponents1: laws and definitions)	Working on introductory algebra
	Maths focus 2: inverses of functions	Teaching focus 1: explaining key ideas in trigonometry	Teaching focus 1 (explaining and exponents)	Working on introductory algebra (cont.)
16h00 - 16h30	Support time	Support time	Support time	Support time
	Day 2			
08h30	Maths focus 3: Transformation of functions	Tutorial session: Feedback on homework and further support	Maths focus 4: Sequences and series	Working with functions
	Tea	Maths focus 4: trigonometric equations	Tea	Tea
	Maths focus 3 (cont.)	Tea	Maths focus 5: Exponents 2:	Working with functions

			exponential growth,	
	Lunch	Maths focus 5: Applications of trigonometry involving measurements	Lunch	Feedback on homework
	Teaching focus 2: Analysing Grade 10 learners exam responses	Lunch	Tutorial session: feedback on homework,	Lunch
	Wrap up	Maths focus 6: Application of trig involving calculations	Guest speaker,	Working with integers,
16h00 – 16h30	Support time	Wrap up	Wrap up.	Wrap up.

From these four sessions, the maths teaching focus was structured by what was called the Maths Teaching Framework (MTF). It is in the MTF that what counts as explanation is made explicit. Of course, there were implicit messages about a good and a full explanation co-constituted by both the teachers and the course presenter with respect to the mathematical content they were handling in all of the course sessions. I have chosen, for the purpose of this chapter and its focus on criteria transmited by the course for explanation and acts of explaining, to provide a full decription of the framework and its elaboration as used in TM1.6 and 1.8. This is then followed by the two examples of how teachers engaged with the notion of explanation coming from TM1.6 and 1.7. From the discussion of the MTF, its elaboration and the two examples it is possible to read criteria for both explanation and acts of explaining transmited by the course. The full table showing how I used the documents from the course to read criteria for explanation and acts of explaining is found in Apendix B1. The next discussion is focused on the MTF.

6.2.2 The Maths Teaching Framework

The MTF is an idea that has advanced in development over time in the project. It was a moving target that my study was seeking to understand. As reflected in the course presenter's comments below, the framework has since changed and the project now has different versions of it, however, the version I used for this interview and this chapter is the very first one presented to teachers in 2012 and 2013.

8 P: *I think it's important, just before we carry on, to remember that this framework has evolved since 2012 right. So, I mean, this is TM 1.2, which is about March 2013, and so the fact that stuff has shifted its quite possible that I will say things now that we might have not even known at that time. So I think, I mean, when I look at the prompts that are on this explanation column, they not even the same any more compared to the current version. We didn't even have the expanded version of the explanations column in 2013, the expanded version only arrived late 20, 2013* (softly, and not sure). *Err, the guys who did this course I don't think they did the expanded version at all.*

Object of learning teaching x to y		
Examples and representations	Explanations and questions	Learner activity
What examples and representations will I use? At the start of the lesson For questioning For further explanation How will these help to build the key concepts and skills?	What kinds of explanations (and related questions) will I use? What? How? Why? When? How will these help to build the key concepts and skills?	What work will the learners do? e.g., listening, answering questions, copying from the board, solving a problem, discussing their thinking with others, explaining their thinking to the class How will these help to build the key concepts and skills?
Coherence: Are there coherent connections between the object of learning, and my choices of examples and explanations? Sequencing: Will the sequencing of examples & explanations be appropriate for learners to learn?		

Figure 6:1 – MTF handout for planning, observing, and reflecting

The first row has the object of learning. In this row, the teacher has to articulate what they are teaching and what skill or capability they are seeking to have developed in learners by the end of the lesson or what it is learners are to know and know how to do. The next row is divided into three columns: the first column presents ideas related to selection and use of examples in a lesson. The second column specifies the four types of explanations offered by the course. The last column presents types of learner activities.

Under 'examples', first column, the teacher is asked to decide on the examples she will use, how the examples will be represented and sequenced and how these will help to build the key concepts and skills in relation to the object of learning. These are pedagogical decisions. The teacher needs to be clear about the usefulness of each example and the mathematical skill or capability each example seeks to develop in relation to the object of learning in the lesson. The teachers' subject matter knowledge (SMK) is required to decide on the mathematical skill and how the collective examples come together to build this.

With regard to 'explanation', second column, the teacher is asked to think about the kinds of explanations she will use, the actual words she will utter to make the content clear and accessible to learners. The "what", "how", "why" and "when" interrogative questions are presented in this column as explanation criteria for teachers to think about as they plan and reflect on their lessons. In effect, these are the criteria the course stated and transmited for an explanation. I will dwell more on this column since the concern is to find criteria for what the course presented for explanation and acts of explaining.

The last column (third column) focuses on learner activity; here teachers are asked to plan and reflect on what work learners will do. Learners as the audience also contribute to the construction of the explanation and this column describes their actions in the lesson, such as answering questions, writing and listening.

The last row of the MTF refers to coherence. There is a message transmitted that says for explanations to be coherent there has to be connections between what is to be learnt, examples used and the explanations offered. The Maths Teaching Framework (MTF) is used with teachers to plan, observe and reflect on lessons. This is the actual picture, with the MTF taken from TM1.6 teaching session, presented in Figure 6:2 below, taken from the PowerPoint slides (Slide number 2) presented to teachers.

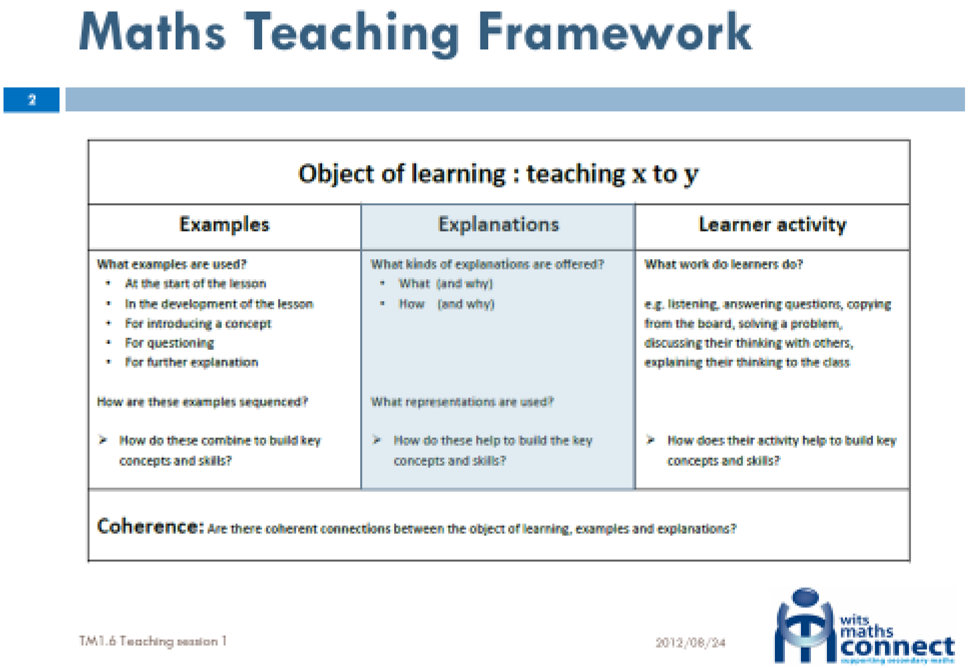


Figure 6:2 – The Maths Teaching Framework (WMCS PD TM1.6 pptslide 2)

From the MTF, my attention is focused on capturing how “explanation and acts of explaining” in mathematics are themselves explained. The concern is to find criteria transmitted for explanation and acts of explaining. There are three guiding questions for the discussion and analysis of the documents: 1) what is stated? 2) what is elaborated? and 3) what is exemplified? I then bring what is stated, elaborated and exemplified as explanation into alignment with Ruben’s criterias of explanation. I also discuss the criteria for acts of explaining in relation to what emerged in the literature. I start off by discussing criteria transmitted for explanation and then move on to discuss criteria transmitted for acts of explaining.

6.3 Explanation

The MTF, which was the framework used generally across most teaching sessions, stated the explanation criteria. The MTF was used in sessions that focused on functions, trigonometry, and algebra and so the elaborations and exemplifications of the explanation are in relation to the mathematics focused on in the particular session. I decided that the elaboration was anything that sought to define the stated criteria. For the exemplification, I looked at documents that sought to do an application of the stated and elaborated criteria. For explanation, the elaboration came from TM1.6. The exemplification of the explanation was taken from sessions 1.6, 1.7 and 1.8.

6.3.1 What is stated as explanation criteria?

What was stated as explanation has already been mentioned in the discussion of the MTF, specifically the middle column. I bring it in again here as shown in Figure 6:3 below.

Explanations
What kinds of explanations are offered? • What (and why) • How (and why) What representations are used? ➢ How do these help to build the key concepts and skills?

Figure 6:3 – Statement of explanation[12]

In the figure above, there is a question posed and it says: what kinds of explanation are offered? This question is followed by two bullets, the first stating "what" and "why" and the second stating "how" and "why". There is an omission of the "when" interrogative question in the PowerPoint slide, whereas it was mentioned in the handout. Also the four interrogative questions (what, how, why and when) were listed as four distinct questions in the handout whereas in the PowerPoint slide there is a grouping of some of the interrogative questions into two bullets.

There are thus two kinds of explanation that are differentiated that is the "what" and "how" explanations. The "why" is a sub-explanations for the "what" and "how" explanations. The course presenter's comments below confirmed this distinction.

[12] The notion of representations included towards the end of the explanations column is important to highlight the idea earlier noted in Chapter 3 that representations are found in both explanation and acts of explaining.

8 *P: I think at this point the big issue that we were distinguishing, at the time, is that there are how explanations and there are why explanations* [...] *I think the key thing was a how explanation as we said. Err sorry, a what explanation and a how explanation. And both of them had whys attached to it. And a what was related to, to what you do, to what the thing is and a how was related to how you do it…*

Using Ruben's four criteria of explanation to look at the course presenter's comment, the "what" explanation agrees with elements and form, while the "how" explanation tie in with process. The course presenter notes that the "why" explanation is attached to the "what" and "how" explanations. The "why" attachment explanation concurs with goal at the level of justifications. Thus the "what" and "how" explanations with their "why" and "when" attachments were the criteria the course stated and transmitted to teachers. These interrogative particles are in relation to the mathematics content and thus SMK. The following discussion is focused on how the "what" and "how" explanations were elaborated.

6.3.2 What is elaborated as explanation?

The elaboration of the two types of explanation stated above were obtained from TM 1.6. The examples used comprise trigonometry, Euclidean geometry, and algebra, even though TM 1.6 predominantly focused on trigonometry. Slide Number 4 in Figure 6:4 below illustrates the elaboration of the explanation criteria.

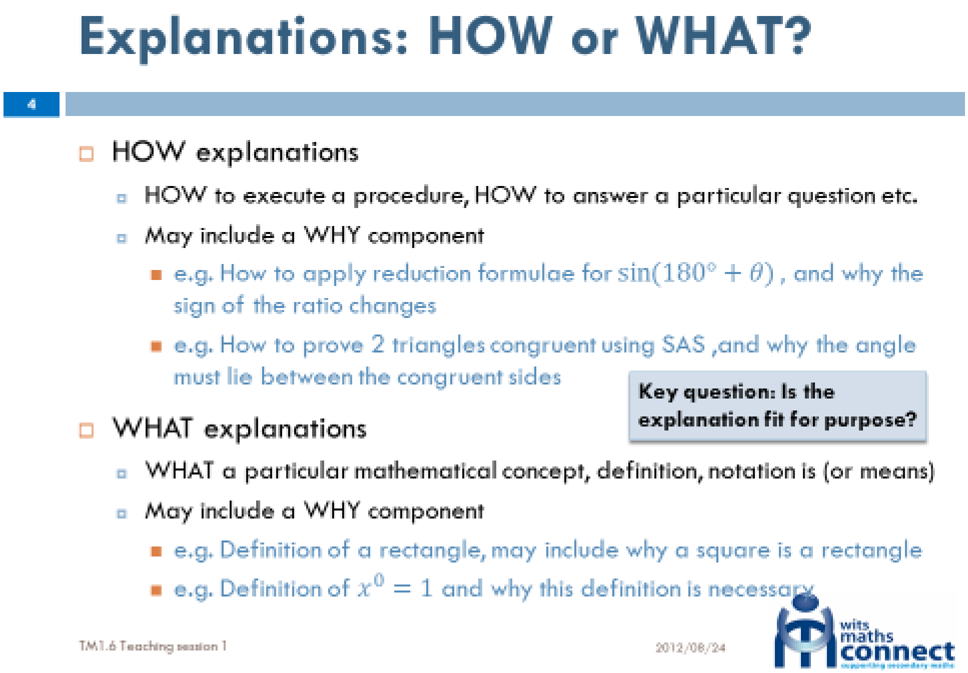


Figure 6:4 – Elaboration of explanations (WMCS PD TM1.6 ppt slide 4)

The "how" explanation is elaborated as the execution of the procedure; application of formulae, as well as providing a proof for congruency. The use of the word *may* indicates that the "why" explanation is not decisively attached to the "how", meaning there may be instances in the execution of a procedure which may not require or have a "why" explanation. The "what" explanation is elaborated as the meaning of particular concepts, definitions or notation. Similalry the "why" attachment is a wavering supplement because of the word *may*. In the follow up interview the course presenter elaborated further on the slide in this way:

10 P: …the how explanation was, how you execute a procedure. Typical example was the distributive law, completing the square, using the quadratic formula kind of stuff… versus the what explanation being what this thing is, and getting at the concept, the, the notation, the definition, whatever…

The elaborations of the "what" explanations with their "why" attachments concurs with Ruben's critieria of explanation elements and form. The "what" explanation is defined as *what this thing is, getting at the concept, notation, definition*.

In Ruben's terms, the course presenter's phrase: "*what this thing is*" is describing the elements that constitute the object. When one provides an explanation of *what the thing is* made of and its characteristics, that is talk about matter and form. To define the form the elements take requires making explicit the laws and definitions that govern or bring the elements together into a particular arrangement which is the structure of the object; in the course presenter's words that is "*notation, definition, and getting at the concept*". In relation to teacher knowledge categories the "what" explanation is SMK.

The "how" explanation as seen above pertains to *how to execute a procedure* and the examples the course presenter provides to elaborate are the distributive law, completing the square and using the quadratic formula. This refers to process according to Ruben's criteria and SMK in relation to Shulman's knowledge categories. The course presenter's elaboration of the "why" component of the explanation is shown below.

10 P: ...and typically working from the assumption that many times teachers' explanations are how explanations not what and that the why part of the explanation often doesn't get attention. And in many cases you can pass the test or exam without having the why. So teachers don't, maybe, feel the need to do that. So we wanted to foreground the importance of having a justification for what gets done and why it gets done, err. I think that those are the overarching things that come in the beginning.

The justification part for what gets done and why it gets done is what Ruben refers to as the goal. Viewed from the WMCS course perspective, the goal is to know why the "what" is what it is and why the method holds. By answering the "why" aspect of both "what" and "how" explanations the purpose is made clear. The "why" attachment is also SMK.

One of the things the course presenter mentions is the fact that teachers' explanations in mathematics are often "how" explanations which is a result of the way assessments are fashioned, and so the course wanted to foreground the justification aspect for what gets done in mathematics.

At this point of the interview I ask the course presenter about the "when" sub-explanation. In response to this, the course presenter acknowledges that the "when" aspect which was mainly related to the "how" part of the explanation was backgrounded and hasn't been foregrounded ever since.

13 I: ....You have not spoken about the "when", and there is a "when" put in brackets there [pointing at writing on the slide].

14 P: ...*the "when" was also, like, so it was kinda like, when you use this, that's what it was related to. I mean something that we haven't, we didn't foreground much in the course if I remember, and we certainly haven't foregrounded it since. It was kinda like related to procedures of when you use it and when you don't. And it kind of links I suppose to over-generalisation when procedures get used inappropriately…*

As already noted, the course presenter emphasises that the "when" explanation was an attachment to the "how" explanation. In relation to the "how" explanation the "when" attachment was referring to misapplication of a procedure which often is an overgeneralisation of a rule. The course presenter is explicit that the "when" attachment was *related to procedures of when you use it and when you don't* and there is no mention of the "when" attachment in relation to the "what" explanation. The course presenter is highlighting that there are permissible bounds over which procedures are applicable, and so the "when" attachment was supposed to highlight those as part of the "how" explanation. The interrogative question "when" refers to a time when a procedure must begin and stop. The "when" explanation in relation to a "what" explanation is not necessary since the what in itself is expounded through a process. So, the "when" explanation in relation to the "how" also caters for the "what".

The next interesting question is how the course operationalised the statements and elaborations of explanation criteria. The way the course put into use its definitions of "what" and "how" explanations was through elaborations which offered indicators for "what" and "how" explanations as already discussed in Figure 6:4 in this section. The course used the concept of the distributive property as an exemplification of the explanation to which I now turn.

6.3.3 What is exemplified as explanation?

The concern is to find exemplifications of a "what" and a "how" explanation. I present three instances from the course documents that I considered to be exemplifications of explanation criteria. The first is the slide with the title: *what mathematics is emphasised?* The second instance is an exemplification of the explanation for the distributive property of number. The last instance is an activity from the independent work handout, which required teachers to critique the provided explanation. I begin with what mathematics is emphasised.

6.3.3.1 What mathematics is emphasised?

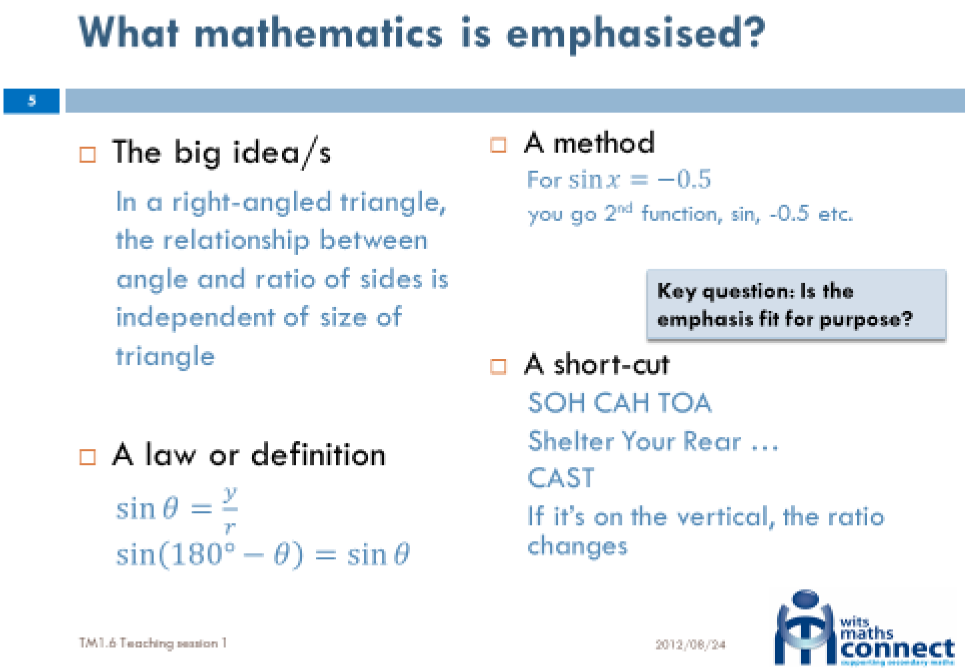


Figure 6:5 – Exemplification of explanation (taken from TM1.6 ppt slide 5)

In this slide shown in Figure 6:5 above, there are four indicators mentioned, which highlight the mathematics that is emphasised, and those are: 1) the big idea; 2) a method; 3) a law or definition; and 4) a short cut. The content focus was trigonometry instead of algebriac expression. However, the four indicators (big idea, method, short cut, and law) are used in the following slide, taken from TM session 1.8, which is an illustration of how the explanations were exemplified in algebra through the use of the distributive property.

6.3.3.2 Distributive property

The slide in Figure 6:6 below is an exemplification of the "how" and "why" explanation within the context of algebriac expressions. There is now an additional indicator, of the mathematics that is emphasised, to the ones earlier mentioned in the trigonometry slide making it a sum of five, viz. 1) the big idea; 2) law; 3) method; 4) short cut; and 5) illustration.

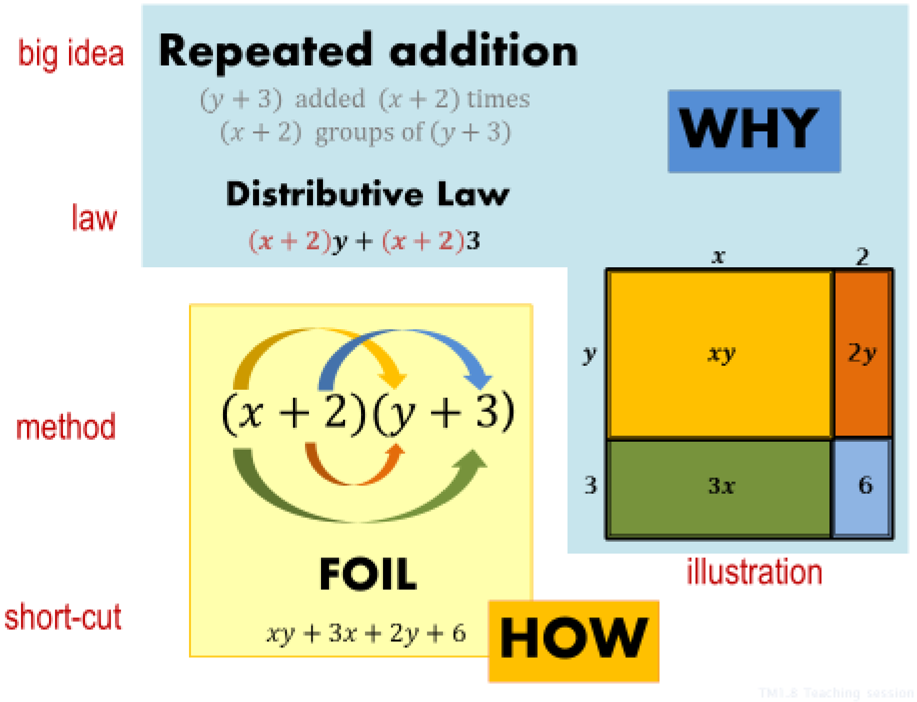


Figure 6:6 – Exemplification of explanation (taken from TM1.8 ppt slide 11)

According to the slide, the *law* is the distributive property and the *big idea* is the notion of repeated addition. The rectangular *illustration* as well as the big idea (repeated addition), are shaded in blue and labelled as the "why" explanation. In the yellow box the "how" explanation is constituted of the *method* (distributive arrows) and the *short cut*, which is the FOIL method. When the indicators are viewed using the "what" and "how" criteria of explanation, one can see that the big idea and the law are "what" kinds of explanation; while the method, illustration and short-cut are "how" kinds of explanation. However, the illustration is included in the "why" explanation because of the role it plays as a justification for "why" the method and short cut work. In this way, it becames a different level of a "how" explanation. Similarly, with the notion of repeated addition. Repeated addition is a description of a method performed step-by-step, and in essence a "how" explanation. Since it is used here to provide a justification, as a "why" explanation, for the distributive property, then the notion of repeated addition is a different level of a "how" explanation. Solidified here is the notion that a "why" explanation for the "what" and "how" is a "how" explanation. For a Grade Nine learner, the notions of repeated addition and area are computational processes learners have encountered in lower grades and so can be considered prior knowledge. This prior knowledge is invoked here to serve as justifications for the distributive property because they both are about the notion of multiplication.

What is becoming clear from this slide is that there is no way of bringing a mathematical concept into existence without the use of representations (algebraic and geometric) and accompanying computational processes (adding groups, calculating area). In addition, the distributive property as the example used in this slide makes clear that the "what" and "how" distinctions are fluid. The course presenter admitted this by highlighting that the boundaries are blurred between the "what" and "how" explanations.

18 *P: Big idea, law, definition, method or short cut. Err, these are different in a sense. If you like are four examples. Are what. What is the mathematics they are emphasising? Is it a big idea, a method, or a short cut, etc.? They are whats. This is probably one of the reasons that this how and what distinction has probably disappeared. That it slides around. Err, it was partly to say that there are different things that we might want to explain, and we might do them in different ways. And err, so some are, there might be some literature that speaks negatively of short cuts and some of those things and wants the conceptual. And we saying, well that's not the practice of our teachers. These are all parts of the practice.*

According to the course presenter the big idea, definition, method and short cut are all "what" kinds of explanations. Even though the method and short-cut can be thought of as "how" type of explanation. This is true because, as already noted, the "how" is a demonstration of the "what"; the "how" is an application of the "what".

The course presenter acknowledges the blurring of boundaries between the "what" and "how" explanations by stating that *it slides around*, and as the project continued the "what" and "how" distinctions have disappeared. As shown in the example using the distributive property, it becomes difficult to separate the two ("what" and "how"), especially because they both contain the "why" attachment. It is in the explanation of the "why" that the distinction between the "what" and "how" explanations becomes blurred. The "why" is a justification for a "what" through a "how". The concept, which is a "what", that required a "why" justification was the distributive property, the "why" justification for the distributive property was presented as repeated addition and the rectangular array which are "how" explanations.

What	How	Why	When
Distribution of multiplication Over addition	- Method - FOIL	- Repeated addition - Geometric illustration	- Method applicable generally where products - FOIL limited to product of two binomials

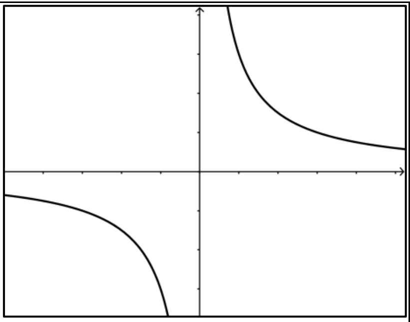
Figure 6:7 – The four interrogative particles in relation to the distributive property.

As shown in Figure 6:7 above, the use of the FOIL method as a process for multiplying polynomials had presented an opportunity to adress the "when" attachment even though not addressed. The last exemplification of the explanation criteria is taken from the independent work activity.

6.3.3.3 Exemplification of explanation by critiquing and extending a given explanation

The following scenario was presented in TM1.7 independent work. In this scenario, a learner's question is posed and it is a why question seeking for a justification. The learner's question is seeking to know *why does the hyperbola have two graphs for one equation?* There is a teacher's response in the scenario, which provides an explanation to the learner's question. In this independent work activity, teachers participating in the course were expected to critique the explanation provided by elaborating and extending the example and the explanation using the criteria for explanation. This means teachers had to pay attention to the "what", "how", as well as the "why" and "when" criteria of explanation.

A Grade 10 learner asked his teacher the following question:
Why does the hyperbola have two graphs for one equation?



The teacher responded:

Let's look at an example of $xy = 12$. If x and y are both positive, then the product is positive and so the graph is in Quadrant I. If x and y are both negative, then the product is still positive but the graph is in quadrant III. That's why you have two graphs.

Task: Think about the teacher's response by considering the example and the explanation.

Figure 6:8 – Hyperbola scenario taken from TM1.7

The course presenter elaborated in the follow up supporting interview the intentions of the task. The intentions of the task were firstly for teachers to identify that the explanation is inadequate because it does not answer the why question. Secondly, teachers were expected to correct or expand the explanation by providing the missing information to make it a correct response to the question posed by the learner, which requires a justification for why there are two pieces for a hyperbola, and not two graphs.

Furthermore, in terms of the structure of the equation, teachers were expected to note that the explanation should rather maintain the form $y = \frac{12}{x}$, because it makes visible that the independent variable x is in the denominator and the implications this has for an input of zero.

Lastly, the teachers were expected to focus on correct mathematical language use that its not two graphs but two pieces of the same graph with a discontinuity.

34 P: *I mean what the explanation does do is to separate working with positive inputs and working with negative inputs. But it doesn't deal with the idea of, which is the crux, an input of zero either for x or y or both. Then the equality of xy equals to twelve doesn't hold or if you go the standard form of twelve over x. We gonna get a zero denominator. It was those issues that I was wanting teachers to get at. This explanation is inadequate if you like. It doesn't get at the why. It's kinda telling me some of the what. Positive time positive they gonna be positives. It doesn't get at the issue of discontinuity of*

asymptote. Of the fact that if you wanna go as x tends to zero from the left, that the y values are getting larger and larger negative and change to zero from the right the values are getting larger and larger positive. None of those issues is dealt with in this explanation. So it was the justification linked to input and output that for me was important for me here. And that was gonna focus on did the teachers appreciate the mathematics? Oh I mean, the other thing is why you have two graphs kinda show the error, in a sense this is not two graphs this is one graph with two pieces. And so that other issue was picking up on the mathematics that's why you have two graphs. So from that point there is an error in the explanation that it's not, its two pieces of the same graph and that's how we talk about it. Because the graph is of the entire function so I mean there is correct stuff and something that's not. I don't think that is majorly incorrect, it's just a wrong way of talking about it.

We see that the course presenter actually expected teachers to use the criteria for explanation, this is evident in his talk as he refers to the "why" and "what", to engage with the task. The course presenter expected teachers to engage with the different forms $xy = 12$ and $y = \frac{12}{x}$. The form $xy = 12$ does not show the effect and meaning of having the input $x = 0$ compared to the starndard form $y = \frac{12}{x}$. This is all at the level of the "what", which in Ruben's terms is getting at understanding the meaning of the elements that constitute the equation and the meaning of the form of the equation in order to understand the form of the graph.

To account for the discontinuity, the discussion of x and its relationship to y is highlighted as necessary for noting in the explanation that as x tends to zero from the left, the y values are getting larger and larger negative. As x tends to zero from the right the y values are getting larger and larger positive.

The course presenter was alluding to a particular way of talking about the graph of the hyperbola, where, instead of referring to it as two separate graphs it is rather *two pieces of the same graph and that's how we talk about it, because the graph is of the entire function*. At this particular point, the course presenter is emphasising a way of perceiving the *entire* structure of the graph.

In a nutshell, the course used the distributive property and the independent work scenario as exemplification for the explanation criteria. Under the "what" explanation, the big idea and law were highlighted; for the "how" explanation the method, short cut and illustration; under the "why" attachment, the big idea and illustration were highlighted.

To summarise, in this section I discussed criteria transmitted by the course using documents and the follow up supporting interview with the course presenter. To find the criteria I looked at what was stated, elaborated and exemplified as criteria for explanation. I found that the four interrogative particles, viz. "what", "how", "why" and "when" were stated as criteria of explanation. I found that the elaborations of the stated criteria were that the "what" referred to definition of a concept and law; the "how" referred to method used to execute a procedure. The "why" and "when" were meant as attachments to the two kinds of explanation, the "what" and "how". However, there were no elaborations and exemplifications of the "when" attachment except for the course presenter's elaboration in the follow up interview. I found that five indicators in relation to the distributive property were used to exemplify the explanation criteria and those were: definition, law, illustration, method, and short cut. I dovetailed the findings with Ruben's criteria of explanation as shown in Table 6-3 below.

Table 6-3: Criteria for explanation

Stated	Elaborated	Exemplified	Ruben
What	Definition, big idea, concept, notation	Distributive law, big idea	Matter and Form
How	How to execute a procedure	Illustration, method, short cut, big idea	Process
Why	Justification	Big idea, and illustration	Goal
When	Overgeneralisation and misapplication of rules	----	Process

The elaboration recorded in this table for the "when" explanation was taken from the follow up supporting interview with the course presenter. According to the course presenter's elaboration, the "when" explanation was in relation to the "how" explanation and hence the relation with Ruben's third criterion of explanation: process. The "how" explanation in particular encompasses the "what" and "why" in the sense that the big idea, which was repeated addition, is a description of a process, and so is the illustration a process of computing area. Again, the point that the process is an application of a concept is supported by this example. I now proceed to a discussion of what was legitimated as acts of explaining in the course.

6.4 Acts of explaining

Acts of explaining are processes of transforming one's own understanding of mathematical ideas and concepts into pedagogical actions. The "what" and the "how" explanations as provided by the course are both about the mathematical content and specifically they are criteria of explanation. The two interrogative particles transmitted by the course as criteria for explanation, specifically the "what" and "how", can be used alongside Shulman's knowledge categories to classify the explanation and acts of explaining. Borrowing from the language of the course, explanation is a "what" kind of knowledge whereas acts of explaining is a "how" kind of knowledge. The acts of explaining are pedagogical decisions about mathematics in relation to the learners.

In this part of the chapter, an analysis of what was legitimated as criteria for acts of explaining is presented using the three guiding questions used earlier in the foregoing section, namely what is stated, elaborated and exemplified as criteria for acts of explaining. I begin with what was stated.

6.4.1 What is stated as acts of explaining?

I obtained the statements of criteria for acts of explaining from three sources. The PPT slide concerned with how to make mathematics available to learners and the two versions of the MTF (handout and PPT slide).

The title of the slide from TM 1.6 teaching session is "*how is the mathematics made available to learners*". I read this as acts of explaining, because ways of making mathematics available to learners are necessarily acts of explaining, and not the explanation. Ways of making mathematics available to learners are the methodologies of enacting the explanation. The next slide shown in Figure 6:9 below is focused on acts of explaining in the sense that it is mainly about pedagogical choices the teacher needs to make. For instance, the teacher has to make mathematical decisions on examples and their representations to use as well as language and terminology to use in her explanation.

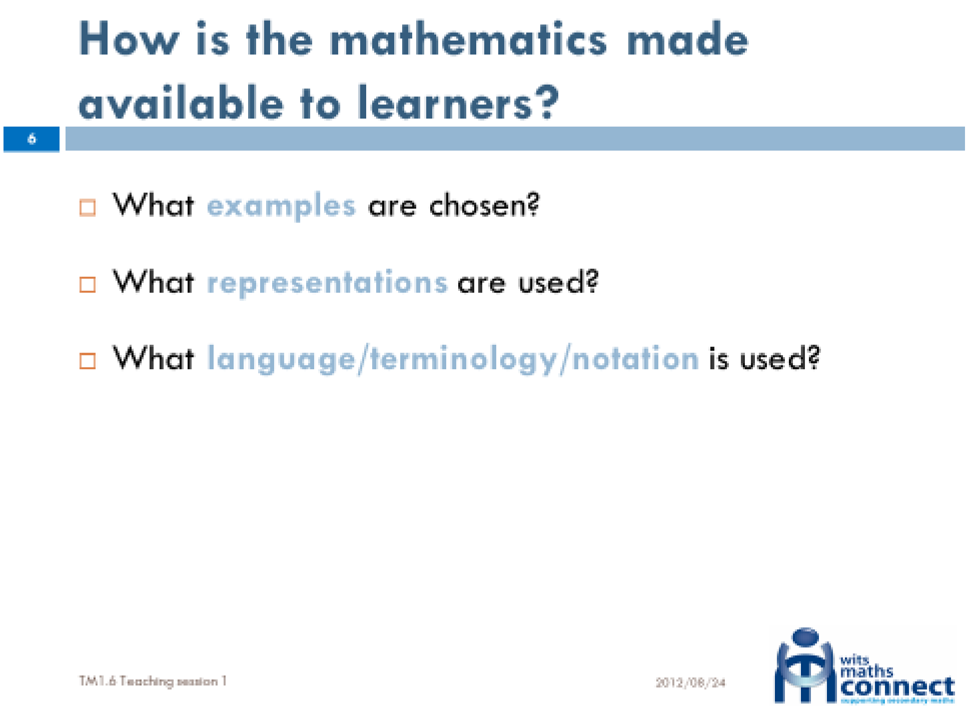


Figure 6:9 – Statement of acts of explaining (WMCS PD TM1.6 ppt slide 6)

The selection of examples and their representations; the language, terminology and notation are all pedagogical decisions in the acts of explaining which the teacher has to think about in planning and in the delivery of the lesson. The ppt slide is connected to the first column of the MFT handout discussed earlier.

The first column had examples and representations. In this column, specific questions were asked. From the three roles of the MTF as a planning, observing and reflecting tool, it is clear from the nature of questions asked that in this case it was intended as a planning tool. The questions are "*what examples and representations will I use?*" This question requires teachers to think more carefully about the selection of examples and their representations. The teacher is expected to be clear about the use and purpose of each selected example and its representation. Whether it will be used at the beginning of the lesson: "*at the start of the lesson*", or "*for questioning*", or "*for further explanation*". I bring in the MTF again in Figure 6:10 below.

Object of learning teaching x to y		
Examples and representations	Explanations and questions	Learner activity
What examples and representations will I use? At the start of the lesson For questioning For further explanation How will these help to build the key concepts and skills?	What kinds of explanations (and related questions) will I use? What? How? Why? When? How will these help to build the key concepts and skills?	What work will the learners do? e.g. listening, answering questions, copying from the board, solving a problem, discussing their thinking with others, explaining their thinking to the class How will these help to build the key concepts and skills?
Coherence: Are there coherent connections between the object of learning, and my choices of examples and explanations? Sequencing: Will the sequencing of examples & explanations be appropriate for learners to learn?		

Figure 6:10 – Mathematics Teaching Framework

As already shown in the previous discussion, based on explanation, the MTF was given to teachers as a handout and in the PPT presentation. In the powerpoint slide, the questioning is in the present tense for a lesson that is being observed. Teachers are supposed to observe examples used at the beginning of the lesson, as the lesson developed. Examples used "*for introducing a concept*", for questioning and for further explanation. The first five bullets in this column require teachers to comment individually on each example while the last two points in the column required teachers to look across all examples used and comment on sequencing and the collective set of examples used in the lesson on how they come together to "*build key concepts and skills*".

PPT slide for MTF from teaching session 1.6

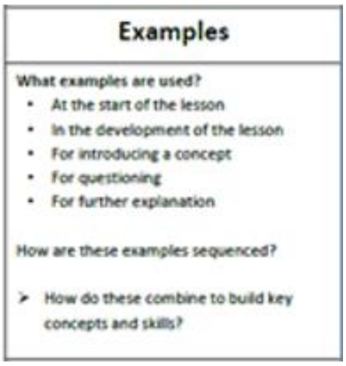


Figure 6:11 – Statement of examples in the MTF

There are two additional points in relation to examples that are added in the ppt slide version of the MTF. Firstly that the lesson develops through the use of examples and secondly that concepts are introduced through the help and use of examples. The PPT slide and the MTF handout refer to examples and representations separately from each other. The choice of the representation that will be used for the example is an inseparable activity and so I chose to refer to examples and their representations.

The next discussion is focused on how the course operationalised its definitions of examples, representations, and language through elaborations and exemplifications of acts of explaining.

6.4.2 What is elaborated as acts of explaining?

I discuss an activity from the independent work based on trigonometry. The activity is interesting in the sense that it requires a production of the explanation. This supports the fact that the explanation is a product of the acts of explaining. The trigonometry task requires that teachers work with a set of criteria to produce the explanation. The criteria are put down as decisions teachers must make before writing up the explanation. The set of criteria are elaborations of the acts of explaining.

Figure 6:12 is the task given to teachers to work on and produce a written explanation as part of their independent work for TM1.6. Even though the activity requires teachers to produce a written explanation, the majority of points listed in the text are actualy criteria for acts of explaining.

Teaching task - Producing a written explanation

In this task, you are required to produce a written explanation. Please do the task individually.

Assume you are required to explain the following to a Grade 10 learner: **Why is $\boldsymbol{tan\ 45° = 1}$?**

Decide on the following before you write up your explanation:

- What is the most important aspect (or aspects) that learners must learn from the explanation?
- Where will you start?
- What sequence will you follow?
- What representations will you use?
- What important terminology will you use? Are there any words you will avoid?
- Will your explanation help learners to understand why $\boldsymbol{tan(-45°) \neq 1}$ and other similar questions?

Figure 6:12 – Producing a written explanation (homework task from TM 1.6)

The criteria transmitted by the course in this task for acts of explaining foregrounds pedagogical decisions, such as selecting the focus, the order of ideas, the examples to be used, and the terminology to be emphasised. The last criterion is self evaluation for the teacher to reflect on the explanation if it will accomplish that which it is intended to achieve. In this task, there are pedagogical decisions that need to be made in relation to how to make the mathematics (why is tan 45^0 =1?) available to learners. In this list, there are only two bullets which do not quite fit neatly into the acts of explaining. The first and the last bullets are better fitting in the criteria for explanation as will be shown in supporting follow up interview responses of the course presenter below. I discuss each one.

26 *P: The first bullet was about the most important aspects that learners must learn from the task, which for me was relating to whats the fundamental mathematics. What's the mathematical thing here kinda of thing, and it was about, if you want, the object of learning in the language that we use now. What's this thing that you want the learners to use? So probably now that object of learning is much more common in our language with teachers I would probably change that to object of learning.*

The course presenter's clarification of the first bullet is that it is the object of learning. The selection of the object of learning is both a pedagogical as well as mathematical decision. Thus, it fits in both explanation and acts of explaining. According to Ruben's criteria the object of learning is the goal which the teacher wishes to achieve.

26 *P: Where will you start and what sequence will you follow? Was kinda of linked with are you gonna start with the calculator maybe to prove that? Let me just show that it's easy, that its easy, then explain. Or are gonna draw a diagram first? Are you gonna start with the definition of tan as tan thetha is y over x. Some teachers, because its tan, work with gradient and so they focus on tan of and link it up with gradient. So that was part of the "where will you start?"*

According to the course presenter the issue is the selection of an appropriate representation and making links between representations. The course presenter highlights four options that teachers can choose to start with, and those include the use of a calculator, stating the definition, drawing a diagram, or using the concept of gradient. However, the calculator, the diagram and the concept of the gradient all rely on and elaborate the definition.

26 *P: Looking at the sequencing is kinda wanting teachers to like to pay attention to what's the next thing I'm gonna say. And in a sense, often what you get when you ask for an explanation you don't get the history of that explanation that you want. You kinda of get the final product. So it's kinda of if they've drawn a diagramme, you have no clue what they've drawn first. So sometimes we ask them to draw speech bubbles and say this is where I started, then I did this, then I said this, then that. So that's one way of trying to capture the history when you seeing the final product only.*

The course presenter here highlights the importance of sequencing ideas and the importance of being purposeful for each and every step, noting that what comes to be presented as the explanation is often the final product. As a way of capturing the journey and the history of where the final product comes from, the course presenter emphasises the notion of sequencing of ideas. Specifically, the course presenter encourages teachers to draw speech bubles tracing each and every step, and how the steps followed each other. Sequencing is a useful way of establishing whether or not the steps connect and whether the entire explanation is coherent. Sequencing is an act of explaining.

26 *P: "What representations will you use?" Well some of that was just related to would you just use a right angled triangle not linked to the Cartesian plane for example. Or would you use a right angled triangle in the Cartesian plane? Or would you not use a*

right triangle at all? Would you use, err this diagramme, which I am trying to explain for the purposes of the tape (draws), where you got that arm that rotates right around. So there is no triangle kinda of a thing. Are they gonna use that? So that would be [...] *In terms of graphical representations, are they gonna use the tan graph? And say if you look on the graph of tan of, at 45 degrees, is one. So I think trig gives a lot of opportunities to use a lot of representations and that is why it lends itself quite well. You can say this one. It's not like there is one obvious one that is best. I think that's helpful in terms of working on explanations.*

The notion of representations is an important component of acts of explaining, because the example is represented in someway. This is because mathematical ideas are not tangible, but are represented since they are abstract notions. The notion of representations is also part of the explanation because the representation reveals the structure or the form of the object being dealt with. The course presenter enumerates the options teachers might choose from to provide the why explanation. Those include the right angled triangle on its own, outside of the set of axes and in the set of axes. The rotating arm in the set of axes and the graphical representation of the tan of 45 degrees. All these representations are choices teachers have and can chose to justify why tan 45 degrees equals one.

26 *P: "What important terminology will you use are there any words you will avoid?" So some of those things would be stuff like talking about tan is y over x as opposed to tan of thetha is y over x. Or they gonna use y over x, because that kinda connects to gradient. Or they gonna use delta y over delta x. Or they gonna use opposite over adjacent. Or they gonna use whatever. Err, so that would be some of the terminology. Would they talk about, erm, would they emphasize a right angle if they use a triangle? For some it might be, they would avoid using the gradient, because gradient is not generalizable. So I know that I saw examples, maybe it was a different course, where teachers focused completely on gradient and said its rise over run, so you rising as much as you running. And so it's obviously 45 degrees, because you've got an isosceles triangle. Do you wanna focus on that isosceles triangle as your explanation and use those words when they are not generalisable to why is sine of 45 degrees, not one, whatever? And so it's how local versus how global, if you want. Can this explanation generalise if you change the ratio, if you change the angle?... So I mean, so that actually* *goes back to the previous question. So I thought rise over run might come in. Someone might choose to avoid it, and someone might want to include it because they trying to build links between inclination, the trig and the function work. Are they gonna use words like asymptote?*

Language is an important criterion in the acts of explaining. It is the means through which the explanation is indeed made available to learners. What is highlighted by the course presenter here is the importance of avoiding the use of mathematical language that is not generalisable to other trigonometric ratios. The course presenter highlights some of the phrases used when talking about the trigonometric ratio where the angle is omitted "*tan is y over x as opposed to tan of thetha is y over x*". The course presenter highlights some of the reasons for avoiding certain concepts in the provision of the explanation, specifically the gradient, because its not generalisable to other trig ratios. The words "local" and "global" are used by the course presenter as a way of deciding what words are appropriate for the explanation. For instance, if the word is local, then its not good to use in the explanation, because it will not generalise. The course presenter goes on to highlight some of the terminology such as "opposite over adjacent", "gradient", "isosceles triangle" and "asymptote".

26 *P: "Will the explanation help learners understand why tan of negative 45 degrees is not equal to one?" So it's kinda of what I was saying now, in terms of how local is this explanation? And can, so I also used the word powerful explanation in some of the work related to trig. Stuff related to rotating an arm in a Cartesian plane around the center or origin, whatever you want to call it. And then being able to talk about what's going on with the ratios, vertical, horizontal, or whatever. And that kind of representation and explanation, and err, justification, would be generalisable to any angle because you just see it as reasoning by continuity if you want. As the angle changes and you could pick and choose the ratio depending on which ever one you wanted. And that means you could use the same kind of explanation regardless of ratio or angle. And that for me was what I wanted, and that's in a sense why I deliberately picked tan, because it is the one where you can go to gradient.*

As already noted, this last bullet is an evaluation of the explanation. The course presenter repeats what he has already mentioned in the previous bullet about representations, terminology and generalisability. It is an important step, because it requires that the teacher takes a step back and evaluate the choices made for representations, language and the sequencing of ideas. The judgement must be made with the learner in mind, because the explanation is prepared for the learner. Also the why question, which is necessarily a criterion of explanation, is used in this last point. The idea, judging from the course presenter's response to this point, is that different representations can be used to answer the why question. This supports the discussion that has taken place already based on explanation and the why question.

All these are questions aimed at making teachers think about their pedagogical decisions. The teacher has to decide on the object of learning, the representation to use to introduce the object of learning. On sequencing the course, the presenter mentioned the importance of having a history basically a starting point, then thinking carefully about the next thing to say by writing or drawing speech bubbles of how one develops to the final product. The last question ties in with the question asked in the MTF on how the explanation help learners build key concepts and skills. If the explanation is local then the knowledge is not transferable to other mathematical scenarios, while for a generalisable explanation, the knowledge is transferable. These are considerations teachers have to factor in as they think about and plan the production of the explanation. The next discussion, which I now turn to, is focused on the exemplification of acts of explaining.

6.4.3 What is exemplified as acts of explaining?

The tasks which I use to exemplify acts of explaining are from TM 1.8. The first task is taken from a slide in TM 1.8, and demonstrates a way of selecting and sequencing examples such that learners are able to notice and learn something. The second task is focused on critiquing and evaluating a teachers' selection and generation of examples.

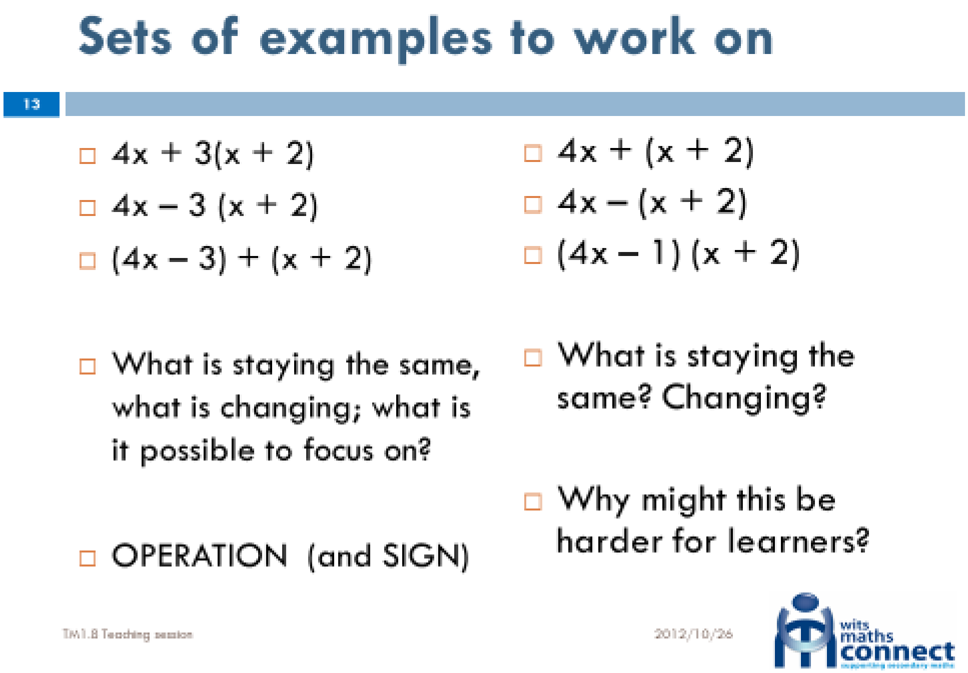


Figure 6:13 – Exemplification of acts of explaining criteria

The slide presents "*sets of examples to work on*". The slide has two columns, each column has three algebraic expressions. The next two bullets in each column are questions higlighting things teachers are supposed to observe in relation to the set of examples.

I now bring in another scenario to illustrate how the course exemplified the criteria for acts of explaining. The reason I bring in this scenario is because it also required teachers to work with examples but from a different perspective than the foregoing slide. Teachers are required to analyse the selected examples, critique and extend them. The activity teachers engaged in in the foregoing slide was observation of what is changing and staying the same.

2. Choosing examples in Grade Nine Algebra

A Grade Nine teacher is setting a revision task at the end of the section on exponents. She has given her learners Ex 3.8 from Classroom Mathematics Grade Nine and checked their answers. Now she wants to give six to eight further examples for them to work on. She makes up the three examples shown below.

Simplify:

1) $\frac{(2a^2)^2}{2b}$
2) $a^2(-b)^2$
3) $\frac{(4a)^{-1}}{(-a)^{-2}}$

Critique the teacher's choice of examples and think of other examples you would suggest.

Figure 6:14 – Exemplification of acts of explaining from TM1.7.

In this task, teachers can produce a positive or negative criticism of the choice of examples presented in the task depending on their reasoning. Teachers are expected to apply all the criteria transmitted thus far pertaining to selection, sequencing, representations, connections and coherence of examples. Teachers are supposed to question the starting example if it is appropriate, and ask whether the starting, the collective set of examples develops key knowledge and skills and if not suggest alternative examples. Since this is a revision activity for a concept that has already been taught, teachers might want to see examples which assess all the laws and definitions of exponents.

To this end, acts of explaining that have been focused on is the use of correct mathematical language, the correct way of talking about the mathematical object. The use of and choice of examples and their representations.

To summarise, I have discussed the criteria transmitted by the course for acts of explaining using course documents and the supporting follow up interview I conducted with the course presenter. I found that the course stated the criteria for acts of explaining as ways of making the mathematics available through the use of examples and their representations, terminology and mathematical language. I found that the course elaborated these criteria through the use of an activity which required teachers to produce a written explanation. I also found that the course exemplified the acts of explaining criteria through an activity which required teachers to critique and extend a given set of examples.

Table 6-4: Criteria for acts of explaining

Stated	Elaborated	Exemplified	SMK/ PCK
Examples	Where to start and what sequence to follow?	Choosing examples in Grade Nine Critique a set of examples	PCK
Representations	What representations will be used?	Suggest other representations	PCK
Language	Important terminology, Words to avoid	----	PCK
Object of learning	Selection of the object of learning	----	PCK

In Table 6-4 above, I have blurred the boundaries between examples and representations because the representation is of the example. The selection of the object of learning I have interpreted as Ruben's fourth criterion of explanation and that is goal. This is because the selection of the object of learning is the main objective at which all the examples and their representations as well as language use and other acts of explaining are aimed. In the assigning of Shulman's knowledge categories to the acts of explaining criteria, I limited myself to PCK, even for criteria such as representations and language where SMK is needed. The reasons why I did this is that PCK relies on SMK and as stated earlier in Chapter 2 that there are huge penalties that PCK suffers when SMK is weak. In this vein, the selection of appropriate language and representations for pedagogical purposes is PCK. Because of the close relation between PCK and SMK all the criteria in the acts of explaining are reliant on SMK, while they are predominantly pedagogical decisions.

6.5 Conclusion

The aim in this chapter was to find criteria transmitted by the WMCS TM1 course for explanation and acts of explaining. To be specific, the research questions I sought to answer in this chapter were phrased as follows: What is legitimated as an explanation, and acts of explaining in the WMCS course? What constitutive criteria of 'explanation' and 'acts of explaining' are privileged? What messages about 'quality' of both explanation and acts of explaining are transmitted? I have shown from the documents and the interview I conducted with the course presenter that for explanation there are explicit criteria transmitted by the course for what constitutes an explanation. The first main criterion that is explicit is the distinction between the what and the how explanations. The what and the how explanations are categorised further in the course slides, where a what explanation is about the big idea, or a definition, concept, notation; and the how explanation is about a method, law or a short cut. These are types of explanation that are transmitted as valid and thus part of a good explanation. Furthermore, from the messages conveyed by the course it was possible to separate acts of explaining and the explanation. The message of the course on acts of explaining is implicit. Acts of explaining became visible when the course materials refer to the learner and imply pedagogical decisions such as selection and sequencing of representations, examples, word use, and terminology. Also pertaining to the act of explaining is the consideration of how to make sure of coherence in the explanation offered.

I have shown how the distinctions made by the course for a what and a how explanation help resolve the mathematics battles over conceptual and procedural knowledge/understanding. The distinction made by the course between the what and how as the main kinds of explanation both with sub-explanations of why illuminates the interplay between conceptual and procedural understanding/ knowledge.

In relation to teacher knowledge, one can conclude that what one explains in an explanation is their knowledge of “what”, “how”, “why”, and “when”. This serves to highlight that SMK, as part of teacher knowledge, plays a significant role in the production of an explanation as well as PCK for acts of explaining. A satisfactory explanation for learners is one that convinces them as to why the definition, law, a procedure works; and this kind of knowledge required here on the part of the teacher is SMK more than it is pedagogical. PCK is needed to decide on the kinds of examples; representations, their sequencing and words to use in the offering of an explanation and these are what have been classified as acts of explaining as they have learners in view. Therefore, the notions of SMK as the actual explanation and acts of explaining as PCK simply reinforce each other. Table 6-5 below summarises the findings of this chapter.

Table 6-5: Criteria for explanation and acts of explaining

	Stated	Elaborated	Exemplified	PCK/ SMK
Explanation	What	Definition, big idea, concept, notation	Distributive law, big idea	SMK
	How	How to execute a procedure	Illustration, method, short cut, big idea	SMK
	Why	Justification	Big idea, and illustration	SMK
	When	Overgeneralisation and misapplication of rules	----	SMK
Acts of explaining	Examples	Where to start and what sequence to follow?	Choosing examples in Grade Nine Critique a set of examples	SMK and PCK
	Representations	What representations will be used?	Suggest other repersentations	SMK and PCK
	Language	Important terminology, Words to avoid	----	SMK and PCK
	Goal	Selection of the OOL	----	PCK

Ruben's criteria of explanation and Shulman's knowledge categories were used in this chapter to read and interprete the crteria transmitted by the course. Ruben's criteria revealed that the goal was explicitly mentioned in the acts of explaining for the course even though it's a criterion of explanation in Ruben's criteria. Also Ruben's criteria revealed that the notion of representations holds both the matter and the form of the example. If the representation used is geometric the matter will differ from a symbolic representation and so will the form. So while it is a pedagogic decision to select representations it is also a criterion of explanation because of form and structural properties made available by the representation. Hence the blurred bounderies between examples and their representations. The coding system which I use in the next chapter took this discussion into consideration as already indicated in the methodology chapter. I now turn to the analysis of teachers' practices.

