On the packing chromatic numbers of some connected spanning subgraphs of $\mathbb{Z}^3$

A thesis submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfillment of the requirements for the degree of Doctor of Philosophy

by

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Signed

De Villiers Lessing

2025-05-28, at Stellenbosch.

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Abstract

Let G be a graph. A function $\pi : V(G) \rightarrow 1, 2, ..., k$ is referred to as a packing colouring of order k, $k \in \mathbb{Z}^+$, if $\pi(v) = \pi(u)$, where $u, v \in V(G)$, implies that $d(u, v) > \pi(u)$. The minimum order of a packing colouring of a graph G is referred to as the packing chromatic number of G, and is denoted by $\chi_\rho(G)$.

The *infinite square lattice*, denoted by $\mathbb{Z}^2$, is defined as the Cartesian product of $\mathbb{Z}$ and $\mathbb{Z}$, that is, $\mathbb{Z}\Box\mathbb{Z}$, where $\mathbb{Z}$ is the *two-way infinite path*. Similarly, the *infinite cubic lattice*, denoted by $\mathbb{Z}^3$, is defined as the Cartesian product of $(\mathbb{Z}\Box\mathbb{Z})\Box\mathbb{Z}$.

In this thesis we consider the following question: What is the minimum proportion of edges that must be removed from $\mathbb{Z}^3$ to obtain a connected spanning subgraph for which a finite packing colouring exists?

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Firstly, I wish to express my heartfelt gratitude to my supervisors, Prof. Elizabeth Jonck and Prof. Johannes H. Hattingh, for their invaluable guidance and unwavering encouragement throughout this journey. Your insights, feedback, and perspectives have been instrumental in shaping this thesis - one that I am both proud of and deeply grateful for.

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To my wife, Natassia, and our daughters, Mianke, Katryn, and Lienke - your belief in my perseverance far exceeded my own. During moments of frustration, when weeks of work seemed futile, your unwavering support carried me forward to what once felt like an impossible achievement. I am eternally grateful. To my father, Piet, my mother, Kotie, my brothers, Gerhard and De Wet, and my sister, Mandie - your belief in me, your encouragement, and your words of support have meant more than I can express.

In loving memory of Peanut, Lila, and Charlie.

” 1 I lift up my eyes to the mountains - where does my help come from? 2 My help comes from the Lord, the Maker of heaven and earth. 3 He will not let your foot slip - he who watches over you will not slumber; 4 indeed, he who watches over Israel will neither slumber nor sleep. 5 The Lord watches over you - the Lord is your shade at your right hand; 6 the sun will not harm you by day, nor the moon by night. 7 The Lord will keep you from all harm - he will watch over your life; 8 the Lord will watch over your coming and going both now and forevermore.”

– Psalm 121

List of Tables

List of Figures

Table of Contents

List of Symbols

Symbol	Description
$\approx$	Approximately equal to.
$\Delta N_i(x)$	The boundary of the closed neighbourhood of radius i about vertex x.
$G \square H$	The Cartesian product of two graphs G and H.
$N_i(x)$	The closed neighbourhood of radius i about vertex x.
$\pi(v)$	The colour assigned to a vertex v.
H^δ	H is a δ-subgraph.
$d(u,v)$	The distance between two vertices u and v.
$E(G)$	The edge set of graph G.
$\mathbb{Z}^2$	The infinite square lattice.
$\mathbb{Z}^3$	The infinite cubic lattice.
$\Delta(G)$	The maximum degree of graph G.
$\delta(G)$	The minimum degree of graph G.
$\mathbb{Z}_{\geq 0}$	The non-negative integers.
A_i	An i-packing.
$\chi_\rho(G)$	The packing chromatic number of a graph G.
$\mathbb{Z}^+$	The positive integers.
θ_H	The proportion of vertices of graph H that have degree 4.
$\mathbb{Z}^3_i$	An i-regular connected spanning subgraph of $\mathbb{Z}^3$.
$\mathbb{Z}^3_{i'}$	An i-regular connected maximal subgraph of $\mathbb{Z}^3$.
$\mathbb{Z}$	The two-way infinite path.
$V(G)$	The vertex set of graph G.

Chapter 0

Introduction

Packing colourings arose from regulations introduced by the United States Federal Communication Commission. In particular, when assigning the same broadcasting signal to two distinct radio stations, care should be taken to ensure that these radio stations are located sufficiently far apart in order for the broadcast signal of either station to not interfere with the broadcast signal of the other.

Packing colourings were first introduced as *broadcast colourings* by Goddard et al. in [16]. Brešar et al. were the first to use the term *packing colouring* in [4].

We refer the reader to [7] for all terms not defined in this thesis.

Definition 1. Let $G = (V(G), E(G))$ be a simple graph. A function $\pi : V(G) \to 1$, 2, ..., k is referred to as a *packing colouring* of *order* k, $k \in \mathbb{Z}^+$, if $\pi(v) = \pi(u)$, where $u, v \in V(G)$, implies that $d(u, v) > \pi(u)$. A packing colouring of order k may also be referred to as a *packing-k-colouring.* The minimum order of a packing colouring of a graph G is referred to as the *packing chromatic number* of G and is denoted by $\chi_\rho(G)$.

Definition 2. A subset of vertices, denoted by A, of some graph G, is said to be an *i-packing*, $i \in \mathbb{Z}^+$, if for every pair of distinct vertices $u, v \in A$, $d(u, v) > i$.

A packing colouring of order k is therefore a partition of $V(G)$ into k disjoint sets A_i, such that

$$\bigcup_{i=1}^{k} A_i = V(G),$$

and for every pair of vertices $u, v \in A_i$, $d(u,v) > i$.

Definition 3. Let G be a graph. For $x \in V(G)$ and a positive integer i, the *closed neighbourhood* of x, denoted by $N_i(x;G)$, is defined by

$$N_i(x;G) = \{y \in V(G) : d(x,y) \leq i\}.$$

If G is clear, we may simply write $N_i(x)$.

