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Optimal Stopping Problems and American Options

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, in fulfilment of the requirements for the degree of Master of Science.

March 23, 2005

University of the Witwatersrand, Johannesburg

Declaration

I declare that, apart from the assistance acknowledged, this research is my own unaided work. It is being submitted in fulfilment of the requirements for the degree of Master of Science at the University of the Witwatersrand and has not been submitted for any other degree or examination at any other university.

Nadia Uys
23rd day of March 2005.

Abstract

The superharmonic characterization of the value function is proved, under the assumption that an optimal stopping time exists. The fair price of an American contingent claim is established as an optimal stopping problem. The price of the perpetual Russian option is derived, using the dual martingale measure to reduce the dimension of the problem. American barrier options are discussed, and the solution to the perpetual American up-and-out put is derived. The price of the American put on a finite time horizon is shown to be the price of the European put plus an early exercise premium, through the use of a local time-space formula. The optimal stopping boundary is characterised as the unique increasing solution of a non-linear integral equation. Finally, the integral representation of the price of an American floating strike Asian call with arithmetic averaging is derived.

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Chapter 1

Introduction

The aim of this work is to bring together the theory of optimal stopping problems relevant to mathematics of finance. The emphasis is on explaining the approach to solving these problems through the examples in the text. Throughout this work, it is assumed that the reader is familiar with basic probability theory. Concepts such as probability spaces, filtrations, stopping times, Brownian motion and stochastic processes are used without definition. The first chapter of [20] contains a good review of these prerequisites. Necessary results are also included in an appendix.

Suppose $X = (X_t)_{t\geq 0}$ is a stochastic process on the probability space $(\Omega, \mathcal{F}, P)$. An optimal stopping problem may be formulated as

$$V(t,x) = \sup_{\tau} \mathbb{E}_{t,x}(G(X_\tau)) \tag{1.1}$$

where τ ranges over stopping times of X. The solution to (1.1) consists of the *value function* V, and an *optimal stopping time* τ_*, at which the supremum is attained. Typically, a cost function may be associated with running the process X. In addition, there may be a reward function, dependent on the level of X, which will be realised upon termination of X. These two functions are incorporated into the *gain function* G. It is desirable to maximize the payoff, i.e. to get the maximum reward at minimum cost. The question that an optimal stopping problem seeks to answer is whether to continue running the process and incur more cost, in expectation of a greater reward later, or whether to terminate the process and accept the current reward. It is important to note that (1.1) attempts to answer this question in terms of the expected value and not over any particular sample path.

Two approaches to solving optimal stopping problems have been developed. The first is *Snell's envelope* [49], and the second is Dynkin's *superharmonic characterization of the value function* [10], which requires X to be a Markov process. Chapter 2 deals with Dynkin's approach for general Markov processes, although later chapters work only with diffusions.

The state space of the underlying process X is divided into two regions: the

continuation region C, which is open, and (as its name suggests) is the region in which it is optimal to continue running X; and the *region of instantaneous stopping* D, which is the region where it is optimal to terminate the process and receive the reward. The region D is closed, and its boundary b is called the *optimal stopping boundary*.

Finding the value function V at the same time as finding the unknown boundary b leads to a *free boundary problem*. Then V and b can be characterized as the solution to a system of partial differential equations. An explicit solution to a free boundary problem rarely exists, but the existence, uniqueness and asymptotic behaviour of the solution may be analysed.

The process X (and hence the optimal stopping problem) may be defined on a finite or an infinite time horizon. The infinite time horizon is easier to deal with (if G is time-homogeneous), since the free boundary problem for V is then a system of ordinary differential equations, and the optimal stopping boundary b degenerates to a constant. In the finite horizon case, b is a function of time, and V satisfies a system of partial differential equations.

The link between optimal stopping problems and mathematics of finance is through American contingent claims. The holder of such a claim may choose when to exercise his right. Exercising will result in a reward dependent on the stock price (and possibly the path of the stock price), which is modelled by geometric Brownian motion. Since we include discounting, there is a cost associated with not exercising the claim. Chapter 3 rigorously sets up a the model in which pricing (on a finite or an infinite time horizon) takes place. It is then shown, using standard no arbitrage pricing theory, that the price of an American contingent claim is the solution to an optimal stopping problem.

Techniques for solving optimal stopping problems are examined in Chapters 4–6 by discussing specific examples of American contingent claims. Chapters 4 and 5 deal with the perpetual Russian option and the perpetual barrier option, respectively. Chapter 6 analyses the American put on a finite time horizon.

The Russian option is an example of a problem in which the gain function is a function of the maximum value attained by the stock price over the life of the option. The solution to this problem demonstrates how a change of measure can be used to achieve a reduction in the dimension of a problem. The maximality principle developed by Peskir [34] is also discussed briefly.

The American barrier option is dealt with as an optimal stopping problem with two boundaries (one of which — the barrier — is known). The effect of the position of the barrier on the solution is discussed.

The price of an American put option is greater than that of an otherwise identical

European option, due to the added advantage provided by the early exercise feature. An integral representation of the American put price — first presented by Kim [23], Jacka [18] and Carr et al. [6] — expresses it as the sum of the corresponding European option price and an early exercise premium. Chapter 6 proves the regularity of the American option price, and derives the early exercise premium representation, using a local time-space formula developed by Peskir [36]. The optimal stopping boundary is shown to be a solution to a non-linear integral equation. It is shown that the optimal stopping boundary is the unique continuous increasing solution to this equation.

The final chapter derives the integral representation for the price and optimal stopping boundary for a floating strike Asian call option with the American early exercise feature. The proof that the optimal stopping boundary is the unique continuous increasing solution to an integral equation is left to a paper being written with G. Peskir.

Chapter 2

The General Theory of Optimal Stopping Problems

2.1 Introduction

This chapter gives the background of optimal stopping problems and deals with the techniques used to solve the specific problems dealt with in later chapters. Section 1 presents a brief history of optimal stopping problems. The next section deals with Markov processes. Section 3 proves the superharmonic characterisation of the value function using Markovian methods. The final section presents a proof of the stochastic solution to the Dirichlet problem, and hence derives the stochastic solution to the Poisson problem.

A modern formulation of the optimal stopping problem is

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\Big(G\left(X_\tau\right)\Big) \tag{2.1}$$

where $(X_t)_{t\geq 0}$ is some stochastic process, and $\mathcal{T}$ is a set of stopping times of X. The function V is called the *value function*, and the function G is called the *reward function*, or *gain function*. A solution to the optimal stopping problem consists of:

1. the value function V; and
2. a stopping time τ such that the supremum in (2.1) is attained (if such a stopping time exists), for each initial state x of X.

Any stopping time that maximizes $\mathbb{E}_x(G(X_\tau))$ is called an optimal stopping time. The most important optimal stopping time is the first one.

The study of optimal stopping problems started with the development of sequential analysis techniques in statistical inference. Sequential analysis is described in [53] as "a method of statistical inference whose characteristic feature is that the number of observations required by the procedure is not determined in advance of

the experiment." The stopping time of the experiment is therefore random, and depends on the observations previously made.

In conjunction with the Statistical Research Group of Columbia University, Wald [52] developed the sequential probability ratio test. This was a method of sequential analysis that proved very successful in reducing the number of observations required, while preserving the reliability of a test. The results of sequential analysis were linked to stochastic processes by Wald and Wolfowitz [54], and this was generalised by Arrow, Blackwell and Girshick [1]. Following their work, in 1952 Snell [49] formulated the general optimal stopping problem for a random sequence, and characterised the value function as the minimal supermartingale dominating the reward sequence. This supermartingale is known as *Snell's envelope*. Snell concentrated on general stochastic sequences and semimartingales, and hence techniques developed from this theory are often called "martingale methods".

In 1963 Dynkin [10] considered the special case of the optimal stopping problem for Markov processes. He formulated the optimal stopping problem as in (2.1), and established the *superharmonic characterisation of the value function*. Dynkin proved that if the stochastic process X is a Markov process, then the value function is the least superharmonic function that dominates the gain function. He also proved that the optimal stopping time is the first instant at which the value function is not greater than the gain function. Methods of solving optimal stopping problems that developed from Dynkin's approach are called "Markovian methods".

Optimal stopping problems are linked to free boundary problems. This connection was independently discovered by Mikhalevich [30], Chernoff [7], Lindley [26] and McKean [28]. To set up a free boundary problem that can be solved, an extra condition is needed. The *principle of smooth fit* provides this condition. It was first adopted by Mikhalevich [30], and was studied in greater depth by Shiryaev [46]. McKean [28] studied the American option as an optimal stopping problem and van Moerbeke [51] extended the results. A more complete history of optimal stopping and related problems can be found in [35].

2.2 Stopping Times and Markov Processes

2.2.1 The Shift Operator

The shift operator is useful in defining the (strong) Markov property below.

Definition 2.1. Let Ω be some space of functions from $[0, \infty)$ into $\mathbb{R}$. The shift operator $\theta_t : \Omega \to \Omega$ is defined by

$$(\theta_t(\omega))(s) = \omega(t + s) \tag{2.2}$$

for $\omega \in \Omega$. (Typically, we regard $\omega \in \Omega$ as a sample path of some stochastic process.)

Suppose that $X = (X_t)_{t\geq 0}$ is a stochastic process on the probability space $(\Omega, \mathcal{F}, P)$. The following useful results are given without proof.

Theorem 2.2. *If t and s are deterministic times, then*

$$X_t \circ \theta_s = X_{s+t}. \tag{2.3}$$

If τ and σ are stopping times with $\sigma \leq \tau$, and τ a hitting time, then

$$\tau = \sigma + \tau \circ \theta_\sigma. \tag{2.4}$$

If τ and σ are stopping times, then

$$X_\tau \circ \theta_\sigma = X_{\sigma+\tau\circ\theta_\sigma}. \tag{2.5}$$

2.2.2 The Measure P_x

Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion under the measure P. Thus each B_t is a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$, and $B_0 = 0$ under P. Now define $X_t = x + B_t$, for all $0 \leq t < \infty$. Then X_t is a random variable on the same probability space. Moreover, we see that $X_0 = x$ under P. Now, the general fact is that each process can be realised on a canonical space (see [41, page 122]). This means that there exists a probability measure Q on the measurable space $(C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)))$, such that the co-ordinate process $\pi = (\pi_t)_{t\geq 0}$ (defined by $\pi_t(x) = x(t)$, for $t \geq 0$) on this space, and the process X, have the same finite dimensional distributions. Since X depends on x, so does Q, and we may write P_x instead of Q. In this way we obtain a family of probability measures $(P_x)_{x\in\mathbb{R}}$ on $(C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)))$, so that the co-ordinate process π starts at x under P_x.

2.2.3 The (Strong) Markov Property

The future behaviour of a Markov process is not dependent on its past, but only on its current value.

Definition 2.3. If a process $X = (X_t)_{t\geq 0}$ is equipped with the filtration $(\mathcal{F}_t)_{t\geq 0}$, with $\mathcal{F} = \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$, then X has the (strong) Markov property if any of the following three equivalent conditions hold:

$$\mathbb{E}_x\left(f(X_{\tau+h}) \mid \mathcal{F}_\tau\right) = \mathbb{E}_x\left(f(X_{\tau+h}) \mid X_\tau\right) \tag{2.6}$$

$$\mathbb{E}_x\left(f(X_{\tau+h}) \mid \mathcal{F}_\tau\right) = \mathbb{E}_{X_\tau}\left(f(X_h)\right) \tag{2.7}$$

$$\mathbb{E}_x\left(Y \circ \theta_\tau \mid \mathcal{F}_\tau\right) = \mathbb{E}_{X_\tau}(Y) \tag{2.8}$$

for all x, all stopping times τ, all $h > 0$, any bounded Borel-measurable function f, and any (bounded) $\mathcal{F}$-measurable random variable Y.

2.2.4 The Infinitesimal Generator

It is very useful to be able to associate a second order partial differential operator with a stochastic process. This is what the infinitesimal generator does. We will deal only with twice differential functions f, except possibly at one point (see Chapter 6). The following definitions and results are taken from [31, Chapter 7].

The characteristic operator is most useful in understanding optimal stopping problems since it deals with first passage times of shrinking domains about a point.

Definition 2.4. Let $X = (X_t)_{t\geq 0}$ be a (time-homogeneous) Itô diffusion in $\mathbb{R}^n$. The characteristic operator $\mathcal{A}_X$ of X is defined by

$$\mathcal{A}_X f(x) = \lim_{U \downarrow x} \frac{E^x\big(f(X_{\tau_U})\big) - f(x)}{E^x(\tau_U)} \tag{2.9}$$

where the U's are open sets U_k decreasing to the point x, in the sense that $U_{k+1} \subset U_k$ and $\bigcap_k U_k = \{x\}$, and $\tau_U = \inf\{t > 0 | X_t \notin U\}$ is the first exit time from U for X_t. The set of functions for which the limit exists for all $X \in \mathbb{R}^n$ (and all $\{U_k\}$) is denoted by $\mathcal{D}_{\mathcal{A}}$. If $E^x(\tau_U) = \infty$ for all open U such that $x \in U$ we define $\mathcal{A}_X f(x) = 0$.

The characteristic operator corresponds with the infinitesimal generator defined below, for twice differentiable functions, $f \in C^{1,2}$.

Definition 2.5. Let $X = (X_t)_{t\geq 0}$ be a (time-homogeneous) Itô diffusion in $\mathbb{R}^n$. The *infinitesimal generator* $\mathbb{L}_X$ of X_t is defined by

$$\mathbb{L}_X f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x(f(X_t)) - f(x)}{t} \tag{2.10}$$

for $x \in \mathbb{R}^n$. The set of functions $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that the limit exists at x is denoted by $\mathcal{D}_{\mathbb{L}}(x)$, while $\mathcal{D}_{\mathbb{L}}$ denotes the set of functions for which the limit exists for all $x \in \mathbb{R}^n$.

We can write down the formula for the generator $\mathbb{L}_X$ of an Itô diffusion.

Theorem 2.6. *Let $X = (X_t)_{t\geq 0}$ be the Itô diffusion with stochastic differential equation*

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dB_t. \tag{2.11}$$

If $f \in C^{1,2}$ then $f \in \mathcal{D}_{\mathbb{L}}$ and

$$\mathbb{L}_X f(x) = \sum_i b_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{2.12}$$

2.3 The Superharmonic Characterisation

First we define what it means for a function to be superharmonic.

Definition 2.7. A function V is called *superharmonic* if

$$\mathbb{E}_x\left(V(X_\sigma)\right) \leq V(x) \tag{2.13}$$

for all x, and for every stopping time σ. If the process X has the (strong) Markov property and is adequately regular (as is assumed below), this is equivalent to saying that the process $(V(X_t))_{t\geq 0}$ is a *supermartingale* under P_x, for each x.

Dynkin's superharmonic characterisation of the value function for Markov processes is captured in the four statements of the following theorem.

Theorem 2.8. *Let*

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\left(G(X_\tau)\right) \tag{2.14}$$

where $X = (X_t)_{t\geq 0}$ is a (strong) Markov process, started at x under P_x, and $\mathcal{T}$ is a set of stopping times. Suppose that the stopping time $\tau_* \in \mathcal{T}$ is optimal (i.e. $V(x) = \mathbb{E}_x(G(X_{\tau_*}))$). Then (under certain regularity conditions):

(i) *The value function V is the smallest superharmonic function which dominates the gain function G.*

(ii) *$\tau_D \leq \tau_*$ P_x-a.s., where the stopping time τ_D is defined by*

$$\tau_D = \inf\{t \geq 0 \mid X_t \in D\} \tag{2.15}$$

and where $C = \{x \mid V(x) > G(x)\}$ and $D = C^c = \{x \mid V(x) = G(x)\}$.

(iii) *The stopping time τ_D defined in (2.15) is optimal.*

(iv) *The process $(V(X_{t\wedge\tau_D}))_{t\geq 0}$ is a martingale.*

The proofs of the statements will be given under the following assumptions:

1. there exists an optimal stopping time τ_* (at which the supremum is attained); and
2. all functions are adequately regular.

Assumption 1 is reasonable, since we will approach a problem under this assumption when dealing with specific examples. If no optimal stopping time exists, this will become apparent, and then the structure of the problem will require deeper analysis. Under this assumption, V is finite.

Proof. (i) First we show that V is a superharmonic function. By (2.8) and (2.5), we have

$$\begin{aligned}\mathbb{E}_x\left(V(X_\sigma)\right) &= \mathbb{E}_x\left(\mathbb{E}_{X_\sigma}(G(X_{\tau_*}))\right) = \mathbb{E}_x\left(\mathbb{E}_x(G(X_{\tau_*})\circ\theta_\sigma \mid \mathcal{F}_\sigma)\right) \\ &= \mathbb{E}_x\left(G(X_{\sigma+\tau_*\circ\theta_\sigma})\right) \leq \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\left(G(X_\tau)\right) = V(x)\end{aligned} \tag{2.16}$$

for all x, and for any stopping time σ. The inequality arises because $\sigma + \tau_* \circ \theta_\sigma$ is a stopping time.

Next we show that V is the smallest superharmonic function that dominates G. Clearly V dominates G, since

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\left(G(X_\tau)\right) \geq \mathbb{E}_x\left(G(X_0)\right) = G(x). \tag{2.17}$$

Let $\hat{V}$ be any superharmonic function dominating G. Then

$$\mathbb{E}_x\left(G(X_\tau)\right) \leq \mathbb{E}_x\left(\hat{V}(X_\tau)\right) \leq \hat{V}(x) \tag{2.18}$$

for every stopping time τ. Hence

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\left(G(X_\tau)\right) \leq \hat{V}(x) \tag{2.19}$$

for all x; showing that $V \leq \hat{V}$. Thus V is the least superharmonic function that dominates G.

(ii) We show that $\tau_D \leq \tau_*$ where τ_D is defined in (2.15). First note that $V(X_{\tau_*}) = G(X_{\tau_*})$ P_x-a.s., for all x. Indeed, if for some x we have $P_x(V(X_{\tau_*}) > G(X_{\tau_*})) > 0$, then $\mathbb{E}_x(G(X_{\tau_*})) < \mathbb{E}_x(V(X_{\tau_*})) \leq V(x)$, since $(V(X_t))_{t\geq 0}$ is a supermartingale. This contradicts the fact that τ_* is optimal. It thus follows that $\tau_D \leq \tau_*$ P_x-a.s., since τ_D is the first time that $V = G$.

(iii) We will now show that τ_D is optimal. Since $V(X_{\tau_*}) = G(X_{\tau_*})$ (as shown above), we have

$$V(x) = \mathbb{E}_x\left(G(X_{\tau_*})\right) = \mathbb{E}_x\left(V(X_{\tau_*})\right) \leq \mathbb{E}_x\left(V(X_{\tau_D})\right) = \mathbb{E}_x\left(G(X_{\tau_D})\right) \leq V(x) \tag{2.20}$$

where the first inequality arises because $(V(X_t))_{t\geq 0}$ is a supermartingale and $\tau_D \leq \tau_*$ (from above). Thus τ_D is optimal.

(iv) We will now show that $(V(X_t))_{t\geq 0}$ is a martingale in the continuation region C. To do this, take any stopping time $\sigma \leq \tau_D$. Then from (2.4), (2.5) and (2.8), we have

$$\begin{aligned}E_x\left(V(X_\sigma)\right) &= \mathbb{E}_x\left(\mathbb{E}_{X_\sigma}\left(G(X_{\tau_D})\right)\right) = \mathbb{E}_x\left(\mathbb{E}_x\left(G(X_{\tau_D})\circ\theta_\sigma \mid \mathcal{F}_\sigma\right)\right) \\ &= \mathbb{E}_x\left(\mathbb{E}_x\left(G(X_{\sigma+\tau_D\circ\theta_\sigma}) \mid \mathcal{F}_\sigma\right)\right) = \mathbb{E}_x\left(G(X_{\tau_D})\right) = V(x)\end{aligned} \tag{2.21}$$

for all x. Now

$$E_x\left(V(X_{\tau_D})|\mathcal{F}_{t\wedge\tau_D}\right) = E_x\big(V(X_{t\wedge\tau_D+\tau_D\circ\theta_{t\wedge\tau_D}})|\mathcal{F}_{t\wedge\tau_D}\big) \tag{2.22}$$
$$= E_x\left(V(X_{\tau_D})\circ\theta_{t\wedge\tau_D}|\mathcal{F}_{t\wedge\tau_D}\right) = E_{X_{t\wedge\tau_D}}(V(X_{\tau_D})) = V(X_{t\wedge\tau_D})$$

where the third equality is a result of the strong Markov property of X and the final equality is a result of (2.21). Thus $(V(X_{t\wedge\tau_D}))_{t\geq 0}$ is a martingale. □

If we assume that an optimal stopping time exists, then it is possible to link the optimal stopping problem to a Dirichlet problem in which the boundary is not known (i.e. a free boundary problem). It is interesting to note that this free boundary problem arises directly from the fact that $(V(X_t))_{t\geq 0}$ is a martingale in the continuation region. This fact is proved above in the special case when X is a Markov process, but the result extends to more general processes, using Snell's envelope.

Since $(V(X_t))_{t\geq 0}$ is a martingale in the continuation region C, the definition of the infinitesimal generator gives

$$\mathbb{L}_X V(x) = 0 \tag{2.23}$$

for all $x \in C$. Also, by the definition (2.15) of τ_D, we have $V = G$ on D, and hence at the boundary; so that

$$V(X_{\tau_D}) = G(X_{\tau_D}). \tag{2.24}$$

Thus we have the free boundary problem

$$\mathbb{L}_X V = 0 \quad \text{in } C \tag{2.25}$$
$$V|_{\partial C} = G. \tag{2.26}$$

This system can often be solved using the principle of smooth fit, which states that the value function should join the gain function smoothly at the boundary, so that

$$\left.\frac{\partial V}{\partial x}\right|_{\partial C} = \frac{\partial G}{\partial x}\,. \tag{2.27}$$

This is an extra condition that we include in order to solve the problem and in practice, including this condition gives the right result. Why the principal of smooth fit should work is still obscure, but it seems intuitive that the boundary should be smooth.

2.4 Properties of the value function

We will show that the value function V is $C^{1,2}$ in the continuation region C. This argument relies on the coefficients of the infinitesimal generator being $C^{1,2}$. Recall

that if the process $X = (X_t)_{t\geq 0}$ satisfies

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t \tag{2.28}$$

then the infinitesimal generator of X is

$$\mathbb{L}_X = b(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}. \tag{2.29}$$

Let $X \in C$ and let $U \subset C$ be an open ball such that $x \in G$. Such a set exists since C is open. By [12, Theorem 0.6, page 230], there exists a function f that is continuous on $\bar{U}$ and $C^{1,2}$ on U and which solves

$$\frac{\partial f}{\partial t} + \mathbb{L}_X f = 0 \quad \text{in } U \tag{2.30}$$

$$f\,|_{\partial U} = V. \tag{2.31}$$

Now compose f with the process X and apply Itô to get

$$f(X_t) - f(x) = \int_0^t \left(\frac{\partial f}{\partial t}(X_s) + \mathbb{L}_X f(X_s)\right) ds + \int_0^t \frac{\partial f}{\partial x}\,\sigma(X_s)\,dB_s. \tag{2.32}$$

Let

$$\tau_U = \inf\{t \geq 0 \mid X_t \in \partial U\} \tag{2.33}$$

and apply the optional stopping theorem to get

$$f(X_{\tau_U}) - f(x) = \int_0^{\tau_U} \left(\frac{\partial f}{\partial t}(X_s) + \mathbb{L}_X f(X_s)\right) ds + \int_0^{\tau_U} \frac{\partial f}{\partial x}\,\sigma(X_s)\,dB_s. \tag{2.34}$$

By (2.30) and (2.31) we have

$$V(X_{\tau_U}) - f(x) = \int_0^{\tau_U} \frac{\partial f}{\partial x}\,\sigma(X_s)\,dB_s \tag{2.35}$$

and by taking expectations we have

$$V(x) = f(x). \tag{2.36}$$

Since x and U were arbitrary, the properties of f extend to V, and V is $C^{1,2}$.

Now let us show that V is convex if the gain function G is convex. Let $\lambda \in (0,1)$. By the definition of V we have

$$\begin{aligned} &\lambda V(t,x) + (1-\lambda)V(t,y) \\ &= \sup_{0\leq\tau\leq T-t} \lambda\mathbb{E}_x\Big(G(X_\tau)\Big) + \sup_{0\leq\tau\leq T-t} (1-\lambda)\mathbb{E}_y\Big(G(X_\tau)\Big) \\ &\geq \sup_{0\leq\tau\leq T-t} \Big(\lambda\mathbb{E}_x\Big(G(X_\tau)\Big) + (1-\lambda)\mathbb{E}_y\Big(G(X_\tau)\Big)\Big). \end{aligned} \tag{2.37}$$

Now by the convexity of G, and because X is Markovian, we have

$$\begin{aligned}(2.38)\qquad &\lambda\mathbb{E}\Big(G(X_\tau)\Big) + (1-\lambda)\mathbb{E}_y\Big(G(X_\tau)\Big)\\ &= \mathbb{E}_1(\lambda G(xX_\tau) + (1-\lambda)G(yX_\tau))\\ &\geq \mathbb{E}_1\Big(G((\lambda x + (1-\lambda)y)X_\tau)\Big)\\ &= \mathbb{E}_{\lambda x+(1-\lambda)y}\Big(G(X_\tau)\Big).\end{aligned}$$

Thus

$$\begin{aligned}(2.39)\qquad &\lambda V(t,x) + (1-\lambda)V(t,y)\\ &\geq \sup_{0\leq\tau\leq T-t}\left(\mathbb{E}_{\lambda x+(1-\lambda)y}\Big(G(X_\tau)\Big)\right)\\ &= V(t,\lambda x + (1-\lambda)y)\end{aligned}$$

and this proves that V is convex.

2.5 Free Boundary Problems

2.5.1 The Dirichlet Problem

Motivated by the link between optimal stopping problems and free boundary problems, it is instructive to consider the stochastic solution to the Dirichlet problem.

Theorem 2.9. *Let C be an open set with a regular boundary (see [31, page 172]). Suppose a function $V \in C^2(C)\cap C(\bar{C})$ solves the system*

$$(2.40)\qquad \mathbb{L}_X V = 0 \quad \text{in } C$$

$$(2.41)\qquad V|_{\partial C} = G.$$

Then V is given by

$$(2.42)\qquad V(x) = \mathbb{E}_x\left(G(X_{\tau_D})\right)$$

where $(X_t)_{t\geq 0}$ is the Markov process corresponding to the infinitesimal generator $\mathbb{L}_X$, and the stopping time τ_D is defined by

$$(2.43)\qquad \tau_D = \inf\left\{t \geq 0 \mid X_t \notin C\right\}.$$

[Only if the boundary is regular is the stopping time τ_D also the stopping time $\tau_* = \inf\{t \geq 0 | X_t = \partial C\}$, which is the implied stopping time required in $V(x) = \mathbb{E}_x(G(X_{\tau_D}))$.]

Proof. We will show that the function V defined by (2.42) solves the system (2.40)–(2.41). Let $U \subset C$ be an open ball around the point $x \in C$, and define

$$\tau_U = \inf\{t \geq 0 \mid X_t \notin U\}. \tag{2.44}$$

From (2.42), we have

$$V(X_{\tau_U}) = \mathbb{E}_{X_{\tau_U}}(G(X_{\tau_D})). \tag{2.45}$$

By the (strong) Markov property for X, and (2.8), it follows that

$$\mathbb{E}_{X_{\tau_U}}(G(X_{\tau_D})) = \mathbb{E}_x\left(G(X_{\tau_D}) \circ \theta_{\tau_U} \mid \mathcal{F}_{\tau_U}\right). \tag{2.46}$$

Also, from (2.5) and (2.4), we have

$$G(X_{\tau_D}) \circ \theta_{\tau_U} = G(X_{\tau_D} \circ \theta_{\tau_U}) = G(X_{\tau_U + \tau_D \circ \theta_{\tau_U}}) = G(X_{\tau_D}). \tag{2.47}$$

Thus, it follows from (2.46) and (2.47) that

$$V(X_{\tau_U}) = \mathbb{E}_x\left(G(X_{\tau_D}) \mid \mathcal{F}_{\tau_U}\right) \tag{2.48}$$

and hence taking expectations yields

$$\mathbb{E}_x\left(V(X_{\tau_U})\right) - V(x) = 0. \tag{2.49}$$

From the definition of the infinitesimal generator, it follows that

$$\mathbb{L}_X V(x) = 0 \tag{2.50}$$

for all $x \in C$. Clearly $V(X_{\tau_D}) = G(X_{\tau_D})$, and thus the function V, defined in (2.42), solves the system (2.40)–(2.41). □

2.5.2 The Poisson Problem

It is useful to see the method of deriving the stochastic solution to the Poisson problem from the solution to the Dirichlet problem.

Theorem 2.10. *Let C be an open set with a regular boundary (see [31, page 172]). Suppose a function $V \in C^2(C) \cap C(\bar{C})$ solves the system*

$$\mathbb{L}_X V = -g \quad \text{in } C \tag{2.51}$$

$$V|_{\partial C} = 0. \tag{2.52}$$

Then V is given by

$$V(x) = \mathbb{E}_x\left(\int_0^{\tau_D} g(X_t)\, dt\right) \tag{2.53}$$

where $(X_t)_{t \geq 0}$ is the Markov process corresponding to the infinitesimal generator $\mathbb{L}_X$, and the stopping time τ_D is defined by

$$\tau_D = \inf\left\{t \geq 0 \mid X_t \notin C\right\}. \tag{2.54}$$

Proof. Let the process $Y = (Y_t)_{t\geq 0}$ be defined by

$$Y_t = y + \int_0^t g(X_s)\, ds. \tag{2.55}$$

Then the process $Z = (Z_t)_{t\geq 0}$ given by

$$Z_t = (X_t, Y_t) \tag{2.56}$$

is a two dimensional Markov process. Let

$$G(z) = G(x, y) = y. \tag{2.57}$$

Then, from (2.53), we have

$$V(x) = \mathbb{E}_x \left(\int_0^{\tau_D} g(X_s)\, ds \right) = \mathbb{E}_{x,y} \left(G(X_{\tau_D}, Y_{\tau_D}) - y \right). \tag{2.58}$$

Now define

$$\tilde{V}(z) = \mathbb{E}_z \left(G(Z_{\tau_D}) \right). \tag{2.59}$$

From the stochastic solution to the Dirichlet problem above, we have

$$\mathbb{L}_Z \tilde{V}(z) = 0 \tag{2.60}$$

in the region C. We know that

$$\mathbb{L}_Z = \mathbb{L}_X + g\frac{\partial}{\partial y}. \tag{2.61}$$

Also notice that

$$\tilde{V}(z) = V(x) + y. \tag{2.62}$$

From (2.60), we have

$$\mathbb{L}_X \tilde{V} + g\frac{\partial}{\partial y}\tilde{V} = 0. \tag{2.63}$$

Thus

$$\mathbb{L}_X (V + y) + g\frac{\partial}{\partial y}(V + y) = 0 \tag{2.64}$$

which gives

$$\mathbb{L}_X V = -g \tag{2.65}$$

proving the theorem. □

Chapter 3

American Contingent Claims

3.1 Introduction

The aim of this chapter is to establish the relationship between the arbitrage-free price of an American contingent claim and the solution to an optimal stopping problem. The environment in which pricing will take place is defined in Section 2. Section 3 derives the risk-neutral martingale measure $\tilde{P}_T$ for a finite time horizon $T \in [0, \infty)$. The family of measures $(\tilde{P}_T)_{0 \leq T < \infty}$ is then extended to a measure $\tilde{P}$ on an infinite time horizon. Section 4 describes investment strategies and defines the concept of arbitrage, and Section 5 deals with pricing American contingent claims.

3.2 The Market Model

Let the process $W = (W_t)_{t \geq 0}$, with $W_t = (W_t^1, \ldots, W_t^d)$, be a standard d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, equipped with the filtration $(\mathcal{F}_t)_{t \geq 0}$. The latter is the usual augmentation of the natural filtration

$$\mathcal{F}_t^W = \sigma\left(W_t \mid 0 \leq t < \infty\right) \tag{3.1}$$

generated by the Brownian motion W (see [41, page 172]). Letting

$$\mathcal{N} = \left\{F \subseteq \Omega \mid \exists\, G \in \mathcal{F}^W \text{ with } F \subseteq G \text{ and } P(G) = 0\right\} \tag{3.2}$$

be the collection of P-null subsets of Ω, where

$$\mathcal{F}^W = \mathcal{F}_\infty^W = \sigma\Big(\bigcup_{t \geq 0} \mathcal{F}_t^W\Big), \tag{3.3}$$

the filtration $(\mathcal{F}_t)_{t \geq 0}$ is defined by

$$\mathcal{F}_t = \sigma\left(\mathcal{F}_t^W \cup \mathcal{N}\right). \tag{3.4}$$

It is assumed that

$$\mathcal{F} = \sigma\Big(\bigcup_{t\geq 0} \mathcal{F}_t\Big). \tag{3.5}$$

The augmented filtration is necessary, since the filtration generated by a Brownian motion does not satisfy the usual conditions [41, page 172]. These are required, to ensure that the first entrance time (or début time) of the process into an open set is a stopping time (see [41, page 183]).