Explanation	CODE	MEANING
	M1	Reading a string of symbols
	M2	Definition of elements
	F1	Sub-structure of an algebraic expression
	F2	Structure of whole algebraic expression
	P1	Steps of a method
	P2	Derivation of rules
	G1	Statement of the goal for the lesson
	G2	Answer to "why" question or a justification using the word "because"

Acts of explaining		
	Rx	Examples and their representations
	Rn	Language and terminology
	Rv	Re-voicing
	Rf	Finishing teachers' sentence
	Rt	Gestures
	Re	Evaluating
	Rc	Chorusing
	Rg	Regulating

CHAPTER 7 VIDEO ANALYSIS

7.1 Introduction

In the previous chapter, criteria for what was transmitted as explanation and acts of explaining in the WMCS course were obtained. The criteria obtained in the literature and presented by the course for explanation and acts of explaining were then used to code the transcripts of the teachers' lessons and from the coding process the criteria expanded. In this chapter, I present the results of that coding, and so a full description of the explanation and acts of explaining across the two teachers' lessons. In so doing, I begin to build an answer to research question 3, which I expressed as follows:

3. What is legitimated as explanation and acts of explaining in algebra in the classroom?

To answer this question, I first provide an overview of what each teacher taught by looking across the segments in each lesson. I organise each lesson into main activities, some of which have a number of segments within them, and list the examples provided so as to illuminate the lesson content. I move on to show evidence of how the analysis was done by selecting three segments from each teachers' lessons, presenting the transcript excerpt for each and through this show the coding for both explanation and acts of explaining. This is followed by a discussion of what was constituted as explanation for each teacher and a discussion of what was constituted as acts of explaining for each teacher.

Both teachers in this study introduced the notion of algebraic expressions to Grade Nine learners in similar ways by first paying attention to what the words "algebraic expression" refer to in mathematics. Lesson one (L1) for each teacher began with a discussion that focused on naming, identifying and defining the elements that compose an algebraic expression (e.g., constant, coefficient, variable, term). Once this discussion had occurred both teachers moved to the procedural activity of operating on algebraic expressions (e.g. adding, subtracting, and multiplying). For Teacher A the discussion on operations on algebraic expressions begins in L1, and continues through lessons 2 (L2) and 3 (L3). For Teacher B, operating on algebraic expressions begins in L2 and continues through L3. Hence Teacher B spends more time discussing the 'what' of algebraic expressions before moving to the 'how'. I begin the discussion with Teacher A.

7.2 Teacher A

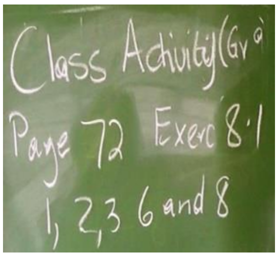
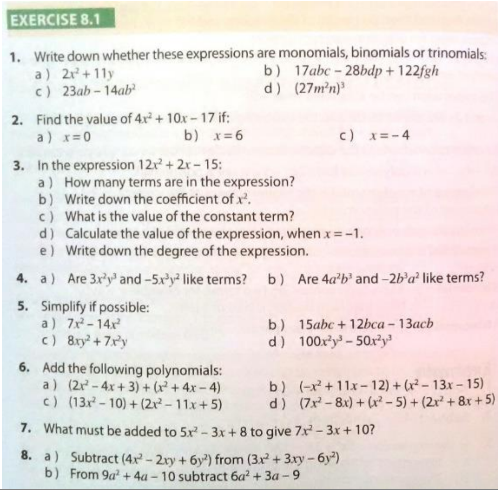
As noted, Teacher A introduced the notion of algebraic expressions and related terminology in L1 and then moved on to operating on (adding, subtracting and multiplying) these for the remaining time across the lessons.

7.2.1 Teacher A Lesson 1-3

In L1, there were four main activities. Activity 1 was a discussion of algebraic expressions with two segments in it, Sg 1.1 was a call out of associative words with the notion of “algebraic expressions”, and in Sg 1.2, the teacher provided an example of a polynomial with four terms and this was used to discuss terminology like coefficient, constant, term, like terms, exponent. Activity 2 was simply a mark in the lesson where the teacher was explicit about the goal of the lesson, which was stated as “add and subtract expressions”. Activity 3 had four segments in it, each focused on a different algebraic expression, or set of expressions, where the operations of adding and subtracting were in focus. Activity 4 was recorded as one segment as it was the provision of homework from a textbook for learners to do for the next lesson. Table 7-1 below shows the segments, activities and examples used in L1 for Teacher A.

I have highlighted segment (Sg) 1.1, 1.2 and 2 in the table as I will use them in this chapter to evidence the coding and analysis Teacher A’s lesson transcripts.

Table 7-1: Activities, segments, and examples used in Teacher A L1

<table>
<tr><td colspan="4">Lesson: 1 (00028) Topic: add and subtract expressions</td></tr>
<tr><th>Sg</th><th>Time</th><th>Activities</th><th>Example</th></tr>
<tr><td>1.1</td><td>00:00</td><td>Discussion on “algebraic expression”</td><td>No written example</td></tr>
<tr><td>1.2</td><td>02:46</td><td>Example of algebraic expression</td><td>$3x^2 + 3xy - 6y^2 + 4$</td></tr>
<tr><td>2</td><td>06:44</td><td>Add and subtract like terms</td><td>Statement of goal</td></tr>
<tr><td>3.1</td><td>09:09</td><td>Example 1</td><td>$5x + 5y - 3x + 2y$</td></tr>
<tr><td>3.2</td><td>13:14</td><td>Example 2</td><td>“add $(8x^2 + 4x - 6)$ and $(-3x^2 + 6x + 4)$”</td></tr>
<tr><td>3.3</td><td>19:28</td><td>Example 3</td><td>“subtract $(7x^2 + 4)$ from $(15x^2 - 2)$”</td></tr>
<tr><td></td><td></td><td></td><td>“subtract 4 from 6”</td></tr>
<tr><td>3.4</td><td>25:56</td><td>Example 4</td><td>“subtract $(-3x^2 + 6x + 4)$ from $(8x^2 + 4x - 6)$”</td></tr>
<tr><td>4</td><td>33:29
34:33</td><td>Class activity then end of lesson

</td><td>Platinum Grade 9 CAPS

EXERCISE 8.1
1. Write down whether these expressions are monomials, binomials or trinomials:
a) $2x^2 + 11y$ b) $17abc - 28bdp + 122fgh$
c) $23ab - 14ab^2$ d) $(27m^2n)^3$
2. Find the value of $4x^2 + 10x - 17$ if:
a) $x = 0$ b) $x = 6$ c) $x = -4$
3. In the expression $12x^2 + 2x - 15$:
a) How many terms are in the expression?
b) Write down the coefficient of x^2.
c) What is the value of the constant term?
d) Calculate the value of the expression, when $x = -1$.
e) Write down the degree of the expression.
4. a) Are $3x^2y^3$ and $-5x^3y^2$ like terms? b) Are $4a^2b^3$ and $-2b^3a^2$ like terms?
5. Simplify if possible:
a) $7x^2 - 14x^2$ b) $15abc + 12bca - 13acb$
c) $8xy^2 + 7x^2y$ d) $100x^2y^3 - 50x^2y^3$
6. Add the following polynomials:
a) $(2x^2 - 4x + 3) + (x^2 + 4x - 4)$ b) $(-x^2 + 11x - 12) + (x^2 - 13x - 15)$
c) $(13x^2 - 10) + (2x^2 - 11x + 5)$ d) $(7x^2 - 8x) + (x^2 - 5) + (2x^2 + 8x + 5)$
7. What must be added to $5x^2 - 3x + 8$ to give $7x^2 - 3x + 10$?
8. a) Subtract $(4x^2 - 2xy + 6y^2)$ from $(3x^2 + 3xy - 6y^2)$
b) From $9a^2 + 4a - 10$ subtract $6a^2 + 3a - 9$</td></tr>
</table>

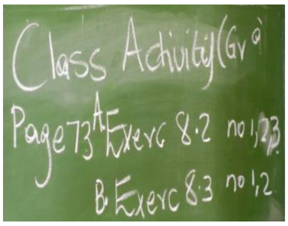
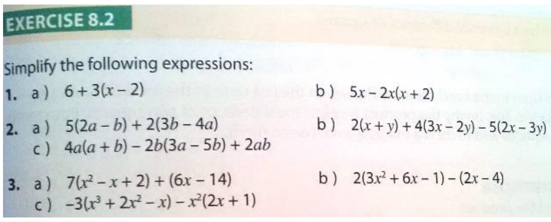
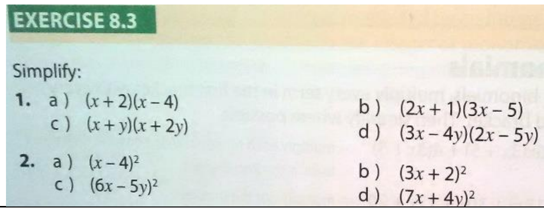
The whole of L2 was taken up by going over the homework items above. There is thus one main activity in the lesson. The activity in L2 consisted of the teacher or learners calling out answers to particular homework questions, or writing answers to these on the board. Each of the homework items has been segmented for analysis, including sub-questions. There were segments where the answer was called out. For example, these involved the identification of constant, number of terms and coefficient, where there was nothing written on the board. These were Sg 1, 3 and 5. For homework items that required an answer to be shown, learners were required by the teacher to go to the board to write these (Sg 2, 4, 6-9). The example in Sg 2 and Sg 4 were quadratic expressions for which values of the unknown were given to evaluate by substitution. The examples in Sg 6-9 are focused on the addition of quadratic expressions. The quadratic expressions are composed of two or three negative and positive terms. The range of homework activity and segments dealt with in L2 is shown below in Table 7-2.

Table 7-2: Activities, segments, and examples used in Teacher A L2

		Lesson: 2 (00029) Topic: Going over homework	
Sg	**Time**	**Activity**	**Example**
1	00:00	Teacher calling out answers	No written example
2	02:06	Learners answer on the board	(a) $4 \times 0^2 + 10 \times 0 - 17$
3	04:02	Learner answer on the board	(b) $4x^2 + 10x - 17$
4	05:47	Learner answer on the board	(c) $4 \times (-4^2) + \ 10 \times (-4) - 17$
5	08:47	Teacher calling out answers	No written example
6	09:47	Learner answer on the board	$12(-1^2) + 2(-1) - 15$
7	11:06	Teacher calling out answers	No written example
8	11:46	Learner answer on the board	$(2x^2 - 4x + 3) + (x^2 + 4x - 4)$
9	14:43	Learner answer on the board	$(-x^2 + 11x - 12) + (x^2 - 13x - 15)$
11	19:00	Learner answer on the board	$(13x^2 - 10) + (2x^2 - 11x + 5)$
12	24:00	Learner answer on the board	$(7x^2 - 8x) + (x^2 - 5) + (2x^2 + 8x + 5)$

In L3 there were four main activities. The first activity was focused on finishing corrections to homework items. The examples used were a subtraction of two quadratic expressions. Activity two was the statement of the goal as "Multiplying expressions". Activity three was multiplication of expressions demonstrated through eight examples. The first example (Sg3.1) is a one term example where multiplication is distributed over the addition of two terms. The next four examples used were similar in structure because they are all expressions with two terms. The first term is a "simple term" (5) or a "product term" ($5x$), see Sg3.3: $\underline{5 + 2x(3x + 2)}$ and Sg3.4: $\underline{5x + 3x(x + 1)}$. The second term is a "complex term" (e.g. ($2x(3x + 2)$ and $-3x(x + 1)$), because it involves distribution of multiplication over the terms added inside the bracket (Banerjee & Subramaniam, 2012). Sg 3.6 is constituted of two complex terms separated by a subtraction sign. The last two examples are multiplication of two binomials. I have highlighted Sg3.7, because I use it to evidence the coding and analysis of the teaching of the distributive law. Activity four was the provision of classwork/homework (HW). Table 7-3 below shows the segments, main activities and examples used in L3 for Teacher A.

Table 7-3: Activities, segments, and examples used in Teacher A's L3

		Lesson: 3 (00030 and 00003) Topic: Multiplying expressions	
Sg	Time	Activity	Example
1.1	00:00	Answer to previous HW	$(3x^2 + 3xy - 6y^2) - (4x^2 + 2xy + 6y)$
1.2	06:47	Answer to previous HW	$(9a^2 + 4a - 10) - (6a^2 + 3a - 9)$
2	18:23	Statement of the goal	Multiplying expressions
3.1	19:00	Example 1:	"multiply $(3x + 1)$ by 2"
3.2	21:43	Example 2:	$5 + 2(3x + 2)$
3.3	24:44	Example 3:	$5 + 2x(3x + 2)$
3.4	28:55	Example 4:	$5x + 3x(x + 1)$
3.5	32:59	Example 5:	$5x - 3x(x + 1)$
3.6	40:56	Example 6:	$2x(x + y) - 2y(3x - 5y)$
3.7	48:25	Example 7:	$(x + 1)(x + 2)$
3.8	54:24	Example 8:	$x + 1)(3x - 2)$
4	01:03:54	Class activity written on the board  	8.2  EXERCISE 8.2 Simplify the following expressions: 1. a) $6 + 3(x - 2)$ b) $5x - 2x(x + 2)$ 2. a) $5(2a - b) + 2(3b - 4a)$ b) $2(x + y) + 4(3x - 2y) - 5(2x - 3y)$ c) $4a(a + b) - 2b(3a - 5b) + 2ab$ 3. a) $7(x^2 - x + 2) + (6x - 14)$ b) $2(3x^2 + 6x - 1) - (2x - 4)$ c) $-3(x^3 + 2x^2 - x) - x^2(2x + 1)$ 8.3  EXERCISE 8.3 Simplify: 1. a) $(x + 2)(x - 4)$ b) $(2x + 1)(3x - 5)$ c) $(x + y)(x + 2y)$ d) $(3x - 4y)(2x - 5y)$ 2. a) $(x - 4)^2$ b) $(3x + 2)^2$ c) $(6x - 5y)^2$ d) $(7x + 4y)^2$

In overview, Teacher A first introduced the words 'algebraic expressions' and discusses what these refer to in mathematics. She then moved to identification of like terms before introducing the main goal – adding and subtracting expressions. L2 and L3 continued with this operational activity, including multiplication in L3. For addition and subtraction, learners thus were offered a range of examples of polynomials with two, three or four terms, some with negative coefficients, and some where terms differed both by letters used, and by different powers of the variable, though this did not go beyond the power of two. For multiplication, the range of examples differed in the position of brackets, thus distinguishing the difference between the product of two binomials (Sg 3.7, 3.8) from those expressions where only a monomial was to be multiplied (Sg 3.1 – 3.6) and thus different applications of distribution.

In the description of the three lessons, I have attempted to foreground the content covered, as this provides the backdrop to moving on to what mathematics was in focus as Teacher A built her explanations.

I now move on to discuss the transcript excerpts highlighted in L1 and L3, and so the explanation and acts of explaining in these lessons.

7.2.2 Teacher A transcript excerpts

I selected Sg 1.1, 1.2 and 2 from L1 and Sg 3.7 from L3 as examples of how the analysis and coding of each segment was done. In selecting these segments, the aim was to show an example where a "what" and a "how" explanation was given as well as the goal. I selected Sg 1.1 and 1.2, because they were introductory segments to the notion of algebraic expressions and as such the talk within them was focused on constitutive elements and structure which are both features of a "what" kind of explanation. In Sg 1.1, a discussion of the notion of algebraic explanation is provided, and in Sg 1.2 an example of an algebraic expression is provided. Sg 2 was simply the statement of the goal. For an example of procedural activity, or to put it in the language of the course, for the "how" explanation, I chose Sg 3.7 from L3 because in it was a discussion comparing and linking two methods of multiplication which are the distributive property and the short-cut method known as FOIL. Another reason for choosing Sg 3.7 is that the distributive law was also taught by Teacher B and the explanation of it was first given in the WMCS PD course that both Teacher A and Teacher B attended. In this way, an opportunity for contrasting between the two teachers and the WMCS PD course was made possible. Table 7-4 below is a transcript excerpt from L1 Sg 1.1, which lasted for a duration of 02 minutes 46 seconds.

In the first column, there are speaking turns, which have been numbered and labelled as T. the next two columns are codes for explanation (E) and acts of explanining (AoE). This is followed by the transcript. The transcript indicates speaking turns in the conversation between teacher and individual learners who the teacher calls by names (V- Vuyo, M-Malusi, R- Raphael). I have italicised learners names, because they are not the original names they are pseudonyms added by the researcher. Also italicised are words inside the brackets, which describe the actions and moves the teacher makes, to disntinguish them from the teacher's and learners' utterances. For collective learner responses, I used Lrns to indicate that the response comes from more than one learner and for individual learners whose names were not called out I used L.

Table 7-4: Excerpt from Teacher A L1 Sg 1.1 – Discussion on Algebraic expression

T	E	AoE	Transcript	
1	G1	Rg	T:	*Ok algebraic expressions. What do you understand about algebraic expressions? Ok! No choir! If you have an answer you put up your hand. What do you understand about algebraic expressions?* [Pause]
2			T:	*Grade 9s?* [pause]. *Algebraic expressions what comes to your mind?*
3			T:	*Vuyo*?
4			V:	*Mam they are terms with alphabets combined together*
5		Rv	T:	*Terms with?*
6			V:	*Alphabets*
7	F2	Rv	T:	*Alphabets combined together?*
8			V:	*Yes.*
9		Rg	T:	*Ok! Malusi*?
10	F1		M:	*Algebraic terms Ma'am, are separated by a plus or a minus* [looking at a book]. *Algebraic terms Ma'am are separated Ma'am* [smiling].
11		Rg	T:	*Where are you reading that from? No. I just want. There are no wrong or right answers. I just want those ideas. What do you understand by algebraic expressions? I'm not looking for correct or wrong answers. What comes to your mind if you see algebraic expression?*
12		Rv Rt	T:	*So he said terms* [pointing at one learner]. *Alphabets* [pointing at a different learner]. *Anything else?*
13			L:	*Equations.*
14	F2	Rv	T:	*Equations.*
15			L:	*Variables.*
16	M1	Rv	T:	*Variable…*
17		Rg	T:	*…okay I said no choir, so put up your hand. Mhm. So that thought that comes to your mind that is what I'm looking for.*
18			L:	*Coefficient.*
19	M1	Rv	T:	*Coefficient.*
20			T:	*Anything else? Raphael?*
21			R:	*Constant.*
22	M1	Rv	T:	*Constant.*
23			L:	*Integers.*
24	M1	Rv	T:	*Integers. Integers, yes there are numbers there but you are going back now to numbers.*
25			L:	*Exponents.*
26	M1	Rv	T:	*Exponents…* [Pause]

1. **Why is this a segment?**

I have classified this as a segment based on a discussion of what is an "algebraic expression". The discussion is based on a "*what*" question and proceeds through spontaneous answers offered by learners. The focus of attention is on the phrase *algebraic expression* and words associated with it. There is no example of an actual algebraic expression visible at this stage. The question in column T number 1 "*What do you understand about algebraic expressions?*" is repeated throughout the segment in turns 2, 11, 12, 17, and 20.

The segment ends when the teacher shifts learner attention from the spontaneous discussion of the words to focus on an example. This specific example and the developing explanation is discussed in segment 1.2.

I now discuss segment 1.1 in terms of the four criteria for explanation.

2. What criteria of explanation are present and how?

a) Are both the elements of matter present? (M1 and M2)

In the sub-unit above there is calling out of constituent elements such as "terms", "alphabets", "variables", "coefficient", "constant", "integers" and "exponent" see turns 12, 14, 16, 19, 22, 26. For example, in T6, the learners associate algebraic expressions with "alphabets". All these utterances I have coded as M1 because they are at the level of naming elements. There are no definitions yet, and thus no presence of M2.

b) Are both types of form present? (F1 and F2)

In the call out of spontaneous answers there is mention of "equation", which is a structure having two expressions separated by an equality sign. Talk in this segment is also focused on the form or the structuring of the elements. For instance, in T4 learners mention things like "*Ma'am they are terms with alphabets combined together*", and the description of terms being combined to form an expression repeated in T7. In T10, however, the emphasis is on how terms are separated in an expression: "*Algebraic terms Ma'am, are separated by a plus or a minus.*"

The first focus on combination suggests that operations such as addition and subtraction bring or join the elements together while the T10 utterance suggests that the operation in fact separates the elements into individual terms. When the utterance isolates and identifies individual terms it was coded F1 as in T10. When the utterance refers to the structure of the entire expression then it was coded F2 as it described the structuring of the terms. Thus, in this segment, both types of form are present.

c) **Are both kinds of process present if any? (P1 and P2)**

At this stage of the lesson, there is no procedural activity done, so far it is a calling out of associative terms with the notion of algebraic expression.

d) Is the goal stated and justified? (G1 and G2)

From the beginning of the lesson, one can assume that the lesson has something to do with algebraic expressions. For this segment the goal is not yet explicitly stated for learners to know what the purpose of the lesson as a whole will be. But the purpose of the discussion is explicitly stated as what learners know concerning algebraic expressions. Utterances from turns 1, 2, 11, and 17 lead one to assume that the purpose of the segment is to check learners' spontaneous or previous understanding of the words "algebraic expression". T1 – "*Ok algebraic expressions. What do you understand about algebraic expressions?*", T2 – "*algebraic expressions what comes to your mind?*", T11 – "*No I just want, there are no wrong or right answers I just want those ideas what do you understand by algebraic expressions. I'm not looking for correct or wrong answers. What comes to your mind if you see algebraic expression?*" and T17 – "*Mhm, so that thought that comes to your mind that is what I'm looking for*".

e) Is anything absent and why

As the goal here was to elicit learners' prior meanings, nothing can be said to be missing. Teacher A did not elaborate on the difference between an equation and an expression as this would not have meaning without first elaboration of what is an expression.

3. What criteria are privileged and thus the achieved explanation?

The privileged criteria from the analysis above is mainly matter with little mention of form. The achieved explanation was that words such as coefficient, variable, term, that are associated with the notion of algebraic explanations were called out. There was no elaboration of any the words offered, except when the teacher rejected integers as part of the associative words, see T24.

4. What is the nature of pedagogy and learner participation?

As discussed in Chapter 5, the ways in which pedagogy and communication in the classroom proceeded was through acts of explaining, with three categories. The first category was in relation to an example being offered (Rx) and language use (Rn). The second category was about communication strategies such as re-voicing (Rv), finishing the teacher's sentence (Rf), gesturing (Rt), chorusing and evaluating learners' contributions (Re). The third category was the regulative talk (Rg). In this segment, the teacher initiated the discussion by asking a question and there was no accompanying written example, and hence no Rx code; at the level of language there was no attention given to meaning of the words that were called out and so the absence of Rn codes. From the communicative techniques re-voicing was the only means of communication used to acknowledge learner inputs as acceptable or not acceptable. For example, the teacher first re-voices learner input of ‘integer’ in T24 before stating that it is not acceptable. The teacher also regulated the source from where learners’ responses come from by continuously reminding learners that she wanted their thinking not what was written in the textbook, and hence the significant amount of Rg codes. Figure 7:1 below shows the frequency for explanation and acts of explaining codes.

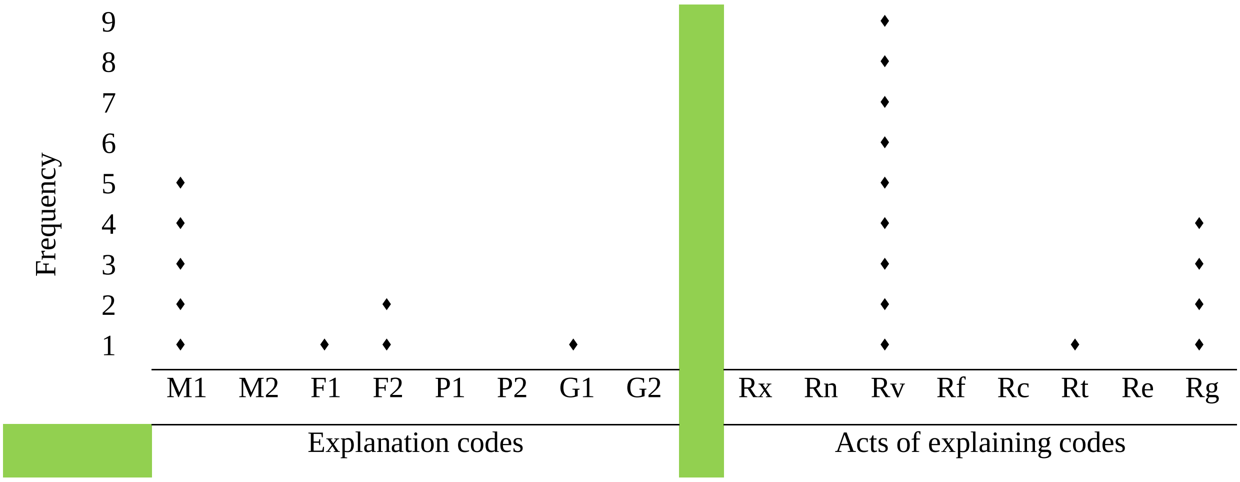
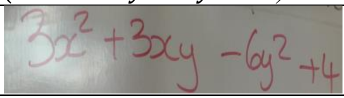


Figure 7:1 – Codes frequency for Teacher A L1 Sg 1.1 – Discussion on Algebraic Expression

Following the discussion of the notion of algebraic expressions the next move the teacher made was to introduce an example of an algebraic expression: $3x^2 + 3xy - 6y^2 + 4$ and she used this to assist learners to identify the associative words called out in the transcript excerpt in Sg 1.1. Table 7-5 below is the transcript excerpt for Sg 1.2 - exemplification of an algebraic expression which lasted for a duration of 4 minutes 38 seconds.

Table 7-5: Excerpt from Teacher A L1 Sg 1.2 – Example of algebraic expression $\mathbf{3x^2 + 3xy - 6y^2 + 4}$

T	E	AoE	Transcript	
1	M1	Rx	T:	*Okay now let's use one example of an algebraic expression* [walking to the desk, then to the board to write]. …*Minus six y squared plus four* ($3x^2 + 3xy - 6y^2 + 4$)*. So that's, is that an expression?*
2				
3			Lrns:	*Yes Ma'am.*
4			T:	*Is that an expression?*
5			Lrns:	*Yes Ma'am.*
6			T:	*How do you know it's an expression?*
7	F2		T:	*Okay how many terms are in that expression?*
8			Lrns:	*Two, three, four Ma'am.*
9	F2	Rg	T:	*Choir again* [learners automatically put their hands up]. *How many terms?*
10			L:	*Four.*
11	F2	Rv	T:	*There are four terms. How did you see the terms? How did you see that there are four terms? Yes.*
12			L:	*Because they are separated by a plus and a minus.*
13	F1 G2	Rv Rf	T:	*Because they are separated by either a plus or …*
14			Lrns:	*A minus.*
15		Rf	T:	*Then that is when we say it's a term because they are separated by a plus or…*
16			Lrns:	*…a minus.*
17	M1 F1		T:	*Okay, now let's look at the first term. What is the coefficient of the first term? The coefficient of the first term?*
18			T:	*You say its x.*
19			L:	*x to the exponent of two.*
20		Rv Re	T:	*He says x to the exponent of two, do you agree?*
21			Lrns:	*Yes.*
22	M1	Rg	T:	*What is the coefficient? His hand is still up.*
23			L:	[Not audible].
24	M2	Rg	T:	*Okay. Maybe we should define what is a coefficient? Close the books we are revising what you've learnt last year. What is a coefficient?*
25			L:	*Three. Three is a coefficient.*
26	M1	Rv	T:	*Three. So three there is the coefficient. So what is a coefficient?*
27			L:	*It's a number, a number next to x.*
28	M2	Rv	T:	*A number next to x. The number next to x. Okay let me check what is the coefficient of y squared?*
29			L:	*y squared.*
30			T:	*Yes.*
31			L:	*Negative six.*

32	M2	Rv Rf	T:	*Negative six. So coefficient is a number next to a …*
33			Lrns:	*Variable.*
34	M1	Rv	T:	*Variable. Now what is, err? You just said variable, cause I was going to ask what are the letters.*
35	M1		T:	*So in that case how many variables?*
36			L:	*We have two.*
37		Rv Rf	T:	*Two. Which is …?*
38			L:	x *and* y.
39		Rv Rf	T:	x *and…*
40			Lrns:	*…* y .
41	F1		T:	*Now what else can you say about that expression?* [Pause] *Are there like terms?*
42			Lrns:	*No.*
43			T:	*But what else can you say about the expression?*
44			L:	*There's an exponent Ma'am.*
45	M1	Rv	T:	*Yes there's an exponent.* [Learner enters and talks to teacher – pause]. *Okay and what is the highest exponent?*
46			L:	*Two.*
47		Rf Rt	T:	*The highest exponent is* [holding up two fingers]…
48			Lrns:	*…two.*
49	M1	Rf	T:	*And what else can you say about the expression? We have a constant which is…*
50			L:	*…four.*
51	M2	Rf	T:	*Remember a constant is a number without a …*
52			Lrns:	*Variable.*
53	M1		T:	*That's a constant.*

1. Why is this a segment?

The talk is focused on identifying and defining the elements that compose the algebraic expression ($3x^2 + 3xy - 6y^2 + 4$) written on the board. Learner attention is no longer focused on offering spontaneous responses but limited to what the example offers and what they are able to identify as they study the expression in detail. The teacher draws learner focus in T41, 43 and 49 by repeatedly uttering the question: "*and what else can you say about the expression?*" This is the main activity that the attention was focused on for this segment. After the definition of a constant, the attention shifts from the example of an algebraic expression to the next activity, which was the statement of the goal for the entire lesson (Sg2). I now move into a discussion of Sg 1.2 in terms of the four criteria of explanation.

2. What criteria of explanation are present and how?

- **Are both elements of matter present? (M1 and M2)**

In the segment, there is identification of elements such as coefficient (T17-33), exponent (T44-48), variable (34-40), and constant (49-53) which I coded as M1. Of the elements that were identified from the example of an algebraic expression, only two were defined, and that is, the coefficient and the constant. In T32-33 a coefficient is defined as "*a number next to a variable*" and in T51-52 a constant is defined as "*Remember a constant is a number without a variable*". These utterances I coded as M2 and thus both the elements of matter were present.

- **Are both types of form present? (F1 and F2)**

The question the teacher poses in T11 requires a study of the structure, "*how did you see the terms?*" The question is basically looking to establish a principle for observing structure specifically of terms in the example. The discussion is from T11 to T16 and the established principle is the utterance of the teacher in T15 "*Then that is when we say it's a term because they are separated by a plus or a minus*". I have coded this as F1, because the focus is on individual terms. The teacher directs learners' attention to the first term of the example in T17, "*Okay now let's look at the first term. What is the coefficient of the first term? The coefficient of the first term?*" By doing this the teacher has drawn attention to an individual term to study the elements that compose the individual term. I have coded this utterance as F1 because the talk is focused on the structure of the individual term.

In T7-11 the discussion is based on the question from T7 which is "*ok how many terms are in that expression?*" and so the emphasis is on the structure of the entire expression and I coded as F2. Therefore in this segment both types of form are present.

- **Are both kinds of process present if any? (P1 and P2)**

There is no procedural activity for this example because the activity is at the level of identifying elements. Also, the expression has no like terms, which would have required some operational activity to be performed. In T41, Teacher A asks: "*are there like terms?*" Learners correctly respond by saying no.

- **Is there a goal stated and justified?**

There is no explicitly stated goal for the lesson and for the segment so far. From the actions that have taken place one can conclude that for this segment the goal is to identify, name and define elements that make up the algebraic expression.

- **Is anything absent and why?**

The definition given for coefficient is missing some detail, because in T51, the definition is given as "*a coefficient is a number next to a variable*". Essentially, a coefficient has been described according to its position in terms of where it appears. The functional definition is not given so that learners know what it does and its relation to the variable, that it is not just "*a number next to a variable*" but a number which multiplies the variable. From the four elements spoken about in this segment, definitions of exponent and variable were not given, these were simply identified in the expression.

3. What criteria are privileged and thus the achieved explanation?

From the four criteria for explanation, matter and form are privileged. The achieved explanation for this segment was defining the elements: constant and coefficient, though the latter was partial. Also the achieved explanation is that terms of the expression and like terms were spoken about and identified.

4. What is the nature of pedagogy and learner participation

The teacher leads the discussion by asking questions such as "How many terms are in that expression?" which has to do with form. "What is the coefficient of the first term?" which has to do with constitutive element. "What else can you say about that expression?" in which case learners have to study the expression for more elements to identify and structure. The teacher mainly re-voices (Rv) learner responses to her questions and also requires learners to finish her sentences (Rf).

Figure 7:2 below is an illustration showing the frequency for explanation and acts of explaining codes for segment 1.2 of Teacher A's L1. The explanatory talk was mainly focused on elements and their definitions, there was also a considerable amount of talk focused on form. There is no procedural talk, because the aim was to identify, define and name elements that compose an expression. From the acts of explaining the most dominant communicative technique is re-voicing followed by a requirement to finish the teacher's sentence.

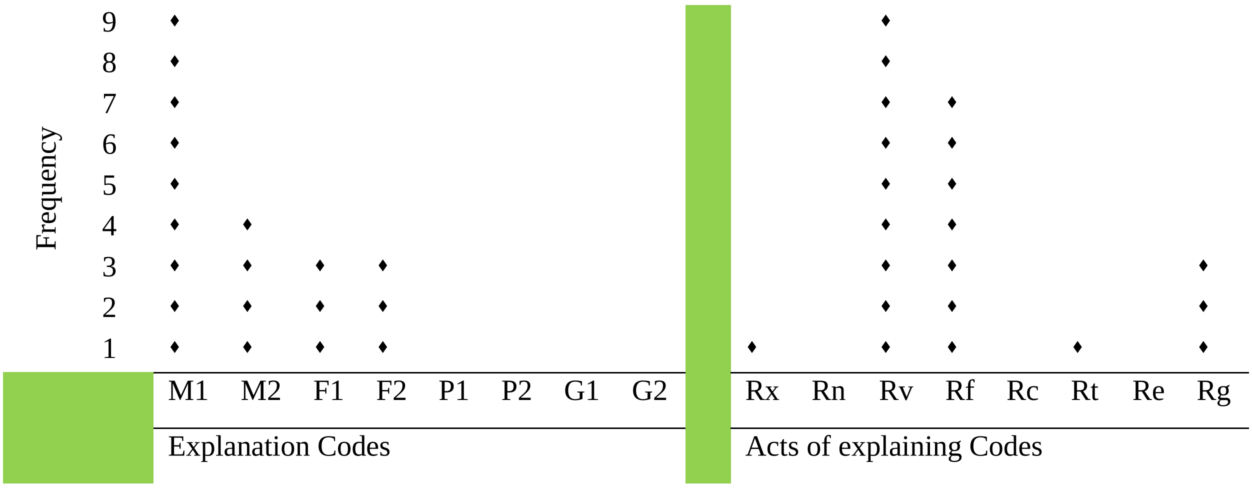
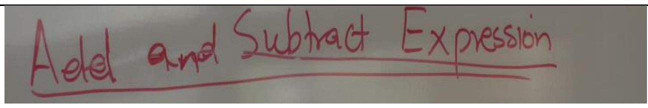


Figure 7:2 – Codes frequency for Teacher A L1 Sg1.1 – Example of algebraic expression $\boldsymbol{3x^2 + 3xy - 6y^2 + 4}$

Subsequent to the discussion of the example of an algebraic expression the teacher writes on the board the goal for the lesson shown in Table 7-6 below.

Table 7-6: Excerpt from Teacher A L1 Sg 2 – Statement of the goal for the lesson

T	E	AoE	Transcript	
1	G1	Rg	T:	*Okay! What we going to do today will be adding. So we said there are four terms from there. So we want to add and subtract expressions* [written on the board and underlined].
2				
3	G1	Rg	T:	*Okay! Add and subtract expressions.*

Following the discussion and exemplification of algebraic expressions was the statement of the goal for the lesson. Here, the teacher states clearly that the goal of the lesson is to add and subtract algebraic expressions. The knowledge learners are supposed to have by the end of the lesson is how to add and subtract algebraic expressions. The statement of the goal for the entire lesson comes after the teacher has checked learners' pre-existing ideas about the concept of *algebraic expressions* through a spontaneous discussion where associative words were called out and through an example where the words were linked to identified symbolic forms. I have chosen to make the statement of the goal a separate segment on its own to highlight its significance as a declaration of that which needs to be explained and learnt in the lesson. Figure

7:3 below shows the frequency for explanation G1, which was the statement of the goal and acts of explaining codes Rg which was the regulative talk for Segment 2.

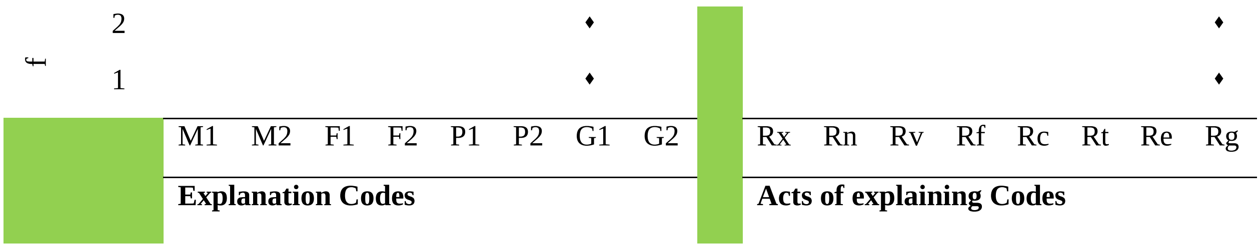
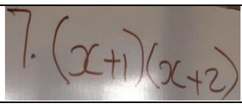
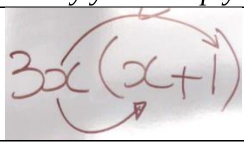
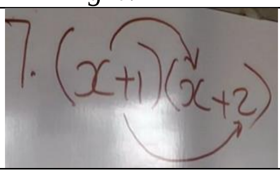
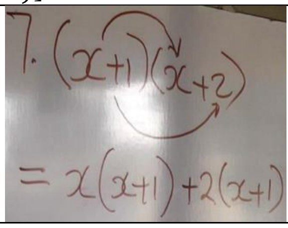
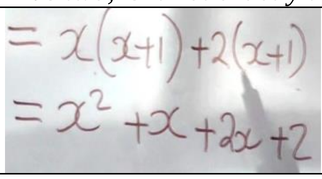
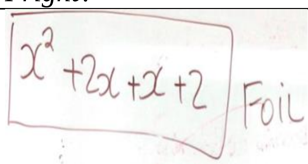


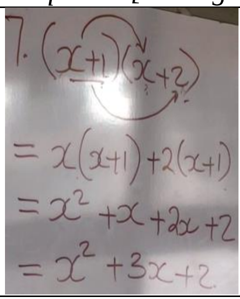
Figure 7:3 – Codes frequency for Teacher A L1 Sg2 – Statement of the goal for the lesson

After the goal for the lesson has been written on the board the teacher moves straight on to the procedural activity. The emphasis on the work done after the statement of the goal was on identification of like terms and how to add and subtract like terms. After the activity of adding and subtracting algebraic expressions, and towards the end of the lesson, a lengthy classwork activity was given from the textbook. Learners were to complete the activity at home. The entire L2 was based on providing answers to these homework items. This extended into L3 and answering L1 homework was only completed in the first 19 minutes of L3. In L3, following the completion of answers to the homework items that remained, the goal was introduced as multiplication of algebraic expressions. After the introduction of the goal, six examples are dealt with, notably Example 7. In Example 7, learners encounter the product of two binomials for the first time, because, as described earlier, in the previous examples the factors multiplied were monomial and binomial. Table 7-7 below is the transcript excerpt of Example 7 from Teacher A's L3.

Table 7-7: Excerpt from Teacher A L3Sg3.7 – Example 7$(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\mathbf{2})$

T	E	AoE	Transcript	
1	M1	Rx	T:	*And one last example* [writing on the board $(x+1)(x+2)$]. *x plus one into x plus two.*
2				
3	M1 F2 F1	Rg	T:	*Okay x plus one into x plus two. We are now multiplying into the bracket. We are now multiplying a binomial with a binomial. Two expressions. Yes Khaya?*
4			K:	*We first say x multiply by x, and then you multiply x by two.*
5	P1		T:	*Is it? Okay is it not you multiply x by x.*
6			K:	*It's going to be x squared. And then you multiply x by two. It's going to be two x.*
7	P1	Rv	T:	*Two x.*
8			K:	*And then you multiply one by x.*
9	P1	Rv	T:	*One by x, aha.*
10			K:	*And then it's going to be one x. And then you multiply one by two.*
11	P1	Rv	T:	*One x which is x aha.*
12			K:	*And then you multiply one by two, it's going to be two.*
13		Re	T:	[Written on the board]. *Do you agree?*
14				
15			Lrns:	*Yes Ma'am.*
16		Re	T:	*Do you agree?*
17			Lrns:	*Yes .*
18	G2	Re	T:	*Why do you agree?*
19			L:	[Not audible]
20	M1 P1	Rg	T:	*We are still using the same distributive law. This* [referring to the answer offered by Khaya] *is correct but I want us to follow the same method that we are using here* [drawing arrows on Example 4], *when we say you multiply three x by x and multiply three x by one.*
21				
22	M1 P1	Re Rt	T:	*Using the same method so that means we are now multiplying this* [pointing at $(x+1)$]. *Let's take this bracket* [referring to $(x+1)$] *as one term. That means you are now multiplying this* [drawing curved distributive arrows from $(x+1)$ to x] *bracket by x and the same bracket* [drawing curved distributive arrow from $(x+1)$ to 2] *by two. Am I right?*
23				
24			Lrns:	*Yes.*

25	P1	Rf	T:	*It's still the same distributive law that we have been doing. So how do we write this? It will be …*
26			L:	*x squared*
27	M1 P1	Rt	T:	*No. I mean when we multiply. We multiply x plus one by x, and we are multiplying x plus one by two* [pointing]. *So we are saying we are multiplying x plus one by x. Let's put x plus one, we multiplying it by x* [written $x(x+1)$]. *x plus one we multiply it by two* [written $2(x+1)$].
28				
29			J:	[not audible]
30	M1		T:	*Okay, let's say we had two into x plus one. Let me take this example* [referring to Example 4]. *Three x, we had three x into x plus one. Am I right?*
31			Lrns:	*Yes*
32	P1 M1		T:	*And we said we multiply that ($3x$) by x and we also multiply that ($3x$) by one, and we said that is three x squared plus three x. Now we are still taking the same bracket Joseph. So the same bracket, the bracket is x plus one. Are we together? The bracket is x plus one. You take the first bracket you multiply the second one. So why we are doing this distributive law, we distribute the first bracket amongst the terms inside the second bracket and that is what we distributed if we distributed x plus one. So x must have x plus one and two should have x plus one. Then if you multiply that you'll get exactly what Khaya got here* [pointing at $x^2+2x+x+2$]. *So let's multiply. Okay let's multiply. What is x times x?*
33			Lrns:	*x squared.*
34	P1	Rv	T:	*It's x squared.*
35			L:	*And x times x … ?*
36	P1	Rv	T:	*x times x?*
37			T:	*x times one? Plus x and then two times x?*
38			L:	*Two x.*
39	P1	Rf	T:	*Two times one?*
40			L:	*Two.*
41		Re	T:	*Plus two, is it not exactly the same?*
42				
43			Lrns:	*It's the same.*
44	P1	Rg	T:	*But the method that he used, he used FOIL* [written on the board]. *Am I right?*
45				
46			K:	*Yes ma'am.*

47	P1		T:	*Explain to them what is FOIL?*
48			K:	*F means First, O means Outer, and I means Inner, and L means Last.*
49			Lrns:	*Repeat.*
50			K:	*F means First, O means Outer, and I means Inner, and L means Last.*
51	P1 F2 F1	Rg	T:	*Okay so this method* [pointing at Khaya's answer] *is a short method for this* one [pointing at her answer]. *This is the distributive law* [pointing at her own answer]. *We distributed the first bracket amongst the terms in the second bracket. So this* [pointing Khaya's answer] *is a shorter method of doing the distributive law. But you can get to the same answers. Now what he did, he multiplied the first by first that is the F, he multiplied x by x and he got x squared. And he also said the outer, x and two. And then the inner is one and x. And the last terms in an expression is one and two and you get that* [pointing at Khaya's answer]. *Okay so I'll give you one more, where you decide whether you use distributive law or you use FOIL. It's up to you which one you use. Okay! Are we done here* [*pointing at* $x^2 + x + 2x + 2$]*?*
52			Lrns:	*No.*
53	P1	Rg	T:	*Are we done?*
54			Lrns:	*No.*
55	P1		T:	*What is the final answer?*
56			K:	*x squared plus three x plus two.*
57	P1	Rv	T:	*x squared [writing the rest on the board].*
58				

1. **Why is this a segment?**

I classified this as a segment based on the discussion of one example focused on multiplication of two binomials. The attention is focused on one example, even though a previous example was referred to. Two different methods are used to find the product of two binomials. The first method was a short-cut method for multiplying two binomials called the FOIL method offered by a learner, whom I chose to call Khaya. The second method was offered by the teacher using the distributive property. The teacher makes a different move by introducing the next example, which I have numbered Sg3.8, and this move marks the end of the current segment, Sg 3.7, and the next example becomes the beginning of a new segment.

2. **What criteria of explanation are present and how?**

   a. **Are both the elements of matter present? (M1 and M2)**

The reading of a string of symbols is done at the beginning of the segment when the teacher introduces the example by reading through it. The reading of elements as a string of symbols also occurs throughout the segment in talk that is concerned with process. The teacher reads the string of symbols she has written on the board in T1 at the beginning of the segment "*and one last example* [writing on the board $(x+1)(x+2)$] *x plus one into x plus two*." I have coded this as M1 because the elements are read in isolation from each other as a string of symbols. The reading of a string of symbols occurs throughout the segment because talk about process involves elements; in T20 the teacher is talking about the process of multiplication and the elements are read as individual symbols: "*we say you multiply three x by x and multiply three x by one*". Talk about process requires a mention of the elements because it is process carried out on the elements. Only M1 is present, because there are no definitions of elements at this level, where all the elements that constitute an algebraic expression have already been called out, identified and some defined in the initial segments of L1.

b. **Are both types of form present? (F1 and F2)**

In T3, the teacher talks about the form of the expression T3 – "*Okay x plus one into x plus two, we are now multiplying into the bracket we are now multiplying a binomial with a binomial, two expressions,*" and I have coded this as F2 even though it is a description of the process of multiplication. The ways in which the teacher talks about the elements that are multiplied, the factors, is such that their structural properties are made visible in the talk. I have also coded T51 as F2, because the question the teacher poses requires that learners look at the entire expression in order to answer it "*Okay! Are we done here* [pointing at $x^2+x+2x+2$]?" learners have to study the expression and decide on the remaining steps of the process that still need to be performed.

I have assigned the code F2 to T3 as well as the F1 code. I have already discussed the F2 code. The F1 code refers to the last part of the sentence where the teacher says "*two expressions*". Both types of form are present in this segment.

c. **Are both kinds of process present if any? (P1 and P2)**

In T51 the teacher elaborates on the difference between FOIL and distributive law. T51 – "*Okay so this method* [pointing at Khaya's answer] *is a short method for this one* [pointing at her answer]. *This is the distributive law* [pointing at her own answer]. *We distributed the first bracket amongst the terms in the second bracket. So this* [pointing Khaya's answer] is *a shorter method of doing the distributive law. But you can get to the same answers. Now what he did, he multiplied the first by first that is the F, he multiplied x by x and he got x squared. And he also said the outer, x and two. And then the inner is one and x. And the last term in an expression is one and two and you get that* [pointing at Khaya's answer]. *Okay so I'll give you* *one more, where you decide whether you use distributive law or you use FOIL. It's up to you which one you use. Okay! Are we done here* [pointing at $x^2 + x + 2x + 2$]?" The teacher clarifies each letter in FOIL. The teacher also gives learners the option to choose between the method shown by Khaya and the distributive law she has demonstrated. There is only one kind of process present, the reason for this is that the steps are simply stated there are no derivations and so the absence of P2 codes.

d. **Is the goal stated and justified? (G1 and G2)**

The goal that was stated for L3 at the beginning of the lesson was that the activity they will be engaged in is multiplication of expressions and this example adds towards the goal of the lesson. Within the segment the achieved goal is that the teacher first elaborates on the FOIL method offered by the learner. The teacher highlights that it is a short-cut method for the distributive property which she then demonstrates.

e. **Is anything absent and why?**

The teacher presented the distributive property as an alternative option of a method to the FOIL method. She did not highlight the distributive law as a general rule for multiplication of expressions where the FOIL method cannot be used. For instance, the FOIL method cannot be performed when it comes to multiplication of a binomial with a trinomial. In introducing the distributive property, the teacher missed the opportunity to highlight the limitations of FOIL as restricted and applicable to the product of two binomials and not useful for algebraic expressions involving multiplication of factors with more than two terms. Another missed opportunity for explanation is pertaining to the distributive property. The distributive property has been demonstrated to learners via an example. However, the reasons for why it is possible to multiply in the manner shown have not been given. One may argue that the explanation for the distributive property given is appropriate for the grade level of the learners. However, as argued under the criteria for process, the missing derivations of the distributive law are responsible for the absence of P2 codes.

3. **What criteria are privileged and thus the achieved explanation?**

In this segment, the criterion that is privileged is process. The achieved explanation is that the FOIL method has been demonstrated by the learner and the distributive property by the teacher; and the two methods were compared to show that they produce the same answer. The teacher gives learners a choice between FOIL and the Distributive law to use as a method for multiplying expressions in the remaining examples.

4. **What is the nature of pedagogy and learner participation?**

Generally, the discussion is teacher led and learners respond to the teacher's questions and finish the teacher's sentences. The teacher re-voices (Rv) and evaluates (Re) learner responses. A learner contributes to the discussion in this example by offering and elaborating on the FOIL method.

In this segment, the discussion was focused on the process. The dominant communicative technique is re-voicing as well as evaluation of learner contributions. After the discussion of what learners understand by the phrase algebraic expressions in the introductory segments of L1, the focus in the remaining part of L1, L2 and L3 was on practicing the techniques of adding, subtracting, and multiplying expressions. Figure 7:4 below presents a general picture of the coding of all the segments across the three lessons, following the discussion of the first two introductory segments in L1.

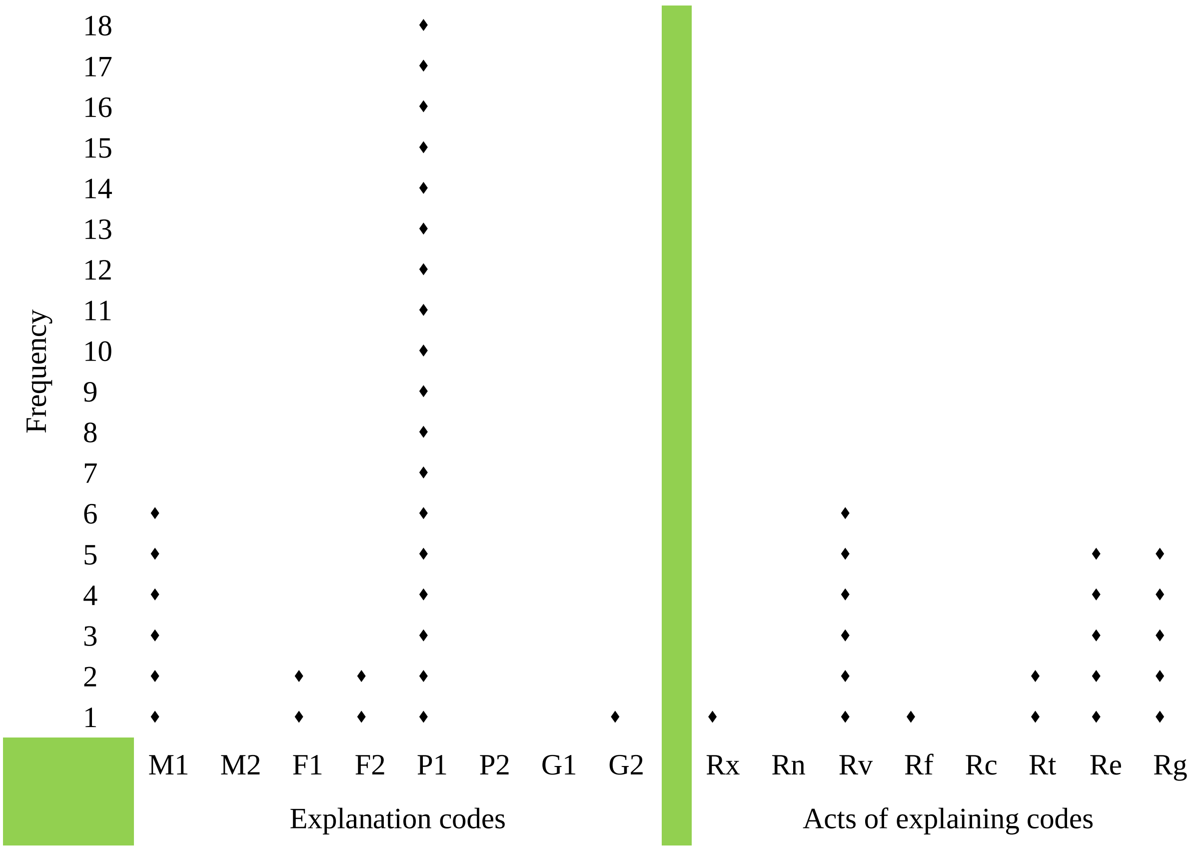


Figure 7:4 – Codes frequency for Teacher A L3 Sg3.7 – Example 7: $(x+1)(x+2)$

Having illustrated how the coding of extracts was done using four excerpts from Teacher A, I can now move on to a discussion and description of what was privileged as explanation in instruction across the three lessons for Teacher A.

7.2.3 Explanation for Teacher A

What criteria for explanation were privileged in Teacher A's lessons? What is the teacher's achieved explanation across the three lessons? Before these questions can be answered, the analysis needs to extend to all excerpts across the three lessons. What is visible in the four excerpts discussed so far is that explanation of elements and form are present, though there were some limitations in the definitions provided. The goals were stated, and processes were outlined, typically without derivation. From the transcript excerpts presented, we have seen in Sg3.7 from L3, which is an instance of how the codes in segments dealing with examples manifested, that there was a dominance of M1 and P1 codes. This pattern was evident from L1 immediately after the initial segments. In order to illustrate this and answer the question of what criteria of explanation are privileged across the three lessons and thus the achieved explanation for Teacher A, I present Table 7-8 below which displays the coding summary for all three lessons.

When reading the table from the bottom, the table is partitioned according to lessons, L1 – L3, labelled on the last row. Indicated for each lesson is the activity and the duration for each segment. The frequency for the eight explanation codes are mapped for all the segments and totalled. Figure 7:5 below is a key for reading Table 7-8 for Teacher A which shows the frequency for explanation and acts of explaining codes.