Definition 4. Let G be a graph. For $x \in V(G)$ and a positive integer i, the *boundary* of $N_i(x)$, denoted by $\Delta N_i(x)$, is defined by

$$\Delta N_i(x) = \{y \in V(G) : d(x,y) = i\}.$$

Definition 5. The *Cartesian product* of two graphs G and H, denoted by $G\square H$, is the graph with vertex set $V(G) \times V(H)$ whose vertices (g,h) and (g',h') are adjacent if and only if $gg' \in E(G)$ and $h = h'$ or $hh' \in E(H)$ and $g = g'$.

Definition 6. The *infinite square lattice*, denoted by $\mathbb{Z}^2$, is defined as the Cartesian product of $\mathbb{Z}$ and $\mathbb{Z}$, that is, $\mathbb{Z}\square\mathbb{Z}$, where $\mathbb{Z}$ is the *two-way infinite path*. Similarly, the *infinite cubic lattice*, denoted by $\mathbb{Z}^3$, is defined as the Cartesian product of $(\mathbb{Z}\square\mathbb{Z})\square\mathbb{Z}$.

Definition 7. Let G be a graph. The *minimum degree* of G, denoted by $\delta(G)$, is the smallest degree among all the vertices of G. The *maximum degree* of G, denoted by $\Delta(G)$, is the largest degree among all the vertices of G.

For a given graph G and $k \in \mathbb{Z}^+$, Goddard et al. showed in [16] that determining the packing k-colouring of G is NP-complete for $k = 4$, even when restricted to planar graphs. Furthermore, Fiala and Golovach demonstrated in [12] that the packing colouring problem remains NP-complete for trees.

The study of the packing chromatic number of infinite graphs has been extensive. For a comprehensive survey on the progress of research in this area, see Brešar et al. in [3].

We introduce an informal definition of the infinite hexagonal lattice. Let C be a cycle of six vertices arranged in the shape of a hexagon (see Figure 0.1).

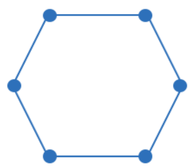

Figure 0.1: The construction of a hexagonal tiling starting from a cycle C.

Next, place a copy of C along each of its edges such that each copy shares exactly one edge (and two vertices) with the original cycle. This process is repeated iteratively until every edge of C is shared by exactly two hexagons.

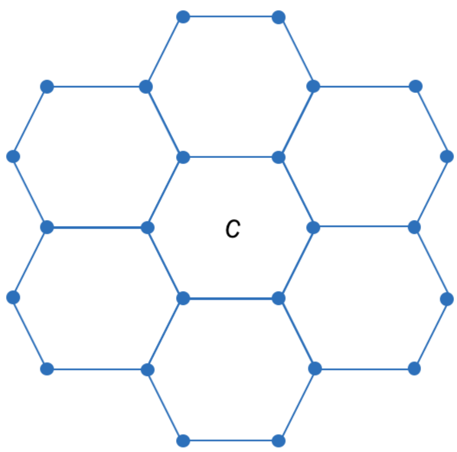

Figure 0.2: The iterative construction of the hexagonal lattice.

By continuing this construction indefinitely, as illustrated in Figure 0.2, we ensure that every edge in the resulting graph is shared by exactly two hexagons, thereby forming the infinite hexagonal lattice H.

Bounds on the packing chromatic number of the infinite hexagonal lattice H were initially introduced by Brešar et al. in [4], who established that $6 \leq \chi_\rho(H) \leq 8$. The upper bound was later improved to 7 by Fiala et al. in [13], where they constructed a packing colouring of H using seven colours. Finally, Korže and Vesel confirmed in [20] that $\chi_\rho(H) = 7$.

The packing chromatic number of the infinite triangular lattice T, which is constructed by adding all edges of the form $[(i, j), (i + 1, j - 1)]$ to $\mathbb{Z}^2$, was conjectured to be infinite by Brešar et al. in [4]. This was later confirmed by Finbow and Rall in [14].

The packing chromatic number of infinite product graphs has also been widely studied. Goddard et al. established in [16] that $9 \leq \chi_\rho(\mathbb{Z}^2) \leq 23$. This lower bound was improved to 10 by Fiala et al. in [13] and further strengthened to 12 through computational methods by Ekstein et al. in [11]. Later, Soukal and Holub refined the upper bound to 17 in [32]. Martin et al. employed a SAT-solver in [25] to show that $13 \leq \chi_\rho(\mathbb{Z}^2) \leq 15$, and, ultimately, Subercaseaux and Heule proved in [33] that $\chi_\rho(\mathbb{Z}^2) = 15$.

Additionally, and serving as the primary motivation for this thesis, Finbow and Rall investigated the packing chromatic number of $\mathbb{Z}^3$ in [14], proving that it is infinite. They employed an intriguing technique that we will also leverage in this work.

The study of packing chromatic numbers has developed into a rich field of research. The packing chromatic numbers of products of paths and cycles have been explored in [16], [30], and [31]. Research has also extended to Sierpiński-type graphs [2], [6], [9], [19], generalized Sierpiński graphs in [2], [21], and Sierpiński triangle graphs in [17] and [21].

Finally, we direct the reader's attention to additional studies on the packing chromatic numbers of specific classes of graphs: [1], [10], [15], [22], [23], [24], [26], [27], [28], [34], [35] and [36].

In this thesis, we investigate how adjustments to the size of $\mathbb{Z}^3$ impact the packing chromatic number of the resultant graph. Moreover, we address the following question: What is the minimum proportion of edges that need to be removed from $\mathbb{Z}^3$ to produce a connected spanning subgraph of $\mathbb{Z}^3$ for which a finite packing colouring exists?