Consider a market in which $d+1$ assets are traded continuously. These assets include d risky assets and a *bond* (or *bank account*). The value of the bond follows the process $B = (B_t)_{t\geq 0}$, which is determined by

$$dB_t = r_t B_t\, dt \qquad (B_0 = 1). \tag{3.6}$$

Thus

$$B_t = \exp\left(\int_0^t r_s\, ds\right). \tag{3.7}$$

The process $A = (A_t)_{t\geq 0}$, given by

$$A_t = \frac{1}{B_t} = \exp\left(-\int_0^t r_s\, ds\right) \tag{3.8}$$

is known as the *discount factor*.

The d risky assets will also be referred to as *stocks*. Together they follow the vector process $S = (S_t)_{t\geq 0}$, where $S_t = \left(S_t^1, \ldots, S_t^d\right)^{\mathrm{T}}$, and the value of the ith stock satisfies the stochastic differential equation

$$dS_t^i = S_t^i\Big(\left(\mu_t^i + \delta_t^i\right)\, dt + \sum_{j=1}^{d} \sigma_t^{i,j}\, dW_t^j\Big). \tag{3.9}$$

Initially, it will be assumed that the *risk free rate* r, the *drift* process μ, the *dividend* process δ, and the *volatility* process σ, are $\mathcal{B}\otimes\mathcal{F}$–measurable, adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, and uniformly bounded in $(t,\omega) \in [0,T]\times\Omega$, for every finite $T > 0$. In addition, the matrix σ_t will be assumed to be non-singular for a.a. $t \in [0,T)$, when $T > 0$. In order to price perpetual instruments, the measurablity requirements will need to be strengthened. Similarly to [21], a financial market will be defined in the following way.

Definition 3.1. A financial market consists of

1. a probability space $(\Omega, \mathcal{F}, P)$;

2. a d-dimensional Brownian motion $(W_t)_{0\leq t<\infty}$ defined on $(\Omega, \mathcal{F}, P)$, where $(\mathcal{F}_t)_{t\geq 0}$ is the augmentation of the filtration generated by the Brownian motion;

3. a measurable and adapted risk-free rate process r, satisfying $\int_0^T |r_t|\, dt < \infty$ a.s., for every finite $T > 0$;

4. a measurable and adapted d-dimensional drift process μ, satisfying $\int_0^T \|\mu_t\|\, dt < \infty$ a.s., for every finite $T > 0$;

5. a measurable and adapted d-dimensional dividend process δ, satisfying $\int_0^T \|\delta_t\|\, dt < \infty$ a.s., for every finite $T > 0$;

6. a measurable and adapted $d \times d$ matrix-valued volatility process σ, satisfying $\sum_{i=1}^d \sum_{j=1}^d \int_0^T (\sigma_t^{i,j})^2\, dt < \infty$ a.s., for every finite $T > 0$, and with σ_t non-singular, for a.a. $t \in [0, T)$, when $T > 0$; and

7. a vector of positive, constant initial stock prices $S_0 = \left(S_0^1, \ldots, S_0^d\right)^{\mathrm{T}}$.

This market will be denoted by $\mathcal{M} = (r, \mu, \delta, \sigma, S_0)$.

It should be noted that, by [21, Theorem 6.6, page 24], since the dimension of the driving Brownian motion is equal to the number of risky assets, and since the volatility matrix σ_t is non-singular, for a.a. $t \in [0, T)$ where $T > 0$, the market $\mathcal{M}$ is complete. A complete market has the property that all contingent claims can be replicated by trading in the underlying stocks and the bond.

3.3 The Martingale Measure

3.3.1 Finite Time Horizon

The famous result of Black and Scholes [4] and Merton [29] owes much of its success to the absence of the drift parameter μ in the option pricing formula. The drift of a stock price process is more difficult to estimate than its volatility, which may be estimated from an arbitrarily small interval of continuous measurements (cf. [48, page 63]). The method of eliminating the drift through a change of measure was first identified in discrete time by Harrison and Krepps [16], and was extended to continuous time by Harrison and Pliska [17].

Let

$$\theta_t = \sigma_t^{-1} \left(\mu_t - r_t \underline{\mathbf{1}}\right) \tag{3.10}$$

for $0 \leq t < \infty$, where $\underline{\mathbf{1}}$ denotes the vector of ones. It is clear that the process θ is measurable, adapted and bounded, owing to the assumptions on the other processes. Thus, the process $\tilde{Z} = (\tilde{Z}_t)_{0 \leq t < \infty}$, defined by

$$\tilde{Z}_t = \exp\left(-\sum_{i=1}^{d} \int_0^t \theta_s^i \, dW_s^i - \frac{1}{2} \int_0^t ||\, \theta_s ||^2 \, ds \right) \tag{3.11}$$

for $0 \leq t < \infty$, is a martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$, since the conditions in Definition 3.1 ensure that the the Novikov condition

$$\mathbb{E}\left(\exp\left(\frac{1}{2} \int_0^t ||\, \theta_s ||^2 \, ds \right) \right) < \infty \tag{3.12}$$

for $0 \leq t < \infty$, is satisfied (see [21, page 199]). Here $\mathbb{E}(\cdot)$ represents expectation with respect to P.

The random variable $\tilde{Z}_T$ may be used as a Radon-Nikodym derivative to define another measure on $\mathcal{F}_T$, where $0 \leq T < \infty$ is fixed, by setting

$$\tilde{P}_T(A) = \mathbb{E}\left(\tilde{Z}_T \mathbf{1}_A \right) \tag{3.13}$$

for $A \in \mathcal{F}_T$. As a result of the martingale property of $\tilde{Z}$, the collection of probability measures $(\tilde{P}_T)_{0 \leq T < \infty}$ satisfies the consistency condition

$$\tilde{P}_T(A) = \tilde{P}_t(A) \tag{3.14}$$

for $A \in \mathcal{F}_t$ and $0 \leq t \leq T$. This follows from

$$\begin{aligned} \tilde{P}_t(A) &= \mathbb{E}\left(\tilde{Z}_t \mathbf{1}_A \right) = \mathbb{E}\left(\mathbb{E}\left(\tilde{Z}_T \mid \mathcal{F}_t \right) \mathbf{1}_A \right) \\ &= \mathbb{E}\left(\mathbb{E}\left(\tilde{Z}_T \mathbf{1}_A \mid \mathcal{F}_t \right) \right) = \mathbb{E}\left(\tilde{Z}_T \mathbf{1}_A \right) = \tilde{P}_T(A) \end{aligned} \tag{3.15}$$

for all $A \in \mathcal{F}_t$ (the third last equality follows from the $\mathcal{F}_t$-measurability of $\mathbf{1}_A$).

It is pointed out in [48] that the measures P_T and $\tilde{P}_T$ are equivalent for every finite $T \geq 0$. If only finite time horizon problems were to be considered, then the consistent collection of measures $(\tilde{P}_T)_{0 \leq T < \infty}$ would be sufficient. However, when considering problems on an infinite time horizon, it is necessary to extend this collection in order to obtain a single measure $\tilde{P}$, defined for an infinite time horizon. This will be dealt with in the next section.

The new measure $\tilde{P}_T$ can be used to define a new Brownian motion. Define the process $\tilde{W} = (\tilde{W}_t)_{0 \leq t \leq T}$, where $\tilde{W}_t = \left(\tilde{W}_t^1, \ldots, \tilde{W}_t^d \right)^{\mathrm{T}}$, by setting

$$\tilde{W}_t^i = W_t^i - \int_0^t \theta_s^i \, ds \tag{3.16}$$

for $1 \leq i \leq d$ and $0 \leq t < \infty$.

By Girsanov's theorem (see Theorem B.5), since the process $\tilde{Z}$ of (3.11) is a positive martingale, $\tilde{W}$ is a d-dimensional Brownian motion on $(\Omega, \mathcal{F}_T, \tilde{P}_T)$, for each fixed $T \in [0, \infty)$ (see [20, Section 3.5]).

3.3.2 Infinite Time Horizon

When pricing perpetual instruments, it is important that $\tilde{W}$ be a d-dimensional Brownian motion on the entire interval $[0, \infty)$. It is thus necessary for the collection of measures $(\tilde{P}_T)_{0 \leq T < \infty}$ to be extended to obtain a measure $\tilde{P}$, defined on $\mathcal{F}_\infty^W$ (see (3.3)), in such a way that $\tilde{P}$ restricted to any $\mathcal{F}_T^W$ agrees with $\tilde{P}_T$. This extension is discussed in [20], [21], [19] and [48], and the results in [32] are useful in this context. However, it does not exist in general, and so we restrict our attention to the measurable space $(C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)))$ (see [20, page 49]). Let $W : [0, \infty) \times C(\mathbb{R}_+) \to \mathbb{R}$ be the coordinate mapping process on this space, defined by $W_t(\omega) = \omega(t)$, for all $t \geq 0$ and $\omega \in C(\mathbb{R}_+)$. By [20, Corollary 2.11, page 55], there exists a probability measure P on $(C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)))$ such that W equipped with its natural filtration $(\mathcal{F}_t^W)_{t \geq 0}$ is a Brownian motion on $(C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)), P)$.

The Brownian motion $\tilde{W}$, which we aim to construct, must be accompanied by a filtration which satisfies the usual conditions and measures all the processes in the market model. To resolve any technical difficulties caused by these requirements, the assumption is made in [19] that all the market processes are progressively measurable with respect to the natural filtration of W.

Following the argument in [48], consider the algebra $\mathcal{A} = \bigcup_{t \geq 0} \mathcal{F}_t^W$, and define a set function $\tilde{P}$ on its elements, by setting $\tilde{P}(A) = \tilde{P}_t(A)$, if $A \in \mathcal{F}_t^W$, where $\tilde{P}_t(A)$ is defined in (3.13). It has been shown that the collection of measures $(\tilde{P}_t)_{0 \leq t < \infty}$ forms a consistent family. Thus $\tilde{P}$ is uniquely defined on $\mathcal{A}$, and is a finitely additive set function. The aim is to use the Caratheodory extension theorem [41, Theorem 5.1, page 93] to extend $\tilde{P}$ to $\mathcal{F}_\infty^W = \mathcal{B}(C(\mathbb{R}_+))$. In order to apply the theorem, it remains to be shown that $\tilde{P}$ is countably additive.

The result in [41, Lemma 4.3, page 92] shows that if $\tilde{P}(A_n) \to 0$ as $n \to \infty$, for each sequence $(A_n)_{n \geq 1}$ of sets from $\mathcal{A}$ with $A_n \downarrow 0$, then the set function $\tilde{P}$ is countably additive. This can be verified in the present case, because we are working in $(\Omega, \mathcal{F}) = (C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)))$, which is a Borel space, and thus the results in [32, Theorem 4.1, page 141] and [32, Theorem 4.2, page 143] hold. Therefore, according to the Caratheodory extension theorem, $\tilde{P}$ admits a unique extension, also denoted by $\tilde{P}$, to a probability measure on the σ-algebra $\mathcal{F}_\infty^W = \sigma(\mathcal{A}) = \mathcal{B}(C(\mathbb{R}_+))$.

Now that the existence and uniqueness of the measure $\tilde{P}$ on $\mathcal{F}_\infty^W$ has been shown, it is possible to define a Brownian motion on $(\Omega, \mathcal{F}_\infty^W, \tilde{P})$. Recall the process $\tilde{W}$ defined in (3.16). Let $0 \leq t_1 < \cdots < t_n \leq t$ be given. Then

$$\tilde{P}\Big((\tilde{W}_{t_1}, \ldots, \tilde{W}_{t_n}) \in \Gamma\Big) = \tilde{P}_t\Big((\tilde{W}_{t_1}, \ldots, \tilde{W}_{t_n}) \in \Gamma\Big) \tag{3.17}$$

where $\Gamma \in (\mathbb{R}^{d \times n})$. Thus the finite dimensional distributions of the process $\tilde{W}$ are the same under the two measures. That means that since $\tilde{W}$ is a Brownian motion

under $\tilde{P}_t$, it must be a Brownian motion under $\tilde{P}$ as well.

Recall that it is required that the Brownian motion $\tilde{W}$ be accompanied by a filtration satisfying the usual conditions. The filtration $(\mathcal{F}_t^W)_{t\geq 0}$ is not right continuous, and so it does not qualify. The filtration $(\mathcal{F}_t)_{t\geq 0}$, defined in (3.4) above, does satisfy the usual conditions, but it is not a suitable choice of filtration with which to endow $(\Omega, \mathcal{F}_\infty^W)$, since there is no guarantee that the measures P and $\tilde{P}$ will be equivalent on $(\mathcal{F}_t)_{t\geq 0}$.

The measures P and $\tilde{P}$ are clearly equivalent when restricted to $\mathcal{F}_T^W$. However, viewed as probability measures on $\mathcal{F}_\infty^W$, P and $\tilde{P}$ are equivalent if and only if the martingale $\tilde{Z}$ of (3.11) is uniformly integrable. The following example from [20, page 193] illustrates this point.

Suppose $d = 1$, and let $\theta_t = \alpha$ be a non-zero constant. Then the P-martingale

$$\tilde{Z}_t = \exp\left(\alpha W_t - \frac{1}{2}\alpha^2 t\right) \tag{3.18}$$

for $0 \leq t < \infty$, is not uniformly integrable. The law of large numbers implies that

$$\tilde{P}\left(\lim_{t\to\infty} \frac{1}{t} W_t = \alpha\right) = \tilde{P}\left(\lim_{t\to\infty} \frac{1}{t} \tilde{W}_t = 0\right) = 1 \tag{3.19}$$

and that

$$P\left(\lim_{t\to\infty} \frac{1}{t} W_t = \alpha\right) = 0. \tag{3.20}$$

In particular, the P-null event $\left\{\lim_{t\to\infty} \frac{1}{t} W_t = \alpha\right\}$ is in $\mathcal{F}_T$, as defined in (3.4), for every $0 \leq T < \infty$, so $\tilde{P}$ and $\tilde{P}_T$ cannot agree on $\mathcal{F}_T$, and the consistency condition fails.

A better choice of filtration is suggested in [19]. Let $\mathcal{M}_t$ denote the augmentation under $\tilde{P}$ of $\mathcal{F}_t^W$, i.e.

$$\mathcal{M}_t = \sigma\left(\mathcal{F}_t^W \cup \mathcal{N}_t\right) \tag{3.21}$$

where

$$\mathcal{N}_t = \left\{F \subseteq C_d(\mathbb{R}_+) \mid \exists\, G \in \mathcal{F}_t^W \text{ with } F \subseteq G \text{ and } \tilde{P}(G) = 0\right\}. \tag{3.22}$$

Define

$$\tilde{\mathcal{F}}_t = \mathcal{M}_{t+} = \bigcap_{\varepsilon>0} \mathcal{M}_{t+\varepsilon} \tag{3.23}$$

for $0 \leq t < \infty$. Since the measures P and $\tilde{P}$ are equivalent when restricted to $\mathcal{F}_t^W$, they are also equivalent on $\tilde{\mathcal{F}}_t$. The filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ obviously satisfies the usual conditions.

The process $\tilde{W}$ is a standard Brownian motion on the probability space $(C_d(\mathbb{R}_+), \mathcal{B}(C_d(\mathbb{R}_+)), \tilde{P})$, endowed with the filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$. Consequently

$$\tilde{P}(\tilde{W}_t \in A) = P(W_t \in A) \tag{3.24}$$

for $0 \leq t \leq \infty$, and $A \in \mathcal{B}(C_d(\mathbb{R}_+))$.

The stock price process defined in (3.9) can be written in vector form as

$$dS_t = S_t\left((\mu_t + \delta_t)\, dt + \sigma_t\, dW_t\right). \tag{3.25}$$

Then, using (3.16) and (3.10), we get

$$\begin{aligned} dS_t &= S_t\left((\mu_t + \delta_t)\, dt + \sigma_t(d\tilde{W}_t + \theta_t\, dt)\right) \\ &= S_t\left((\mu_t + \delta_t)\, dt + \sigma_t\, d\tilde{W}_t - \sigma_t\sigma_t^{-1}\left(\mu_t - r_t\,\underline{\mathbf{1}}\right) dt\right) \\ &= S_t\left((r_t\,\underline{\mathbf{1}} + \delta_t)\, dt + \sigma_t\, d\tilde{W}_t\right). \end{aligned} \tag{3.26}$$

The change of measure has therefore eliminated the need to estimate the drift rate of S.

Consider now the *discounted stock price processes* $(A_t S_t^i)_{t\geq 0}$. If the stocks pay no dividends, then

$$\begin{aligned} d\left(A_t S_t^i\right) &= A_t\, dS_t^i + S_t^i\, dA_t \\ &= A_t S_t^i\left(r_t\, dt + \sum_{i=1}^{d} \sigma_t^{i,j}\, d\tilde{W}_t^{\,j}\right) + S_t^i A_t\left(-r_t\, dt\right) = S_t^i \sum_{i=1}^{d} \sigma_t^{i,j}\, d\tilde{W}_t^{\,j} \end{aligned} \tag{3.27}$$

for $i = 1, \ldots, d$ (recalling (3.8)). In integral form, this is

$$A_t S_t^i = S_0^i + \sum_{j=1}^{d} \int_0^t S_t^i \sigma_t^{i,j}\, d\tilde{W}_t^{\,j} \tag{3.28}$$

for $i = 1, \ldots, d$. Therefore, the discounted stock price processes are Itô integrals, and consequently also local martingales, under the measure $\tilde{P}$. Similarly, if the stock pays dividends, then the process $\left(\exp(-\int_0^t \delta_s^i ds) A_t S_t^i\right)_{t\geq 0}$ is a local martingale. This is the justification for referring to $\tilde{P}$ as the martingale measure.

3.4 Investment

3.4.1 Portfolios

In a later section, contingent claims will be priced by replicating the payoff with stocks and bonds. This will be achieved by following some investment strategy;

in other words, by investing in a portfolio of stocks and the bank account. This portfolio at time t will be represented by

$$\pi_t = (\beta_t, \gamma_t) \tag{3.29}$$

where β_t is the fraction of the portfolio invested in the bank account at time t, and the ith component of the d-dimensional vector γ_t is the fraction of the portfolio invested in the ith stock at time t. Any component of the portfolio may have a negative value, which will represent borrowing from the bank account, or short selling in the case of a stock. The portfolio is constrained by

$$\beta_t + \sum_{i=1}^{d} \gamma_t^i = 1. \tag{3.30}$$

Since later chapters will deal with infinite horizon problems, the *portfolio process* $\pi = (\pi_t)_{t\geq 0}$ is an $\tilde{\mathcal{F}}_t$-progressively measurable functional, such that

$$\int_0^t |\beta_u|\, dB_u < \infty \tag{3.31}$$

$$\int_0^t ||\gamma_u S_u||^2 du < \infty \tag{3.32}$$

hold a.s. under $\tilde{P}$. It is also assumed that the processes $(\beta_t)_{t\geq 0}$ and $(\gamma_t)_{t\geq 0}$ have bounded variation.

At any time t, the investor's capital X_t^π can be represented by

$$X_t^\pi = \beta_t B_t + \sum_{i=1}^{d} \gamma_t^i S_t^i. \tag{3.33}$$

The process $X^\pi = (X_t^\pi)_{t\geq 0}$ is known as the *wealth process corresponding to the portfolio process* π.

Only *self financing* strategies will be considered, with the class of self financing strategies denoted by SF. These are strategies such that the wealth process satisfies the integral equation

$$X_t^\pi = X_0^\pi + \int_0^t \beta_u\, dB_u + \sum_{i=1}^{d} \int_0^t \gamma_u^i\, dS_u^i. \tag{3.34}$$

In differential form, this becomes

$$dX_t^\pi = \beta_t\, dB_t + \sum_{i=1}^{d} \gamma_t^i\, dS_t^i. \tag{3.35}$$

The role of the technical conditions (3.31) and (3.32), as pointed out in [48], are to ensure the existence of the Lebesgue-Stieltjes integral $\int_0^t |\beta_u|\, dB_u$, and the almost sure existence of the stochastic integrals $\int_0^t \gamma_u^i\, dS_u^i$ for each $t > 0$ and $i = 1, \ldots, d$.

Since Itô's formula and (3.33) give

$$dX_t^\pi = \beta_t\, dB_t + B_t\, d\beta_t + \sum_{i=1}^{d} \left(\gamma_t^i\, dS_t^i + S_t^i\, d\gamma_t^i\right) \tag{3.36}$$

the consequence of (3.35) is that

$$B_t\, d\beta_t + \sum_{i=1}^{d} S_t^i\, d\gamma_t^i = 0. \tag{3.37}$$

Changes in wealth occur solely due to changes in the values of the underlying assets, and not due to changes in their weightings in the portfolio.

It is pointed out in [48] that many problems include a capital inflow or outflow. An example of such an inflow would be the dividends paid on a stock. An outflow is often called consumption. These capital flows are modelled by right continuous non-decreasing processes $C = (C_t)_{t\geq 0}$ and $D = (D_t)_{t\geq 0}$, where $C_0 = D_0 = 0$, and C_t and D_t are $\tilde{\mathcal{F}}_t$-measurable for $t \geq 0$. The processes C and D can be interpreted respectively as cumulative outflow and inflow.

The capital at time t is now

$$X_t^\pi = X_0^\pi + \int_0^t \beta_u\, dB_u + \sum_{i=1}^{d} \int_0^t \gamma_u^i\, dS_u^i - C_t + D_t \tag{3.38}$$

which gives the result

$$B_t\, d\beta_t + \sum_{i=1}^{d} S_t^i\, d\gamma_t^i = -dC_t + dD_t. \tag{3.39}$$

When a dividend paying stock is being considered, the process D_t can be written as $D_t = \gamma_t S_t \delta_t$. For notational simplicity, we will focus on trading strategies consisting of portfolio-consumption pairs (π, C), with the corresponding wealth process denoted by $X^{\pi,C}$. Consumption is included to provide a framework for pricing contingent claims with payoff structures including a continuous stream of payments.

Consider the wealth process $X^{\pi,C}$. From (3.6), (3.26) and (3.38), it follows that

$$
\begin{aligned}
dX_t^{\pi,C} &= \beta_t\, dB_t + \sum_{i=1}^{d} \gamma_t\, dS_t - dC_t \\
&= \beta_t(r_t B_t\, dt) + \sum_{i=1}^{d} \gamma_t^i S_t^i \left(r_t\, dt + \sum_{j=1}^{d} \sigma_t^{i,j}\, d\tilde{W}_t^j \right) - dC_t \\
&= r_t X_t^{\pi,C}\, dt + \sum_{i=1}^{d}\sum_{j=1}^{d} \sigma_t^{i,j}\gamma_t^i S_t^i\, d\tilde{W}_t^j - dC_t.
\end{aligned} \tag{3.40}
$$

Recall from (3.28) that the discounted stock-price processes are local martingales. Now consider the *discounted wealth process* $A_t X_t^{\pi,C}$, for $0 \le t < \infty$. It follows from Itô's formula that

$$
d(A_t X_t^{\pi,C}) = \sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\sigma_t^{i,j}\gamma_t^i S_t^i}{B_t}\, d\tilde{W}_t - \frac{dC_t}{B_t} = \sum_{i=1}^{d}\sum_{j=1}^{d} \sigma_t^{i,j}\gamma_t^i S_t^i A_t\, d\tilde{W}_t - A_t\, dC_t. \tag{3.41}
$$

3.4.2 Arbitrage and Admissible Portfolios

The central assumption in mathematical finance is that no arbitrage opportunities exist in the market. The following definition of an arbitrage can be found in [48].

Definition 3.2. A strategy $(\pi, C) \in SF$ is called an arbitrage (or realizing an arbitrage possibility) on $[0,T]$, if $\tilde{P}$-a.s.

$$
X_0^{\pi,C} \le 0 \text{ and } X_T^{\pi,C} \ge 0 \tag{3.42}
$$

and, with positive $\tilde{P}$-probability,

$$
X_T^{\pi,C} > 0. \tag{3.43}
$$

An arbitrage is an investment opportunity in which there is no possibility of loss, and in which the expected gain is strictly positive. This is not realistic in a properly functioning market. This section will focus on describing the class of self-financing investment strategies that do not allow arbitrage opportunities.

Definition 3.3. Given initial capital $X_0^{\pi,C} = x \ge 0$, and a finite time horizon $T > 0$, a pair of portfolio and consumption processes $(\pi, C) \in SF$ is *admissible* on $[0,T]$, for the initial capital $x \ge 0$, written $(\pi, C) \in \mathcal{A}(T,x)$, if

$$
X_t^{\pi,C} \ge 0 \tag{3.44}
$$

for every $0 \le t \le T$, holds $\tilde{P}$-a.s. We also introduce the notation

$$
\mathcal{A}(\infty, x) = \bigcap_{T>0} \mathcal{A}(T,x). \tag{3.45}
$$

It follows from (3.41) that if the initial capital is $X_0^{\pi,C} = x$, then

$$A_t X_t^{\pi,C} + \int_0^t A_s \, dC_s = x + \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^t \sigma_s^{i,j} \gamma_s^i S_s^i A_s \, d\tilde{W}_s^j. \tag{3.46}$$

The right-hand side of (3.46) is a $\tilde{P}$-local martingale, whereas the left-hand side is, for every $(\pi, C) \in \mathcal{A}(T, x)$, a non-negative process. Therefore, the left-hand side is a supermartingale (see Theorem B.7), for which the optional sampling theorem yields

$$\tilde{\mathbb{E}}_T \left(A_\tau X_\tau^{\pi,C} + \int_0^\tau A_t \, dC_t \right) \leq x \tag{3.47}$$

for every finite $\tilde{\mathcal{F}}$-stopping time τ. The symbol $\tilde{\mathbb{E}}_T(\cdot)$ denotes expectation with respect to $\tilde{P}_T$, where $0 \leq T \leq \infty$ and $\tilde{P}_\infty = \tilde{P}$.

Inequality (3.47) is called the *budget constraint* in [21], which also points out the interpretation that the expected discounted terminal wealth plus the expected discounted consumption cannot exceed the initial capital. This interpretation gives an intuitive idea of why admissible strategies preclude arbitrage. This is stated in the following lemma.

Lemma 3.4. *No strategy $(\pi, C) \in \mathcal{A}(T, x)$ is an arbitrage.*

The proof follows directly from the budget constraint (3.47). If (π, C) were an arbitrage, then (3.47) would lead to a contradiction.

3.5 American Contingent Claims

A derivative security, or contingent claim, is a financial instrument that derives its value from some other underlying security. Thus, introducing a derivative security into the market does not change the number of sources of risk.

The party selling such a contract agrees to pay the buyer a terminal payoff of f_τ at the exercise time $\tau \in [0, T]$, and a continuous stream of payments $(g_t)_{0 \leq t \leq \tau}$. The contract is valid on $[0, T]$, where $T \in [0, \infty]$ is the expiration time (or maturity) of the contract. When $T = \infty$, the contingent claim is called *perpetual*.

The processes $f = (f_t)_{0 \leq t \leq T}$ and $g = (g_t)_{0 \leq t \leq T}$ are assumed to be non-negative, progressively measurable functionals satisfying

$$\tilde{\mathbb{E}}_T \left(\sup_{0 \leq s \leq t} f_s + \int_0^t g_s \, ds \right)^p < \infty \tag{3.48}$$

for some fixed $p > 1$ and for every $0 < t < \infty$, where $0 < T \leq \infty$. The process f is assumed to have continuous sample paths. When the contract is perpetual, the convention

$$f_\infty = \overline{\lim_{t \to \infty}} f_t \tag{3.49}$$

is followed.

A European contingent claim has a fixed exercise time $\tau = T$. In the case of an American contract, the buyer may choose an exercise time $\tau \in [0, T]$. The decision to exercise at time t must be based solely on information available at time t, since it is not possible to predict the future. Therefore, the exercise time τ must be a stopping time. In other words, the criteria upon which the buyer bases the decision to exercise at time t should not depend on information that is not available at that time. The following definition is based on Definition 5.1 of [19].

Definition 3.5. A (x, f, g)-*hedge of American type* is a strategy $(\pi, C) \in \mathcal{A}(T, x)$, for $0 \leq T \leq \infty$, satisfying:

1. $X_0^{\pi,C} = x$.
2. The function $H_t = C_t - \int_0^t g_s \, ds$ is non-decreasing, for $t \in [0, T]$.
3. The inequality $X_t^{\pi,C} \geq f_t$, holds for all $t \in [0, \tau]$, where $\tau = \inf\{0 \leq s \leq T \mid X_s = f_s\}$.

On $[0, \tau]$, the hedging strategy (π, C) super-replicates the payoff stream from the contingent claim. It is necessary for the hedging strategy to provide at least enough to cover the claim if it is exercised. It is required in [19] that the function H be continuous. The concept of a hedging strategy is used in defining the fair, or rational, price of a contingent claim as in the following definition from [48].

Definition 3.6. Let $\Pi(x, f, g)$ be the set of (x, f, g)-hedges from $\mathcal{A}(T, x)$. The value

$$\mathbb{C}_T(f, g) = \inf\{x \geq 0 \mid \Pi(x, f, g) \neq \emptyset\} \tag{3.50}$$

is called the *hedge price* of the American contingent claim.

The hedge price of a contingent claim is the smallest amount of initial capital needed to construct a hedging strategy. The problems to which the pricing of a contingent claim reduces are identified in [48] as:

1. determining the investment cost $\mathbb{C}_T(f, g)$;
2. finding the (x, f, g)-hedging strategy for which $x = \mathbb{C}_T(f, g)$; and
3. finding the optimal exercise time.

These problems are solved in Theorem 3.7 below. It states that the hedge price of an American contingent claim is the solution to an optimal stopping problem, and is proved by constructing a hedge portfolio. The theorem holds for both finite and infinite time horizon problems, as long as the appropriate measurablity conditions, as described in previous sections, hold.

Theorem 3.7. *There exists a hedging strategy (π, C), with corresponding wealth process $X = (X_t)_{0\le t\le T}$, that satisfies*

$$X_t = \operatorname*{ess\,sup}_{t\le\tau\le T} \tilde{\mathbb{E}}_T\left(f_\tau \exp\left(-\int_t^\tau r_u\,du\right) + \int_t^\tau g_s \exp\left(-\int_t^s r_u\,du\right) ds \,\middle|\, \mathcal{F}_t \right) \tag{3.51}$$

$\tilde{P}_T$-a.s., for every fixed $t \in [0,T]$, where $0 \le T \le \infty$. The hedge price $\mathbb{C}_T(f,g)$ of an American contingent claim is given by

$$\begin{aligned} \mathbb{C}_T(f,g) &= u_0 \\ &= \sup_{0\le\tau\le T} \tilde{\mathbb{E}}_T\left(f_\tau \exp\left(-\int_0^\tau r_u\,du\right) + \int_0^\tau g_s \exp\left(-\int_0^s r_u\,du\right) ds \right) \end{aligned} \tag{3.52}$$

for $0 \le T \le \infty$.

Proof. It is first shown that the process X is a wealth process corresponding to a hedging strategy (π, C), which we construct. The expression for the price then follows since $u_0 = X_0$, and thus u_0 is the initial wealth needed to hedge the claim using strategy (π, C). It is shown that u_0 is the minimal amount needed for a hedging strategy to exist, and thus u_0 is the hedge price of the claim.