Act – Activity	**Eg –** Example	**AE –** Algebraic Expressions
Seg – Segment	**HW –** Homework	**Ans –** Answer
L – Lesson	**Tpc –** Topic	**LA –** Learner Answer

Figure 7:5 – Key for reading table 7-8 Teacher A

Table 7-8: Frequency for explanation across the three lessons for Teacher A

EXPLANATION	**G_2**				4	1	2	1		3												1		1	1	2	1	1	1	1	1		**21**
	G_1	1		2																			3										**6**
	P_2																																**0**
	P_1				10	15	20	5			1	1	1				1	2	1	3	6	1		5	11	4	7	7	17	18	5		**131**
	F_2	2	3		1	2	3	1		6				1								1		1		3				2			**26**
	F_1		3		3		1				1		2					1		4	1	1		2					1	2			**23**
	M_2		4		2											2								1									**9**
	M_1	5	9		1	1	4	2			2		2	5	1	2	2	1	2	2	6	2		6	4	1	4	1	6	6	4		**81**
	Act	AE	Eg	tpc	Eg 1	Eg 2	Eg 3	Eg 4	HW	Ans	LA	LA	LA	Ans	LA	Ans	LA	LA	LA	LA	LA	LA	tpc	Eg 1	Eg 2	Eg 3	Eg 4	Eg 5	Eg 6	Eg 7	Eg 8	H W	
	Seg	1.1	1.2	2	3.1	3.2	3.3	3.4	4	1	2	3	5	6	7	8	9	10	11	12	1.1	1.2	2	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	4	
	Time	00:00	02:46	06:44	09:09	13:14	19:28	25:56	33:29	00:00	02:06	04:02	05:47	08:47	09:47	11:06	11:46	14:43	19:00	24:00	00:00	06:47	18:23	19:00	21:43	24:44	28:55	32:59	40:56	48:25	54:24	01:03:54	Total
	L	L1: ADDING AND SUBTRACTING ALGEBRAIC								L2: ANSWERS TO HOMEWORK											L3: MULTIPLYING ALGEBRAIC EXPRESSIONS												

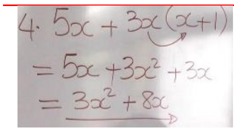
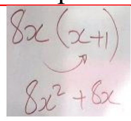
I outline what is visible across the table in terms of the four criteria of explanation below.

With respect to matter, across the three lessons there are 81 utterances coded as M1, while there are only nine utterances coded as M2. The reason for this is that after the initial excerpts, where M2 was present, the elements are not repeatedly defined. As the lesson advances it is expected that learners already know what each symbol and notation in the expression refers to and hence no need to repeatedly define elements. The high numbers for M1 indicate that in most of the explanatory talk the elements are continuously read as a string of symbols. Between M1 and M2 we see that M1 was the most privileged criterion of explanation, thus it can be concluded that the achieved explanation was at the level of reading a string of symbols.

It was also observable across the three lessons that attention to form was present, with a total of 22 utterances coded as F1, while there were 24 utterances coded as F2. In the three lessons the explanation of form had to do with terms (F1), identification of structure of the individual term, and how to identify like terms and group like terms in an expression. There was also emphasis on which expression to write first when subtracting expressions (F2). Explanation in relation to form across the three lessons was based on the identification of structure in the final answer (F2) and a decision to stop operational activity. There was no clear cut privileging of F1 over F2 codes, as there was no significant difference between F1 and F2 counts. However, the low frequency of form in relation to M is troubling as it points to a lack of explicit talk given to algebraic structure. Each expression dealt with in the example in each activity has its own structure, and so it is expected that there would be a study of structure (F2) for each expression before the operational activity. In this way, F codes are expected to show larger numbers than what is currently depicted. The achieved explanation at the level of form was that structure was spoken about at both levels.

With regard to process, across the three lessons there were a total of 131 utterances coded as P1, while there was an absence of utterances coded as P2. In L1, the procedural activity was concerned with addition and subtraction of expressions where steps of a process were outlined. One of the steps that was outlined was an explanation of how to open a bracket with a negatively signed number – this is multiplication. This kind of talk to refer to the process of multiplication was carried through to L3. Consider the transcript excerpt below in Table 7-9 taken from Teacher A L3Sg3.4

Table 7-9: Teacher A L3Sg3.4

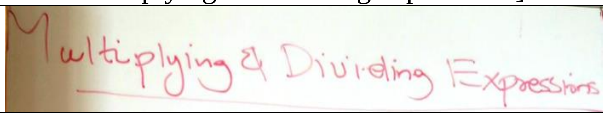
T	E	AoE	Transcript	
47	P1	Re	T:	*Would it be the same answer if we open the bracket?*
48			Lrns:	*No.*
49	P1 F2	Re Rg	T:	*If we open the bracket here, we will be having eight x squared plus eight x* [written], *which is not the same as what we have there.* Teacher and learners  Joseph's answer  *So we must use BODMAS, so we have to use BODMAS, so that is how important BODMAS is* [writing example five on the board] .

The notion of "opening a bracket" is colloquial talk in mathematics classrooms, used to refer to the process of distributing multiplication over addition.

This operational activity was carried over to L2. In L2, the process learners were also engaged in was evaluation of expressions by substituting given values into the expression. In L3, the main process the teacher and learners were engaged in was multiplication of expressions. The explanation was mainly based on distributing what is 'outside the bracket' to all the terms inside the bracket. The importance of BODMAS as a mnemonic for remembering the order of operations was emphasised. The FOIL method was described as a short cut for the distributive law. There was no mention of the FOIL method's limitations and weaknesses as a short-cut for multiplication of expressions, nor that it is indeed a short cut of multiplication by distribution of multiplication over addition. As a result, there is no attention to how steps in a process or procedure are derived. The most privileged type of process is P1 with an absence of P2 codes. It is a limitation not to have derivation of steps and the methods used, because this means there was no "why" explanation for the steps for operating on expressions.

Also observable across the three lessons is the statement of the goal and justifications given to some aspects of the goal, process, form or matter, and this occurred in L1 and L3, where the goal of the lesson was explicitly stated. L2 was the continuation of work from L1, and so the entire lesson was devoted to offering of corrections to homework items. In L3, the teacher connects the goal by noting the sequence of topics in the textbook. Consider below Table 7-10.

Table 7-10: Statement of the goal taken from Teacher A L3Sg2T1-5

T	E	AoE	Transcript	
1	G1	Rg	T:	*So what are we left with now that we know adding and subtracting expressions what are we left with?*
2			Lrns:	*Multiplying and dividing expressions* [reading from their textbooks].
3	G1	Rg	T:	*Multiplying and dividing expressions so that is our next topic, so write todays date and write multiplying and diving expressions* [writing on the board multiplying and dividing expressions] .
4				
5	G1	Rg	T:	*Multiplying and diving expression* [pause]. *Okay, we will start by looking at multiplying expressions, so we will first look at multiplying expressions.*

At the level of G2 codes, the explanation required reasons by asking the why question at the level of P1. There are instances in the procedural activity where a reason for why operational activity on a specific expression had to stop was required. And so, while the steps were not derived, there was attention given to some why features of the process.

The criteria then are how to read out strings of symbols and what steps to perform in the procedure. M1 and P1 codes were the most privileged criteria of explanation, and thus the achieved explanation across the three lessons for Teacher A. While there was attention to structure, attention to definition of terms and derivation of steps were largely absent. Having described the criteria for explanation in all three of Teacher A's lessons, what then of criteria for acts of explaining across the lessons?

7.2.4 Acts of explaining for Teacher A

What criteria for acts of explaining are privileged in Teacher A's lessons? What can be concluded about the teacher's acts of explaining from the coding used across the three lessons?

What is visible in the four excerpts in this chapter is that acts of explaining concerning selection and use of examples and symbolic representation is present. The code for mathematical language in this study was described as explicit naming and identification of what things are called mathematically, as well as mathematical terminology. I acknowledged that the codes used for the interactive pattern (Rv, Rf, Re, Rc, Rg) are all dependent on spoken language as much as gestures (Rt) are dependent on symbolic language. And so, while language is the main activity of communicating the explanation, the naming of mathematical entities and mathematical terminology, I have given a separate code Rn. Viewed from this understanding, the use of mathematical language is absent in Teacher A's practice. In the next chapter, I discuss possible reasons for this absence. Re-voicing was the most dominant mode of communicating the mathematics. The question that the summary of coding for all three lessons needs to answer is whether this pattern from the four transcript excerpts is maintained or not? In Table 7-11, the coding of all segments across the three lessons is presented for acts of explaining.

Table 7-11: Frequency for acts of explaining across the three lessons for Teacher A

EXPLAINING	**Rg**	4	3		5	4	6	4	1	4	3	2	2	4	1		2	4	2	2	2	3	3	7	9		4	6	7	5	5	1	**109**
	Re						3	7			2		2	2	1		3	2	2	4	5	5		1	1		6	4	5	5	2		**62**
	Rc																																**0**
	Rt	1											1				1		1	2					1	2	2	2		2			**15**
	Rf		7		4	3	2	3								1			1	2	3			3		1	2		1	1			**34**
	Rv	9	9		10	14	11	1		4				2							2			6	10	6	8	4	13	6			**115**
	Rn																																**0**
	Rx		1		2	1	2	1			1	1	1	1	1		1	1	1	1	1	1		1	1	1	1	1	1	1	1		**26**
	Act	AE	Eg	tpc	Eg 1	Eg 2	Eg 3	Eg 4	H W	Ans	LA	LA	LA	Ans	LA	Ans	LA	LA	LA	LA	LA	LA	tpc	Eg 1	Eg 2	Eg 3	Eg 4	Eg 5	Eg 6	Eg 7	Eg 8	H W	
	Seg	1.1	1.2	2	3.1	3.2	3.3	3.4	4	1	2	3	4	5	6	7	8	9	10	12	1.1	1.2	2	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	4	
	Time	00:00	02:46	06:44	09:09	13:14	19:28	25:56	33:29	00:00	02:06	04:02	05:47	08:47	09:47	11:06	11:46	14:43	19:00	24:00	00:00	06:47	18:23	19:00	21:43	24:44	28:55	32:59	40:56	48:25	54:24	01:03:54	Total
	SU	L1: ADDING AND SUBTRACTING EXPRESSIONS								L2: SOLUTIONS TO HOMEWORK											L3: MULTIPLYING ALGEBRAIC EXPRESSIONS												

What is visible across the three lessons is outlined below according to the three categories of acts of explaining namely language and examples (Rn, Rx), modes of communication (Rv, Rf, Rt, Rc, Re) and regulative talk (Rg).

The examples used across the three lessons were symbolic representations and a total of 26 examples were used and discussed in class. There were no utterances coded as Rn across the three lessons. While definitions of constitutive elements and form were present, there was no explicit attention given to the meaning of mathematical terms. There was no deliberate attention focused on what things are called in mathematics and why. In Teacher A, class names for elements were taken as shared knowledge, possibly because all learners were in possession of a mathematics textbook where they could look up definition of terms – I elaborate more on this discussion in the next chapter.

Across the three lessons, Teacher A has predominantly used re-voicing to communicate the knowledge with 115 utterances coded as Rv. The second highest communicative technique to re-voicing was evaluation of learner responses with a total of 62 utterances. There was finishing of the teacher's sentences, with a total of 34 utterances coded as Rf. The fact that re-voicing and evaluation of learners' responses are the most dominant criteria for communicative techniques in the acts of explaining is an indication that learners participate and contribute to classroom talk and explanation. Also, it is an indication that learners participate in the use of language to talk about mathematics, even though the mathematical language is not adequately emphasised in terms of what things are called and why.

The Rg codes have the highest number of utterances because management and maintenance of classroom order occurs throughout. Order is maintained while dealing with examples, and specific rules are upheld in the communication for example chorusing of responses to teachers' questions is forbidden, while finishing the teacher's sentence (Rf) is allowed.

Criteria for acts of explaining that have been privileged in Teacher A's three lessons in the first category of language and examples presented in symbolic form. In the second category, re-voicing has been privileged and the regulative talk is dominant. We can see that the pattern depicted in the four transcript excerpts used for evidencing in this chapter has been maintained across the three lessons. There is a clear relationship between criteria for explanation and acts of explaining as earlier argued in Chapter 3. The acts of explaining are actions taken to produce the explanation. And so, in the three lessons for Teacher A the examples and their symbolic representations, and the act of re-voicing and evaluating learners' responses are actions taken to talk about constitutive elements, form, process and goals. I pick up on the discussion of how the nature of acts of explaining further illuminate the nature of explanation offered in the next chapter. I now turn to a discussion of the second teacher and that is Teacher B.

7.3 Teacher B

In this section, I provide a description of what was privileged as explanation and acts of explaining in Teacher B's practice. I first begin with a discussion of segments, activities and examples used across the three lessons. I then proceed to a presentation of particular cases to evidence the coding process for explanation and acts of explaining. This is followed by a discussion of that which was constituted as explanation and acts of explaining for Teacher B.

7.3.1 Teacher B Lesson 1-3

The summary and structure of L1 is such that the lesson has six main activities within it. The first activity is what the teacher calls a 'mental maths' activity, which has three segments to it. In the first segment, the teacher calls the four expressions out verbally; in the second segment, the teacher walks around monitoring each learner's book and the last segment is a discussion of answers to the mental maths activity. The expressions called out verbally are: the addition of like terms ($a + a$), the second is multiplication ($2a$), the third is squaring (a^2), and the fourth is a square root ($\sqrt{a}$). The four have the same variable, whose value has been given as 4. The activity is evaluation of each expression, by substituting the given value. The second activity is based on a discussion of the notion of a variable, which is a segment on its own. The third activity has three segments to it, focused on terms and polynomials such as monomial, binomial, and trinomial. The segment focused on monomial has a side segment within it focused on a discussion of a coefficient. The segment focused on a discussion of a trinomial also has a side segment within it, focused on multiplication. Activity number four is the statement and justification of the goal for the lesson.

Activity five is focused on the powers of a variable, and has a side segment within it focused on operations that create terms. The last activity, activity six, is the provision of homework by the teacher. In the instruction of the homework activity, the number of terms and degree of polynomial is required. Interestingly, the last expression in the list of homework items is not a polynomial, because the exponents for the variable are negative numbers, instead of positive whole numbers.

I have highlighted Sg 2 and 4, because I use them in the next section, where I present the transcript excerpts to evidence the coding of the data. Table 7-12 below shows the examples used in L1 for Teacher B.

Table 7-12: Activities, segments, and examples used in Teacher B L1

		Lesson: 1 Topic: algebraic expressions	
Sg #	**Time**	**Activity**	**Example**
1.1, 1.2, 1.3	00:00	Mental Maths	If $a = 4$ then calculate: $a + a$, $2a$, a^2, $\sqrt{a}$
2	13:54	Definition of a variable	$If\ x = 4\ then\ x + x,\ emptybox + 2 = 5,\ 2emptybox = 8$
3.1	19:20	Monomial	(a) x, (b) $3x$, (c) $3x + 2x = 5x$
3.1.1	22:28	Discussion of coefficient,	$10 \times x = 10x$
3.2	24:18	Binomial	$3x + 2y$
3.3 3.3.1	26:01	Trinomial and polynomial with a side based on multiplication	$3x + 2y + z$
4	33:18	Goal	No written example
5.1 5.1.1	35:22	Powers of a polynomial with a side based on terms	$2x^3 + 4x^2 - 2x + 7$, $7 + 2x + 4x^2 + 2x^3$ and $2x^3y + 4x^2y^2 - 2xy^3 + 3$
6	41:07 – 48:06	Classwork: How many terms are in the following polynomials? Does each have ascending or descending powers of x?	(a) $x^3 + 3x^2 + 3x + 1$, (b) $x^4 + 2x^3y + 7x^2y^2 - xy^3 + y^4$, (c) $x^5 + x^3 - 19$, (d) $x^{-2} + 30x^{-1} + 3$

There are five main activities in L2. The first activity is the marking and checking of homework from the previous day. There are six segments within this activity, the first five are made up of the homework examples and the sixth segment is a response to a learner’s question in relation to the homework. Activity 2 is the mental maths activity. There are seven segments within this activity. The first two segments the teacher calls out the mental maths examples, and then checks each learner’s book if they have correctly translated the words into symbolic expressions. The mental maths activity is composed of five algebraic expressions, where the first and last are expressions where terms are added or subtracted while the middle three are one term expressions because of the multiplication. The third activity is the statement of the goal for the lesson which is written as “Algebraic expressions - Group like terms”. There are five segments within Activity Four, which are examples of expressions focused on identifying, grouping and adding like terms. The last activity is the homework activity, where the teacher writes on the board algebraic expression for learners to practice adding and subtracting like terms. Table 7-13 below shows the activities, segments and examples used in L2 for Teacher B.

Table 7-13: Activities, segments, and examples used in Teacher B L2

		Lesson 2	Algebraic expressions - Group like terms
Sg #	**Time**	**Activity**	**Example**
1.1, 1.2	00:00	Answers to HW	$x^3 + 3x^2 + 3x + 1$
1.3	09:46		$x^4 + 2x^3y + 7x^2y^2 - xy^3 + y^4$
1.4	11:52		$x^5 + x^3 - 1$
1.5, 1.6	12:50		$9x^{-2} + 30x^{-1} + 3$ Learner question based on (b)
2.1-2.7	13:59 28:16	Mental Maths	(1) $2m + 2m + 2m + 2m$, (2) $c \times d \times a \times b$, (3) $2 \times x \times y \times 3$, (4) $c \times c \times c$, (5) $7x - 5y + 1$
3	28:48	Goal	No written example
4.1	29:42	Example 1	$5xyz; 3xyz; 2zyx; 5xy$
4.2	34:48	Example 2	$x, 5x, 5y, 3x$
4.3	36:22	Example 3	$7y^2, 7, 3y^3, 4y^2$
4.4	38:42	Example 4	$(2a \times b) + (3a \times b) - (a \times b)$
4.5	40:26	Example 5	$(3a \times 2) + (-4b \times -2) + (a \times -2)$
5	42:24 46:30	Homework	1. Is it possible to simplify? (yes/no) (a) $x + 2x$, (d) $4x + 4y + z$, (b) $a + 3b$, (e) $-7a + 2a$, (c) $10a - a$, (f) $5a + 2b + 3b - 4a$ Simplify $(3a \times 2b) + (6b \times c) + (a \times b) + (b \times c)$ $(4y \times 2) + (2 \times 5) + (3 \times 4) + (4 \times 4y)$ $(6b \times d) - (4a \times 3c) + (7a \times -2c) + (-3a \times -2c)$ $(6m \times n) + (3m \times n) + (4n \times -p) + (3n \times -p)$

L3 has five main activities to it. First the teacher checks and marks learners' books and then provides answers to number one only of the homework from the previous day. In activity two the teacher moves straight into the day's lesson which they call the "new work"; this discussion is composed of four segments with geometric representations. In the four geometric examples, the activity is to find area which is then connected to multiplication of algebraic expressions using the distributive law. The third activity is the statement of the goal for the lesson as "multiplying a monomial and a binomial". The fourth activity for the lesson is comprised of two examples, which are segments in themselves based on using the distributive rule to multiply a monomial and a binomial. The last activity of the lesson is based on the teacher providing homework and restating the goal of the lesson. I have highlighted Sg2.4, because I use it to evidence the analysis and coding of transcript excerpts. Table 7-14 below shows the activities, segments and examples for L3.

Table 7-14: Activities, segments, and examples used in Teacher B L3

		Lesson 3: Multiplying a monomial and a binomial	
Sg #	Time	Activity	Example
1	00:00	corrections	Is it possible to simplify? (yes/no) (a) $x + 2x$, (d) $4x + 4y + z$, (b) $a + 3b$, (e) $-7a + 2a$, (c) $10a - a$, (f) $5a + 2b + 3b - 4a$
2.1	06:40	Example 1	Square of unit length 4
2.2	13:09	Example 2	Rectangle $l = 3;\ b = 1$
2.3	14:26	Example 3	Rectangle $l = (x + 2);\ b = 3$
2.4	15:40	Example 4	Square + rectangle, Square $l = x$, Rectangle $l = x + 2;\ b = x$,
3	20:06	Goal	No example
4.1	20:42	Example 5	$2(2x + 3)$
4.2	21:37	Example 6	$2x(-x - 4)$
5	22:55 25:09	Homework:	Simplify: (a) $5(3x - 1)$, (b) $6(3x - 2) + 2(x + 3)$, (c) $3(3x - 2x) - (4x - 2x)$, (d) $-7(-3x + 4y)$

In overview, in L1 examples of a concept like variable, monomial, binomial, trinomial, polynomial, degree of polynomial are provided. There are no examples of a process as yet. The teacher takes detailed, lengthy side steps in Sg 3.1.1, 3.3.1, and 5.1.1 to clarify the concepts coefficient, terms, and multiplication. In L2, the examples presented are examples of a process of grouping and adding like terms. In L3, the examples are first presented in geometric form. The aim is to link the process of multiplication using an area model to the process of multiplying using the distributive law. These, interestingly, are simultaneously of a concept of multiplication as well as examples of a process.

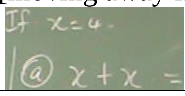
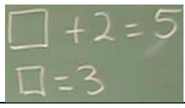
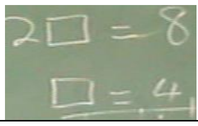
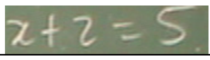
Mental Maths activity is used frequently as a warm up exercise by Teacher B. It is a verbal exercise, where learners have to listen, convert the verbal utterances into written symbols and words, and provide answers to the mathematical problem. It appears as if the mental maths activities are done as a supplement to the lesson that can be omitted when time does not allow. In L3, for instance, there is no Mental Maths, perhaps because the duration of the lesson is short, and so subordinate to the curriculum work that Teacher B needs to cover.

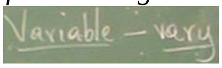
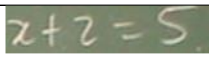
I have again attempted to foreground the content covered in the lessons for Teacher B, so that this can serve as a backdrop to a discussion of how she built her explanations and acts of explaining. Before I present the explanation and acts of explaining for Teacher B, a discussion of the examples of transcript excerpts highlighted in L1 and L3 to show the coding is first undertaken.

7.3.2 Teacher B transcript excerpts

I selected examples of how the coding and analysis of each segment was done from L1 and L3. From L1 I chose Sg2 because it comes after Mental Maths activity, and informally marks the beginning of the lesson. Sg2 is an interesting fragment of the lesson because it has a lot of elaborations with examples, it also incorporates most of the codes from both explanation and acts of explaining. Sg 2 also shows the amount of detail Teacher B goes into, to provide an explanation of the concept of a variable. I then jumped to Sg7 of the same lesson and included it in the exemplification process, because it is at this point that Teacher B gives a statement of the goal, which she also goes into great lengths to justify. These two segments (Sg2, Sg7) only provide examples of how an element is defined and how the goal for the lesson is introduced and justified. Still needed is an example of how the procedural activity is done. For this I chose Sg 3.7 from L3, because the teacher starts with multiplication of expressions using the area model and then links it to multiplication of expressions using the distributive law. Table 7-15 below is Sg 2 from Teacher B’s L1.

Table 7-15: Excerpt from Teacher B L1 Sg2 – Discussion of variable

T	Exp	AoE	Transcript	
1	M1	Rg	T:	*Right! Now I want us to look at* a*. Where there is* a *can I use* x*?*
2			Lrns:	*No.*
3			T:	*Where there is* a *can I have said calculate* x *plus* x*?*
4			Lrns:	*No.*
5	G2		T:	*Why not?*
6			Lrns:	*Because we are solving* a*.*
7	P1 M1	Rx Rv	T:	*Because we solving* a*. Okay!* [Approaching the board]. *Let me say to you I did not say* a *plus* a *but I said* x *plus* x [writing on the board]. *And then I say if* x [while writing] *is equal to four* [moving away from the board]. 
8	P1		Lrns:	*Four plus four.*
9			T:	*It's gonna be…?*
10			Lrns:	*…four plus four.*
11	M1	Rv	T:	*Four plus four. Right! So can I use* y*. If I say if* y *is equal to four and then I say* y *plus* y*.*
12			Lrns:	*Yes.*
13			T:	*Will I still be right to use y?*
14			Lrns:	*Yes.*
15	M1		T:	*Can I use any other alphabet?*
16			Lrns:	*Yes.*
17			T:	*I can use any other alphabet right?*
18			Lrns:	*Yes.*
19			T:	*Okay.*
20		Rx		[Teacher erasing the board, then writing $\square + 2 = 5$] 
21	M1 P1		T:	*So now I don't use the alphabet. I choose to use a box. What is inside the box?* [Moving away from the board]
22			Lrns:	*Three.*
23		Rv	T:	*Its three, three is inside the box. We all agree*. [Approaching the board to write ☐ equal to 3 in the next line] 
24	M1 P1	Rx	T:	*So now we say the box stands for three. Right! And then when I say* [writing on the board] *two times a box is equal to eight.*  *What is in the box?*
25			Lrns:	*Four.*
26	P1		T:	*Four is in the box* [writing in the next line $\square = 4$]. 
27	M1 F2	Rt	T:	*So the box stands for four. So, is there a difference between this* [pointing at $\square + 2 = 5$] *and* x *plus two is equal to five* [writing on the board $x + 2 = 5$]*?* 

28			Lrns:	*No.*
29	M1	Rt	T:	*There is no difference. So when you were a child the boxes were used. But now you are an adult the alphabets are used. Okay! So it means here again* [pointing to 2 times ◻ =] *I can write this as what?*
30			Lrns:	*Two x equals to eight.*
31	P1 M1	Rv Rn	T:	*Two x is equals to eight* [writing on the board]. *Or I can write it as two g, two h. Yah! Right now. But when we use it in maths, do we still call it an alphabet?* 
32			Lrns:	*No.*
33	M2	Rn	T:	*We don't call it an alphabet* [writing on the board the word variable]. *We call it a variable.* 
34		Rc	Lrns:	[learners reading from the board] *variable*.
35	G2 M2	Rn	T:	*We call it a variable. Okay. But there is a reason why it's called a variable. So this word variable comes from an English word* [writing the word vary next to variable] 
36		Rn	T:	*That means vary… very… vary, I don't care how you pronounce it. So what does? If I say to you eh these learners they are doing, they vary in their vernacular languages. What am I saying?*
37			L:	*They are different.*
38	M2	Rv Rn Rt	T:	*Different! Very good, my baby. Different! So these learners are doing different vernaculars. Right! So this word* [pointing at variable] *variable comes from the word vary* [pointing at vary]. *Which means varying, different, changing. So look here* [pointing at $x + 2 = 5$] *for instance. So when you look here x is what?*
39			Lrns:	*Three.*
40		Rt	T:	*You see that?* [pointing at ] *x is three. But when you look here* [pointing at $2x = 8$ and writing $x = 4$]
41			Lrns:	*Four.*
42		Rv	T:	*x is four. So does it mean if I come to class and say what is the value of x can you tell me?*
43			Lrns:	*No.*
44	M1 M2	Rn Rt Rc	T:	*You can't tell me before I give you the sum. Can you see that? Now before I say if a is equal to four* [pointing at a $=4$ on the board] *then calculate a plus a. If I come to class and say calculate a plus a. Okay. Now which means that there is no one value of a what? Of a variable, hence we make use of this word* [pointing at variable]. *Which in English comes from the word vary which means the value is forever changing which means we don't have one value of a. The value depends on what the equation is. The value depends on different other things. The sum that you are given. The question you are given. So what do we call an alphabet in maths?*
45			Lrns:	*A variable.*
46	M1	Rv Rn	T:	*We call it a variable.*
47		Rn Rc	T:	*What do we call an alphabet in maths?*

48			Lrns:	*Variable.*
49	M2 G2	Rv	T:	*We call it a variable because it stands for any number. Okay.*
50			19:00	[Erasing the board and then looking at notebook]
51	G2		T:	*Right! Instead of using boxes, because now you are grown up, you are high school kids so we make use of variables*

1. **Why is this a segment?**

I have categorised this as a segment, because one mathematical idea is discussed from beginning to end. Attention in this segment is focused on the notion of variable. At the beginning of this segment, attention shifts from providing answers to mental maths items to focus on letters used in the expression and so the beginning of this segment. The teacher uses different examples to elaborate the notion of variable. Teacher B also draws on an everyday English meaning of the word "vary". The segment ends when attention shifts from variable to focus on polynomials. I now discuss the segment in relation to the four criteria for explanation.

2. **What criteria of explanation are present and how?**

   a) **Are both elements of matter present? (M1 and M2)**

In T1, the teacher starts the segment with the utterance which I coded M1: "*Right! Now I want us to look at a. Where there is a can I use x*?" The talk in this utterance is focused on the letter a, which is an element. Teacher B uses examples to bring into focus what she wants to elaborate in this segment. In T7 the teacher says, "*Let me say to you I did not say a plus a but I said x plus x* [writing on the board]. *And then I say if x* [while writing] *is equal to four* [moving away from the board]." I have coded this as M1, because individual elements are in focus and symbols are read in isolation from each other.

The teacher connects the notion of an "alphabet" or letter to ideas learners have seen and worked with before. T21 "*So now I don't use the alphabet. I choose to use a box. What is inside the box?*" T29 "*So when you were a child the boxes were used. But now you are an adult the alphabets are used. Okay*!" The teacher explains how the notion of an unknown in an expression has always existed before alphabets were introduced to symbolise unknowns.

The denotation of the word variable is unpacked in T38 by highlighting its roots in the English word "vary" and the meaning of the word "vary": "*So this word* [pointing at variable] *variable comes from the word vary* [pointing at vary]. *Which means varying, different, changing*." I have coded this utterance as M2 because a definition of a variable is given as "…varying, different, changing."

The teacher adds extra detail to give reasons for the use of a letter in the definition of a variable she provides in T49 "*We call it a variable because it stands for any number. Okay.*" I coded as M2 because the teacher expresses a definition of a variable which is an element. In this segment, therefore, both elements of matter are present.

b) Are both types of form present? (F1 and F2)

Even though the focus of the segment is providing a definition of a variable, the examples Teacher B uses are equations. The specific utterance that I have coded as F2 is when the teacher requires a comparison between "$\square + 2 = 5$" and "$x + 2 = 5$" in T27 "*So the box stands for four. So, is there a difference between this* [pointing at $\square + 2 = 5$] *and x plus two is equal to five* [writing on the board $x + 2 = 5$]?" to compare the two expressions. Learners will have to look at the form of each equation. There were no F1 codes, and so only one type of form is present in this segment.

c) Are both kinds of process present if any? (P1 and P2)

The process learners are engaged in is solving equations by inspection. There are two kinds of equations where the value of the unknown is required that is $x + 2 = 5$ and $2x = 8$. Procedural activity is not necessarily the focus in this segment, but a means used to make sense of the notion of a variable, and so one kind of process is present and these utterances are coded P1.

d) Is the goal stated and justified? (G1 and G2)

The goal is not yet stated for the entire lesson but the achieved goal for the segment is that the notion of variable is discussed in detail. Thus, there is no G1 code. However, in T35 the why question was asked seeking a reason for the name of the element which I coded as G2. "*We call it a variable. Okay. But there is a reason why it's called a variable. So this word variable comes from an English word* [writing the word vary across variable]."

e) Is anything absent and why?

The teacher uses the context of equations with an unknown as examples to define notion of variable. This is one context, because the meaning of the variable is context dependent. In fact, the contexts/example used have no variability as the word suggests, because it is a linear equation, and so there is only one value that will make the equation true. However, at Grade

Nine level it can be argued that it is sufficient to say a letter is called a variable without getting into too much detail of what that means, as the information may confuse rather than clarify.

3. What criteria are privileged and thus the achieved explanation?

The most privileged explanatory talk in this segment is focused on constitutive elements. This goes without saying, because the focus of the discussion is itself a constitutive element: a variable. There is a considerable amount of P1 type procedural talk because the examples used to explain the notion of a variable are operated on. However, this talk is backgrounded as the explanatory talk is focused on elements in the segment. Three main points emerge as the achieved explanation of the notion of a variable (and so M1 and M2):

- That any letter can be used to stand for a number in mathematics. This is the simplest definition of a variable is that the letter is a symbol for a number, the value of which is unknown.
- That it is similar to their use of a box in primary school, in this way the teacher connects the notion back to ideas that are familiar to learners.
- We call the letter a variable because it "varies", changes, it stands for any number. From this definition, the meaning of variable is related to its use in everyday English.

4. What is the nature of pedagogy and learner participation?

As with segments from Teacher B already discussed, the most dominant mode of communication is re-voicing (Rv) followed by gesturing (Rt) with a minimal amount of chorusing. The teacher pays special attention to what an alphabet in mathematics is called and the reasons for the name, hence, the significant amount of codes focused on mathematical language (Rn). Figure 7:6 below gives a picture of the distribution of where most of the explanatory talk is situated as well as acts of explaining.

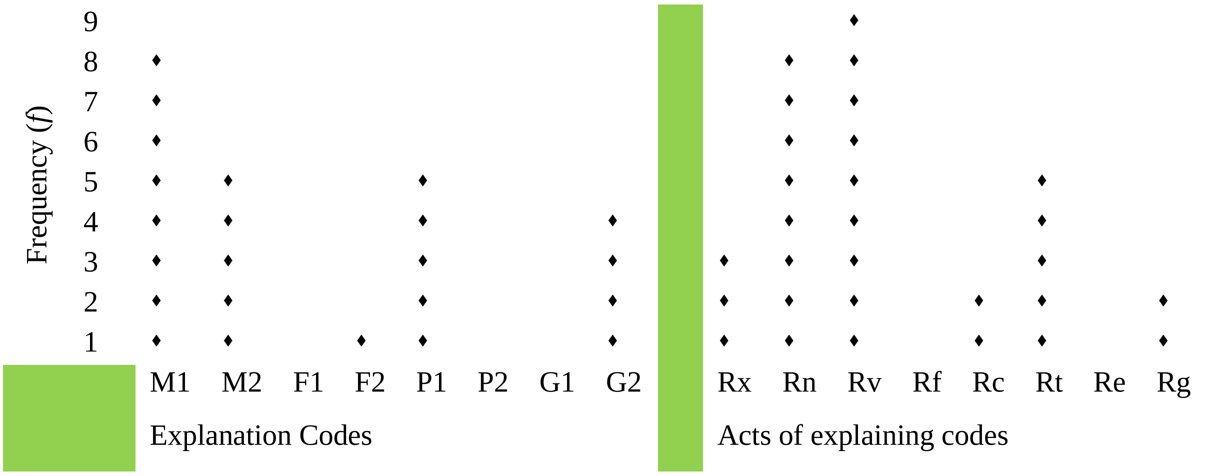
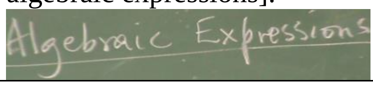


Figure 7:6 – Codes frequency for Teacher B L1 Sg2 – Discussion on Variable

After the discussion of a variable the teacher discussed polynomials, where she defined mathematical terminology, such as monomial, binomial, trinomial, and polynomial. Following the discussion of polynomials was the statement of the goal, which I present in Table 7-16 below.

Table 7-16: Excerpt from Teacher B L1 Sg9 – Statement of the goal

T	Exp	AoE	Transcript	
1	G1 G2	Rg Rn	T:	*Right! So now I want us to look at the last thing in this section. Because, when this whole section, when we deal with it we say algebraic expressions* [writing on the board and underlining: algebraic expressions]. 
2			Lrns:	*algebraic expressions* [reading from the board]
3	G1 G2	Rg Rn	T:	*Okay, okay, okay, so algebraic expressions. Why expressions? If I say give me an English expression. Give me any English expression.*
4			Lr:	*'to experience'*
5		Rn Rt	T:	*That is not an expression. It means, it means a short statement of whatever you want to say. When you say… yes?* [pointing at a learner]
6			Lr:	*'My name is Goliath'.*
7			T:	*That is an expression.*
8			Lrns:	[clapping hands]
9	M1 F2	Rn Rg	T:	*So, that is an expression. Right! So it means a short sentence, a short sentence in English. Okay. I love you. That is an expression. But in maths, okay guys* [learners making a noise], *the expressions are different than in English. So when I say a plus b, that's an expression, but it's a mathematical expression. Because this section we are, eeh. This chapter is under we call it under algebra so that's why we call it algebraic expressions. But it means it's a sentence*

				that is said mathematically. If I say if a is equal to four what is a plus a? That's an expression, but it's a mathematical expression, do you get that?
10			Lrns:	*Yes.*
11	G1		T:	*So, when we deal with this chapter we are dealing with how we express ourselves in maths, in maths. How we express ourselves in maths. That is what we mean.*

1. **Why is this a segment?**

In this segment, the teacher states the goal and clarifies what the words "algebraic expressions" refer to in mathematics. The teacher makes an attempt to explain what is meant by the topic: *algebraic expressions*. Teacher B did not introduce the topic at the beginning of the lesson by writing the words on the board, underlining and then carrying on to the first example, as is typical in many lessons including Teacher A's lessons. Instead, the topic comes after a number of examples associated with the concept of algebraic expressions have been dealt with; and so the teacher links the topic to the work that has come before by referring to examples used in the lesson so far, and to the everyday ordinary English usage of the term: *expression*. The segment ends when the teacher introduces an example and thus shifts the attention from the statement of the topic to an example.

2. **What criteria of explanation are present and how?**

   a) **Are both the elements of matter present?** (M1 and M2)

The teacher reads the string of symbols in a previously used example in T9 "*If I say if a is equal to four what is a plus a?*" I have coded this as M1, and there are no utterances that define an element and so the absence of M2 codes.

   b) **Are both types of form present? (F1 and F2)**

In this segment, the teacher is discussing the notion of algebraic expressions and in T9 she teacher provides an example of an expression "*So when I say a plus b that's an expression, but it's a mathematical expression.*" I have coded this as F2, because it is talk that looks at the whole expressions. There is no talk that looks at the individual term, and so no F1 is present.

   c) **Are both kinds of process present if any? (P1 and P2)**

There is no procedural activity since the main activity is the statement and justification of the goal

   d) **Is the goal stated and justified? (G1 and G2)**

The goal for the entire lesson is stated with examples to clarify the meaning of the words "algebraic expression". The examples are everyday English and mathematical examples. The goal is also justified through highlighting what "algebraic expressions" refers to mathematically. In T9, the teacher stressed that an algebraic expression is a mathematical sentence or statement "…*Because this session we are, eeh this chapter is under, we call it, under algebra. So that's why we call it algebraic expressions. But it means it's a sentence that is said mathematically*…"

e) Is anything absent and why?

So far, the image that has been offered to learners in connection with an algebraic expression is that it should involve symbols. The teacher missed an opportunity to highlight that the verbal statements uttered for the mental maths activity at the beginning of the lesson, before they are translated to symbolic written form, are algebraic expressions. In addition, it would have been useful to highlight that later on in the lessons they will be using geometric figures to express algebraic notions. To know that algebraic expressions can be represented by the symbolic, geometric and word forms would have better illuminated the notion of algebraic expressions for learners.

3. What criteria are privileged and thus the achieved explanation?

The privileged criteria for explanation are both G1 and G2. The achieved explanation was focused on the similarity between an English expression and a mathematical expression. The achieved explanation is that the topic "algebraic expressions" was defined in T9 as "*a sentence that is said mathematically*". More so, the why question for the topic was asked, and hence the presence of G2 codes.

4. What is the nature of pedagogy and learner participation?

The dominant code in the acts of explaining is focused on the language and how the teacher went about unpacking the meaning of the words used for the topic. There are reasons given for the word "algebra" and the word "expression". Hence the presence of Rn codes. Learners read what is written on the board and offered examples of algebraic expressions in response to the teacher's questions. Presented below is Figure 7:7 showing the frequency of codes for explanation and acts of explaining.

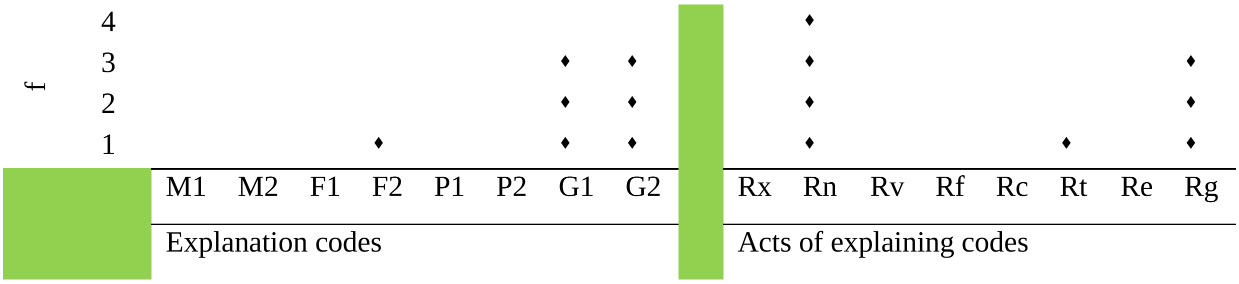
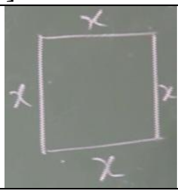


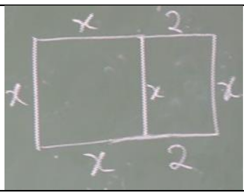
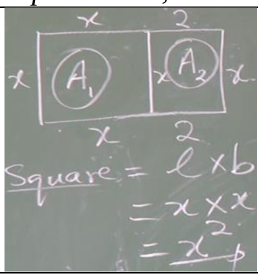
Figure 7:7 – Codes frequency for Teacher B L1 Sg9 – Statement of the goal

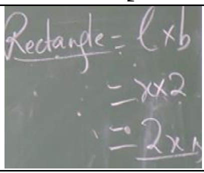
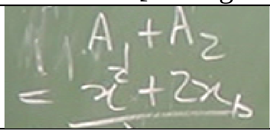
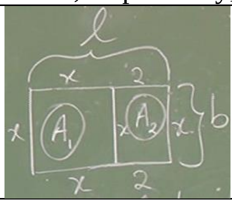
The discussion after the statement of the goal was focused on the degree of a polynomial, and then homework items were written on the board, which require learners to first identify the number of terms in the expression. Secondly, learners had to study the numerical powers of variable x, and conclude if the terms are in descending or ascending powers of x. In L2, the teacher goes over homework. The homework was followed by mental maths activity, which was followed by the statement of the goal as 'adding like terms' and a sequence of five examples accompanied by a homework exercise to end the lesson.

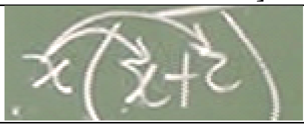
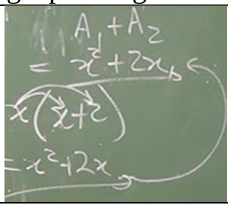
In L3, the teacher marks the homework, and there is no mental maths activity. Instead, the teacher goes into Example 1 after checking and marking homework. The activity in the first three examples is finding area of the rectangle. The following transcript excerpt, Table 7:17 below, shows how the notion of distributive law was dealt with and introduced in Example 4 of Teacher B L3 Sg2.5.

Table 7-17: Excerpt from Teacher B L3 Sg2.5 – Example 4

T	E	AoE	Transcript	
1	M1 F2	Rg	T:	*Now, when we take the very same rectangle and then* [pause and take a look at notebook in her hand]. *Oh, let me redraw it* [Drawing a square of length x].
2				
3	M1	Rt	T:	*So, if I have x here, I have here. What do I have here* [pointing with finger on the opposite side not labelled]*?*
4			Lrns:	x.
5	M1	Rt	T:	*What do I have here?* [Pointing at the last side not labelled]
6			Lrns:	x.
7	F1 M1	Rv Rt	T:	*I have x. Now when I decide to extend it and then I say I add two here. So what is it going to be here?* [Pointing at side opposite length two]

8				
9			Lrns:	*Two.*
10		Rv Rt	T:	*It's going to be two. What is it going to be here?* [pointing]
11			Lrns:	x.
12	M1	Rt	T:	*It's going to be* x *can you see that?* [Pointing at the other two sides parallel to x]
13	F1	Rt Rn	T:	*This* [pointing at the first shape drawn before extension]. *What do we call this shape here?*
14			Lrns:	*Square.*
15	F1	Rv Rn Rt	T:	*It's a square. But what do we call this one* [pointing at the second shape extended next to the square]?
16			Lrns:	*Rectangle.*
17	F2 F1 P1	Rv Rt	T:	*It's a rectangle. Can you see that? So all the sides are equal here* [pointing at square], *but here* [pointing at rectangle] *opposite sides are equal. Okay, so let's say now I want the area of both these figures. I want the area of both of them. So what am I gonna do? I have to find the area of this one, area one* [writing A_1 inside the square]. *I have to find the area of the rectangle* [writing A_2 inside the rectangle].
18	P1	Rg	T:	*So I have to find these two areas and then I bring them together. So let's start by the area of the square* [writing square]. *So what is the area of the square? What do we say?*
19			Lrns:	*Length times breadth*
20	P1	Rv	T:	*Length times breadth* [writing square = $\mathrm{l} \times \mathrm{b}$]. *So what is our length?*
21			Lrns:	x
22	P1		T:	*What is our breadth?*
23			Lrns:	x [teacher writing $x \times x$]
24			T:	*So what is it?* x *times* x *is what?*
25			Lrns:	x *to the power two.*
26	P1	Rv	T:	x *to the exponent two, so it is* x *squared.* [Writing x^2]
27				
28	P1		T:	*But are we finished to find the area?*
29			Lrns:	*No.*
30	P1	Rg Rt	T:	*So I have to find the area of this second part. Okay! So now this is the area of the square and that is the area of the rectangle.* [Writing rectangle]. *So length times breadth again. So what is our length?* [Pointing up and down on the length with her finger]
31			Lrns:	x
32	P1	Rv	T:	x *is our length. What is our breadth?*
33			Lrns:	*Two*

34	P1	Rv	T:	*Two* [writing $x \times 2$]. *So what do we have here? x times two. Two times x. x times two or two times x*?
35			Lrns:	*Two x*
36	P1	Rv	T:	*It's two x* [writing $2x$].
37				
38	P1	Rf	T:	*So therefore, the area, the combined area, area one plus area* two [writing A_1+A_2] *is going to be x squared plus what…?*
39			Lrns:	Two x.
40		Rv	T:	Plus two x [writing $x^2 + 2x$]
41				
42	P1 F2 P2 G2	Rg Rt Rn	T:	*That's my area right? But now when we let's say we go back now. We want to multiply. But we don't want to multiply because they are, they might be long when we using area. So we want to multiply a monomial and a binomial. So now we take now this whole thing. So what is it* [putting a set bracket along the length and along the breadth] *going to be called? So this is called what? Length and this breadth* [putting the labels, respectively].
43				
44	M1	Rt	T:	*So it means what are we going to have here* [pointing at b]?
45			Lrns:	x.
46	M1 P1	Rf Rt	T:	*We are going to have x multiply by what* [pointing at the length]*…?*
47			Lrns:	*Two x, x two.*
48		Re	T:	*Is it two x?*
49			Lrns:	[No response].
50		Rt	T:	*What is this* [pointing at the length]?
51		(Re)	Lr:	*Two x, x two, x to the power two*
52			T:	[teacher shakes her head]
53			Lr2:	*x plus two*
54		Re Rt	T:	*It is* [with excitement pointing at the learner who provided the answer]?
55			Lr2:	*x plus two.*
56	F1 P1 G2	Rv Re Rt	T:	*x plus two! Very good! So that is x plus two* [writing $x(x + 2)$]. *So we are multiplying x plus two, because we have added the two, er, the two. This we have x and then we have two and then we multiply by this x* [pointing at x], *which is our breadth. Okay.*
57	M1 P1		T:	*Now, let us see if we multiply like that we say x times x* [drawing the distribution curve between x and x in the expression $x(x + 2)$]. *Then what is the answer?*
58			Lrns:	*x squared.*

59	P2 M1	Rt	T:	*And then, x times two* [pointing with finger] *like that* [drawing the distribution curve from x to 2]. *We are using the distributive property.*
60				
61			Lrns:	*x times two.*
62			Lr:	*Two x.*
63	P2	Rv Rt	T:	*Two x* [written on the board is $x^2 + 2x$]. *Is it not the same as that?* [Drawing a pointing curve between the two solutions]
64				
65			Lrns:	*It's the same.*
66	P2	Rv	T:	*It is the same thing. Okay? So that is how we multiply when we multiply a monomial and a binomial. But even when you use the area formula it does come out. Okay.*

1. **Why is this a segment?**

I have marked this as a segment centered on one example where a discussion of multiplication using rectangles and using the distributive property transpired. The discussion began with a square. A rectangle was then attached to the square forming a larger rectangle when combined. The area of both partitions (original square and attached rectangle) and the area of the whole figure was found and compared. The teacher thus demonstrates the distributive property and moved on to abandon rectangles as a way of multiplying algebraic expressions. The segment ends when the teacher shifts attention by introducing and stating the goal for the lesson. I now discuss this segment in terms of the four criteria for explanation.

2. **What criteria of explanation are present and how?**
    a) **Are both the elements of matter present? (M1 and M2)**

The sides of the figure and their size are the elements that compose the figure. In T3 and T7, the teacher talks about these elements and I have coded them as M1. T3 – "*So if I have x here I have here, what do I have here* [pointing with finger on the opposite side not labeled]" and in T7 the teacher extends the figure: "*I have x. Now when I decide to extend it and then I say I add two here. So what is it going to be here*? (Pointing at side opposite length two)". The teacher refers to individual parts of the figure, and so I have coded these utterances as M1. There are no M2 codes, because there are no definitions of elements provided, and so only one criterion of matter is present, and that is M1.

b) Are both types of form present? (F1 and F2)

In T13 and 15, the teacher points at one part of the figure and requires the name for it "[Pointing at the first shape drawn before extension] *this what do we call this shape here?*" T15 – "*It's a square*. *But what do we call this one* [pointing at the second shape attached to the square]." I have coded these utterances as F1, because they are looking at the sub-structure of the bigger structure, both the square and the rectangle are partitions that together make up the bigger rectangle. In T42, the teacher requires that learners move beyond looking at the parts of the figure but to look at the entire figure "*…So now, we take, now, this whole thing. So what is it* [putting a set bracket along the length and along the breadth] *going to be called? So this is called what? Length and this breadth* [putting the labels respectively]*"* I have coded this utterances as F2, because they examine the structure of the entire figure. In this segment both types of form are present.

c) Are both kinds of process present if any? (P1 and P2)

The teacher shifts from talking about the properties of the figure to talk about the process of finding area. T17 – **"***Okay, so let's say now I want the area of both these figures. I want the area of both of them. So what am I gonna do? I have to find the area of this one area* one [writing A_1 inside the square]. *I have to find the area of the rectangle* [writing A_2 inside the rectangle]." I have coded this as P1, because the focus is now on process. In T42 the teacher mentions that the area method might be long, so a shorter method is necessary and this is the justification for shifting to the use of the distributive law. T42 – "*We want to multiply but we don't want to multiply because they are, they might be long when we using area*." In T59, the teacher introduces the distributive property. T59 –" *And then x times two* (pointing with finger) like that (drawing the distribution curve from x to *2*). *We are using the distributive property*." From T59, I have coded as P2, because the multiplication method now shown is justified by linking it back to the geometric representation earlier used.

Area is another representation of multiplication that provides a good illustration of the distributive property. Without this visual illustration, the distributive property is left in symbolic form, and so not fully derived as the derivation needs to be done algebraically. Teacher B connects the area representation of multiplication to illustrate the distributive property. The area representation of multiplication is used as a justification for the application of the distributive property. Learners can be convinced as to why it is possible to distribute multiplication over addition, because the distributive property is illustrated by the area model.

The teacher emphasises this by showing that both the area method and "arrows" (distributive property) method give the same answer. I coded these utterances, which link the two representations of multiplying expressions as P2. This particular segment from Teacher B is in fact the only example, across the three lessons, of a case where an attempt is made to justify the steps of the process and hence the presence of P2 codes and so both kinds of process are present in this segment. The area model is an illustration, which assists in meaning making.

d) Is the goal stated and justified? (G1 and G2)

The achieved goal for this segment is that the teacher illustrates for learners how to find the product of algebraic expressions using an area model, and she then uses the same example to demonstrate the distributive property.

e) Is anything absent and why?

There is no explicit justification given for why the answers are the same when using the area model and the distributive law. There is no explicit mention of the fact that the underlying principle in both the distributive law and the area model is multiplication, where essentially you are multiplying when finding the area (area formula is a product of two factors). For this reason, it is possible to connect it to the distribution of multiplication over addition, which is also a product of two factors.

3. What criteria are privileged and thus the achieved explanation?

The privileged criteria for explanation for this segment are P1, M1 and P2. The achieved explanation is that the teacher gets to translate from the area model of multiplication to the distributive property. The teacher links the two representations and methods of multiplication by showing that they both produce the same answer.

4. What is the nature of pedagogy and learner participation?

The most dominant mode of communication in this segment is gesturing (Rt), followed by re-voicing (Rv). Learners answer teacher questions, the teacher evaluates learner responses, and requires learners to finish her sentences. Figure 7:8 below shows the distribution of codes for both explanation and acts of explaining codes.

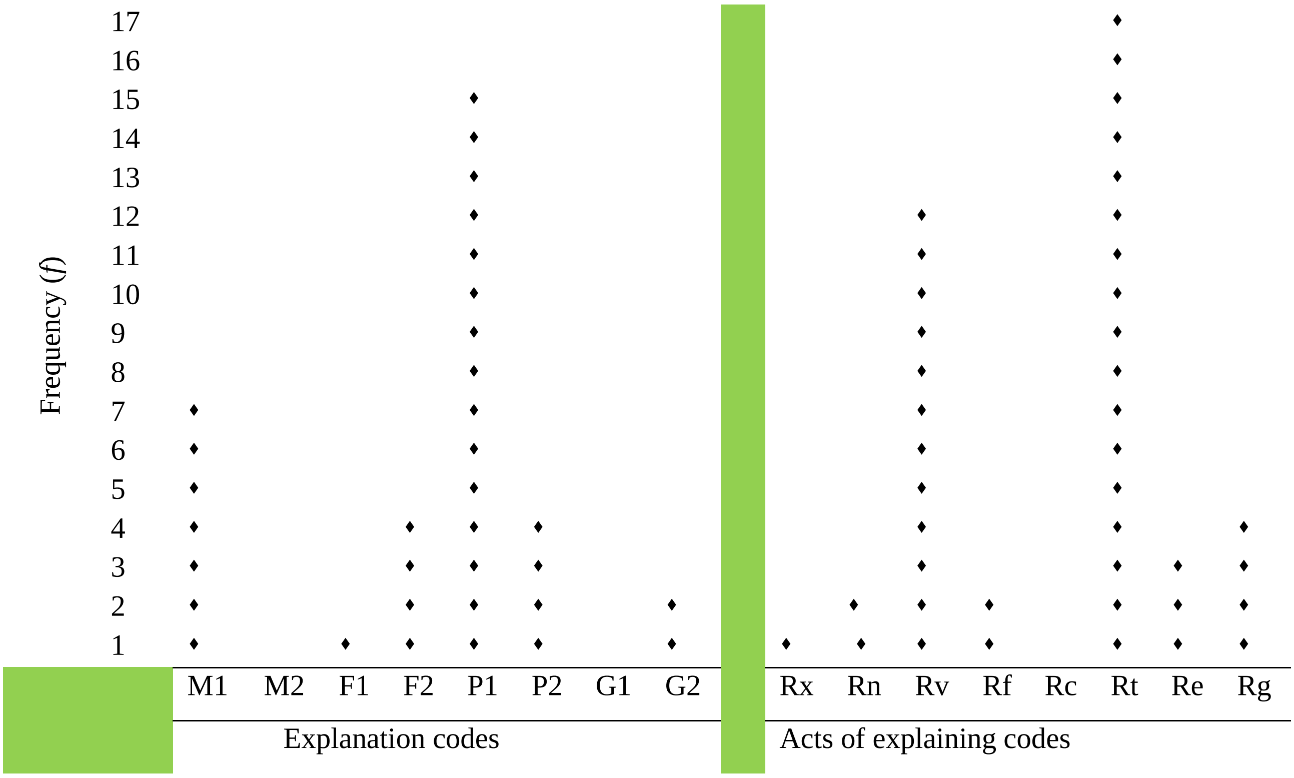


Figure 7:8 – Codes frequency for Teacher B L3 Sg5 - Example 4 area and DP

At this point of the lesson, Teacher B writes the goal of the lesson, however, she is distracted because the next subject teacher enters and asks to sit inside, and so Teacher B rushes through two more examples using the distributive property and not the area model with rectangles. After these two examples, she then writes the homework on the board and that is the end of the three consecutive lessons for Teacher B. I now proceed into a discussion of what were the privileged criteria and the achieved explanation across the three lessons for Teacher B.