It is impractical - indeed, impossible - to consider every subgraph of $\mathbb{Z}^3$ in answering this question. Therefore, we focus on specific classes of subgraphs that help streamline our approach to finding a suitable answer. In particular, we examine various cases of regular connected spanning subgraphs of $\mathbb{Z}^3$ and determine whether these graphs are finitely packing colourable.

In our first chapter, we consider the possibility of constructing a two regular connected spanning subgraph of $\mathbb{Z}^3$. Additionally, we will consider arbitrary three regular connected spanning subgraphs of $\mathbb{Z}^3$. Specifically, for arbitrary three regular connected spanning subgraphs of $\mathbb{Z}^3$, we investigate whether χ_ρ is finite.

In our second chapter, we will continue our investigation by examining whether arbitrary four and five regular connected spanning subgraphs of $\mathbb{Z}^3$ are finitely packing colourable.

In our third chapter, our aim is to determine or estimate the minimum proportion of edges that must be removed so that the resulting connected spanning subgraph of $\mathbb{Z}^3$, denoted by F, can be coloured with a finite packing colouring. Second, we seek specific examples of connected spanning subgraphs F of $\mathbb{Z}^3$ where

$$\deg(x) \in \{\delta(F),\, \Delta(F)\},$$

$x \in V(F)$ arbitrary, $\delta(F) = 3$ and $\Delta(F) = 4$. The motivation for identifying such specific examples will become clearer in later discussions.

For arbitrary three regular connected spanning subgraphs of $\mathbb{Z}^3$, denoted by $\mathbb{Z}^3_3$, we want to show that

$$a \leq \chi_\rho(\mathbb{Z}_3^3) \leq b,$$

where a, $b \in \mathbb{Z}^+$.

For the cases $\mathbb{Z}_4^3$ and $\mathbb{Z}_5^3$, our intuition suggests that χ_ρ is infinite - which is based on multiple four regular connected spanning subgraphs of $\mathbb{Z}^3$ that was constructed and analysed for their feasibility to receive a finite packing colouring. In order to show this, we will utilise the technique introduced by Finbow and Rall in [14], which requires estimating the order and size of the subgraph induced by $N_i(x)$, $i \in \mathbb{Z}^+$, where $x \in V(\mathbb{Z}_4^3)$ or $x \in V(\mathbb{Z}_5^3)$ respectively.

Generation of Subgraphs and Estimation of Properties: We begin by exploring the feasibility of generating all possible subgraphs of the subgraph induced by $N_k(x)$, where $k \in \{1, 2, .., 5\}$, and $x \in V(\mathbb{Z}^3)$, after which we can then filter the generated subgraphs to identify those that are either four of five regular, which are relevant to our study. These specific subgraphs serve as the foundation for applying the least squares method to estimate the order and size of the subgraph induced by $N_k(x)$.

As an alternative strategy we will also consider the use of a sampling approach.

Sampling Approach: Generating all possible subgraphs provides comprehensive insight into a graph's structural characteristics but is computationally intensive for large graphs. This computational challenge necessitates the need for an efficient alternative that still captures the essence of the graph's structure. We will adopt a sampling approach to explore a representative subset of all possible subgraphs, reducing computational demands.

Chapter 1

Two and three regular connected spanning subgraphs of $\mathbb{Z}^3$

Our objective in this chapter is to determine whether we can find a two regular connected spanning subgraph of $\mathbb{Z}^3$. Additionally, we investigate whether $\chi_\rho(\mathbb{Z}_3^3) \leq p$, where $p \in \mathbb{Z}^+$ and $\mathbb{Z}_3^3$ an arbitrary three regular connected spanning subgraph of $\mathbb{Z}^3$.

1.1 Two regular connected spanning subgraphs of $\mathbb{Z}^3$

We observe that the graph $\mathbb{Z}_2^3$, defined as an arbitrary two regular connected spanning subgraph of $\mathbb{Z}^3$, exists if and only if $\mathbb{Z}^3$ is Hamiltonian - that is, if and only if $\mathbb{Z}^3$ contains a Hamiltonian cycle.

In general, determining whether a given graph contains a Hamiltonian cycle is an NP-complete problem [18]. Consequently, the only known approach to verifying the existence of a Hamiltonian cycle in an arbitrary graph is exhaustive search. While various search algorithms have been developed to identify some or all Hamiltonian cycles in a given graph [29], we believe these methods remain insufficient to make this feasible for $\mathbb{Z}^3$. Moreover, our intuition suggests that $\mathbb{Z}^3$ does not possess such a cycle.

Conjecture 1.1.1. $\mathbb{Z}^3$ *is non-Hamiltonian.*

As an alternative to our initial intent of constructing a two regular connected spanning subgraph of $\mathbb{Z}^3$, we now construct a connected spanning subgraph H of $\mathbb{Z}^3$, where H is a two-way infinite path. Begin by selecting an arbitrary vertex $x \in V(\mathbb{Z}^3)$. From this point, construct a spiral centered at x, adding edges as needed while remaining within the same layer as x. This forms a square spiral around x, which terminates at one of its four corners - specifically, in our example, at the vertex labeled y (see Figure 1.1). Throughout this thesis, the term *layer* of $\mathbb{Z}^3$ refers specifically to a horizontally oriented subgraph $\mathbb{Z}^2$ within $\mathbb{Z}^3$.

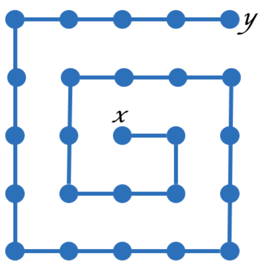

Figure 1.1: The construction of a square spiral around a vertex x.

Next, we extend the construction parallel to the z-axis by connecting y to an adjacent vertex directly above or below it. This choice is arbitrary; in this case, we select the vertex directly above y, denoted h (see Figure 1.2).

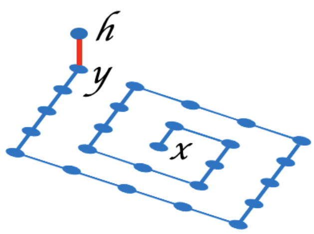

Figure 1.2: The addition of an edge parallel to the z-axis.