Define $Q = (Q_t)_{0\le t\le T}$ by setting

$$Q_t = A_t f_t + \int_0^t A_s g_s\,ds \tag{3.53}$$

and let

$$u_t = \sup_{t\le\tau\le T} \tilde{\mathbb{E}}_T(Q_\tau). \tag{3.54}$$

We will now show, following the argument of [48], that the process $Y^* = (Y_t^*)_{0\le t\le T}$ with

$$Y_t^* = \operatorname*{ess\,sup}_{t\le\tau\le T} \tilde{\mathbb{E}}_T(Q_\tau \mid \mathcal{F}_t) \tag{3.55}$$

is a non-negative supermartingale which has an RCLL modification $Y = (Y_t)_{0\le t\le T}$. First, we show that

$$\tilde{\mathbb{E}}_T(Y_t^* \mid \mathcal{F}_s) = \operatorname*{ess\,sup}_{t\le\tau\le T} \tilde{\mathbb{E}}_T(Q_\tau \mid \mathcal{F}_s). \tag{3.56}$$

Now

$$\tilde{\mathbb{E}}_T(Q_\sigma \mid \mathcal{F}_t) \le \operatorname*{ess\,sup}_{t\le\tau\le T} \tilde{\mathbb{E}}_T(Q_\tau \mid \mathcal{F}_t) \tag{3.57}$$

where σ is a stopping time such that $t \leq \sigma \leq T$. Thus

$$\tilde{\mathbb{E}}_T(Q_\sigma \mid \mathcal{F}_s) \leq \tilde{\mathbb{E}}_T\left(\operatorname*{ess\,sup}_{t\leq\tau\leq T} \tilde{\mathbb{E}}_T(Q_\tau \mid \mathcal{F}_t) \mid \mathcal{F}_s\right) \tag{3.58}$$

by the tower property of conditional expectations. It follows that

$$\operatorname*{ess\,sup}_{t\leq\sigma\leq T} \tilde{\mathbb{E}}_T(Q_\sigma \mid \mathcal{F}_s) \leq \tilde{\mathbb{E}}_T\left(\operatorname*{ess\,sup}_{t\leq\tau\leq T} \tilde{\mathbb{E}}_T(Q_\tau \mid \mathcal{F}_t) \mid \mathcal{F}_s\right) \tag{3.59}$$

i.e.

$$\tilde{\mathbb{E}}_T(Y_t^* \mid \mathcal{F}_s) \geq \operatorname*{ess\,sup}_{t\leq\tau\leq T} \tilde{\mathbb{E}}_T(Q_\tau \mid \mathcal{F}_s). \tag{3.60}$$

Let

$$\rho_n = \inf\{n \leq s \leq T \mid Y_s = Q_s\}. \tag{3.61}$$

Now construct the sequence $(\tau_n)_{n\geq 1}$ of stopping times by setting

$$\tau_1 = \rho_1 \tag{3.62}$$

$$\tau_{n+1} = \begin{cases} \tau_n & \text{if } \tilde{\mathbb{E}}_T(Q_{\tau_n} \mid \mathcal{F}_t) \geq \tilde{\mathbb{E}}_T(Q_{\rho_{n+1}} \mid \mathcal{F}_t) \\ \rho_{n+1} & \text{otherwise.} \end{cases} \tag{3.63}$$

Then, using a simple induction argument, we have

$$\tilde{\mathbb{E}}_T(Q_{\tau_n} \mid \mathcal{F}_t) = \max_{k\leq n} \tilde{\mathbb{E}}_T(Q_{\rho_k} \mid \mathcal{F}_t) \tag{3.64}$$

which increases to Y_t^* as $n \to \infty$. Hence, by the monotone convergence theorem, we have

$$\begin{aligned} \tilde{\mathbb{E}}_T(Y_t^* \mid \mathcal{F}_s) &= \tilde{\mathbb{E}}_T(\lim_{n\to\infty} \tilde{\mathbb{E}}_T(Q_{\tau_n} \mid \mathcal{F}_t) \mid \mathcal{F}_s) \\ &= \lim_{n\to\infty} \tilde{\mathbb{E}}_T(Q_{\tau_n} \mid \mathcal{F}_s) \leq \operatorname*{ess\,sup}_{t\leq\tau\leq T} \tilde{\mathbb{E}}_T(Q_\tau \mid \mathcal{F}_s). \end{aligned} \tag{3.65}$$

This, together with (3.60), proves (3.56).

Since $s \leq t$, we have

$$\tilde{\mathbb{E}}_T(Y_t^* \mid \mathcal{F}_s) = \operatorname*{ess\,sup}_{t\leq\tau\leq T} \tilde{\mathbb{E}}_T(Q_\tau \mid \mathcal{F}_s) \leq \operatorname*{ess\,sup}_{s\leq\tau\leq T} \tilde{\mathbb{E}}_T(Q_\tau \mid \mathcal{F}_s) = Y_s^* \tag{3.66}$$

and thus Y^* is a supermartingale.

We now show that Y^* has an RCLL modification. By [27, Theorem 3.1, page 57], since the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and contains the $\tilde{P}$-null subsets of the sample space, Y^* has a right continuous modification Y if and only if $(\tilde{\mathbb{E}}_T(Y_t^*))_{t\geq 0}$

is right continuous. Then, by [27, Theorem 3.2, Corollary 1, page 61], Y has RCLL paths.

We will now show that $(\tilde{\mathbb{E}}_T(Y_t^*))_{t\geq 0}$ is a right continuous function. Similarly to above, we can show

$$\tilde{\mathbb{E}}_T(Y_t^*) = \sup_{t\leq\tau\leq T} \tilde{\mathbb{E}}_T(Q_\tau). \tag{3.67}$$

Now let $(t_n)_{n\geq 1}$ be a sequence such that $t_n \in (t, t+1)$ and $t_n \downarrow t$. Since Y^* is a supermartingale, we have

$$\tilde{\mathbb{E}}_T(Y_t^*) \geq \tilde{\mathbb{E}}_T(Y_{t_n}^* \mid \mathcal{F}_t) \tag{3.68}$$

for all $n \geq 1$. It follows that

$$\tilde{\mathbb{E}}_T(Y_t^*) \geq \lim_{n\to\infty} \tilde{\mathbb{E}}_T(Y_{t_n}^*). \tag{3.69}$$

To show the reverse inequality, fix $\varepsilon > 0$, and choose a stopping time σ_ε, such that $t \leq \sigma_\varepsilon \leq T$ $\tilde{P}$ a.s., with

$$\tilde{\mathbb{E}}_T(Y_t^*) \leq \tilde{\mathbb{E}}_T(Q_{\sigma_\varepsilon}) + \varepsilon \tag{3.70}$$

and

$$\tilde{P}(\sigma_\varepsilon > t) = 1. \tag{3.71}$$

Such a stopping time exists, firstly, by the definition of the supremum and since Q is a right continuous non-negative process, there exists a stopping time such that (3.70) holds. Now suppose the essential supremum in (3.55) is attained at some stopping time τ such that $t \leq \tau < T$ (i.e. $Y_t^* = \tilde{\mathbb{E}}_T(Q_\tau|\mathcal{F}_t)$). Since Q is right continuous, by the same argument as above, $(\tilde{\mathbb{E}}_T(Q_t|\mathcal{F}_t))_{t\geq 0}$ is right continuous. Thus, the right continuity of Q and the definition of the essential supremum ensure that for any $\varepsilon > 0$, there exists a stopping time $\sigma_\varepsilon > \tau$ such that $\tilde{\mathbb{E}}_T(Q_\tau|\mathcal{F}_t) < \tilde{\mathbb{E}}_T(Q_{\sigma_\varepsilon}|\mathcal{F}_t) + \varepsilon$, i.e. $\tilde{\mathbb{E}}_T(Y_t^*) < \tilde{\mathbb{E}}_T(Q_{\sigma_\varepsilon}) + \varepsilon$. If the essential supremum is attained at T then set $\sigma_\varepsilon = T$. Thus we have $\tilde{P}(\sigma_\varepsilon > t) = 1$.

Now define a sequence $(\sigma_n)_{n\geq 1}$ of stopping times such that $t_n \leq \sigma_n \leq T$, by setting

$$\sigma_n = \begin{cases} \sigma_\varepsilon & \text{if } \sigma_\varepsilon \geq t_n \\ t+1 & \text{otherwise.} \end{cases} \tag{3.72}$$

Then

$$\begin{aligned} &\left|\tilde{\mathbb{E}}_T(Q_{\sigma_\varepsilon}) - \tilde{\mathbb{E}}_T(Q_{\sigma_n})\right| \\ &= \left|\tilde{\mathbb{E}}_T\left((Q_{\sigma_\varepsilon} - Q_{\sigma_n})1_{\{\sigma_\varepsilon<t_n\}} + \tilde{\mathbb{E}}_T((Q_{\sigma_\varepsilon} - Q_{\sigma_n})1_{\{\sigma_\varepsilon\geq t_n\}}\right)\right| \\ &= \left|\tilde{\mathbb{E}}_T\left((Q_{\sigma_\varepsilon} - Q_{t+1})1_{\{\sigma_\varepsilon<t_n\}}\right)\right| \leq \tilde{\mathbb{E}}_T\left((Q_{\sigma_\varepsilon} + Q_{t+1})1_{\{\sigma_\varepsilon<t_n\}}\right) \end{aligned} \tag{3.73}$$

which tends to 0 as $n \to \infty$ since $t_n \to t$ and $\sigma_\varepsilon > t$ a.s. by (3.71). The above inequality follows since Q is non-negative. Hence

$$\tilde{\mathbb{E}}(Y_t^*) \leq \tilde{\mathbb{E}}_T(Q_{\sigma_\varepsilon}) + \varepsilon = \lim_{n\to\infty} \tilde{\mathbb{E}}_T(Q_{\sigma_n}) + \varepsilon \leq \lim_{n\to\infty} \tilde{\mathbb{E}}_T(Y_{t_n}^*) + \varepsilon. \tag{3.74}$$

Now ε is arbitrary, thus

$$\tilde{\mathbb{E}}_T(Y_t^*) \leq \lim_{n\to\infty} \tilde{\mathbb{E}}_T(Y_{t_n}^*) \tag{3.75}$$

and so

$$\tilde{\mathbb{E}}_T(Y_t^*) \leq \lim_{n\to\infty} \tilde{\mathbb{E}}_T(Y_{t_n}^*) \tag{3.76}$$

and $(\tilde{\mathbb{E}}_T(Y_t^*))_{t\geq 0}$ is a right continuous function. Thus, there exists an RCLL modification Y of Y^*.

From (3.67) it follows that

$$u_t = \tilde{\mathbb{E}}_T(Y_t) \tag{3.77}$$

holds for any given $t \in [0, T]$. In particular, we have

$$u_0 = Y_0 \tag{3.78}$$

$$u_T = Y_T = Q_T \tag{3.79}$$

$\tilde{P}$ a.s.

The process Y is the Snell envelope of Q, or the least supermartingale with RCLL paths which majorizes it (see [21]). By Theorem 2.8 of Chapter 1, the stopping time

$$\sigma_t = \inf\{t \leq s \leq T \mid Y_s = Q_s\} \tag{3.80}$$

is optimal for (3.67) and thus

$$u_t = \tilde{\mathbb{E}}_T(Q_{\sigma_t}) \tag{3.81}$$

for all $0 \leq t \leq T$.

In order to show that Y is in class(D) (see Definition B.8 and Lemma B.9), we refer to [9] for the following argument. Define the stochastic process Γ_t by

$$\Gamma_t = \tilde{\mathbb{E}}_T\left(\sup_{0\leq s\leq T} |Q_s| \,\middle|\, \mathcal{F}_t\right) \geq \tilde{\mathbb{E}}_T(Q_t \mid \mathcal{F}_t) = Q_t \tag{3.82}$$

for all $0 \le t \le T$. Since Γ is a martingale majorizing Q, we have that $\Gamma_t \ge Y_t$. So we obtain, with p as in (3.48), that

$$
\begin{aligned}
(3.83) \qquad \tilde{\mathbb{E}}_T\left(\left(\sup_{0\le s\le T} Y_s\right)^p\right) &\le \tilde{\mathbb{E}}_T\left(\left(\sup_{0\le s\le T} \Gamma_s\right)^p\right) = \tilde{\mathbb{E}}_T\left(\sup_{0\le s\le T} \Gamma_s^p\right) \\
&\le \left(\frac{p}{p-1}\right)^p \tilde{\mathbb{E}}_T\left(\Gamma_T^p\right) = q^p\, \tilde{\mathbb{E}}_T\left(\Gamma_T^p\right) \\
&= q^p\, \tilde{\mathbb{E}}_T\left(\left(\tilde{\mathbb{E}}_T\left(\sup_{0\le s\le T} |Q_s| \,\middle|\, \mathcal{F}_T\right)\right)^p\right) \\
&\le q^p\, \tilde{\mathbb{E}}_T\left(\tilde{\mathbb{E}}_T\left(\left(\sup_{0\le s\le T} |Q_s|\right)^p \,\middle|\, \mathcal{F}_T\right)\right) \\
&= q^p\, \tilde{\mathbb{E}}_T\left(\left(\sup_{0\le s\le T} |Q_s|\right)^p\right) < \infty
\end{aligned}
$$

where the second inequality follows from the Doob's maximal inequality (see Definition B.10) and the fact that $1/p + 1/q = 1$; and the third inequality follows from Jensen's inequality, since $p > 1$ implies that x^p is convex for $x \ge 0$. Now the fact that $Y_\tau \le \sup_{0\le s\le T} Y_s$, for any stopping time such that $0 \le \tau \le T$ a.s., implies that

$$
(3.84) \qquad \tilde{\mathbb{E}}_T(Y_\tau) \le \tilde{\mathbb{E}}_T\left(\sup_{0\le s\le T} Y_s\right) < \infty
$$

and we see that the family $(Y_\tau)_{0\le\tau\le T}$ is uniformly integrable (see [14, Example 4, page 351], and thus Y is in class(D), by Lemma B.9.

Thus, the Doob-Meyer decomposition (see Theorem B.11) yields

$$
(3.85) \qquad Y_t = u_0 + M_t - \Lambda_t
$$

where $M = (M_t)_{0\le t\le T}$ is a (uniformly integrable) martingale with RCLL paths and $\Lambda = (\Lambda_t)_{0\le t\le T}$ is a non-decreasing (uniformly) integrable process, with $\tilde{\mathbb{E}}_T(\Lambda_T) = u_0 - \tilde{\mathbb{E}}_T(Q_T)$ and $M_0 = \Lambda_0 = 0$.

By the martingale representation theorem (see Theorem B.6), the martingale M can be written in the form

$$
(3.86) \qquad M_t = \sum_{j=1}^{d} \int_0^t \psi_s^j \, dW_s^j
$$

where $t \in [0, T]$, and the integrands are measurable adapted processes satisfying

$$
(3.87) \qquad \sum_{j=1}^{d} \int_0^T (\psi_s^j)^2 \, ds < \infty
$$

$\tilde{P}$ a.s.

Now introduce the adapted process $X = (X_t)_{0\le t\le T}$ with

$$X_t = \frac{1}{A_t}\left(Y_t - \int_0^t A_s g_s\, ds\right). \tag{3.88}$$

It satisfies (3.51) since

$$\begin{aligned} X_t &= \frac{1}{A_t}\left(\operatorname*{ess\,sup}_{t\le\tau\le T} \tilde{\mathbb{E}}_T\left(A_\tau f_\tau + \int_0^\tau A_s g_s\, ds \middle| \mathcal{F}_t\right) - \int_0^t A_s g_s\, ds\right) \\ &= \operatorname*{ess\,sup}_{t\le\tau\le T} \tilde{\mathbb{E}}_T\left(\frac{A_\tau}{A_t} f_\tau + \int_t^\tau \frac{A_s}{A_t} g_s\, ds \middle| \mathcal{F}_t\right) \end{aligned} \tag{3.89}$$

by the definition (3.8) of A_t, and since both $1/A_t$ and $\int_0^t A_s g_s\, ds$ are $\mathcal{F}_t$-measurable.

From the representations of Y and M, given in (3.85) and (3.86), we can rewrite X to get

$$\begin{aligned} X_t &= \frac{1}{A_t}\left(u_0 + M_t - \Lambda_t - \int_0^t A_s g_s\, ds\right) \\ &= \frac{1}{A_t}\left(u_0 + \sum_{j=1}^d \int_0^t \psi_s^j\, d\widetilde{W}_s^j - \Lambda_t - \int_0^t A_s g_s ds\right). \end{aligned} \tag{3.90}$$

Thus

$$A_t X_t + \int_0^t A_s g_s\, ds + \Lambda_t = u_0 + \sum_{j=1}^d \int_0^t \psi_s^j\, d\widetilde{W}_s^j. \tag{3.91}$$

Now let γ_t^j be such that

$$\psi_t^j = \sum_{i=1}^d \sigma_t^{(j,i)} \gamma_t^j S_t^j A_t \tag{3.92}$$

and let C_t be such that

$$C_t = \int_0^t g_s\, ds + \int_0^t B_s\, d\Lambda_s. \tag{3.93}$$

Then

$$A_t X_t + \int_0^t A_s\, dC_s = u_0 + \sum_{j=1}^d \int_0^t \sum_{i=1}^d \sigma_s^{(j,i)} \gamma_s^j S_s^j A_s\, d\widetilde{W}_s^j \tag{3.94}$$

and by comparing this to (3.46), it is clear that X is the wealth process corresponding to the portfolio-consumption pair (π, C), defined by (3.92) and (3.93).

The strategy (π, C) is a hedging strategy if conditions (1)–(3) of Definition 3.5 are satisfied. Condition (1) is obvious. Now

$$H_t = C_t - \int_0^t g_s \, ds = \int_0^t B_s \, d\Lambda_s \tag{3.95}$$

and thus $(H_t)_{0 \le t \le T}$ is a non-decreasing function since B and Λ are non-negative processes and Λ is non-decreasing. Thus condition (2) is satisfied.

Also, suppose the essential supremum in (3.51) is realised at $\tau = t$. Then

$$X_t = f_t + \int_0^t \frac{A_s}{A_t} g_s \, ds \ge f_t. \tag{3.96}$$

Thus $X_t \ge f_t$ for all $0 \le t \le T$ and condition (3) is satisfied. It follows that the portfolio-consumption pair (π, C) is a hedging strategy.

The second part of the theorem follows from the first part. Let $x \ge 0$ be any number for which there exists a hedging strategy. By the budget constraint (3.47), and (3.95) and (3.96) above, we have

$$\tilde{\mathbb{E}}_T(Q_\tau) = \tilde{\mathbb{E}}_T \left(A_\tau f_\tau + \int_0^\tau A_s g_s \, ds \right) \le \tilde{\mathbb{E}}_T \left(A_\tau X_\tau + \int_0^\tau A_s \, dC_s \right) \le x. \tag{3.97}$$

Recalling from (3.54) that

$$u_t = \sup_{t \le \tau \le T} \tilde{\mathbb{E}}_T(Q_\tau) \tag{3.98}$$

we get

$$u_0 \le x. \tag{3.99}$$

Thus

$$u_0 \le \mathbb{C}_T(f, g) \tag{3.100}$$

which proves the theorem, since u_0 is the initial capital required for the hedging strategy defined by (3.92) and (3.93). □

Recall points (1)–(3) following Definition 3.6. From the above theorem we see that the hedge price and the optimal exercise time of an American contingent claim are the solution to the optimal stopping problem (3.51). The optimal hedging strategy is given by (3.92) and (3.93).

Chapter 4

The Russian Option

4.1 Introduction

The Russian option pays the holder the maximum price achieved by the underlying asset over the life of the option, discounted at some rate $\lambda > 0$. The option was first proposed and priced in [44]. That paper describes it as "reduced regret", since the holder will experience less regret at having missed exercising at the maximum value of the underlying asset than the holder of an American put. Of course, the holder may still regret having exercised too early. The discounting factor affects the "reduced regret" feature of the option, since the holder will suffer for holding the option longer than necessary. The discounting factor is required when pricing the option on an infinite time horizon, to ensure that the problem has a finite solution. The problem has a finite solution if and only if $r + \lambda > \mu$ where r is the risk-free rate of interest, λ is the rate of discount applied to the option, and μ is the drift rate of the underlying asset. We will in fact assume that

$$\lambda > r - \frac{1}{2}\sigma^2 > 0 \tag{4.1}$$

where σ is the volatility of the underlying asset.

The problem was first solved as an optimal stopping problem for a two dimensional Markov process in [44]. In [45] the same solution was derived by reducing the problem to one dimension by means of a change-of-measure theorem. Both of these approaches will be discussed.

In the next section, the problem is formulated as a two-dimensional optimal stopping problem. Section 3 describes the solution to the two-dimensional problem, although the proof of optimality is left to the one-dimensional formulation described in Section 4 and proved in Section 5.

4.2 Formulation of the Problem

The market consists of a riskless bank account process $B = (B_t)_{t\geq 0}$ satisfying

$$dB_t = rB_t\,dt \qquad (B_0 = 1) \tag{4.2}$$

with $r \geq 0$, and a single risky asset $S = (S_t)_{t\geq 0}$ satisfying

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t \qquad (S_0 = x) \tag{4.3}$$

where W is a Brownian motion under the unique martingale measure P.

The strong solution of (4.3) is given by

$$S_t = x\exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \tag{4.4}$$

and we denote by $(\mathcal{F}_t)_{t\geq 0}$ the augmentation of the filtration generated by the asset price process S (see Chapter 3).

Recall that the Russian option pays the maximum price achieved by the underlying asset over the life of the option, discounted at rate $\lambda > 0$. The payoff function is thus

$$f_t = e^{-\lambda t}M_t \tag{4.5}$$

where $M = (M_t)_{t\geq 0}$ is the maximum process, given by

$$M_t = \left(\max_{0\leq u\leq t} S_u\right) \vee m \tag{4.6}$$

with $m \geq x > 0$ given and fixed.

From the general theory of option pricing (see Chapter 3), the price of the Russian option is the solution to the two-dimensional optimal stopping problem

$$V(x,m) = \sup_{0\leq\tau<\infty} \mathbb{E}_{x,m}\left(e^{-r\tau}f_\tau\right) = \sup_{0\leq\tau<\infty} \mathbb{E}_{x,m}\left(e^{-(r+\lambda)\tau}M_\tau\right) \tag{4.7}$$

where τ is an $(\mathcal{F}_t)_{t\geq 0}$ stopping time. The symbol $\mathbb{E}_{x,m}(\cdot)$ denotes the expected value under $P_{x,m}$ (see Section 2.2.2).

Notice that the process $Z = (Z_t)_{t\geq 0}$ given by

$$Z_t = (S_t, M_t) \tag{4.8}$$

is a two-dimensional Markov process. The state space of Z is the cone in $\mathbb{R}^2$ between $x > 0$, $m > 0$ and $m \geq x$. It is necessary to introduce a fixed m into the definition (4.6) of the process M, to enable Z to have an initial point anywhere in its state

space. If M_t was simply defined as the maximum up to time t of the asset price, Z would be forced to start on the diagonal. The introduction of m thus preserves the Markovian structure of Z.

The infinitesimal generator of Z is given by

$$\mathbb{L}_Z = \begin{cases} \mathbb{L}_S & \text{for } x < m \\ \frac{\partial}{\partial m} = 0 & \text{at } x = m \end{cases} \tag{4.9}$$

where

$$\mathbb{L}_S = rs\frac{\partial}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2}{\partial s^2} \tag{4.10}$$

is the infinitesimal generator of S. In [34] one can find a technical explanation of this fact. A point to consider is that the first co-ordinate of $Z_t = (S_t, M_t)$ changes only off the diagonal, and the second co-ordinate changes only on the diagonal.

The optimal stopping problem (4.7) can be reformulated in Mayer form. This is convenient, as the solution to the problem arises naturally in this form. Set $z = (x, m)$ and define

$$G(z) = G(x, m) = m. \tag{4.11}$$

Then

$$V(z) = V(x, m) = \sup_{0 \le \tau < \infty} \mathbb{E}_z \left(e^{-(\lambda + r)\tau} G(Z_\tau) \right). \tag{4.12}$$

Now let $\tilde{Z} = (\tilde{Z}_t)_{t \ge 0}$ be the process with infinitesimal generator

$$\mathbb{L}_{\tilde{Z}} = \mathbb{L}_Z - (\lambda + r)I \tag{4.13}$$

where I is the identity operator. The process $\tilde{Z}$ corresponds to Z killed at rate $\lambda + r$ (see [41, pages 269–272]). Thus

$$V(z) = \sup_{0 \le \tau < \infty} \mathbb{E}_z \left(G(\tilde{Z}_\tau) \right). \tag{4.14}$$

The theory of killed processes will not be discussed here because we are only interested in the infinitesimal generator of the process.

4.3 The Initial Solution

Under certain regularity conditions, the solution to an optimal stopping problem characterizes both the value function and a boundary dividing the state space of the

process into two regions. The first region is the region of continuation, denoted by C, and given by

$$C = \big\{(x, m) \mid V(x, m) > G(x, m)\big\}. \tag{4.15}$$

The second region is the region of instantaneous stopping, denoted by D, and given by

$$D = \big\{(x, m) \mid V(x, m) = G(x, m)\big\}. \tag{4.16}$$

The optimal stopping boundary is the boundary between these two regions.

It is useful to consider the behaviour of the process Z, to gain some insight into the shape of the region of continuation. When Z is close to the diagonal, there is a high probability that it will hit the diagonal, which will result in a higher payoff to the option holder. It is therefore unlikely for it to be optimal to stop the process close to the diagonal. However, if Z is far from the diagonal, there might be an optimal stopping opportunity. This is a result of the discounting rate λ. There is a high probability that the process will take so long to reach the diagonal that any increase in the payoff will be insufficient to cover the decrease resulting from discounting. It would therefore be optimal to stop the process if it was far enough from the diagonal. This suggests that at each level m, there is a boundary value $g(m)$, such that if the process reaches $x = g(m)$, then it is optimal to stop.

It is possible to set up a system of equations which characterize the value function V and the optimal stopping boundary g, by recognising that the solution to the optimal stopping problem (4.14) also solves the following free boundary problem:

$$\mathbb{L}_{\tilde{Z}} V(z) = 0, \quad \text{for } x > g(m) \tag{4.17}$$

$$V(z)\big|_{x=g(m)} = m \tag{4.18}$$

$$\frac{\partial V}{\partial x}(z)\bigg|_{x=g(m)} = 0 \tag{4.19}$$

where $V(z)$, for $z = (x, m)$, is given by (4.14), and $\mathbb{L}_{\tilde{Z}}$ is given by (4.13).

These equations arise from the Mayer formulation of the optimal stopping problem (4.14), as a result of the Markov property of $\tilde{Z}$. If the region of continuation were known, equations (4.17) and (4.18) would form a Dirichlet problem corresponding to (4.14). Equation (4.18) is a result of instantaneous stopping at the boundary — when the process is stopped, the option holder receives $G(z) = m$.

Equation (4.19) is an additional condition needed to find the optimal stopping boundary, and arises from the principle of smooth fit. (This is the principle that the value function should join the gain function smoothly at the boundary.) Since only the first co-ordinate of $\tilde{Z}$ is changing near the boundary, only the derivative of

V with respect to x needs to be considered. Since the boundary is only a function of m, smooth fit dictates that this derivative is set to zero.

At this point the above system is merely a reasonable guess at the optimal solution to the problem (4.7). The system is solved in [44] using standard methods. This solution will not be considered here, and neither will the proof of its optimality, since another formulation of the problem will now be discussed.

It is worth noting, however, that in the solution of (4.17)–(4.19), the optimal stopping boundary takes the form

$$g(m) = \alpha m. \tag{4.20}$$

This is linear, and suggests that some reduction in the dimension of the problem is possible. The functional form (4.20) is derived as a solution to a first-order nonlinear differential equation which has many solutions. Each of these solutions will also lead to a solution to the free boundary problem (4.17)–(4.19), and thus there is no unique solution to this problem. Since $\tilde{Z}$ is a Markov process, the optimal solution is the least supermartingale that dominates the gain function. It is shown in [34] that in the absence of discounting, this corresponds to the maximal solution of (4.20) that stays below the diagonal. This is known as the maximality principle. It is shown in [33] that any solution larger than $g(m) = \alpha m$ crosses the diagonal, which suggests that the maximality principle might hold in the presence of discounting, as is conjectured in [34].

4.4 The Dual Martingale Measure

Recall that in a complete market, the equivalent martingale measure P is the unique measure under which the discounted asset price process $(S_t/B_t)_{t\geq 0}$ is a martingale. Thus

$$\mathbb{E}\left(\frac{S_t}{B_t}\right) = \frac{S_0}{B_0} \tag{4.21}$$

and so

$$\mathbb{E}\left(\frac{S_t}{B_t}\frac{B_0}{S_0}\right) = 1. \tag{4.22}$$

It is possible to use this to define a new measure $\tilde{P}_t$ on $\mathcal{F}_t$, by setting

$$\tilde{P}_t(A) = \mathbb{E}\left(\frac{S_t}{B_t}\frac{B_0}{S_0}1_A\right) = \mathbb{E}\Big(\exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right)1_A\Big) \tag{4.23}$$

for all $A \in \mathcal{F}_t$.

Now define the measure $\tilde{P}$ on $\sigma\left(\bigcup_{t\geq 0}\mathcal{F}_t\right)$ in such a way that its restrictions satisfy $\tilde{P}\mid_{\mathcal{F}_t} = \tilde{P}_t$, for all $t \geq 0$ (see Chapter 3). Also introduce the process $\tilde{W} = (\tilde{W}_t)_{t\geq 0}$ given by

$$\tilde{W}_t = W_t - \sigma t. \tag{4.24}$$

This is a Brownian motion under $\tilde{P}$, by the Girsanov theorem (see [20, pages 190–193] and Theorem B.5).

The stochastic differential equation (4.3) is given in terms of $\tilde{W}_t$ by

$$dS_t = rS_t\,dt + \sigma S_t\,d(\tilde{W}_t + \sigma t) = (r+\sigma^2)S_t\,dt + \sigma S_t\,d\tilde{W}_t. \tag{4.25}$$

Thus, by Itô's formula, the stochastic differential equation satisfied by the process $(B_t/S_t)_{t\geq 0}$ is

$$d\left(\frac{B_t}{S_t}\right) = \frac{1}{S_t}\,dB_t + \left(-\frac{B_t}{S_t^2}\right)\,dS_t + \frac{1}{2}\left(\frac{2B_t}{S_t^3}\right)d\langle S,S\rangle_t = -\sigma\frac{B_t}{S_t}d\tilde{W}_t. \tag{4.26}$$

The absence of a drift term implies that $(B_t/S_t)_{t\geq 0}$ is a local martingale with respect to $\tilde{P}$. The measure $\tilde{P}$ is called the *dual martingale measure* in [48].

It is possible to reformulate (4.7) in terms of $\tilde{P}$. Consider

$$B_0\mathbb{E}_{x,m}\left(\frac{f_\tau}{B_\tau}\right) = S_0\mathbb{E}_{x,m}\left(\frac{f_\tau}{S_\tau}\frac{S_\tau}{S_0}\frac{B_0}{B_\tau}\right) = S_0\tilde{\mathbb{E}}_{x,m}\left(\frac{f_\tau}{S_\tau}\right). \tag{4.27}$$

We may thus write (4.7) as

$$V(x,m) = \sup_{0\leq\tau<\infty}\mathbb{E}_{x,m}\left(e^{-r\tau}f_\tau\right) = S_0\sup_{0\leq\tau<\infty}\tilde{\mathbb{E}}_{x,m}\left(\frac{f_\tau}{S_\tau}\right) \tag{4.28}$$

since $B_0 = 1$.

Now define the process $\psi = (\psi_t)_{t\geq 0}$, by setting

$$\psi_t = \frac{M_t}{S_t} = \frac{(\max_{0\leq u\leq t} S_u)\vee m}{S_t}. \tag{4.29}$$

Noting that $\psi_0 = m/x$, it follows that

$$\psi_t = \frac{(\max_{0\leq u\leq t} S_u)\vee x\psi_0}{S_t}. \tag{4.30}$$

The problem (4.28) can be reformulated in terms of ψ in the following way:

$$V(\psi_0) = x\sup_{0\leq\tau<\infty}\tilde{\mathbb{E}}_{\psi_0}\left(e^{-\lambda\tau}\psi_\tau\right). \tag{4.31}$$

The optimal stopping problem has thus been reduced to a one dimensional problem involving the process ψ.