7.3.3 Explanation for Teacher B

What criteria for explanation are privileged in Teacher B's lessons? What has Teacher B achieved as explanation across the three lessons? To answer these the results of the coding process need to extend beyond the selection presented here. In this section, I provide a summary of the analysis and the results of the coding for explanation across the three lessons for Teacher B. From the three excerpts I selected for presentation, what is visible is that explanation of elements and form are present, though with some limitations. The goal for the entire section on algebraic expression was stated, with reasons even though limited. To extend the analysis beyond the selected segments, I present Table 7-19, showing the coding of all segments across the three lessons.

The table is partitioned according to lessons, the third lessons appears on the next page. The activity, the segment, and the duration for each segment is shown in the table. The frequency for the eight codes for explanation is indicated for each segment, and totalled for each code. Table 7-18 below is a key explaining the abbreviations used in Table 7-19 particularly the segmentation.

Table 7-18: Key for reading table and table

Act	**MM**	**V**	**Mo**	**Cf**	**Bi**	**Tr**
Activity	Mental Maths	Variable	Monomial	Coefficient	Binomial	Trinomial
sd	**tpc**	**DP**	**HW**	**Eg 1**	**LQ**	**Seg**
Side	Topic	Distributive Property	homework	Example 1	Learner Question	Segment

Table 7-19: Frequency for explanation across the three lessons for Teacher B

EXPLANATION	**G2**			2	4	2	1	2	1			3		2	2	
	G1											3				
	P2															
	P1	3	5	5	5	3	5	3	1	13				2		
	F2					3		5	1		6	1	2	7		1
	F1				1	4		3	1	4				1	8	1
	M2				5	3										
	M1	3	5	1	8	8	3	4	1	7			1	1	5	
	Act	MM	MM	MM	V	Mo	Cf	Bi	Tr	Sd	Tr	tpc	DP	Sd	DP	HW
	Seg	1.1	1.2	1.3	2	3	4	5	6.1	6.2	6.1	7	8.1	8.2	8.1	9
	Time	00:00	03:35	11:15	13:54	19:20	22:28	24:18	26:01	26:35	31:44	33:18	35:22	35:50	37:58	41:07
	L	L1: ALGEBRAIC EXPRESSIONS **(48:05minutes)**														

EXPLANATION	**G2**		1			1			2				1		5	2	3		1	
	G1													1						
	P2																			
	P1		1						2		3	2	1			3	4	9	8	
	F2		5	3	2	1			2	2		**1**			1	1		1	2	
	F1		1	2	2	1			1		1	1	1		11	8	4	1	1	
	M2														1					
	M1		1	2	2	1	3	11	2	3	1	1	4		16	3	6	2		
	Act	HW	Eg 1	Eg 2	Eg 3	Eg 4	LQ	MM	Eg 1	Eg 2	Eg 3	Eg 4	Eg 5	tpc	Eg 1	Eg 2	Eg 3	Eg 4	Eg 5	HW
	Seg	1.1	1.2	1.3	1.4	1.5	1.6	2.2	2.3	2.4	2.5	2.6	2.7	3	4.1	4.2	4.3	4.4	4.5	6
	Time	00:00	07:32	09:46	11:10	11:52	12:50	17:55	25:05	26:08	27:11	27:41	28:16	28:48	29:42	34:48	36:22	38:42	40:26	42:24
	L	L2: ADDING ANS SUBTRACTING ALGEBRAIC EXPRESSIONS (**46:30 minutes**)																		

EXPLANATION	G2	2					2			1		**40**
	G1							1			1	**6**
	P2					2	4					**6**
	P1	8		6	3	2	15		4	5		**111**
	F2	9		8	3	2	4		1	1	1	**76**
	F1	1		1			1			1		**62**
	M2											**9**
	M1	7		5	6	2	7		2	3		**137**
	Act	HW	HW	Eg 1	Eg 2	Eg 3	Eg 4	tpc	Eg 5	Eg 6	HW	
	Seg	1.1	1.2	2.1	2.2	2.3	2.4	3	4.1	4.2	5	
	Time	00:00	04:24	06:40	13:09	14:26	15:40	20:06	20:42	21:37	22:55	**Total**
	L		L3: MULTIPLYING ALGEBRAIC EXPRESSIONS (**25:09**)									

I discuss what come to be the achieved explanation across the three lessons for Teacher B in terms of the four explanatory criteria.

Pertaining to matter, across the three lessons, there were a total of 137 utterances coded as M1, while there were only nine utterances coded as M2. In L1 and L2, there was first a call out of mental maths activities, from which learners were to listen and translate the verbal utterances into written symbolic form. All these were coded as M1, because the verbal utterances were string of symbols. The notion of a variable was elaborated by linking it to previous expressions and knowledge of it in earlier grades. The notion of a variable was also elaborated by linking the word to everyday use and by stating and showing through examples that it is called a variable because the value changes. From the elements that compose an expression, a variable was the only one that was defined extensively. Explanatory talk related to elements predominantly occurs through the reading of a string of symbols across the three lessons, and so the achieved explanation is by reading of a string of symbols.

It is noticeable, across the three lessons, that there were a total of 62 utterances for F1 and 76 codes for F2. There was also talk about structure of individual terms where the rules of identifying terms and what separates terms were elaborated. There was also talk about the structure of the entire expression/figure where words such as mono, bi, tri, poly, square, and rectangle were elaborated. The difference of 14 between F1 and F2 codes shows that an attempt to talk about the structure of the symbolic expression or geometric figure was made across the three lessons. For this reason, it can be concluded that F2 codes were the most privileged criterion of explanation and hence the achieved explanation.

In relation to process there were a total of 111 utterances coded as P1, while there were only six utterances coded as P2. The reason for this huge disparity is because in L1 and L2 there are no derived steps for the processes performed. In L2, the procedural activity was focused on adding like terms and in L3 the procedural activity was focused on multiplication of expressions. In L3, specifically, the process of finding area was used as a justification for the distributive law of multiplication. Evidently, across the three lessons, the privileged criterion of explanation is P1 and so the achieved explanation at this level is gaining fluency at performing the steps of a process with justifications in some cases.

With regards to goal, across the three lessons there was a total of six utterances concerned with the statement of the goal. The goal was stated and justified in L1 by linking the words "algebraic expressions" to everyday English usage. In L2, the goal was introduced immediately as grouping and adding like terms. In L3 the goal was introduced after four examples and was justified by linking to area model of multiplication. The why question was asked in relation to the name given to an element – variable; in relation to names given to expressions and figures – mono, bi, tri, poly, square, rectangle; and specifically in L3 the why question was asked in relation to steps of the process. In total, there was a total of 40 utterances concerned with asking of a why question and the provision of reasons. This indicates that Teacher B made a significant attempt at providing justifications and answers to the why question across the three lessons.

Overall, the M1 and P1 are the most privileged criteria of explanation and thus the achieved explanation. I now move on to discuss the acts of explaining across the three lessons.

7.3.4 Acts of explaining for Teacher B

What is visible in the three excerpts presented earlier is that there was a selection of examples, representations and careful use of mathematical language. Acts of explaining also included ways of communicating the content, such as re-voicing learners' contributions, gesturing, finishing teacher's sentences, evaluating learners' contributions, and chorusing. The three excerpts illustrate that the communication in the classroom was regulated in terms of sequencing the content. There was also regulation of behaviour in the interaction. From the selection of segments presented in this chapter, what comes through as dominant criteria for acts of explaining at the level of communicative techniques was re-voicing and gesturing. The question I am seeking to answer in this section is what criteria of acts of explaining are privileged in Teacher B's lessons, and what can be concluded about Teacher B's acts of explaining and the coding used across the three lessons? To answer this question for the three lessons, I include a summary of the coding across the three lessons. Table 7-20 below is the coding of all excerpts across the three lessons.

Table 7-20: Frequency for acts of explaining across the three lessons for Teacher B

EXPLAINING	**Rg**	4	7	6	2	6		4	1	8	4	3		5	4	3	5	4	3	1	4	3	11	2	3	1	1	1	1	3	4	4	5	2	1
	Re		3			1				10								1	1	4	4	1		3	2			1		11	1	3		3	
	Rc				2	1															1														
	Rt	1	2	2	5	6	2	4		13	1	1	1	2	2			6	3	1	2	2	1	1	2	1				14	3	5	4	2	
	Rf		5	7		3	1	7			1			2	3			1	1	1	1	1					3			7		2	3	3	
	Rv	1	4	14	9	7	1	2		11	3		1	3	7	1		5	3	1	2	2	2	2	1	2	1	1		7	5	4	5	6	
	Rn				8	3		4			8	4	1		4			5												2					
	Rx				3	1			1	1			1					1						1						4		1	1	1	
	Act	M M	M M	M M	V	Mo	Cf	Bi	Tr	Sd	Tr	tpc	DP	Sd	DP	H W	H W	Eg 1	Eg 2	Eg 3	Eg 4	LQ	M M	Eg 1	Eg 2	Eg 3	Eg 4	Eg 5	tpc	Eg 1	Eg 2	Eg 3	Eg 4	Eg 5	H W
	Seg	1.1	1.2	1.3	2	3	4	5	6.1	6.2	6.1	7	8.1	8.2	8.1	9	1.1	1.2	1.3	1.4	1.5	1.6	2.2	2.3	2.4	2.5	2.6	2.7	3	4.1	4.2	4.3	4.4	4.5	6
	Time	00:00	03:35	11:15	13:54	19:20	22:28	24:18	26:01	26:35	31:44	33:18	35:22	35:50	37:58	41:07	00:00	07:32	09:46	11:10	11:52	12:50	17:55	25:05	26:08	27:11	27:41	28:16	28:48	29:42	34:48	36:22	38:42	40:26	42:24
	L	L1: ALGEBRAIC EXPRESSIONS **(48:05minutes)**															L2: ADDING ANS SUBTRACTING ALGEBRAIC EXPRESSIONS (**46:30 minutes**)																		

EXPLAINING	**Rg**	8		8	1		4	2		1	4	**139**
	Re	3		4	2		3			1		**62**
	Rc			1								**5**
	Rt	15		4	1	1	17		2	2		**131**
	Rf	6		1	1	2	2		1			**65**
	Rv	9		9	8	3	12			5		**159**
	Rn			7	2		2					**50**
	Rx			1	1	1	1		1	1		**22**
	Act	H W	H W	Eg 1	Eg 2	Eg 3	Eg 4	tpc	Eg 5	Eg 6	H W	
	Seg	1.1	1.2	2.1	2.2	2.3	2.4	3	4.1	4.2	5	
	Time	00:00	04:24	06:40	13:09	14:26	15:40	20:06	20:42	21:37	22:55	**Total**
	L		L3: MULTIPLYING ALGEBRAIC EXPRESSIONS **(25:09)**									

I discuss what the privileged criteria from the acts of explaining came to be in terms of the three categories, viz. examples and language; communicative techniques; and regulative talk.

Across the three lessons, examples are present in verbal, symbolic, and geometric representations. There was some attention given to mathematical language even though with inaccuracies, and I provide an elaboration of this in the next chapter. Teacher B defined and provided reasons for names of elements by drawing on their everyday use to clarify their mathematical meaning (variable). Explanations and definitions were offered for words such as polynomial, binomial, monomial, trinomial, degree of polynomial, distributive property, square, rectangle, algebraic expressions, like terms. There was a total of 22 examples used to teach the content and 50 utterances that I have coded as Rn. A substantial amount of utterances focused on definition of words indicates that the teacher payed attention to mathematical language, and what things refer to mathematically.

The interactional pattern was mainly through teacher questions, and re-voicing of learner responses, totalling 159 counts. Teacher B used gestures by pointing at learners, her own writing on the board and signing with her fingers for learners to read. Gesturing had a total of 131 counts. There was evaluation of learners' responses, 62 counts, as well as finishing teacher's sentence sitting at 65 counts. There was some chorusing in Teacher B's classroom in the sense that leaners were required to chant together some of the answers to her questions. There were specific instances where Teacher B required the volume of the chorus response to be louder when she was discussing the notion of a variable. The statistics table above indicates that the most dominant and privileged criteria for acts of explaining was re-voicing, followed by gesturing.

The regulative talk is dominant, with 139 counts, because it is the one that drives the lesson forward, to change from one example to the next and also to maintain order in terms of speaking turns. Overall, the most privileged criteria for acts of explaining were re-voicing, regulative talk and gesturing.

7.4 Conclusion

In this chapter, I sought to answer research question 2, which asked: what was privileged as explanation and acts of explaining in instruction for each teacher. I first described the content of the three consecutive lessons each teacher taught by presenting the activities, segments, and examples dealt with across the three lessons for each teacher. I then moved on to evidence the coding and analysis of the lessons by presenting a selection of transcript segments from each teacher. In discussing what came to be the privileged criteria of explanation and acts of explaining in both teachers' practices I provided a summary of the coding and analysis for the three lessons for both teachers. What came through as a dominant practice for both teachers across the three lessons was that procedural activity (P1) and reading a string of symbols (M1) are privileged. Nevertheless, both teachers made attempts to address structure/form and provide some justifications. In the next chapter I contrast the two teachers' explanations and acts of explaining with each other, so as to more carefully answer the third research question.

CHAPTER 8 THE PRIVILEGED EXPLANATION AND ACTS OF EXPLAINING

8.1 Introduction

In the previous chapter, we have seen how both teachers similarly introduced and progressed with the topic of algebraic expressions. Both teachers focused attention to what the words "algebraic expression" refer to in mathematics. For Teacher A, words associative with the notion of algebraic expressions were called out and identified from an example of an expression. On the other hand, Teacher B defined the meaning of "algebraic expression" by noting its everyday English meaning, and related that to its mathematical meaning. Both teachers proceeded to elaborate the notion of algebraic expressions through operational activity focused on adding, subtracting and multiplying algberiac expressions. The specific research question I am seeking to answer in this chapter is:

3.1 What constitutive criteria of 'explanation' and acts of explaining are privileged in teachers' practices?

The teaching approach adapted by the teachers for this topic is almost similar. However, the amount of time spent discussing the matter, form and process of algebraic expressions differed and thus illuminates what each teacher emphasised and privileged. Teacher B spent the whole of L1, 48 minutes 06 seconds, discussing the matter and form of algebraic expressions, while Teacher A's discussion of the matter and form[13] of algebraic expressions was limited to the first two segments of L1, 06 minutes 44 seconds. Teacher B began work foregrounding operational activity in L2, while Teacher A began this work in Sg3 of L1.

In this chapter, I focus on the kinds of matter, form, and types of process that were deliberated on in both teachers' practices. I also look at the manner of introducing the goal for both teachers. It is worthy to note that the time dedicated for the discussion of matter and form were different, as was their time spent on procedural activity different. In total, the three lessons for Teacher A amounted to (L1 00:34:33 + L2 00:24:00 + L3 01:03:54 = 02:02:27), 2 hours 2 minutes and 27 seconds, while for Teacher B the total amount of time across the three lessons was (L1 00:48:06 + L2 00:46:30 + L3 00:25:09 = 01:59:45) 1 hour 59 minutes and 45 seconds. Of this time, almost two hours, 1h55m43s, for Teacher A was spent on procedural activity, and

[13] The discussion based on matter and form of algebraic expressions is not limited to the first lesson for Teacher B neither is it limited to the first two segments of L1 for Teacher A. However, what I am highlighting here is the fact that more of the discussion based on matter and form occurred in the introductory part of the topic, and snippets of it were found backgrounded in a discussion focused on form or process as the lessons proceeded.

1h11m39s for Teacher B. I consider the nature of procedural activity focused on by both teachers that is adding, subtracting and multiplying algebraic expressions. I also reflect on the two teachers' ways of introducing the goal. To judge the teachers' privileging of criteria for explanation, I pay special attention to the two teachers' use of level two codes as they express the quality of the explanation. I particularly consider where and how Level 2 codes surfaced. I also include in this chapter a discussion of what was privileged as acts of explaining across the two teachers. Lastly, I briefly discuss, the relationship between explanation and acts of explaining and then conclude the chapter.

8.2 The teachers privileging of criteria for explanation and acts of explaining.

In constructing what was privileged as explanation and acts of explaining across the two teachers, I consider the total number of occurrences for explanation and acts of explaining criteria. Table 8-1 and 8-2 below show the combined frequency for explanation and acts of explaining for Teacher A and Teacher B, respectively. I begin with explanation codes.

Table 8-1: Codes frequency across the three lessons for Teacher A

EXPLAINING	**Rg**	4	3		5	4	6	4	1	4	3	2	2	4	1		2	4	2	2	2	3	3	7	9		4	6	7	5	5	1	109
	Re						3	7			2		2	2	1		3	2	2	4	5	5		1	1		6	4	5	5	2		62
	Rc																																0
	Rt	1											1				1		1	2					1	2	2	2		2			15
	Rf		7		4	3	2	3								1			1	2	3			3		1	2		1	1			34
	Rv	9	9		10	14	11	1		4				2							2			6	10	6	8	4	13	6			115
	Rn																																0
	Rx		1		2	1	2	1			1	1	1	1	1		1	1	1	1	1	1		1	1	1	1	1	1	1	1		26
EXPLANATION	**G2**				4	1	2	1		3												1		1	1	2	1	1	1	1	1		21
	G1	1		2																			3										6
	P2																																0
	P1				10	15	20	5			1	1	1				1	2	1	3	6	1		5	11	4	7	7	17	18	5		131
	F2	2	3		1	2	3	1		6				1								1		1		3				2			26
	F1		3		3		1				1		2					1		4	1	1		2					1	2			23
	M2		4		2											2								1									9
	M1	5	9		1	1	4	2			2		2	5	1	2	2	1	2	2	6	2		6	4	1	4	1	6	6	4		81
	Act	AE	Eg	tpc	Eg 1	Eg 2	Eg 3	Eg 4	H W	Ans	LA	LA	LA	Ans	LA	Ans	LA	LA	LA	LA	LA	LA	tpc	Eg 1	Eg 2	Eg 3	Eg 4	Eg 5	Eg 6	Eg 7	Eg 8	H W	
	Sg	1.1	1.2	2	3.1	3.2	3.3	3.4	4	1	2.1	2.2	2.3	3.1	3.2	3.3	4.1	4.2	4.3	4.4	1.1	1.2	2	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	4	
	Time	00:00	02:46	06:44	09:09	13:14	19:28	25:56	33:29	00:00	02:06	04:02	05:47	08:47	09:47	11:06	11:46	14:43	19:00	24:00	00:00	06:47	18:23	19:00	21:43	24:44	28:55	32:59	40:56	48:25	54:24	01:03:54	02:02:27
	L	L1: ADDING AND SUBTRACTING ALGEBRAIC EXPRESSIONS								L2: ANSWERS TO HOMEWORK											L3: MULTIPLYING ALGEBRAIC EXPRESSIONS												Total

Table 8-2: Codes frequency across three lessons for Teacher B

EXPLAINING	**Rg**	4	7	6	2	6		4	1	8	4	3		5	4	3
	Re		3			1				10						
	Rc				2	1										
	Rt	1	2	2	5	6	2	4		13	1	1	1	2	2	
	Rf		5	7		3	1	7			1			2	3	
	Rv	1	4	14	9	7	1	2		11	3		1	3	7	1
	Rn				8	3		4			8	4	1		4	
	Rx				3	1			1	1			1			
EXPLANATION	**G2**			3	4	2	1	2	1			3		2	2	
	G1											3				
	P2															
	P1	3	5	5	5	3	5	3	1	13				2		
	F2					3		5	1		6	1	2	7		1
	F1				1	4		3	1	4				1	8	1
	M2				5	3										
	M1	3	5	1	8	8	3	4	1	7			1	1	5	
	Act	M M	M M	M M	V	Mo	Cf	Bi	Tr	Sd	Tr	tpc	DP	Sd	DP	H W
	Sg	1.1	1.2	1.3	2	3	4	5	6.1	6.2	6.1	7	8.1	8.2	8.1	9
	Time	00:00	03:35	11:15	13:54	19:20	22:28	24:18	26:01	26:35	31:44	33:18	35:22	35:50	37:58	41:07
	L	L1: ALGEBRAIC EXPRESSIONS **(48:05minutes)**														

EXPLAINING	**Rg**	5	4	3	1	4	3	11	2	3	1	1	1	1	3	4	4	5	2	1
	Re		1	1	4	4	1		3	2			1		11	1	3		3	
	Rc					1														
	Rt		6	3	1	2	2	1	1	2	1				14	3	5	4	2	
	Rf		1	1	1	1	1					3			7		2	3	3	
	Rv		5	3	1	2	2	2	2	1	2	1	1		7	5	4	5	6	
	Rn		5												2					
	Rx		1						1						4		1	1	1	
EXPLANATION	**G2**		1			1			2				1		5	2	3		1	
	G1													1						
	P2																			
	P1		1						2		3	2	1			3	4	9	8	
	F2		5	3	2	1			2	2		1			1	1		1	2	
	F1		1	2	2	1			1		1	1	1		11	8	4	1	1	
	M2														1					
	M1		1	2	2	1	3	11	2	3	1	1	4		16	3	6	2		
	Act	H W	Eg 1	Eg 2	Eg 3	Eg 4	LQ	M M	Eg 1	Eg 2	Eg 3	Eg 4	Eg 5	tpc	Eg 1	Eg 2	Eg 3	Eg 4	Eg 5	H W
	Sg	1.1	1.2	1.3	1.4	1.5	1.6	2.1	2.2	2.3	2.4	2.5	2.6	3	4.1	4.2	4.3	4.4	4.5	5
	Time	00:00	07:32	09:46	11:10	11:52	12:50	17:55	25:05	26:08	27:11	27:41	28:16	28:48	29:42	34:48	36:22	38:42	40:26	42:24
	L	L2: ADDING ANS SUBTRACTING ALGEBRAIC EXPRESSIONS (**46:30 minutes**)																		

For Teacher B the third lesson goes into the next page.

EXPLAINING	**Rg**	8		8	1		4	2		1	4	139
	Re	3		4	2		3			1		62
	Rc			1								5
	Rt	15		4	1	1	17		2	2		131
	Rf	6		1	1	2	2		1			65
	Rv	9		9	8	3	12			5		159
	Rn			7	2		2					50
	Rx			1	1	1	1		1	1		22
EXPLANATION	**G2**	2					2			1		40
	G1							1			1	6
	P2					2	4					6
	P1	8		6	3	2	15		4	5		111
	F2	9		8	3	2	4		1	1	1	76
	F1	1		1			1			1		62
	M2											9
	M1	7		5	6	2	7		2	3		137
	Act	HW	HW	Eg 1	Eg 2	Eg 3	Eg 4	tpc	Eg 5	Eg 6	HW	
	Sg	1.1	1.2	2.1	2.2	2.3	2.4	3	4.1	4.2	5	
	Time	00:00	04:24	06:40	13:09	14:26	15:40	20:06	20:42	21:37	22:55	01:59:45
	L		L3: MULTIPLYING ALGEBRAIC EXPRESSIONS **(25:09)**									Total

8.2.1 Explanation for both teachers

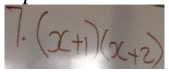
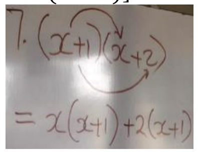
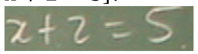
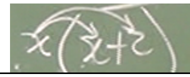
What do the numbers in the tables above tell us about the two teachers' privileging of explanation criteria? I discuss the two teachers privileging of explanation criteria according to the eight codes: M1, M2, F1, F2, P1, P2, G1 and G2, and in light of research in the field.

8.2.1.1 Reading a string of symbols or naming an element – M1

As discussed in Chapter 4, researchers of mathematics education have argued that it is important that signs and symbols become transparent (Radford and Puig, 2007). By transparent, they mean that there needs to be an explanation of what the symbol represents (representational character of signs) and to what it refers (referential character of signs). There was an attempt in the two teachers' practices to provide explanations of symbols, although in different ways and with some inaccuracies. While Radford and Puig recognise the need to offer explanations of symbols, Barnejee and Subramaim (2012) emphasise that symbols are the constituent elements that make up the structure of the algebraic expressions. The description of symbols offered by Barnejee and Subramaim is in agreement with Ruben's presentation of the definition of matter as offered by Aristotle. In the teachers' practices, there was awareness created for learners to be acquainted with the elements that comprise an algebraic expression. This is seen in the initial segments of the first lessons (L1) of both teachers. As already discussed in the previous chapter, Teacher A created an awareness of elements and symbols through an activity where associative terms with the notion of "algebraic expressions" were called out, and also through identification of elements from a given example. Teacher B, on the other hand, created awareness by naming and discussing the mathematical meaning of names for symbols. The mathematical meaning of names of symbols contributes to the understanding of structure. I elaborate further on this relationship in the discussion focused on M2.

In Chapter 4, the act of reading "a string of symbols" (p.2) was described by Sfard and Linchevski (1994) as "a meaningless way" of talking about elements as they were arguing that "symbols do not speak for themselves" (p.3), but that they need to be unpacked. As shown in the methodology, the language used by Sfard and Linchevski is the description of the code M1. The results of the analysis show that there was a total of 81 occurrences for Teacher A and 137 occurrences for Teacher B in the M1 code. This indicates that there was a naming of elements and a reading of a string of symbols. As shown in Table 8-3 below, there are three ways in which the reading of a string of symbols took place in both the teachers' lessons. Firstly, when an example was introduced there was a reading of a string of symbols that compose the expression/figure. Secondly, in the unpacking of structure, symbols were mentioned and thirdly, in the operational activity the string of symbols was read.

Table 8-3: M1 – Reading of a string of symbols

	Examples and their representations	Structure of expressions	Operational activity
Teacher A	1 M1 Rx T: *And one last example* [writing on the board: $(x+1)(x+2)$] *x plus one into x plus two.* 2 	3 M1 F2 F1 Rg T: *Okay x plus one into x plus two. We are now multiplying into the bracket. We are now multiplying a binomial with a binomial. Two expressions. Yes Khaya?*	27 M1 P1 Rt T: *No. I mean when we multiply. We multiply x plus one by x, and we are multiplying x plus one by two* [pointing]. *So we are saying we are multiplying x plus one by x. Let's put x plus one, we multiplying it by x* [written $x(x+1)$]. *x plus one we multiply it by two* [written $2(x+1)$]. 28 
	L3Sg3.7T1-2	L3Sg3.7T3	L3Sg3.7T27-28
Teacher B	24 M1 P1 Rx T: *… Right! And then when I say* [writing on the board] *two times a box is equal to eight.*  *What is in the box?*	27 M1 F2 Rt T: *… So, is there a difference between this* [pointing at $\square + 2 = 5$] *and x plus two is equal to five* [writing on the board $x + 2 = 5$]? 	56 P1 M1 T: *Now let us see if we multiply like that we say x times x* [drawing the distribution curve between x and x in the expression $x(x+2)$]. *Then what is the answer?* 57 Lrns: *x squared* 58 T: [teacher shakes her head] 59 M1 P2 Rt T: *And then x times two* [pointing with finger] *like that* [drawing the distribution curve from x to 2]. *We are using the distributive property.* 60 
	L1Sg2T24	L1Sg2T27	L3SgT56-60

Thus, the reading of a string of symbols is ever present talk in the mathematics classroom for both teachers. The reason for this is that all the talk, whether it be focused on explanation of matter, form, or process, is with respect to the algebraic expression currently under use for explanatory purposes. The issue is that with the reading of a string of symbols in the presentation of the example and in operational activity, talk that emphasises the structure is compromised. In the historical section, Thomas Hobbes the philosopher was quoted that he protested the total symbolisation of algebraic notions and the disregard for preservation of words. The double labour of first reducing the symbols to words and secondly attending to the ideas the symbolic words signify is manifest in the teachers' practices as they struggle to talk about algebraic ideas. The rhetoric way of talking about algebraic ideas is lost, and the ways in which algebraic ideas are spoken about is through a reading of a meaningless string of symbols, as Sfard and Linchevski (1994) once noted.

The explanation of constitutive elements in the two teachers' practices was also through definition of elements.

8.2.1.2 Definition of a constitutive element – M2

As a reminder, the M2 code was concerned with the definition of the elements. Concerning the definitions, Vinner (1991), who is not discussed in the literature review but useful to highlight here, has provided constructs which are called formal concept definition (verbal definition which are accurate statements of the basic properties of the concept), personal concept definition (this refers to learners' reconstruction of the definition), and concept image (framed according to how the concept has been experienced, mental pictures and impressions and from this a personal construction is obtained). He argues that to do mathematics, one does not need the formal concept definition, but needs the concept image to form the personal concept definition. In a sense, the personal concept definition and the concept image are related, and should be checked against the formal concept definition. The notion of a coefficient is one such example where the lack of a formal concept definition does not inhibit learners from doing the mathematics. Indeed where this concept, coefficient, is concerned, as will be shown again in this section, both teachers provided definitions to learners that enable the doing of mathematics instead of formal concept definitions. The constructs of formal definition and personal concept image are useful to understand the nature of definitions provided by both teachers. In the teaching process, the use of precise mathematical language and vocabulary has been stressed to be of greatest importance in the provision of definitions, because it is a language that the learner will then use to communicate mathematics verbally. (For a more exhaustive and fuller description of definitions in mathematics education see Vinner, 1991).

Both teachers made an attempt to define some of the elements that compose an algebraic expression. Interestingly, both teachers have a small number of counts for M2, which is a total of nine counts for Teacher B and seven counts for Teacher A over the three lessons. Even though the occurrences of M2 seem to be small for both teachers, the duration taken for the occurrence differs as does the locations where they are found in the teachers' lessons.

Teacher A provided definitions for brackets, exponents, constant, and coefficient while Teacher B provided definitions for variable, and coefficient. For Teacher A, the four counts of the M2 code first appears in Sg 1.2, where a coefficient and a constant are defined. In the definition of a coefficient and constant there is mention of the word "variable", but itself is not defined. In Sg 5 of L2 there are two occurrences of the M2 code, where the degree of the polynomial is defined as "*the highest exponent*". Lastly, in Sg 3.1 of L3 brackets ( ) and multiplication sign $\times$ are defined as referring to "*the same thing*", which constitutes one occurrence of M2 code.

On the other hand, Teacher B had five occurrences of the M2 code in Sg 2 of L1. As already shown and discussed in the previous chapter, the entire segment is focused on the notion of a variable. Sg3 of L1 has three occurrences of the M2 code, where the teacher restated the definition of a variable, and also defines a number next to a variable, without providing the mathematical name for it. I did not include this particular case, because both the elements it deals with are covered in the previous segment Sg 2 of L1 and in Sg4.1 of the next lesson. Sg4.1 of L2 has one occurrence of the M2 code, where Teacher B defines and names a coefficient. I have chosen to include a discussion of the three elements that were defined in the teachers' practices, and those are coefficient, constant and variable. I first begin with the coefficient, because both teachers provided a definition of coefficient in similar ways.

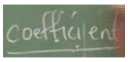
8.2.1.2.1 Coefficient

In the historical account of algebra in Chapter 4, I noted that Descartes (1596-1650 CE) differentiated between symbols denoting variables (x, y) and between symbols denoting coefficients/ parameters (a, b, c); conventions which have not always been followed. In fact, we see Teacher B in the beginning of the segment on variable (L1Sg2) challenging exactly this very fact of having predetermined symbols to denote variables or parameters. The meaning of the algebraic expression depends on one being able to distinguish between variables and coefficients. I present below the specific instances where each teacher defined the notion of a coefficient.

32	M2	Rv Rf	T:	*Negative six. So coefficient is a number next to a*
33			Lrns:	*Variable*

Teacher A L1Sg1.2T32-33

Sg1.2 from L1 of Teacher A with a total of 53 turns is the main segment specifically dedicated to the discussion of matter of algebraic expressions and only two elements are defined and that is a coefficient and a constant. As already noted earlier in the discussion of Sg1.2, the M2 code arises after Teacher A observes that learners are unable to identify a coefficient. Below is the presentation of the utterance concerning the definition of a coefficient for Teacher B.

72	M2	Rn Rf	T:	*…the number that is next to the variable by the way we give it a special name. We call it what* [writing coefficient on the board]*?*
73				
74			Lrns:	*Coefficient* [reading off the board]

Teacher B L2Sg4.1T72-74

The definitions of a coefficient for both teachers shows the same imprecisions, because they do not highlight the operational relation, namely that a coefficient is a number that multiplies the variable. Also, with regards to its notation, it can be denoted using letters, which are referred to as parameters instead of variables. For instance m and c in $y = mx + c$ are parameters while x and y are variables. The definition provided by both teachers is in terms of the position of the coefficient in relation to the variable. The definition does not accurately capture the function of the coefficient in the term and in the expression. The role of the coefficient is multiplication regardless of its position, in front of, or after the variable. In future grades, understanding the concept of a coefficient is going to be useful and important in the solving non-factorisable quadratic equations and matrices.

8.2.1.2.2 Constant

In my view, the notion of a constant is taken for granted, and its definition trivialised, due to its deceivingly straightforward or obvious meaning. However, a discussion probing its relation to coefficients would improve its understanding. The similarity between constants and coefficients is that they are all fixed, non-varying quantities in the expression. The difference is that a constant does not contain a variable in an algebraic expression, because it is the coefficient of a variable raised to the exponent zero, x^0. Hence, the constant term in the expression $x^2 + qx + r$, where p and q are coefficients is r. The transcript excerpt below is taken from Teacher A L1Sg1.2T51-53, where a definition of a constant was offered and coded as M2.

51	M2	Rf	*T:*	*Remember a constant is a number without a*
52			*Lrns:*	*Variable*
53	M1		*T:*	*That's a constant.*

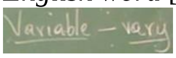
Teacher A L1Sg1.2T51-53

As shown in the transcript exerpt above, the definition of a constant is mentioned as a reminder. The definition of a constant offered by Teacher A above is missing the detail of algebra where letters stand for numbers, where the distinguishing factor would be to clarify what kind of number. In the case of a constant, a letter stands for a fixed number.

The definition of a constant is important knowledge in understanding constant functions such $f(x) = 8$, which is a horizontal straight line parallel to the *x*-axis. Later on in calculus, the understanding of a constant is important for understanding why derivatives of constants are zero, because derivatives measure change and since constants do not change but are fixed, then the derivative of a constant is zero. These are understandings of a constant that learners will need in Grade Nine for the topic of straight line functions. Further, a discussion probing the constants relation to its English antonym variables would enhance understanding. A variable varies, while a constant does not change. The notion of quantities that are changing and others not, is important to distinguish.

8.2.1.2.3 Variable

We see from the transcript excerpt below that the M2 code in Teacher B L1Sg2 is preceded by a why question seeking a reason for the name, something that does not occur in Teacher A's seven incidences of the M2 code. Teacher B mentions that variable is constructed from the word vary. The word "vary" is stated as the root word for variable and so the meaning of "variable" is derived from the meaning of the word vary. This is the main difference between Teacher A and Teacher B at the level of providing definitions. For Teacher A, the meaning of the words given to elements were not elaborated, and the reason for this could be that learners have textbooks, which provide the definitions of terminology. I pick up on this point in the discussion of the Rn code (mathematical language and terminology) later, when discussing teachers privileging of the acts of explaining.

35	G2 M2	Rn	T:	*We call it a variable. Okay. But there is a reason why it's called a variable. So this word variable comes from an English word* [writing the word vary next to variable] 

Teacher B L1Sg2T35

On the notion of a variable the literature emphasised that the meaning of a variable is dependent on the context in which it is expressed. Teacher B missed this point in her explanation of the notion of a variable. Usiskin (1999), in particular, provides examples of five contexts that illuminate different meanings of the notion of a variable. A linear equation with one unknown was one of the context used by Teacher B, and also highlighted by Usiskin. In such a linear equation, the role played by the letter/alphabet is that of an unknown, unique value; and so the notion of variability is meaningless within this context.

The other contexts highlighted in the literature to illuminate the context dependent nature of the notion of a variable is formulae such as the area formula, where each letter is an unknown $A = lb$. Teacher B used this context, but fell short of highlighting that A, l and b in the formula are variables because they are unknown values for area, length, and breadth. A more appropriate context to highlight the notion of variability was the equation $y = mx$; a more suitable example to highlight the notion of variability, where y is the value of the function and x is the argument of the function while m is a parameter.

The variable is the only element Teacher B defines extensively. The time spent by Teacher B discussing the notion of a variable was five minutes 24 seconds, while the entire segment which involves identification as well as some definitions of elements in Teacher A took four minutes. Thus, it is clear that Teacher B privileged the explanation of a variable over other elements of algebraic expressions. Operation signs and numerals are symbols in arithmetic as well as algebra. Variables are the main distinguishing elements of symbolic algebra, and hence Teacher B's emphasis and effort to provide an explanation of the variable. We see the importance she places on the variable as a distinguishing element of symbolic algebra from the statement she used to justify the use of symbols towards the end of Sg2.

52	G2	Rg	T:	*Right! Instead of using boxes, because now you are grown up, you are high school kids, so we make use of variables.*

Teacher B L1Sg2T52

Mathematics education researchers have agreed that the use of boxes or blank spaces in numeric equations are a 'transitional language' to algebra (Radford, 2000). Teacher B in the last speaking turns of Sg2 appears to be appealing to the same notion as she tries to provide a definition of a variable and highlight it as the most distinguishing element of symbolic algebra.

Teacher A presented more definitions of elements (coefficient, constant, exponent, and brackets and multiplication sign), while Teacher B presented only two (variable and coefficient). Teacher B and Teacher A had similarities and differences in how they both made use of the M2 code. The similarity in the practices of both teachers is that the definition of a coefficient emerges in the lessons incidentally. This could mean that the concept has been covered in the preceeding grade, and so it is proper to go over it in passing. The major difference is that Teacher B presents a more elaborate definition of a variable, while in Teacher A the notion of a variable is simply noted in passing that letters are variables. Teacher B defined the variable not only by providing its name, but also the meaning of the name and its function in the expression, albeit with imprecision. Another difference between Teacher A and Teacher B is that Teacher B's use of the M2 code was preceded by a why question something that doesn't happen in Teacher A. Teacher B stressed the importance of knowing what things are called in mathematics and why the naming. In her definition of a variable she paid attention to mathematical language by showing how it connects to everyday English language. In the incidental definition of a coefficient, she also highlighted the naming, what things are called. Teacher B's use of the M2 code connects with the Rn code, language, due to the emphasis on terminology, and what things are called in mathematics. The similarity across the two teachers definitions of a coefficient provided is a ‘concept image’, rather than a ‘formal concept definition’, as described by Vinner (1991). The major limitation in both teachers’ definitions of elements that compose an algebraic expression is that the notions of variable, coefficient and constant are discussed in isolation, without highlighting the relations, similarities and differences between them. This discussion was important, as these are the “main” constituent elements of an algebraic expression.

While M1 was merely a ‘meaningless string of symbols’, utterances pertaining to M2 advances the meaningless talk about elements by bringing in definitions of elements, and thus making sense of symbols. The level two code points to the mathematical quality of the explanation, because it does more than just a reading of a string of symbols. It brings into focus the relation of the element to other elements and thus illuminates its meaning in the term and in the expression. For both teachers, the level two code in relation to matter was less privileged compared to the M1 code. The statistics table show that for Teacher A M1 = 81 and M2 = 7 and for Teacher B M1 = 137 and M2 = 9.

8.2.1.3 Structure of a term – F1

The explanatory talk about structure in F1 refers to term structures which are constituents to the algebraic expression. This refers to individual units which are determined by the principles regulating the construction of a term creating a division of the expression into terms. Barnerjee and Subramaniam (2012), discussed in the literature review, defined terms as units of the algebraic expression demarcated by the + or – sign. Both teachers discussed how to identify terms and what operations “separate” or “combine” terms.

As shown in the literature review in Chapter 4, Barnerjee and Subramaniam’s (2012) work focused on terms, they presented two varieties of terms. The first is what they call *simple term* and the second is a *complex term*. The definition they provide for a simple term is that it is a constant term, in this example $14 + 5x + 2(3x + 2)$, where the constant term or simple term is 14. For a complex term, there were two descriptions and that is a *product term* and a *bracket term*. In the example, I provided $5x$ is an example of a product term, because 5 and x are factors which make the product $5x$. In other cases of an algebraic expression, x would be considered a product term, because the two factors which are elements that make up the product term can be written as $1 \times x$. A complex term includes what they call the bracket term, for instance $2(3x + 2)$ in the example I provided is a bracket term.

Both teachers discussed the form of the sub-structures that make up the entire structure of the expression being dealt with. This was through identifying like terms, collecting and grouping like terms. The literature also emphasised actions such as recognising, noticing and identification of structure. Specifically, Rüede (2013) argued that the identification of structure means recognition of individual terms that make up the expression. Both teachers assisted learners to recognise the structure of a term, and both teachers made use of different expressions to provide explanations of algebraic structure. By asking questions such as: "*how many terms are in that expression?*", "*How do you see terms?*", "*Are there like terms?*" Below is a selection of expressions used by both teachers and it is apparent that the expressions were a combination of simple and complex terms.

A selection from Teacher A	A selection from Teacher B
$3x^2 + 3xy - 6y^2 + 4$	$2m + 2m + 2m + 2m$
$(9a^2 + 4a - 10) - (6a^2 + 3a - 9)$	$7 + 2x + 4x^2 + 2x^3$
$2x(x + y) - 2y(3x - 5y)$	$2x^3y + 4x^2y^2 - 2xy^3 + 3$
$(x + 1)(x + 2)$	$(2a \times b) + (3a \times b) - (a \times b)$
	Rectangle: $l = (x + 2)$; $b = 3$

Figure 8:1 – Simple and complex terms

The coding I used did not separate between different kinds of terms, as reasoned by Barnerjee and Subramaniam, because the aim was not to differentiate kinds of terms, but the value of their work is that it enabled me to distinguish talk that refers to a sub-structure, a term, from talk that refers to the whole structure which is the F2 code. The total number of utterances coded as F1 for Teacher A was 23 and for Teacher B 62.

8.2.1.4 Structure of whole expression/figure – F2

Sfard and Linchevski (1994) were focused on how learners perceive/conceive of algebraic expressions structurally as objects, as well as operationally as a process. They argue that the two ways of perceiving mathematical objects are complementary, because there exists a strong interplay between process and structure. I will pick up shortly on this discussion when discussing form and process. For now, I want to narrow my discussion to their explanation of a structural perception of algebraic objects, because it is related to the ways in which I have defined the F2 code. The structural perception refers to a viewing and understanding of the whole algebraic expression. The authors note that the structural perception is context dependent. The structural perception was described as seeing the mathematical object as a product instead of a computational process.

In the two teachers' practices, there was talk focused on the structure of the entire expression/geometric figure (monomial, binomial, trinomial, square, and rectangle). I comment here on how both teachers assisted learners recognise structure through their explanations and the types of structures both teachers used. The types of representations used to present the structures were symbolic and geometric, and so the explanatory talk in relation to structure was found in the symbolic and geometric forms. In the procedural activity, there was a requirement to understand structure in order to be able to know when to stop the operational activity. For symbolic and operational ways of working with structure, I have examples from both teachers but for geometric representations I only have examples from Teacher B. I discuss with excerpts from the transcripts to show how F2 related explanatory talk manifested itself in the three contexts concerning: (1) symbolic expressions; (2) procedural activity; and in (3) geometric forms. I begin the discussion with structure of symbolic representations.

8.2.1.4.1 Structure of symbolic expressions

Algebraic expressions have different structures (Hoch & Dreyfus, 2004) according to their mathematical property. Rüede (2013) mentions that Hoch and Dreyfus (2010) provide suggestions of general algebraic structures that are central to secondary education, namely: the difference of two squares: $a^2 - b^2$; the quadratic trinomial which can also be expressed as a product of two linear factors $a^2 + 2ab + b^2$; the expression where the multiplication of a is distributed over the addition of three unlike terms $ab + ac + ad$; the linear equation: $ax + b = 0$; and the quadratic equation which can also be expressed as a product of two linear factors: $ax^2 + bx + c = 0$. The structure of the algebraic expression is thus dependent on the class of expressions to which it belongs. Therefore, structure understood in this way is in fact a mathematical property of the expression. The class of expressions dealt with by each teacher differ. The structure of symbolic expressions were studied and compared in both teachers' practices. Teacher B used the linear equation structure to exemplify and define the notion of a variable, while Teacher A used linear factors of a quadratic expression to teach structural differences between expressions.

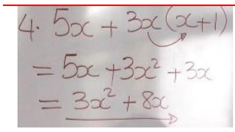
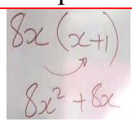
Teacher B compares structural similarities between arithmetic and algebraic expressions. In the transcript excerpt below taken from Teacher B L1Sg2T27, we see that learners need to look at both expressions $\square + 2 = 5$ and $x + 2 = 5$ and make a comparison and perceive the two expressions as the same expressions.

27	F2	Rt	T:	*So the box stands for four. So, is there a difference between this* [pointing at $\square + 2 = 5$] *and x plus two is equal to five* [writing on the board $\mathrm{x} + 2 = 5$]? 

Teacher B L1Sg2T27

Teacher A required learners to perceive structural differences in the answers. The first answer was a misconception offered by a learner (named Joseph and abbreviated as J in the transcript), and the second was the correct method offered by learners. The teacher requires learners to compare the two symbolic expressions, and perceive why J’s answer is incorrect.

43	F2	Re Rt	T:	*Okay, now is it the same as this one?* [pointing at Joseph’s answer written on opposite side]
44			Lrns:	*No.*
45		Re	T:	*You said it would be the same thing*
46			L:	*He said it’s the same thing.*
47	P1	Re	T:	*Would it be the same answer if we open the bracket?*
48			Lrns:	*No.*
49	P1 F2	Re Rg	T:	*If we open the bracket here we will be having eight x squared plus eight x* [written], *which is not the same as what we have there.*

Teacher and learners	Joseph’s answer
	

So we must use BODMAS, so we have to use BODMAS, so that is how important BODMAS is. [writing example five on the board]

Teacher A L3Sg3.4T43-49

The requirement here by the teacher is for learners to see that the expression $8x^2 + 8x$ is not an equivalent expression of the original expression and so there is an error in the transformation. For learners to see this, they have to understand the structure of both symbolic expressions $3x^2 + 8x$ and $8x^2 + 8x$ by comparing them. Teacher A wants her learners to see the difference between the structures: $5x + 3x(x + 1)$ and $(5x + 3x)(x + 1)$. Another way of seeing/testing the differences in the structures was to choose a value for x say 2 and substitute, where the correct transformation would yield the same answer to the original expression, while the incorrect transformation would yield a different answer. This specific excerpt is also an example of how operational activity is related to structure. There the discussion of structure of symbolic expressions overlaps and connects to operational activity as illustrated the discussion of this transcript excerpt from Segment 3.4.

Overall for Teacher B the explanation of the words mono, bi, tri, poly, for monomial, binomial, trinomial and polynomial were dealt with separately in detail as individual segments and exemplified with symbolic expressions. Teacher B emphasised the name and its mathematical meaning. Teacher A on the other hand relied on the textbook that each learner in her class was in possession of to provide definitions to the words: expression, monomial, binomial and trinomial. I provide evidence of how Teacher A refers learners explicitly to the textbook in the discussion concerned with language and terminology. In the discussion which follows, I continue to illustrate how structure was highlighted by both teachers in procedural activity.

8.2.1.4.2 Structure in the Operational activity

Talk related to form of the entire expression was present in procedural activity. The operational structuring was used to check if it was still possible to transform the expression or end the procedural activity. The two teachers' ways of dealing with procedural activity were similar, and so I only present examples from Teacher A as they are representative of how Teacher B also dealt with structure in the procedural activity. The response that would be elicited after the several transformations have been performed on the expression was that there are no like terms, and so the expression is in its simplest form. I present Figure 8:2 below, a snapshot from Sg3 of Teacher A's first lesson followed by speaking turns 50 -54 of the same segment.

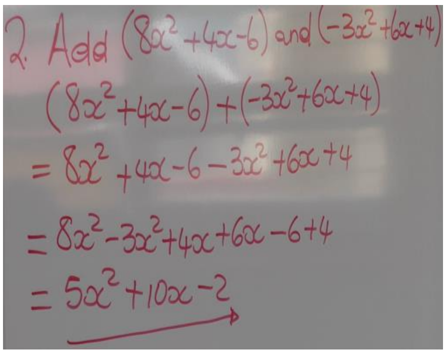


Figure 8:2 – Teacher A L1Sg3.2

In Sg 3.2 of Teacher A's L1 the teacher asks the question: "*and then?*" when she arrives at the transformed expression $5x^2 + 10x - 2$. The conversation proceeds in the following manner.

50			T:	*And then?*
51			L:	*It remains*
52	G2	Rv	T:	*It remains like this. Why?*
53			L:	*Because there are no like terms*
54	F2	Rv	T:	*Because there are no like terms. We accept this as the final answer. So we accept that as the final answer.*

Teacher A L1Sg3.2

After the teacher and learners establish that there are no like terms and so the expression $5x^2 + 10x - 2$ should be accepted as the final answer the transformations stop and the segment comes to an end. Looking for like terms is studying the structure of the entire expression and hence the code F2.

The identification and recognition of structure was performed in geometric representations in L3 of Teacher B.

8.2.1.4.3 Structure of geometric representations

As already noted, Teacher B also incorporated geometric representations. Four examples in L3 out of the six examples are geometric representations while the last two are symbolic representations of expressions. Teacher B required the name of the figure and its "special" property. Learners have to look at the entire figure and perceive that it is a rectangle and mention its "special" property and so the F2 code.

6	F2	Rv	T:	*We call it a rectangle. What is special about a rectangle?*
7			Lrns:	*All sides are equal.*
8		Re	T:	*Are they all equal?*
9			Lrns:	*No.*
10			Lrns:	*Two sides are equal.*
11	F2	Rv Re Rt	T:	*Two, two sides that are parallel to each other* [pointing at learner], *very good. Two sides that are opposite each other, opposite and parallel to each other. Right so which means if this side is three what is this side?*
12			Lrns:	*Three.*

Teacher B L3Sg2.2T6-12

I have illustrated how talk related to algebraic structure (F2) manifested itself in the two teachers' practices. Specifically, I have demonstrated through transcript excerpts how structure was highlighted in symbolic expressions, in the procedural activity and in geometric expressions.

There was a total of 26 utterances for Teacher A and 76 utterances for Teacher B coded as F2. The privileging of explanation criteria between F1 (23) and F2 (26) for Teacher A is unclear, because the small difference between F1 and F2 is three counts. For Teacher B the difference between F1 (62) and F2 (76) codes is 14 counts, and so a larger difference, but does not warrant a claim that F2 codes have been more privileged than F1 codes. In essence, it is safe to conclude pertaining to form that both F1 and F2 codes have been privileged by both teachers in the sequence of three lessons considered.

To this end, I have discussed matter and form. The close relationship between matter and form was clarified in the literature in two ways. Firstly, researchers argued that the act of unpacking the symbolism reveals meaning and structure, and therefore, symbols display structure. The significance of this statement is that for structure to be seen, there needs to be understanding and meaning making of the syntax (Novotna & Hoch, 2008). This manifested itself in the two teachers' practices through talk focused on elements, the naming and defining of elements that compose the algebraic expression. Secondly, I considered both matter and form to be conceptual explanations, because in talking about the arrangement or pattern the elements take, properties and laws that govern the structuring or arrangement of the elements are mentioned and dealt with. In essence, what has been classified as conceptual knowledge in the literature can be understood here as matter and form.

While the discussion above highlights the interplay between matter and form, there is an interactional relationship between form and process that was noted in the discussion on F2, in particular the section 8.2.1.4.2, which focused on structure in operational activity. This demonstrates that while the four (matter, form, process, goal) are distinct criteria of explanation, there is a back and forth relationship between them. The next discussion is focused on process (P1 and P2).

8.2.1.5 Steps of a method – P1

The second way of perceiving mathematical objects according to Sfard and Linchevski (1994) earlier cited in the literature and in the F2 discussion above was operational. This is the computational aspect where an expression is seen as a sequence of steps to be performed. The instruction which accompanies the expression is key in terms of determining which computation is appropriate. On the other hand Kieran (2007) has enumerated the kinds of actions involved in the operational activity in her definition of the second activity of school algebra, which she calls transformational activity. The list includes "collecting like terms, factoring, expanding, substituting, adding and multiplying polynomial expressions, and exponentiation with polynomials, solving equations, simplifying expressions, working with equivalent expression and equations and so on" (Kieran, 2007, p.714).

On instructions which accompany the algebraic expression, Chazan (1996) argues that the cryptic instructions such as "simplify", "solve", "multiply", "factorise", "expand" suggest the type of manipulation that needs to be done. Indeed, the steps of a procedure that were executed by both teachers involved multiplying or expanding and simplifying.

All utterances concerned with transmission of rules/ steps of a method for manipulation of expressions, I coded as P1. At this level of process, both teachers presented steps of a method, and so P1 talk was prevalent and dominant. Interestingly, while Teacher B has always exceeded the number of utterances by almost double those of Teacher A for other codes, for the P1 code, Teacher A has a total of 134 utterances, while Teacher B has 126. The multiplication process for the distributive law was presented in different ways. Teacher A emphasised the method, while Teacher B used and linked different representations to elaborate the meaning of the method through the geometric illustration and so the presence of P2 codes which I discuss next.

8.2.1.6 Derived and justified steps of a method – P2

Only Teacher B had utterances which I could code as P2. These were as a result of the representation and approach she used to introduce the notion of distributive law. Teacher B used geometric representations to justify why it was possible to draw arrows and distribute the multiplication over the addition or subtraction. Teacher B takes learners through a familiar method of multiplication which is the area model. She then, in the fourth example of using rectangles, poses the question seeking a shorter way to enable them to abandon the area model. As highlighted in the previous chapter, a weakness in her explanation is the fact that she does not highlight the common principle of multiplication between the two methods. Instead, her justification is based on the reasoning that the two methods produce the same answer, and the second method is shorter and quicker to perform. I coded these utterances as P2, because the symbol manipulation method of multiplication using the distributive arrows was derived, to some degree, from the rectangles method of calculating area. Consider transcript excerpt below taken from Teacher B L3Sg2.5T42.

42	P1 F2 P2 G2	Rg Rt Rn	T:	*That's my area right? But now when we, let's say we go back now. We want to multiply. But we don't want to multiply because they are, they might be long when we using area. So we want to multiply a monomial and a binomial. So now we take now this whole thing. So what is it* [writing a bracket along the length and along the breadth] *going to be called? So this is called what? Length and this breadth* [writing the labels on length and breadth, respectively].

Teacher B L3Sg2.5T42

Teacher B had 11 utterances coded as P2 while Teacher A has none, and so between P1 and P2, P1 is the most dominant and most privileged criteria of explanation by both teachers. It is appropriate to have P1 dominating, because most of the work in school algebra is in the transformational activity, as Kieran has already noted in Chapter 4. While most of the work and the algebra curriculum is weighted more towards transformational activities, it is inappropriate to have a complete absence of the P2 code, as this implies that the rules are not derived, and there is no understanding of where procedures come from, and why they work.

I promised to come back to the discussion about the interplay between structure and process and I revisit the two authors, and that is Sfard and Linchevski (1994) and Kieran (2013), who have articulated with clarity their positions in relation to structure/concepts and operations/ procedures in the learning of elementary algebra.

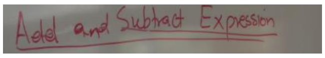
Early on Sfard and Linchevski (1994) stated that the constructs operational and structural are inseparable, while different aspects of the same algebraic expression. They went on to emphasise that it is rather a duality instead of a dichotomy that is being dealt with, which has continued to be the case with literature focusing on procedural and conceptual knowledge. As they argue their theory of reification, Sfard and Linchevski put forth that operational activity comes first, while structural thinking is being developed and later reified. From this argument, it seems as though it is an unreasonable requirement to look for talk related to structure of the algebraic expression early on in the lessons. However, the empirical evidence presented here disputes their argument of operational perception taking precedence over structural. Both Teacher A and Teacher B have more elaborations of matter and form in the introductory parts of their lessons. It seems then, that the perception (structural or operational) that will take precedence over the other is context dependent and subject to the design and priorities of the algebraic task (how targeted is it to teaching structural perception). Since this is not initial work on algebraic expressions, learners have done operational activity on algebraic expressions in Grade Eight, it might be the case that the procedural activity preceded the structural.