Since the initial spiral was constructed in a clockwise direction, we now proceed in a counterclockwise direction, beginning from h, to construct a second spiral that mirrors the first.

We continue this process, constructing up to p spirals, where $p \in \mathbb{Z}^+$ is odd. The resulting structure outlines a square prism (see Figure 1.3).

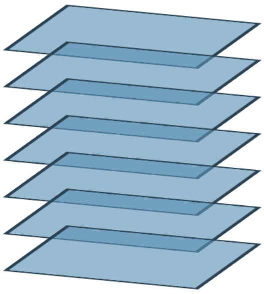

Figure 1.3: The construction of a square prism A of height p.

Observe that the first spiral (as well as all subsequent odd-numbered spirals) terminates at a vertex on a corner of the spiral, whereas even-numbered spirals terminate at the center of their respective spirals. The choice of p spirals ensures that we can exit the p^{th} spiral, allowing for the construction of additional square prisms adjacent to the initial one.

Furthermore, once the first spiral is constructed, any additional spirals added above or below it must adhere to the outline of the original spiral to maintain the structure of a square prism.

The construction of the first spiral also determines which corners of the p^{th} spiral can serve as exit points from the current square prism. Specifically, two possible exit options exist, as illustrated in Figure 1.4.

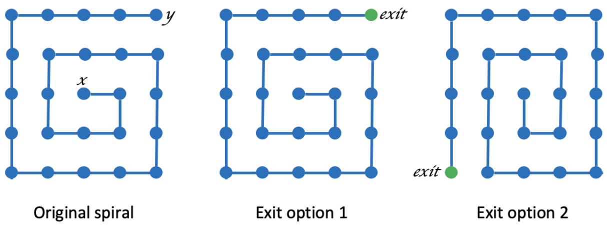

Figure 1.4: Exiting the current square prism to construct the next square prism.

Following the same procedure used to construct prism A, we extend the structure by adding additional prisms above, below, and adjacent to prism A. When constructing new square prisms adjacent to A, we encounter cases where the path being constructed terminates at the center of the p^{th} spiral, specifically for even-numbered prisms. To prevent this, we employ a zig-zag pattern, as illustrated in Figure 1.5, ensuring that the path exits from one of the corners of the final spiral.

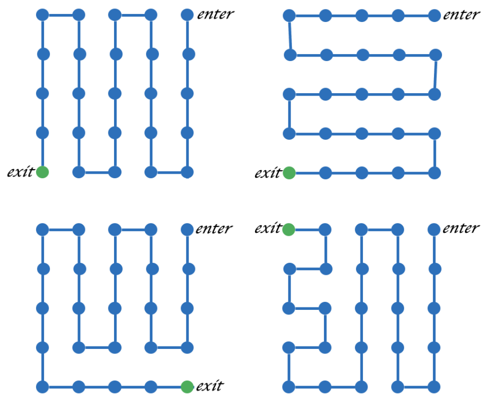

Figure 1.5: Strategies in exiting the final spiral of a square prism contruction.

This iterative process ultimately constructs a two-way infinite path H, which serves as a connected spanning subgraph of $\mathbb{Z}^3$. Finally, it follows from the fact that $\chi_\rho(\mathbb{Z}) = 3$ that $\chi_\rho(H) = 3$.

1.2 Three regular connected spanning subgraphs of $\mathbb{Z}^3$

In our investigation of three regular connected spanning subgraphs of $\mathbb{Z}^3$, we first obtain two distinct subgraphs. Label these as $\mathbb{Z}^3_{3_a}$ and $\mathbb{Z}^3_{3_b}$. We intend to construct $\mathbb{Z}^3_{3_a}$ and $\mathbb{Z}^3_{3_b}$, respectively, so that they represent minimal and maximal three regular connected spanning subgraphs of $\mathbb{Z}^3$. We then aim to show that

$$a \leq \chi_\rho(\mathbb{Z}^3_3) \leq b,$$

where $\mathbb{Z}^3_3$ is an arbitrary three regular connected spanning subgraphs of $\mathbb{Z}^3$ and $a, b \in \mathbb{Z}^+$. We start by considering whether we can show that $\chi_\rho(\mathbb{Z}^3_{3_a}) \leq a$ and $\chi_\rho(\mathbb{Z}^3_{3_b}) \leq b$.

All colourings presented in the chapters that follow were produced iteratively in Microsoft Excel. In each case the distance constraint of the packing colouring was verified based on the colouring applied. No SAT-solver, script, or other heuristic search tool was used. The annotated workbooks, included in the supplementary material, see Appendix A, allow the reader to reproduce the verification steps applied in exactly the same environment.

The construction of $\mathbb{Z}^3_{3_a}$: Select an arbitrary vertex $x \in V(\mathbb{Z}^3)$. Label its neighbours on the same layer as x as x_1, x_2, x_3, and x_4, as illustrated in Figure 1.6.

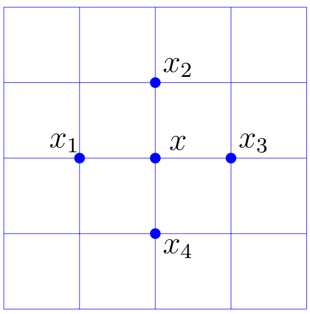

Figure 1.6: An arbitrary vertex x and its neighbours x_1, x_2, x_3, and x_4 within $\mathbb{Z}^3$.

Now, observe pattern ϕ_1 which incorporates the labeled vertices as shown in Figure 1.7.