4.4.1 The Process $\psi = (\psi_t)_{t\geq 0}$

Since the theory of optimal stopping is well developed for Markov processes, it is desirable for $\psi = (\psi_t)_{t\geq 0}$ to be Markov under $\tilde{P}$. We have

$$
\begin{aligned}
\psi_{t+h} = \frac{M_{t+h}}{S_{t+h}} &= \frac{(\max_{0\leq u\leq t+h} S_u) \vee m}{S_{t+h}} = \frac{(\max_{0\leq u\leq t+h} S_u) \vee \psi_0 x}{S_{t+h}} \\
&= \frac{(\max_{0\leq u\leq t} S_u) \vee (\max_{t\leq u\leq t+h} S_u) \vee \psi_0 x}{S_{t+h}} \\
&= \frac{S_t}{S_{t+h}} \left(\frac{(\max_{0\leq u\leq t} S_u) \vee \psi_0 x}{S_t} \right) \vee \frac{(\max_{t\leq u\leq t+h} S_u)}{S_{t+h}} \\
&= \frac{S_t\psi_t \vee (\max_{t\leq u\leq t+h} S_u)}{S_{t+h}} \\
&= \frac{S_t\psi_t \vee \left(\max_{t\leq u\leq t+h} S_t \exp\left((r+\frac{1}{2}\sigma^2)(u-t) + \sigma(\tilde{W}_u - \tilde{W}_t) \right) \right)}{S_t \exp\left((r+\frac{1}{2}\sigma^2)h + \sigma(\tilde{W}_{t+h} - \tilde{W}_t) \right)} \\
&= \frac{x\psi_t \vee \left(\max_{t\leq u\leq t+h} x \exp\left((r+\frac{1}{2}\sigma^2)(u-t) + \sigma(\tilde{W}_u - \tilde{W}_t) \right) \right)}{x \exp\left((r+\frac{1}{2}\sigma^2)h + \sigma(\tilde{W}_{t+h} - \tilde{W}_t) \right)}.
\end{aligned} \tag{4.32}
$$

The process $(\tilde{W}_u - \tilde{W}_t)_{u\geq t}$ is independent of $\mathcal{F}_t$ and has the same distribution as $\tilde{W}$. Thus

$$
\tilde{\mathbb{E}}_{\psi_0} \left(f(\psi_{t+h}) \mid \mathcal{F}_t \right) = \tilde{\mathbb{E}}_{\psi_t} \left(f(\psi_h) \right) \tag{4.33}
$$

for each bounded measurable function f. So ψ is a Markov process.

In order to solve the problem using the developed theory, it is necessary to find the infinitesimal generator of ψ. We now show that the infinitesimal generator of this process on functions $h \in C^2$ takes the form

$$
\mathbb{L}_\psi = \begin{cases} (\sigma^2 - r)\psi \frac{\partial}{\partial \psi} + \frac{\sigma^2}{2}\psi^2 \frac{\partial^2}{\partial \psi^2} & \text{for } x > 1 \\ \frac{\partial}{\partial \psi} = 0 & \text{when } x = 1. \end{cases} \tag{4.34}
$$

This can be seen by applying the Itô formula to ψ to get

$$
d\psi_t = d\left(\frac{M_t}{S_t} \right) = (\sigma^2 - r)\psi_t \, dt - \sigma\psi_t \, d\tilde{W}_t + \frac{dM_t}{S_t} \tag{4.35}
$$

(Note, during the above calculations, that M is a finite variation process.) Applying the Itô formula to $f(\psi_t)$, for a smooth function f, yields the relation

$$
f(\psi_t) = f(\psi_0) + \int_0^t \mathbb{L} f(\psi_u) du - \sigma \int_0^t f'(\psi_u)\psi_u d\tilde{W}_u + \int_0^t f'(\psi_u) \frac{dM_u}{S_u} \tag{4.36}
$$

where the differential operator is

$$
\mathbb{L} = (\sigma^2 - r)\psi \frac{\partial}{\partial \psi} + \frac{\sigma^2}{2}\psi^2 \frac{\partial^2}{\partial \psi^2}. \tag{4.37}
$$

It is clear that $\psi_t \geq 1$, for all $t \in \mathbb{R}_+$. This suggests that a closer analysis of the behaviour of ψ at the boundary $\{1\}$ is needed. It will now be shown that

$$\int_0^t 1_{\{\psi_u=1\}} du = 0. \tag{4.38}$$

By the Fubini theorem, we have

$$\tilde{\mathbb{E}}\left(\int_0^\infty 1_{\{\psi_t=1\}}\, dt\right) = \int_0^\infty \tilde{\mathbb{E}}\big(1_{\{\psi_t=1\}}\big)\, dt = \int_0^\infty \tilde{P}\big(\psi_t = 1\big)\, dt \tag{4.39}$$

which is equal to zero since, as a result of the properties of the Brownian motion $\tilde{W}$ (specifically, B_t has a normal distribution), ψ_t has a distribution which is absolutely continuous with respect to Lebesgue measure.

Property (4.38) implies that ($\tilde{P}$-a.s.) the process ψ spends zero time in the set $\{1\}$, and so we have an instantly reflecting — or non-sticky — boundary. The process ψ is thus a diffusion with instant reflection at $\{1\}$. It then follows that $\frac{\partial}{\partial \psi}\Big|_{\psi=1} = 0$, and so we have verified (4.34).

Once again, it is possible to reformulate the optimal stopping problem in Mayer form. Let $(\tilde{\psi}_t)_{t\geq 0}$ be the process with infinitesimal generator

$$\mathbb{L}_{\tilde{\psi}} = \mathbb{L}_{\psi} - \lambda I \tag{4.40}$$

where I is the identity operator. The process $\tilde{\psi}$ corresponds to ψ killed at rate λ and thus we can write

$$\tilde{\psi}_t = \exp(-\lambda t)\psi_t. \tag{4.41}$$

Also let

$$\tilde{\mathbb{L}} = \mathbb{L} - \lambda I. \tag{4.42}$$

Now reformulate (4.31) to get

$$V(\psi_0) = x \sup_{0\leq\tau<\infty} \tilde{\mathbb{E}}_{\psi_0}\left(\tilde{\psi}_\tau\right). \tag{4.43}$$

4.5 The Alternative Solution

It will be assumed that the continuation region takes the form

$$C = \left\{\psi_0 \in \mathbb{R} \,\middle|\, 1 \leq \psi_0 < \hat{\psi}\right\} \tag{4.44}$$

for some constant $\hat{\psi} > 1$. This is the assumption that the optimal stopping boundary is a constant in $\tilde{\psi}$-space. (This corresponds to a boundary of the form (4.20) in

(x, m)-space.) This is reasonable for a one-dimensional problem for a Markov process on an infinite time horizon. Given assumption (4.44), it is possible to set up a system of equations in such a way that solving it will give both the value function and the optimal stopping boundary.

Theorem 4.1. *If a number $\hat{\psi}$ and a function $\hat{V} : [1, \infty) \to \mathbb{R}$ (where $\hat{V} \in C^2\big([1, \hat{\psi}) \cup (\hat{\psi}, \infty)\big) \cup C^1(\{\hat{\psi}\})$) solve the free boundary problem*

$$\tilde{\mathbb{L}}\hat{V}(\psi_0) = 0 \quad \textit{for} \;\; 1 \leq \psi_0 < \hat{\psi} \tag{4.45}$$

$$\hat{V}'(1+) = 0 \tag{4.46}$$

$$\hat{V}(\psi_0) = \psi_0 \quad \textit{for all} \;\; \psi_0 \geq \hat{\psi} \tag{4.47}$$

$$\hat{V}'(\hat{\psi}-) = 1 \tag{4.48}$$

where $\tilde{\mathbb{L}}$ is the differential operator defined in (4.42), then $\hat{V}$ coincides with the value function V of the optimal stopping problem (4.43), and

$$\tau_0 = \inf\left\{t \geq 0 \mid \tilde{\psi}_t \geq \hat{\psi}\right\} \tag{4.49}$$

is the optimal stopping time.

Before the theorem is proved, consider how the system was derived. If we assume that an optimal stopping time does exist then, just as in (4.17)–(4.19), it is possible to identify a Dirichlet problem corresponding to the optimal stopping problem (4.43). Equations (4.45)–(4.47) form what would be a Dirichlet problem for the process $\tilde{\psi}$ (or ψ killed at rate λ), if the region of continuation was known. Equations (4.45) and (4.46) arise from the infinitesimal generator defined in (4.40) (and (4.34)). Equation (4.47) is a result of instantaneous stopping beyond the boundary.

In order to find the optimal stopping boundary, the extra condition given by (4.48) is needed. It arises from the principle of smooth fit. Since the process under consideration is a diffusion, and the gain function is the identity function, the value function should join the gain function smoothly at the optimal stopping boundary. Their derivatives must therefore be equal at the boundary. The intuition behind the principle of smooth fit in this case is that the value function is the least superharmonic function dominating the gain function. It is easy to visualise this because the gain function is the identity function. If $\hat{V}'(\hat{\psi}-) > 1$ then the value function would have to drop below the gain function before the optimal stopping boundary is reached. If $\hat{V}'(\hat{\psi}-) < 1$ then we would be able to find a superharmonic function dominated by V. The above system thus seems to be a good guess for the solution to the problem (4.43). The following proof, from [45], shows that this is indeed the case.

Proof. We will first show that

$$\tilde{\mathbb{E}}_{\psi_0}(\tilde{\psi}_\tau) \leq \hat{V}(\psi_0) \tag{4.50}$$

for all stopping times τ, such that $0 \leq \tau < \infty$ a.s. We will then show that the stopping time τ_0 defined in (4.49) is finite $\tilde{P}$-a.s., and that

$$\tilde{\mathbb{E}}_{\psi_0}(\tilde{\psi}_{\tau_0}) = \hat{V}(\psi_0) \tag{4.51}$$

for all $\psi_0 \geq 1$.

By applying Itô's formula to $\hat{V}(\tilde{\psi}_t)$, we get

$$\hat{V}(\tilde{\psi}_t) = \hat{V}(\psi_0) + \int_0^t \tilde{\mathbb{L}}\hat{V}(\tilde{\psi}_u)\,du - \int_0^t \sigma\tilde{\psi}_u\hat{V}'(\tilde{\psi}_u)\,d\tilde{W}_u + \int_0^t \hat{V}'(\tilde{\psi}_u)\frac{dM_u}{S_u}. \tag{4.52}$$

By (4.45) and (4.47), the inequality

$$\tilde{\mathbb{L}}\hat{V}(\psi_0) \leq 0 \tag{4.53}$$

holds for all $\psi_0 \geq 1$. Also, the last integral in (4.52) is zero, since dM_t/S_t only increases when $\tilde{\psi}_t = 1$, and $\hat{V}'(1+) = 0$, by (4.46).

The integral

$$I_t = -\int_0^t \sigma\tilde{\psi}_u\hat{V}'(\tilde{\psi}_u)\,d\tilde{W}_u \tag{4.54}$$

is a local martingale (since it is a stochastic integral with respect to a Brownian motion). It is uniformly bounded from below, since

$$-\hat{V}(\psi_0) \leq \hat{V}(\tilde{\psi}_t) - \hat{V}(\psi_0) \leq I_t. \tag{4.55}$$

Therefore, $(I_t)_{t\geq 0}$ is a supermartingale (see Theorem B.7). Hence

$$\tilde{\mathbb{E}}_{\psi_0}(I_\tau) \leq \tilde{\mathbb{E}}_{\psi_0}(I_0) = 0 \tag{4.56}$$

implying, together with (4.52) and (4.53), that

$$\tilde{\mathbb{E}}_{\psi_0}\left(\hat{V}(\tilde{\psi}_\tau)\right) \leq \hat{V}(\psi_0) + \tilde{\mathbb{E}}_{\psi_0}(I_\tau) \leq \hat{V}(\psi_0) \tag{4.57}$$

for any $\psi_0 \geq 1$ and any finite stopping time τ. This, together with $\psi_0 \leq \hat{V}(\psi_0)$, proves (4.50).

If $\psi_0 \geq \hat{\psi}$, then $\tau_0 = 0$, which is finite, and (4.51) is clearly satisfied. Thus, suppose $\psi_0 < \hat{\psi}$. First consider the finiteness of the stopping time τ_0, defined in (4.49), for all $\psi_0 \geq 1$. Note that, for integral $T \geq 1$,

$$\begin{aligned}
\tilde{P}_{\psi_0}\left(\max_{0\leq t\leq T}\tilde{\psi}_t \geq \hat{\psi}\right) &\geq \tilde{P}_{\psi_0}\left(\max_{0\leq u\leq t\leq T}\frac{S_u}{S_t} \geq \hat{\psi}\right) \\
&= \tilde{P}_{\psi_0}\left(\max_{0\leq u\leq t\leq T}\exp\left(\left(\tilde{W}_u - \tilde{W}_t\right)\sigma + \left(r + \frac{\sigma^2}{2}\right)(u-t)\right) \geq \hat{\psi}\right) \\
&\geq \tilde{P}_{\psi_0}\left(\max\left(\tilde{W}_1 - \tilde{W}_0, \ldots, \tilde{W}_T - \tilde{W}_{T-1}\right) \geq K\right)
\end{aligned} \tag{4.58}$$

where $K = \left(\log\hat{\psi} - (r + \frac{\sigma^2}{2})\right)/\sigma$. The first inequality is a result of $\tilde{\psi} \geq \psi$ by (4.41) and $\psi_0 < \hat{\psi}$. For any real K, the last probability tends to 1 as $T \to \infty$. Thus, the process $\tilde{\psi}$ is almost surely unbounded, which proves that τ_0 is finite.

It now remains to show

$$\tilde{\mathbb{E}}_{\psi_0}\left(\tilde{\psi}_{\tau_0}\right) = \hat{V}(\psi_0). \tag{4.59}$$

From (4.52) it follows that

$$\hat{V}(\tilde{\psi}_{t\wedge\tau_0}) = \hat{V}(\psi_0) + \int_0^{t\wedge\tau_0} \tilde{\mathbb{L}}\hat{V}(\tilde{\psi}_u)\,du + I_{t\wedge\tau_0}. \tag{4.60}$$

If $\psi_0 < \hat{\psi}$, then

$$\tilde{\mathbb{L}}\hat{V}(\tilde{\psi}_u) = 0 \tag{4.61}$$

for $u \leq t \leq \tau_0$. Since $\hat{V}$ is increasing, and by the definition (4.49) of τ_0 we have $\tilde{\psi}_{t\wedge\tau_0} \leq \hat{\psi}$, it follows that

$$\hat{V}(\hat{\psi}) - \hat{V}(\psi_0) \geq \hat{V}(\tilde{\psi}_{t\wedge\tau_0}) - \hat{V}(\psi_0) = I_{t\wedge\tau_0} \geq -\hat{V}(\psi_0). \tag{4.62}$$

Therefore the process $(I_t)_{t\geq 0}$ is a local martingale, uniformly bounded above and below, making it a uniformly integrable martingale. Hence

$$\tilde{\mathbb{E}}_{\psi_0}\left(I_{\tau_0}\right) = \tilde{\mathbb{E}}_{\psi_0}\left(I_0\right) = 0 \tag{4.63}$$

by the optional sampling theorem (see [20, page 19]). Thus

$$\tilde{\mathbb{E}}_{\psi_0}\left(\tilde{\psi}_{\tau_0}\right) = \hat{V}(\psi_0) \tag{4.64}$$

which proves the statement (4.51). □

From Theorem 4.1, we see that to find a closed-form solution for the price of the Russian option, the system of equations (4.45)–(4.48) must be solved. The price of the option is then $x\hat{V}(\phi_0)$. Theorem 4.2 now shows how the system is solved.

Theorem 4.2. *The rational price of the perpetual Russian option, with payoff function $f = (f_t)_{t\geq 0}$ from (4.5), is given by*

$$V(\psi_0) = x\begin{cases} \frac{\hat{\psi}}{x_2 - x_1}\left((x_2 - 1)\left(\frac{\psi_0}{\hat{\psi}}\right)^{x_1} + (1 - x_1)\left(\frac{\psi_0}{\hat{\psi}}\right)^{x_2}\right) & \text{for } 1 \leq \psi_0 < \hat{\psi} \\ \psi_0 & \text{for } \psi_0 \geq \hat{\psi}. \end{cases} \tag{4.65}$$

This can be written more compactly as

$$V(\psi_0) = x\begin{cases} \hat{\psi}\frac{x_2\psi_0^{x_1} - x_1\psi_0^{x_2}}{x_2\hat{\psi}^{x_1} - x_1\hat{\psi}^{x_2}} & \text{for } 1 \leq \psi_0 < \hat{\psi} \\ \psi_0 & \text{for } \psi_0 \geq \hat{\psi} \end{cases} \tag{4.66}$$

where x_1 and x_2 are the roots of the quadratic equation

$$y^2 - Ay - B = 0 \tag{4.67}$$

where

$$A = 1 + \frac{2r}{\sigma^2} \qquad \textit{and} \qquad B = \frac{2\lambda}{\sigma^2}. \tag{4.68}$$

Then x_1 and x_2 are given explicitly by

$$x_1 = \frac{A}{2} - \sqrt{\left(\frac{A}{2}\right)^2 + B} \tag{4.69}$$

and

$$x_2 = \frac{A}{2} + \sqrt{\left(\frac{A}{2}\right)^2 + B}. \tag{4.70}$$

Finally, $\hat{\psi}$ is given by

$$\hat{\psi} = \left|\frac{x_2}{x_1}\frac{x_1 - 1}{x_2 - 1}\right|^{\frac{1}{x_2 - x_1}}. \tag{4.71}$$

Proof. To derive (4.65), the system of equations (4.45)–(4.48) must be solved. Assume that the solution of (4.45) takes the form $V(\psi) = \psi^y$. Then the quadratic equation $y^2 - Ay - B = 0$ is obtained, with $A = 1 + \frac{2r}{\sigma^2}$ and $B = \frac{2\lambda}{\sigma^2}$. Finding the roots of this equation, we get

$$x_1 = \frac{A}{2} - \sqrt{\left(\frac{A}{2}\right)^2 + B} \tag{4.72}$$

$$x_2 = \frac{A}{2} + \sqrt{\left(\frac{A}{2}\right)^2 + B}. \tag{4.73}$$

Note that $x_1 < 0$ and $x_2 > 1$.

In the region $1 < \psi < \hat{\psi}$, we can write the solution $V(\psi)$ to (4.45) in the form

$$V(\psi) = C_1\psi^{x_1} + C_2\psi^{x_2} \tag{4.74}$$

where C_1 and C_2 are constants. Because (4.45) is linear, a linear combination of the two particular solutions, ψ^{x_1} and ψ^{x_2} is also a solution to (4.45). The three remaining equations in the system (4.45)–(4.48) give the three conditions necessary to find C_1, C_2 and $\hat{\psi}$. These are, $V(\hat{\psi}) = \hat{\psi}$, which implies

$$C_1\hat{\psi}^{x_1} + C_2\hat{\psi}^{x_2} = \hat{\psi} \tag{4.75}$$

$V'(\hat{\psi}-) = 1$, which implies

$$C_1 x_1 \hat{\psi}^{x_1-1} + C_2 x_2 \hat{\psi}^{x_2-1} = 1 \tag{4.76}$$

and $V'(1+) = 0$, which implies

$$C_1 x_1 + C_2 x_2 = 0. \tag{4.77}$$

Solving for C_1 in (4.77) yields

$$C_1 = -\frac{x_2}{x_1} C_2. \tag{4.78}$$

Substituting (4.78) into (4.75), we get

$$C_2 = \frac{\hat{\psi} x_1}{x_1 \hat{\psi}^{x_2} - x_2 \hat{\psi}^{x_1}}. \tag{4.79}$$

Thus

$$C_1 = \frac{\hat{\psi} x_2}{x_2 \hat{\psi}^{x_1} - x_1 \hat{\psi}^{x_2}}. \tag{4.80}$$

Finally, by substituting C_1 and C_2 into (4.76), we get

$$\hat{\psi} = \left| \frac{x_2}{x_1} \frac{x_1 - 1}{x_2 - 1} \right|^{\frac{1}{x_2 - x_1}} \tag{4.81}$$

thus completing the proof. □

Chapter 5

Barrier Options

5.1 Introduction

A barrier option is a financial instrument that has some feature of its payoff structure triggered if the price of the underlying asset reaches a particular barrier level. Barrier options can be classified as either 'up' or 'down', depending on whether the price of the underlying asset is initially above or below the barrier level, and hence must move either down (in the case of a 'down' option) or up (in the case of an 'up' option) to reach the barrier. A barrier option can also be referred to as either 'in' or 'out'. An 'out' option will cease to exist if the underlying asset price hits the barrier level. This is called 'knocking-out'. In the case of an 'in' option, the underlying asset price must reach the barrier level for the option to 'knock-in'. Once an option has 'knocked-in', it behaves like a vanilla option. Before 'knocking-in', the option has a zero payoff. Barrier options can thus be classified as 'up-and-in', 'up-and-out', 'down-and-in' or 'down-and-out'. They can also be either put or call options.

This chapter focuses on the pricing of American barrier options. Section 1 formulates the problem in the case of the 'up-and-out' put, and the solution, as derived in [22], is given in Section 2. For completeness, the final section analyses the 'down-and-out' put.

5.2 Formulation of the Problem

As before, the riskless bank account process $B = (B_t)_{t\geq 0}$ satisfies

$$dB_t = rB_t\,dt \qquad (B_0 = 1) \tag{5.1}$$

with $r \geq 0$, and the asset price process $S = (S_t)_{t\geq 0}$ satisfies

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t \qquad (S_0 = x) \tag{5.2}$$

where $x > 0$, and the process $W = (W_t)_{t\geq 0}$ is a Brownian motion under the risk-neutral martingale measure Q.

Karatzas and Wang [22] price a perpetual American 'up-and-out' put with strike price $K > 0$ and barrier level $h > K$. Such an option can be exercised at any time, and pays $(K - S_t)^+$ on exercise, provided the stock price has not reached the barrier level h. If the stock price reaches the barrier, then the option immediately becomes worthless. The payoff of the option at time t is thus

$$Y_t = (K - S_t)^+ 1_{\{t<\tau_h\}} \tag{5.3}$$

for $0 \leq t < \infty$, where

$$\tau_h = \inf\{t \geq 0 \mid S_t = h\} \tag{5.4}$$

is the first time the stock price hits the barrier level h, and is therefore the instant at which the option is 'knocked-out' and becomes worthless.

From the general theory of American option pricing in Chapter 3, the arbitrage-free price of this option is given by

$$\begin{aligned} V(x) &= \sup_\tau \mathbb{E}_x \left(e^{-r\tau} (K - S_\tau)^+ 1_{\{\tau<\tau_h\}} \right) \\ &= \sup_\tau \mathbb{E}_x \left(e^{-r\tau} (K - S_\tau)^+ 1_{\{\max_{0\leq u\leq\tau} S_u<h\}} \right). \end{aligned} \tag{5.5}$$

The solution to this optimal stopping problem will give the price of the option and the optimal exercise time.

5.3 The Solution

First, assume the existence of a stock price $b < K$ at which it is optimal to exercise the option, so that the continuation region is

$$C = \{x \in (0, \infty) \mid x > b\} \tag{5.6}$$

and the stopping region D is

$$D = \{x \in (0, \infty) \mid x \leq b\}. \tag{5.7}$$

Now consider the structure of the problem. At first glance it seems that the optimal stopping problem (5.5) is two-dimensional, with a non-continuous gain function. On closer inspection, however, it becomes clear that the indicator function is so simple that the problem degenerates and becomes one dimensional in the following way:

$$V(x) = \sup_{0\leq\tau\leq\tau_h} E_x \left(e^{-r\tau} (K - S_\tau)^+ \right) \tag{5.8}$$

where τ_h is defined in (5.4). This is an optimal stopping problem that has two boundaries — the barrier level h can be seen as a second optimal stopping boundary.

Once the barrier is hit, the most the option holder can receive is zero. It is thus optimal to stop the process at h, since the situation cannot improve for the option holder after that point.

Since the gain function is continuous on $(0, h)$, the theory of Chapter 2 can be used to set up a free boundary problem corresponding to (5.5). Now recall Theorem 2.8. Since the process $(V_t)_{t\geq 0}$ (where $V_t = V(S_t)$) is a supermartingale, we have

$$\mathbb{L}_S V(x) - rV(x) \leq 0 \tag{5.9}$$

for all x, where $-rV$ is introduced because of the exponential factor in the gain function, and where

$$\mathbb{L}_S = \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} \tag{5.10}$$

is the infinitesimal generator of S. However, the process $(V_t)_{t\geq 0}$ is a martingale in the continuation region. Thus

$$\mathbb{L}_S V(x) - rV(x) = 0 \tag{5.11}$$

for $b < x < h$. Instantaneous stopping beyond the boundaries gives

$$V(x) = (K - x)^+ \tag{5.12}$$

for $0 < x \leq b$, and

$$V(x) = 0 \tag{5.13}$$

for $h \leq x < \infty$. We also know that the value function dominates the gain function, and thus

$$V(x) > (K - x)^+ \tag{5.14}$$

for $b < x < h$. The principle of smooth fit gives the condition

$$V'(b+) = -1. \tag{5.15}$$

Smooth fit is only applied at the boundary b (and not at h). This is the case because h is a fixed boundary and not a free boundary and therefore does not need an extra condition to define it.

Theorem 5.1. *If a number $b \in (0, K)$ and a convex decreasing function $\hat{V} : \mathbb{R}_+ \to \mathbb{R}$ (where $\hat{V} \in C((0,\infty)) \cap C^1((0,\infty)\backslash\{h\}) \cap C^2((0,\infty)\backslash\{b,h\})$) solve the system*

$$\mathbb{L}_S\hat{V}(x) < r\hat{V}(x) \qquad \text{for } 0 < x < b \tag{5.16}$$

$$\mathbb{L}_S\hat{V}(x) = r\hat{V}(x) \qquad \text{for } b < x < h \tag{5.17}$$

$$\hat{V}(x) > (K - x)^+ \qquad \text{for } b < x < h \tag{5.18}$$

$$\hat{V}(x) = (K - x)^+ \qquad \text{for } 0 < x \leq b \tag{5.19}$$

$$\hat{V}(x) = 0 \qquad \text{for } h \leq x < \infty \tag{5.20}$$

$$\hat{V}'(b+) = -1 \qquad \text{(smooth fit)} \tag{5.21}$$

where $\mathbb{L}_S$ is the differential operator defined in (5.10), then $\hat{V}$ coincides with the value function V of the optimal stopping problem (5.5), and

$$\tau_b = \inf\{t \geq 0 \mid S_t \leq b\} \tag{5.22}$$

is the optimal stopping time.

Proof. A pair $(b, \hat{V})$ that satisfies the system (5.16)–(5.21) clearly solves the optimal stopping problem for $x \geq h$. The proof therefore concentrates on a fixed $x \in (0, h)$.

Since a function $\hat{V}$ that solves the above system has the properties that $\hat{V}$ and $\hat{V}'$ are continuous and bounded on $(0,\infty)\backslash\{h\}$, and $\hat{V}''$ is continuous and bounded on $(0,\infty)\backslash\{b,h\}$, Itô's rule can be applied in the standard form to $e^{-rt}\hat{V}(S_t)$ as follows:

$$\begin{aligned} &d\left(e^{-rt}\hat{V}(S_t)\right) \\ &\quad = e^{-rt}\left(rS_t\hat{V}'(S_t) - r\hat{V}(S_t) + \frac{\sigma^2}{2}S_t^2\hat{V}''(S_t)\right)dt + e^{-rt}\sigma S_t\hat{V}'(S_t)dW_t \end{aligned} \tag{5.23}$$

where $\hat{V}(S_0) = \hat{V}(x)$ (see [31, page 57, Exercise 4.8]). This is well defined by the boundedness assumptions on $\hat{V}$. Now recall τ_h as defined in (5.4). Since $\tau_h < \infty$ a.s., it follows from (5.23) that

$$\begin{aligned} &\hat{V}(x) - e^{-r(\tau\wedge\tau_h)}\hat{V}(S(\tau\wedge\tau_h)) + \sigma\int_0^{\tau\wedge\tau_h} e^{-ru}S_u\hat{V}'(S_u)dW_u \\ &\quad = -\int_0^{\tau\wedge\tau_h} e^{-ru}\left(\frac{\sigma^2}{2}S_u^2\hat{V}''(S_u) + rS_u\hat{V}'(S_u) - r\hat{V}(S_u)\right)du \geq 0 \end{aligned} \tag{5.24}$$

for all stopping times τ, where the inequality arises as a result of (5.16) and (5.17). Thus

$$\hat{V}(x) - e^{-r(\tau\wedge\tau_h)}\hat{V}(S_{\tau\wedge\tau_h}) + \sigma\int_0^{\tau\wedge\tau_h} e^{-ru}S_u\hat{V}'(S_u)dW_u \geq 0. \tag{5.25}$$

The P_x-expectation of the stochastic integral is zero because the convexity of $\hat{V}$ implies that $\mathbb{E}(\int_0^\infty e^{-2rt}(S_t\hat{V}'(S_t))^2\,dt) < \infty$ and thus $\{\int_0^t e^{-ru}S_u\hat{V}'(S_u)\,dW_u\}_{t\geq 0}$ is a martingale. Thus

$$\begin{aligned}\hat{V}(x) &\geq \mathbb{E}_x\left(e^{-r(\tau\wedge\tau_h)}\hat{V}(S_{\tau\wedge\tau_h})\right)\\ &= \mathbb{E}_x\left(e^{-r\tau}\hat{V}(S_\tau)1_{\{\tau<\tau_h\}}\right) + \mathbb{E}_x\left(e^{-r\tau_h}\hat{V}(S_{\tau_h})1_{\{\tau\geq\tau_h\}}\right)\\ &= \mathbb{E}_x\left(e^{-r\tau}\hat{V}(S_\tau)1_{\{\tau<\tau_h\}}\right)\\ &\geq \mathbb{E}_x\left(e^{-r\tau}(K-S_\tau)^+1_{\{\tau<\tau_h\}}\right)\end{aligned} \tag{5.26}$$

where the third equality is a result of (5.20) and $S_{\tau_h} = h$, and the last inequality arises from (5.18) and (5.19).

Now substitute $\tau = \tau_b$, as defined in (5.22), into (5.24). Then, if $x > b$, by (5.17) and (5.20), we have

$$\begin{aligned}\hat{V}(x) &= \mathbb{E}_x\left(e^{-r(\tau_b\wedge\tau_h)}\hat{V}(S_{\tau_b\wedge\tau_h})\right)\\ &= \mathbb{E}_x\left(e^{-r\tau_b}\hat{V}(S_{\tau_b})1_{\{\tau_b<\tau_h\}}\right) + \mathbb{E}_x\left(e^{-r\tau_h}\hat{V}(S_{\tau_h})1_{\{\tau_b\geq\tau_h\}}\right)\\ &= \mathbb{E}_x\left(e^{-r\tau_b}\hat{V}(S_{\tau_b})1_{\{\tau_b<\tau_h\}}\right)\\ &= \mathbb{E}_x\left(e^{-r\tau_b}(K-S_{\tau_b})^+1_{\{\tau_b<\tau_h\}}\right)\end{aligned} \tag{5.27}$$

and, if $x \leq b$, then $\tau_b = 0$ and by (5.19) we have

$$\hat{V}(x) = (K-x)^+ = \exp(-r\tau_b)(K-S_{\tau_b})^+1_{\{\tau_b<\tau_h\}}. \tag{5.28}$$

Thus, it follows from (5.26), (5.27) and (5.28) that

$$\hat{V}(x) = \sup_\tau \mathbb{E}_x\left(e^{-r\tau}(K-S_\tau)^+1_{\{\tau<\tau_h\}}\right) = V(x) \tag{5.29}$$

and the supremum is attained at τ_b. □

The system (5.17)–(5.21) can be solved to obtain closed form solutions for both V and b. This is shown in the following theorem.

Theorem 5.2. *The equation*

$$1 + \beta\left(\frac{b}{K}\right) = \beta + \left(\frac{b}{h}\right)^\beta \tag{5.30}$$

has a unique solution $b \in (0, K)$, where

$$\beta = 1 + \frac{2r}{\sigma^2}. \tag{5.31}$$

Define

$$V(x) = \begin{cases} K - x & \text{for } 0 < x \leq b \\ x\left(\frac{K-b}{b}\right)\frac{(h/x)^{\beta}-1}{(h/b)^{\beta}-1} & \text{for } b < x < h \\ 0 & \text{for } h \leq x < \infty. \end{cases} \tag{5.32}$$

Then V is a convex decreasing function, and the pair (b, V) solves the system (5.16)–(5.21).