While I may be tempted to support the argument for precedence, Kieran (2013) on the other hand clarified that the separation that has always existed in mathematics education literature concerning processes and concepts is misleading because the processes are informed by mathematical principles and properties, which are in themselves conceptual. At this level, both Kieran (2013) and Sfard and Linchevski (1994) seem to agree. The agreement is broken when Kieran goes on to argue the co-emergence of the procedural and conceptual in the teaching and learning of algebra, while Sfard and Linchevski argued for the precedence of operational over structural.

In my study, I have shown and argued that there is merit in having the categories structure and process separated for analytic purposes. Also, the separation enables one to see which category dominates talk in the mathematics classroom and to understand its appropriateness or inappropriateness.

8.2.1.7 Statement of the goal – G1

Ball et al. (2008) have highlighted the importance of stating and clarifying goals to parents. The explanations literature on the other hand emphasised the importance of having clearly stated goals for the lesson in terms of what is to be learnt. In the two teachers' practices, the goal was introduced at different times of the lesson. For Teacher A, it was the first thing in L1 and L3, followed by a sequence of examples, while for Teacher B, the goal was introduced after a number of examples have been dealt with in L1 and L3. For L2, Teacher B introduced the goal immediately, before any examples were dealt with, while for Teacher A L2 was comprised of solutions to homework. I present from L1 for both teachers transcript excerpts of how they went about stating and clarifying the goal.

1	G1	Rg	T:	*Okay! What we going to do today will be adding. So we said there are four terms from there. So we want to add and subtract expressions* [written on the board and underlined].
2				
3	G1	Rg	T:	*Okay! Add and subtract expressions.*

Teacher A L1Sg2T1-3

We see that Teacher A stated and wrote the goal on the board. The goal she stated was limited to the activity they were going to be engaged with on that day, which is "add and subtract expression". On the other hand, Teacher B stated a broader goal that goes beyond a day's work and that is "algebraic expressions", see Teacher B L1Sg9T3 below.

3	G1 G2	Rg Rn	T:	*Okay, okay, okay, so algebraic expressions. Why expressions? If I say give me an English expression. Give me any English expression.*

Teacher B L1Sg9

Teacher B does not only state the goal and write it as algebraic expressions without clarifying the mathematical and everyday English meaning of the words. There is a 'why' question asked, something that does not happen with Teacher A. The goal as a criterion of explanation is important to state in the lesson and its statement must be accompanied by reasons and justification. The connections Teacher B makes are important for the explanation, because it is not just a a statement of the mathematical goal, but is connected to other activities that are outside of the mathematics classroom. While for Teacher A, the statement of the goal was limited to its relevence to the the current activity on which they are about to embark. The next discussion is focused on the why question, and is a response to it.

8.2.1.8 Why question and a response to it – G2

In the explanations literature, Chapter 3, the goal was defined as the purpose for which a thing is done. I have argued in the methodology that the purpose can also be understood as a justification in the form of a 'why' question, and a response to it. For explanation, I have argued that the 'why' question, when used in a careful manner, brings about depth and quality of explanation. This aspect of the explanation is understood as having the knowledge of what to do and why. In the literature, this has broadly been known as relational understanding (Skemp, 1978), which can be correlated to conceptual understanding. The G2 code is the only code in the level two codes, which is expected to be present in M, F, P and G1 codes, since it offers the reasons and justifications for elements, form, process and goals.

The ‘why’ question was never such for its own sake. Instead, it was a question about something. In the transcripts, there were reasons sought for naming the element/expression and so this is a ‘why’ question asked in relation to M. The ‘why’ question was asked in relation to form. A ‘why’ question was also asked in relation to the process, for carrying out the step and lastly a ‘why’ question was asked in relation to the goal or the topic of the lesson. Teacher B has a total of 44 counts, while teacher Teacher A has a total of 22 counts for the G2 code. I present the tables (see Tables 8-4 and 8-5) for both teachers which show how the G2 code was used in relation to M1, M2, F1, F2, P1, P2, G1 codes. Teacher A’s use of the why question is in relation to F1, F2 and P1 only, while for Teacher B there is a presence of the why question in M1, M2, F1, F2, P1, P2 and G1.

Table 8-4: Teacher A’s use of the G2 code

G1														
P2														
P1		1		1	1				1		1	1		1	1
F2		1	1	1		3	1	1		2					
F1	1	2											1		
M2															
M1															
Sg	1.2	3.1	3.2	3.3	3.4	1	1.2	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8
			L1			L2					L3				

It is apparent from the table above showing Teacher A’s use of the G2 code in relation to other explanation codes, that the ‘why’ question in her three lessons was in relation to F1, F2 and P1 only. There was no ‘why’ question asked in relation to M1, M2, P2 (which is absent in Teacher A’s practice) and G1. This is to say that the goals were not justified explicitly, so that learners know the reasons for the topic. The steps of a procedure were not derived as already shown in the discussion on process. The names of the elements were not justified beyond knowing what the element is called, and its function in the expression.

Table 8-5 shows Teacher B’s use of the G2 code in relation to other codes.

Table 8-5: Teacher B’s use of the G2 code

G1							3														
P2																				1	
P1	3		1	1	1	1		2		1	1	2			1					1	
F2					1	1								1	1	2		1	3		1
F1					1										3		2				
M2		2																			
M1		2	1						2				1								
Sg	1.3	2	3	4	5	6.1	7	8.2	8.1	1.2	1.5	2.2	2.5	2.6	4.1	4.2	4.3	4.5	1.1	2.4	4.2
					L1									L2						L3	

For Teacher B, there was a G2 code for each code: M1, M2, F1, F2, P1, P2, G1. However, the spread is concentrated on F1, F2 and P1 across the three lessons and thus similar to Teacher A. In terms of the privileging, it is not easy to tell in the case of the G2 code as it is a code that runs through other codes. Nevertheless, I can argue that there needs to be more deliberate use of the G2 code for matter, structure, process and goal at G1 level.

The privileged explanation criteria

To this point, I have presented a discussion for each code in an attempt to answer the sub-research question of what was privileged as explanation for both teachers. When considering the eight codes as a collective instead of specific singular codes, it is clear that M1 and P1 are the most dominant, and thus are privileged criteria of explanation for both teachers.

In the discussion of the eight codes, I have reasoned that the level two codes speak to the quality of the explanation and I will reiterate the original definitions of the level two codes just to substantiate this point. Briefly, the M2 code discusses the elements that highlight the operational relations between elements. F2 is talk about understanding structure for the entire algebraic expression. In Sfard and Linchevski’s (1994) terms, it is the highest level required to understand algebraic objects by reaching the process of reasoning structurally. On the other hand, P2 is about understanding the underlying reasons for carrying out rules or steps of a procedure. It is about derivation of steps and justification for the rules or method performed. The G2 code is a ‘why’ question as well as a response to it, running through all the criteria of explanation.

The results of the explanation from both teachers demonstrated that there needs to be special attention committed to studying means of teaching procedural steps in such a way that they are derived and justified. This is a point for PD programmes to purposefully consider. Also, there needs to be special attention aimed at providing accurate definitions of elements and study of structure. Furthermore, the why question needs to be the centre of the explanation. I present below in bullet form the main findings of what was privileged as explanation.

8.2.2 Main findings on what was privileged as explanation across the two teachers:

1. There was a dominance of M1 and P1 codes, suggesting a dominance of talk about elements without elaboration and procedural steps without derivation. These are what Skemp (1978) referred to as rules without reasons.
2. There were imprecisions in the definition (M2) provided by both teachers some of which might have been in view of learners' grade level and cognitive ability.
3. The over-representation of P1 meant that there was a significant lack of derivation and justification of steps P2.
4. The 'why' question brought depth to the explanation in different parts (elements, form, process and goal) of the explanation.
5. With regard to the development of the explanation, and also the codes, for both teachers in the first lesson, the talk was mainly centred on explanation of matter and form. Once both teachers were satisfied with the development of learner understanding, then the focus shifted more towards the explanation of process but without derivation.

The question I was seeking to answer at the beginning of the chapter was: "What constitutive criteria of explanation are privileged in teachers' practices?" From the discussion of the codes and main findings, I can say that M1 and P1 are the most previledged criteria of explanation in both teachers' practices.

I turn now to reflect further on the weighting of the explanation codes as these manifested in the two teachers lessons, and whether and how this was appropriate or not.

The data presented in this study shows that the lesson in its early stages will have some presence of M2 codes, and as the lesson develops the M2 codes diminish and completely disappear, because the content of the lesson has progressed and what was new knowledge has now become old, known concepts. In view of this, it is expected and acceptable that M2 codes will not enjoy the same level of occurrence as M1 codes.

In the F codes, both F1 and F2, for both teachers had almost the same number of counts, with F2 having slightly more than F1; and I have already indicated that there was no outright privileging of neither F1 nor F2 for both teachers. While this was the case, it is in fact expected that there should be deliberate study of structure for each example introduced in the lesson and so the question proposed by Sfard and Linchevski (1994), such as “what do you see” is important; and should find its way into algebra lessons and mathematics lessons in general for each example introduced as it seems to have occurred for both teachers.

As already noted with regards to P codes, P1 was notably more dominant than P2 codes for Teacher B, while for Teacher A P1 was the only code in the P codes present. Again, as in the M codes, it is expected that P1 codes will be more dominant, however, a complete absence of P2 codes suggests limited explanations offered. And so there needs to be more deliberate inclusion of P2 codes in the form of linking multiple representations as demonstrated in Teacher B’s L3 and derivation of steps in general.

In the case of G codes, G1 was the statement of the goal for the lesson, while G2 was focused on the reasons, justification and answer to the why question. The expectation was that the G2 codes would be surfacing more frequently than they did as a way of providing reasons and justifications for most of the moves the teacher makes.

I have merely presented here what can be thought of as the ideal use of the criteria for explanation. However, a more robust presentation would be possible if each criteria of the eight in all is studied in detail. For this study, the aim was to investigate the criteria for explanation in algebra, and in this chapter, the aim was to find out which criteria of explanation are privileged in the teachers’ practices. From the findings, it is possible to postulate that the condition and status of mathematics education in South Africa, particularly for disadvantaged communities, as earlier noted in Chapter 1, can be accounted for from the dominant and most priviledged criteria of explanation. There seems to be a correlation between M1 and P1 dominance and poor performance in Mathematics.

I now turn to a discussion asking what has been privileged as criteria of acts of explaining in the teachers’ practices with a view to link to the overall study.

8.2.3 Acts of explaining for both teachers

I discuss what was privileged as acts of explaining criteria by referring to the statistics table and by drawing on and referring back to literature. I have sectioned the discussion of the two teachers privileging of criteria for acts of explaining according to: examples (Rx), language (Rn), communicative strategies (Rv, Rf, Rt, Re, Rc,), and regulation (Rg).

8.2.3.1 Rx – What was exemplified and what representation was used

A good explanation is not only dependent on appropriate mathematical language used for definitions of elements, expression and process, but a good explanation is also generated through a purposeful selection of good examples. The selection of good examples and creation of tasks depends on the teachers' subject matter knowledge and pedagogical content knowledge. In Chapter 3, I discussed three kinds of example use as posited by Zaslavsky (2014) and those were: 1) sponteneuos example use; 2) evoked example production; and 3) example provisioning. In the three lessons I observed from each teacher, there was predominantly example provisioning.

Zaslavsky's discussion on examples in Chapter 3 went on to articulate the notion of an example space and its internal structure. The internal structure of the example space focused on how the collection of examples are interrelated, and can be associated with the particular idea they are seeking to develop. The particular concept the example spaces in the teachers' lessons were addressing is the notion of algebraic expressions in Grade Nine, and operations on these. I looked into the internal structure of the example spaces both teachers provided in the lessons, as worked out examples, as well as in the homework as practice examples.

The nature of examples dealt with in both teacher's lessons were mainly symbolic representations, and so require elaborations limited to algebra while in teacher Teacher B's Lesson 3 there are geometric representations used, which require elaborations in geometry first before dealing with the algebraic aspect of the example. There were in total 26 examples used and elaborated in class for Teacher A and 22 for Teacher B. Table 8-6 below is a presentation of the examples dealt with in L1 for both teachers.

Table 8-6: Lesson examples from L1 for Teacher A and Teacher B

Teacher A examples from L1	Teacher B examples from L1
Lesson - How many terms? $3x^2 + 3xy - 6y^2 + 4$ - Simplify: $5x + 5y - 3x + 2y$ - "add $(8x^2 + 4x - 6)$ and $(-3x^2 + 6x + 4)$" - "subtract $(7x^2 + 4)$ from $(15x^2 - 2)$" and - "subtract 4 from 6" - "subtract $(-3x^2 + 6x + 4)$ from $(8x^2 + 4x - 6)$"	MM activity - If $a = 4$ then calculate: $a + a$, $2a$, a^2, $\sqrt{a}$ Lesson - $If\ x = 4\ then\ x + x, \square + 2 = 5, 2\square = 8$ - Monomial examples (a) x, (b) $3x$, (c) $3x + 2x = 5x$ - Example of a coefficient: $10 \times x = 10x$ - Example of a binomial: $3x + 2y$ - Example of a trinomial: $3x + 2y + z$ - $2x^3 + 4x^2 - 2x + 7$ - $7 + 2x + 4x^2 + 2x^3$ - $2x^3y + 4x^2y^2 - 2xy^3 + 3$ } Ascending or descending powers of x

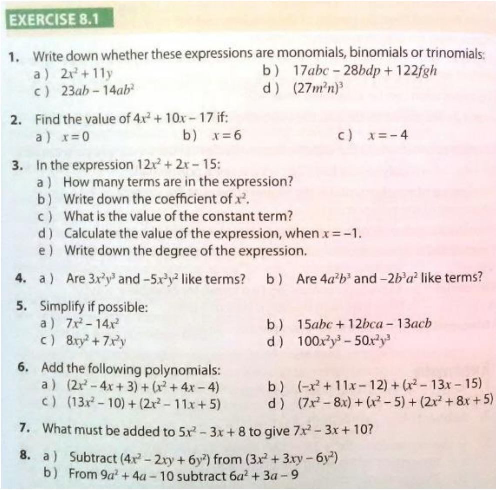
The table above contains the examples dealt with in the first lesson by both teachers. There was what Teacher B called "mental maths" activity before the actual lesson, and this was composed of an example where different expressions ($a + a$; $2a$; a^2; $\sqrt{a}$), with the same variable were evaluated ($a = 4$). In the lesson, Teacher B exemplified ideas and concepts which were matter and form such as variable ($x + 2 = 5$; $\square + 2 = 5$; $2x = 8$; $2\square = 8$), monomial (x; $3x$; and $3x + 2x = 5x$), binomial ($3x + 2y$), and trinomial ($3x + 2y + z$), powers of a variable ($2x^3 + 4x^2 - 2x + 7$ and $2x^3y + 4x^2y^2 - 2xy^3 + 3$) individually as separate segments. Teacher A did not necessarily provide examples of specific matter and form, but from one expression ($3x^2 + 3xy - 6y^2 + 4$) required learners to identify elements such as constant, coefficient, first term. This meant that each element was discussed briefly and the attention was focused on too many other elements, but the strength of the approach was that one expression was studied in its entirety. For Teacher A, the examples progressed from an example of elements to examples foregrounding the process of adding and subtracting like terms in an expression, consider the last four remaining expressions.

The Table 8-7 below shows algebraic expressions provided in L1 for the homework for both teachers.

Table 8-7: Homework examples from L1 for Teacher A and Teacher B

<table>
<tr><th>Teacher A examples from L3</th><th>Teacher B examples from L3</th></tr>
<tr><td>

“Class Activity page 72 Exercise 8.1, number 1,2,3,6 and 8”



EXERCISE 8.1

1. Write down whether these expressions are monomials, binomials or trinomials:
   a) $2x^2 + 11y$ b) $17abc - 28bdp + 122fgh$
   c) $23ab - 14ab^2$ d) $(27m^2n)^3$

2. Find the value of $4x^2 + 10x - 17$ if:
   a) $x = 0$ b) $x = 6$ c) $x = -4$

3. In the expression $12x^2 + 2x - 15$:
   a) How many terms are in the expression?
   b) Write down the coefficient of x^2.
   c) What is the value of the constant term?
   d) Calculate the value of the expression, when $x = -1$.
   e) Write down the degree of the expression.

4. a) Are $3x^2y^3$ and $-5x^3y^2$ like terms? b) Are $4a^2b^3$ and $-2b^3a^2$ like terms?

5. Simplify if possible:
   a) $7x^2 - 14x^2$ b) $15abc + 12bca - 13acb$
   c) $8xy^2 + 7x^2y$ d) $100x^2y^3 - 50x^2y^3$

6. Add the following polynomials:
   a) $(2x^2 - 4x + 3) + (x^2 + 4x - 4)$ b) $(-x^2 + 11x - 12) + (x^2 - 13x - 15)$
   c) $(13x^2 - 10) + (2x^2 - 11x + 5)$ d) $(7x^2 - 8x) + (x^2 - 5) + (2x^2 + 8x + 5)$

7. What must be added to $5x^2 - 3x + 8$ to give $7x^2 - 3x + 10$?

8. a) Subtract $(4x^2 - 2xy + 6y^2)$ from $(3x^2 + 3xy - 6y^2)$
   b) From $9a^2 + 4a - 10$ subtract $6a^2 + 3a - 9$

</td><td>

“Classwork: How many terms are in the following polynomials? Does each have ascending or descending powers of x?”

(a) $x^3 + 3x^2 + 3x + 1$
(b) $x^4 + 2x^3y + 7x^2y^2 - xy^3 + y^4$
(c) $x^5 + x^3 - 19$
(d) $x^{-2} + 30x^{-1} + 3$

</td></tr>
</table>

Teacher A had more practice examples for learners, such that the whole lesson the next day is completely focused on feedback. In the homework activity, there are four expressions that involve identification of structure, and two expressions which involve an evaluation of expression. In the remaining part of the homework, there is addition and subtraction of six expressions, while homework items for Teacher B are only four expressions, where learners have to identify the number of terms and degree of the polynomial, the last example, as already noted in the previous chapter, is not a polynomial ($x^{-2} + 30x^{-1} + 3$). The teacher is the only knowledge resource (Adler, 2000), and so learners depend on what Teacher B provides.

The examples used by Teacher A in the lesson and in homework activity are expressions where operations such as adding, substituting, and simplifying have to be performed and identification of elements and structure such as number of terms, coefficient, and constant term were undertaken. On the other hand, Teacher B used examples in the lesson and in the homework activity, which were mainly concerned with identification of structure and elements.

The example sets in the lesson as well as homework activity, for both teachers, accumulates in relation to the object of learning (Adler & Ronda, 2015). In relation to acts of explaining, both teachers made a deliberate selection and sequencing of the example set according to the knowledge and skills they wanted their learners to obtain from the lesson. Table 8-8 is a presentation of examples dealt with in L2 for both teachers.

Table 8-8: Lesson examples from L2 for Teacher A and Teacher B

Teacher A examples from L2	Teacher B examples from L2
h/w corrections - $4 \times 0^2 + 10 \times 0 - 17$ - $4x^2 + 10x - 17$ - $4 \times (-4^2) + 10 \times (-4) - 17$ - $12(-1^2) + 2(-1) - 15$ - $(2x^2 - 4x + 3) + (x^2 + 4x - 4)$ - $(-x^2 + 11x - 12) + (x^2 - 13x - 15)$ - $(13x^2 - 10) + (2x^2 - 11x + 5)$ - $(7x^2 - 8x) + (x^2 - 5) + (2x^2 + 8x + 5)$	h/w corrections: identify number of terms and degree of polynomial - $x^3 + 3x^2 + 3x + 1$ - $x^4 + 2x^3 y + 7x^2 y^2 - xy^3 + y^4$ - $x^5 + x^3 - 1$ - $9x^{-2} + 30x^{-1} + 3$ MM activity: Simplify - (1) $2m + 2m + 2m + 2m$, (2) $c \times d \times a \times b$, (3) $2 \times x \times y \times 3$, (4) $c \times c \times c$, (5) $7x - 5y + 1$ Lesson: add like terms - $5xyz; 3xyz; 2zyx; 5xy$ - $x, 5x, 5y, 3x$ - $7y^2, 7, 3y^3, 4y^2$ - $(2a \times b) + (3a \times b) - (a \times b)$ - $(3a \times 2) + (-4b \times -2) + (a \times -2)$

After the corrections, Teacher B moved into the mental maths activity, which was composed of five expressions, the first of which was an addition of like terms, while the next three were a multiplication, and the last one could not be operated on because it has no like terms. The lesson had five expressions which were examples of terms. And so it can be concluded that from Teacher B's lessons, the examples were examples of structure and process. Because of the lengthy homework exercise Teacher A gave to the learners in L1, all of L2 was taken up by provision of answers to the homework. Only eight items are written on the board, the first four of which are evaluations of expressions, while the remaining four are addition of expressions. And so, the processes that are exemplified are substitution and addition of like terms. Only Teacher B has homework items provided in this lesson which are shown in Table 8-9. The set of examples used by both teachers in L2 were concerned with recognition of structure and operational activity of algebraic expressions.

Table 8-9: Homework examples from L2 for Teacher A and Teacher B

Teacher A examples from L2	Teacher B examples from L2
	Homework 1. Is it possible to simplify? (yes/no) (a) $x + 2x$, (d) $4x + 4y + z$, (b) $a + 3b$, (e) $-7a + 2a$, (c) $10a - a$, (f) $5a + 2b + 3b - 4a$ 2. Simplify a) $(23a \times 2b) + (6b \times c) + (a \times b) + (b \times c)$ b) $(4y \times 2) + (2 \times 5) + (3 \times 4) + (4 \times 4y)$ c) $(6b \times d) - (4a \times 3c) + (7a \times -2c) + (-3a \times -2c)$ d) $(6m \times n) + (3m \times n) + (4n \times -p) + (3n \times -p)$

There are two tasks to the homework activity Teacher B provided for learners. In the first task, six algebraic expressions are provided, where learners have to determine if the expressions can be simplified and answer yes or no. To do this, learners have to identify like terms. And so the examples are examples of form/ structure and process. In the second task learners have to simplify within the term and then the expression, and there are four expressions. Also, these four are examples of form and process.

Table 8-10: Lesson examples from L3 for Teacher A and Teacher B

Teacher A examples from L3	Teacher B examples from L3
h/w corrections - $(3x^2 + 3xy - 6y^2)$ – $(4x^2 + 2xy + 6y)$ - $(9a^2 + 4a - 10)$ – $(6a^2 + 3a - 9)$ Lesson - "multiply $(3x + 1)$ by 2" - $5 + 2(3x + 2)$ - $5 + 2x(3x + 2)$ - $5x + 3x(x + 1)$ - $5x - 3x(x + 1)$ - $2x(x + y) - 2y(3x - 5y)$ - $(x + 1)(x + 2)$ - $(x + 1)(3x - 2)$ Multiply out and simplify	h/w corrections Is it possible to simplify? (yes/no) (a) $x + 2x$, (d) $4x + 4y + z$, (b) $a + 3b$, (e) $-7a + 2a$, (c) $10a - a$, (f) $5a + 2b + 3b - 4a$ Lesson - Square: $l = 4cm$ - Rectangle: $l = 3cm$; $b = 1cm$ - Rectangle: $l = (x + 2)$; $b = 3$ - Square, then rectangle, Square $l = x$, Rectangle $l = x + 2$; $b = x$, - $2(2x + 3)$ - $2x(-x - 4)$

In Table 8-10, the first two expressions in Teacher A's L3 are left over from the lengthy homework exercise, which was issued in L1 and they involve subtraction of expressions, and so are examples of structure and process. The lesson is composed of eight examples, which are examples of the process of multiplication as well as structure. Teacher B, on the other hand, has six expressions in her lesson, which are examples of the process of multiplication using geometric structures. In Chapter 3 the notion of using multiple representations was advocated for by researchers as a useful strategy, which assists learners in making sense of the explanation (Wilkie, 2016; Zazkis, 2016).

Table 8-11: Homework examples from L3 for Teacher A and Teacher B

Teacher A examples from L3	Teacher B examples from L3
"Class Activity Gr9 Page 73 Exercise 8.2 no 1, 2, 3 Exercise 8.3 no 1, 2" 8.2 EXERCISE 8.2 Simplify the following expressions: 1. a) $6+3(x-2)$ b) $5x-2x(x+2)$ 2. a) $5(2a-b)+2(3b-4a)$ b) $2(x+y)+4(3x-2y)-5(2x-3y)$ c) $4a(a+b)-2b(3a-5b)+2ab$ 3. a) $7(x^2-x+2)+(6x-14)$ b) $2(3x^2+6x-1)-(2x-4)$ c) $-3(x^3+2x^2-x)-x^2(2x+1)$ 8.3 EXERCISE 8.3 Simplify: 1. a) $(x+2)(x-4)$ b) $(2x+1)(3x-5)$ c) $(x+y)(x+2y)$ d) $(3x-4y)(2x-5y)$ 2. a) $(x-4)^2$ b) $(3x+2)^2$ c) $(6x-5y)^2$ d) $(7x+4y)^2$	Homework Simplify: (a) $5(3x-1)$ (b) $6(3x-2)+2(x+3)$ (c) $3(3x-2x)-(4x-2x)$ (d) $-7(-3x+4y)$

In Table 8-11, there are 18 expressions in the homework for Teacher A, that learners have to experience compared to four algebraic expressions written on the board for Teacher B. All are examples of the process of multiplication and structure.

As already noted, the explanation is the explanation of something, viz. that which needs to be explained. In the case of the topic of algebraic expressions, this was achieved through the selection, sequencing and use of examples. Mathematics is an abstract subject, and so is algebra so the ideas, the concepts, and the processes performed in algebraic expressions come into sight through symbolic representations and examples. Examples play a pivotal role in the production of an explanation in terms of illuminating ideas, concepts and procedures. In relation to acts of explaining the selection and sequencing of examples are decisions that the teacher makes prior the delivery of the lesson, in this way the act of selecting and choosing examples is example provisioning. As already noted, the examples all accumulate to address the object of learning which was: algebraic expressions. Zaslavski (2014) mentioned the notion of internal structure of the example set. Internally each example had a specific idea that it was exemplifying such as coefficient, constant, binomial, polynomial, etc. All these, though individually dealt with at first, were concepts that were collectively required in order to deal with the topic of algebriac expressions. I have provided a limited analysis of example space, due to the scope of the study. The acts of explaining also include pedagogic decisions about the language that will be used to talk about the examples to convey the explanation.

8.2.3.2 Rn – Language use

Access to the explanation is made possible through language and so the teachers' use of language is a transmission of criteria on how learners should in turn use mathematical language. The teachers' explanation of the mathematical language depends on their subject matter knowledge as well as pedagogic skills (Ball et al., 2008).

The language of mathematics was used in the teachers' lessons to describe methods and their steps, to explain notation of the elements which compose the expression and for definitions of mathematical terms. There is one point which I wish to highlight from Boulet (2007), noted earlier in Chapter 3. The point is related to reading of a string of symbols. When it comes to notation, the author makes an example of a decimal number, nevertheless, the argument is relevant for thinking about language used by teachers in algebraic expressions and mathematics generally:

> Consider for example, the case of decimal numbers. Teachers who refer to 0.37 by mentioning each symbol, *zero point three seven*, and not as the number *thirty-seven hundredths*, are then obligated to teach a rule to connect this number to its equivalent as a percentage: "Move the decimal over two places to the right." However, *thirty-seven hundredths* is *thirty-seven percent*; there is no need to perform any kind of operation on the symbols, such as moving the decimal point, to view the number in decimal notation as a percentage. [...] To regard numbers as lists of discrete symbols may account for the difficulties students have with decimal numbers (Boulet, 2007, p.6).

The reading of a string of symbols is a language that describes the expression's notational form rather than its meaning. The point here is that PD programmes have a role to play in supporting teachers to use the correct mathematical language in their explanations, and specifically in the language used in the reading of algebraic expressions and mathematical texts in general. Novotna and Hoch (2008) mentioned in the literature review, argue that symbol sense is the ability to interprete symbolic expressions. In this view, the reading of a string of symbols points to a lack of symbol sense. PD programmes have a role to play in terms of teaching the mathematics vocabulary, which implies learning the definitions of elements, terms, and expressions.

Teacher B appears to be more sensitive and aware of the difficulties learners' encounter in understanding and making sense of the algebraic language. Only Teacher B paid attention to mathematical terminology, what things are called, and the meaning of words in everyday English language versus their mathematical meanings. Teacher A's language remained at the formal level, without meaning or elaboration of words. Teacher B took nothing for granted, she focused attention to the meaning of words such as variable, monomial, binomial, trinomial, polynomial, degree of polynomial, by ensuring that learners know what these elements are called and the mathematical meaning of their names. Teacher B also defined in detail mathematical terminology, such as square, rectangle, constant, and algebraic expressions in detail with examples. On the other hand, Teacher A took all these words as though their meaning was shared. She only defines a coefficient when she realises that learners are unable to identify the coefficient of the first term in L1 Sg1.2, and in the same segment reminds learners of the definition of a constant. The fact that each learner in teacher Teacher A's class is in possession of a textbook makes it possible for learners to study for themselves definition of terms and meaning of mathematical words. We see this at the beginning of L2, when the teacher asks in Teacher A L2Sg1T1-5, see transcript excerpt below:

1		Rg	T:	*You read the book?*
2			Lrns:	*Yes*
3	F2	Rg	T:	*What is a monomial, binomial and what is a trinomial?*
4			Lrns:	*Yes*
5	G2		T:	*Because it is given there in your textbook* [teacher in front of class holding an open book up for learners to see picture from the textbook page 70 and 71].

Teacher A L2Sg1T1-5

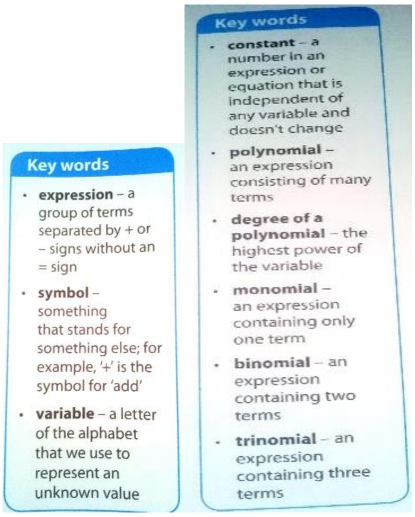


Figure 8:3 – Key words from learner textbook: Classroom Mathematics Grade 9 p. 70-71.

In the textbook, the key word boxes are positioned at the margins of the textbook separated from the main text in the middle. Figure 8:3 above shows the key word boxes from the textbook. Total number of utterances coded as Rn for Teacher A are 0 and for Teacher B are 50, and so Teacher B privileged the use of language in her acts of explaining more than Teacher A.

8.2.3.3 Rc, Rv, Re, Rf and Rt – Modes of communication

Re-voicing (Rv) was one of the ways of teaching learners to communicate mathematics, because the act of re-voicing is actually repeating the learner's input in order to highlight both correct and incorrect learner ideas. In the teachers' practices, re-voicing was the most dominant communicative technique for both teachers. Teacher A had a total of 115 counts and Teacher B a total of 159 counts for the Rv code. Also, the act of re-voicing has been noted in the literature as a defining feature of teacher led interaction with learners.

Both teachers expected their learners to know the end of the sentence and finish it off (Rf) for some of the utterances. Teacher A 34 Teacher B 65.

Chorusing as a communicative technique was used infrequently by teacher Teacher B. She required learners to chant in unison the name of an alphabet in mathematics in L1. She would ask repeatedly "what do we call an alphabet in mathematics"; her learners would respond in chorus, and if the response was not loud enough, she would require them to chant it again.

Teacher A gave specific instructions in the first lesson that she did not want chorus responses. Learners were instead instructed to put up their hands. As a result, teacher A had zero for the Rc code, while Teacher B had a total of five counts. This is the second difference in the two teachers' acts of explaining. The notion of chorusing may come across as a frivolous criterion of the acts of explaining, however, it may have some benefits for learners, because it forces them to agree and align themselves with the specific idea under consideration.

From the taxonomy of gestures presented by Roth (2001) which I discussed in Chapter 3, the deictic gesture is the one that surfaced in the empirical data of the study. Deictic gestures were defined as prevalent in the development stage for children in the process of acquiring language. This might be an indication that the teachers' own mathematical language is still in need of development into formal mathematical language. The frequency counts also show that Teacher B (131 counts) used gestures much more than Teacher A (15 counts). In fact, for Teacher B, the Rt code was the second highest amongst the communicative strategies. Gestures (Rt) are not necessarily a strength of the communication of knowledge, but rather an indication of ambiguous language used to refer to elements and expressions. In the event that a learner is not looking at the teacher, but only listening to what is said, then they lose important communication about a concept or a process, because they will not make sense of the particles 'this' 'that' 'those', 'here' 'there'. The use of these indexical/deictic words also limits the process of generalisation, because the meaning they point to becomes dependent on the current context. Should a different expression be used where the arrangement of the elements is not the same as that used in the specific example, then learners will not recognise the expression or be able to execute the procedure.

Both teachers required learners to engage with other learners' contributions and evaluate (Re) the correctness of the contribution by agreeing or disagreeing. By allowing learners to evaluate and provide reasons for their opinions and views, an opportunity for a common understanding was created, through resolving the disagreement. Both teachers provided opportunities for learners to assess their own work, Teacher A and Teacher B had the same total of 62 counts in the Re code. Both Teacher A and Teacher B taught learners to do mathematics, but also to be certain of its correctness without requiring the teacher to legitimate it at all times. Learners became arbiters of the correctness of the solution presented, and thus engaged with each other's contributions by evaluating their contributions. For learners to be engaged meaningfully in this activity of evaluating each others contributions, they need to be in possession of the criteria of explanation.

In the main, the act of re-voicing learner contributions was the most privileged criterion for acts of explaining for both teachers as far as communicative strategies are concerned. For Teacher B, this was followed by Rt while for Teacher A, the second privileged mode of communication was Re, and so in essence, the order of preference/privileging for Teacher A is: Rv, Re, Rf, Rt, while for Teacher B it is: Rv, Rt, Rf, Re.

The act of re-voicing, finishing teachers sentence, chorusing, evaluating, and the use of gestures are all actions which were performed by the teachers in this study to communicate the explanation at the level of matter, form, process, and goal. The communicative techniques are very important in the development of a full explanation in instruction.

8.2.3.4 Rg – regulation: social norms and sociomathematical norms

Mathematical practice is defined or composed of a wide range of practices and culture, counted among them is the notion of socio-mathematical norms responsible for creating a favourable environment for the teaching and learning of mathematics (Güven & Dede, 2017 drawing from Cobb). While these might come across as having nothing to do with the explanation, in fact they set the tone for interaction, and what comes to be the constituted as explanation depends on what has been established as social and socio-mathematical norms.

Rg codes manages the social aspect of teaching which involves setting up and maintaining classroom order through established socio-mathematical norms and social norms, which regulate the social interaction. By the time I got to video lessons in the classroom for both teachers, a culture has been shaped and built, because these are activities that occur at the very beginning. Güven and Dede (2017) argue that these depend on the teacher, the learners, as well as the context and its culture.

The norm in Teacher A’s class is that learners go to the board to present their work. The environment atmosphere is stress-free and learners engage each other’s contributions. As already noted in the methodology, the Rv, Rf, Re, Rc, Rt, codes are a reflection of the already established socio-mathematical norms for communicating the explanation.

The statistics table show that both teachers focused attention on manners and regulation of behaviour, where these are social norms. The dominance of regulation in both teachers’ talk might be misinterpreted as a struggle to maintain order and paint a picture of rowdy, uncontrollable classrooms, while in fact, the role of regulative talk was to drive the sequence and structure of classroom communication. For instance, the movement from one example to the next is regulated by the teacher. Overall, Teacher A had a total of 109 counts for social and socio mathematical norms, Rg code, while Teacher B had 139 counts.

8.2.4 Main findings for acts of explaining

1. In both teachers' practices there was a dominance of re-voicing and regulative talk;
2. There was a lack in talk that uses mathematical language in the description of symbolic expressions used to exemplify the mathematics; and
3. Gestures are known to contribute to the communication of the mathematics, however, the use of ambiguous referents was prevalent here because of gestures, thus compromising the use of mathematical language.

8.3 The relationship between explanation codes and acts of explaining codes

The quality of the explanation predominantly depends on the quality of examples selected, and the accuracy of the language used. I will mainly focus on how Rn, and Rt codes from the acts of explaining influence the quality of explanation codes. It can be concluded from what is dominant across both explanation and acts of explaining that the Rv and Rf is predominantly re-voicing and finishing of teacher's sentences pertaining to process at Level 1 (P1). I consider below in Figure 8:4 Teacher A's use of both explanation and acts of explaining codes.

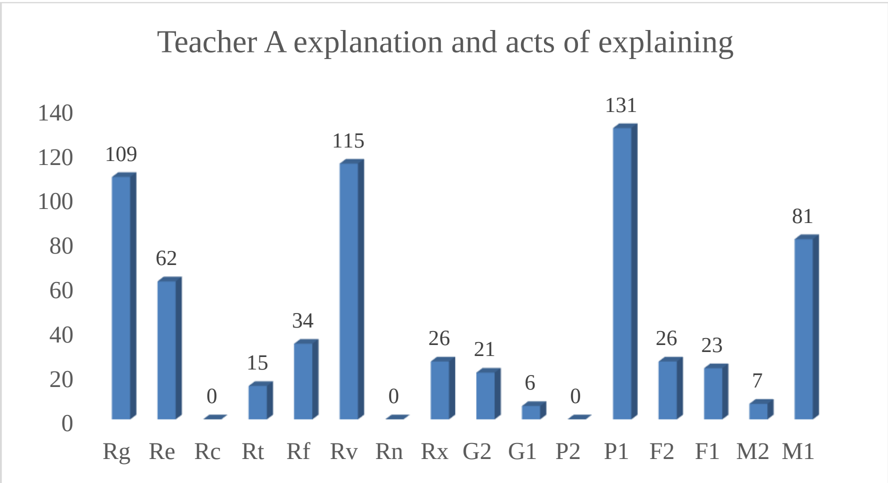


Figure 8:4 – Bar graph for Teacher A

There are three codes with a zero, namely Rc (chorusing), Rn (language) and P2 (derived steps of a process). The Rn code, if it received deliberate focus, would have made the presence of P2 codes possible. For instance, an inquiry could have been made as to what is the distributive property, and its meaning. In this way, not only the meaning of words but their mathematical properties, the form, and the process would have been interrogated. This is a possible use of Rn codes that goes beyond the M2 (definition of element) code, which is concerned with what names are given to elements. While having argued above that the presence of a textbook affected the Rn code, the presence of the Rt (gestures) code, even though minimal, has also impacted on the absence of the Rn code. Figure 8:5 below show the distribution of occurrences for explanation and acts of explaining codes for Teacher B.

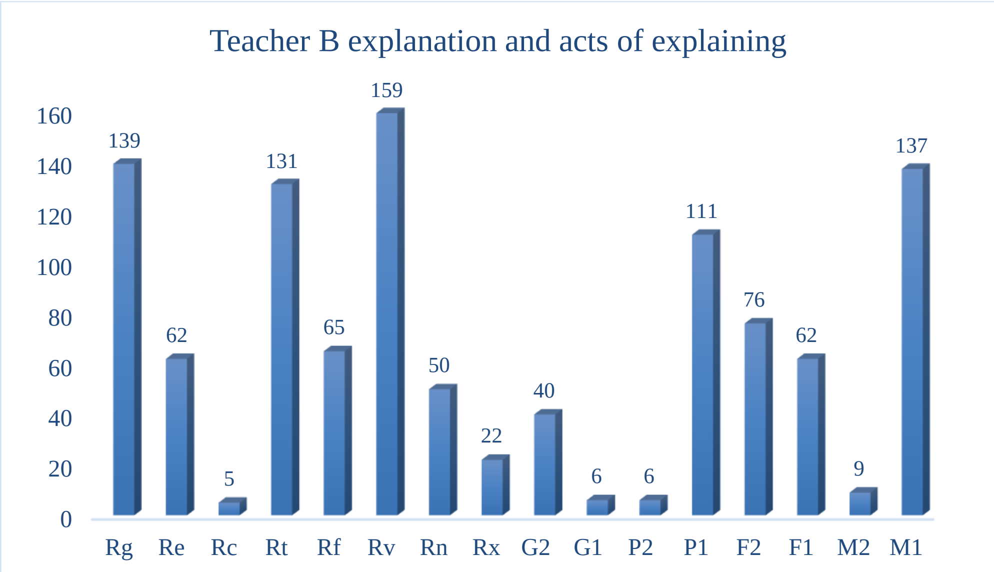


Figure 8:5 – Bar graph for Teacher B

While there is a presence of each code for Teacher B, Rt codes were extensive and frequent, and thus limiting the naming and proper use of language for F codes.

In view of the lessons I observed, it is possible to conclude that when the aim was drill practice of a routine, then P1 will be most dominant, which is appropriate. However, Teacher B's L3 shows that task design, and specifically the use of geometric representations, allows for justified rules/steps (P2) to be part of the explanation, which directly impacts and connects with Rx (examples and their representations) codes. As argued earlier, once explanation codes at P2 level are incorporated, then that means there will be more components of level 2 explanation possibly at F2 and G2. For instance, when Teacher B was connecting the arrows to the area model in teaching the distributive property, the structure of the entire expression is studied, and there are reasons given for why it is possible to connect the two multiplication methods, even though imprecise.

There are important criteria of explanation and acts of explaining which, when their relationship to each other is compared, illuminate the quality of the explanation. The discussion in this chapter has illuminated that the definition of the element (M2) depends on language (Rn), and the representation (Rx) used to express the algebraic expression. The recognition and identification of structure (F1, F2) in the algebraic expression is facilitated through naming and definitions (Rn) of terms and whole structure. The derivation of rules (P2), as already mentioned, is dependent on the use of mathematical language (Rn) to name the laws and illustrate through various representations (Rx) the underlying principles/concepts. Use of mathematical language (Rn) and representations (Rx) is important in providing justifications (G2) and in stating goals (G1) for the lesson.

Table 8-12: The explanation as the product and object of the acts of explaining

	A of E	Rn, Rx
Object Product	E	M2, F1, F2, P2, G1, G2
	A of E	Rt
	E	M1, P1

The foregoing discussion in this chapter has highlighted that frequent use of gestures (Rt) by the teacher are a disadvantage to language use (Rn), which affects definition of elements (M2), structure (F1, F2), process (P2), and goal (G1, G2). Gestures in themselves are not a problem, the problem is the use of deictic terms, as already shown by Roth (2001). Gestures as the act of explaining will be beneficial to the explanation if accompanied by correct naming of elements/objects (Rn) dealt with instead of ambiguous referents. In the table above, M1 and P1 are on the disadvantage side of the explanation, because they do not necessarily illuminate the quality of the explanation. I have argued in the foregoing discussion that the use of P1 is appropriate if the method has already been derived. The literature has already emphasised the need to bring sense into the symbols, and the disadvantages of talk about symbols as a meaningless string of incomprehensible things.

Finally, in Chapter 3, I noted that the explanation is both an object and a product of the acts of explaining. Table 8-12 illustrates this. The notion of the explanation being the object of the acts of explaining needs to be consciously and deliberately foregrounded. The ways in which this can be achieve is if teachers are in possession of the criteria of explanation. In this way, the criteria of explanation became the purpose for which the acts of explaining are carried out.

Since the explanation is the product of the acts of explaining, it is limited to what the acts of explaining were able to achieve. The aim is for the explanation to be the object of the acts of explaining, from the planning process and in the implementation, so that the product that is obtained at the end is good quality explanations. Again, this can only be achieved if teachers are in possession of the criteria for explanation.

8.4 Conclusion

The research question I was seeking to answer at the beginning of this chapter was: what constitutive criteria of explanation and acts of explaining are privileged in teachers' practices?

I compared both teachers' practices using the eight criteria of explanation and the eight criteria of acts of explaining. The discussion in this chapter showed through frequency counts that the privileged criteria for explanation are reading a string of symbols (M1) and steps of a method (P1) for both teachers. The discussion also showed that there was a lack of emphasis on level two codes in the explanation for both teachers. At the same time, level two codes did appear, but infrequently.

For acts of explaining, the discussion showed that examples and their representations are acts of explaining that are predominantly pre-planned and observable in lesson delivery. On the other hand, lesson delivery depends on the language that is in use. What the discussion has also shown in this chapter is that modes of communicating the explanation are norms and regulations that reflect the context and its culture. Thus, the most dominant criteria in the acts of explaining were, in the main, re-voicing (Rv) and regulative talk (Rg). Also highlighted in the discussion for this chapter is that there was a lack in the use of specialised mathematical language, which was mainly replaced by the use of gestures and ambiguous referents, such as "this", "that".

The conceptual framework enabled me to draw a picture of teachers' explanation and acts of explaining, and thus notice what the privileged criteria were. I discussed the manner in which some of the explanation and acts of explaining criteria are interrelated, and influence each other positively and negatively to produce quality explanations. I suggested from the evidence I obtained from the analysis that learner poor performances are due in part to the level (M1 and P1). and thus the quality of explanations offered in the classroom.

In Chapter 6, there were criteria transmitted by the course for explanation and acts of explaining. In the next chapter, I compare what was privileged as criteria for explanation and acts of explaining in the teachers' practices with what was privileged in the course.

CHAPTER 9 WMCS PD AND THE TEACHERS

9.1 Introduction

In the preceding two chapters I presented the analysis of the classroom data and what came to be the privileged criteria for explanation and acts of explaining in the two teachers' practices. The framework illuminated the privileged criteria by considering the total number of utterances for each code in the explanation and acts of explaining. From this, I saw that the reading of a string of symbols (M1) and the steps of a process (P1) were the most privileged criteria of explanation and the most previledged criteria for acts of explaining was re-voicing (Rv) and regulation (Rg). There were similarities and differences in the teachers' explanation and acts of explaining. The comparison of both teachers' practices also illuminated the relationship between explanation and acts of explaining. Which was that the explanation is the object, as well as the product of the acts of explaining. In this chapter, I compare what teachers privileged to that which was privileged in the WMCS PD course as criteria for explanation and acts of explaining. Specifically, I answer research Question 3.2:

3.2 In what ways are teachers' privileging practices similar and or different from what was privileged as explanation and acts of explaining in the WMCS course?

The aim in the beginning of the study was to understand teacher take up from a PD programme by looking at what are explanations in the WMCS PD and in the teachers' practices. As already noted, teacher take up as the focus of the study has been backgrounded for a focus that sought to develop the criteria for explanation and acts of explaining. The interest is on what is illuminated through the conceptual framework in the PD and in teachers' practices, and what this means for research and practice. In other words, the purpose of doing the comparison is more to see what the criteria highlight as propositions for research and practice, instead of assessing PD impact.

In my investigation of the WMCS PD course, I analysed what was stated, elaborated and exemplified as criteria for explanation and acts of explaining from the PowerPoint slides and course handouts. Two kinds of explanations were stated in the MTF framework of the WMCS PD course, namely *what* and *how* explanations, both of which could have a *why* and or *when* sub-explanation accompanying them. I revisit and discuss them now in relation to the teachers' privileged practices. As shown before, both teachers privileged, in the main, the explanation of matter at M1 level and process at P1 level.

In Chapter 6, I found that the WMCS PD course transmitted explicit criteria of what in this study I call acts of explaining; and those were discussed as ways of making mathematics available to learners, such as selection and use of examples, representations and mathematical language. From the teachers' practices, examples and their representations as well as language are acts of explaining that are predominantly pre-planned and executed in the delivery of the lesson. Modes of communicating (re-voicing, gestures, finishing sentence, chorusing, evaluating) the knowledge were discussed in the previous chapter as socio-mathematical norms that reflect the context and its culture. The regulation of behaviour and classroom control was also part of the acts of explaining. The dominant criteria in the acts of explaining were, for the most part, re-voicing and regulative talk.

As noted earlier the course, as part of exemplification of explanation, presented the distributive property. Both teachers taught the distributive property. I thus compare the ways in which the course presented the notion of distributive property and how the two teachers taught and communicated the same topic.

The difference between chapter 6, 7 and 8 is that in Chapter 6, I did not develop a coding system to analyse the constitution of criteria for explanation and acts of explaining in the course; instead I presented what was stated, elaborated and exemplified in the documents and connected these with Ruben's criteria and Shulman's knowledge categories. In Chapter 7 and 8, a coding system was developed because of the nature of the data. This difference in the type of data and analysis carried out across the course and the teachers' practices was the main difference accounting for the additional codes in the acts of explaining namely the modes of communication (Rv, Rf, Rt, Re, Rc) and the regulative talk (Rg), as these are only possible to discern through discursive data rather than document analysis.

To do the comparison, I first looked in terms of explanation and secondly in terms of acts of explaining in the WMCS PD and in the teachers' practices. I also related the criteria from the course with Ruben's criteria, which I used to analyse teachers' practices. I then discuss, briefly, the consequences of having a PD programme focused on explanation and acts of explaining for teacher learning. I begin with a comparison of teachers' explanations with what was transmitted as explanation in the WMCS PD course.

9.2. Criteria for explanation in WMCS PD and in teachers' practices

From the WMCS PD course, there were explicit statements of what constitutes an explanation. Specifically, the WMCS PD course transmitted that there are two types of explanation, namely the *what* explanation and the *how* explanation; both with possibilities of a *why* and *when* justification kinds of sub-explanations. It is necessary to consider the meaning of these words in order to understand their application as criteria for an explanation. I have already noted in Chapters 3 and 6 that the words: *what, how*, *why* and *when* are interrogative particles within the context of language.

The word *what* is investigative in nature and is used interrogatively to determine the identity, facts and qualities of the object in question. On the other hand, the word *how* is also used interrogatively to determine the means by which something is accomplished, the method, and the procedure. At the same time, the word *what* reveals both the identity and the qualities as well as the process which is a *how*, and hence the close relationship between procedures and concepts, which has already been alluded to. I have already shown how Kieran (2013) critiques the dichotomising between concepts and processes.

The *why* question is also an interrogative word seeking the reason, purpose, justification, or motive for the statement, or process. Now considering its use in the WMCS PD course, the *why* question was an attachment explanation in relation to the *what* explanation interrogating the fact or quality of the object in question. In relation to the *how* explanation, the *why* attachment explanation interrogates the means used to accomplish the procedure or method, specifically the derivation of the process, where the process comes from, and why it works.

The *when* sub-explanation is likewise an interrogative particle concerning the time at which something occurs or will occur, in which case referring to the future. In the case of the WMCS PD course, it is with regards to *when* the *what* and the *how* of the object in question will be used. This requires knowing the structure, and the substantive and syntactic organisation of knowledge in the discipline of mathematics as earlier argued by Shulman (1986). The course presenter had clarified in the follow up supporting interview that the *when* attachment explanation did not receive attention. Nevertheless, in the supporting interview the course presenter elaborated that the *when* was concerned with appropriate application of procedures, and so related to a *how* explanation. To repeat and remind, I bring in the course presenter's utterance below:

14 *P: I mean the "when" was also like, so it was kinda like, when you use this, that's what it was related to. I mean, something that we haven't, we didn't foreground much in the course if I remember, and we certainly haven't foregrounded it since.*

The *when* sub-explanation, even though stated, was neither elaborated nor exemplified for both the *what* and *how* explanations. Whereas the *why* attachment explanation was elaborated and exemplified for both the *what* and *how* explanations using the distributive law.

Within the context of a study focused on explanation, the interrogative particles are epistemic because their function is to inquire about the knowledge to be included in the explanation. Shulman, discussed in Chapter 2, presented types of teacher knowledge from which I chose, driven by the focus of the study, to focus on SMK and PCK. Briefly, SMK was stated as more than just knowledge of the facts and concepts of the discipline but knowledge of how the facts and concepts are organised as well as knowledge of what are accepted truths, their definitions, and the syntax of the discipline. From the WMCS PD point of view, knowledge of facts and concepts is the *what* explanation. Seen from Shulman's perspective both the conceptual and procedural knowledge is knowledge of the subject matter. I begin the comparison of the WMCS PD course and teachers with the *what* explanation.

9.2.1. The *what* explanation

The discussion that took place in chapters 7 and 8, revealing that both teachers attended to the *what* explanation by focusing on M1 and M2, F1 and F2 as well as G1. Both teachers enumerated the elements that compose an algebraic expression and defined the terminology of some of the elements in the algebraic expression. Both teachers discussed the structure of the entire expression and the structure of individual terms within the expression. In the teachers' practices, there was a statement of the goal for the lesson. This was an explanation at the level of the *what*. In Chapter 6, criteria for explanation were obtained from what was stated in the Maths Teaching Framework (MTF), elaborated and exemplified in the PPT slides and handouts. Table 9-1 is the row of findings for the *what* explanation taken from the summary table in Chapter 6.

Table 9-1: The what explanation

Stated	Elaborated	Exemplified	Ruben	PCK/ SMK
What	Definition, big idea, concept, notation	Distributive law, big idea	Matter and Form	SMK

I am studying the course and the teachers not in themselves, but in relation to building the theoretical clarification of explanation not for comparison in itself nor for take up. Thus, I make use of the elaboration slide from the course in order to discuss the *what* explanation, which I then connected with Ruben's criteria. Figure 9.1 below is a PPT slide showing the elaboration of both the *how* and *what* explanation.

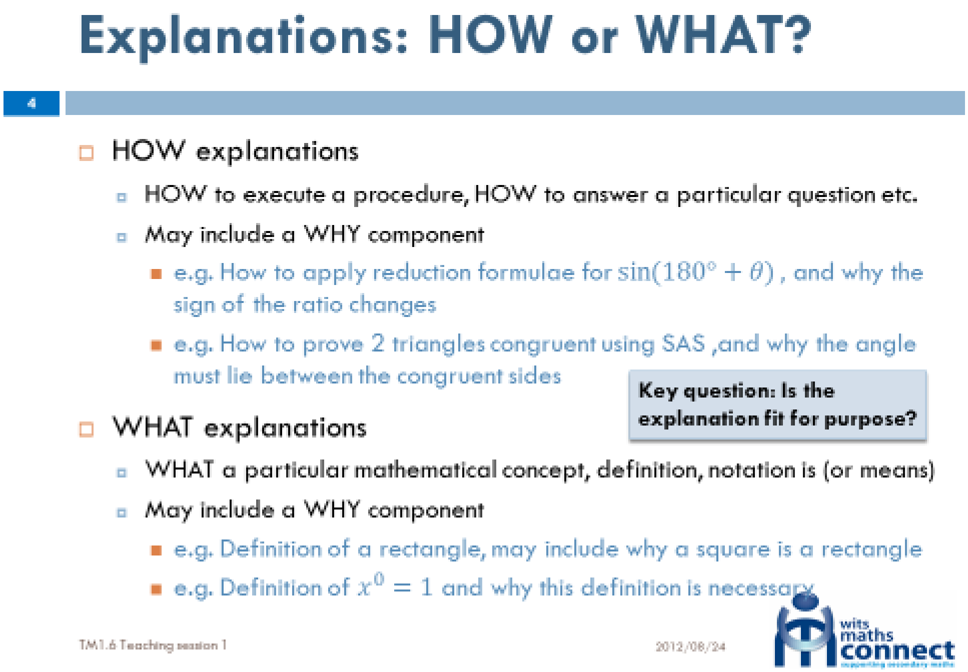


Figure 9:1 – Elaboration of *what* and *how* explanations

The *what* explanation as presented in the course intended to give information such as: definitions (M2, F2), explanation of notation (M1 and M2), and so it is essentially an explanation of constitutive elements (M1, M2) and structure (F1, F2), because a definition of a concept includes structure.