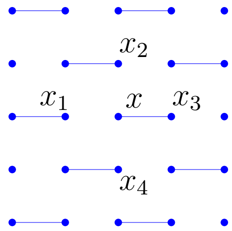

Figure 1.7: A representative tile of pattern ϕ_1

By aligning pattern ϕ_1 such that the vertices in Figure 1.7 match those in Figure 1.6, a tiling of pattern ϕ_1 around x is formed, covering every vertex in x's layer with ϕ_1. We refer to this layer of $\mathbb{Z}^3_{3_a}$ as the *Central Layer*.

Consider now patterns ϕ_2 and ϕ_3.

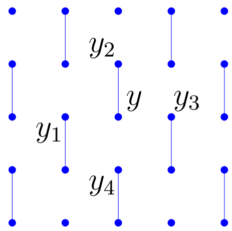

Figure 1.8: A representative tile of pattern ϕ_2.

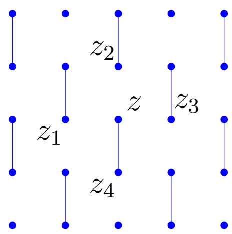

Figure 1.9: A representative tile of pattern ϕ_3.

We proceed as follows for the remaining layers of $\mathbb{Z}^3$. For the layer above the Central Layer we create a tiling of pattern ϕ_2, proceeding as we had for pattern ϕ_1, with x adjacent with y, x_1 adjacent with y_1 and so on. We call this layer *Layer 1*. For the layer above *Layer 1* we create a tiling of pattern ϕ_3, again proceeding as we had for pattern ϕ_2, with y adjacent with z, y_1 adjacent with z_1 and so on. We call this layer *Layer 2*.

We repeat this pattern as we had for Layer 1 and Layer 2, covering all layers above the Central Layer. All layers below the Central Layer is a mirror image of those layers above the Central Layer, completing the construction of $\mathbb{Z}^3_{3_a}$.

$\mathbb{Z}^3_{3a}$ **is connected:** A key observation in the construction of $\mathbb{Z}^3_{3a}$ is that the tilings of patterns ϕ_1, ϕ_2, and ϕ_3 retain all the vertical edges of $\mathbb{Z}^3$. This retention ensures direct paths of transition between layers. Specifically, the Central Layer is the only layer that allows left-to-right travel (and vice versa), whereas all other layers permit only up-and-down travel (and its reverse).

Consider Figure 1.10, which depicts the distances from x to the vertices arranged about x within the Central Layer, noting that x is representative in terms of the ability to travel from an arbitrary vertex of the Central Layer to all those vertices located about the selected vertex.

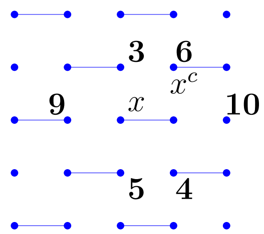

Figure 1.10: An illustration of the distances travelled from x to those vertices about x.

We illustrate the travel from vertex x to vertex x^c in Figure 1.11 below.

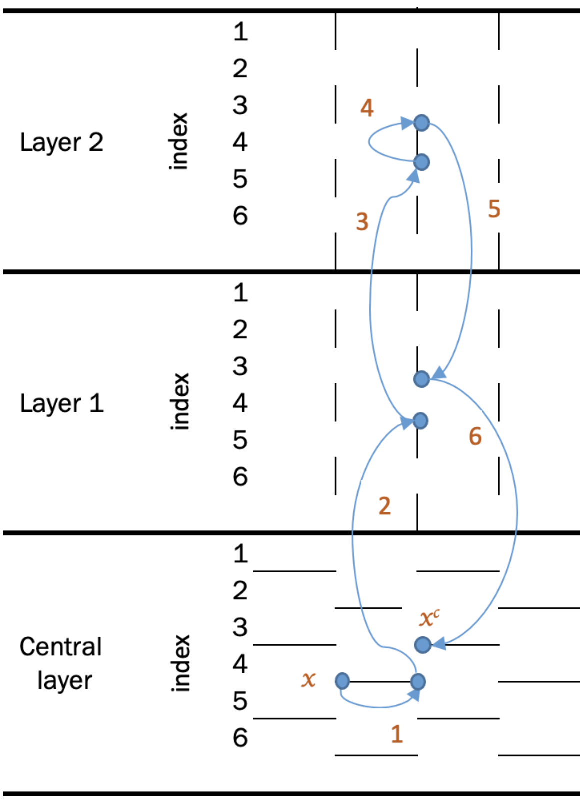

Figure 1.11: A depiction of travel from vertex x to vertex x^c.

Finally, consider Figure 1.12, which depicts the distances from y to the vertices arranged about y within Layer 1, noting that y is representative, in terms of the ability to travel, from an arbitrary vertex of an arbitrary layer (other than the Central Layer) to all those vertices located about the selected vertex.

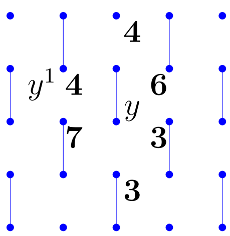

Figure 1.12: An illustration of the distances travelled from y to those vertices about y

We illustrate the travel from vertex y to vertex y^1 in Figure 1.13 below.

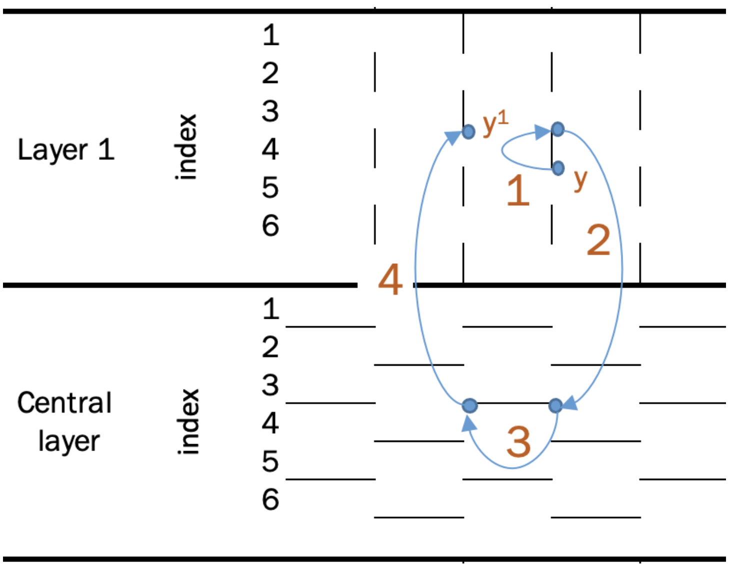

Figure 1.13: A depiction of travel from vertex y to vertex y^1.