Proof. We will first derive equation (5.30) and show that it has a unique root. Let $\hat{V}$ be a convex decreasing function satisfying the conditions of Theorem 5.1. Then, by (5.17), $\hat{V}$ satisfies

$$\frac{\sigma^2}{2}x^2\hat{V}''(x) + rx\hat{V}'(x) - r\hat{V}(x) = 0 \tag{5.33}$$

in the region $b < x < h$. The general solution of (5.33) is

$$\hat{V}(x) = Ax^{-\gamma_+} + Bx^{-\gamma_-} \tag{5.34}$$

for real constants A and B, where

$$\gamma_+ = \frac{2r}{\sigma^2} \tag{5.35}$$

and

$$\gamma_- = -1 \tag{5.36}$$

are the roots of the quadratic equation

$$\frac{\sigma^2}{2}\gamma(\gamma+1) - r\gamma - r = 0. \tag{5.37}$$

Since $\hat{V}$ is continuous, we have $\hat{V}(b+) = K - b$ and $\hat{V}(h-) = 0$, by (5.19) and (5.20). Hence, by (5.34), we have the simultaneous equations

$$Ab^{-\gamma_+} + Bb^{-\gamma_-} = K - b \tag{5.38}$$

$$Ah^{-\gamma_+} + Bh^{-\gamma_-} = 0. \tag{5.39}$$

Solving them yields

$$A = \left(\frac{K}{b} - 1\right)\left(b^{-\beta} - h^{-\beta}\right)^{-1} \tag{5.40}$$

$$B = -Ah^{-\beta} \tag{5.41}$$

where β is defined in (5.31). Now, by (5.21), we have $\hat{V}'(b) = -1$, and substituting (5.40) and (5.41) into this equation gives (5.30).

To see that (5.30) has a unique solution $b \in (0, K)$, note that

$$F(u) = \beta - 1 + \left(\frac{u}{h}\right)^{\beta} - \beta\left(\frac{u}{K}\right) \tag{5.42}$$

is convex for $0 \leq u < \infty$, since

$$F''(u) = \beta(\beta - 1)\left(\frac{u}{h}\right)^{\beta-2} > 0 \tag{5.43}$$

for $0 < u < \infty$. Also

$$F(0) = \beta - 1 > 0 \tag{5.44}$$

and

$$F(K) = \left(\frac{K}{h}\right)^{\beta} - 1 < 0. \tag{5.45}$$

Thus F has exactly one root in the interval $(0, K)$.

Now consider the function V defined in (5.32). We derive V by setting

$$V(x) = K - x \tag{5.46}$$

for $0 \leq x \leq b$, and by setting

$$V(x) = 0 \tag{5.47}$$

for $h \leq x$, and lastly setting

$$V(x) = Ax^{-\beta+1} + Bx \tag{5.48}$$

where A and B are defined in (5.40) and (5.41). Thus V satisfies (5.17)–(5.21) by construction. It is easy to see that V also satisfies (5.16) and $V \in C((0,\infty)) \cap C_b^1((0,\infty)\backslash\{h\}) \cap C_b^2((0,\infty)\backslash\{b,h\})$.

Now consider

$$V'(x) = \begin{cases} -1 & \text{for } 0 \leq x \leq b \\ \frac{K-b}{b}\frac{(h/x)^{\beta}(1-\beta)-1}{(h/b)^{\beta}-1} & \text{for } b < x < h \\ 0 & \text{for } x \leq h. \end{cases} \tag{5.49}$$

Then $V'(x) < 0$ on $(0, h)$ since $\beta - 1 < 0$ and thus V is decreasing on $(0, h)$. In the interval (b, h), the relation

$$\frac{\sigma^2}{2}x^2V''(x) = r\left(V(x) - xV'(x)\right) > 0 \tag{5.50}$$

holds since $V(x) > 0$ and $-xV'(x) > 0$, showing that V is strictly convex on (b, h). Thus V satisfies all the conditions stipulated for a solution to the system (5.16)–(5.21). □

It is interesting to note that $b > \hat{b} = \frac{\beta-1}{\beta}K$, where $\hat{b}$ is the optimal stopping point for the vanilla perpetual American put, since substituting $\hat{b}$ into the convex function F of (5.42) gives

$$F(\hat{b}) = \left(\frac{\beta-1}{\beta}K\right)^{\beta} > 0 \tag{5.51}$$

while $F(0) > 0$ and $F(K) < 0$. The relationship $b > \hat{b}$ is to be expected since the possibility of knocking out makes it preferable to lock in a lower profit, rather than run the risk of losing the option and any future possibility of profit.

Now consider the behaviour of the boundary point b as the barrier $h \downarrow K$. The boundary point satisfies the equation

$$1 + \beta\left(\frac{b}{K}\right) = \beta + \left(\frac{b}{h}\right)^{\beta}. \tag{5.52}$$

Let h tend to K from above. Since the function

$$\hat{F}(h) = \beta - 1 + \left(\frac{b}{h}\right)^{\beta} - \beta\left(\frac{b}{K}\right) \tag{5.53}$$

is continuous for $h > 0$, it is clear that the boundary point b tends to the unique solution of the equation

$$1 + \beta\left(\frac{b}{K}\right) = \beta + \left(\frac{b}{K}\right)^{\beta}. \tag{5.54}$$

The solution to this equation is $b = K$, and thus $b \uparrow K$ as $h \downarrow K$.

This also shows that the problem (5.5), with $0 < h \leq K$, becomes trivial in the sense that it is optimal to exercise instantaneously. To see this, observe that $b \uparrow K$ as $h \downarrow K$ implies that

$$\sup_{\tau} \mathbb{E}_x\left((K - S_\tau)^+ 1_{\{\tau<\tau_K\}}\right) = (K-x)^+. \tag{5.55}$$

Now suppose that $h < K$, and let $x \in (0, h)$ be fixed. Then

$$\begin{aligned}(K-x)^+ &\leq \sup_{\tau} \mathbb{E}_x\left((K - S_\tau)^+ 1_{\{\tau<\tau_h\}}\right) \\ &\leq \sup_{\tau} \mathbb{E}_x\left((K - S_\tau)^+ 1_{\{\tau<\tau_K\}}\right) = (K-x)^+\end{aligned} \tag{5.56}$$

since $\tau_h < \tau_K$ implies that $1_{\{\tau<\tau_h\}} \leq 1_{\{\tau<\tau_K\}}$. Thus

$$\sup_{\tau} \mathbb{E}_x\left((K - S_\tau)^+ 1_{\{\tau<\tau_h\}}\right) = (K-x)^+ \tag{5.57}$$

proving the claim.

5.4 The Down-and-Out Put ($x > h$)

This type of option has the same payoff

$$Y_t = (K - S_t)^+ 1_{\{t<\tau_h\}} \tag{5.58}$$

as the up and out put already considered. However, the initial stock price x is now above the barrier h. It is possible to determine the option price by considering the nature and behaviour of the option, in conjunction with the theory of optimal stopping problems outlined in Chapter 2. The relationships between the strike price K of the option, the barrier level h, and the optimal stopping point $\hat{b}$, of the vanilla perpetual American put influence both the price of the barrier option and the level of the boundary point b. There are a number of cases to consider.

5.4.1 Case 1: $\hat{b} < K < h$

In this case the option has zero value, since it knocks out before ever being in-the-money.

5.4.2 Case 2: $h < \hat{b} < K$

In this case the price of the barrier option is the same as that of the vanilla American put, since the option is exercised at $\hat{b}$ before the barrier is hit.

5.4.3 Case 3: $\hat{b} < h < K$

In order to price this option analytically, a slight adjustment to the payoff (5.58) is needed. Let us now assume that the option may be exercised instantaneously if the stock price hits the barrier h. Thus

$$Y_t = (K - S_t)^+ 1_{\{t\leq\tau_h\}}. \tag{5.59}$$

Since the option knocks out in-the-money, this assumption is both useful and reasonable.

The optimal stopping point for this option is then h. This becomes clear when we consider that, in the absence of a barrier, the option holder would not exercise until the stock price reached the level $\hat{b}$, which is below h. It would therefore not be optimal to exercise the option above the level h. However, after the stock price reaches h, the option knocks out and becomes worthless, and so the option holder cannot wait for the stock price to reach $\hat{b}$, but must exercise at h.

Since the boundary point is known, the option price is

$$V(x) = \mathbb{E}_x\left(e^{-r\tau_h}\left(K - S(\tau_h)\right)\right) = \mathbb{E}_x\left(e^{-r\tau_h}\left(K - h\right)\right) \tag{5.60}$$
$$= \mathbb{E}_x\left(e^{-r\tau_h}\right)(K-h) = \left(\frac{h}{x}\right)^{\beta-1}(K-h)$$

for $x \geq h$, where the last inequality follows from a result in [41, page 18]. Thus

$$V(x) = \begin{cases} 0 & \text{for } x < h \\ \left(\frac{h}{x}\right)^{\beta-1}(K-h) & \text{for } x \geq h. \end{cases} \tag{5.61}$$

Chapter 6

The American Put

6.1 Introduction

This chapter considers the problem of pricing an American put option with strike price K. Such an option gives the holder the right to sell a unit of the underlying stock for the price K, at any time before maturity of the option at time T. The option price and the optimal exercise time of the option form the solution to an optimal stopping problem. If T is infinite then the problem is one-dimensional, and closed form solutions for the option price V, and the optimal stopping boundary b, can be obtained (see [18, page 4]). However, when T is finite, the problem of finding V and b is dependent on the time to maturity, and is therefore two-dimensional. This section reviews the history of the pricing of the American put. Section 2 formulates the problem, and Section 3 shows that the optimal stopping boundary is the unique solution of an integral equation.

It was Bensoussan [3], and later Karatzas [19], that first used no-arbitrage methods to show that the price of the American put is the solution to an optimal stopping problem. This work followed that of McKean [28], who was the first to derive a free boundary problem for the 'discounted' American call with gain function $G(x) = e^{-\beta t}(x - K)^+$. He expressed V in terms of b, so that b solves a countable system of nonlinear integral equations (see [28, page 39]). The existence and uniqueness of a solution to this system was left open.

McKean's work was later extended by van Moerbeke [51], who derived a single non-linear integral equation for b. The existence and uniqueness of a solution to this integral equation was proved by van Moerbeke for a general optimal stopping problem, although these results are merely indicated for the discounted American call. A disadvantage of van Moerbeke's result is that his integral equation involves both b and its derivative b'. This makes it difficult to solve numerically.

During the early 1990s, Kim [23], Jacka [18] and Carr et al. [6] independently arrived at a more tractable nonlinear integral equation for b, that also has a clear

economic meaning. When the superharmonic function V is decomposed into its harmonic and potential parts, it is possible to identify the early exercise premium paid by the option holder for the privilege of being able to exercise early. The new integral equation for b arises from the early exercise premium representation of V, and can be called the 'free boundary equation'. The uniqueness and regularity of the solution of this integral equation remained open, until Peskir [37] showed that the free boundary equation alone does indeed characterize the optimal exercise boundary b of the American put.

6.2 Formulation of the Problem

This section sets up the American put problem, and introduces a change-of-variable formula with local times on curves, first established in [36]. This formula will be used in the proof of the main result in the next section.

The arbitrage-free price of the American put option at time $t \in [0, T]$ is given by

$$V(t,x) = \sup_{0 \le \tau \le T-t} \mathbb{E}_{t,x}\left(e^{-r\tau}(K - X_{t+\tau})^+\right) \tag{6.1}$$

where τ is a stopping time. The stock price process $X = (X_t)_{t \ge 0}$ satisfies

$$dX_{t+s} = rX_{t+s}ds + \sigma X_{t+s}dB_s \qquad (X_t = x) \tag{6.2}$$

under $P_{t,x}$, where the process $B = (B_s)_{s \ge 0}$ is a standard Brownian motion starting at zero. The strong solution of (6.2) is given by

$$X_{t+s} = x \exp\left(\sigma B_s + \left(r - \frac{\sigma^2}{2}\right)s\right) \tag{6.3}$$

whenever $t > 0$ and $x > 0$ are given and fixed. The infinitesimal generator of the (strong) Markov process X is given by

$$\mathbb{L}_X = rx\frac{\partial}{\partial x} + \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2}. \tag{6.4}$$

Standard Markovian arguments (as outlined in Chapter 2) suggest that V from (6.1) solves the following free boundary problem of parabolic type:

$$V_t + \mathbb{L}_X V = rV \quad \text{in} \quad C \tag{6.5}$$

$$V(t,x) = (K - x)^+ \quad \text{for} \quad x = b(t) \tag{6.6}$$

$$V_x(t,x) = -1 \quad \text{for} \quad x = b(t) \quad \text{(smooth fit)} \tag{6.7}$$

$$V(t,x) > (K - x)^+ \quad \text{in} \quad C \tag{6.8}$$

$$V(t,x) = (K - x)^+ \quad \text{in} \quad D \tag{6.9}$$

where the continuation region C and the stopping region $D = C^c$ are defined as follows:

$$C = \{(t,x) \in [0,T) \times (0,\infty) \mid x > b(t)\} \tag{6.10}$$

$$D = \{(t,x) \in [0,T) \times (0,\infty) \mid x \leq b(t)\} \tag{6.11}$$

and $b : [0,T] \to \mathbb{R}$ is the optimal stopping boundary. It follows that the stopping time

$$\tau_b = \inf\{0 \leq s \leq T - t \mid X_{t+s} \leq b(t+s)\} \tag{6.12}$$

is optimal for (6.1) and that the supremum is attained at this time.

A change-of-variable formula for continuous semimartingales is derived in [36]. Since X solves the stochastic differential equation

$$dX_t = rX_t\,dt + \sigma X_t\,dB_t \tag{6.13}$$

the following simplified version of the formula is sufficient. Let $c : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function of bounded variation, and let $F : \mathbb{R}_+^2 \to \mathbb{R}$ be a continuous function satisfying

$$F \text{ is } C^{1,2} \text{ on } C_1 \cup C_2 \tag{6.14}$$

$$F_t + \mathbb{L}_X F \text{ is locally bounded} \tag{6.15}$$

$$x \mapsto F(t,x) \text{ is convex} \tag{6.16}$$

$$t \mapsto F_x(t, b(t)\pm) \text{ is continuous.} \tag{6.17}$$

where C_1 and C_2 are given by

$$C_1 = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} \mid x > c(t)\} \tag{6.18}$$

$$C_2 = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} \mid x < c(t)\}. \tag{6.19}$$

Then the following change-of-variable formula holds

$$\begin{aligned} F(t,X_t) =& F(0,X_0) + \int_0^t (F_t + \mathbb{L}_X F)(s,X_s) 1_{\{X_s \neq c(s)\}}\,ds \\ &+ \int_0^t F_x(s,X_s)\sigma X_s 1_{\{X_s \neq c(s)\}}\,dB_s \\ &+ \frac{1}{2}\int_0^t \big(F_x(s,X_s+) - F_x(s,X_s-)\big) 1_{\{X_s = c(s)\}}\,d\ell_s^c(X) \end{aligned} \tag{6.20}$$

where $\mathbb{L}_X$ is the infinitesimal generator of X, defined in (6.4), and $\ell_s^c(x)$ is the local time of X at the curve c:

$$\ell_s^c(X) = P - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\int_0^s 1_{\{c(u)-\varepsilon < X_u < c(u)+\varepsilon\}} \sigma^2 X_u^2\,du. \tag{6.21}$$

The symbol $d\ell_s^c(X)$ refers to Lebesgue-Stieltjes integration with respect to the continuous increasing function $s \mapsto \ell_s^c(X)$.

It has recently been noted by Peskir [37] that "even if F_t is to diverge when the boundary c is approached within C_1, this deficiency is counterbalanced by a similar behaviour of F_{xx} through (6.15), and consequantly the first integral in (6.20) is still well-defined and finite. [When we say in (6.15) that $F_t + \mathbb{L}_X F$ is locally bounded, we mean that $F_t + \mathbb{L}_X F$ is bounded on $K \cap (C_1 \cup C_2)$ for each compact set in $\mathbb{R}_+ \times \mathbb{R}$]. The condition (6.16) can further be relaxed to the form where $F_{xx} = F_1 + F_2$ on $C_1 \cup C_2$ where F_1 is non-negative and F_2 is continuous on $\mathbb{R}_+ \times \mathbb{R}$. This will be referred to below as the relaxed form of (6.14)–(6.17). For more details on this and other extensions see [36]."

6.3 The Result and Proof

Remember that the distribution function of a standard normal random variable is

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx. \tag{6.22}$$

The main result may now be stated as follows:

Theorem 6.1. *The optimal stopping boundary in the American put problem (6.1) can be characterized as the unique continuous increasing solution $b : [0,T] \to \mathbb{R}$ to the nonlinear integral equation*

$$\begin{aligned} K - b(t) &= e^{-r(T-t)} \mathbb{E}_{t,b(t)}\big((K - X_T)^+\big) + rK \int_0^{T-t} e^{-ru} P_{t,b(t)}(X_{t+u} \le b(t+u)) du \\ &= e^{-r(T-t)} \int_0^K \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{K-z}{b(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)\right) dz \\ &\quad + rK \int_0^{T-t} e^{-ru} \Phi\left(\frac{1}{\sigma\sqrt{u}}\left(\log\left(\frac{b(t+u)}{b(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)u\right)\right) du \end{aligned} \tag{6.23}$$

for all $t \in [0,T]$.

The arbitrage-free price of the American put (6.1) admits the following 'early exercise premium' representation:

$$\begin{aligned} V(t,x) &= e^{-r(T-t)} \mathbb{E}_{t,x}\big((K - X_T)^+\big) + rK \int_0^{T-t} e^{-ru} P_{t,x}(X_{t+u} \le b(t+u)) du \\ &= e^{-r(T-t)} \int_0^K \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{K-z}{x}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)\right) dz \\ &\quad + rK \int_0^{T-t} e^{-ru} \Phi\left(\frac{1}{\sigma\sqrt{u}}\left(\log\left(\frac{b(t+u)}{x}\right) - \left(r - \frac{\sigma^2}{2}\right)u\right)\right) du \end{aligned} \tag{6.24}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$.

Proof. 1. Since the statements of Chapter 2 (Theorem 2.8) were proved assuming that an optimal stopping time exists, it is necessary to ensure the existence of an optimal stopping time for (6.1), before these results can be used. The gain function

$$\tilde{G}(t,x) = e^{-rt}G(x) = e^{-rt}(K-x)^+ \tag{6.25}$$

is bounded and continuous on $[0,T] \times \mathbb{R}_+$, and hence it is possible to apply a version of [47, Theorem 3, page 127] for a finite time horizon. By statement (2) of that theorem, an optimal stopping time exists for (6.1). The results of Chapter 2 may therefore be applied to (6.1). By statements (i), (ii) and (iii) of Theorem 2.8, the state-space of the two-dimensional Markov process (t, X_t) is divided into two regions: the continuation region

$$C = \{(t,x) \in [0,T) \times (0,\infty) \mid V(t,x) > G(x)\} \tag{6.26}$$

and the stopping region

$$D = \{(t,x) \in [0,T) \times (0,\infty) \mid V(t,x) = G(x)\} \tag{6.27}$$

so that the optimal stopping time of (6.1) is

$$\tau_*(t,x) = \inf\{0 \le s \le T-t \mid X_{t+s} \in D\}. \tag{6.28}$$

The strong Markov property of X ensures that the value function V is $C^{1,2}$ in C and the convexity of the gain function $x \mapsto G(x)$, given by (6.25), implies that $x \mapsto V(t,x)$ is convex (see Section 2.4), so that the following properties of the value function V can be used in the proof:

(6.29) V is $C^{1,2}$ on C (and $C^{1,2}$ on D)

(6.30) $x \mapsto V(t,x)$ is decreasing and convex with $V_x(t,x) \in [-1,0]$

(6.31) $t \mapsto V(t,x)$ is decreasing with $V(T,x) = (K-x)^+$.

The following properties of V and b will be proved:

(6.32) V is continuous on $[0,T] \times \mathbb{R}_+$

(6.33) $t \mapsto b(t)$ is increasing and continuous

(6.34) $0 < b(0+) < K$ and $b(T-) = K$

(6.35) $x \mapsto V(t,x)$ is C^1 at $b(t)$.

Property (6.35) is the smooth fit condition.

2. It is now shown that the continuation region takes the form

$$C = \{(t,x) \in [0,T) \times (0,\infty) \mid x > b(t)\}. \tag{6.36}$$

(i) If the option was exercised at any point (t,x), with $x \geq K$ and $0 \leq t < T$, then the payoff would be zero. It is therefore intuitive that all points (t,x) with $x \geq K$ and $0 \leq t < T$ belong to the continuation region C. This is verified by considering the stopping times $\tau_\varepsilon = \inf\{0 \leq s \leq T-t \mid X_{t+s} \leq K - \varepsilon\}$, for $0 < \varepsilon < K$, and noting that $P_{t,x}(0 < \tau_\varepsilon < T-t) > 0$, if $x \geq K$ and $0 \leq t < T$. The strict inequality implies that $\mathbb{E}_{t,x}\left(e^{-r\tau_\varepsilon}(K - X_{t+\tau_\varepsilon})^+\right) > 0$, and thus $V(t,x) > 0 = G(x)$, so that (t,x) must belong to the continuation region C.

(ii) It is shown in [18] that the solution to (6.1) in the case of an infinite horizon gives the constant optimal stopping point $\hat{b} = ((\beta-1)/\beta)K$ where $\beta = 2r/\sigma^2 + 1$. We see that $0 < \hat{b} < K$ and, for all $x \leq \hat{b}$, we have

$$\begin{aligned} \sup_{0\leq\tau\leq T-t} \mathbb{E}_{t,x}\left(e^{-r(t+\tau)}(K - X_{t+\tau})^+\right) &\leq \sup_{0\leq\tau<\infty} \mathbb{E}_{t,x}\left(e^{-r(t+\tau)}(K - X_{t+\tau})^+\right) \\ &= (K-x)^+ = K - x. \end{aligned} \tag{6.37}$$

Thus, all points (t,x), with $0 < x \leq \hat{b}$ and $0 \leq t < T$, belong to the stopping region D.

(iii) Let $0 \leq t < T$ be given and fixed and set

$$b(t) = \sup\{y \geq \hat{b} \mid V(t,y) = G(y)\}. \tag{6.38}$$

Since $x \mapsto V(t,x)$ is convex on $(0,\infty)$, for each $0 \leq t \leq T$ given and fixed, by (6.30), we have $V(t,y) \leq G(y)$, for all $0 < y < b(t)$. The supremum in (6.38) is attained, since the continuity of $x \mapsto V(t,x)$ follows from its convexity, and so $V(t,b(t)) = G(b(t))$. Thus $(t,x) \in [0,T) \times (0,b(t)]$ implies that $V(t,x) \leq G(x)$, and it follows that $(t,x) \in D$. Hence the stopping region D equals

$$D = \{(t,x) \in [0,T) \times (0,\infty) \mid x \leq b(t)\} \tag{6.39}$$

and similarly the continuation region C is

$$C = \{(t,x) \in [0,T) \times (0,\infty) \mid x > b(t)\}\ . \tag{6.40}$$

3. To show that the boundary $t \mapsto b(t)$ in (6.39) and (6.40) is increasing on $[0,T]$, recall that $t \mapsto V(t,x)$ is decreasing on $[0,T]$, for each $x \in (0,\infty)$. Hence, if (t,x) belongs to C, for some $x \in (0,\infty)$, and we take any other $0 \leq t' < t \leq T$, then

$$V(t',x) - G(x) \geq V(t,x) - G(x) > 0 \tag{6.41}$$

showing that (t', x) belongs to C as well; and it follows that $t \mapsto b(t)$ is increasing on $[0, T]$.

4. Let us now show that the value function $(t, x) \mapsto V(t, x)$ is continuous on $[0, T) \times (0, \infty)$. It is enough to prove that

(6.42) $x \mapsto V(t, x)$ is continuous at x_0 and

(6.43) $t \mapsto V(t, x)$ is continuous at t_0 uniformly over $x \in [x_0 - \delta, x_0 + \delta]$

for each $(t_0, x_0) \in [0, T) \times (0, \infty)$ and some $\delta > 0$ small enough (δ may depend on x_0). See Lemma B.1 for a justification of this statement.

It has been noted that the continuity of $x \mapsto V(t, x)$ follows from the fact that the map is convex. Therefore it remains to establish (6.43).

Fix arbitrary $0 \le t_1 < t_2 \le T$ and $x \in (0, \infty)$. Let $\tau_1 = \tau_*(t_1, x) \in (0, T - t_1)$ denote the optimal stopping time for $V(t_1, x)$, and set $\tau_2 = \tau_1 \wedge (T - t_2)$. Now write

$$S_t = \exp(\sigma B_t + \gamma t) \tag{6.44}$$

with $\gamma = r - \sigma^2/2$, and note that, since $t \mapsto V(t, x)$ is decreasing on $[0, T]$, it follows that

$$\begin{aligned} 0 \le V(t_1, x) - V(t_2, x) &\le \mathbb{E}\left(e^{-r\tau_1}(K - xS_{\tau_1})^+\right) - \mathbb{E}\left(e^{-r\tau_2}(K - xS_{\tau_2})^+\right) \\ &\le \mathbb{E}\left(e^{-r\tau_2}\left((K - xS_{\tau_1})^+ - (K - xS_{\tau_2})^+\right)\right) \le x\mathbb{E}\left((S_{\tau_2} - S_{\tau_1})^+\right) \end{aligned} \tag{6.45}$$

where we use $\tau_2 \le \tau_1$ and the fact that $(K-y)^+ - (K-z)^+ \le (z-y)^+$, for $y, z \in \mathbb{R}$. The symbol $\mathbb{E}$ denotes expectation with respect to the measure P under which B is a standard Brownian motion.

Now set $Z_t = \sigma B_t + \gamma t$, and recall that the stationarity and independence of the increments of $Z = (Z_t)_{t \ge 0}$ (which together imply the strong Markov property) imply that $(Z_{\tau_2 + t} - Z_{\tau_2})_{t \ge 0}$ is a version of Z; i.e. the two processes have the same finite dimensional distributions. We can see that $\tau_1 - \tau_2 \le t_2 - t_1$ by looking at the various cases. If $t_1 + \tau_1 \le t_2$, then $\tau_1 - \tau_2 \le t_2 - t_1 - \tau_2 \le t_2 - t_1$. If $t_1 + \tau_1 \ge t_2$ then either $\tau_1 \le T - t_2$ and $\tau_2 = \tau_1$, or $\tau_1 \ge T - t_2$ and $\tau_2 = T - t_2$. In the first case, $\tau_1 - \tau_2 = \tau_1 - \tau_1 = 0 < t_2 - t_1$. In the second case, we have $T - \tau_1 \ge t_1$ which implies that $T - \tau_1 - t_2 \ge t_1 - t_2$ and so $\tau_1 - \tau_2 \le t_2 - t_1$.

Now using that $\tau_1 - \tau_2 \le t_2 - t_1$, the fact that $(Z_{\tau_2 + t} - Z_{\tau_2})_{t \ge 0}$ is a version of Z

implies that

$$
\begin{aligned}
(6.46)\qquad \mathbb{E}\left((S_{\tau_2}-S_{\tau_1})^+\right) &= \mathbb{E}\left(\mathbb{E}\left((S_{\tau_2}-S_{\tau_1})^+\big|\,\mathcal{F}_{\tau_2}\right)\right)\\
&= \mathbb{E}\left(S_{\tau_2}\mathbb{E}\left(\left(1-\frac{S_{\tau_1}}{S_{\tau_2}}\right)^+\Bigg|\,\mathcal{F}_{\tau_2}\right)\right)\\
&= \mathbb{E}\left(S_{\tau_2}\mathbb{E}\left((1-\exp(Z_{\tau_1}-Z_{\tau_2}))^+\big|\,\mathcal{F}_{\tau_2}\right)\right)\\
&= \mathbb{E}(S_{\tau_2})\mathbb{E}\left((1-\exp(Z_{\tau_1}-Z_{\tau_2}))^+\right)\\
&\le \mathbb{E}(S_{\tau_2})\mathbb{E}\left(1-\inf_{0\le t\le t_2-t_1}\exp(Z_{\tau_2+t}-Z_{\tau_2})\right)\\
&= \mathbb{E}(S_{\tau_2})\mathbb{E}\left(1-\inf_{0\le t\le t_2-t_1}\exp(Z_t)\right)\\
&=: \mathbb{E}(S_{\tau_2})L(t_2-t_1)
\end{aligned}
$$

where we also use the fact that $Z_{\tau_1}-Z_{\tau_2}$ is independent of $\mathcal{F}_{\tau_2}$. Since $B_t\to 0$ as $t\to 0$, it is easily seen that $L(t_2-t_1)\to 0$ as $t_2-t_1\to 0$. Combining (6.45) and (6.46), we get, by the martingale property of $(\exp(\sigma B_t-(\sigma^2/2)t)_{t\ge 0}$, that

$$
(6.47)\qquad 0\le V(t_1,x)-V(t_2,x)\le x\mathbb{E}(S_{\tau_2})L(t_2-t_1)\le xe^{rT}L(t_2-t_1)
$$

and (6.43) becomes evident.

5. In order to prove that the smooth fit condition (6.35) holds, fix a point $(t,x)\in(0,T)\times(0,\infty)$ lying on the boundary b, so that $x=b(t)$. Then $x<K$ and, for all $\varepsilon>0$ such that $x+\varepsilon<K$, we have

$$
(6.48)\qquad \frac{V(t,x+\varepsilon)-V(t,x)}{\varepsilon}\ge\frac{G(x+\varepsilon)-G(x)}{\varepsilon}=-1.
$$

Hence, taking the limit in (6.48) as $\varepsilon\downarrow 0$, we get

$$
(6.49)\qquad \frac{\partial^+V}{\partial x}(t,x)\ge G'(x)=-1
$$

where the right-hand derivative in (6.49) exists (and is finite), by virtue of the convexity of the mapping $x\mapsto V(t,x)$ on $(0,\infty)$.

To prove the reverse inequality, fix $\varepsilon>0$ such that $x+\varepsilon<K$, and consider the stopping time $\tau_\varepsilon=\tau_*(t,x+\varepsilon)$, being optimal for $V(t,x+\varepsilon)$. Then we have

$$
\begin{aligned}
(6.50)\qquad & V(t,x+\varepsilon)-V(t,x)\\
&\le \mathbb{E}\left(e^{-r\tau_\varepsilon}(K-(x+\varepsilon)S_{\tau_\varepsilon})^+\right)-\mathbb{E}\left(e^{-r\tau_\varepsilon}(K-xS_{\tau_\varepsilon})^+\right)\\
&\le \mathbb{E}\left(e^{-r\tau_\varepsilon}\left((K-(x+\varepsilon)S_{\tau_\varepsilon})-(K-xS_{\tau_\varepsilon})\right)1_{\{(x+\varepsilon)S_{\tau_\varepsilon}<K\}}\right)\\
&= -\varepsilon\mathbb{E}\left(e^{-r\tau_\varepsilon}S_{\tau_\varepsilon}1_{\{(x+\varepsilon)S_{\tau_\varepsilon}<K\}}\right)
\end{aligned}
$$

where S_t is defined as in (6.44).

To verify that $\tau_\varepsilon \to 0$ P-a.s. as $\varepsilon \downarrow 0$, consider the stopping time

$$\sigma_\varepsilon = \inf\{0 < s \le T - t \mid (x+\varepsilon)S_s = x\}. \tag{6.51}$$

Then $\sigma_\varepsilon \ge \tau_\varepsilon$ P-a.s., since $s \mapsto b(s)$ is increasing on $[t,T]$. Now $S_s = x/(x+\varepsilon)$ means that

$$\mathbb{E}\left(e^{-r\tau_\varepsilon} S_{\tau_\varepsilon} 1_{\{(x+\varepsilon)S_{\tau_\varepsilon} < K\}}\right) \to 1 \tag{6.52}$$

as $\varepsilon \downarrow 0$, by the regularity and continuity of paths of S. Combining (6.50) and (6.52), we see that

$$\frac{\partial^+ V}{\partial x}(t,x) \le G'(x) = -1 \tag{6.53}$$

which, together with (6.48), gives

$$\frac{\partial^+ V}{\partial x}(t,x) = G'(x) = -1. \tag{6.54}$$

This establishes (6.35).

6. We now show that the boundary b is continuous on $[0,T]$ and that $b(T) = K$.

(i) First we show that b is right-continuous on $[0,T]$. Fix $t \in (0,T]$, and consider a sequence $t_n \downarrow t$ as $n \to \infty$. Since b is increasing, the right-hand limit $b(t+)$ exists. Since $(t_n, b(t_n)) \in \bar{D}$, for all $n \ge 1$, and $\bar{D}$ is closed, it follows that $(t, b(t+)) \in \bar{D}$. Hence, by (6.39), we see that $b(t+) \le b(t)$. The reverse inequality follows from the fact that b is increasing on $[0,T]$, thus proving the claim.

(ii) Next we show that b is left-continuous as well. Suppose that at some point $t_* \in (0,T)$ the function b makes a jump. Without loss of generality, let us assume that $b(t_*-) < b(t_*)$. Fix a point $t' < t_*$ close to t_*, and consider the half-open region $R \subset C$, being a curved trapezoid formed by the vertices $(t', b(t'))$, $(t_*, b(t_*-))$, (t_*, x') and (t', x'), with any x' fixed arbitrarily in the interval $(b(t_*-), b(t_*))$. Clearly $x' < K$, since b is increasing and $b(t) < K$, for all $0 < t < T$. Note that (t_*, x') belongs to the stopping region $\bar{D}$.