The elaboration of a *what* explanation provided in the course PPT slide emphasises definitions. The specific examples provided in the PPT slide are definition of a rectangle and $x^0 = 1$. The definition of a rectangle is a *what* explanation at the level of form, specifically F2 because for learners to know that a square is a rectangle they have to consider the structure of a square and rectangle and compare the constitutive elements in both. And so, the definition of a rectangle is about structure of the entire figure and thus the F2 code. The definition $x^0 = 1$ is derived from laws of exponents concerned with division of exponents. The definition is derived from the principle that $\frac{x^a}{x^b}$, where $x \neq 0, a$ & $b \neq 0$, by the law for diving exponents is x^{a-b}, when $a = b$ then the numerator and the denominator are the same value and so the quotient is 1. From this, we see that while the PPT slide puts emphasis on definitions, these are not necessarily definitions of elements as I have used M2; but they are mathematical principles, which have to do with structure and process and thus definitions of structure (F1, F2) and processes (P1, P2).

In the teachers' practices the M1 code highlighted the nature of the talk that dominates in the classroom. The talk that dominated in the teachers' practices comprised of reading the string of symbols. On the other hand, the M2 code, which is the definition of the element, was present in both teachers' practices. In the PPT slide, the definition is concerning a law derived from exponents. What this means is that the notion of definitions is not limited to the elements, but there are definitions of laws and principles, where the elements already take a particular form as shown in $x^0 = 1$.

There are two ways in which the notion of goal was defined. The first was defined to mean the goal of the lesson as stated by the teacher (G1), and the second was referring to justification (G2) – I discuss this under the *why* explanation. For the *what* discussion, I limit myself to the discussion of G1. In other words, the objective that the lesson was seeking to achieve. The goal for the lesson in Teacher B was stated and written on the board as "multiplication of a monomial and a binomial" after Sg 2.4. In Teacher A's class, the goal was stated and written on the board at the beginning of the lesson as "multiplication of expressions". See Figure 9:2 below. The course in the MTF emphasised the idea of stating the object of learning and bringing it into focus for learners. The notion of object of learning is related to the notion of goal, because the aim is to use examples and representations, which collectively accumulate to address the object of learning or the goal of the lesson. However, literature that uses the concept of object of learning has a broader conceptualisation than the goal for the lesson (Moalosi, 2014). In connection with determining the goal of the lesson, Sfard and Linchevski (1994) have argued that the structural perception should be the ultimate goal in the teaching and learning of school algebra.

G1 for Teacher A L3 G1 for Teacher B L3

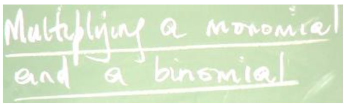


Figure 9:2 – G1 L3 for Teacher A and Teacher B

The summary Table 9-2 is a combination of the ways in which the codes derived from Ruben's framework have been defined and particularised with the *what* explanation criterion as transmited by the course.

Table 9-2: Summary table for the what explanation

	Stated	**Elaborated**	**Exemplified**	**Ruben**	**PCK/ SMK**
Course content	What	Definition, big idea, concept, notation	Distributive law, big idea	Matter and Form	SMK
Codes	**Description of codes**			**Ruben**	
M1	String of symbols			Matter	
M2	Variable, coeffient, constant, brackets, exponent, multiplication sign			Matter	
F1	Seeing terms, adding terms, simple term, product term, complex term			Form	
F2	Expression, monomial, binomial, trinomial, polynomials, square, rectangle Symbolic structure, geometric structure, structure in operational activity			Form	
G1	Statement of the goal for the lesson			Goal	

The notion of notation refers to the definition of elements, the talk focused on individual elements, their arrangement, and the mathematical conventions. Teachers offered explanations of exponential notation in the expression. As shown in the previous two chapters there was a study of the powers of the algebraic expression. The talk about notation includes elements (M1) and definitions of elements (M2).

There is a relationship between M2 codes and F1 codes in the sense that the definition of the element highlights the relational aspect of the element to other elements and in a way moves towards understanding of structure of a term, depending on the nature of the polynomial. I have discussed in the previous chapter by bringing in the debates from the mathematics education literature in connection with conceptual/procedural; structural/operational; relational/instrumental understanding. I have argued how it is important to consider the purpose of the activity. If the purpose is to teach a definition, then the *what* is foregrounded even if some operational activity is used in the development of the definition. Ruben's criteria helped me to illuminate the meaning of the *what* explanation. I now discuss the *how* explanation.

9.2.2 The How explanation

Talk focused on the operational activity was clearly the most dominant component of the explanation for each teacher at P1 level. The explanation was focused on steps of a method/short-cut and the sequence in which they should be executed and how to judge the final answer in order to stop operational activity. This in the language of the WMCS PD course was the *how* explanation, which in the language of the current was the P1 and P2 explanatory talk. In the PPT slide shown above, the *how* explanation is described as knowing 'how to execute' and the specific examples that are given is knowing 'how to prove two triangles congruent' and why; also 'how to apply reduction formulae'. The specific examples provided in the elaboration slide are geometric and in trigonometry. Nevertheless, they highlight that the "how" explanation refers to application of procedures such as the distributive law in algebra.

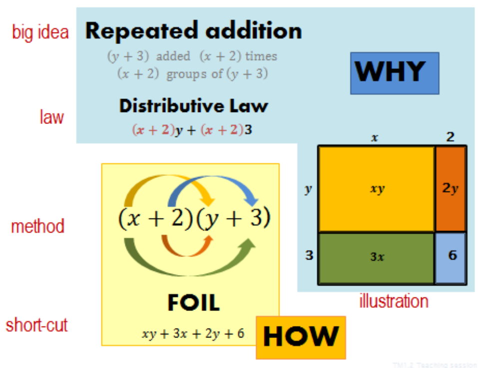
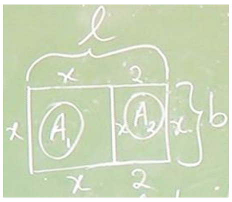
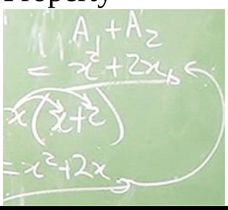
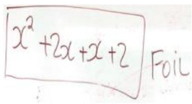
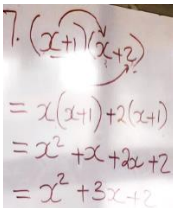
In the WMCS PD course, the distributive property was one of the topics that were used to exemplify specifically the *how* explanation accompanied by a *why* sub-explanation. Both teachers taught the distributive property in L3. As noted already, there was no specific exemplification of the *what* and *why*. Then again, the WMCS PD course presenter had already acknowledged, in the follow up interview, that the distinction between *what* and *how* '*slides around*' (IT18), and so was not continued. The course presenter was admitting the swaying, unclear boundaries between *what* and *how* and in fact indirectly acknowledging that the distinction between concepts and procedures is not in opposition. I bring in the summary table, Table 9-3 here, from Chapter 6, which illustrates the findings of criteria for the *how* explanation.

Table 9-3: The “how” explanation

Stated	Elaborated	Exemplified	Ruben	PCK/ SMK
How	How to execute a procedure	Illustration, method, short cut, big idea	Process	SMK
Why	Justification	Big idea, and illustration	Goal, Process	SMK
When	Overgeneralisation and misapplication of rules	----	Process	SMK & PCK

I show the approaches used by both teachers in Lesson 3 for the distributive property and contrast them to what the WMCS PD course exemplified as a *how* explanation. The method and the mnemonic short cut called FOIL were presented in the PPT slide as *how* explanations and these were accompanied by two versions of a *why* sub-explanation for the distributive property. I discuss both versions of a *why* sub-explanation because they were pertaining to the *how* explanation. The first was the notion of repeated addition as the ‘big idea’ behind the distributive property. The second was the area model as an illustration for the distributive property, see the first column of Table 9-4 below.

Table 9-4: Distributive property in WMCS PD and in teacher’s practices

<table>
<tr><th>WMCS PD Course</th><th>Teacher B</th><th>Teacher A</th></tr>
<tr>
<td>
</td>
<td>Area model

Connect to Distributive Property
</td>
<td>FOIL

Distributive Property
</td>
</tr>
</table>

As already noted in Chapter 4, the distributive law is one of the properties of number that relate to the operations of addition and multiplication. Other properties of number which involve the addition and multiplication are associative and commutative properties. These properties of number are a foundation of algebra and are what permits learners to do algebra and they are the underlying principle to the distributive law.

The distributive property in particular is one of the basic properties of real numbers, which is important to understand. The distributive property is not a major concern within the context of arithmetic, but is a serious issue within the context of algebra, where variables $(x + y)$ are used instead of number $(2 + 3)$. The word distribute means to allocate, assign, allot, ration, and hand out in English. The distributive law, unlike associative and commutative properties of number, is the only property which combines two binary operations of multiplication and addition. Distributive law is used to expand expressions $p(x + 2 + y + z) = px + 2p + yp + zp$, and also to combine terms that have a common factor; $(x^2 + 1)p + (x^2 + 1)q = (x^2 + 1)(p + q)$. The significance this has for the study is that the concept of the distributive property is an example of the nature of the relationship between concepts and procedures.

There are two principles behind the teaching of the distributive property that were highlighted in the WMCS PD course as *why* sub-explanations. The first is what the WMCS PD called the 'big idea', namely the repeated addition and the second is the geometric illustration for multiplication of expressions. The FOIL method is a short cut trick for expanding algebraic expressions, which have binomials as factors. However, the underlying principle is the distributive property ,which learners are supposed to know to avoid over reliance on a short cut method which does not generalise. The geometric illustration and the notion of repeated addition are essentially two multiplication structures. Below, I discuss briefly the big idea from which the method and short cut are derived as well as the geometric representation for the distributive property.

9.2.2.1 The big idea - repeated addition

Repeated addition is the process of adding the same quantity over and over in order to find the answer to a multiplication problem. This is not to say multiplication is a sub-category of addition. The inverse operation for multiplication and addition are different, just as the identity elements for both addition and multiplication are different. Moreover, the notion of multiplication as repeated addition works in the case of natural numbers, whereas for fractions the notion of multiplication is considered to be scaling. In repeated addition multiplication is here used to combine equal groups. For example, 3×5 will be $5 + 5 + 5$, which gives a sum of 15. This method still works if we decompose 5 into 2 and 3, in which case $3(2 + 3)$ will be $(2 + 3) + (2 + 3) + (2 + 3)$, which by the commutative law of addition gives $2 + 2 + 2 + 3 + 3 + 3$, and the addition of like terms yields $6 + 9$, which is 15. Learners should be able to observe that they can either add 2 and 3 to have 5, which is then multiplied by 3 or multiply each numeral inside the bracket by 3 to get the answer. For instance $3 \times 2 + 3 \times 3$ will give $6 + 9$ which is 15. When letters are used instead of numerals, because the distributive property of number is generalisable for real numbers $a(x + y)$, then by repeated addition the sum of $x + y$ must be added a times: $a(x + y) = \underbrace{(x + y) + (x + y) \ldots + (x + y)}_{a \text{ times}}$.

Then group like terms by the commutative law of addition to obtain the following result: $a(x + y) = \underbrace{x + x + \cdots + x}_{a \text{ times}} + \underbrace{y + y + \cdots + y}_{a \text{ times}}$, where x and y are each added a times.

Learners at this point should be able to make the abstraction that $\underbrace{x + x + \cdots + x}_{a \text{ times}}$ is the same as $a \times x$, similarly with $\underbrace{y + y + \cdots + y}_{a \text{ times}}$ is the same as $a \times y$. The multiplication of a is distributed over the addition of x and y. This is the *why* sub-explanation to the *how* explanation, which was presented by the WMCS PD project as the "big idea" behind the distributive property. See Figure 9:3 below, which is an extraction from the overhead clip.

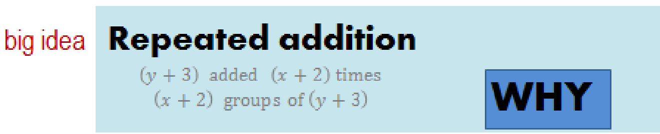


Figure 9:3 – Big idea for distributive property

The distributive property is a reliable principle for the multiplication of algebraic expressions such as: $(y + 3)(x + 2) = (y + 3)x + (y + 3)2$, or even $(a + b)(w + x + y) = (a + b)w + (a + b)x + (a + b)y$. The last expression is a multiplication of a binomial and a trinomial, and so to find the product of a trinomial and a binomial, the short cut method (FOIL) presented in the course PPT slide and by Teacher A would not work, because its usefulness is restricted to finding the product of two binomials. The geometric illustration was also presented by the WMCS as a *why* sub-explanation to the distributive property.

9.2.2.2 The geometric illustration/ the area model of multiplication

An alternative *why* sub-explanation of the *how* explanation based on the distributive property, presented by the WMCS PD was the geometric illustration. The geometric illustration is the use of rectangles to demonstrate the idea of multiplication by finding the area of the rectangle given the length of sides. The two binomials which are multiplied $(x + 2)(y + 3)$ each element in each binomial $x, 2, y$ and 3 takes up a certain amount of length as shown in the geometric illustration used by the WMCS PD course in Figure 9:4 below.

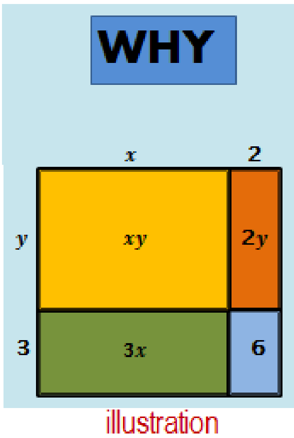


Figure 9:4 – Rectangular array for distributive property from WMCS PD

The definition for the area of a rectangle highlights a multiplication relationship between length and breadth of a rectangle: $A = l \times b$. The multiplication of two line segments is taken as the area of the rectangle they bound. The length and the breadth represent the factors: $(x + 2)\ (y + 3)$, while the area represents the product: $xy + 3x + 2y + 6$. In this specific example, the area model is a visual aid for multiplication, where the product of two numbers $(x + 2)$ and $(y + 3)$ are represented as the rectangular region made up of the four colour coded rectangles. So, drawing from the use of multiple representations to explain multiplication assists learners in making sense of the concepts factors, multiplication, and the distributive law. Future topics in mathematics, such as integration rely on the concept of finding area for polygons. However, the area model as a multiplication structure has its own limitations when it comes to the multiplication of negative numbers. I noted in the historic account of algebraic notions that the advancement in understanding of algebra and number was constrained by the geometric expressions algebra once took.

Teacher B used the notion of rectangles to teach the multiplication of expressions. In the beginning part of L3 we see Teacher B highlighting that the topic of area of rectangles was taught independently in Grade 8. What she is now in L3 seeking to do is to bring these two by demonstrating the relationship that exists between algebra and geometry. This is shown in the transcript excerpt taken from Teacher B L3 Sg 2.1.

11	*Rg Rn*	*T:*	*Okay so you just draw it* [referring to the figure on the board - teacher walks around while learners draw the figure]. *Then from your grade 8 work in geometry I want you to tell me what is the name given to that figure?* [Pause]
			Teacher B L3Sg2.1T11

Teacher B presented the geometric illustration as a *why* explanation for the method. With the first three example the process of finding area is coded P1, the P1 code changes to P2 when Teacher B connects the process of finding area to the process of multiplying using arrows.

Sg 2.1: Sg 2.2 Sg 2.3 Sg 2.4

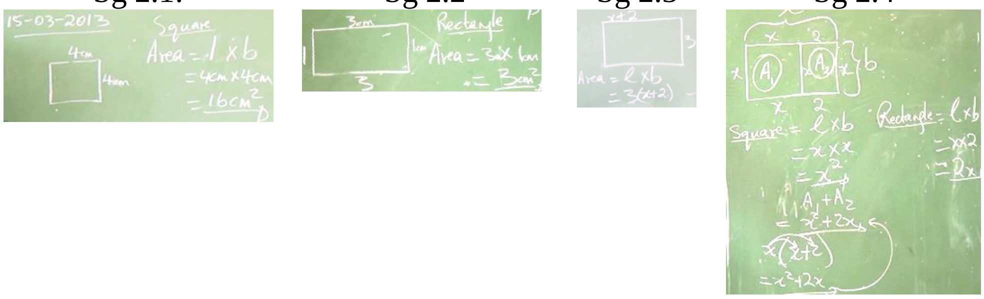


Figure 9:5 – First four examples from Teacher B L3

From Figure 9:5 above, we see that numerals are first used to demonstrate the idea of a four by four square and three by one rectangle in Sg 2.1 and 2.2. Here, Teacher B does not then go on to illustrate the individual one by one squares that make up the 16 cm^2 and 3cm^2. Had she taken this approach with the first or even second example, the notion of multiplication as repeated addition would have emerged, because for learners to find the area of the first two rectangles they would have had to count and add the individual one by one squares, as shown in Figure 9:6 below.

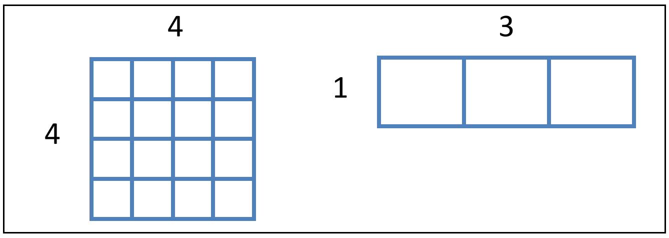


Figure 9:6 – Extension of Sg 2.1 and 2.2

In segment 2.4 of L3, Teacher B was now confronted with the question: why the idea of finding area for a rectangle is multiplication? This is a question she could have addressed in Sg 2.1 and 2.2. Teacher B's attempt in answering this question is seen in Sg 2.4, where she was met with a challenge to connect the method of rectangles to the procedural activity of applying the distributive property. Teacher B provides a justification for why they are now going to abandon the area model and go on to the distributive property. She does this by demonstrating that the steps for multiplying using arrows are validated by the fact that the two processes, rectangles and arrows, provide the same answer. Thus it is sensible to continue with the "shorter" method of arrows since the area model is a "longer" process of multiplying expressions. This is found in the beginning part of speaking Turn 42 from Teacher B L3 Sg 2.4.

42	*P1 F2 G2*	*Rg*	*T:*	*That's my area. Right! But now when we, let's say we go back now. We want to multiply, but we don't want to multiply because the area, might be long when we use the area. So we want to multiply a monomial and a binomial…*

Teacher B L2Sg2.4T42

From the above we can appreciate that even though Teacher B might have appropriated the approach for teaching multiplication of algebraic expressions from the WMCS PD project, however, at the level of what to say in the explanation she is experiencing a challenge.

In Teacher A's class, a discussion of the short cut method for distributive law called FOIL was discussed, and the distributive law was presented by the teacher as the law to be used for multiplication of two binomials. From this, it appears that Teacher B complied with what the WMCS PD course presented as a *why* sub-explanation for the distributive law, specifically the appeal to rectangles to illustrate the distributive law, even though not a key idea. Table 9-5 summarises this.

Table 9-5: Summary for the *how* explanation

	Stated	**Elaborated**	**Exemplified**	**Ruben**	**PCK/ SMK**
	How	How to execute a procedure	Illustration, method, short cut, big idea	Process	SMK
Code	**Description**			**Ruben**	**SMK**
P1	Steps of a process			Process	SMK
P2	Derived steps				

What follows is a discussion of the sub-explanations in the WMCS PD course, and in the teachers' practices.

9.2.3 The *why* and *when* sub-explanations

The second way in which the notion of goal was defined was that it is the purpose and justification for the form and process. I used it to include the justification of matter at Level 2 and G1. What is the objective that the explanation was seeking to achieve for each criterion? This is to say the goal is embedded within the criteria of matter, form and process. In other words, there is a goal or purpose for each move. In the classroom data, I coded all the utterances which included the words: *why* and *because* as G2. The WMCS PD course explicitly presented an example of a why (G2) aspect of the explanation in relation to the *how* explanation, but nothing in relation to the *what* explanation. I have shown in the previous chapters that the *why* question when posed in relation to M was asking for the meaning of the word or notation, for example, when the teacher asks why is it called a variable, the question requires a reason for the name of the element. In relation to the P code, the why explanation was reasons for the steps performed.

The *when* sub-explanation has been invisible from the course and so not clear what exactly are its elaborations, and exemplifications beyond just stating. However, the meaning of *when* is referring to a time or a circumstance. In my discussion of M2 and the specific definitions for coefficient, constant and variable offered by the teachers, I noted a possible use of the when explanation, such as noting and stating the importance of understanding the concept and how it impacts future understanding. The *when* explanation was not only invisible in the WMCS PD course but in the teachers practices. Table 9-6 summarises this.

Table 9-6: Summary table for *why* and *when* explanation

Stated	**Elaborated**	**Exemplified**	**Ruben**	**PCK/ SMK**
Why	Justification	Big idea, and illustration	Goal, Process	SMK
When	Overgeneralisation and misapplication of rules	----	Process	SMK
Code	**Description**		**Ruben**	**SMK**
G2	Justification, the why question, answer to the why question		Goal	SMK

9.2.4 Summary points for explanation

To sum up the discussion on *what* and *how* explanations and their *why* and *when* sub-explanations, I first comment on the frequency counts and secondly on the false dichotomy as argued by Kieran (2013) and what this means for criteria of explanation.

9.2.4.1 A comment on the frequency counts

For the frequency counts, I include Table 9-7, which is a presentation of the frequency counts for the *what* and *how* explanations as well as the *why* sub-explanation.

Table 9-7: Frequency counts for the *what* and *how* explanations

Explanation	Teacher A	Teacher B
what = M1+M2+F1+F2+G1	81+7+23+26 = 137 + (-81) = **56+2**	137+9+62+76 = 284 + (-137) = **147+3**
how = P1 + P2	131 + 0 = 131	111 + 6 = 117
why = G2	21	40

Understood from the WMCS PD the M and F codes are a *what* explanation while the P code is the *how* explanation. The reading of a meaningless string of symbols was a way of talking about the elements in the teachers' lessons, but not included as a description of the *what* explanation in the WMCS PD course. So if we subtract the M1 counts in the *what* explanation for both teachers since it is of a lower quality to the *what* explanation, then we see that the *what* explanation has smaller counts than the *how* explanation for Teacher A. On the other hand, Teacher B still has more counts than the *how* even after the subtraction of the M1. The strength I suppose about the codes, specifically M1 in this case, is that they present what is than what should be in the classroom. In the WMCS PD course what was presented were necessarily criteria of what a good explanation should include, and what it should look like. However, in the coding of the data, the talk about elements was considered an explanation, and it didn't matter if the talk referred to elements as a meaningless string of symbols, or provided names and definitions; it was the way in which elements were spoken about. Viewed from this perspective, the *what* explanation reduces to 147 for Teacher B and 56 for Teacher A. The G1 code I can argue that it fits within the *what* explanation, because it is concerned with understanding and explaining what is to be learnt.

For the *how* explanation, both the P1 and P2 codes were emphasised in the course, by stating the method and its short cut and by providing the geometric illustration and the big idea. I have shown that from the two teachers, Teacher B practice was more aligned with the course approach to teaching the distributive law only by using the geometric illustration, while teacher A's practice was aligned by using FOIL method and the application of the distributive property, without necessarily highlighting the underlying principle of repeated addition or using the geometric illustration. Teacher A has more counts, 131, on the *how* explanation than Teacher B who has a total of 117.

The *why* sub-explanation matched up with the G2 code. Teacher A had a total of 21 while Teacher B had a total of 40. I now move on to comment on the relationship between the explanation criteria and link this up to the debates in the field about the false dichotomy between concept and procedures.

9.2.4.2 A comment on the false separation

To have a blurring of boundaries between criteria means there are boundaries, while to say the separation is false, means there should not be any separation. The course seems to have been faced with this challenge of defining the blurred boundaries and so abandoned the *what* and *how* separation as earlier noted by the course presenter. However, the classroom data shows the benefits of having the separation in order to judge what became privileged as criteria. I suppose the separation in the classroom data was possible, since the aim was to analyse and so the *what* and *how* or conceptual procedural are analytically separable. The opening discussion in this chapter on interrogative particles highlighted that the *what* evinces the *how*. It is no wonder then that there was a blurring of boundaries. The relationship between *what* and *how* explanations needs to be studied thoroughly before one can come to a conclusion to continue or discontinue the separation.

The fact that the WMCS PD course proposed in its statement that the *what* and *how* explanations will possibly have *why* and *when* sub-explanations; further highlight the relationship between the conceptual and procedural. To begin with, the *what* explanation is already at a conceptual level, because it is an explanation of the definitions and concepts. When accompanied by the why explanation which seeks for the reasons behind the concepts, the explanation is still conceptual, even though procedures might be used to elaborate the reasons. The *how* explanation can be conceptual because when the *why* component is considered for the *how*, then the concepts and their definitions are invoked in the derivation of the method.

The distributive property slide was meant to be an exemplification of a *how* explanation accompanied by a *why* sub-explanation. However, the *what* was accommodated, because the answer to what is the distributive property mentions the big idea of repeated addition and area of rectangles model. Table 9.8 below is an illustration of the conceptual procedural relationship as separate analytical categories.

Table 9-8: The conceptual/procedural separation

Algebra literature	**Conceptual**		**Procedural**
Explanations literature	Matter and Form	Goal	Process
The WMCS PD course	What	Why/when	How

I now move on to discuss and compare teachers' criteria for acts of explaining with those transmitted in the WMCS PD course.

9.3 Criteria for acts of explaining in WMCS PD and in teachers practice

As a reminder, the course did not make the distinction between explanation and acts of explaining but transmitted explicit criteria for what in my study I call acts of explaining. There were PPT slides, which I considered as statements of acts of explaining rather than the explanation itself, and PPT slides, which I considered to be elaborations and exemplifications of acts of explaining.

For acts of explaining, the WMCS PD course presented a slide asking “how is the mathematics made available to learners?” I classified the overhead clip as a statement of the acts of explaining. There were two ways in which the exemplification of acts of explaining occurred. The first was exemplifications as to how to produce an explanation which stipulated steps to consider and undertake in the preparation of an explanation. The second was a task aimed at critiquing a given explanation. Between the requirement to produce an explanation and critique an explanation, I chose to include the slide that offered criteria for how to produce an explanation because it offered criteria for acts of explaining. These were necessarily steps taken to produce the explanation. They concentrated on how to choose examples and their representations and what language to use in the explanation. In addition to these, the classroom data, because it was the actual teaching process captured, extended these criteria to include communicative strategies used in the act of explaining and maintenance of classroom order.

Figure 9.7 below is taken from Chapter 6, and it is what the course stated and transmitted as criteria for acts of explaining. The three questions posed as ways of making mathematics available to learners are what I have coded in the teachers practices as Rx examples and their representations, and Rn the use of mathematical language.

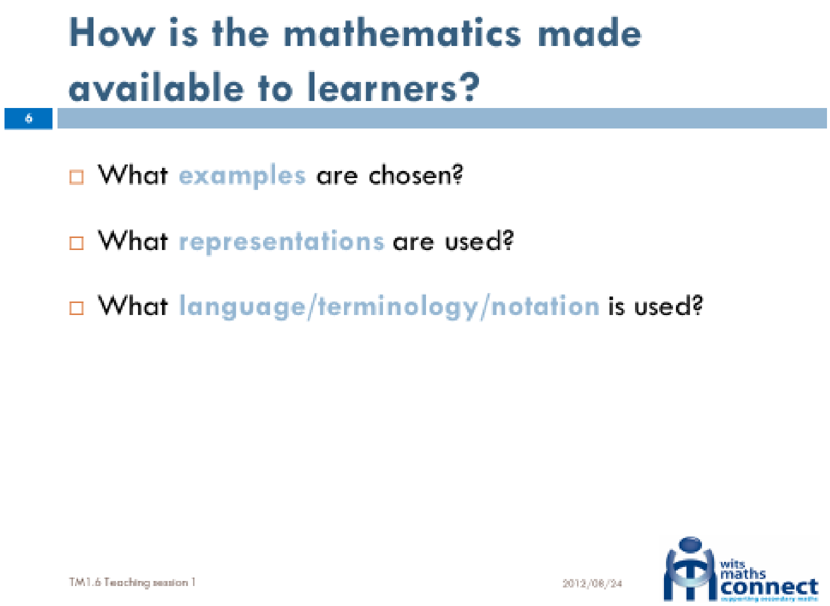


Figure 9:7 – Acts of explaining from WMCS PD course

Already the question in Figure 9:7 starts with an interrogative particle "how" which, as discussed earlier, seeks to know the methodology and process of making mathematics available to learners. The mathematics has been interrogated in terms of the what, how, why and when and now the issue in Figure 9:7 is about the acts of explaining, and so the interrogative particle "how" is at the level of the pedagogy instead of criteria for explanation.

To add on the three questions posed in Figure 9:7 above, the WMCS PD presented criteria for how to produce an explanation in a homework task that teachers were supposed to complete. I classified this as an elaboration of the process of producing a written explanation. Figure 9:8 shows the criteria from the homework task.

Teaching task - Producing a written explanation

In this task, you are required to produce a written explanation. Please do the task individually.

Assume you are required to explain the following to a Grade 10 learner:

Why is $\boldsymbol{tan\ 45° = 1}$?

Decide on the following before you write up your explanation:

- What is the most important aspect (or aspects) that learners must learn from the explanation?
- Where will you start?
- What sequence will you follow?
- What representations will you use?
- What important terminology will you use? Are there any words you will avoid?
- Will your explanation help learners to understand why $\boldsymbol{tan}(-\boldsymbol{45°}) \neq \boldsymbol{1}$ and other similar questions?

Figure 9:8 – Teaching task producing a written explanation

The criteria provided for teachers to consider as they produce the explanation are pedagogical and mathematical decisions at the level of concepts that learners must learn from the explanation. Pedagogical and mathematical decisions in terms of the examples to start with and proceed as well as their representations. The mathematical language to use and avoid using. The last post is about evaluation of the explanation if it was successful, and achieved its purpose.

I contrast the criteria provided in the WMCS PD for acts of explaining with what come through as acts of explaining from the teachers' practices. There were exemplification of what the

WMCS PD course means by examples, I also include these in the discussion of examples as an act of explaining.

9.3.1 Examples and their representations used (Rx)

The mathematics in both teachers' practice was made available through the use of examples. The examples for Teacher A were all symbolic representations and for Teacher B the examples used in the third lesson included geometric representations. See Table 9-9 below.

Table 9-9: Examples and their representations

Stated	**Elaborated**	**Exemplified**	**Ruben**	**SMK/PCK**
Examples	Where to start and what sequence to follow?	Choosing examples in Grade 9 Critiquing a set of examples	----	SMK and PCK
Representations	What representations will be used?	Suggest other representations	Matter and Form	SMK and PCK

The course presented examples in the PowerPoint presentation for the teaching of the distributive property. The two teachers reported in this study also presented examples of the concept of distributive property in L3, where the goal was multiplication of expressions. From the course there was a PPT slide, which transmitted criteria on how to produce an explanation, four of the points listed were related to examples. I use the four points pertaining to examples from the course to compare with the examples both teachers used in L3. I present in Table 9-10 below the course PPT slide pertaining to examples for the distributive property alongside the examples both teachers used in their third lesson for the distributive property.

Table 9-10: Examples in the WMCS PD and in teachers' practices

WMCS PD	Teacher A L3	Teacher B L3
□ 4x + (x + 2) □ 4x − (x + 2) □ (4x − 1) (x + 2) □ What is staying the same? Changing? □ Why might this be harder for learners?	- "multiply $(3x + 1)$ by 2" - $5 + 2(3x + 2)$ - $5 + 2x(3x + 2)$ - $5x + 3x(x + 1)$ - $5x - 3x(x + 1)$ - $2x(x + y) - 2y(3x - 5y)$ - $(x + 1)(x + 2)$ - $(x + 1)(3x - 2)$	- Square of unit length 4 - Rectangle $l = 3;\ b = 1$ - Rectangle $l = (x + 2);\ b = 3$ - Square + rectangle: Square $l = x$, Rectangle $l = x + 2$; $b = x$, - $2(2x + 3)$ - $2x(-x - 4)$

The first two examples from the WMCS PD for the distributive law have two terms made up of multiplicative units. The first multiplicative unit is the same for both examples and that is $4x$. The second multiplicative unit is $(x + 2)$ for the first and $-(x + 2)$ for the second. Teacher A started with an example showing distribution of multiplication of a single term into the addition of two terms. Teacher A gradually increased the level of difficulty in her examples by introducing more terms (simple and complex) into the expression and finally came to multiplication of two binomials. On the other hand, Teacher B started with a square of four centimetres in length and gradually increased the level of difficulty as she progressed through the examples by introducing variables and later abandoning the geometric representations to work with symbolic representations.

Teacher B remained at the multiplication of a monomial and a binomial,[14] i.e. $x(x + 2)$, while Teacher A progressed to multiplication of two binomials $(x + 1)(x + 2)$, as in the WMCS PD course $(4x - 1)(x + 2)$.

I now discuss the examples in terms of criteria transmitted by the course as to where to start, how to sequence examples, and representations to use. The most important aspect learners should learn from the explanation is multiplication of polynomials, specifically the distributive property. The starting example for both teachers is fairly simple, and there is a clear sequence and development of examples in L3 for both teachers. On the question of representations Teacher B used both algebraic and geometric representations. In terms of alignment in the

[14] This was the last lesson in the sequence of three lessons, and so Teacher B might have continued in the subsequent lessons to multiplication of binomial with a binomial and so forth to fulfil the curriculum requirements for Grade Nine.

selection and sequencing of examples, Teacher A has a better selection and sequencing of examples that aligns with the exemplification provided in the WMCS PD course. Both teachers seem to have appropriated differently from the course, Teacher A took up the notions of selection and sequencing better, while Teacher B took up the use of representations better. Table 9-11 summarises this.

Table 9-11: Summary table for examples and their representations

Stated	**Elaborated**	**Exemplified**	**Ruben**	**SMK/PCK**
Examples	Where to start and what sequence to follow?	Choosing examples in Grade Nine Critiquing a set of examples	----	SMK and PCK
Representations	What representations will be used?	Suggest other repersentations	Matter and Form	SMK and PCK
Code	**Description**		**Ruben**	**SMK PCK**
Rx	Examples and their representations		----	PCK

9.3.2 Language, terminology and notation used (Rn)

The language is the most pivotal and essential way of making mathematics available to learners. As Achinstein (1983) has already noted that explanation is a speech act, the examples have to be accompanied by talk and so the explanation depends on what is said because writing of mathematical notation, symbols and representation on its own is meaningless. It needs to be elaborated through language use. The WMCS PD course posed the following question in one of the bullets in Figure 6:12 of Chapter 6: "*What important terminology will you use? Are there any words you will avoid?*" Teachers used language to clarify notation and also used language to provide definitions for mathematical terminology (distributive property, FOIL, square, rectangle, variable, coefficient, constant, exponent, terms, like terms, monomial, binomial, trinomial, polynomial, ascending, descending, powers of x, algebraic expressions). More than just terminology and definitions of terms, the study has highlighted the importance of the actual words used in constructing sentences for the explanation of matter, form, process, and goals, especially the words used to refer to and talk about algebraic expressions. Table 9-12 is a summary of the aspect of language and terminology.

Table 9-12: Language and terminology

Stated	**Elaborated**	**Exemplified**	**Ruben**	**SMK/ PCK**
Language	Important terminology, Words to avoid	----	----	PCK
Code	**Description of code**		**Ruben**	**SMK/ PCK**
Rn	Language use and terminology		----	PCK

In the course handout there was elaboration of language used in the explanation, where a selection of words to use and avoid in the explanation was highlighted.

9.3.3 Communicative strategies (Rv, Rf, Rt, Rc, Re)

The use of communicative approaches by both teachers was another way of making the mathematics available to learners by re-voicing leaner responses and making them part of the knowledge transmission conversation. To make learners part of the conversation, the teachers often required the act of finishing the sentence from learners. The teachers used gestures and required learners to evaluate each other's ideas. Criteria concerning explanatory communicative strategies through which mathematics could be made available to learners is the kind of data that cannot be extracted from document data and a follow up interview from the WMCS PD course. This is a limitation of the study, and it is a point for consideration in future research to study the pedagogy of the PD programme and conclude what criteria for explanation and acts of explaining are modelled, and hence transmitted to teachers.

9.3.4 Regulative talk – Rg

An environment that will enable the making of mathematics available to learners needs to be created in classroom practice. This occurred through organised and systematically ordered sequencing of information and maintenance of order in the classroom. In the WMCS PD course, the Rg code is absent like the Rv, Rf, Rt, Rc, Re codes, because they are discursively observed, and cannot be derived from the document data. As already noted, this is a limitation of the study.

In summary, what is apparent is that ways of teaching the acts of explaining such as examples and language were explicit, while ways of communicating were probably inferred by teachers from observing the course presenter. Modes of communicating are context specific and depend on the culture of the school and so it would be a challenge to teach these to teachers. However, learner participation in the knowledge construction depends on the use of productive modes of communicating the knowledge. I now move to a discussion looking across both explanation and acts of explaining.

9.4 The four interrogative particles

I comment here on the use of the four interrogative particles beyond themselves as criteria for explanation. There are three ways in which these can be used. One refers to the ways in which they have been already used in the WMCS PD course as criteria for the explanation. Two refers to the ways in which the four interrogative particles can be used as guiding principles for the acts of explaining. Three is about the two interrogative particles the “what” and “how” being used to refer to explanation and acts of explaining, respectively. I will discuss here the two extensions of the ways in which the interrogative particles can be used to refer to both explanation and acts of explaining.

The “what” and “how” interrogative particles and their accompanying justification interrogative particles “why” and “when” can be used in the pedagogical sense to refer to the acts of explaining, and the pedagogical decisions on how to make the mathematics available. Meanwhile, in the WMCS PD, they were meant as interrogative particles investigating the SMK of algebraic expressions. The “what” can be asked for Rx, Rn, communicative modes and Rg, similarly the “how” can be asked for Rx, Rn, communicative modes and Rg. But this kind of interrogative investigation would occur, predominantly, at the level of planning the lesson.

The “what” and “how” can be seen to talk about the explanation and acts of explaining, respectively. As earlier defined in Chapter 3, the explanation is the knowledge conception and the acts of explaining are the methodology of enacting the knowledge conception, and so the “what” and “how” in the broadest sense. The above discussion on acts of explaining is, in view of this, essentially a “how” and the discussion on explanation is a “what”. Table 9.13 below illustrates this relationship of explanation and acts of explaining to the interrogative particles.

Table 9-13: Explanation and acts of explaining

Explanation (WHAT)		**Acts of explaining (HOW)**	
What	Elements, Form and Goal	What	Rx and representations, Rn
How	Process	How	Rv, Rt, Rf, Rc
Why	Justification	Why	Re
When		When	Rg

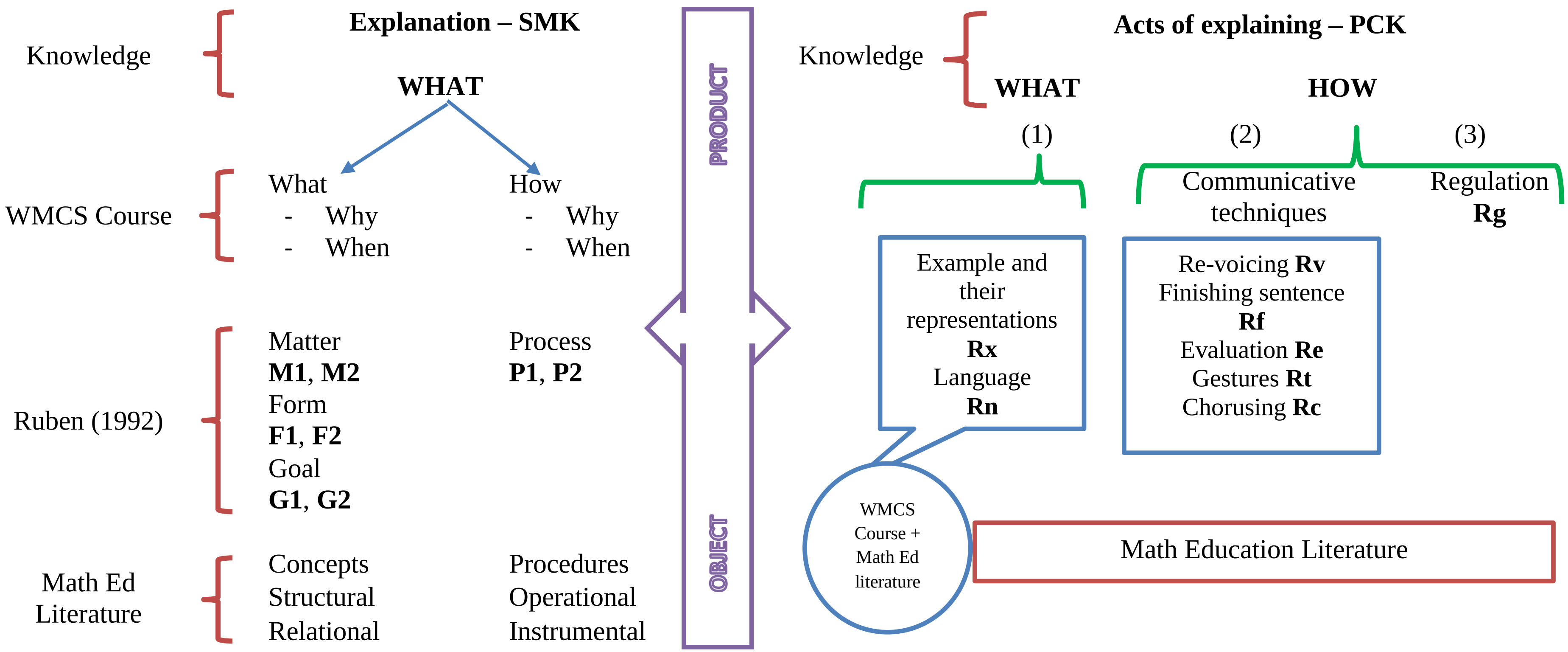


Figure 9:9 – Explanation and acts of exaplaining

Based on Figure 9:9 above, idealistically, the teacher decides on what actions to take based on what she is seeking to achieve in the explanation. The notion of the explanation as the object infers that the acts of explaining should gaze upon, and be guided by the explanation to constitute themselves (AoE). When viewed in this way, the role of explanation, as the object, is to govern what comes to be constituted as the acts of explaining. On the other hand, since the acts of explaining have the explanation as their object, the actions taken produce the explanation to indicate that the AoE were successful. The explanation is the product that the teacher produces in the classroom. However, in order to have the explanation as the product it (explanation) must first be the object of the teacher's acts of explaining.

9.5 Conclusion

In this chapter, the aim was to compare what was privileged as explanation and acts of explaining in teachers' practices and in the course. What emerges in the comparison is that there was some alignment of what was constituted as explanation and acts of explaining in the WMCS PD project and in the teachers' practices. For the explanation, Teacher B seems to have taken up the notion of a "what" explanations, specifically definitions more so than Teacher A.

Despite the fact that the WMCS PD course did not explicitly name the acts of explaining as such, explicit criteria with regards to acts of explaining at the level of pedagogical decisions were transmitted. These included choice and use of examples, representations and mathematical language including terminology and notation. From the two teachers practices I can conclude, that they (Teacher A and Teacher B) selectively took up from the WMCS PD course.

The account I have provided in this thesis thus far is actually an account of written (WMCS PD) and spoken (teachers') explanation, as well as written (WMCS PD) and enacted (teachers') acts of explaining. The criteria from Ruben and the course allowed me to see and describe:

1. teacher practices;
2. relationships between criteria for explanation; (concepts versus processes);
3. relationships between criteria for acts of explaining;
4. relationships between criteria for explantion and acts of explaining; and
5. take up.

In the next chapter, I provide the conclusion of the study.

CHAPTER 10 DISCUSSION AND CONCLUSIONS

10.1 Introduction

This study has added to the understanding of the notion of explanation and acts of explaining in mathematics education literature by introducing from philosophy first the separation between explanation and acts of explaining, and secondly the constitutive criteria for explanation which I successfully applied in the teachers' practices. Through the process of empirical application, the ideas of explanation and explaining have been extended. The notion of separating between explanation and acts of explaining is a new preoccupation, because explanation and acts of explaining are integrated in the mathematics education literature. In the current chapter, my undertaking is to track back and highlight critical points in the conceptual and empirical journey, and how I attained them. To reach this objective, I begin the current chapter with a short narrative of my initial ideas, and how these shifted. I describe and reflect on the moves, changes and decisions I made while developing the focus of the study. Typically, in the initial stages of the study, I was preoccupied with formulating questions, linking these to the literature, and developing the focus of the study as well as tools of analysis. The middle stages were dedicated to analysis in terms of how I then used the tool obtained from the literature. After sharing the narration of the initial and middle stages of my journey, I then proceed to record the story of what I found as a result of having been in possession of the tool by returning back to the research questions. Lastly, I briefly discuss the implications and limitations of the findings. I begin the story with a brief account of the first and critical part of my journey which was shifting the focus.

10.1.1 Shifting the focus – the first and critical part of my conceptual journey

I originally set out to study the notion of teacher take-up from a PD programme, and so guided by related literature, I initially located the study within the research practice of teacher professional development. Within this literature there are studies that looked at the ways in which teachers take-up from a PD programme. At the beginning, I chose to adopt the word take-up because it highlighted the notion of knowledge enacted in different contexts – PD and the classroom and that teachers selectively learn/ take-up from a PD according to their needs. The concepts of explanation as a teaching focus and algebra as a content focus were both initially intended as means to be used to study teachers' take-up from the WMCS PD course into their own classroom practices. However, fairly early on, I faced a substantial challenge. I could not describe what the criteria for explanation were to have taken up because the concept of explanation and its related criteria were unclear. I considered the education and mathematics education literature, because this was where I anticipated to find the constitutive criteria of explanation. I also looked into PD literature hoping that in the descriptions of the content of PD programmes there would be criteria for explanation. The descriptions in the education and mathematics education literature were mainly of actions taken in the process of explaining, for example, the notions of selecting and sequencing examples and their representations. There were ideas that were mentioned as part of the criteria for explanation such as connections and coherence, however these are not actions and neither are they criteria for explanation. Coherence and connections are ideas used to judge the acts of explaining as well as the explanation. The question of what is an explanation in general, its elements or constitutive criteria was unclear in this literature. The study of teachers' take-up of algebraic explanations from a PD required that the constitutive criteria of explanation from the literature be known. Since these were not clear from the education and mathematics education literature, it then became necessary to first find out what these were.

With this consideration in mind, the original focus of the study shifted slightly because a more conceptual base was needed. Thus, the empirical setting and data I had obtained from PD and teachers became resources for studying the notion of explanation in the context of mathematics teacher education in South Africa. This meant that the focus on what teachers learn and take-up from a PD programme was backgrounded as the theorisation of the notion of explanation took centre stage. In terms of research questions, the journey of the literature informed the language I then used to phrase the research questions.

In the interaction with the philosophical literature, in Chapter 3, a separation was made between explanation and acts of explaining (Ruben, 1992), which I adopted as a methodological device and language to phrase the questions. I will discuss more on the separation in the literature section that follows. Consequently, the first research question had to be added and phrased as: what are the constitutive criteria of explanation and acts of explaining? The answer to this question is a brief discussion of the literature journey I embarked on. The second and the third research questions are empirical results, demonstrating how I applied the criteria. The fourth research question is about the interrelation between the empirical results obtained from PD and classroom and what they illuminate as implications for PD, classroom and the mathematics education field. I present below what came to be the research questions for the study. The overall purpose of the study was to answer the following research question: how do the WMCS PD course and teachers present criteria for explanation and acts of explaining algebraic expressions in Grade Nine? The research question was elaborated by four critical questions, which I use to then summarise the key findings of the study.

1. What are the constitutive criteria of explanation and acts of explaining?
2. What is legitimated as an explanation, and acts of explaining in algebra in the WMCS course? (Chapter 6)

2.1 What constitutive criteria of explanation and acts of explaining are privileged? (Chapter 6)

2.2 What messages about 'quality' of both explanation and acts of explaining are transmitted? (Chapter 6)

3. What is legitimated as explanation and acts of explaining in algebra in the classroom? (Chapter 7)

3.1 What constitutive criteria of explanation and acts of explaining are privileged in teachers' practices? (Chapter 8)

3.2 In what ways are teachers privileging practices similar and or different from what was privileged as explanation and acts of explaining in the WMCS course (Chapter 9)

4. What and how do these empirical applications and their inter-relation further illuminate about the criteria and about PD and mathematics teaching?

What follows is a brief account of the journey I undertook into the literature in a quest to find constitutive criteria for explanation. The journey illuminates the conceptual contribution of the study and also adds to the clarification of the slight shift in the research focus. The specific research question the journey of the literature answers is the first question phrased as: what are the constitutive criteria for explanation and acts of explaining?

10.2 Journey of the literature and research focus – Tools from the literature

I began the journey by first looking into the WMCS PD course since it had an explicit focus on and transmitted criteria of explanation to teachers. There were constitutive criteria that I found in the course materials, although not sufficiently elaborated to constitute an analytic tool. The reason for this was that at the time when I collected data from the WMCS PD course the notion of explanation was being developed, and was in the process of its development. Ever since then, more work on the criteria has been done. I then turned to the education and mathematics education literature to search for explanation criteria because this was where I expected to find more particularised criteria for explanation in algebra. Indeed, I found Leinhardt's work which was focused on offering criteria for *effective instructional explanations*. As discussed in Chapter 3, Leinhardt (1987) distinguishes explanations which occur in the classroom for instructional purposes from self-explanations, common explanations and disciplinary explanations. Just to reiterate, for effective instructional explanations, Leinhardt says there should be a goal where the important problem is established. A rich array of examples and their representations appropriate to the situation need to be explicitly mapped. Leinhardt continues to say that an effective instructional explanation includes information about the use and limits of use for a concept or a procedure and in this way access to the principles of the discipline is opened up. Leinhardt included the notions of interconnections between ideas, prior knowledge, examples, error, and misuse of procedures as important components of an effective instructional explanation. The last element Leinhardt argues for as a component of effective instructional explanations is the notion of assumptions, which needs to be made clear.

I found that Leinhardt had a different question to the one I was asking. Leinhardt was concerned to find features that appeared to be predominantly representative of what effective instructional explanations tend to do. Accordingly, the components: access to the principles of the discipline; interconnections; and assumptions are judgements of what an effective instructional explanation should achieve, whereas the components: goal; examples and their representations are general to any explanation, effective or not. Even though our questions were not exactly the same, the commonality was that they are both about explanation and so some (goal, examples and their representations) of the criteria Leinhardt provided were useful because they provided something to build from. However, before embarking on the question of what an effective explanation should achieve, I first needed to know its constitutive elements, the *what* of the explanation in general and so Leinhardt's criteria were limiting in terms of the objectives of my study. This was the point in the literature which made me realise that there was no suitable tool within the field to talk about explanation and use in the analysis for my study, hence the journey to the philosophical literature to first conceptually clarify the notion of explanation.

In the philosophical literature, I found Aristotle's four criteria of explanation as interpreted by Ruben (1992). Ruben's aim in the first chapter of the book was to provide definitions of terms and clarify what the book is about (explanation) and not about (acts of explaining). From this discussion, I adopted the distinction Ruben made between explanation and acts of explaining, because it served as an organisational structure for looking at and understanding what was provided in the literature, in the PD and classroom data. Acts of explaining were of a noticeably different nature from the explanation in the sense that actions were pedagogic moves and decisions the teacher made to produce the explanation. Ruben thus provided criteria for explanation while some of the criteria for acts of explaining I obtained from the mathematics education literature and data.

As discussed in Chapter 3, the four criteria of explanation as offered by Ruben were matter, form, motion originator, and goal. These needed to be re-interpreted for a mathematics education context, and there were two ways in which I went about this. The first step was to compare Ruben's criteria with criteria provided by Leinhardt in education and mathematics education research and organise according to explanation and acts of explaining. I showed in Chapter 3 that there was a connection between Leinhardt's goal and representations with Ruben's goal and form respectively for explanation criteria. As demonstrated before, the notion of representations fitted in both the criteria for explanation and acts of explaining. To recapitulate, the representation provides the structure of the expression and in this way fits under the criterion "form" in the explanation; also, the representation is a representation of the selected example – it is a way of bringing abstract mathematical objects into existence so that they can be perceived through the eyes, and in this way made concrete. The notion of selecting examples and their representations is a criterion in the acts of explaining.

The second step was to look into the literature concerned with PD as the empirical field from which I obtained data (Chapter 2) and the teaching and learning of school algebra (Chapter 4), because this was the content focus for the study.

As I went into the search of the literature pertaining to PD, I had interest in both the overall content as well as the mathematical content of PD programmes, specifically if they offered any constitutive criteria for explanation of algebra. As already discussed in Chapter 2, Shalem, Sapire, and Sorto (2014) in a similar fashion segmented the criteria of an explanation into SMK (conceptual and procedural,) and PCK (everyday, diagnostic and multiple). The difference with how it has been conducted as part of this study is that the six criteria of explanation they provide are accompanied by a Likert scale to judge the quality (partial, full, incomplete, and absent). In my own study, the quality judgement is within the explanation criteria, due to the two levels within each criterion. Also, the six criteria of explanation they provide are specific to the activity of analysing or providing explanation for learner error on assessments (Shalem, Sapire and Sorto; 2014). The difference is that the SMK and PCK categories I associated with explanation and acts of explaining respectively, while for DIPIP both SMK and PCK constitute two types of explanation.

From the PD literature, what I also found to be relevant was the list of 16 '*works of teaching'* that Ball et al. (2008) presented, because they were empirically obtained. The fact that the works of teaching were obtained by observing classroom practice made them extremely relevant to this study, because they highlight some of the realities involved in the provision of an explanation in the classroom. Mosvold (2016) argued that the list was unstructured because there was no theoretical backing for the work. Selling, Garcia and Ball (2016) presented work that begins to structure the list according to what they call *mathematical objects*[15] for task developers. Given that I had a particular organisational structure and a different focus from task developers, I continued and demonstrated how the list of 'works of teaching' could be structured by adopting and using the separation Ruben made in the philosophical literature between explanation and acts of explaining. Furthermore, Ruben's distinction between explanation and acts of explaining aligned with Shulman's knowledge categories called SMK and PCK, respectively. I organised the list into goals/purposes, examples and their representations, maths content and justifications. The notion of goal as described in the works of teaching corresponded with Ruben's goal. Under maths content, representations, equivalences tied in with form, because these were about structure of mathematical expressions. Still, under maths content, the notion of definitions and notation I classified as constitutive elements and thus described as matter. In the 16 works of teaching the why question was highlighted and I matched it up with Ruben's goal at the level of justifications. Ruben's criteria helped me to classify some of the works of teaching developed by Ball and colleagues. The works of teaching that I classified as acts of explaining were evaluating learner claims, using language, examples, and their representations. As already noted, the notion of representation fitted under explanation, because it carries the form, as well as acts of explaining, because it is the way in which abstract mathematical ideas find expression and are exemplified.

[15] They use the word object differently from how it has always been used in the field to refer to mathematical instructional objects instead of a mathematical representation such as number, polygon or function.

I had the four criteria of explanation, as presented by Ruben, to structure my reading of the historical account of algebraic expressions and mathematics education literature focused on teaching and learning of school algebra. I found that Ruben's notion of matter could be redescribed within the context of algebraic expressions to refer to the notion of signs and symbols and their notation, where the notion of form was spoken about in the literature as structure of algebraic expressions, while motion originator could be interpreted as process and goal as the topic and justification.

The literature in the historical section highlighted that symbolic algebra was necessarily a practice of antiquity mathematicians, because the pictographs in the early hieroglyphic writings were symbolic representations of mathematical notions. Therefore, the algebraic expressions progressed from pictographic symbols, to rhetoric, then syncopation, to the symbolic expressions of the modern era. In this progression, I realised that the metaphoric ideas carried by the pictographic symbols as well as the words in the rhetoric expressions are essential expressions in the constitution and production of an explanation, which have since been lost. The modern symbolic expression of algebraic notions were advantageous in terms of brevity, however, there is a resultant burden of labour to translate the symbols to words and ideas they signify. The modern symbolic expressions of algebraic notions have proved to be useful in the advancement of the discipline, however, a preservation of the pictographic and rhetoric expressions to use alongside modern symbols would have played a complementary role in the provision of explanation.