It is clear then that we are able to travel from an arbitrary vertex $x \in V(\mathbb{Z}^3_{3a})$ to all those vertices located about x, hence $\mathbb{Z}^3_{3a}$ is connected.

$\mathbb{Z}^3_{3a}$ is finitely packing colourable: Before proceeding, we identify additional layers for clarity. Let Layer 3 be the layer above Layer 2, where vertex h in Layer 3 is adjacent to vertex z in Layer 2. For the sake of simplicity, we refer to the vertices x, y, z, and h as the central vertices of the Central Layer and Layers 1, 2, and 3, respectively. Extending this pattern, we define the vertices i, j, k, ℓ and m as the central vertices of Layers 4, 5, 6, 7, and 8.

Similarly, we define the layer below the Central Layer as Layer $1'$, with its central vertex denoted as y'. Following the same pattern, Layers $2'$, $3'$, $4'$, $5'$, $6'$, $7'$ and $8'$ have central vertices z', h', i', j', k', ℓ' and m'.

To avoid ambiguity, we highlight the path $P = (m, \ell, k, j, i, h, z, y, x, y', z', h', i', j', k', \ell', m')$. This finite path, as the reader will observe, suffices to demonstrate that $V(\mathbb{Z}^3_{3a})$ is finitely packing colourable since we only need to reference a finite number of layers, each represented by its respective central vertex. Finally, just as we extended from x to y to z, and so forth, the path P can naturally be extended infinitely in both directions.

In order to find a finite packing colouring of $V(\mathbb{Z}^3_{3a})$, we approach the respective layers of $\mathbb{Z}^3_{3a}$ systematically. Consider colour pattern ϕ_4 on a subset of vertices of the Central Layer as depicted in Figure 1.14. We cover every vertex of the Central Layer by making use of a tiling of colour pattern ϕ_4.

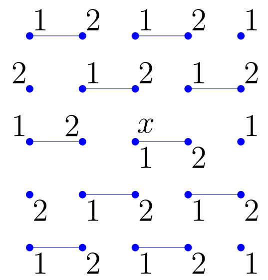

Figure 1.14: A representative tile of colour pattern ϕ_4.

Consider next colour pattern ϕ_5 on a subset of vertices of Layer 1 as depicted in Figure 1.15. We cover every vertex of Layer 1 by making use of a tiling of colour pattern ϕ_5.

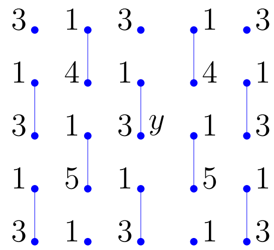

Figure 1.15: A representative tile of colour pattern ϕ_5.

Consider next colour pattern ϕ_6 on a subset of vertices of Layer 2 as depicted in Figure 1.16. We cover every vertex of Layer 2 by making use of a tiling of colour pattern ϕ_6.

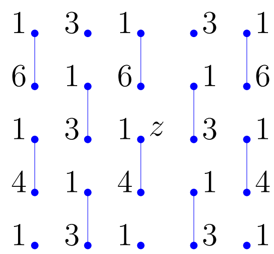

Figure 1.16: A representative tile of colour pattern ϕ_6.

We cover every vertex of Layer 3 with a tiling of colour pattern ϕ_4, adjusting as is necessary for the vertical orientation of the edges of Layer 3. Consider next colour pattern ϕ_7 on a subset of vertices of Layer 4 as depicted in Figure 1.17. We cover every vertex of Layer 4 by making use of a tiling of colour pattern ϕ_7.

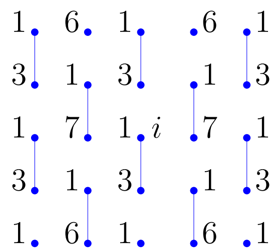

Figure 1.17: A representative tile of colour pattern ϕ_7.

Consider next colour pattern ϕ_8 on a subset of vertices of Layer 5 as depicted in Figure 1.18. We cover every vertex of Layer 5 by making use of a tiling of colour pattern ϕ_8.

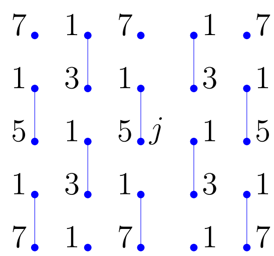

Figure 1.18: A representative tile of colour pattern ϕ_8.

In a manner similar to our approach with the Central Layer, Layer 1 and subsequent layers up to Layer 5, we colour every vertex in the layers above the Central Layer. This is achieved by sequentially applying the ordered sequence of patterns ϕ_4, ϕ_5, ϕ_6, ϕ_4, ϕ_7, and ϕ_8. Next, for Layers $3'$ through $8'$, we employ a different ordered sequence: ϕ_4, ϕ_7, ϕ_8, ϕ_4, ϕ_5, and ϕ_6, successfully covering every vertex. This strategy effectively covers all vertices in the layers below the Central Layer, with the exception of Layers $1'$ and $2'$. The remaining task is to address the colouring of vertices in these two layers. Consider next colour pattern ϕ_9 on a subset of vertices of Layer $1'$ as depicted in Figure 1.19. We cover every vertex of Layer $1'$ by making use of a tiling of colour pattern ϕ_9.