From (6.29) the value function V is $C^{1,2}$ in C. Note also that the gain function G is C^2 in $R \subset C$, so that by the Leibnitz-Newton formula, using $V(t,b(t)) = G(t,b(t))$, and by (6.35), it follows that

$$V(t,x) - G(x) = \int_{b(t)}^{x} \int_{b(t)}^{u} \left(V_{xx}(t,v) - G_{xx}(v)\right) dv\, du \tag{6.55}$$

for all $(t,x) \in R$. Moreover, the strong Markov property implies that the value function V solves the equation

$$V_t + \mathbb{L}_X V = rV \tag{6.56}$$

in C. Using (6.56) and also using the fact that $t \mapsto V(t,x)$ and $x \mapsto V(t,x)$ are decreasing, by (6.30) and (6.31), so that $V_t < 0$ and $V_x < 0$ in C, we obtain

$$V_{xx}(t,x) = \frac{2}{\sigma^2 x^2}\left(rV(t,x) - V_t(t,x) - rxV_x(t,x)\right) \geq \frac{2}{\sigma^2 x^2} r(K-x)^+ \geq c > 0 \tag{6.57}$$

for each $(t,x) \in R$, where $c > 0$ is small enough. Hence by (6.55), and using the fact that $G_{xx} = 0$ in R, we get

$$V(t',x') - G(x') \geq \frac{c(x' - b(t'))^2}{2} \to \frac{c(x' - b(t_*-))^2}{2} > 0 \tag{6.58}$$

as $t' \uparrow t_*$. This implies that $V(t_*, x') > G(x')$, which contradicts (t_*, x') belonging to the stopping region D. Thus $b(t_*-) = b(t_*)$, which shows that b is continuous at t_*, and thus also on $[0,T]$.

(iii) We finally note that the method of proof from part (ii) also implies that $b(T) = K$. To see this, let $t_* = T$, and suppose that $b(T) < K$. Then, repeating the argument presented above, we arrive at a contradiction with the fact that $V(T,x) = G(x)$, for all $x \in [b(T), K]$.

7. Summarizing the facts proved in subsections 1–6 above, we may conclude that the following hitting time is optimal in problem (6.1):

$$\tau_*(t,x) = \inf\{0 \leq u \leq T-t \mid X_{t+u} \leq b(t+u)\} \tag{6.59}$$

(the infimum of an empty set being equal to $T-t$), where the boundary b has the following properties:

$$b : [0,T] \to (0,K] \quad \text{is continuous and increasing} \tag{6.60}$$

$$b(T) = K. \tag{6.61}$$

Standard arguments based on the strong Markov property lead to the following free boundary problem for the value function V and the boundary b:

$$V_t + \mathbb{L}_X V = rV \quad \text{in} \quad C \tag{6.62}$$

$$V(t,x) = (K-x)^+ \quad \text{for} \quad x = b(t) \tag{6.63}$$

$$V_x(t,x) = -1 \quad \text{for} \quad x = b(t) \quad \text{(smooth fit)} \tag{6.64}$$

$$V(t,x) > (K-x)^+ \quad \text{in} \quad C \tag{6.65}$$

$$V(t,x) = K - x \quad \text{in} \quad D \tag{6.66}$$

where the continuation region C is defined in (6.40), and the stopping region D is defined in (6.39).

8. We will now derive (6.24) and then show that b is a unique solution to (6.23), so that the free boundary is uniquely characterized by (6.23). From the results proved above, we see that, for any $(t,x) \in (0,T) \times (0,\infty)$ given and fixed, the function $F : (0, T-t) \times (0,\infty) \to \mathbb{R}$, defined by

$$F(s,y) = e^{-rs} V(t+s, xy) \tag{6.67}$$

satisfies (6.14)–(6.17) (in the relaxed form), where $C_1 = C$, and $\bar{C}_2 = D$ and $c = b/x$. Applying (6.20) to $F(s, S_s)$, where S_s is defined as in (6.44), we get

$$\begin{aligned} F(s,S_s) =& F(0,1) + \int_0^s (F_t + \mathbb{L}_S F)(u, S_u) 1_{\{xS_u \neq b(t+u)\}}\, du \\ &+ \int_0^s F_x(u, S_u) \sigma S_u 1_{\{xS_u \neq b(t+u)\}}\, dB_u \\ &+ \frac{1}{2} \int_0^s \left(F_x(u, S_u+) - F_x(u, S_u-)\right) 1_{\{xS_u = b(t+u)\}}\, d\ell_u^{b/x}(S). \end{aligned} \tag{6.68}$$

Noting that $\mathbb{L}_S = \mathbb{L}_X$, and recalling (6.67), we see that this becomes

$$\begin{aligned} e^{-rs} V(t+s, X_{t+s}) =& V(t,x) + M_s^b \\ &+ \int_0^s e^{-ru} (-rV + V_t + \mathbb{L}_X V)(t+u, X_{t+u}) 1_{\{X_{t+u} \neq b(t+u)\}}\, du \\ &+ \frac{1}{2} \int_0^s e^{-ru} \left(V_x(t+u, X_{t+u}+) - V_x(t+u, X_{t+u}-)\right) 1_{\{X_{t+u} = b(t+u)\}}\, d\ell_u^b(X) \end{aligned} \tag{6.69}$$

where

$$M_s^b = \int_0^s e^{-ru} V_x(t+u, X_{t+u}) \sigma X_{t+u} 1_{\{X_{t+u} \neq b(t+u)\}} dB_u \tag{6.70}$$

and $(M_s^b)_{0 \le s \le T-t}$ is a martingale under $P_{t,x}$, so that $\mathbb{E}_{t,x}(M_s^b) = 0$, for each $0 \le s \le T-t$.

The last term of (6.69) is zero, as a result of the smooth fit condition (6.35). By (6.62) and (6.40), we see that $V_t + \mathbb{L}_X V - rV = 0$, for $x > b(t)$. Since $b(t) < K$, for all $0 \le t < T$, using (6.66) and (6.39) gives $-rV + V_t + \mathbb{L}_X V = -rK$, for $x \le b(t)$. Therefore, letting $s = T - t$, we get

$$e^{-r(T-t)} V(T, X_T) = V(t,x) - rK \int_0^{T-t} e^{-ru} 1_{\{X_{t+u} < b(t+u)\}}\, du + M_{T-t}^b. \tag{6.71}$$

Rearranging (6.71), and taking the $P_{t,x}$-expectation, gives

$$V(t,x) = e^{-r(T-t)} \mathbb{E}_{t,x} \left(V(T, X_T)\right) + rK \int_0^{T-t} e^{-ru} P\left(X_{t+u} \le b(t+u)\right)\, du. \tag{6.72}$$

We have now derived (6.24), since $V(T, X_T) = (K - X_T)^+$, by (6.31). Substituting $x = b(t)$ into (6.72) gives (6.23).

9. It is now shown that the boundary b is the unique increasing continuous solution to (6.23). The following proof is due to G. Peskir and appears in [37]. It is included here as an introduction to the techniques used in Chapter 7, and for the complete proof of the integral representation of the American put option to appear in one place. Let us assume that a continuous increasing function $c : [0,T] \to (0,K)$ solving (6.23) is given. We will show that c must then coincide with the optimal stopping boundary b.

(i) In view of (6.24), let us introduce the function

$$U^c(t,x) = e^{-r(T-t)}\mathbb{E}_{t,x}\left(G(X_T)\right) + rK\int_0^{T-t} e^{-ru}P_{t,x}(X_{t+u} \le c(t+u))\,du. \tag{6.73}$$

Using (6.3), this can be written more explicitly as

$$U^c(t,x) = e^{-r(T-t)}U_1^c(t,x) + rKU_2^c(t,x) \tag{6.74}$$

where U_1^c and U_2^c are defined as follows:

$$U_1^c(t,x) = \int_0^K \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{K-z}{x}\right) - \gamma(T-t)\right)\right) dz \tag{6.75}$$

$$U_2^c(t,x) = \int_t^T e^{-r(v-t)}\Phi\left(\frac{1}{\sigma\sqrt{v-t}}\left(\log\left(\frac{c(v)}{x}\right) - \gamma(v-t)\right)\right) dv \tag{6.76}$$

for all $(t,x) \in [0,T) \times (0,\infty)$, upon setting $\gamma = r - \sigma^2/2$ and $v = t+u$.

Writing $\varphi = \Phi'$, we then have

$$\frac{\partial U_1^c}{\partial x}(t,x) = -\frac{1}{\sigma x\sqrt{T-t}}\int_0^K \varphi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{K-z}{x}\right) - \gamma(T-t)\right)\right) dz \tag{6.77}$$

$$\frac{\partial U_2^c}{\partial x}(t,x) = -\frac{1}{\sigma x}\int_t^T \frac{e^{-r(v-t)}}{\sqrt{v-t}}\varphi\left(\frac{1}{\sigma\sqrt{v-t}}\left(\log\left(\frac{c(v)}{x}\right) - \gamma(v-t)\right)\right) dv \tag{6.78}$$

for all $(t,x) \in (0,T)\times(0,\infty)$, where the interchange of differentiation and integration is justified by standard means. From (6.77) and (6.78) we see that $\partial U_1^c/\partial x$ and $\partial U_2^c/\partial x$ are continuous on $[0,T) \times (0,\infty)$, which, in view of (6.74), implies that U_x^c is continuous on $[0,T) \times (0,\infty)$.

(ii) In accordance with (6.24), define a function $V^c : [0,T) \times (0,\infty) \to \mathbb{R}$, by setting

$$V^c(t,x) = \begin{cases} G(x) & \text{if } x \le c(t) \\ U^c(t,x) & \text{if } x > c(t) \end{cases} \tag{6.79}$$

for $0 \le t < T$. Note that, since c solves (6.23) with $x = c(t)$, we have that V^c is continuous on $[0,T) \times (0,\infty)$; i.e. $V^c(t,x) = U^c(t,x) = G(x)$, for $x = c(t)$ and

$0 < t < T$. Let C_1 and C_2 be defined in terms of c in the following way:

$$C_1 = \{(t,x) \in [0,T) \times \mathbb{R} \mid x > c(t)\} \tag{6.80}$$

$$C_2 = \{(t,x) \in [0,T) \times \mathbb{R} \mid x < c(t)\}. \tag{6.81}$$

Standard arguments based on the Markov property (see Section 2.4 and apply [11, Theorem 5.9, page 158]) show that V^c (i.e. U^c) is $C^{1,2}$ on C_1 , and that

$$V_t^c + \mathbb{L}_X V^c = rV^c \text{ in } C_1. \tag{6.82}$$

Moreover, since U_x^c is continuous on $[0,T) \times (0,\infty)$, we see that V_x^c is continuous on $\bar{C}_1$. Finally, since $0 < c(t) < K$, for $0 < t < T$, we see that V^c (i.e. G) is $C^{1,2}$ on $\bar{C}_2$.

(iii) Summarizing the preceding conclusions, we see that, for any $(t,x) \in [0,T) \times (0,\infty)$ given and fixed, the function $F : [0, T-t) \times (0,\infty) \to \mathbb{R}$, defined by

$$F(s,y) = e^{-rs} V^c(t+s, xy) \tag{6.83}$$

satisfies (6.14)–(6.17) (in the relaxed form), so that (6.20) can be applied. In this way we get

$$\begin{aligned} e^{-rs} V^c(t+s, X_{t+s}) &= V^c(t,x) \\ &+ \int_0^s e^{-ru} \left(V_t^c + \mathbb{L}_X V^c - rV^c\right)(t+u, X_{t+u}) 1_{\{X_{t+u} \neq c(t+u)\}}\, du \\ &+ M_s^c + \frac{1}{2} \int_0^s e^{-ru} \Delta_x V_x^c(t+u, c(t+u))\, d\ell_u^c(X) \end{aligned} \tag{6.84}$$

where we define

$$M_s^c = \int_0^s e^{-ru} V_x^c(t+u, X_{t+u}) \sigma X_{t+u}\, dB_u \tag{6.85}$$

and we set

$$\Delta_x V_x^c(v, c(v)) = V_x^c(v, c(v)+) - V_x^c(v, c(v)-) \tag{6.86}$$

for $t \leq v \leq T$. Moreover, it is easily seen from (6.77) and (6.78) that $(M_s^c)_{0 \leq s \leq T-t}$ is a martingale under $P_{t,x}$, so that $\mathbb{E}_{t,x}(M_s^c) = 0$, for each $0 \leq s \leq T-t$.

(iv) Setting $s = T-t$ in (6.84), and then taking the $P_{t,x}$-expectation, and using that $V^c(T,x) = G(x)$, for all $x > 0$, together with the fact that V^c satisfies (6.82) in C_1, we get

$$\begin{aligned} e^{-r(T-t)} \mathbb{E}_{t,x}(G(X_T)) &= V^c(t,x) \\ &+ \int_0^{T-t} e^{-ru} \mathbb{E}_{t,x}(H(t+u, X_{t+u}) 1_{\{X_{t+u} \leq c(t+u)\}})\, du \\ &+ \frac{1}{2} \mathbb{E}_{t,x} \left(\int_0^{T-t} e^{-ru} \Delta_x V_x^c(t+u, c(t+u))\, d_u \ell_u^c(X) \right) \end{aligned} \tag{6.87}$$

for all $(t,x) \in [0,T) \times (0,\infty)$, where $H = G_t + \mathbb{L}_X G - rG = -rK$, for $x \le c(t)$. From (6.87) we thus see that

$$V^c(t,x) = e^{-r(T-t)}\mathbb{E}_{t,x}(G(X_T)) + rK\int_0^{T-t} e^{-ru}P_{t,x}(X_{t+u} \le c(t+u))\,du - \mathbb{E}_{t,x}\left(\frac{1}{2}\int_0^{T-t} e^{-ru}\Delta_x V_x^c(t+u,c(t+u))\,d_u\ell_u^c(X)\right) \tag{6.88}$$

for all $(t,x) \in [0,T) \times (0,\infty)$. Comparing (6.88) with (6.73), and recalling the definition of V^c in terms of U^c and G, we find by (6.23) that

$$\mathbb{E}_{t,x}\left(\int_0^{T-t} e^{-ru}\Delta_x V_x^c(t+u,c(t+u))d_u\ell_u^c(X)\right) = 2\,(U^c(t,x) - G(x))\,1_{\{x\le c(t)\}} \tag{6.89}$$

for all $0 \le t < T$.

(v) From (6.89) we see that if we prove that

$$x \to V^c(t,x) \text{ is } C^1 \text{ at } c(t) \tag{6.90}$$

for each $0 \le t < T$ given and fixed, then it will follow that

$$U^c(t,x) = G(x) \tag{6.91}$$

for all $0 < x \le c(t)$. On the other hand, if we know that (6.91) holds, then using the general fact

$$\frac{\partial}{\partial x}\Big(U^c(t,x) - G(x)\Big)\Big|_{x=c(t)} = V_x^c(t,c(t)+) - V_x^c(t,c(t)-) = \Delta_x V_x^c(t,c(t)) \tag{6.92}$$

for all $0 \le t < T$, we see that (6.90) holds too (since U_x^c is continuous). The equivalence of (6.90) and (6.91) just explained then suggests that, instead of dealing with the equation (6.89), in order to derive (6.90) above (which was the content of an earlier version of this proof), we could rather concentrate on establishing (6.91) directly. [To appreciate the simplicity and power of the probabilistic argument to be given shortly below, one may differentiate (6.89) with respect to x, compute the left-hand side explicitly (taking care of a jump relation), and then try to prove the uniqueness of the zero solution to the resulting (weakly singular) Volterra integral equation, using any of the known analytic methods (see e.g. [50]).]

(vi) To derive (6.91), first note that standard arguments based on the Markov property (or a direct verification) show that U^c is $C^{1,2}$ on C_2 and that

$$U_t^c + \mathbb{L}_X U^c - rU^c = -rK \text{ in } C_2. \tag{6.93}$$

Since F in (6.83), with U^c instead of V^c, is continuous and satisfies (6.14)–(6.17) (in the relaxed form), we see that (6.20) can be applied, just as in (6.84), and this

yields

$$e^{-rs}U^c(t+s,X_{t+s}) = U^c(t,x) - rK\int_0^s e^{-ru}1_{\{X_{t+u}\le c(t+u)\}}\,du + \tilde{M}_s^c \tag{6.94}$$

using (6.82) and (6.94), and the fact that $\Delta_x U_x^c(t+u,c(t+u))=0$, for all $0\le u\le s$, since U_x^c is continuous. In (6.94) we have

$$\tilde{M}_s^c = \int_0^s e^{-ru}U_x^c(t+u,X_{t+u})\sigma X_{t+u}1_{\{X_{t+u}\ne c(t+u)\}}\,dB_u \tag{6.95}$$

and $(\tilde{M}_s^c)_{0\le s\le T-t}$ is a martingale under $P_{t,x}$.

Next note that (6.20) applied to F in (6.83), with G instead of V^c, yields

$$e^{-rs}G(X_{t+s}) = G(x) - rK\int_0^s e^{-ru}1_{\{X_{t+u}<K\}}\,du + M_s^K + \frac{1}{2}\int_0^s e^{-ru}d\ell_u^K(X) \tag{6.96}$$

using $G_t + \mathbb{L}_X G - rG = -rK$, on $(0,K)$, and $G_t + \mathbb{L}_X G - rG = 0$ on (K,∞), and $\Delta_x G_x(t+u,K)=1$ for $0\le u\le s$. In (6.96) we have

$$\begin{aligned} M_s^K &= \int_0^s e^{-ru}G'(X_{t+u})\sigma X_{t+u}1_{\{X_{t+u}\ne K\}}\,dB_u \\ &= -\int_0^s e^{-ru}\sigma X_{t+u}1_{\{X_{t+u}<K\}}\,dB_u \end{aligned} \tag{6.97}$$

and $(M_s^K)_{0\le s\le T-t}$ is a martingale under $P_{t,x}$.

For $0<x\le c(t)$ consider the stopping time

$$\sigma_c = \inf\{0\le s\le T-t \mid X_{t+s}\ge c(t+s)\}. \tag{6.98}$$

Then, using $U^c(t,c(t)) = G(c(t))$, for all $0\le t<T$, since c solves (6.24), and $U^c(T,x)=G(x)$, for all $x>0$, by (6.73), we see that $U^c(t+\sigma_c,X_{t+\sigma_c}) = G(X_{t+\sigma_c})$. Hence, from (6.94) and (6.96), using the optional sampling theorem, we find

$$\begin{aligned} &U^c(t,x) \\ &= \mathbb{E}_{t,x}\left(e^{-r\sigma_c}U^c(t+\sigma_c,X_{t+\sigma_c})\right) + rK\,\mathbb{E}_{t,x}\left(\int_0^{\sigma_c} e^{-ru}1_{\{X_{t+u}\le c(t+u)\}}\,du\right) \\ &= E_{t,x}\left(e^{-r\sigma_c}G(X_{t+\sigma_c})\right) + rK\,\mathbb{E}_{t,x}\left(\int_0^{\sigma_c} e^{-ru}1_{\{X_{t+u}\le c(t+u)\}}\,du\right) \\ &= G(x) - rK\,\mathbb{E}_{t,x}\left(\int_0^{\sigma_c} e^{-ru}1_{\{X_{t+u}<K\}}\,du\right) \\ &\quad + rK\,\mathbb{E}_{t,x}\left(\int_0^{\sigma_c} e^{-ru}1_{\{X_{t+u}\le c(t+u)\}}\,du\right) \\ &= G(x) \end{aligned} \tag{6.99}$$

since $X_{t+u}<K$ and $X_{t+u}\le c(t+u)$ for all $0\le u\le\sigma_c$. This establishes (6.91), and thus (6.90) as well, as explained above.

(vii) Consider the stopping time

$$\tau_c = \inf\{0 \le s \le T-t \mid X_{t+s} \le c(t+s)\}. \tag{6.100}$$

Note that, using (6.82) and (6.90), we can rewrite (6.84) as

$$\begin{aligned} &e^{-rs}V^c(t+s, X_{t+s}) \\ &\quad = V^c(t,x) + \int_0^s e^{-ru} H(t+u, X_{t+u}) 1_{\{X_{t+u} \le c(t+u)\}}\, du + M_s^c \end{aligned} \tag{6.101}$$

where $H = G_t + \mathbb{L}_X G - rG = -rK$ for $x \le c(t)$, and $(M_s^c)_{0 \le s \le T-t}$ is a martingale under $P_{t,x}$. Thus $E_{t,x}(M_{\tau_c}^c) = 0$, so that, after inserting τ_c in place of s in (6.101), it follows, upon taking the $P_{t,x}$-expectation, that

$$V^c(t,x) = \mathbb{E}_{t,x}\left(e^{-r\tau_c}(K - X_{t+\tau_c})^+\right) \tag{6.102}$$

for all $(t,x) \in [0,T) \times (0,\infty)$, where we use that $V^c(t,x) = G(x) = (K-x)^+$, for $x \le c(t)$, or $t = T$. Comparing (6.102) with (6.1), we see that

$$V^c(t,x) \le V(t,x) \tag{6.103}$$

for all $(t,x) \in [0,T) \times (0,\infty)$.

(viii) Let us now show that $c \ge b$, on $[0,T]$. For this recall that, by the same arguments as for V^c, we have

$$e^{-rs}V(t+s, X_{t+s}) = V(t,x) + \int_0^s e^{-ru} H(t+u, X_{t+u}) 1_{\{X_{t+u} \le b(t+u)\}}\, du + M_s^b \tag{6.104}$$

where $H = G_t + \mathbb{L}_X G - rG = -rK$, for $x \le b(t)$, and $(M_s^b)_{0 \le s \le T-t}$ is a martingale under $P_{t,x}$. Fix $(t,x) \in (0,T) \times (0,\infty)$ such that $x < b(t) \wedge c(t)$, and consider the stopping time

$$\sigma = \inf\{0 \le s \le T-t \mid X_{t+s} \ge b(t+s)\}. \tag{6.105}$$

Inserting σ in place of s in (6.101) and (6.104), and taking the $P_{t,x}$-expectation, we get

$$\mathbb{E}_{t,x}\left(e^{-r\sigma}V^c(t+\sigma, X_{t+\sigma})\right) = G(x) - rK\,\mathbb{E}_{t,x}\left(\int_0^\sigma e^{-ru} 1_{\{X_{t+u} \le c(t+u)\}}\, du\right) \tag{6.106}$$

and

$$\mathbb{E}_{t,x}\left(e^{-r\sigma}V(t+\sigma, X_{t+\sigma})\right) = G(x) - rK\,\mathbb{E}_{t,x}\left(\int_0^\sigma e^{-ru}\, du\right). \tag{6.107}$$

Hence, by (6.103), we see that

$$\mathbb{E}_{t,x}\left(\int_0^\sigma e^{-ru} 1_{\{X_{t+u} \le c(t+u)\}}\, du\right) \ge \mathbb{E}_{t,x}\left(\int_0^\sigma e^{-ru}\, du\right) \tag{6.108}$$

and it follows, by the continuity of c and b, that $c(t) \geq b(t)$, for all $0 \leq t \leq T$.

(ix) Finally, let us show that c must be equal to b. For this, assume that there exists $t \in (0,T)$, such that $c(t) > b(t)$, and pick $x \in (b(t), c(t))$. Under $P_{t,x}$, consider the stopping time τ_b, from (6.12). Inserting τ_b in place of s in (6.101) and (6.104), and taking the $P_{t,x}$-expectation, we get

$$\mathbb{E}_{t,x}\left(e^{-r\tau_b}G(X_{t+\tau_b})\right) = V^c(t,x) - rK\,\mathbb{E}_{t,x}\left(\int_0^{\tau_b} e^{-ru}1_{\{X_{t+u}\leq c(t+u)\}}\,du\right) \tag{6.109}$$

and

$$\mathbb{E}_{t,x}\left(e^{-r\tau_b}G(X_{t+\tau_b})\right) = V(t,x). \tag{6.110}$$

Hence, by (6.103), we see that:

$$\mathbb{E}_{t,x}\left(\int_0^{\tau_b} e^{-ru}1_{\{X_{t+u}\leq c(t+u)\}}\,du\right) \leq 0 \tag{6.111}$$

and it follows, by the continuity of c and b, that such a point x cannot exist. Thus c must be equal to b, and the proof is complete. □

The above result of Peskir [37] is stronger than that of Jacka [18]. Jacka [18] characterizes the optimal stopping boundary b as the unique continuous increasing function that solves

$$\begin{aligned} &K - x \\ &\quad = e^{-r(T-t)}\mathbb{E}_{t,x}\left((K - X_T)^+\right) + rK\int_0^{T-t} e^{-ru}P_{t,x}\left(X_{t+u} \leq b(t+u)\right)\,du \end{aligned} \tag{6.112}$$

for each $x \leq b(t)$. Jacka's result allows a candidate $\tilde{b}$ for the optimal stopping boundary to be found by substituting $\tilde{b}(t)$ into (6.112), and solving the equation (numerically). In order to identify this candidate as the optimal stopping boundary, it must then be verified that $\tilde{b}$ solves (6.112), for every $x \leq \tilde{b}(t)$. The above result shows that the candidate $\tilde{b}$ is indeed the optimal stopping boundary b and no further verification is needed.

Chapter 7

The American Asian Option

We show that the optimal stopping boundary for the early exercise Asian call option with floating strike can be characterized as the unique solution of a nonlinear integral equation arising from the early exercise premium representation (an explicit formula for the arbitrage-free price in terms of the optimal stopping boundary). The key argument in the proof relies upon a local time-space formula. This paper will be published as [39].

7.1 Introduction

According to financial theory (see e.g. [21] or Chapter 3 of this dissertation) the arbitrage-free price of the *early* exercise *Asian call option* with *floating strike* is given as V in (7.1) below where I_τ/τ denotes the *arithmetic* average of the stock price S up to time τ. The problem was first studied in [15] where approximations to the value function V and the optimal boundary b were derived. The main aim of the present paper is to derive exact expressions for V and b.

The optimal stopping problem (7.1) is three-dimensional. When a change-of-measure theorem is applied (as in [45] and [25]) the problem reduces to (7.9) and becomes two-dimensional. The problem (7.9) is more complicated than the well-known problems [37] and [38] since the gain function depends on time in a nonlinear way. From the result of Theorem 3.1 below it follows that the free-boundary problem (7.10)-(7.14) characterizes the value function V and the optimal stopping boundary b in a unique manner. Our main aim, however, is to follow the train of thought initiated by Kolodner [24] where V is initially expressed in terms of b, and b itself is then shown to satisfy a nonlinear integral equation. A particularly simple approach for achieving this goal in the case of the American put option has been suggested in [23], [18], [6] and we will take it up in the present paper. We will moreover see (as in [37] and [38]) that the nonlinear equation derived for b cannot have other solutions. The key argument in the proof relies upon a local time-space formula (see [36]).

The latter fact of uniqueness may be seen as the principal result of the paper. The same method of proof can also be used to show the uniqueness of the optimal stopping boundary solving nonlinear integral equations derived in [15] and [55] where this question was not explicitly addressed. These equations arise from the early exercise Asian options (call or put) with floating strike based on *geometric* averaging. The early exercise Asian *put* option with floating strike can be dealt with analogously to the Asian call option treated here. For financial interpretations of the early exercise Asian options and other references on the topic see [15] and [55].

7.2 Formulation of the problem

The arbitrage-free price of the early exercise Asian call option with floating strike is given by the following expression:

$$V = \sup_{0<\tau\le T} \mathbb{E}\Big(e^{-r\tau} \Big(S_\tau - \frac{1}{\tau}\, I_\tau \Big)^+ \Big) \tag{7.1}$$

where τ is a stopping time of the geometric Brownian motion $S = (S_t)_{0\le t\le T}$ solving:

$$dS_t = rS_t\, dt + \sigma S_t\, dB_t \qquad (S_0 = s) \tag{7.2}$$

and $I = (I_t)_{0\le t\le T}$ is the integral process given by:

$$I_t = a + \int_0^t S_s\, ds \tag{7.3}$$

where $s > 0$ and $a \ge 0$ are given and fixed. (Throughout $B = (B_t)_{t\ge 0}$ denotes a standard Brownian motion started at zero.) We recall that $T > 0$ is the expiration date (maturity), $r > 0$ is the interest rate, and $\sigma > 0$ is the volatility coefficient.

By the change-of-measure theorem it follows that:

$$V = \sup_{0<\tau\le T} \mathbb{E}\Big(e^{-r\tau} S_\tau \Big(1 - \frac{1}{\tau}\, X_\tau \Big)^+ \Big) = s \sup_{0<\tau\le T} \widetilde{\mathbb{E}}\Big(\Big(1 - \frac{1}{\tau}\, X_\tau \Big)^+ \Big) \tag{7.4}$$

where following [45] and [25] we set:

$$X_t = \frac{I_t}{S_t} \tag{7.5}$$

and $\widetilde{P}$ is defined by $d\widetilde{P} = \exp(\sigma B_T - (\sigma^2/2)T)\, dP$ so that $\widetilde{B}_t = B_t - \sigma t$ is a standard Brownian motion under $\widetilde{P}$ for $0 \le t \le T$. By Itô's formula one finds that:

$$dX_t = (1 - rX_t)\, dt + \sigma X_t\, d\widehat{B}_t \qquad (X_0 = x) \tag{7.6}$$

under $\widetilde{P}$ where $\widehat{B} = -\widetilde{B}$ is a standard Brownian motion and $x = a/s$. The infinitesimal generator of X is therefore given by:

$$\mathbb{L}_X = (1 - rx)\,\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\,x^2\frac{\partial^2}{\partial x^2}\,. \tag{7.7}$$

For further reference recall that the strong solution of (7.2) is given by:

$$S_t = s\exp\left(\sigma B_t + \left(r - \frac{\sigma^2}{2}\right)t\right) = s\exp\left(\sigma\widetilde{B}_t + \left(r + \frac{\sigma^2}{2}\right)t\right) \tag{7.8}$$

for $0 \le t \le T$ under P and $\widetilde{P}$ respectively. When dealing with the process X on its own, however, note that there is no restriction to assume that $s = 1$ and $a = x$ with $x \ge 0$.

Summarizing the preceding facts we see that the early exercise Asian call problem reduces to solving the following optimal stopping problem:

$$V(t,x) = \sup_{0<\tau\le T-t} \widetilde{\mathbb{E}}_{t,x}\left(\left(1 - \frac{1}{t+\tau}\,X_{t+\tau}\right)^+\right) \tag{7.9}$$

where τ is a stopping time of the diffusion process X solving (7.6) above and $X_t = x$ under $\widetilde{P}_{t,x}$ with $(t,x) \in [0,T]\times[0,\infty\rangle$ given and fixed.

Standard Markovian arguments indicate that V from (7.9) solves the following free-boundary problem of parabolic type:

$$V_t + \mathbb{L}_X V = 0 \quad \text{in } C \tag{7.10}$$

$$V(t,x) = \left(1 - \frac{x}{t}\right)^+ \quad \text{for } x = b(t) \text{ or } t = T \tag{7.11}$$

$$V_x(t,x) = -\frac{1}{t} \quad \text{for } x = b(t) \text{ (smooth fit)} \tag{7.12}$$

$$V(t,x) > \left(1 - \frac{x}{t}\right)^+ \quad \text{in } C \tag{7.13}$$

$$V(t,x) = \left(1 - \frac{x}{t}\right)^+ \quad \text{in } D \tag{7.14}$$

where the continuation set C and the stopping set $S = \bar{D}$ are defined by:

$$C = \{\,(t,x) \in [0,T\rangle\times[0,\infty\rangle \mid x > b(t)\,\} \tag{7.15}$$

$$D = \{\,(t,x) \in [0,T\rangle\times[0,\infty\rangle \mid x < b(t)\,\} \tag{7.16}$$

and $b : [0,T] \to I\!R$ is the (unknown) optimal stopping boundary, i.e. the stopping time:

$$\tau_b = \inf\{\,0 \le s \le T - t \mid X_{t+s} \le b(t+s)\,\} \tag{7.17}$$

is optimal in (7.9) (i.e. the supremum is attained at this stopping time). It follows from the result of Theorem 3.1 below that the free-boundary problem (7.10)-(7.14) characterizes the value function V and the optimal stopping boundary b in a unique manner (proving also the existence of the latter).