In the presentation of literature concerned with teaching and learning of school algebra, I used the four criteria of explanation to highlight the main ideas researchers focused on and thus a particularisation of Ruben's criteria of explanation within the mathematics education context.

The main debates in the field of mathematics education are whether or not there is a separation between concepts and procedures or which one precedes the other. The understanding my study contributes, which is not the case in the polarisation in the literature, is that the criteria are four parts of an explanation and all of them useful in the provision of an explanation. Kieran (2013) emphasised the close relationship between concepts and procedures by highlighting two stages where procedures are conceptual so as to render the separation old fashioned. That procedures are conceptual in their stage of elaboration and updating. Indeed, the notion of representations is one criterion which serves as an example of what Kieran is referring to. The notion of representations did not only fit in both explanation and acts of explaining, but within explanation, the notion of representation enabled a discussion about matter, form and process. Following on the concepts and procedures debate in the field, the four parts of an explanation could be classified as concepts and procedures. From the Aristotelean framework I interpreted matter, form and goal to be referring to concepts, while motion originator referred to procedures. The constitutive elements, which are the symbols and their definitions; the structure/form of the symbols are mathematical concepts. The statement of the goal is the concept that needs to be learnt.

To this end, I have discussed the constitutive criteria for explanation. I have described the reason that made me go to the philosophical literature to look for the criteria. I also described how the separation in the philosophical literature assisted me to structure and organise my reading of the education and mathematics education literature as well as how I used the descriptions in the mathematics education literature to particularise the explanation criteria obtained from philosophy. I now discuss the criteria for acts of explaining.

The criteria for acts of explaining were a compilation I gathered from across the literature chapters concerned with PD, explanations, and from the empirical data. By drawing together common categories across literature and data I constructed criteria for acts of explaining which had three groups. The first group was examples and their representations, coded as Rx and language use, coded as Rn. The second group was communicative techniques composed of re-voicing, coded as Rv, finishing utterance, coded as Rf, evaluation, coded as Re, gestures, coded as Rt, chorusing, coded as Rc. The last group was regulative talk, coded as Rg.

Examples and their representations Rx

The example and its representation has consequences for what the explanation will become, however, the representation of the example and the example itself is not the explanation, but carries the potential for what eventually becomes the explanation. Without the accompanying explanatory talk, the example on its own offers no explanation while possessing ‘explanatory power’ (Zaslavski, 2014). The act of selecting and deciding on which examples and what representation to use in the lesson is a pedagogic decision, which is why I classified the examples and their representations as acts of explaining instead of the explanation. I have already discussed the notion of representations and how they fit in both explanation as form F2 and acts of explaining as the example Rx.

Communicative techniques

From the empirical data it was clear that there are types of strategies used by teachers in the study to communicate algebraic expressions in their lessons; these I classified as acts of explaining. The strategies used to communicate the mathematics in both teachers' classrooms reveal the nature of conversation. Speech is one of the ways in which mathematics and the explanation are communicated to learners through the use of verbal language, Rn, and non-verbal language in the form of gestures, Rt. In the explanation, language is the main ingredient, in the sense that there is no way of conveying the explanation outside of language. The provision of a definition for a mathematical word/term is a commonly used method in the teaching process. Language is a means of communicating the mathematical vocabulary which is a very important component in learning any language.

Other criteria obtained from the data that I classified as acts of explaining were re-voicing, Rv. The communication or classroom conversation was characterised of teacher take up of learners' ideas as meaningful contributions towards the construction of the explanation through re-voicing or the teacher echoing of learner responses. The nature of the interaction between teacher and learners was such that the teachers emitted unfinished utterances as a way of keeping learners involved and part of the construction of the explanation. The learners' role was to show that they follow and understand the teacher by finishing the utterance, Rf. From this interaction the explanation was also communicated through the use of public evaluation of learner contributions, Re. The nature of communication and interaction included the notion of chorusing, Rc. The notion of regulative talk, Rg, refers to the use of social norms and socio-mathematical norms, which outline what is acceptable behaviour and acceptable mathematical answer or response, respectively. The acts of explaining were the moves (the performances) the teacher and learners carried out to produce the explanation.

To this end, I have presented a narrative of the slight shift in the research focus for this study linked to the journey of the literature. Shifting the original focus of the thesis enabled me to develop criteria for the study by finding criteria for explanation in general and using them to organise different sets of criteria in the mathematics education literature and thus make a modest conceptual contribution to the field. The primary decision I made was to adopt the separation Ruben made between explanation and acts of explaining. The separation enabled me to structure and organise what I found in the literature and the data. I went ahead with this refocused study, even though there were consequences as a result of having shifted the focus of the study. For instance, in Chapter 5, I provided a rationale for using only two teachers instead of the original five teachers. I also had to abandon the original theoretical orientation I had chosen for the study. I return to the discussion on participants in the limitations section. In the next section, I discuss the journey of using the criteria in the data and developing the coding system.

10.3 The journey of developing the coding system using the criteria

The development of a coding system was achieved through a constant dialogic process between the tools obtained from the literature and the empirical data. Through the analysis process the language provided in the literature was further refined and developed into the coding system. This process was undertaken with the classroom data, because it provided extensive and substantive discursive data compared to the document analysis in Chapter 6.

The four criteria of explanation were elaborated by the empirical data. As the criteria entered into a conversation with the empirical data, I began to see that each criterion from the four split into two levels, Level 1 and Level 2. The levels were hierarchical for matter, form and process, in the sense that Level 1 explanations in matter, form and process were low level explanations, while Level 2 were more challenging explanations requiring a deeper and more accurate understanding of the mathematical principles in question. Goal Level 1 and Level 2 were not necessarily related in a hierarchical manner. G1 was the statement of the goal for the entire lesson while G2 was the why question and its answer. G1 codes occurred at a particular point of the lesson, while G2 codes had a continuous presence throughout the lesson. The G2 code was a code that accompanied the M, F, and P codes at level 1 and 2.

For acts of explaining, the conversation between literature and empirical texts demonstrated that there is continuous regulation of what comes to be communicated as explanation through the acts of explaining by means of examples and their representations (Rx), language (Rn), modes of communication (Rv, Rf, Rt, Re, Rc) and general regulative talk (Rg). The coding system for both explanation and acts of explaining is the tool and the main contribution of the study which came out of a nonlinear, messy, dialogic process between literature and the data. I bring in again Figure 10:1, which I presented in the methodology chapter, because it encapsulates the entire process of developing the tool.

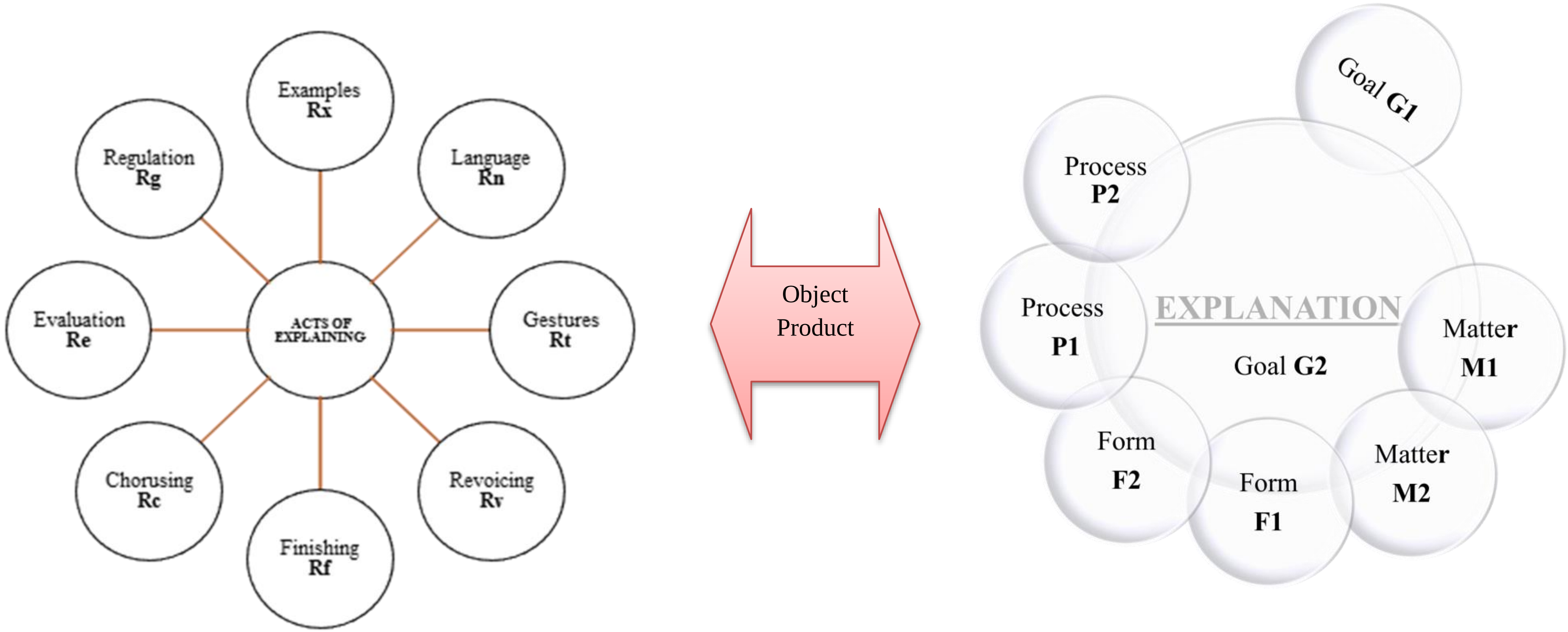


Figure 10:1 – Tools from literature and data for explanation and acts of explaining

To this end, I have discussed the slight shift in the research focus and how I obtained the tools from the literature. In the narrative I illustrated that there was much presented in the literature to build from in the frameworks of Ball and Leinhardt, in order to talk about the notion of explanation. However, there was no exact tools which sufficiently brought the "what" of explanation satisfactorily into focus to constitute an analytic tool. Consequently, I demonstrated how I developed the tool by telling the story of the journey through the literature. The journey answered the first research question which was conceptual in nature. In what follows I present the empirical findings as a result of having been in possession of the tool.

10.4 Empirical Findings and development of the constitutive criteria

In this discussion I return to the rest of the research questions to show how the constitutive criteria were elaborated in the WMCS PD course and the classroom.

Critical research Question 2 had two accompanying questions, shown below:

2. What is legitimated as an explanation, and acts of explaining in algebra in the WMCS course?

2.1 What constitutive criteria of explanation and acts of explaining are privileged?

2.2 What messages about 'quality' of both explanation and acts of explaining are transmitted?

For the second research question I collected document data from the WMCS PD project. From the empirical texts presented in Chapter 6, I read what was stated, elaborated and exemplified as explanation and acts of explaining in the WMCS PD TM1 course in which teachers reported in this study participated. The course was very clear that there are two types of explanation, the "what" and "how", which could possibly have a "why" and "when" sub-criteria to it. For acts of explaining, the course enumerated ways of making the mathematics available to learners and those were selection of examples and their representations; and the use of mathematical language and terminology.

The messages about quality in the explanation criteria were exemplified in the approach presented by the WMCS PD on how to teach the distributive property. Outlined in this respect was the use of different representations, that is, geometric and algebraic and this is form and which connects with a *what* explanation. Also outlined in this regard was the importance of knowing the underlying principles, such as the notion of repeated addition as the big idea behind the distributive property. This is a process, and so a how explanation with derivation, because the distributive property is shown to be a concept derived from the notion of repeated addition. In terms of the criteria the different representations and the distributive property are the "what" and "how" explanations and in relation to the Ruben's criteria these are form, F2, and process, P2 respectively. The level 2 explanations were the messages about quality transmitted by the course. The *when* sub-explanation was merely stated, without any elaboration and exemplification.

In the acts of explaining the messages about quality were shown in the exemplification, where an explanation to a learner's question about two pieces of the hyperbola was presented and teachers had to engage with the explanation to judge its quality. This process presented opportunities to judge the quality of the explanation from a mathematical standpoint. By using the criteria presented in the acts of explaining slide, I could ask what examples and representations are used or would be used in the explanation? Representations appear in both explanation and acts of explaining codes, because their selection is a pedagogic decision while their use allows for different forms, F2, of the algebraic expression to be dealt with. What mathematical language and terminology is used, or would be used or avoided in the explanation? These questions were messages the course transmitted about how to use the criteria in the acts of explaining to judge and produce the explanation.

The first activity of presenting an approach for teaching the distributive property was in essence modelling for teachers what a quality explanation comprises. At the same time, the activity of critiquing an explanation presented teachers with an opportunity to think about how to use the criteria in the acts of explaining to first critique the explanation and then produce a better explanation to the one offered.

The findings from this research question demonstrate that the course too made a separation between explanation and acts of explaining. The findings also confirm what has been documented in the literature about the role and importance of multiple representations in the provision of the explanation. By making the why question a sub-explanation to the two broad explanations – what and how – the findings confirm the importance of providing justifications throughout the explanation. The inclusion of the why question adds to the quality of the explanation. Also, the *what* and *how* types of explanations can be mistakenly taken to be in agreement with the debates of separating between concepts and procedures. However, the fact that the *why* and *when* are sub-explanations accompanying the *what* and *how*, blurs the boundaries between concepts and procedures.

The third critical research question was as follows:

3. What is legitimated as explanation and acts of explaining in algebra in the classroom? (Chapter 7)

3.1 What constitutive criteria of explanation and acts of explaining are privileged in teachers' practices? (Chapter 8)

3.2 In what ways are teachers privileging practices similar and or different from what was privileged as explanation and acts of explaining in the WMCS course? (Chapter 9)

To answer this question I reported on the teaching practices of two teachers whom I called Teacher A and Teacher B. In Chapter 7, I presented the classroom data for both teachers. Using the coding system earlier developed from the four criteria of explanation, which I adopted from Ruben's criteria and integrated with Ball, Thames and Phelp (2008) and Leinhardt's (1987, 1989) criteria. I analysed the classroom data by working with each utterance to determine the constitutive criteria of explanation and acts of explaining for the teachers. I only coded the teachers' utterances and excluded learner talk because my interest was on teachers' explanations, however these occur and are co-constituted in the interaction between teacher and learners.

From the criteria that were legitimated as explanation and acts of explaining, specific criteria were privileged over others. For the explanation, the talk about elements at M1 and process at P1 were the most dominant, and therefore the most privileged criteria for both teachers. In the acts of explaining the communicative technique of re-voicing, Rv, and the regulative talk, Rg, were the most dominant and so privileged in both teachers' practices. In Chapter 8 I have already argued that the four criteria of explanation, especially form and process, have blurred instead of solid boundaries. This was more apparent in the discussion of form and process. Kieran (2013) had already argued against the polarisation between concepts and procedures; that procedures are conceptual in their stage of elaboration, which is the introductory stage of the process – this can be seen when the notion of repeated addition is used to elaborate the distributive property. The discussion, as already shown in Chapter 9, straddles between form and process. I continued to keep *form* and *process* categories separate analytically and indeed the data demonstrated that the explanation of a concept is offered by demonstrating a process because mathematical concepts are embedded in procedural activity. Since a concept is developed through a process, the two are inseparable. To refer back to the work of Kilpatrick et al. (2001), the concepts and procedures are interwoven.

As discussed earlier, teachers recontextualising practices of what they learnt as criteria for explanation and acts of explaining was backgrounded even though initially intended as the main focus of the study. The study of teachers' practices was ultimately for the purpose of further developing the constitutive criteria of explanation and acts of explaining. Nevertheless, there are conclusions that I am able to make about how teachers' practices aligned or not with what was transmitted in the course. In Chapter 9, I compared the criteria privileged in the course and in teachers' practices. The comparison is important, because specific PD recommendations are drawn later in the chapter. The course transmitted explicit criteria of explanation and acts of explaining even though the course did not explicitly use the term 'acts of explaining'. There were clear and explicit criteria I could classify as acts of explaining from the slide entitled "*how is the mathematics made available to learners"*.

What emerges from the comparison was that Ruben's criteria for explanation elaborated the meaning of the interrogative particles, used in the course as criteria for explanation namely the what, how, why and when. The "what" explanation is in tune with matter and form and the "how" explanation concurs with process. The goal at G1 level matched up with "what" explanation while the goal at G2 level with "why" sub-explanation. The "when" criteria of explanation was not elaborated nor exemplified, and nor did it surface in the empirical classroom data. I demonstrated that the interrogative particles were not uniquely criteria for explanation, but that they could be used to interrogate the acts of explaining. The only categories that corresponded in the acts of explaining were examples and their representations and language. There was no way of making a comparison for communicative techniques, since the forms of data collected from the course did not afford this opportunity. I pick up on this discussion in the limitations section.

The last research question was posed as follows:

4. What and how do these empirical applications and their inter-relation further illuminate about the criteria and about PD and mathematics teaching?

I discuss below the meaning of the results in the form of implications/recommendations and limitations of the study. I start with implications.

10.5 Implications

These findings have implications for PD and classrooms as settings, where the empirical texts were obtained. Even though the setting worked out to be resources to study the notion of explanation and acts of explaining, the constitutive criteria obtained allows me to speak back to PD programmes, teachers, and the mathematics education community at large.

The study demonstrated that there is merit in separating the words explanation and explaining which highlights the different meanings and clarifies the process of analysing teaching. Secondly, there is value in having explicit criteria for explanation and acts of explaining in PD programs. As already noted, when the notion of explanation is spoken about in the literature, the verb and the noun are used interchangeably, without making any demarcation. As a result, the criteria of explanation that are provided are mixed between acts of explaining and explanation, with the effect of diminishing focus on explanation itself as product.

The four interrogative particles transmitted by the course as criteria for explanation needed more elaboration and clarification in order for teachers to hear messages about the quality of the explanation particularly at Level 2. Messages concerning explanation at Level 2 are not coming through clearly enough, especially at the level of matter and form. The derived steps, P2, were intended, and we have seen how one teacher took this up into her practice. In a nutshell, the implication the results have for PD programme are that in order for teachers to hear and notice important messages about explanation at Level 2, then there needs to be clear communication in terms of the criteria of explanation and acts of explaining.

Equally the “when” explanation was only stated without elaboration and exemplification as a result it was absent in teachers’ practices. This means that messages that the course foregrounds have a better chance of being in focus for teachers and present in their practices if explicitly experienced in the course.

The importance of the interactional pattern as well as communicative techniques (Rv, Rf, Rt, Re, Rc) cannot be trivialised, for the reason that it is the use of spoken and non-spoken language (gestures) to communicate the technical specialised language of mathematics. In the PD, ways of making the mathematics available to learners were limited to examples and their representations and language. It is likely that had I worked with discursive data then I would have been in a better position to tell what and how communicative techniques were used in the course. A practical application from the study would be to have a more targeted focus addressing productive and conducive communicative techniques which support the production of quality explanations. For example, PD programme needs to interrogate and critically evaluate the usefulness of re-voicing learner contributions as a means of communicating, participating, and generating the explanation. Even though the result of what I observed in teachers' practices cannot be related directly as a consequence of what transpired in the course, there is a strong sense, as suggested by the results, that PD programmes need to have an explicit and targeted focus on the explanation and acts of explaining.

There are practical applications the findings have for teachers in the classroom. The importance of planning cannot be overstated as a precondition for producing quality explanations with a specific focus on Level 2 codes. The tool produced in this study can be a great resource for teachers to use in their preparations to interrogate and be clear about their explanations and acts of explaining. When teachers have criteria for explanation and acts of explaining, they will be able to notice what they focused on and what they do not focus on; and justify their actions and choices. For example, teachers will be aware of the dominance of M1 and P1 in their explanations, as well as Rv and Rg in the acts of explaining and decide if it is appropriate or not. Teachers could learn to push the explanation to have P2 Level, which are derived steps for the procedural activity. From the criteria, teachers could learn to be more careful in the use of spoken language, in particular, to guard against the use of ambiguous referents and to incorporate more of the mathematical language by naming and defining. Teachers could learn about the importance of selecting quality examples and their representations that will open up opportunities to learn in depth about the particular topic. These are well known and well researched aspects in the literature, however, my study through the use of the criteria developed for explanation and acts of explaining brings a collective understanding of these tenets, which are often dealt with separately in the literature. Although Leinhardt brought together the notions of examples and their representations, and goal; the notions of matter, process, language and communicative techniques are absent in Leinhardt's criteria.

The mathematics education community contributed in the re-interpretation of Aristotle's criteria of explanation, and so the findings of the study lead to a number of suggestions for the field. There are additional criteria for both explanation and acts of explaining to the criteria which the literature offered that are important to consider when looking at practice. The separation and additional criteria were possible to achieve because of having sought out conceptual clarification of the notion of explanation from the field of philosophy. The study adds to the debates in the field about concepts and procedures. Concepts from the criteria are matter (M1 and M2), form (F1 and F2), and goal (G1 and G2); while procedures are process (P1 and P2). In the analysis, I demonstrated the benefits of having concepts and procedures separated because it enabled me to see what was in focus and what was not. Teachers do focus on some of the criteria but all level two criteria are backgrounded or missing. In the problem statement, I highlighted the notion of poor performance. The fact that teachers' explanations predominantly emphasise M1 and P1 is an indication that learners are not afforded the opportunity to learn the substance of mathematics. Possibilities and opportunities to imitate better definitions, recognise and study structure and derive rules are not opened up in the teachers' explanations. Certainly a lesson that I have learnt, even as I was surveying literature within the field, is that examples, language, symbols, structure and process are concepts that are treated separately and individually in the literature. However, the work of providing explanations requires an intermingling of these concepts.

10.6 Limitations

The outcomes of the study are promising in terms of what they have put forth as criteria for explanation and acts of explaining, however, there are limitations that need to be highlighted as potential for future research. I discuss below limitations pertaining to the criteria, PD, the sample, explanations and the mathematics content dealt with.

- **Criteria**

Language, examples, symbols, structure and process have largely been dealt with in isolation and individually in the literature (Boulet, 2007; Zaslavsky, 2014; Radford, 2000; Banerjee & Subramaniam, 2012; Ruede, 2013; Kieran, 2013), the strength of this approach in the literature is that detailed and focused ways of understanding these notions has been possible. In the current study the criteria I worked with for explanation and acts of explaining have brought these notions together collectively as necessary components in the production of an explanation. This bringing together meant that the in-depth understanding and discussion of each notion could not be undertaken. A potential future research direction would be to explore the "what and how" of each notion, for example language: the role language plays in the production of explanation criteria. The description and discussion of these notions relies on what has been done in the field. In this way, the collective approach was both a strength and a limitation of the study.

- **PD**

I made the comparison between criteria presented in the course and in teachers' practices. I was limited in terms of how much I can attribute as course impact. However, I did draw an alignment between Teacher B's use of multiple representations to teach the distributive property to the manner in which it was presented in the course. The alignment should be taken with caution, since there was no pre PD lessons I observed from the teachers to confidently support the alignment. There are specific designs in the field for studies which are interested in measuring PD impact (Sztajn, Borko and Smith, 2016) and my study had a different design to these. Furthermore only document data was considered from the WMCS PD project. In the discussion of the finding I highlighted that I could not say whether or not the acts of explaining, such as communicative techniques and regulation, were in the delivery of the course and in what manner, because I did not have the kind of data that allows for observation of the course presenters pedagogy. Therefore, the conclusions linking and aligning teachers' practice with what the course presented should be read and taken with caution – Chapter Nine to be specific. For future studies, discursive data is especially important for a study that seeks to understand acts of explaining from which the explanation is obtained.

- **Participants**

I had purposefully selected the original sample of five teachers for a recontextualisation study. Since I had shifted the focus of the study from the participants' data, I only used two teachers to elaborate the criteria. Two teachers limits the potentials of elaborating and illuminating the criteria. An exploration of more teachers of diverse expertise in teaching mathematics would be an interesting development. In fact, Leinhardt's earlier work in the 80s and 90s explored expert and novice teachers' explanations, yet did not go far enough. To take this work further, an exploration of expert and novice teachers' explanations and acts of explaining would contribute more informatively to the constitutive criteria. One way of doing this would be to explore how expert, novice and preservice teachers who are informed of the constitutive criteria compare to those who are not.

- **Research on explanations in the field of mathematics education**

Leinhardt and others who came before me studied explanations from classroom practice (Andrews, 2009; Charalambous et al, 2011; Esmonde, 2009; Leinhardt, 1987, 1989, 2009, 2010). Stacey and Vincent (2009) have already conducted work that looks at what textbooks for Grade Nine in Australia offer as explanation (explanation was classified according to modes of reasoning presented in the explanation). I studied written explanations from the course hand outs and presentations slides, other texts where explanations are offered such as textbooks, curricular and assessment documents were not in focus. van de Sande and Greeno (2010) explored the notion of *explaining with* instead of *explaining to* using the perspectival theory... Learner generated explanations have been the focus for Levinson (2010) using a framing that characterised two types of explanations: practically based (PB) and mathematically based (MB) explanations. Work remains to be done in terms of determining how to teach pre- and in-service teachers the skill of explaining mathematics using a set of clear criteria which demarcate between the explanation and acts of explaining. The tools obtained from the study are well suited to extend the work focused on explanation.

- **Algebra**

The generalisability of the criteria obtained in the study is further constrained by the fact that only one topic of mathematics was studied – algebraic expressions – a topic regarded as foundational to all of mathematics. Nonetheless, the criteria for explanation and acts of explaining still need to be investigated in other high school mathematical topics such as functions, geometry, trigonometry, and at primary school level to further elaborate and extend the criteria for explanation and acts of explaining in mathematics.

10.7 Conclusion

In this chapter, I have reported what I initially set out to study, which was teachers' recontextualisation practices, and how this focus receded as a focus on the nature of explanation took centre stage. The shift in the focus came after realising that the available literature did not enable me to do what I wanted to do with explanation. I have shared the literature journey I undertook in order to develop the tools I needed for the study. The tool obtained from the literature was further refined and elaborated as it went into a conversation with empirical data. The result of this entire process is the main contribution of the study. I have also reported in this chapter how having the tool allowed me to see what constitutive criteria of explanation and acts of explaining were foregrounded or backgrounded in the WMCS PD and in teachers' practices – and through this, in the end, stated some recommendations for practice and policy. As with all research, this study had limitations that open opportunities for further research.

REFERENCES

Achinstein, P. (1983). *The Nature of Explanation*: Oxford University Press.

Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education*(3), 205 - 224.

Adler, J. (2014). The research and development learning curve continued: Report of the Wits FRF Mathematics Education Chair and the Wits Maths Connect Secondary Project, 2014 University of the Witwatersrand. Johannesburg.

Adler, J. (2017). Mathematics Discourse in Instruction (MDI): A discursive resource as boundary object across practices. In G. Kaiser (Ed.), *Hans Freudenthal award paper from the Proceedings of the 13th International Congress on Mathematics Education, ICME-13*. Hamburg: Germany

Adler, J., Ball, D., Krainer, K., Lin, F.-L., & Novotna, J. (2005). Reflections on an emerging field: Researching Mathematics Teacher Education. *Educational Studies in Mathematics, 60,* 359-381. doi: 10.1007/s10649-005-5072-6

Adler, J., & Reed, Y. (2002). Researching teachers' take-up from a formal in-service professional development programme. In J.Adler & Y. Reed (Eds.), *Challengesof Teacher Development: An investigation of take-up in South Africa* (pp. 36-52). Pretoria: Van Schaik.

Adler, J., & Ronda, E. (2015). A framework for mathematics discourse in instruction: Documenting shifts in practice. *African Journal for Research in Mathematics, Science and Technology Education*. doi: 10.1080/10288457.2015.1089677

Adler, J., & Venkat, H. (2014). Teachers' mathematical discourse in instruction: Focus on examples and explanations. In M. Rollnick, H. Venkat, M. Askew & J. Loughran (Eds.), *Exploring mathematics and science teachers' knowledge: Windows into teacher thinking* (pp. 132-146). London: Routledge.

Andrews, P. (2009). Mathematics Teachers' Didactic Strategies: Examining the Comparative Potential of Low Inference Generic Descriptors Comparative. *Education Review, 53*(4), 1-24.

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics *For the learning of mathematics, 14*(3), 24 - 35.

Avalos, B. (2011). Teacher professional development in Teaching and Teacher Education over ten years. *Teaching and Teacher Education, 27*, 10-20. doi: 0.1016/j.tate.2010.08.007

Ball, D., & Forzani, F. M. (2009). The Work of Teaching and the Challenge for Teacher Education. *Journal of Teacher Education*, *60*(5), 497 - 511 doi: 10.1177/0022487109348479

Ball, D., Hill, H. C., & Bass, H. (2005). Knowing Mathematics for Teaching: Who Knows Mathematics Well Enough To Teach Third Grade, and How Can We Decide? *American Educator*, 1-10.

Ball, D., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What makes it special? *Journal of Teacher Education, 59*, 389. doi: 10.1177/0022487108324554

Banerjee, R., & Subramaniam, K. (2012). Evolution of a teaching approach for beginning algebra. *Educational Studies in Mathematics, 80*(3), 351-367. doi: 10.1007/s10649-011-9353-y

Bernstein, B. (1996). *Pedagogy, Symbolic Control, and Identity: Theory, Research, Critique*. Lanham, Boulder, New York and Oxford: Rowman and Littlefield Publishers inc.

Bills, L., Dreyfus, T., Mason, J., Tsamir, P., Watson, A., & Zaslavsky, O. (2006). Exemplification in Mathematics Education. In J. Novotna (Ed.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*. Prague, Czech Republic: PME.

Blitzer, R. F. (2013). *Precalculus essentials*: Pearson Higher Education.

Borko, H. (2004). Professional Development and Teacher Learning: Mapping the Terrain. *Educational Researcher, 33*(8), 3-15.

Borko, H., Liston, D., & Whitcomb, J. (2007). Genres of empirical research in teacher education. *Journal of Teacher Education, 58* (1), 3-11.

Borko, H., & Livingston, C. (1989). Cognition and Improvisation: Difference in mathematics Instruction by Expert and Novice Teachers. *American Educational Research Journal, 26*(4), 473-498.

Boulet, G. (2007). How Does Language Impact the Learning of Mathematics? Let Me Count the Ways. *Journal of teaching and learning, 5*(1), 1-12.

Brodie, K. (2014). Learning about learner errors in professional learning communities. *Educational Studies in Mathematics, 85*(2), 221-239.

Cajori, F. (1993). *A history of Mathematical notations two Volumes Bound As One, Volume I: Notations in Elementary Mathematics, Volume II: Notations Mainly in Higher Mathematics* New York: Dover Publications.

Caspi, S., & Sfard, A. (2012). Spontaneous meta-arithmetic as a first step toward school algebra. *International Journal of Educational Research, 51*, 45-65.

Cereseto, A. (2015). *What and How do Teachers Learn in Professional Learning Communities? A South African Case study.* (Doctor of Philosophy), University of the Witwatersrand, Johannesburg.

Charalambous, C., Hill, H., & Ball, D. (2011). Prospective teachers' learning to provide instructional explanations: How does it look and what might it take? *Journal of Mathematics Teacher Education, 14*(6), 441-463.

Chazan, D. (1996). "Algebra for All Students?". *Journal of Mathematical Behavior, 15,* 455-477.

Desimone, L. M. (2009). Improving Impact Studies of Teachers' Professional Development: Toward Better Conceptualizations and Measures. *Educational Researcher, 38*(3), 181-199. doi: 10.3102/0013189X08331140

Ding, M., & Li, X. (2014). Transition from concrete to abstract representations: the distributive property in a Chinese textbook series. *Educational Studies in Mathematics, 87*(1), 103-121.

DoE. (2017). *Report on the 2017 National Senior Certificate results: Shaping freedom through education* Pretoria: Department of Basic Education.

Doyal, L., & Harris, R. (1986). *Empiricism, Explanation and Rationality: An introduction to the Philosophy of the Social Sciences*. London and New York: Routledge.

Dreher, A., & Kuntze, S. (2015). Teachers' professional knowledge and noticing: The case of multiple representations in the mathematics classroom. *Educational Studies in Mathematics, 88,* 89-114. doi: 10.1007/s10649-014-9577-8

Duffy, G. G., Roehler, L. A., Meloth, M. S., & Vavrus, L. G. (1986). Conceptualizing Instructional Explanation. *Teaching & Teacher Education, 2*(3), 197 - 214.

Ely, R., & Adams, A. E. (2012). Unknown, placeholder, or variable: what is x? *Mathematics Education Research Journal, 24,* 19-38. doi: 10.1007/s13394-011-0029-9

Ensor, P. (1999). *A study of the recontextualising of pedagogic practices from a South African university preservice mathematics teacher education course by seven beginning secondary mathematics teachers.* (Unpublished Doctor of philosophy), University of London.

Ernest, P. (2000). 'Why Teach Mathematics?'. In S. Bramall & J. White (Eds.), *Why Learn Mathematics?* (pp. 1-14). London Bedford Way Papers.

Ernest, P. (2005). Why Teach Mathematics? *The Mathematical Association, 34*, 28-29. doi: 10.2307/30215766

Esmonde, I. (2009). Explanations in Mathematics Classrooms: A Discourse Analysis. *Canadian Journal of Science, Mathematics and Technology Education, 9*(2), 86-99.

Even, R. (1993). Subject-Matter Knowledge and Pedagogical Content Knowledge: Prospective Secondary Teachers and the Function Concept. *Journal for Research in Mathematics Education, 24*(2), 94-116.

Forman, E. A., McCormick, D. E., & Donato, R. (1998). Learning What Counts as a Mathematical Explanation. *Linguistics and Education, 9*(4), 313-339.

Franke L. M., Kazemi, M., & Battey, D. (2007). Mathematics teaching and classroom practice. In F. K. Lester Jr (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 225-256). Charlotte: NC: Information Age Publishing.

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science and Education as Change*, *10*, 391-408.

Fried, M. N. (2018). Chapter 6 History of Mathematics, Mathematics Education, and the Liberal Arts. In G. Kaiser (Ed.), *Invited Lectures from the 13th International Congress on Mathematical Education, ICME-13 Monographs*. Hamburg, Germany.

Frutiger, A. (1989). *Signs and Symbols: Their design and meaning*. New York: Van Nostrand.

Goldsmith, L. T., Doerr, H. M., & Lewis, C. C. (2014). Mathematics teachers' learning: a conceptual framework and synthesis of research. *Journal of Mathematics Teacher Education, 17*, 5-36. doi: 10.1007/s10857-013-9245-4

Graven, M. (2002). *Mathematics teacher learning, communities of practice and the centrality of confidence.* (Doctor of Philosophy), University of the Witwatersrand.

Graven, M. (2004). Investigating mathematics teacher learning within an in-service community of practice: The centrality of confidence. *Educational Studies in Mathematics, 57*, 177-211.

Groza, V. S. (1968). *A survey of mathematics: elementary concepts and their historical development*. New York: Holt, Rinehart and Winston.

Güven, N. D., & Dede, Y. (2017). Examining social and sociomathematical norms in different classroom microcultures: Mathematics teacher education perspective. *Educational Sciences: Theory & Practice, 17*, 265-292. doi: http://dx.doi.org/10.12738/estp.2017.1.0383

Harper, E. (1987). Ghosts of diophantus. *Educational Studies in Mathematics, 18*(1), 75-90.

Hatch, J. A. (2002). *Doing qualitative research in education setting*. New York: SUNY Press.

Hoch, M., & Dreyfus, T. (2004). *Structure sense in high school algebra: The effect of brackets.* Paper presented at the Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education.

Huillet, D. (2007). *Evolution, through participation in a research group, of Mozambican secondary school teachers' personal relation to limits of functions.* (Unpublished PhD), University of the Witwatersrand, Johannesburg.

Huillet, D., Adler, J., & Berger, M. (2011). Teachers as researchers: Placing mathematics at the core. *Education as Change, 15*(1), 17-32.

Katz, V., & Barton, B. (2007). Stages in the History of Algebra with Implications for Teaching. *Educational Studies in Mathematics, 66*(2), 185-201. doi: 10.1007/s10649-006-9023-7

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning: A Project of NCTM* (pp. 707 - 762). Charlotte: Information Age Publishing.

Kieran, C. (2013). The false dichotomy in mathematics education between conceptual understanding and procedural skills: An example from algebra. In K. R. Leatham (Ed.), *Vital Directions for Mathematics Education Research* (pp. 153-171): Springer.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Kvasz, L. (2006). The History of Algebra and the Development of the Form of its Language. *Philosophia Mathematica, 14*(3), 287-317. doi: 10.1093/philmat/nkj017

Leinhardt, G. (1987). Development of an Expert Explanation: An Analysis of a Sequence of Subtraction Lessons. *Cognition and Instruction, 4*(4), 225-282.

Leinhardt, G. (1989). Math Lessons: A Contrast of Novice and Expert Competence. *Journal for Research in Mathematics Education, 20*(1), 52-75.

Leinhardt, G. (1997). Instructional explanations in history *International Journal of Educational Research, 27*(3), 221-232.

Leinhardt, G. (2009). *Instructional Explanations in the Disciplines*. Boston, MA, US: Springer.

Leinhardt, G. (2010). Introduction: Explaining Instructional Explanations. In M. K. Stein & L. Kucan (Eds.), (pp. 1-5). US: Springer.

Levenson, E. (2010). Fifth-grade students' use and preferences for mathematically and practically based explanations. *Educational Studies in Mathematics, 73*(2), 121-142. doi: 10.1007/s10649-009-9208-y

Levenson, E. (2013). Exploring one student's explanations at different ages: the case of Sharon. *Educational Studies in Mathematics, 83*(2), 181-203. doi: 10.1007/s10649-012-9447-1

Linchevski, L., & Livneh, D. (1999). Structure sense: the relationship between algebraic and numerical contexts. *Educational Studies in Mathematics, 40*(2), 173-196.

Lydon, S., & King, C. (2009). Can a single, short continuing professional development workshop cause change in the classroom? *Professional Development in Education, 35*(1), 63-82. doi: DOI: 10.1080/13674580802264746

Malisani, E., & Spagnolo, F. (2009). From arithmetical thought to algebraic thought: The role of the "variable". *Educational Studies in Mathematics, 71*, 19 - 41.

Maxwell, J. A. (2012). *A realist approach for qualitative research*. Los Angeles, London, New Delhi, Singapore, Washington DC: Sage.

McNeil, N. M., Weinberg, A., Hattikudur, S., Stephens, A. C., Asquith, P., Knuth, E. J., & Alibali, M. W. (2010). A is for apple: Mnemonic symbols hinder the interpretation of algebraic expressions. *Journal of Educational Psychology, 102*(3), 625-634. doi: 10.1037/a0019105

Moalosi, S. (2014). *Enhancing teacher knowledge through an object-focused model of professional development.* (Doctor of Philosophy), University of the Witwatersrand, Johannesburg.

Moravcsik, J. E. (1974). Aristotle on adequate explanations. *Synthese, 28*(1), 3-17. doi: 10.1007/BF00869493

Morrow, W. (1994). Entitlement and Achievement in Education. *Studies in Philosophy and Education, 13*, 33-47.

Mosvold, R. (2016). *The work of teaching mathematics from a commognitive perspective* Paper presented at the Southern African Association for Research in Mathematics, Science and Technology Education (SAARMSTE) 12 - 15 January 2016 Tshwane University of Technology Arts Campus, Pretoria, South africa.

Novotná, J., & Hoch, M. (2008). How structure sense for algebraic expressions or equations is related to structure sense for abstract algebra. *Mathematics Education Research Journal, 20*(2), 93-104.

Opfer, V. D., & Pedder, D. (2011). Conceptualizing Teacher Professional Learning. *Review of Educational Research, 81*(3), 376-407.

Opie, C. (2004). *Doing Educational Research*. London: Sage.

Peng, A. (2007). Knowledge growth of mathematics teachers during professional activity based on the task of lesson explaining *Journal of Mathematics Teacher Education, 10,* 289-299. doi: 10.1007/s10857-007-9041-0

Perry, M. (2000). Explanations of Mathematical Concepts in Japanese, Chinese, and U.S. First-and Fifth-Grade Classrooms. *Cognition and Instruction, 18*(2), 181-207. doi: 10.1207/S1532690XCI1802_02

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics, 42*(3), 237-268.

Radford, L. (2010a). Algebraic thinking from a cultural semiotic perspective. *Research in Mathematics Education, 12*(1), 1-19. doi: 10.1080/14794800903569741

Radford, L. (2010b). The eye as a theoretician: seeing structures in generalising activities. *For the learning of mathematics: An international journal of mathematics education, 30*(2), 2-7.

Radford, L., & Puig, L. (2007). Syntax and Meaning as Sensuous, Visual, Historical forms of Algebraic Thinking. *Educational Studies in Mathematics, 66*(2), 145-164. doi: 10.1007/s10649-006-9024-6

Reddy, V., Visser, M., Winnaar, L., Arends, F., Juan, A., Prinsloo, C. H., & Isdale, K. (2016). TIMSS 2015: Highlights of Mathematics and Science Achievement of Grade 9 South African Learners Human Sciences Research Council.

Riccomini, P. J., Smith, G. W., Hughes, E. M., & Fries, K. M. (2015). The Language of Mathematics: The Importance of Teaching and Learning Mathematical Vocabulary. *Reading & Writing Quarterly, 31*(3), 235-252. doi: 10.1080/10573569.2015.1030995

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*(1), 175-189. doi: 10.1037/0022-0663.91.1.175

Roth, W.-M. (2001). Gestures: Their role in teaching and learning. *Review of Educational Research, 71*(3), 365-392.

Rowland, T. (2012). Contrasting knowledge for elementary and secondary mathematics teaching. *For the learning of mathematics, 32*(1), 16-21.

Ruben, D. H. (1992). *Explaining Explanation*. London and New York: Routledge.

Rüede, C. (2013). How secondary level teachers and students impose personal structure on fractional expressions and equations—an expert-novice study. *Educational Studies in Mathematics, 83*(3), 387-408.

Selling, S. K., Garcia, N., & Ball, D. L. (2016). What Does it Take to Develop Assessments of Mathematical Knowledge for Teaching?: Unpacking the Mathematical Work of Teaching. *Mathematics Enthusiast, 13*(1), 4.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36. doi: 10.1007/BF00302715

Sfard, A., & Linchevski, L. (1994). The Gains and the Pitfalls of Reification — The Case of Algebra. In P. Cobb (Ed.), *Learning Mathematics* (pp. 87-124): Springer Netherlands.

Shalem, Y., Sapire, I., & Sorto, M. A. (2014). Teachers' explanations of learners' errors in standardised mathematics assessments. *Pythagoras, 35*(1), 1-11. doi: http://dx.doi.org/10.4102/pythagoras.v35i1.254

Shulman, L. S. (1986). Those Who Understand: Knowledge Growth in Teaching. *Educational Researcher, 15*(2), 4-14.

Skemp, R. R. (1978). Relational Understanding and Instrumental Understanding. *The Arithmetic Teacher, 26*(3), 9-15.

Stacey, K., & Vincent, J. (2009). Modes of reasoning in explanations in Australian eighth-grade mathematics textbooks. *Educational Studies in Mathematics, 72*(3), 271-288.

Star, J. R. (2005). Reconceptualizing Procedural Knowledge. *Journal for Research in Mathematics Education, 36*(5), 404-411. doi: 10.2307/30034943

Sztajn, P., Borko, H., & Smith, T. (2017). Research on mathematics professional development In J. Cai (Ed.), *Compendium for research in mathematics education* Reston, VA: National Council of Teachers of Mathematics

Usiskin, Z. (1999). Conceptions of School Algebra and Uses of Variables. In B. Moses (Ed.), *Algebraic Thinking, Grades K–12* (pp. 7-13). Reston, VA: National Council of Teachers of Mathematics.

van de Sande, C., & Greeno, J. (2010). A framing of instructional explanations: Let us explain with you. In M. K. Stein & L. Kucan (Eds.), *Instructional Explanations in the Disciplines* (pp. 69-82): Springer.

Venkat, H., & Adler, J. (2012). Coherence and connections in teachers' mathematical discourses in instruction: original research. *Pythagoras, 33*(3), 1-8.

Vermeulen, N. (2007). Does Curriculum 2005 promote successful learning of elementary algebra? *Pythagoras, 66*(December 2007), 14-33.

Vinner, S. (1991). The Role of Definitions in the Teaching and Learning of Mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Dordrecht: Kluwer.

Webster-Wright, A. (2009). Reframing Professional Development Through Understanding Authentic Professional Learning. *Review of Educational Research, 79,* 702. doi: 10.3102/0034654308330970

Wilkie, K. J. (2016). Students' use of variables and multiple representations in generalizing functional relationships prior to secondary school. *Educational Studies in Mathematics, 93,* 333-361. doi: 10.1007/s10649-016-9703-x

Wilson, S. M., & Berne, J. (1999). Teacher learning and the acquisition of professional knowledge: An examination of research on contemporary professional development. *Review of Research in Education, 24,* 173-209.

Wittwer, J., & Renkl, A. (2008). Why instructional explanations often do not work: A framework for understanding the effectiveness of instructional explanations. *Educational Psychologist, 43,* 49-64.

Xu, G.-R. (2011). *Investigating the process of teachers' recontextualisation of the official discourse in the context of Chinese education reform.* (Unpublished Thesis), University of London.

Young, M. (2011b). The future of education in a knowledge society: The radical case for a subject-based curriculum. *Journal of the Pacific Circle Consortium for Education, 22*(1), 21-32.

Zaslavsky, O. (2010). The explanatory power of examples in mathematics: Challenges for teaching In M. K. Stein & L. Kucan (Eds.), *Instructional Explanations in the Disciplines* (pp. 107-128). Boston, MA, US: Springer.

Zaslavsky, O. (2014). Thinking with and through examples. In P. Liljedahl, C. Nicol, S. Oesterle & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 1, pp. 21-34). Vancouver, Canada: PME.

Zazkis, D. (2016). On transitions between representations: The role of contextual reasoning in calculus problem solving. *Canadian Journal of Science, Mathematics and Technology Education, 16*(4), 374-388. doi: 10.1080/14926156.2016.1190042

APPENDICES

A – Ethics Protocol number: 2016ECE006D

Wits School of Education

27 St Andrews Road, Parktown, Johannesburg, 2193 Private Bag 3, Wits 2050, South Africa. Tel: +27 11 717-3064 Fax: +27 11 717-3100 E-mail: enquiries@educ.wits.ac.za Website: www.wits.ac.za

31 May 2016

Student Number: 0217775M

Protocol Number: 2016ECE006D

Dear Nontsikelelo Millicent Luxomo

Application for Ethics Clearance: Doctor of Philosophy

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate has considered your application for ethics clearance for your proposal entitled:

A study of how teachers recontextualise algebraic explanations from a Professional Development project into their classrooms: Does SMK and Context matter?

The committee recently met and I am pleased to inform you that clearance was granted. However, there were a few small issues which the committee would appreciate you attending to before embarking on your research.

The following comments were made:

- Reasons for the need to videotape and audiotape need to be given in the information sheet. Participants also need to be reassured that the tapes will only be used for accuracy in the write up and that clips will not be used in the thesis, conference presentations and journal articles.
- The Principal, teachers, parents and learners need to know that the analyses may be used for conference presentations and journal articles.

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

All the best with your research project.

Yours sincerely,

1

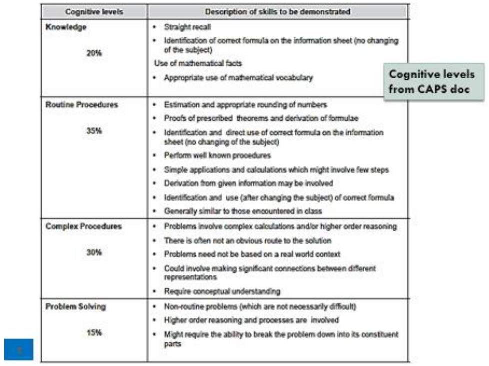
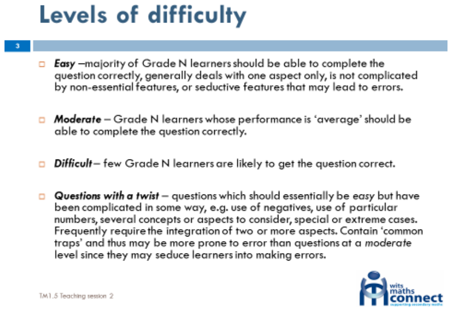
Wits School of Education

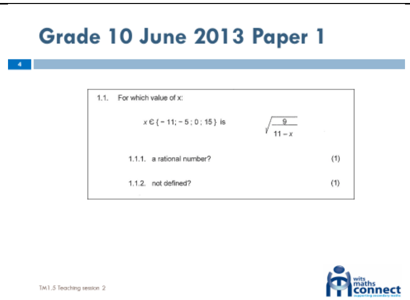
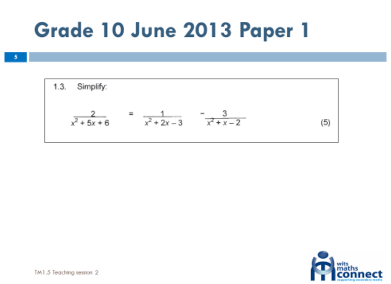
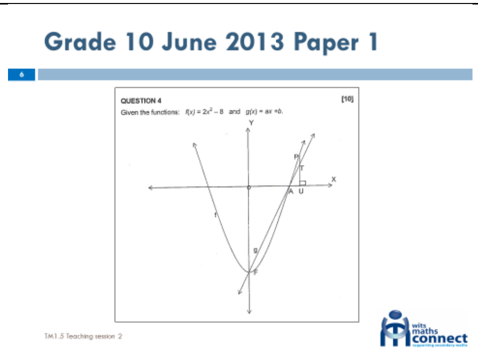
011 717-3416

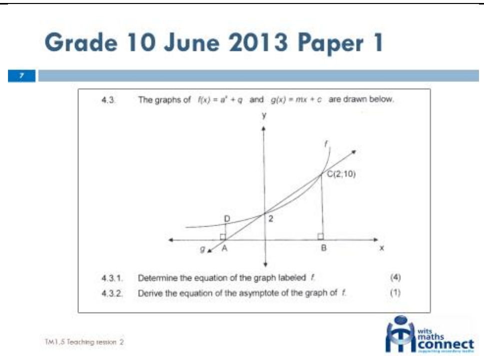
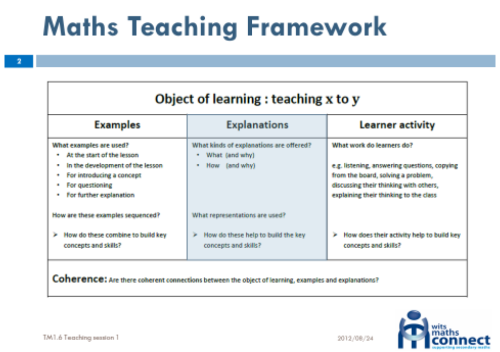
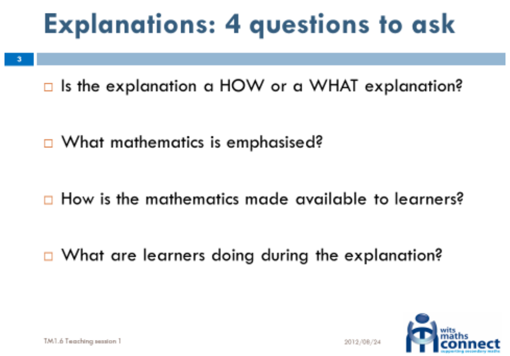
Cc Supervisors: Professor Jill Adler and Professor Yael Shalem

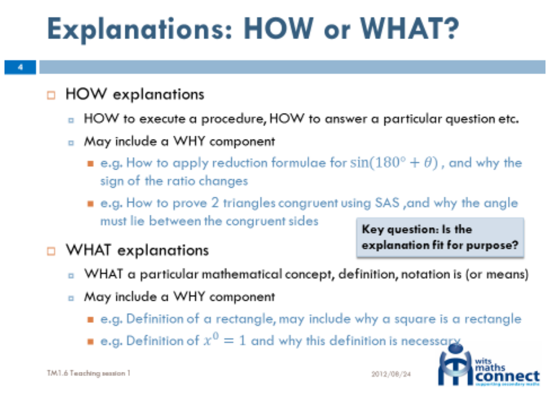
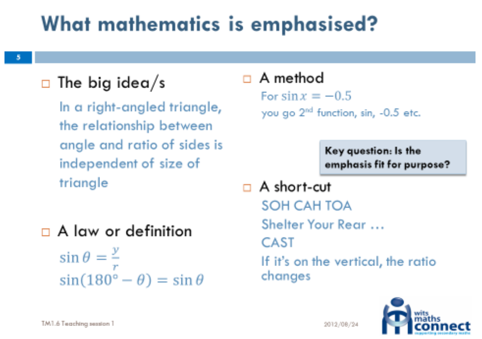
B –WMCS PD Project Data

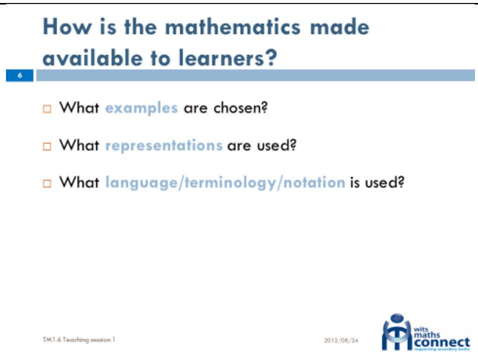
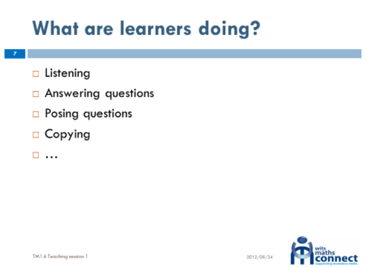
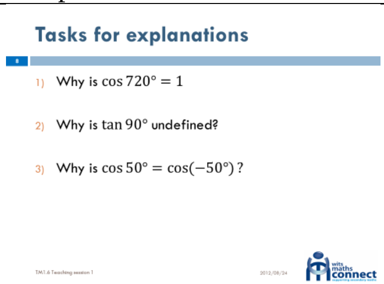
B1 Table for PowerPoint presentations and handouts for TM 1 Sessions 1.5, 1.6, 1.7 and 1.8

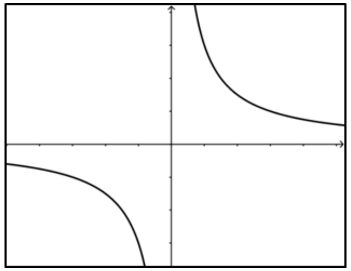
<table>
<tr><th>Source</th><th>E or AoE</th><th rowspan="2">What is written inside the slides or handouts</th><th colspan="3">Explanation and acts of explaining criteria</th></tr>
<tr><th>PPT or handout</th><th></th><th>Stated</th><th>Elaborated</th><th>Exemplified</th></tr>
<tr><td>TM 1.5
PPT slide 2</td><td>AoE</td><td rowspan="4">Choosing and sequencing examples

</td><td></td><td>Knowledge</td><td></td></tr>
<tr><td></td><td>AoE</td><td></td><td>Routine procedure</td><td></td></tr>
<tr><td></td><td>AoE</td><td></td><td>Complex procedure</td><td></td></tr>
<tr><td></td><td>AoE</td><td></td><td>Problem solving</td><td></td></tr>
<tr><td>Slide 3</td><td>AoE</td><td rowspan="4">
</td><td></td><td>Easy – deals with one aspect only</td><td></td></tr>
<tr><td></td><td></td><td></td><td>Questions with a twist – Several concepts or aspects to consider</td><td></td></tr>
<tr><td></td><td></td><td></td><td>Moderate – Examples for average learners</td><td></td></tr>
<tr><td></td><td></td><td></td><td>Difficult – Examples for few learners</td><td></td></tr>
</table>

Slide 4	AoE				Questions from the official Grade 10 June 2013 paper 1 Example exemplified
Slide 5	AoE				Example exemplified
Slide 6	AoE				Example exemplified

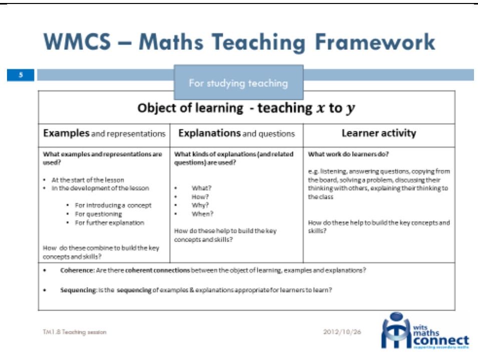
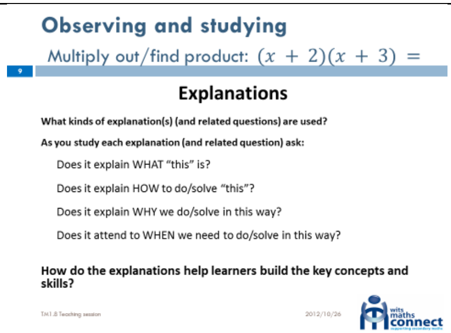
Slide 7	AoE				Example exemplified
Session	**E or AoE**	**What is written inside the slides or handouts**	**Stated**	**Elaborated**	**Exemplified**
TM 1.6 PPT **Slide 2**	E		Object of learning: teaching x to y		
	AoE		Examples		
	E		Explanations		
	AoE		Learner activity		
Slide 3	E & AoE	Explanation slide: Four questions to ask			
			Is the explanation a what or a how?		