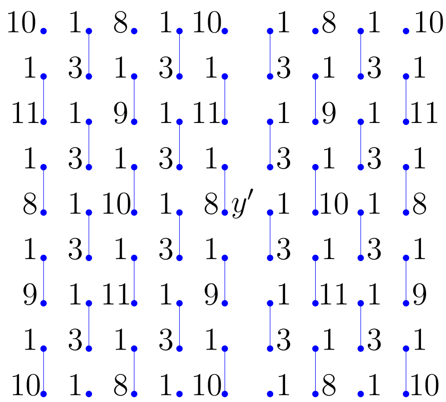

Figure 1.19: A representative tile of colour pattern ϕ_9.

Lastly, consider colour pattern ϕ_{10} on a subset of vertices of Layer $2'$ as depicted in Figure 1.20. We cover every vertex of Layer $2'$ by making use of a tiling of colour pattern ϕ_{10}.

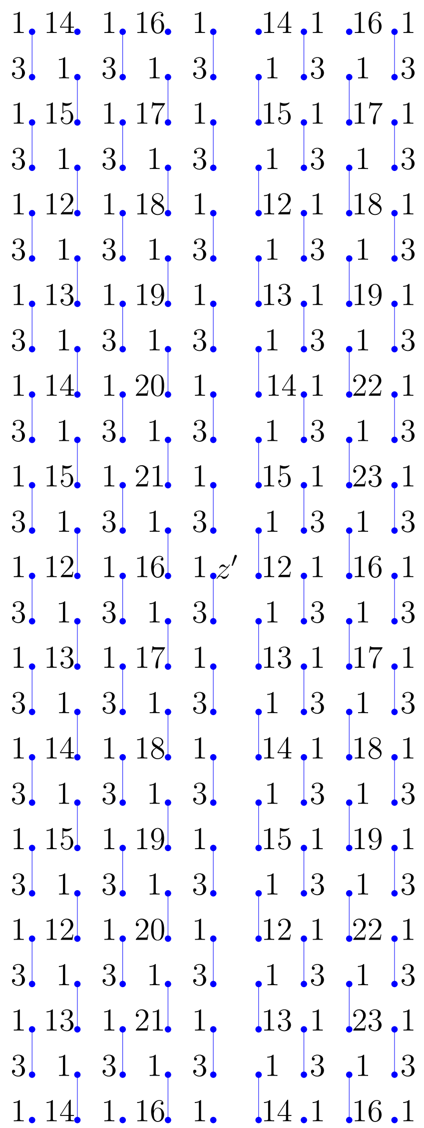

Figure 1.20: A representative tile of colour pattern ϕ_{10}.

Finally, we obtain that

$$\chi_\rho(\mathbb{Z}^3_{3a}) \leq 23.$$

In the construction of $\mathbb{Z}^3_{3a}$, the design inherently increases the complexity of traversing between any arbitrary pair of vertices $x, y \in V(\mathbb{Z}^3_{3a})$. This complexity is particularly pronounced for vertices not positioned on the Central Layer of $\mathbb{Z}^3_{3a}$, as horizontal travel, either left-to-right or vice versa, is facilitated solely through the Central Layer. This structural constraint significantly contributes to a reduction in the number of colours needed to achieve a packing colouring. Proceeding to $\mathbb{Z}^3_{3b}$, we aim to impose minimal restrictions on travel.

The construction of $\mathbb{Z}^3_{3_b}$: Building upon the algorithm used in the construction of $\mathbb{Z}^3_{3_a}$, we commence by selecting an arbitrary vertex $x \in V(\mathbb{Z}^3)$ as an anchor. The Central Layer is covered using a tiling of pattern ϕ_2 (refer to Figure 1.8 for an illustration).

It is important to note that the naming convention established in the construction of $\mathbb{Z}^3_{3_a}$ is retained here. Specifically, the layers positioned above the Central Layer are denoted as Layer 1, Layer 2, etc., whereas those situated below it are referred to as Layer $1'$, Layer $2'$, and so forth. The Central Layer has central vertex x, Layer 1 has central vertex y, Layer $1'$ has central vertex y$'$ and so on.

We cover Layer 1 with a tiling of pattern ϕ_{11} which is depicted in Figure 1.21.

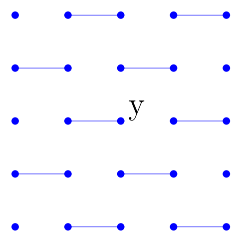

Figure 1.21: A representative tile of pattern ϕ_{11}

Next we cover Layer 2 with a tiling of pattern ϕ_3 (refer to Figure 1.9 for an illustration).

Lastly, we cover Layer 3 with a tiling of pattern ϕ_1 (refer to Figure 1.7 for an illustration). Proceeding as we had with the Central Layer, Layer 1, 2 and 3, we cover all the remaining layers above the Central Layer using a tiling of the ordered sequence of patterns ϕ_2, ϕ_{11}, ϕ_3 and ϕ_1. We cover all the layers below the Central Layer (and in the first instance including the Central Layer) with the ordered sequence of patterns ϕ_2, ϕ_1, ϕ_3 and ϕ_{11}, completing the construction of $\mathbb{Z}^3_{3_b}$. It is clear that $\mathbb{Z}^3_{3_b}$ is connected.

Lastly, and for ease of reference, consider Figure 1.22 and Figure 1.23 which are representative of the distances required to travel between arbitrary vertices located either on layers tiled with patterns ϕ_1, ϕ_2, ϕ_3 or ϕ_{11}.

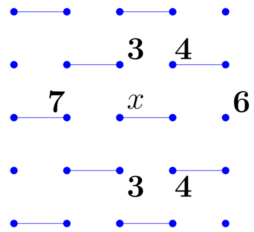

Figure 1.22: An illustration of the distances travelled from some vertex x to those vertices located about x on a layer tiled with pattern ϕ_1 or pattern ϕ_{11} .