7.3 The result and proof

In this section we adopt the setting and notation of the early exercise Asian call problem from the previous section. Below we will make use of the following functions:

(7.18)
$$F(t,x) = \widetilde{\mathbb{E}}_{0,x}\Big(\Big(1-\frac{X_t}{T}\Big)^+\Big) = \int_0^\infty \int_0^\infty \Big(1-\frac{x+a}{Ts}\Big)^+ f(t,s,a)\,ds\,da$$

(7.19)
$$G(t,x,y) = \widetilde{\mathbb{E}}_{0,x}\Big(X_t\, I\big(X_t \le y\big)\Big) = \int_0^\infty \int_0^\infty \Big(\frac{x+a}{s}\Big) I\Big(\frac{x+a}{s} \le y\Big) f(t,s,a)\,ds\,da$$

(7.20)
$$H(t,x,y) = \widetilde{P}_{0,x}\,(X_t \le y) = \int_0^\infty \int_0^\infty I\Big(\frac{x+a}{s} \le y\Big) f(t,s,a)\,ds\,da$$

for $t > 0$ and $x, y \ge 0$, where $(s,a) \mapsto f(t,s,a)$ is the probability density function of (S_t, I_t) under $\widetilde{P}$ with $S_0 = 1$ and $I_0 = 0$ given by:

$$f(t,s,a) = \frac{2\sqrt{2}}{Pi^{3/2}\sigma^3}\,\frac{s^{r/\sigma^2}}{a^2\sqrt{t}}\,\exp\Big(\frac{2Pi^2}{\sigma^2 t} - \frac{(r+\sigma^2/2)^2}{2\sigma^2}\,t - \frac{2}{\sigma^2 a}\,(1+s)\Big) \times \int_0^\infty \exp\Big(-\frac{2z^2}{\sigma^2 t} - \frac{4\sqrt{s}}{\sigma^2 a}\cosh(z)\Big)\sinh(z)\,\sin\Big(\frac{4Piz}{\sigma^2 t}\Big)\,dz \tag{7.21}$$

for $s > 0$ and $a > 0$. For a derivation of the right-hand side in (7.21) see the Appendix below.

The main result of the paper may be stated as follows.

Theorem 7.1. *The optimal stopping boundary in the Asian call problem (7.9) can be characterized as the unique continuous increasing solution $b : [0,T] \to \mathbb{R}$ of the nonlinear integral equation:*

(7.22)
$$1 - \frac{b(t)}{t} = F(T-t, b(t)) - \int_0^{T-t} \frac{1}{t+u}\Big(\Big(\frac{1}{t+u} + r\Big)G(u, b(t), b(t+u)) - H(u, b(t), b(t+u))\Big)\,du$$

satisfying $0 < b(t) < t/(1+rt)$ *for all* $0 < t < T$. *[The solution* b *satisfies* $b(0+) = 0$ *and* $b(T-) = T/(1+rT)$, *and the stopping time* τ_b *from (7.17) is optimal in (7.9).]*

The arbitrage-free price of the Asian call option (7.9) admits the following 'early exercise premium' representation:

(7.23)

$$V(t,x) = F(T-t,x) - \int_0^{T-t} \frac{1}{t+u}\left(\left(\frac{1}{t+u}+r\right)G\left(u,x,b(t+u)\right) - H(u,x,b(t+u))\right)du$$

for all $(t,x) \in [0,T]\times[0,\infty\rangle$. *[Further properties of* V *and* b *are exhibited in the proof below.]*

Proof. The proof will be carried out in several steps. We begin by stating some general remarks which will be freely used below without further mentioning.

1. The reason that we take the supremum in (7.1) and (7.9) over $\tau > 0$ is that the ratio $1/(t+\tau)$ is not well defined for $\tau = 0$ when $t = 0$. Note however in (7.1) that $I_\tau/\tau \to \infty$ as $\tau \downarrow 0$ when $I_0 = a > 0$ and that $I_\tau/\tau \to s$ as $\tau \downarrow 0$ when $I_0 = a = 0$. Similarly, note in (7.9) that $X_\tau/\tau \to \infty$ as $\tau \downarrow 0$ when $X_0 = x > 0$ and $X_\tau/\tau \to 1$ as $\tau \downarrow 0$ when $X_0 = x = 0$. Thus in both cases the gain process (the integrand in (7.1) and (7.9)) tends to 0 as $\tau \downarrow 0$. This shows that in either (7.1) or (7.9) it is never optimal to stop at $t = 0$. To avoid similar (purely technical) complications in the proof to follow we will equivalently consider $V(t,x)$ only for $t > 0$ with the supremum taken over $\tau \geq 0$. The case of $t = 0$ will become evident (by continuity) at the end of the proof.

2. Recall that it is no restriction to assume that $s = 1$ and $a = x$ so that $X_t = (x + I_t)/S_t$ with $I_0 = 0$ and $S_0 = 1$. We will write X_t^x instead of X_t to indicate the dependence on x when needed. It follows that V admits the following representation:

$$V(t,x) = \sup_{0\leq\tau\leq T-t} \widetilde{\mathbb{E}}\left(\left(1 - \frac{x+I_\tau}{(t+\tau)\,S_\tau}\right)^+\right) \tag{7.24}$$

for $(t,x) \in \langle 0,T]\times[0,\infty\rangle$. From (7.24) we immediately see that:

$$x \mapsto V(t,x) \text{ is decreasing and convex on } [0,\infty\rangle \tag{7.25}$$

for each $t > 0$ fixed.

3. We show that $V : \langle 0,T]\times[0,\infty\rangle \to I\!R$ is continuous. For this, using $\sup(f) -$

$\sup(g) \le \sup(f-g)$ and $(z-x)^+ - (z-y)^+ \le (y-x)^+$ for $x, y, z \in \mathbb{R}$, we get:

(7.26)
$$\begin{aligned} V(t,x) - V(t,y) &\le \sup_{0\le\tau\le T-t} \left(\widetilde{\mathbb{E}}\left(\left(1 - \frac{x+I_\tau}{(t+\tau)\,S_\tau} \right)^+ \right) - \widetilde{\mathbb{E}}\left(\left(1 - \frac{y+I_\tau}{(t+\tau)\,S_\tau} \right)^+ \right) \right. \\ &\le (y-x) \sup_{0\le\tau\le T-t} \widetilde{\mathbb{E}}\left(\frac{1}{(t+\tau)\,S_\tau} \right) \le \frac{1}{t}\,(y-x) \end{aligned}$$

for $0 \le x \le y$ and $t > 0$, where in the last inequality we used (7.8) to deduce that $1/S_t = \exp(\sigma \widehat{B}_t - (r+\sigma^2/2)t) \le \exp(\sigma \widehat{B}_t - (\sigma^2/2)t)$ and the latter is a martingale under $\widetilde{P}$. From (7.26) with (7.25) we see that $x \mapsto V(t,x)$ is continuous at x_0 uniformly over $t \in [t_0 - \delta, t_0 + \delta]$ for some $\delta > 0$ (small enough) whenever $(t_0, x_0) \in \langle 0, T] \times [0, \infty\rangle$ is given and fixed. Thus to prove that V is continuous on $\langle 0, T] \times [0, \infty\rangle$ it is enough to show that $t \mapsto V(t,x)$ is continuous on $\langle 0, T]$ for each $x \ge 0$ given and fixed. For this, take any $t_1 < t_2$ in $\langle 0, T]$ and $\varepsilon > 0$, and let τ_1^ε be a stopping time such that $\widetilde{\mathbb{E}}((1 - (X^x_{t_1+\tau_1^\varepsilon})/(t_1+\tau_1^\varepsilon))^+) \ge V(t_1,x) - \varepsilon$. Setting $\tau_2^\varepsilon = \tau_1^\varepsilon \wedge (T - t_2)$ we see that $V(t_2,x) \ge \widetilde{\mathbb{E}}((1 - (X_{t_2+\tau_2^\varepsilon})/(t_2+\tau_2^\varepsilon))^+)$. Hence we get:

$$\begin{aligned} V(t_1,x) - V(t_2,x) &\le \widetilde{\mathbb{E}}\left(\left(1 - \frac{X^x_{t_1+\tau_1^\varepsilon}}{t_1+\tau_1^\varepsilon} \right)^+ \right) - \widetilde{\mathbb{E}}\left(\left(1 - \frac{X^x_{t_2+\tau_2^\varepsilon}}{t_2+\tau_2^\varepsilon} \right)^+ \right) + \varepsilon \\ &\le \widetilde{\mathbb{E}}\left(\left(\frac{X^x_{t_2+\tau_2^\varepsilon}}{t_2+\tau_2^\varepsilon} - \frac{X^x_{t_1+\tau_1^\varepsilon}}{t_1+\tau_1^\varepsilon} \right)^+ \right) + \varepsilon. \end{aligned} \tag{7.27}$$

Letting first $t_2 - t_1 \to 0$ using $\tau_1^\varepsilon - \tau_2^\varepsilon \to 0$ and then $\varepsilon \downarrow 0$ we see that $\limsup_{t_2-t_1\to 0}(V(t_1,x) - V(t_2,x)) \le 0$ by dominated convergence. On the other hand, let τ_2^ε be a stopping time such that $\widetilde{\mathbb{E}}((1 - (X^x_{t_2+\tau_2^\varepsilon})/(t_2+\tau_2^\varepsilon))^+) \ge V(t_2,x) - \varepsilon$. Then we have:

$$V(t_1,x) - V(t_2,x) \ge \widetilde{\mathbb{E}}\left(\left(1 - \frac{X^x_{t_1+\tau_2^\varepsilon}}{t_1+\tau_2^\varepsilon} \right)^+ \right) - \widetilde{\mathbb{E}}\left(\left(1 - \frac{X^x_{t_2+\tau_2^\varepsilon}}{t_2+\tau_2^\varepsilon} \right)^+ \right) - \varepsilon. \tag{7.28}$$

Letting first $t_2 - t_1 \to 0$ and then $\varepsilon \downarrow 0$ we see that $\liminf_{t_2-t_1\to 0}(V(t_1,x) - V(t_2,x)) \ge 0$. Combining the two inequalities we find that $t \mapsto V(t,x)$ is continuous on $\langle 0, T]$. This completes the proof of the initial claim.

4. Denote the gain function by $G(t,x) = (1 - x/t)^+$ for $(t,x) \in \langle 0, T] \times [0, \infty\rangle$ and introduce the continuation set $C = \{\, (t,x) \in \langle 0, T\rangle \times [0, \infty\rangle \mid V(t,x) > G(t,x) \,\}$ and the stopping set $S = \{\, (t,x) \in \langle 0, T\rangle \times [0, \infty\rangle \mid V(t,x) = G(t,x) \,\}$. Since V and G are continuous, we see that C is open and S is closed in $\langle 0, T\rangle \times [0, \infty\rangle$. Standard arguments based on the strong Markov property (cf. [47]) show that the first hitting time $\tau_S = \inf\{\, 0 \le s \le T - t \mid (t+s, X_{t+s}) \in S \,\}$ is optimal in (7.9) as well as that V is $C^{1,2}$ on C and satisfies (7.10). In order to determine the structure of the optimal stopping time τ_S (i.e. the shape of the sets C and S) we will first examine basic properties of the diffusion process X solving (7.6) under $\widetilde{P}$.

5. The state space of X equals $[0,\infty\rangle$ and it is clear from the representation (7.5) with (7.8) that 0 is an entrance boundary point. The drift of X is given by $\mu(x) = 1 - rx$ and the diffusion coefficient of X is given by $\sigma(x) = \sigma x$ for $x \geq 0$. Hence we see that $\mu(x)$ is greater/less than 0 if and only if x is less/greater than $1/r$. This shows that there is a permanent push (drift) of X towards the constant level $1/r$ (when X is above $1/r$ the push of X is downwards and when X is below $1/r$ the push of X is upwards). The scale function of X is given by $s(x) = \int_1^x y^{2r/\sigma^2} e^{2/\sigma^2 y}\, dy$ for $x > 0$, and the speed measure of X is given by $m(dx) = (2/\sigma^2)\, x^{-2(1+r/\sigma^2)}\, e^{-2/\sigma^2 x}\, dx$ on the Borel σ-algebra of $\langle 0,\infty\rangle$. Since $s(0) = -\infty$ and $s(\infty) = +\infty$ we see that X is recurrent. Moreover, since $\int_0^\infty m(dx) = (2/\sigma^2)^{-2r/\sigma^2}\, \Gamma(1+2r/\sigma^2)$ is finite we find that X has an invariant probability density function given by:

$$f(x) = \frac{(2/\sigma^2)^{1+2r/\sigma^2}}{\Gamma(1+2r/\sigma^2)}\, \frac{1}{x^{2(1+r/\sigma^2)}}\, e^{-2/\sigma^2 x} \tag{7.29}$$

for $x > 0$. In particular, it follows that $X_t/t \to 0$ $\widetilde{P}$-a.s. as $t \to \infty$. This fact has an important consequence for the optimal stopping problem (7.9): If the horizon T is infinite, then it is never optimal to stop. Indeed, in this case letting $\tau \equiv t$ and passing to the limit for $t \to \infty$ we see that $V \equiv 1$ on $\langle 0,\infty\rangle \times [0,\infty\rangle$. This shows that the infinite horizon formulation of the problem (7.9) provides no useful information to the finite horizon formulation (such as in [37] and [38] for example). To examine the latter beyond the trivial fact that all points (t,x) with $x \geq t$ belong to C (which is easily seen by considering the hitting times $\tau_\varepsilon = \inf\{\, 0 \leq s \leq T-t \mid X_{t+s} \leq (t+s) - \varepsilon \,\}$ and noting that $\widetilde{P}_{t,x}(0 < \tau_\varepsilon < T-t) > 0$ if $x \geq t$ with $0 < t < T$) we will examine the gain process in the problem (7.9) using stochastic calculus as follows.

6. Setting $\alpha(t) = t$ for $0 \leq t \leq T$ to denote the diagonal in the state space and applying the local time-space formula (cf. [36]) under $\widetilde{P}_{t,x}$ when $(t,x) \in \langle 0,T\rangle \times [0,\infty\rangle$ is given and fixed, we get:

(7.30)

$$\begin{aligned}
G(t+s, X_{t+s}) &= G(t,x) + \int_0^s G_t(t+u, X_{t+u})\, du \\
&\quad + \int_0^s G_x(t+u, X_{t+u})\, dX_{t+u} + \frac{1}{2}\int_0^s G_{xx}(t+u, X_{t+u})\, d\langle X, X\rangle_{t+u} \\
&\quad + \frac{1}{2}\int_0^s \Big(G_x(t+u, \alpha(t+u)+) - G_x(t+u, \alpha(t+u)-) \Big)\, d\ell^{\alpha}_{t+u}(X) \\
&= G(t,x) + \int_0^s \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)} \right) I\big(X_{t+u} < \alpha(t+u)\big)\, du \\
&\quad - \sigma \int_0^s \frac{X_{t+u}}{t+u}\, I\big(X_{t+u} < \alpha(t+u)\big)\, d\widehat{B}_u + \frac{1}{2}\int_0^s \frac{d\ell^{\alpha}_{t+u}(X)}{t+u}
\end{aligned}$$

where $\ell^{\alpha}_{t+u}(X)$ is the local time of X on the curve α given by:

$$\ell^{\alpha}_{t+u}(X) = \widetilde{P}\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^u I\big(\alpha(t+v)-\varepsilon < X_{t+v} < \alpha(t+v)+\varepsilon\big)\, d\langle X,X\rangle_{t+v} \tag{7.31}$$
$$= \widetilde{P}\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^u I\big(\alpha(t+v)-\varepsilon < X_{t+v} < \alpha(t+v)+\varepsilon\big)\, \frac{\sigma^2}{2}\, X^2_{t+v}\, dv$$

and $d\ell^{\alpha}_{t+u}(X)$ refers to the integration with respect to the continuous increasing function $u \mapsto \ell^{\alpha}_{t+u}(X)$. From (7.30) we respectively read:

$$G(t+s, X_{t+s}) = G(t,x) + A_s + M_s + L_s \tag{7.32}$$

where A and L are processes of bounded variation (L is increasing) and M is a continuous (local) martingale. We note moreover that $s \mapsto L_s$ is strictly increasing only when $X_s = \alpha(s)$ for $0 \le s \le T-t$ i.e. when X visits α. On the other hand, when X is below α then the integrand $a(t+u, X_{t+u})$ of A_s may be either positive or negative. To determine both regions exactly we need to examine the sign of the expression $a(t,x) = x/t^2 - (1-rx)/t$. It follows that $a(t,x)$ is larger/less than 0 if and only if x is larger/less than $\gamma(t)$ where $\gamma(t) = t/(1+rt)$ for $0 \le t \le T$. By considering the exit times from small balls in $\langle 0,T\rangle \times [0,\infty\rangle$ with centre at (t,x) and making use of (7.30) with the optional sampling theorem (to get rid of the martingale part), upon observing that $\alpha(t) < \gamma(t)$ for all $0 < t \le T$ so that the local time part is zero, we see that all points (t,x) lying above the curve γ (i.e. $x > \gamma(t)$ for $0 < t < T$) belong to the continuation set C. Exactly the same arguments (based on the fact that the favourable regions above γ and on α are far away from X) show that for each $x < \gamma(T) = T/(1+rT)$ given and fixed, all points (t,x) belong to the stopping set S when t is close to T. Moreover, recalling (7.25) and the fact that $V(t,x) \ge G(t,x)$ for all $x \ge 0$ with $t \in \langle 0,T\rangle$ fixed, we see that for each $t \in \langle 0,T\rangle$ there is a point $b(t) \in [0,\gamma(t)]$ such that $V(t,x) > G(t,x)$ for $x > b(t)$ and $V(t,x) = G(t,x)$ for $x \in [0,b(t)]$. Combining it with the previous conclusion on S we find that $b(T-) = \gamma(T) = T/(1+rT)$. (Yet another argument for this identity will be given below. Note that this identity is different from the identity $b(T-) = T$ used in [15, page 1126].) This establishes the existence of the non-trivial (non-zero) optimal stopping boundary b on a left-neighborhood of T. We will now show that b extends (continuously and decreasingly) from the initial neighborhood of T backward in time as long as it visits 0 at some time $t_0 \in [0,T\rangle$, and later in the second part of the proof below we will deduce that this t_0 is equal to 0. The key argument in the proof is provided by the following inequality. Notice that this inequality is not obvious a priori (unlike in [37] and [38]) since $t \mapsto G(t,x)$ is increasing and the supremum in (7.9) is taken over a smaller class of stopping times $\tau \in [0,T-t]$ when t is larger.

7. We show that the inequality is satisfied:

$$V_t(t,x) \le G_t(t,x) \tag{7.33}$$

for all $(t,x) \in C$. (It may be noted from (7.10) that $V_t = -(1-rx)V_x - (\sigma^2/2)x^2 V_{xx} \le (1-rx)/t$ since $V_x \ge -1/t$ and $V_{xx} \ge 0$ by (7.25), so that $V_t \le G_t$ holds above γ because $(1-rx)/t \le x/t^2$ if and only if $x \ge t/(1+rt)$. Hence the main issue is to show that (7.33) holds below γ and above b. Any analytic proof of this fact seems difficult and we resort to probabilistic arguments.)

To prove (7.33) fix $0 < t < t+h < T$ and $x \ge 0$ so that $x \le \gamma(t)$. Let $\tau = \tau_S(t+h,x)$ be the optimal stopping time for $V(t+h,x)$. Since $\tau \in [0, T-t-h] \subseteq [0, T-t]$ we see that $V(t,x) \ge \widetilde{\mathbb{E}}_{t,x}((1 - X_{t+\tau}/(t+\tau))^+)$ so that using the inequality stated prior to (7.26) above (and the convenient refinement by an indicator function), we get:

(7.34)

$$\begin{aligned}
&V(t+h,x) - V(t,x) - \Big(G(t+h,x) - G(t,x)\Big) \\
&\le \widetilde{\mathbb{E}}\bigg(\bigg(1 - \frac{x+I_\tau}{(t+h+\tau)\,S_\tau}\bigg)^+\bigg) - \widetilde{\mathbb{E}}\bigg(\bigg(1 - \frac{x+I_\tau}{(t+\tau)\,S_\tau}\bigg)^+\bigg) - \bigg(\frac{x}{t} - \frac{x}{t+h}\bigg) \\
&\le \widetilde{\mathbb{E}}\bigg(\bigg(\frac{x+I_\tau}{(t+\tau)\,S_\tau} - \frac{x+I_\tau}{(t+h+\tau)\,S_\tau}\bigg) I\bigg(\frac{x+I_\tau}{(t+h+\tau)\,S_\tau} \le 1\bigg)\bigg) - \frac{xh}{t\,(t+h)} \\
&= \widetilde{\mathbb{E}}\bigg(\frac{x+I_\tau}{S_\tau}\bigg(\frac{1}{t+\tau} - \frac{1}{t+h+\tau}\bigg) I\bigg(\frac{x+I_\tau}{(t+h+\tau)\,S_\tau} \le 1\bigg)\bigg) - \frac{xh}{t\,(t+h)} \\
&= \widetilde{\mathbb{E}}\bigg(\frac{x+I_\tau}{(t+h+\tau)\,S_\tau}\,\frac{h}{t+\tau}\,I\bigg(\frac{x+I_\tau}{(t+h+\tau)\,S_\tau} \le 1\bigg)\bigg) - \frac{xh}{t\,(t+h)} \\
&\le \frac{h}{t}\,\widetilde{\mathbb{E}}\bigg(\frac{x+I_\tau}{(t+h+\tau)\,S_\tau}\,I\bigg(\frac{x+I_\tau}{(t+h+\tau)\,S_\tau} \le 1\bigg)\bigg) - \frac{xh}{t\,(t+h)} \le 0
\end{aligned}$$

where the final inequality follows from the fact that with $Z := (x+I_\tau)/((t+h+\tau)S_\tau)$ we have $V(t+h,x) = \widetilde{\mathbb{E}}((1-Z)^+) = \widetilde{\mathbb{E}}((1-Z)\,I(Z \le 1)) = \widetilde{P}(Z \le 1) - \widetilde{\mathbb{E}}(Z\,I(Z \le 1)) \ge G(t+h,x) = 1 - x/(t+h)$ so that $\widetilde{\mathbb{E}}(Z\,I(Z \le 1)) \le \widetilde{P}(Z \le 1) - 1 + x/(t+h) \le x/(t+h)$ as claimed. Dividing the initial expression in (7.34) by h and letting $h \downarrow 0$ we obtain (7.33) for all $(t,x) \in C$ such that $x \le \gamma(t)$. Since $V_t \le G_t$ above γ (as stated following (7.33) above) this completes the proof of (7.33).

8. We show that $t \mapsto b(t)$ is increasing on $\langle 0, T\rangle$. This is an immediate consequence of (7.34). Indeed, if (t,x) belongs to C and t_0 from $\langle 0, T\rangle$ satisfies $t_0 < t_1$, then by (7.34) we have that $V(t_0,x) - G(t_0,x) \ge V(t_1,x) - G(t_1,x) > 0$ so that (t_0,x) must belong to C. It follows that b cannot be strictly decreasing thus proving the claim.

9. We show that the smooth-fit condition (7.12) holds, i.e. that $x \mapsto V(t,x)$ is C^1 at $b(t)$. For this, fix a point $(t,x) \in \langle 0,T\rangle \times \langle 0,\infty\rangle$ lying at the boundary so that$x = b(t)$. Then$x \le \gamma(t) < \alpha(t)$ and for all $\varepsilon > 0$ such that $x+\varepsilon < \alpha(t)$ we have:

$$\frac{V(t,x+\varepsilon)-V(t,x)}{\varepsilon} \ge \frac{G(t,x+\varepsilon)-G(t,x)}{\varepsilon} = -\frac{1}{t}. \tag{7.35}$$

Letting $\varepsilon \downarrow 0$ and using that the limit on the left-hand side exists (since $x \mapsto V(t,x)$ is convex), we get the inequality:

$$\frac{\partial^+ V}{\partial x}(t,x) \ge \frac{\partial G}{\partial x}(t,x) = -\frac{1}{t}. \tag{7.36}$$

To prove the converse inequality, fix $\varepsilon > 0$ such that $x+\varepsilon < \alpha(t)$, and consider the stopping times $\tau_\varepsilon = \tau_S(t,x+\varepsilon)$ being optimal for $V(t,x+\varepsilon)$. Then we have:

(7.37)

$$\begin{aligned}\frac{V(t,x+\varepsilon)-V(t,x)}{\varepsilon} &\le \frac{1}{\varepsilon}\widetilde{\mathbb{E}}\bigg(\bigg(1-\frac{x+\varepsilon+I_{\tau_\varepsilon}}{(t+\tau_\varepsilon)\,S_{\tau_\varepsilon}}\bigg)^+ - \bigg(1-\frac{x+I_{\tau_\varepsilon}}{(t+\tau_\varepsilon)\,S_{\tau_\varepsilon}}\bigg)^+\bigg)\bigg) \\ &\le \frac{1}{\varepsilon}\,\widetilde{\mathbb{E}}\bigg(\frac{x+I_{\tau_\varepsilon}}{(t+\tau_\varepsilon)\,S_{\tau_\varepsilon}} - \frac{x+\varepsilon+I_{\tau_\varepsilon}}{(t+\tau_\varepsilon)\,S_{\tau_\varepsilon}}\bigg) = -\,\widetilde{\mathbb{E}}\bigg(\frac{1}{(t+\tau_\varepsilon)\,S_{\tau_\varepsilon}}\bigg).\end{aligned}$$

Since each point x in $\langle 0,\infty\rangle$ is regular for X, and the boundary b is increasing, it follows that $\tau_\varepsilon \downarrow 0$ $\widetilde{P}-a.s.$ as $\varepsilon \downarrow 0$. Letting $\varepsilon \downarrow 0$ in (7.37) we get:

$$\frac{\partial^+ V}{\partial x}(t,x) \le -\frac{1}{t} \tag{7.38}$$

by dominated convergence. It follows from (7.36) and (7.38) that $(\partial^+ V/\partial x)(t,x) = -1/t$ implying the claim.

10. We show that b is continuous. Note that the same proof also shows that $b(T-) = T/(1+rT)$ as already established above by a different method.

Let us first show that b is right-continuous. For this, fix $t \in \langle 0,T\rangle$ and consider a sequence $t_n \downarrow t$ as $n \to \infty$. Since b is increasing, the right-hand limit $b(t+)$ exists. Because $(t_n, b(t_n)) \in S$ for all $n \ge 1$, and S is closed, it follows that $(t, b(t+)) \in S$. Hence by (7.16) we see $b(t+) \le b(t)$. Since the reverse inequality follows obviously from the fact that b is increasing, this completes the proof of the first claim.

Let us next show that b is left-continuous. Suppose that there exists $t \in \langle 0,T\rangle$ such that $b(t-) < b(t)$. Fix a point x in $\langle b(t-), b(t)]$ and note by (7.12) that for $s < t$ we have:

$$V(s,x) - G(s,x) = \int_{b(s)}^{x}\int_{b(s)}^{y} \big(V_{xx}(s,z) - G_{xx}(s,z)\big)\,dz\,dy \tag{7.39}$$

upon recalling that V is $C^{1,2}$ on C. Note that $G_{xx} = 0$ below α so that if $V_{xx} \ge c$ on $R = \{\,(u,y) \in C \mid s \le u < t \text{ and } b(u) < y \le x\,\}$ for some $c > 0$ (for all $s < t$

close enough to t and some $x > b(t-)$ close enough to $b(t-)$) then by letting $s \uparrow t$ in (7.39) we get:

$$V(t,x) - G(t,x) \geq c\,\frac{\big(x - b(t)\big)^2}{2} > 0 \tag{7.40}$$

contradicting the fact that (t,x) belongs to $\bar{D}$ and thus is an optimal stopping point. Hence the proof reduces to showing that $V_{xx} \geq c$ on small enough R for some $c > 0$.

To derive the latter fact we may first note from (7.10) upon using (7.33) that $V_{xx} = (2/(\sigma^2 x^2))(-V_t - (1-rx)V_x) \geq (2/(\sigma^2 x^2))(-x/t^2 - (1-rx)V_x)$. Suppose now that for each $\delta > 0$ there is $s < t$ close enough to t and there is $x > b(t-)$ close enough to $b(t-)$ such that $V_x(u,y) \leq -1/u + \delta$ for all $(u,y) \in R$ (where we recall that $-1/u = G_x(u,y)$ for all $(u,y) \in R$). Then from the previous inequality we find that $V_{xx}(u,y) \geq (2/(\sigma^2 y^2))(-y/u^2 + (1-ry)(1/u - \delta)) = (2/(\sigma^2 y^2))((u - y(1+ru))/u^2 - \delta(1-ru)) \geq c > 0$ for $\delta > 0$ small enough since $y < u/(1+ru) = \gamma(u)$ and $y < 1/r$ for all $(u,y) \in R$. Hence the proof reduces to showing that $V_x(u,y) \leq -1/u + \delta$ for all $(u,y) \in R$ with R small enough when $\delta > 0$ is given and fixed.

To derive the latter inequality we can make use of the estimate (7.37) to conclude that

$$\frac{V(u,y+\varepsilon) - V(u,y)}{\varepsilon} \leq -\widetilde{\mathbb{E}}\bigg(\frac{1}{(u+\sigma_\varepsilon)\,M_{\sigma_\varepsilon}}\bigg) \tag{7.41}$$

where $\sigma_\varepsilon = \inf\{\, 0 \leq v \leq T-u \mid X^{y+\varepsilon}_{u+v} = b(u) \,\}$ and $M_t = \sup_{0 \leq s \leq t} S_s$. A simple comparison argument (based on the fact that b is increasing) shows that the supremum over all $(u,y) \in R$ on the right-hand side of (7.41) is attained at $(s, x+\varepsilon)$. Letting $\varepsilon \downarrow 0$ in (3.24) we thus get:

$$V_x(u,y) \leq -\widetilde{\mathbb{E}}\bigg(\frac{1}{(u+\sigma)\,M_\sigma}\bigg) \tag{7.42}$$

for all $(u,y) \in R$ where $\sigma = \inf\{\, 0 \leq v \leq T-s \mid X^x_{s+v} = b(s) \,\}$. Since by regularity of X we find that $\sigma \downarrow 0$ $\widetilde{P}$-a.s. as $s \uparrow t$ and $x \downarrow b(t-)$, it follows from (7.42) that:

$$V_x(u,y) \leq -\frac{1}{u} + \widetilde{\mathbb{E}}\bigg(\frac{(u+\sigma)\,M_\sigma - u}{u\,(u+\sigma)\,M_\sigma}\bigg) \leq -\frac{1}{u} + \delta \tag{7.43}$$

for all $s < t$ close enough to t and some $x > b(t-)$ close enough to $b(t-)$. This completes the proof of the second claim, and thus the initial claim is proved as well.

11. We show that V is given by the formula (7.23) and that b solves equation (7.22). For this, note that V satisfies the following conditions:

$$V \text{ is } C^{1,2} \text{ on } C \cup D \tag{7.44}$$

$$V_t + \mathbb{L}_X V \text{ is locally bounded} \tag{7.45}$$

$$x \mapsto V(t,x) \text{ is convex} \tag{7.46}$$

$$t \mapsto V_x(t, b(t)\pm) \text{ is continuous.} \tag{7.47}$$

Indeed, the conditions (7.44) and (7.45) follow from the facts that V is $C^{1,2}$ on C and $V = G$ on D upon recalling that D lies below γ so that $G(t,x) = 1 - x/t$ for all $(t,x) \in D$ and thus G is $C^{1,2}$ on D. [When we say in (7.45) that $V_t + \mathbb{L}_X V$ is locally bounded, we mean that $V_t + \mathbb{L}_X V$ is bounded on $K \cap (C \cup D)$ for each compact set K in $[0,T] \times \mathbb{R}_+$.] The condition (7.46) was established in (7.25) above. The condition (7.47) follows from (7.12) since according to the latter we have $V_x(t, b(t)\pm) = -1/t$ for $t > 0$.