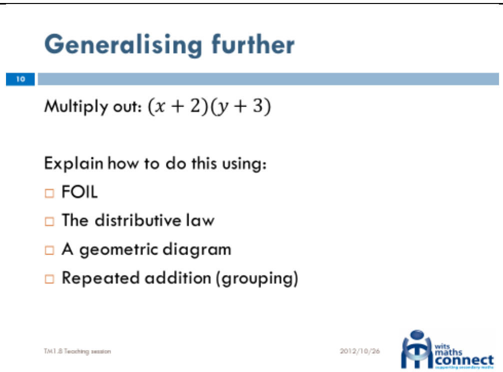
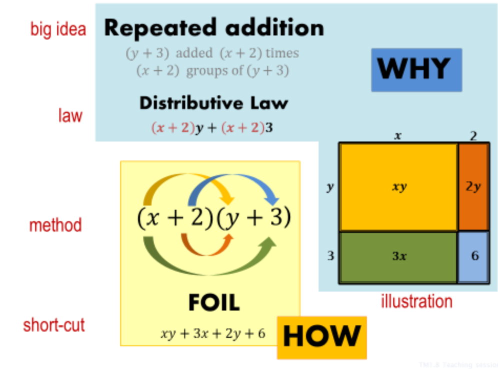
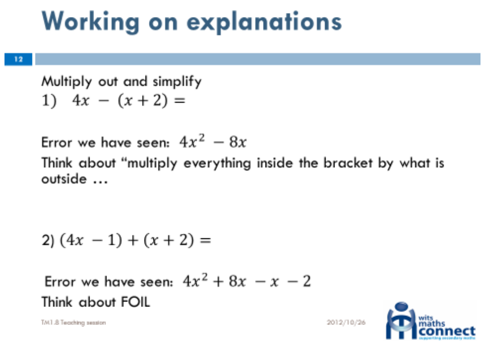
Slide 4	E	Explanations: HOW or WHAT?			
				How to execute procedure, how to answer a particular question May include a why component What a particular mathematical concept, definition, notation is (or means) May include a why component	
Slide 5	E	What mathematics is emphasized?			
				The big idea A method A law or definition A short-cut Is the explanation fit for purpose?	
Slide 6	AoE	How the mathematics is made available to learners?			

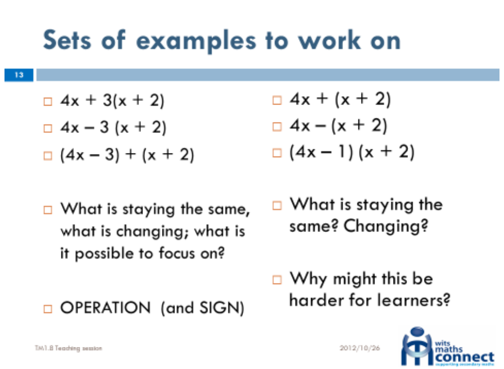
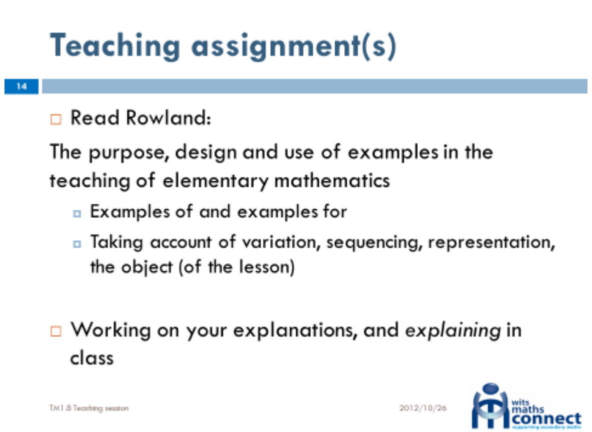
				What examples are chosen? What representations are used? What language, terminology, notation is used?		
Slide 7	AoE	What are learners doing?				
				-Listening -Answering questions -Posing questions -Copying Decide on learner activity		
Slide 8	E & AoE	Independent work				
						
				Stated	Elaborated	**Exemplified**
		Teaching task Independent work task			Explanation	
1.6 Handout	E & AoE	Producing a written explanation for why is $\tan 45^0 = 1$	What is the most important aspect that learners must learn from the explanation		Identify object of learning	

	AoE		Where will you start and what sequence will you follow?		Decide on starting point and sequencing of ideas	
	E & AoE		What representations will you use?		Deciding on representations	
	AoE		What important terminology will you use? Are there any words you will avoid?		Deciding on terminology, words to use or avoid	
Session	**E or AoE**	**What is written inside the slides or handouts**		**Stated**	**Elaborated**	**Exemplified**
1.7 handout	E	Scenario 1 Explanations in functions	Why does the hyperbola have two graphs for one equation?			Question seeking WHY explanation
						
	E		Let's look at an example of $xy = 12$. If x and y are both positive, then the product is positive and so the graph is in quadrant I. If x and y are negative, then the product is still positive but the graph is in quadrant III. That's why you have two graphs.			Teacher's explanation
	AoE		Think about the teachers response by considering the example and the explanation			Required to think about WHAT, HOW

						and WHY Explanation
Handout	AoE	Scenario 2		Stated	Elaborated	Exemplified
		Choosing examples in Gr 9 Algebra	A Grade 9 teacher is setting a revision task at the end of the section on exponents. She has given her learners Ex 3.8 from Classroom Mathematics Grade 9 and checked their answers. Now she wants to give 6 to 8 further examples for them to work on. She makes up the 3 examples shown below.			Scenario described
	AoE		Simplify: 4) $\frac{(2a^2)^2}{2b}$ 5) $a^2(-b)^2$ 6) $\frac{(4a)^{-1}}{(-a)^{-2}}$			Set of examples
	AoE		Critique the teachers choice of examples and think of other examples you would suggest			Evaluate selection of examples
Session	**E or AoE**	**What is written inside the slides or handouts**		**Stated**	**Elaborated**	**Exemplified**
1.8 PPT	E & AoE	MTF Explained for studying teaching		Object of learning		**What**
Slide 5	AoE			Examples and representations		**What and how**
	E			Explanations and questions		**What and how with why and when**

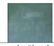
	AoE		Learner activity		**What and how with why and when**
Slide 8 and 9	E	Observing and studying Multiply out/find product: $4x(x + 2)$ or $(x + 2)(x + 3) =$			What kinds of explanation(s) (and related questions) are used?
	E				Does it explain WHAT "this" is?
	E				Does it explain HOW to do/solve "this"?
	E				Does it explain WHY we do/solve in this way?
	E				Does it attend to WHEN we need to do/solve in this way?
	E				How do the explanations help learners build the key concepts and skills?
Slide 10	E	Generalizing further Explain how to do this $(x + 2)(y + 3)$ using:		FOIL, distributive law, geometric diagram, repeated addition (grouping)	

		 			
Slide 11	E	 		Big idea Repeated addition	
	E			Distributive Law	
	E			Geometric illustration	
	E			Method	
	E			Short cut FOIL:	
Slide 12	E	Working on explanations Multiply out and simplify  			1) $4x - (x+2) =$ Error we have seen: $4x^2 - 8x$ Think about "multiply everything inside the bracket by what is outside …
	E				2) $(4x-1)+(x+2) =$ Error we have seen: $4x^2+8x-x-2$

					Think about FOIL
Slide 13	AoE	Set of examples to work on			
					Exemples
Slide 14	AoE			The purpose, design and use of examples in the teaching of elementary mathematics Examples of and examples for Taking account of variation, sequencing, representation, the object (of the lesson)	
	E &AoE			Working on your explanations, and explaining in class	

<table>
<tr>
<td>In Slide 7 and handout</td>
<td>E & AoE</td>
<td>A colleague’s lesson on products of expressions

Studying her examples and explanations</td>
<td>
<table>
<tr><td>Example set</td><td>Comment</td></tr>
<tr><td>PART 1: Application of the exponent laws (correcting homework) – laws on chalkboard

1. $ab^2\ x\ a^3b$ =
2. $ab^2\ x\ ac\ x\ 2a^3b$ =
3. $ab\ x\ ab\ x\ ab$ =
4. $a^2bc\ x\ a^{-3}b^{-2}c$ =</td><td>The teacher called learners to the board to write a solution to each homework example. The teacher worked with the whole class on each solution.
</td></tr>
<tr><td>PART 2: Finding the product of algebraic expressions. Examples.

5 $4\,(x + 2)$ =
6 $4x\,(x + 2)$ =
6.2 Compares 5 and 6
7 $-4x\,(x + 2)$ =
8 $2x\,(3x^2 + 2x - 4)$ =</td><td>The teacher proceeds with each product example in turn, inviting responses from both individual learners and the whole class.
</td></tr>
<tr><td>PART 3: Finding the product of algebraic expressions – distributive law

9. $(x + 2)\,(x + 3)$=</td><td> </td></tr>
<tr><td>Part 4: Class activity
Episode 10 is structured by an exercise with 7 examples
10 $-2x(2+1)$
11 $2(y+3) - y(y-4)$
12 $5a(2+b) + 5a(2+b)$
13 $-3x(2xy + 3xyz - 4y)$
14 $(x+9)(x-2)$
15 $(2x-1)(5x+4)$
16 $(3x-1)^2$</td><td>Remaining class time is spent with learners working on these as the teacher circulates.
</td></tr>
</table>
</td>
<td></td>
<td></td>
<td>Examples from a colleagues lesson</td>
</tr>
<tr>
<td></td>
<td>E & AoE</td>
<td>Discussion using transcript</td>
<td>Watch a video lesson using MTF tool</td>
<td></td>
<td></td>
<td>Examples from videoed lesson</td>
</tr>
</table>

B2 Follow up supporting Interview with course Presenter – schedule and transcript

Interview questions

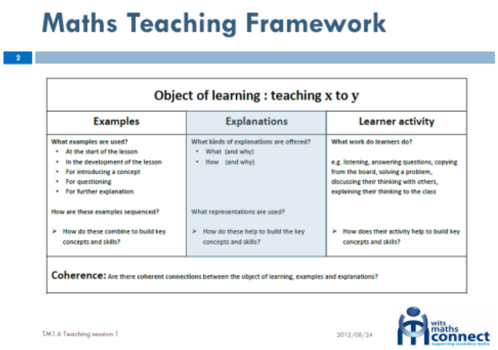
Introduction

The purpose of this interview is to get clarity on some of the documents used to present the teaching and learning of mathematics in the professional development project. The aim of this interview is to get clarity on how you explained explanation in the sessions concerned with teaching explanation.

There are three sources focused on explanation which I would like to get further elaboration on and that is the 1) MTF, 2) PPT slides and 3) independent work handouts.

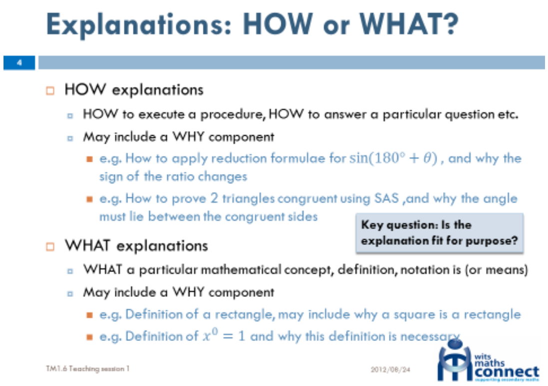
MTF

Question: What criteria about explanation were you intending to transmit?



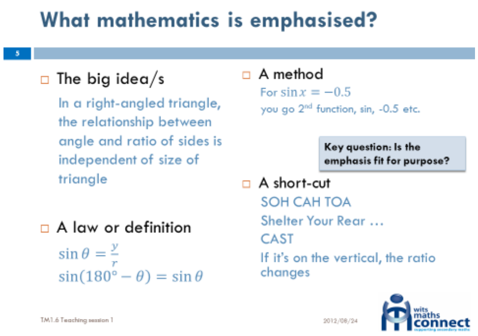
Slide number 4

What messages about explanation were you intending to transmit from this slide?



Slide number 5:

What message about explanation and acts of explaining were you intending to transmit from this slide?



Independent work task from session 1.6

Teaching task: Producing a written explanation

Assume you are required to explain the following to a grade 10 learner: why is tan 45^0 =1?

Decide on the following before you write up your explanation:

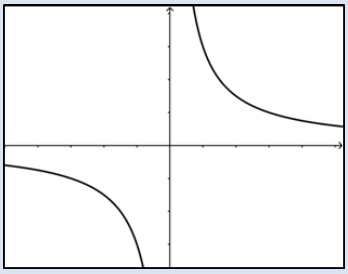
- What is the most important aspect that learners must learn from the explanation
- Where will you start and what sequence will you follow?
- What representations will you use?
- What important terminology will you use? Are there any words you will avoid?
- Will your explanation help learners to understand why $tan\ (-45^0) \neq 1$and other similar questions?

Questions to ask:

1. What were your expectations? What were you intending here? Do you have a memo or notes to share?
2. What came up from teacher responses if you remember?
    a. What kind of explanations did teachers give?
    b. What was inadequate about teachers' explanations? How did you try and supplement them? How did you mediate this? Do you have a memo or notes to share
    c. Was the discussion about selecting, sequencing, or the underlying mathematical principles/ big idea?
    d. How did it relate to the overall what, how, why and when?
        i. Did teachers' pay attention to the four criteria of explanation you gave them when responding to the task?

Scenario one

Scenario one: Why does the hyperbola have two graphs for one equation?



Let's look at an example of $xy = 12$. If x and y are both positive, then the product is positive and so the graph is in quadrant I. If x and y are both negative, then the product is still positive but the graph is in quadrant III. That's why you have two graphs.

Think about the teachers response by considering the example and the explanation

Questions:

1. What were your expectations? What were you intending here? Do you have a memo or notes to share?
2. What came up from teacher responses if you remember?
    a. What kind of explanations did teachers give?
    b. What was inadequate about teachers' explanations? How did you try and supplement them? How did you mediate this? Do you have a memo or notes to share
    c. Was the discussion about selecting, sequencing, or the underlying mathematical principles/ big idea?
    d. How did it relate to the overall what, how, why and when?
        i. Did teachers' pay attention to the four criteria of explanation you gave them when responding to the task?

<u>Interview transcript for WMCS TM1 Course Presenter</u>

1 I: Good afternoon Dr. P

2 P: Hi Ntsiki

3 I: Thank you for being agreeable to do this interview. The purpose of this interview is to find out how you explained explanation in the course

4 P: You assuming I can remember

5 I: Absolutely, because it's quite some time ever since… specifically this TM 1 course happened. So the focus is on those sessions then, teaching sessions specifically. I am aware that the course had more Maths focused sessions compared to teaching sessions and from the teaching session, focus is on the ones that paid attention to explanation. So then it's three main things that I am going to ask you in this interview. The first one is just a question around the MTF framework because there is a framework that the course used to convey the message of explanation. And then, the second part of our discussion will be based on the homework task that you gave to teachers based on trigonometry. And the last part of our discussion will be based on a scenario where a learner ask a question on the hyperbola, why does it have two pieces while it has one equation? Alright are we ready to begin?

6 P: Mhm

7 I: Alright, thank you. So around the Maths Teaching Framework. What criteria about explanations were you intending to transmit when you put that up on your slide. This is taken from slide number two.

8 P: I think it's important just before we carry on to remember that this framework has evolved since 2012 right. So I mean this is TM 1.2 which is about march 2013 and so the fact that stuff has shifted its quite possible that I will say things now that we might have not even known at that time. So I think, I mean, when I look at the prompts that are on this explanation column they not even the same any more compared to the current version. We didn't even have the expanded version of the explanations column in 2013, the expanded version only arrived late 20, 2013 (softly). Err the guys who did this course I don't think they did the expanded version at all. If I remember correctly so I think at this point the big issue that we were distinguishing, at the time, is that there are "how" explanations and there are "why" explanations. I mean we don't even use that language anymore when we talk about explanations, but I think the key thing was a "how" explanation as we said, err sorry a "what" explanation and a "how" explanation and both of them had "whys" attached to it. And a "what" was related to, to what you do, to what the thing is and a "how" was related to how you do it. So if you look at, can we look at the slides from that session?

9 I: Here

10 P: So the how, the "how" explanation was, how you execute a procedure, typical example was the distributive law, completing the square, using the quadratic formula kind of stuff, and we would use those kind of examples in the course because of it being TM1. Versus the "what" explanation being what this thing is, and getting at the concept, the that the notation, the definition, whatever. And typically working from the assumption that many times teachers' explanations are "how" explanations not "what". And that the "why" part of the explanation often doesn't get attention and in many cases you can pass the test or exam without having the "why", so teachers don't maybe feel the need to do that. And so we wanted to foreground the importance of having a justification for what gets done and why it gets done. Err, I think that those are the overarching things that come in the beginning. Err, I mean we can talk about the individual slides I don't know if you wanna go through those?

11 I: Yah, so when you talk about the importance of having a justification…, so from your talk, from what you just said, what a person can deduce then as the main message is that there are two kinds of explanation. There is a "what" and a "how". And then

there are justifications to both of them in the form of a why question. You have not spoken about the “when”, and there is a “when” put in brackets there (pointing at writing on the slide)

12 P: For the “when”.

13 I: Yes

14 P: I mean the “when” was also like, so it was kinda like, when you use this, that’s what it was related to. I mean, something that we haven’t, we didn’t foreground much in the course if I remember, and we certainly haven’t foregrounded it since. It was kinda like related to procedures of when you use it, and when you don’t. And it kinda of links, I suppose, to over-generalization when procedures get used inappropriately. ‘Cause that’s typically “when”, err you see the overgeneralization, not just the definition maybe.

15 I: Alright, if we continue to the next slide then, still trying to find out what were the main messages around the notion of explanation you were trying to teach to teachers? There is this slide ‘what mathematics is emphasized’ so there is something about the ‘big idea’, a ‘method’ and a ‘short cut’. What about these inside the slide would be, in fact, how would you classify these using the criteria, the “whats” and “hows”?

16 P: Well I think, there are contradictions here, I think, because in sense these four are big idea, law, definition, are “what”.

17 I: Okay, so this is basically a “what” slide?

18 P: Err, these are different in a sense, if you like, are four examples are “what”. What is the mathematics they are emphasizing, is it a big idea, a method, or a short cut, etc? They are “whats”. This is probably one of the reasons that this “how” and “what” distinction has probably disappeared. That it slides around. Err it was partly to say that there are different things that we might want to explain and we might do them in different ways. And err, so some are, there might be some literature that speaks negatively of short cuts and some of those things and wants the conceptual and we saying well that’s not the practice of our teachers these are all parts of the practice. Err I mean it’s interesting that big idea is the biggest there, but I don’t think it’s my slide, I wouldn’t have my fonts different, I’m sure it is Jade’s slide.

19 I: No, this is your slide, that’s the trig session (laughs)

20 P: Is that 2.1, 1.2?

21 I: This is 1.6

22 P: Anyway, I’m just joking here, for me these were different kinds of mathematical object if you want. That’s not mathematics that teachers might be explaining. Oh you right this is trig. And so the issue here was to take examples from the context of the session of the…, we now talking about them as units, so TM 1.6 so all the examples here are about trig so we trying to locate it in the trig space, and trig has some good examples some SOHCAHTOA and CAST.

23 I: Alright, okay thank you. I think we can move on to discuss the actual homework task that you gave for the same session, and I have questions here. I don't know if you want to go through the task which was asking teachers to produce a written explanation for why, explanation to grade 10 learners for why tan of 45 degrees is equal to one. And then there are specific bullets which you have given them which I read them as criteria that are being transmitted whether for explanation or acts of explaining, err for teachers to take into consideration. So when you gave this task what were your expectations? What were you intending?

24 P: This part here in the box?

25 I: Yes

26 P: Okay. Part of the rationale behind this if I remember correctly is that we knew at the end of the course that we were gonna ask teachers to produce an explanation a written one, which we were gonna use as some evidence of the learning on the course. It wasn't necessarily gonna be assessed as part of their course mark because it wasn't set out right in the beginning amongst the criteria but it was, if you want, potentially research evidence, to say kind of, a written explanation can people produce? And we felt that we needed to give them an opportunity for them to practice that before they were assessed on it. So if I remember correctly this one was a formative task ….and to see what they would write. Okay. So if we just go through the bullet points there in terms of what the criteria are. The first bullet was about the most important aspects that learners must learn from the task which for me was relating to what's the fundamental mathematics, what's the mathematical thing here kinda of thing and it was about if you want the object of learning in the language that we use now, what's this thing that you want the learners to use. So probably now that object of learning is much more common in our language with teachers I would probably change that to object of learning. Where will you start and what sequence will you follow? Was kinda of linked with are you gonna start with the calculator maybe to prove that let me just show that is easy that it's easy then explain, or are you gonna draw a diagram first, are you gonna start with the definition of *tan* as *tan* thetha is y over x. Some teachers, because it's tan work with gradient and so they focus on *tan* of and link it up with gradient. So that was part of the where will you start and then looking at the sequences is kinda wanting teachers to like to pay attention to what's the next thing I'm gonna say. And in a sense often what you get when you ask for an explanation you don't get the history of that explanation that you want, you kinda of get the final product. So it's kinda of, if they've drawn a diagram, you have no clue what they've drawn first. So sometimes we ask them to draw speech bubbles and say this is where I started, then I did this, then I said this, then that. So that's one way of trying to capture the history when you seeing the final product only. What representations will you use? Well some of that was just related to would you just use a right angled triangle not linked to the Cartesian plane for example. Or would you use a right angled triangle in the Cartesian plane? Or would you not use a right triangle at all. Would you use, err, this diagram which I am trying to explain for the purposes of the tape (draws) where got that arm that rotates, right, around so there is no triangle kinda of a thing. Are they gonna use that, so that would be in terms of graphical representations. Are they gonna use the tan graph and say if you look on the graph of tan, at 45 degrees is one. So I think trig gives a lot of opportunities to use a lot of representations and that is why it lends itself quite well. You can say this one, it's not like there is one obvious one that is best. I think that's helpful in terms of working on explanations. Err, what important terminology will you use, are there any words you

will avoid? So some of those things would be stuff like talking about tan is y over x as opposed to tan of thetha is y over x. Or they gonna use y over x because that kinda connects to gradient, or they gonna use delta y over delta x or they gonna use opposite over adjacent or they gonna use whatever. Err so that would be some of the terminology, would they talk about, erm would they emphasize a right angle if they use a triangle. For some it might be, they would avoid using the gradient, because gradient is not generalizable. So I know that I saw examples, maybe it was a different course where teachers focused completely on gradient and said it's rise over run, so you rising as much as you running, and so it's obviously 45 degrees because you've got an isosceles triangle. Do you wanna focus on that isosceles triangle as your explanation and use those words when they not generalizable to why is sine of 45 degrees not one, whatever, and so it's how local versus how global if you want. Can this explanation generalize if you change the ratio and if you change the angle or it's locked into tan because it's so close to gradient or is it locked into 45 degrees? So I mean… so I, though rise over run might come in, someone might want to include it because they trying to build links between inclination, the trig and the function work. Are they gonna use words like asymptote? Will the explanation help learners understand why tan of negative 45 degrees is not equal to one? So it's kinda of what I was saying now, in terms of how local is this explanation and can. So I also used the word powerful explanation in some of the work related to trig, stuff related to rotating an arm in a Cartesian plane around the center or origin whatever you want to call it. And then being able to talk about what's going on with the ratios vertical, horizontal, or whatever. And that kind of representation and explanation and err justification would be generalizable to any angle. Because you just see it as reasoning by continuity, or if you want, as the angle changes and you could pick and choose the ratio depending on which ever one you wanted. And that means you could use the same kind of explanation regardless of ratio or angle and that for me was what I wanted. And that's in a sense why I deliberately picked tan because it is the one where you can go to gradient. I mean now I should have picked it up but. So it was largely about, it was also reflecting on teachers understanding of mathematics it wasn't just about the ability to justify it was about how does this mathematics fit, what is the, you know, the mathematics that underpins this?

27 I: Alright thank you, so you are saying that this was a formative task?

28 P: It wasn't like we gave them marks or gave them really detailed feedback.

29 I: So you didn't get some, offerings from teachers? What they offered for what they thought, if you are able to remember on this task

30 P: I mean in general what I know from the course is that many teachers didn't bother with the teaching task they just didn't submit. I mean we got records to prove that. And what we didn't do is kinda like go over here is the model answer for the teaching task. But teachers, I hope I'm not misrepresenting this, there would be written comments for those who submitted what they said and they would be able to take those away and hopefully the feedback would be something they would be able to use. But it certainly wasn't, it obviously didn't count for marks. So it was really trying to give them some ideas to think about on explanation in preparation for a task that was coming later.

31 I: So you don't know if in doing the task they paid attention to issues of the four criteria of explanation the "what", "how" …

32 P: I mean, obvious we were hoping that it's a why question, I mean, yah it's essentially a why question rather than a what or a how, it's essentially around a why. So why is it that tan 45 is one, lead you to do whatever you gonna do in terms of the explanation. But I think already that the "how" "what" thing was already slipping around the fact that "why" was kind of sitting in both was, I think we got ourselves maybe a bit distracted by the "how" and "what". The fact that err it hasn't survived into 2014, you should see if you can talk to Emma about that space and that column.

33 I: Alright, let's have a look at the last part of this short interview. The scenario where a learner's question is given, and a teacher's explanation to that question is given. And teachers are asked to think about the examples in the teacher's explanation and the explanation itself.

34 P: So this was kind of like the final task. So when I was talking about the task at the end of the course. This was the one that we were heading towards, and I'm pretty much sure, given the way the course was evolving, that we didn't have this task finalized when the teachers were given that trig task from 1.6. The idea here was we wanted teachers to comment on an explanation as well as, if I think about, I'm pretty sure they had, to elaborate on what can they change and what else could they do. It wasn't just commenting on the, this given piece of explanation, they had to do other stuff as well. So we wanted them to comment on explanation as well as produce their own or an extended version of it. And, err, this issue of, so choosing, so we chose functions because it was the dominant part of the course. We chose the hyperbola because it's something that was, for a TM1 course. It's a function that, attention but, also introduces some new mathematics at the level of the teachers who haven't taught grade 10 who might not be familiar with it. At the same time we wanted to get a sense of the understanding of the teachers own underlying mathematics. So when we put up it had two pieces, it was stuff that we heard teachers talk about before. So they were talking about two graphs, some were talking about two pieces of the graph. And typically when learners see two separate graphs at grade 9, 10 they assume that they come from different functions. In grade 10, the hyperbola and the tan function are the only two with discontinuities, in fact that's the only in the whole high school curriculum they encounter. So it was trying to get at some of that stuff that was pushing them mathematically as well. So now this explanation of products, positive inputs are negative products, positive inputs are positive. There is nothing wrong, I mean there is nothing mathematically wrong with that explanation, but it doesn't explain why there's two pieces aren't connected. So for me this example doesn't get at, don't ask the kids why there's two pieces, partly, I mean what the explanation does do is to separate working with positive inputs and working with negative inputs. But it doesn't deal with the idea of, which is the crux, an input of zero either for x or y or both of them, the equality of $xy = 12$ doesn't hold or if you go the standard form of 12 over x, we gonna get a zero denominator. It was those issues that I was wanting teachers to get at. This explanation is inadequate, if you like, it doesn't get at the why. It's kinda telling me some of the what. Positive times positive they gonna be positives. It doesn't get at the issue of discontinuity of asymptote. Of the fact that if you wanna go as x turns to zero from the left that the y values are getting larger and larger negative and change to zero from the right the values are getting larger and larger positive. None of those issues is dealt with in this explanation. So it was the justification linked to input and output that for me was important for me here. And that was gonna focus on: did the teachers appreciate the mathematics. Err, and so possibly the quality and the feature of an explanation maybe subverted. But obviously overtaken by a task that mathematically

might have been hard as opposed to giving them something really easy. But it also seems to me that when you ask teacher to explain something that is really easy, you don’t necessarily get a decent, a depth in the explanation.

35 I: Okay so

36 P: Oh I mean, the other thing is why you have two graphs, kinda show the error. In a sense this is not two graphs, this is one graph with two pieces. And so that other issue was picking up on the mathematics, that’s why you have two graphs. So from that point there is an error in the explanation, that it’s not, its two pieces of the same graph. And that’s how we talk about it, because the graph is of the entire function. So I mean there is correct stuff and something that’s not. I don’t think that is majorly incorrect, it’s just a wrong way of talking about it, Anna Sfard might disagree.

37 I: Alright so from what you’ve said, I’m hearing that, err, that is what you intended from this task and possibly how you would have mediated the discussion. Was this task discussed or was it also formative assessment?

38 P: It was more summative; so if I remember correctly teachers did it in 1.8 and submitted it, and that is potential research data, so they never got feedback on it.

39 I: So you don’t really know what the teachers’ responses were?

40 P: I can give it to you.

41 I: Okay, that would be helpful. And then the mediation part of all these things that you’ve just said just in case they are far away from the

42 P: Because this was end of the year.

43 I: Yah, alright, thank you, thank you for your time. And now we have come to the end of this short interview, thank you so much.

P: Pleasure (Duration 26:12)

C – Classroom Data for Teacher A and B

C1 Data tables Teacher A lesson 1-3

Teacher A Lesson 1

Sg																											
1.1	**Spontaneous discussion on algebraic expressions**																										
	Turn	1	2	5	7	9	11	12	14	16	17	19	22	24	26												
	Exp	G1			F2				F2	M1		M1	M1	M1	M1												
	AoE		Rg	Rv	Rv	Rg	Rg	Rv Rt	Rv	Rv	Rg	Rv	Rv	Rv	Rv												
1.2	**Discussion of algebraic expressions through an example**																										
	Turn	1	7	9	11	13	15	17	20	22	24	26	28	32	34	35	37	39	41	45	47	49	51	53			
	Exp	M1	F2	F2	F2	F1		M1 F1		M1	M2	M1	M2	M2	M1	M1			F1	M1		M1	M2	M1			
	AoE	Rx		Rg		Rv Rf			Rv	Rg	Rg	Rv	Rv	Rf	Rv		Rv Rf	Rv Rf		Rv	Rv Rf Rt	Rf	Rf				
2	**Goal announced and written on the board**																										
	Turn	1	3																								
	Exp	G1	G1																								
	AoE	Rg	Rg																								
3.1	**Example 1**																										
	Turn	1	4	6	10	11	14	16	18	20	22	26	28	30	32	34	37	39	42	44	46	50	52	54			

	Exp	M1	P1	F1	F1 G2		G2	M2	M2 P1	P1	G2		P1	F1 P1	P1	P1	P1	P1	G2	F2 P1							
	AoE	Rx	Rv	Rv	Rv Rg	Rx		Rf	Rv	Rv		Rg	Rv	Rv Rf	Rf		Rv Rf	Rv		Rv	Rg	Rg	Rg	Rg			
3.2	**Example 2**																										
	Turn	1	3	5	7	9	12	15	17	19	21	23	25	30	32	34	36	39	41	43	47	52	54				
	Exp		M1 P1	P1	P1	P1	P1	F2	P1	P1				P1	P1	P1	P1	P1	P1	P1	P1	G2	F2				
	AoE	Rx	Rg	Rv	Rv				Rv Rg	Rf	Rv	Rv	Rv	Rv Rg	Rv	Rv Rf	Rv Rg		Rv	Rf	Rv	Rv	Rv				
3.3	**Example 3**																										
	Turn	1	4	6	8	10	13	17	20	23	25	27	29	31	33	35	37	39	41	42	43	45	47	49	51	53	55
	Exp	M1	M1 P1	P1	P1	P1	M1 F2	F2 P1			P1 M1	F2 G2 P1	P1	P1		P1	P1		P1	P1		P1		P1	P1	P1	F1
	AoE	Rx	Rg	Rv	Rv	Re	Rx	Rv	Re	Re	Rv		Rv	Rf	Rv		Rf	Rv Rg	Rv Rg		Rg		Rv	Rv	Rg		
	Turn	57	59	61																							
	Exp	P1 G2	P1	P1																							
	AoE	Rg		Rv																							
3.4	**Example 4**																										
	Turn	1	3	7	10	12	14	17	20	22	24	26	28	30	33	35	39	42									

	Exp	F2					P1			P1 G2		P1		P1		M 1	P1	M 1									
	AoE	Rx	Rg	Rg	Re	Re	Rg	Re	Re	Rf	Rf		Rv Rf	Rg	Re	Re		Re									
4	**Class Activy/Homework**																										
	Turn	1																									
	Exp																										
	AoE	Rg																									

Teacher A Lesson 2

Sg										
1	**Learners call out answers to homework**									
	Turn	1	3	5	9	11	13	15	17	19
	Exp		F2	F2 G2	F2 G2	F2		F2	F2 G2	
	AoE	Rg	Rg	Rv	Rv	Rv	Rg		Rv	Rg
2.1	**Learner's Answer to Example 2**									
	Turn	1	2	4	6					
	Exp	M1			P1 M1 F1					
	AoE	Rx Rg	Rg	Rg Re	Re					
2.2	**Learner's Answer to Example 2b**									
	Turn	1	4	6						
	Exp			P1						
	AoE	Rg	Re	Rg						
2.3	**Learner's Answer to Example 2c**									
	Turn	1	3	6	10					
	Exp			M1 P1 F1	M1 F1					
	AoE	Rg	Rg	Re	Re Rt					
3.1	**Answers to Example 3**									
	Turn	1	3	5	7	9				

	Exp	M1 F2	M1	M1	M1	M1				
	AoE	Rx Rg	Rv	Rv Rg	Re Rg	Rg Re				
3.2	**Learner’s Answer to Example d**									
	Turn	1	4	6						
	Exp			M1						
	AoE	Rx Rg	Re							
3.3	**Answers to Example e**									
	Turn	1	3	5						
	Exp	M1	M2	M2 M1						
	AoE	Rx		Rf						
4.1	**Learner’s Answer to Example a**									
	Turn	1	3	6	8	10				
	Exp				M1 P1	M1				
	AoE	Rx Rg	Rg	Re Rt	Re	Re				
4.2	**Learner’s Answer to Example b**									
	Turn	1	3	5	8	10	12			
	Exp					P1 M1	P1 F1			
	AoE	Rg	Rg	Rg	Rg	Re	Re			
4.3	**Learner’s Answer to Example c**									
	Turn	1	4	5	6					

	Exp		P1M1		M1					
	AoE	Rg	Rg Re Rt	Rf	Re					
4.4	**Learner’s Answer to Example d**									
	Turn	1	4	6	11	13	16			
	Exp		P1 F1	F1 P1	P1 F1 M1	F1	M1			
	AoE	Rx Rg	Rg Re	Re Rt	Re Rf	Rv	Re Rt Rf			

Teacher A Lesson 3

Sg																									
1.1	**Learner's answer to homework**																								
	Turn	2	4	5	7	9	11	13	15	17	19	21	23	25	27	29									
	Exp			P1 M 1	M 1	P1	P1 M 1	M1 P1		P1		P1	F1	M 1	M 1										
	AoE	Rg	Rx	Re	Re	Re	Rf	Rv Rf	Rv		Rf	Re				Rg Re									
1.2	**Learner's answer to homework**																								
	Turn	1	4	6	7	9	11	15	16																
	Exp				M 1	F1		F2 G2 P1 M1																	
	AoE	Rg	Rg Re	Rx	Re	Re	Re	Re	Rg																
2	**Goal for the day's lesson**																								
	Turn	1	3	5																					
	Exp	G1	G1	G1																					
	AoE	Rg	Rg	Rg																					
3.1	**Example 1**																								
	Turn	1	3	5	9	11	13	15	18	20	22	24	26	28	32	35	37	39	41	43					
	Exp	M 1	M 1 F1		M 1		M 1	F1	M 2 P1	P1	P1	P1			P1		G2 F2		M 1	M 1					
	AoE	Rg Rx		Rv	Rv	Re	Rv	Rv			Rg	Rv Rf	R g	Rv Rf	Rg	Rg		Rf	Rg	Rg					

3.2	Example 2																								
	Turn	1	3	5	7	9	11	13	15	18	20	21	22	24	26	28	32	34	36	39	41	43	45	47	
	Exp	M1	M1	P1				P1	P1	P1	G2	P1		M1 P1			P1			P1	P1	P1	P1	M1 P1	
	AoE	Rx	Rg		Rv	Rg	Rv	Rv	Rv	Rv Rg Re Rt			Rv	Rv	Rg	Rg	Rg	Rg	Rg	Rv	Rv		Rv	Rg	

3.3	Example 3																								
	Turn	1	3	5	7	9	11	13	16	18	20	24	26	30	32										
	Exp		M1	P1		P1		P1	P1		F2	G2	F2	G2	F2										
	AoE	Rx		Rv	Rv Rf	Rt	Rv	Rt	Rv	Rv			Rv												

3.4	Example 4																								
	Turn	1	3	8	11	14	16	19	21	23	25	27	29	31	34	36	38	41	43	45	47	49			
	Exp		M1	P1 M1			G2		P1		M1	P1			P1		M1	P1			P1	P1			
	AoE	Rx		Rv Rf	Rv	Rg Re		Rv	Rg Re	Rg			Rv	Rv	Rv	Rf Rt	Rv	Rv	Re Rt	Re	Re	Re Rg			

3.5	Example 5																								
	Turn	1	3	5	7	10	14	16	18	21	24	28	31	33	35										
	Exp		M1	P1		P1				P1	P1	P1	G2 P1		P1										

	AoE	Rx		Rv	Rg	Rv	Rg Re Rt	Rg	Re	Rg	Rg Re Rt	Rv Re		Rg	Rv										
3.6	**Example 6**																								
	Turn	1	5	7	9	11	13	15	17	19	21	23	26	29	31	34	36	39	41	43	45	47	49	53	59
	Exp	M1		P1	P1	P1	P1	M1	M1		P1	P1	P1	M1	P1		P1	P1	P1	P1	P1	P1	F1 G2	P1	P1
	AoE	Rx	Rg	Rv	Rv	Rv		Rg	Rv	Rg	Rv	Rv	Rv	Rv Re	Rg	Rg	Re	Rg	Rv	Rg	Rv	Rv	Rf	Rv	Re
		61	64																						
		M1 P1	M1																						
		Re	Rv Re																						
3.7	**Example 7**																								
	Turn	1	3	5	7	9	11	13	16	18	20	22	25	27	30	32	34	36	39	41	44	47	51	53	55
	Exp	M1	F1	P1	P1	P1	P1			G2	M1 P1	M1 P1	P1	M1 P1	M1	P1 M1	P1	P1	P1		P1	P1	P1	P1	P1
	AoE	Rx	Rg		Rv	Rv	Rv	Re	Re	Re	Rg	Re Rt		Rt			Rv	Rv	Rf	Re	Rg		Rg	Rg	
	Turn	57																							
	Exp	P1																							
	AoE	Rv																							
3.8	**Example 8**																								

	Turn	1	2	7	10	12	14	16	18	20															
	Exp	M 1			M 1	M 1 P1	P1	P1	M 1 P1	P1 G2															
	AoE	Rx	Rg	Rg	Rg	Re	Rg	Re		Rg															
4	**Class Activity/Homework**																								
	Turn	1																							
	Exp																								
	AoE	Rg																							

C2 Data tables Teacher B lesson 1-3

Teacher B Lesson 1

Sg																								
1.1	**Calling out mental maths activity**																							
	Turn	2	4	6	8	9	11	13																
	Exp			M1	M1 P1	P1		P1 M1																
	AoE	Rg	Rg	Rg			Rv	Rg Rt																
1.2	**Teacher marking individual books**																							
	Turn	1	2	3	4	7	12	14	16	18	20	21	23	25	26	27	29	31	32					
	Exp		M1 P1			M1	M1 P1					M1	P1		M1	P1	P1							
	AoE	Rg		Rg	Re	Rg	Rf	Rv Rf	Rf	Rg Rf Rt	Rg		Rv	Rv Re		Re Rf Rt	Rv	Rg	Rg					
1.3	**Whole class discussion of mental maths**																							
	Turn	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	33	35	37					
	Exp	M1	P1		P2	P1	P2 G2	P2 G2		P1			F1		P1	P2	P2	P1						
	AoE	Rg Rf	Rv	Rf	Rv Rf		Rv Rg	Rv Rf Rt	Rv	Rv Rf	Rv	Rg	Rv Rg Rt	Rv Rf	Rv	Rv Rf	Rv Rg	Rv	Rv Rg					
2	**Variable**																							
	Turn	1	5	7	8	11	15	20	21	23	24	26	27	29	31	33	35	36	38	40	42	44	46	47

	Exp	M1	G2	M1	P1	M1	M1		M1 P1		M1 P1	P1	F1	M1	P1 M1	M2	G2 M2		M2			M2		
	AoE	Rg		Rx Rv		Rv		Rx		Rv	Rx	Rv	Rt	Rt	Rv Rn	Rn	Rn	Rn	Rn Rv Rt	Rt	Rv	Rn Rt Rc	Rv	Rn Rc
	Turn	49	52																					
	Exp	M2 G2	G2																					
	AoE	Rv Rn	Rg																					
3	**Monomial**																							
	Turn	1	2	3	4	5	6	7	9	10	11	13	15	16	19	21	22	23	25	27	28	30	32	34
	Exp	M1	M2 M1		M1			M1	M2 G2	M1		M1	M1	M2 F1			M1 F1	F1					F1 P1	G2 F2
	AoE	Rg	Rt	Rg	Rt	Rg	Rg		Rv		Rv		Rv	Rg Rt Rc Rn	Rn Rf	Rv	Rx		Rg	Rv	Rt	Re	Rv	Rf Rt
	Turn	36	39																					
	Exp	F2 P1	P1 F2																					
	AoE	Rf	Rv Rt Rn																					
4	**Number next to variable**																							
	Turn	2	3	4	6	8	10																	

	Exp	M1 P1	M1	P1 M1	P1 G2	P1	P1																	
	AoE		Rt	Rt	Rv Rf																			
5	**Binomial**																							
	Turn	1	3	5	7	11	13	15	17	19	21	23	25	27	28	29	31	33	36					
	Exp	M1 F1			G2	M1	P1	P1 F2	F2	M1 G2	P1	F2	F2	F2	M1		F1	F1						
	AoE		Rg	Rg		Rt		Rf Rt	Rf	Rg		Rf		Rv Rn Rf	Rn	Rf	Rn Rf Rt	Rn Rf Rt	Rv Rg					
6.1	**Trinomial**																							
	Turn	1	2	4	6																			
	Exp	F1 M1	P1	G2	F2																			
	AoE	Rx			Rg																			
6.2	**Side**																							
	Turn	1	3	5	7	9	11	13	15	17	19	21	23	25	27	28	32	34	35	37	40	42	46	48
	Exp	M1	M1	M1	M1 P1	M1	P1 M1		F1		F1				M1	P1	P1	P1	F1			P1	P1	P1
	AoE	Rx	Re	Rv Re	Rv Re Rt	Rv Re Rt	Rv Re	Rg Rt	Rt	Rt	Rg	Rt	Rg Re Rt	Re	Rv	Re Rt	Rv	Rv	Rt	Re Rg Rt	Rt	Rg		Rv
	Turn	50	52	54	56	58	60	62	64	66														
	Exp	P1	P1		P1	P1	P1		F1															
	AoE		Rg Rt	Rv	Rg	Rv	Rg	Rc	Re Rt	Rv														

6.1	**Trinomial**																							
	Turn	9	11	13	15	16	18	20	23	24	26	29												
	Exp	F2	F2	F2		F2	F2			F2														
	AoE	Rg Rn	Rg	Rn Rf Rt	Rn Rv Rg	Rn Rg	Rv	Rn	Rn	Rn	Rv Rn	Rg												
7	**Goal: algebraic expressions**																							
	Turn	1	3	5	7	9	11																	
	Exp	G1 G2	G2			G2 G1 F2	G1																	
	AoE	Rg	Rg Rn	Rn Rt	Rn	Rn Rg																		
8.1	**Example 1: Ascending and descending powers**																							
	Turn	1	2	4																				
	Exp	M1	F2	F2																				
	AoE	Rx	Rn	Rv Rt																				
8.2	**Side**																							
	Turn	1	3	5	7	9	10	11	12	14	16													
	Exp	F2	F2	F2	M1 F2	G2 P1 F1	F2 G2			F2	P1 F2													
	AoE	Rg Rt	Rg	Rt Rf	Rg	Rv	Rf	Rv	Rg	Rg	Rv													
8.1	**Example 1: Ascending and descending powers**																							
	Turn	5	7	9	10	12	13	14	16	18	20	22	24											

	Exp	M1 F1	F1	F1	F1		M1 G2	M1 F1		F1	F1	F1 M1	M1 G2											
	AoE	Rg Rt	Rv Rf Rt	Rv Rn	Rn	Rv Rn	Rg	Rg	Rv Rn	Rf	Rv	Rv Rf	Rv Rg											
9	**Classwork**																							
	Turn	1	3	5	6																			
	Exp				F2 F1																			
	AoE	Rg	Rv	Rg	Rg																			

Teacher B Lesson 2

Sg																							
1.1	**Checking homework and signing learner books**																						
	Turn	5	7	8	9	10																	
	Exp																						
	AoE	Rg	Rg	Rg	Rg	Rg																	
1.2	**Doing corrections number 1**																						
	Turn	1	3	5	7	9	11	13	15	17	19	21											
	Exp		F2	F2		F2	F2	P1 G2		F2	M1 F1	F1											
	AoE	Rg	Rg Rx Rt	Re	Rv	Rg Rt	Rv Rn Rt	Rv Rn Rt	Rn	Rv Rn Rf	Rn Rt	Rv Rt Rg											
1.3	**Doing corrections number 2**																						
	Turn	1	3	7	9	11	14	16															
	Exp		M1 F2	F2			F2 M1 F1	F1															
	AoE	Rg		Rg Rt	Rf	Rv Re	Rg Rv Rt	Rv Rt															
1.4	**Doing corrections number 3**																						
	Turn	2	4	6	8	10	12	14															

	Exp	F2				F2 M1 F1		F1 M1															
	AoE	Rg Rt	Re	Re	Re	Re Rv	Rf																
1.5	**Doing corrections number 4**																						
	Turn	2	3	5	7	8	10	12	14	16													
	Exp		F2			M1 F1		G2															
	AoE	Rg	Rg Rt	Re Rv	Re	Rg Rt	Rf Rc	Re Rv	Rg	Re													
1.6	**Learner seeking clarity**																						
	Turn	2	4	7	10	12	13																
	Exp	M1			M1	M1																	
	AoE	Re	Rt	Rg Rf	Rv Rt	Rv Rg	Rg																
2.1	**Calling out items to Mental Maths Activity**																						
	Turn	1	2	4	6	9	11	12	14	15	17	19	20	22	23	25	27	29					
	Exp					M1	M1	M1	M 1	M1	M1	M1	M 1		M 1	M1	M1						
	AoE	Rg	Rg	Rg	Rg	Rt	Rv	Rg				Rg	Rg	Rg	Rg	Rg	Rv	Rg					
2.2	**Answers to Mental Maths Activity number 1**																						
	Turn	1	3	4	6	8	10	12															
	Exp	M1		P1 G2		F1 M1	P1 F2	F2 G2															

	AoE	Rg	Re Rx		Rf	Re Rv Rg	Rv	Re Rt															
2.3	**Answers to Mental Maths Activity number 2**																						
	Turn	1	2	4	6	8	10																
	Exp	M1		M1	M1	F2	F2																
	AoE	Rg Re Rt	Re Rt	Rv	Rg		Rg																
2.4	**Answers to Mental Maths Activity number 3**																						
	Turn	1	3	5	7	9																	
	Exp	M1 P1		P1	P1	F1																	
	AoE	Rg Rt	Rv			Rv																	
2.5	**Answers to Mental Maths Activity number 4**																						
	Turn	1	3	8	10	12	14																
	Exp	M1 P1	F1		G2 F2		P1																
	AoE		Rf	Rg	Rf	Rf	Rv																
2.6	**Answers to Mental Maths Activity number 5**																						
	Turn	1	3	5	7	9																	
	Exp	M1	M1	G2	F1 M1	M1 P1																	
	AoE	Rg			Re Rv																		
3	**Goal**																						
	Turn	1	2																				

	Exp		G1																				
	AoE	Rg	Rx																				
4.1	**Example 1**																						
	Turn	4	6	7	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	45
	Exp	F1 M1	F1 P1	F1		F1		M1		F1	G2 M1	F1		M1	M 1	M1	M1	M1				M1 G2	M 1 F1
	AoE		Rg Rt		Rv Re Rt	Re	Re		Re	Re	Rf	Rg Rt	Rv Re	Rg Rt	Rt	Rt	Rt	Rt	Rt	Re	Rf	Rv	Rt
	Turn			47	49	51	53	55	57	60	62	64	66	68	70	71	72	75	77	80	82	83	85
	Exp			F1			F1 P1	P1 G2 F1 M1		F1	M1 F2	G2		G2	P1		P1 M2		M1	M1	M1		M 1
	AoE			Re	Rt	Rf			Re				Rv	Rf	Re	Rf	Rv Rt Rn	Rf Rn	Rv Rt Rx	Re Rt Rf Rx		Rx	Rv
4.1	**Example 2**																						
	Turn	1	3	5	7	9	11	13	16	18	20	22	25	26	28								
	Exp	M1 F1	P1 M1	F1		F1		F1	P1	P1 G2 F1	F1	M1 F1	G2 F2	F1									
	AoE	Rg			Rv Re	Rv	Rt	Rv			Rg Rt	Rv Rt	Rv	Rg	Rg								
4.2	**Example 3**																						
	Turn	1	3	5	7	9	13	16	18	20	22	25	27	29	31	34							

	Exp		M1	F1		F1	G2 M1	M1	F1	G2 M1	M1 F1 P1 G2	P1		M1 P1	P1								
	AoE	Rg Rx		Rt	Re Rv Rt		Rt	Rt	Re		Rf	Rv Rt	Rv Rf	Re Rv Rg	Rg	Rg							
4.4	**Example 4**																						
	Turn	2	4	6	8	10	12	14	17	19	21	23											
	Exp		M1 P1	P1	P1	P1		P1 M1	F2 P1	P1 F1	P1	P1											
	AoE	Rg Rx		Rv Rf	Rg Rt	Rv Rg	Rg Rt		Rv	Rf Rt	Rv Rf Rt	Rv Rg											
4.5	**Example 5**																						
	Turn	3	5	7	9	11	13	15	17	19	21	23	25	27	30								
	Exp	P1	P1	P1			P1	P1		F2	F1	P1	G2 F2	P1	P1								
	AoE	Rx Rg	Rv Re	Rv Rf	Re	Rv Rf	Rf Rt	Rv Rg	Rv		Rv Re Rt												
5	**Homework**																						
	Turn	4																					
	Exp																						
	AoE	Rg																					

Teacher B Lesson 3

Sg																							
1.1	**Checking and Marking homework**																						
	Turn	1	3	5	6	8	10	12	14	16	18	20	22	24	26	28	30	31	33	35	37	39	41
	Exp					P1 F2	G2	F2 M1	F2	G2	F2	P1	F2 P1	F2	M1	M1	M1	F2 M1	F2 M1 P1	F1 F2		P1 M1	P1
	AoE	Rg	Rg Rt	Rg Rt	Rg Rt	Rt		Rv Rf Rt	Rv Rt	Rf	Rv Rg Rt	Rf	Rt	Rf	Re	Re Rt	Rv Re Rt	Rg Rt	Rg Rt		Rv Rf	Rf	Rv Rt
	Turn	43	45	48	49																		
	Exp	P1	P1																				
	AoE	Rv Rt	Rv	Rv	Rg Rt																		
2.1	**Example 1 - Geometric representation (New lesson starts) (Eg1)**																						
	Turn	1	2	3	5	7	8	10	11	12	13	15	16	17	20	22	24	26	28	30	31	33	35

	Exp			F2	F2			M1 F1	M1	P1	M1		M1	(F1)			F2	F2	F2	F2	P1	P1	P1
	AoE	Rx Rg	Rg		Re Rv	Rv Rg	Rg	Rn	Rn	Rg Rn		Rg	Rg Rn		Re Rn	Rn	Rv	Rt Rf	Rv	Rv Rt		Rg Rt	Re Rn
	Turn	37	39	42	45	47	49																
	Exp		P1	P1	M1	F2	F2																
	AoE	Rv Re Rt	Rv		Rv	Rc	Rv																
2.2	**Example 2 - Geometric representation (Eg2)**																						
	Turn	1	4	6	8	11	13	15	17	19	21	23	25										
	Exp	M1 F2		F2		F2	M1	M1	M1	M1 P1	P1		M1 P1										
	AoE	Rx Rg Rn	Rn Rf	Rv	Re	Rv Re Rt	Rv	Rv	Rv	Rv	Rv	Rv											
2.3	**Example 3 - Geometric representation (Eg3)**																						
	Turn	1	3	7	9	11	13																
	Exp	M1 F2	F2 P1	M1 P1	P2	P2																	
	AoE	Rx		Rv Rf Rt	Rv	Rf	Rv																
2.4	**Example 4 - Geometric representation, translates areas to Distributive law (Eg4)**																						
	Turn	1	3	5	7	10	12	13	15	17	18	20	22	26	28	30	32	34	36	38	42	44	46

	Exp	M1 F2	M1	M1	F2 M1		M1	F2		F2 P1	P1	P1	P1	P1	P1	P1	P1	P1	P1	P1	P1 P2 G2	M1	M1 P1
	AoE	Rx Rg	Rt	Rt	Rv Rt	Rv Rt	Rt	Rt Rn	Rv Rn Rt	Rv Rt	Rg	Rv		Rv		Rg Rt	Rv	Rv	Rv	Rf	Rg Rt	Rt	Rf Rt
	Turn	48	50	53	55	56	58	62	65														
	Exp				F1 P1 G2	P1	P2	P2	P2														
	AoE	Re	Rt	Re Rt	Rv Re Rt		Rt	Rv Rt	Rv														
3	**Statement of the goal (tpc)**																						
	Turn	1	2	3																			
	Exp	G1																					
	AoE		Rg	Rg																			
4.1	**Example 5 (Eg5)**																						
	Turn	1	4	6	9																		
	Exp	P1 M1	P1 M1	P1	P1 F2																		
	AoE	Rx Rf Rt	Rt																				
4.2	**Example 6 (Eg6)**																						
	Turn	1	3	5	7	9	11	13	15	18													

	Exp	M1 F1	P1	P1 M1	P1		M1	P1	P1 F2 G2														
	AoE	Rx			Rv	Rv	Rv Re	Rv Rt	Rt Rv	Rg													
5	**Homework (HW)**																						
	Turn	1	5	7	8																		
	Exp				G1 F2																		
	AoE	Rg	Rg	Rg	Rg																		