Since (3.27)-(3.30) are satisfied we know that the local time-space formula (cf. Theorem 3.1 in [36]) can be applied. This gives:

(7.48)

$$\begin{aligned} V(t+s, X_{t+s}) &= V(t,x) + \int_0^s \big(V_t + \mathbb{L}_X V\big)(t+u, X_{t+u})\, I\big(X_{t+u} \neq b(t+u)\big)\, du \\ &+ \int_0^s \sigma\, X_{t+u}\, V_x(t+u, X_{t+u})\, I\big(X_{t+u} \neq b(t+u)\big)\, dB_u \\ &+ \frac{1}{2} \int_0^s \big(V_x(t+u, X_{t+u}+) - V_x(t+u, X_{t+u}-)\big)\, I\big(X_{t+u} = b(t+u)\big)\, d\ell^b_{t+u}(X) \\ &= \int_0^s \big(G_t + \mathbb{L}_X G\big)(t+u, X_{t+u})\, I\big(X_{t+u} < b(t+u)\big)\, du + M_s \end{aligned}$$

where the final equality follows by the smooth-fit condition (7.12) and $M_s = \int_0^s \sigma X_{t+u}$ $V_x(t+u, X_{t+u})\, I\big(X_{t+u} \neq b(t+u)\big)\, dB_u$ is a continuous martingale for $0 \le s \le T - t$ with $t > 0$. Noting that $(G_t + \mathbb{L}_X G)(t,x) = x/t^2 - (1 - rx)/t$ for $x < t$ we see that (7.48) yields:

(7.49)

$$V(t+s, X_{t+s}) = V(t,x) + \int_0^s \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)}\right) I\big(X_{t+u} < b(t+u)\big)\, du + M_s.$$

Setting $s = T - t$, using that $V(T,x) = G(T,x)$ for all $x \ge 0$, and taking the $\widetilde{P}_{t,x}$-expectation in (7.49), we find by the optional sampling theorem that:

(7.50)

$$\begin{aligned} &\widetilde{\mathbb{E}}_{t,x}\left(\left(1 - \frac{X_T}{T}\right)^+\right) \\ &= V(t,x) + \int_0^{T-t} \widetilde{\mathbb{E}}_{t,x}\left(\left(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)}\right) I\big(X_{t+u} < b(t+u)\big)\right) du. \end{aligned}$$

Making use of (7.18)-(7.20) we see that (7.50) is the formula (7.23). Moreover, inserting $x = b(t)$ in (7.50) and using that $V(t, b(t)) = G(t, b(t)) = 1 - b(t)/t$, we see that b satisfies the equation (7.22) as claimed.

12. We show that $b(t) > 0$ for all $0 < t \le T$ and that $b(0+) = 0$. For this, suppose that $b(t_0) = 0$ for some $t_0 \in \langle 0, T\rangle$ and fix $t \in \langle 0, t_0\rangle$. Then $(t,x) \in C$ for all

$x > 0$ as small as desired. Taking any such $(t,x) \in C$ and denoting by $\tau_S = \tau_S(t,x)$ the first hitting time to S under $\widetilde{P}_{t,x}$, we find by (7.49) that:

(7.51)
$$V(t+\tau_S, X_{t+\tau_S}) = G(t+\tau_S, X_{t+\tau_S}) = \left(1 - \frac{X_{t+\tau_S}}{t+\tau_S}\right)^+ = V(t,x) + M_{t+\tau_S}$$
$$= 1 - \frac{x}{t} + M_{t+\tau_S}.$$

Taking the $\widetilde{P}_{t,x}$-expectation and letting $x \downarrow 0$ we get:

(7.52)
$$\widetilde{\mathbb{E}}_{t,0}\left(1 - \frac{X_{t+\tau_S}}{t+\tau_S}\right)^+ = 1$$

where $\tau_S = \tau_S(t,0)$. As clearly $\widetilde{P}_{t,0}(X_{t+\tau_S} \geq T) > 0$ we see that the left-hand side of (7.52) is strictly smaller than 1 thus contradicting the identity. This shows that $b(t)$ must be strictly positive for all $0 < t \leq T$. Combining this conclusion with the known inequality $b(t) \leq \gamma(t)$ which is valid for all $0 < t \leq T$ we see that $b(0+) = 0$ as claimed.

13. We show that b is the unique solution of the nonlinear integral equation (7.22) in the class of continuous functions $c : \langle 0,T\rangle \to I\!R$ satisfying $0 < c(t) < t/(1+rt)$ for all $0 < t < T$. (Note that this class is larger than the class of functions having the established properties of b which is moreover known to be increasing.) The proof of the uniqueness will be presented in the final three steps of the main proof as follows.

14. Let $c : \langle 0,T] \to I\!R$ be a continuous solution of the equation (7.22) satisfying $0 < c(t) < t$ for all $0 < t < T$. We want to show that this c must then be equal to the optimal stopping boundary b.

Motivated by the derivation (7.48)-(7.50) which leads to the formula (7.53), let us consider the function $U^c : \langle 0,T] \times [0,\infty\rangle \to I\!R$ defined as follows:

(7.53)
$$U^c(t,x) = \widetilde{\mathbb{E}}_{t,x}\left(\left(1 - \frac{X_T}{T}\right)^+\right)$$
$$- \int_0^{T-t} \widetilde{\mathbb{E}}_{t,x}\left(\left(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)}\right) I\big(X_{t+u} < c(t+u)\big)\right) du$$

for $(t,x) \in \langle 0,T] \times [0,\infty\rangle$. In terms of (7.18)-(7.20) note that U^c is explicitly given by:

(7.54)
$$U^c(t,x) = F(T-t,x)$$
$$- \int_0^{T-t} \frac{1}{t+u}\left(\left(\frac{1}{t+u} + r\right) G\big(u,x,c(t+u)\big) - H\big(u,x,c(t+u)\big)\right) du$$

for $(t,x) \in \langle 0,T] \times [0,\infty\rangle$. Observe that the fact that c solves (7.22) on $\langle 0,T\rangle$ means exactly that $U^c(t,c(t)) = G(t,c(t))$ for all $0 < t < T$. We will now moreover show that $U^c(t,x) = G(t,x)$ for all $x \in [0,c(t)]$ with $t \in \langle 0,T\rangle$. This is the key point in the proof (cf. [37] and [38]) that can be derived using a martingale argument as follows.

If $X = (X_t)_{t\geq 0}$ is a Markov process (with values in a general state space) and we set $F(t,x) = \mathbb{E}_x(G(X_{T-t}))$ for a (bounded) measurable function G with $P_x(X_0 = x) = 1$, then the Markov property of X implies that $F(t,X_t)$ is a martingale under P_x for $0 \leq t \leq T$. Similarly, if we set $F(t,x) = \mathbb{E}_x(\int_0^{T-t} H(X_u)\,du)$ for a (bounded) measurable function H with $P_x(X_0 = x) = 1$, then the Markov property of X implies that $F(t,X_t) + \int_0^t H(X_u)\,du$ is a martingale under P_x for $0 \leq t \leq T$. Combining these two martingale facts applied to the time-space Markov process $(t+s, X_{t+s})$ instead of X_s, we find that:

$$U^c(t+s,X_{t+s}) - \int_0^s \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\right) I\big(X_{t+u} < c(t+u)\big)\,du \tag{7.55}$$

is a martingale under $\widetilde{P}_{t,x}$ for $0 \leq s \leq T-t$. We may thus write:

$$U^c(t+s,X_{t+s}) - \int_0^s \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\right) I\big(X_{t+u} < c(t+u)\big)\,du = U^c(t,x) + N_s \tag{7.56}$$

where $(N_s)_{0\leq s\leq T-t}$ is a martingale with $N_0 = 0$ under $\widetilde{P}_{t,x}$.

On the other hand, we know from (7.30) that:

$$\begin{aligned} G(t+s,X_{t+s}) = G(t,x) &+ \int_0^s \left(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\right) I\big(X_{t+u} < \alpha(t+u)\big)\,du \\ &+ M_s + L_s \end{aligned} \tag{7.57}$$

where $M_s = -\sigma \int_0^s (X_{t+u}/(t+u))\, I(X_{t+u} < \alpha(t+u))\, d\widehat{B}_u$ is a continuous martingale under $\widetilde{P}_{t,x}$ and $L_s = (1/2)\int_0^s d\ell^{\alpha}_{t+u}(X)/(t+u)$ is an increasing process for $0 \leq s \leq T-t$.

For $0 \leq x \leq c(t)$ with $t \in \langle 0,T\rangle$ given and fixed consider the stopping time:

$$\sigma_c = \inf\{\, 0 \leq s \leq T-t \mid X_{t+s} \geq c(t+s)\,\}. \tag{7.58}$$

Using that $U^c(t,c(t)) = G(t,c(t))$ for all $0 < t < T$ (since c solves (7.22) as pointed out above) and that $U^c(T,x) = G(T,x)$ for all $x \geq 0$, we see that $U^c(t+\sigma_c, X_{t+\sigma_c}) = G(t+\sigma_c, X_{t+\sigma_c})$. Hence from (7.56) and (7.57) using the optional sampling theorem

we find:

$$
\begin{aligned}
(7.59)\qquad U^c(t,x) &= \widetilde{\mathbb{E}}_{t,x}\Big(U^c(t+\sigma_c, X_{t+\sigma_c})\Big) \\
&\qquad - \widetilde{\mathbb{E}}_{t,x}\bigg(\int_0^{\sigma_c}\bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\bigg) I\big(X_{t+u} < c(t+u)\big)\,du\bigg) \\
&= \widetilde{\mathbb{E}}_{t,x}\Big(G(t+\sigma_c, X_{t+\sigma_c})\Big) \\
&\quad - \widetilde{\mathbb{E}}_{t,x}\bigg(\int_0^{\sigma_c}\bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\bigg) I\big(X_{t+u} < c(t+u)\big)\,du\bigg) \\
&= G(t,x) + \widetilde{\mathbb{E}}_{t,x}\bigg(\int_0^{\sigma_c}\bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\bigg) I\big(X_{t+u} < \alpha(t+u)\big)\,du\bigg) \\
&\quad - \widetilde{\mathbb{E}}_{t,x}\bigg(\int_0^{\sigma_c}\bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\bigg) I\big(X_{t+u} < c(t+u)\big)\,du\bigg) \\
&= G(t,x)
\end{aligned}
$$

since $X_{t+u} < \alpha(t+u)$ and $X_{t+u} < c(t+u)$ for all $0 \le u < \sigma_c$. This proves that $U^c(t,x) = G(t,x)$ for all $x \in [0, c(t)]$ with $t \in \langle 0, T\rangle$ as claimed.

15. We show that $U^c(t,x) \le V(t,x)$ for all $(t,x) \in \langle 0,T] \times [0,\infty\rangle$. For this, consider the stopping time:

$$
(7.60)\qquad \tau_c = \inf\{\, 0 \le s \le T-t \mid X_{t+s} \le c(t+s)\,\}
$$

under $\widetilde{P}_{t,x}$ with $(t,x) \in \langle 0,T]\times[0,\infty\rangle$ given and fixed. The same arguments as those given following (7.58) above show that $U^c(t+\tau_c, X_{t+\tau_c}) = G(t+\tau_c, X_{t+\tau_c})$. Inserting τ_c instead of s in (7.56) and using the optional sampling theorem we get:

$$
(7.61)\qquad U^c(t,x) = \widetilde{\mathbb{E}}_{t,x}\Big(U^c(t+\tau_c, X_{t+\tau_c})\Big) = \widetilde{\mathbb{E}}_{t,x}\Big(G(t+\tau_c, X_{t+\tau_c})\Big) \le V(t,x)
$$

where the final inequality follows from the definition of V proving the claim.

16. We show that $c \ge b$ on $[0,T]$. For this, consider the stopping time:

$$
(7.62)\qquad \sigma_b = \inf\{\, 0 \le s \le T-t \mid X_{t+s} \ge b(t+s)\,\}
$$

under $\widetilde{P}_{t,x}$ where $(t,x) \in \langle 0,T\rangle \times [0,\infty\rangle$ such that $x < b(t) \wedge c(t)$. Inserting σ_b in place of s in (7.49) and (7.56) and using the optional sampling theorem we get:

$$
(7.63)\qquad \widetilde{\mathbb{E}}_{t,x}\Big(V(t+\sigma_b, X_{t+\sigma_b})\Big) = G(t,x) + \widetilde{\mathbb{E}}_{t,x}\bigg(\int_0^{\sigma_b}\bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\bigg)\,du\bigg)
$$

$$
\begin{aligned}
(7.64)\qquad &\widetilde{\mathbb{E}}_{t,x}\Big(U^c(t+\sigma_b, X_{t+\sigma_b})\Big) \\
&= G(t,x) + \widetilde{\mathbb{E}}_{t,x}\bigg(\int_0^{\sigma_b}\bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1-rX_{t+u}}{(t+u)}\bigg) I\big(X_{t+u} < c(t+u)\big)\,du\bigg)
\end{aligned}
$$

where we also use that $V(t,x) = U^c(t,x) = G(t,x)$ for $x < b(t) \wedge c(t)$. Since $U^c \le V$ it follows from (7.63) and (7.64) that:

$$\widetilde{\mathbb{E}}_{t,x}\bigg(\int_0^{\sigma_b} \bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)} \bigg) I\big(X_{t+u} \ge c(t+u)\big)\, du \bigg) \ge 0. \tag{7.65}$$

Due to the fact that $b(t) < t/(1+rt)$ for all $0 < t < T$, we see that $X_{t+u}/(t+u)^2 - (1 - rX_{t+u})/(t+u) < 0$ in (7.65) so that by the continuity of b and c it follows that $c \ge b$ on $[0,T]$ as claimed.

17. We show that c must be equal to b. For this, let us assume that there is $t \in \langle 0, T\rangle$ such that $c(t) > b(t)$. Pick $x \in \langle b(t), c(t)\rangle$ and consider the stopping time τ_b from (7.17). Inserting τ_b instead of s in (7.49) and (7.56) and using the optional sampling theorem we get:

$$\widetilde{\mathbb{E}}_{t,x}\Big(G(t+\tau_b, X_{t+\tau_b}) \Big) = V(t,x) \tag{7.66}$$

$$\begin{aligned} &\widetilde{\mathbb{E}}_{t,x}\Big(G(t+\tau_b, X_{t+\tau_b}) \Big) \\ &= U^c(t,x) + \widetilde{\mathbb{E}}_{t,x}\bigg(\int_0^{\tau_b} \bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)} \bigg) I\big(X_{t+u} < c(t+u)\big)\, du \bigg) \end{aligned} \tag{7.67}$$

where we also use that $V(t+\tau_b, X_{t+\tau_b}) = U^c(t+\tau_b, X_{t+\tau_b}) = G(t+\tau_b, X_{t+\tau_b})$ upon recalling that $c \ge b$ and $U^c = G$ either below c or at T. Since $U^c \le V$ we see from (7.66) and (7.67) that:

$$\widetilde{\mathbb{E}}_{t,x}\bigg(\int_0^{\tau_b} \bigg(\frac{X_{t+u}}{(t+u)^2} - \frac{1 - rX_{t+u}}{(t+u)} \bigg) I\big(X_{t+u} < c(t+u)\big)\, du \bigg) \ge 0. \tag{7.68}$$

Due to the fact that $c(t) < t/(1+rt)$ for all $0 < t < T$ by assumption, we see that $X_{t+u}/(t+u)^2 - (1 - rX_{t+u})/(t+u) < 0$ in (7.68) so that by the continuity of b and c it follows that such a point (t,x) cannot exist. Thus c must be equal to b, and the proof is complete. □

7.4 Remarks on numerics

1. The following method can be used to calculate the optimal stopping boundary b numerically by means of the integral equation (7.22). Note that the formula (7.23) can be used to calculate the arbitrage-free price V when b is known.

Set $t_i = ih$ for $i = 0, 1, \ldots, n$ where $h = T/n$ and denote:

(7.69)
$$J(t, b(t)) = 1 - \frac{b(t)}{t} - F(T-t, b(t))$$

(7.70)
$$K(t, b(t); t+u, b(t+u)) = \frac{1}{t+u}\left(\left(\frac{1}{t+u} + r\right)G(u, b(t), b(t+u)) - H(u, b(t), b(t+u))\right).$$

Then the following discrete approximation of the integral equation (7.22) is valid:

$$J(t_i, b(t_i)) = \sum_{j=i+1}^{n} K(t_i, b(t_i); t_j, b(t_j))\, h \tag{7.71}$$

for $i = 0, 1, \ldots, n-1$. Letting $i = n-1$ and $b(t_n) = T/(1+rT)$ we can solve equation (7.71) numerically and get a number $b(t_{n-1})$. Letting $i = n-2$ and using the values of $b(t_{n-1})$ and $b(t_n)$ we can solve equation (7.71) numerically and get a number $b(t_{n-2})$. Continuing the recursion we obtain $b(t_n), b(t_{n-1}), \ldots, b(t_1), b(t_0)$ as an approximation of the optimal stopping boundary b at points $0, h, \ldots, T-h, T$.

It is an interesting numerical problem to show that the approximation converges to the true function b on $[0, T]$ as $h \downarrow 0$. Another interesting problem is to derive the rate of convergence.

2. To perform the previous recursion we need to compute the functions F, G, H from (7.18)-(7.20) as efficiently as possible. Simply by observing the expressions (7.18)-(7.21) it is apparent that finding these functions numerically is not trivial. Moreover, the nature of the probability density function f in (7.21) presents a further numerical challenge. Part of this probability density function is the Hartman-Watson density discussed in [2]. As t tends to zero, the numerical estimate of the Hartman-Watson density oscillates, with the oscillations increasing rapidly in both amplitude and frequency as t gets closer to zero. The authors of [2] mention that this may be a consequence of the fact that $t \mapsto \exp(2Pi^2/\sigma^2 t)$ rapidly increases to infinity while $z \mapsto \sin(4Piz/\sigma^2 t)$ oscillates more and more frequently. This rapid oscillation makes accurate estimation of $f(t, s, a)$ with t close to zero very difficult.

The problems when dealing with t close to zero are relevant to pricing the early exercise Asian call option. To find the optimal stopping boundary b as the solution to the implicit equation (7.71) it is necessary to work backward from T to 0. Thus to get an accurate estimate for b when $b(T)$ is given, the next estimate of $b(u)$ must be found for some value of u close to T so that $t = T-u$ will be close to zero.

Even if we get an accurate estimate for f, to solve (7.18)-(7.20) we need to evaluate two nested integrals. This is slow computationally. A crude attempt has

been made at storing values for f and using these to estimate F, G, H in (7.18)-(7.20) but this method has not produced reliable results.

3. Another approach to finding the functions F, G, H from (7.18)-(7.20) can be based on numerical solutions of partial differential equations. Two distinct methods are available.

Consider the transition probability density of the process X given by:

$$p(s,x;t,y) = \frac{d}{dy}\,\widetilde{P}(X_t \le y \mid X_s = x) \tag{7.72}$$

where $0 \le s < t$ and $x, y \ge 0$. Since $p(s,x;t,y) = p(0,x;t-s,y)$ we see that there is no restriction to assume that $s = 0$ in the sequel.

4. The *forward equation* approach leads to the initial-value problem:

$$p_t = -((1-ry)\,p)_y + (D\,y\,p)_{yy} \quad (\,t > 0\,,\ y > 0\,) \tag{7.73}$$

$$p(0,x;0+,y) = \delta(y-x) \quad (\,y \ge 0\,) \tag{7.74}$$

where $D = \sigma^2/2$ and $x \ge 0$ is given and fixed (recall that δ denotes the Dirac delta function). Standard results (cf. [13]) imply that there is a unique non-negative solution $(t,y) \mapsto p(0,x;t,y)$ of (7.73)-(7.74). The solution p satisfies the following boundary conditions:

$$p(0,x;t,0+) = 0 \quad (\,0 \text{ is entrance}\,) \tag{7.75}$$

$$p(0,x;t,\infty-) = 0 \quad (\,\infty \text{ is normal}\,). \tag{7.76}$$

The solution p satisfies the following integrability condition:

$$\int_0^\infty p(0,x;t,y)\,dy = 1 \tag{7.77}$$

for all $x \ge 0$ and all $t \ge 0$. Once the solution $(t,y) \mapsto p(0,x;t,y)$ of (7.73)-(7.74) has been found, the functions F, G, H from (7.18)-(7.20) can be computed using the general formula:

$$\widetilde{\mathbb{E}}_{0,x}(g(X_t)) = \int_0^\infty g(y)\,p(0,x;t,y)\,dy \tag{7.78}$$

upon choosing the appropriate function $g : \mathbb{R}_+ \to \mathbb{R}_+$.

5. The *backward equation* approach leads to the terminal-value problem:

$$q_t = (1-rx)\,q_x + D\,x^2\,q_{xx} \quad (\,t > 0\,,\ x > 0\,) \tag{7.79}$$

$$q(T,x) = h(x) \quad (\,x \ge 0\,) \tag{7.80}$$

where $h : I\!R_+ \to I\!R_+$ is a given function. Standard results (cf. [13]) imply that there is a unique non-negative solution $(t,x) \mapsto q(t,x)$ of (7.79)-(7.80). Taking $x \mapsto h(x)$ to be $x \mapsto (1-x/T)^+$ (with T fixed), $x \mapsto x\, I(x \leq y)$ (with y fixed), $x \mapsto I(x \leq y)$ (with y fixed) it follows that the unique non-negative solution q of (7.79)-(7.80) coincides with F , G , H from (7.18)-(7.20) respectively. (For numerical results of a similar approach see [40].)

6. It is an interesting numerical problem to carry out either of the two methods described above and produce approximations to the optimal stopping boundary b using (7.71). Another interesting problem is to derive the rate of convergence.

Appendix A

Appendix to Chapter 7

A.1 Appendix

In this section we derive the explicit expression for the probability density function f of (S_t, I_t) under $\widetilde{P}$ with $S_0 = 1$ and $I_0 = 0$ given in (7.21) above.

Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. With $t > 0$ and $\nu \in \mathbb{R}$ given and fixed recall from [56, page 527] that the random variable $A_t^{(\nu)} = \int_0^t e^{2(B_s + \nu s)} ds$ has the conditional distribution:

$$P\Big(A_t^{(\nu)} \in dy \,\Big|\, B_t + \nu t = x\Big) = a(t, x, y)\, dy \tag{A.1}$$

where the density function a for $y > 0$ is given by:

$$\begin{aligned} a(t, x, y) = {} & \frac{1}{Piy^2} \exp\left(\frac{x^2 + Pi^2}{2t} + x - \frac{1}{2y}\Big(1 + e^{2x}\Big)\right) \\ & \times \int_0^\infty \exp\left(-\frac{z^2}{2t} - \frac{e^x}{y} \cosh(z)\right) \sinh(z) \sin\left(\frac{Piz}{t}\right) dz. \end{aligned} \tag{A.2}$$

This implies that the random vector $\big(2(B_t + \nu t), A_t^{(\nu)}\big)$ has the distribution:

$$P\Big(2(B_t + \nu t) \in dx, A_t^{(\nu)} \in dy\Big) = b(t, x, y)\, dx\, dy \tag{A.3}$$

where the density function b for $y > 0$ is given by:

(A.4)

$$\begin{aligned} b(t, x, y) &= a\left(t, \frac{x}{2}, y\right) \frac{1}{2\sqrt{t}}\, \varphi\left(\frac{x - 2\nu t}{2\sqrt{t}}\right) \\ &= \frac{1}{(2Pi)^{3/2} y^2 \sqrt{t}} \exp\left(\frac{Pi^2}{2t} + \Big(\frac{\nu + 1}{2}\Big)x - \frac{\nu^2}{2}\, t - \frac{1}{2y}\Big(1 + e^x\Big)\right) \\ &\quad \times \int_0^\infty \exp\left(-\frac{z^2}{2t} - \frac{e^{x/2}}{y} \cosh(z)\right) \sinh(z) \sin\left(\frac{Piz}{t}\right) dz \end{aligned}$$

and we set $\varphi(z) = (1/\sqrt{2Pi}) e^{-z^2/2}$ for $z \in \mathbb{R}$ (for related expressions in terms of Hermite functions see [8] and [43]).

Denoting $K_t = \alpha B_t + \beta t$ and $L_t = \int_0^t e^{\alpha B_s + \beta s} ds$ with $\alpha \neq 0$ and $\beta \in I\!R$ given and fixed, and using that the scaling property of B implies:

(A.5)

$$P\left(\alpha B_t + \beta t \leq x, \int_0^t e^{\alpha B_s + \beta s}\, ds \leq y\right) = P\left(2(B_{t'} + \nu t') \leq x, \int_0^{t'} e^{2(B_s + \nu s)}\, ds \leq \frac{\alpha^2}{4}\, y\right)$$

with $t' = \alpha^2 t/4$ and $\nu = 2\beta/\alpha^2$, it follows by applying (A.3) and (A.4) that the random vector (K_t, L_t) has the distribution:

$$P\Big(K_t \in dx, L_t \in dy\Big) = c(t, x, y)\, dx\, dy \tag{A.6}$$

where the density function c for $y > 0$ is given by:

(A.7)

$$\begin{aligned} c(t, x, y) &= \frac{\alpha^2}{4}\, b\left(\frac{\alpha^2}{4}\, t,\, x,\, \frac{\alpha^2}{4}\, y\right) \\ &= \frac{2\sqrt{2}}{Pi^{3/2}\alpha^3}\, \frac{1}{y^2\sqrt{t}} \exp\left(\frac{2Pi^2}{\alpha^2 t} + \Big(\frac{\beta}{\alpha^2} + \frac{1}{2}\Big)x - \frac{\beta^2}{2\alpha^2}\, t - \frac{2}{\alpha^2 y}\Big(1 + e^x\Big)\right) \\ &\quad \times \int_0^\infty \exp\left(-\frac{2z^2}{\alpha^2 t} - \frac{4e^{x/2}}{\alpha^2 y} \cosh(z)\right) \sinh(z)\, \sin\Big(\frac{4Piz}{\alpha^2 t}\Big)\, dz. \end{aligned}$$

¿From (7.8) and (7.3) we see that f satisfies:

$$f(t, s, a) = \frac{1}{s}\, c(t, \log(s), a) = \frac{1}{s}\, \frac{\alpha^2}{4}\, b\left(\frac{\alpha^2}{4}\, t,\, \log(s),\, \frac{\alpha^2}{4}\, a\right) \tag{A.8}$$

with $\alpha = \sigma$ and $\beta = r + \sigma^2/2$. Hence (7.21) follows by the final expression in (A.4).

Appendix B

Important Results

Lemma B.1. *Suppose that* $V : (0,T) \times \mathbb{R} \mapsto \mathbb{R}$ *satisfies*

1. $x \mapsto V(t,x)$ *is continuous for each fixed* t,

2. $t \mapsto V(t,x)$ *is continuous, uniformly on* $[x-\delta(x), x+\delta(x)]$, *for some* $\delta(x) > 0$.

Then V *is continuous on its domain.*

Proof. Let $\varepsilon > 0$ and $(t_0, x_0) \in [0,T) \times (0,\infty)$. Condition 1 shows that we may find $\hat{\delta}_\varepsilon(t_0,x_0) > 0$ such that $|x - x_0| < \hat{\delta}_\varepsilon(t_0,x_0)$ implies that we have $|V(t_0,x) - V(t_0,x_0)| < \varepsilon/2$. Condition 2 shows that there exists $\delta(x_0) > 0$ and $\delta'_\varepsilon(t_0) > 0$ such that $|x - x_0| < \delta(x_0)$ and $|t - t_0| < \delta'_\varepsilon(t_0)$ imply that $|V(t,x) - V(t_0,x)| < \varepsilon/2$. Take $\delta_\varepsilon(t_0,x_0) = \min\{\hat{\delta}_\varepsilon(t_0,x_0), \delta'_\varepsilon(t_0), \delta(x_0)\}$. Then if $\| (t,x) - (t_0,x_0) \| < \delta_\varepsilon(t_0,x_0)$, we will have:

$$\begin{aligned} |V(t,x) - V(t_0,x_0)| &\leq |V(t,x) - V(t_0,x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

and we are done. □

The proofs of the following well-known results can be found in the cited references.

Definition B.2. [20, page 191] A family of probability measures $\{P_T \mid 0 \leq T < \infty\}$ is *consistent* if the consistency condition

$$P_T(A) = P_t(A) \quad \text{for all} \quad A \in \mathcal{F}_t \quad \text{where} \quad 0 \leq t \leq T \tag{B.1}$$

is satisfied for each T such that $0 \leq T < \infty$.

Definition B.3. [20, page 36] Let $X = (X_t, \mathcal{F}_t)_{t\geq 0}$ be a (continuous) process. If there exists a non-decreasing sequence $\{T_n\}_{n=1}^{\infty}$ of stopping times of $\{\mathcal{F}_t\}_{t\geq 0}$ such that $X_t^{(n)} = (X_{t\wedge T_n}, \mathcal{F}_t)_{t\geq 0}$ is a martingale for each $n \geq 1$ and $P(\lim_{n\to\infty} T_n = \infty) = 1$, then we say that X is a (continuous) *local martingale*.

Definition B.4. [21, page 323] Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{X}$ be a non-negative family of random variables defined on $(\Omega, \mathcal{F}, P)$. The *essential supremum* of $\mathcal{X}$, denoted by $\operatorname{ess\,sup} \mathcal{X}$, is a random variable $\tilde{X}$ satisfying

1. for all $X \in \mathcal{X}$, the inequality $X \leq \tilde{X}$ holds a.s., and
2. if Y is a random variable satisfying $X \leq Y$ a.s. for all $X \in \mathcal{X}$, then $\tilde{X} \leq Y$ a.s.

Theorem B.5. (Girsanov) *[20, page 191] Let the process $W = (W_t)_{t\geq 0}$ with $W_t = (W_t^1, \ldots, W_t^d)$ be a standard d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, equipped with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions.*

Let $X = (X_t)_{t\geq 0}$ with $X_t = (X_t^1, \ldots, X_t^d)$ be a vector of measurable, adapted processes with respect to $(\mathcal{F}_t)_{t\geq 0}$ satisfying

$$P\left(\int_0^T (X_t^i)^2\, dt < \infty\right) = 1 \tag{B.2}$$

for $1 \leq i \leq d$ and $0 \leq T < \infty$, such that the process $Z = (Z_t)_{t\geq 0}$ defined by

$$Z_t = \exp\left(\sum_{i=1}^{d} \int_0^t X_s^i\, dW_s^i - \frac{1}{2}\int_0^t ||X_s||^2\, ds\right) \tag{B.3}$$

is a martingale.

Define a process $\tilde{W} = (\tilde{W}_t)_{t\geq 0}$ with $\tilde{W}_t = (\tilde{W}_t^1, \ldots, \tilde{W}_t^d)$ by

$$\tilde{W}_t^i = W_t^i - \int_0^t X_s^i\, ds \tag{B.4}$$

for $1 \leq i \leq d$ and $0 \leq t < \infty$. For each fixed $T \in [0, \infty)$, the process $(\tilde{W}_t, \mathcal{F}_t)_{t\geq 0}$ is a d-dimensional Brownian motion on $(\Omega, \mathcal{F}_T, \tilde{P}_T)$, where the probability measure $\tilde{P}$ is defined by

$$\tilde{P}_T(A) = \mathbb{E}(1_A Z_T) \tag{B.5}$$

for $A \in \mathcal{F}_T$.

Theorem B.6. (Martingale representation theorem) *[42, page 73] Let the process $W = (W_t)_{t\geq 0}$ be a standard d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, equipped with the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by W and augmented to*

satisfy the usual conditions (see (3.1)–(3.5)). If $Y \in L^2(\mathcal{F})$, then there exists a $(\mathcal{F}_t)_{t\geq 0}$-previsible process H with $\mathbb{E}\int_0^\infty |H_s|^2\, ds < \infty$ such that:

$$\mathbb{E}(Y \mid \mathcal{F}_t) = \mathbb{E}(Y) + \int_0^t H_s\, dB_s \tag{B.6}$$

and H is uniquely determined modulo Leb$\times P$*-null subsets.*

Theorem B.7. (The Fatou lemma) *[42, page 22] Let M be a non-negative local martingale such that $\mathbb{E}(M_0) < \infty$. Then M is a supermartingale.*

Definition B.8. [42, page 368] A process X on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is said to be of *class(D)* if the family

$$\{X_T \mid \text{T a finite stopping time}\} \tag{B.7}$$

is uniformy integrable.

Lemma B.9. *[42, page 368] A uniformly integrable martingale M is of class*(D).

Lemma B.10. (Doob's maximal inequality) *[20, page 14] Let $(X_t)_{0\leq t<\infty}$ be a submartingale whose every path is right-continuous, let $[\sigma, \tau]$ be a subinterval of $[0, \infty)$, and let $\alpha < \beta$, $\gamma > 0$ be real numbers. Then*

$$\mathbb{E}\left(\sup_{0\leq\sigma\leq\tau} X_t\right)^p \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}(X_\tau^p) \tag{B.8}$$

where $p > 1$, provided $X_t \geq 0$ a.s. P, for every $t \geq 0$, and $\mathbb{E}(X_\tau^p) < \infty$.

Theorem B.11. (Doob-Meyer decomposition) *[42, page 368] An adapted RCLL process Z is a submartingale of class*(D) *null at 0 if and only if Z may be written*

$$Z = M + A \tag{B.9}$$

where M is a uniformly integrable martingale null at 0, and A is a previsible integrable increasing process null at 0. Moreover, the decomposition (B.9) is unique.

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