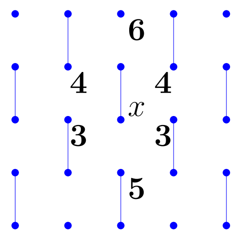

Figure 1.23: An illustration of the distances travelled from some vertex x to those vertices located about x on a layer tiled with pattern ϕ_2 or pattern ϕ_3

Is $\mathbb{Z}^3_{3b}$ finitely packing colourable?: Consider colour pattern ϕ_{12} on a subset of vertices of the Central Layer as depicted in Figure 1.24. We cover every vertex of the Central Layer by making use of a tiling of colour pattern ϕ_{12}.

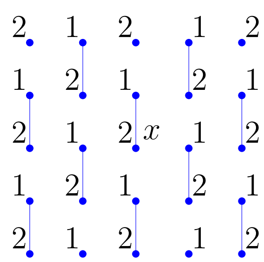

Figure 1.24: A representative tile of colour pattern ϕ_{12}.

Next, consider colour pattern ϕ_{13} on a subset of vertices of Layer 1 as depicted in Figure 1.25. We cover every vertex of Layer 1 by making use of a tiling of colour pattern ϕ_{13}.

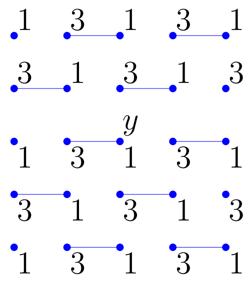

Figure 1.25: A representative tile of colour pattern ϕ_{13}

Next, consider colour pattern ϕ_{14} on a subset of vertices of Layer 2 as depicted in Figure 1.26. We cover every vertex of Layer 2 by making use of a tiling of colour pattern ϕ_{14}.

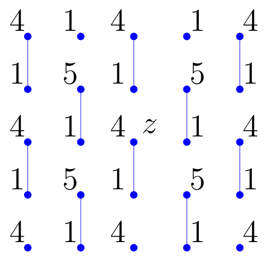

Figure 1.26: A representative tile of colour pattern ϕ_{14}.

We cover every vertex of Layer 3 with a tiling of colour pattern ϕ_4 (see Figure 1.14 for reference, noting that in this case the central vertex is vertex h). Next, consider colour pattern ϕ_{15} on a subset of vertices of Layer 4 as depicted in Figure 1.27. We cover every vertex of Layer 4 by making use of a tiling of colour pattern ϕ_{15}.

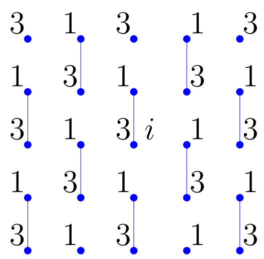

Figure 1.27: A representative tile of colour pattern ϕ_{15}.

We denote the ordered sequence of patterns (ϕ_{12}, ϕ_{13}, ϕ_{14}, ϕ_4 and ϕ_{15}) by γ.

Next, consider pattern ϕ_{16} on a subset of vertices of Layer 5 as depicted in Figure 1.28.

We cover every vertex of Layer 5 by making use of a tiling of colour pattern ϕ_{16}.

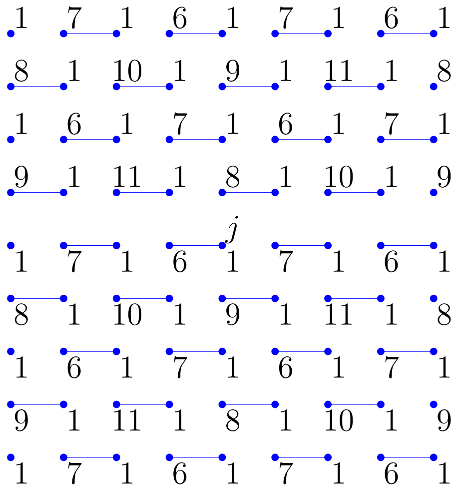

Figure 1.28: A representative tile of colour pattern ϕ_{16}

We cover every vertex of Layers 6 through 10 using ordered sequence γ. For visual reference, please see Figures 1.14, 1.24, 1.25, 1.26 and 1.27, which correspond to each pattern respectively. Next, consider colour pattern ϕ_{17} on a subset of vertices of Layer 11 as depicted in Figure 1.29. We cover every vertex of Layer 11 by making use of a tiling of colour pattern ϕ_{17}.

Figure 1.29: A representative tile of colour pattern ϕ_{17}

Proceeding as we had with Layers 6 through 10, we cover every vertex of Layers 12 through 16 and 18 through 22 utilising γ. We cover every vertex of Layer 17 using a tiling of pattern ϕ_{16}.

For practical reasons, we will need to represent the colour pattern for Layer 23 in a summarised manner. Consider firstly Table 1.1 as an example of a summarised representation of pattern ϕ_{16}, noting that the trivial usage of colours 1, 6 and 7 have been omitted.

8	10	9	11	8
9	11	8	10	9
8	10	9	11	8
9	11	8	10	9

Table 1.1: A summarized representation of colour pattern ϕ_{16}.

Next, consider the summarized representation of colour pattern ϕ_{18} on a subset of vertices of layer 23 as depicted in Table 1.2. We cover every vertex of Layer 23 by making use of a tiling of colour pattern ϕ_{18}.

12	29	14	25	12	40	14	29	12	25	14	41
13	22	15	26	13	24	15	22	13	26	15	24
14	23	12	27	14	34	12	23	14	27	12	34
15	28	13	31	15	35	13	28	15	31	13	35
12	25	14	32	12	36	14	25	12	32	14	37
13	24	15	22	13	26	15	24	13	22	15	26
14	27	12	23	14	38	12	27	14	23	12	42
15	30	13	33	15	39	13	30	15	33	13	43

Table 1.2: A summarized representation of colour pattern ϕ_{18}.

We again employ γ to cover every vertex of Layers 24 through 28, 30 through 34, 36 through 40 and 42 through 46. Layers 29 and 41 are covered with tilings of pattern ϕ_{16} and Layer 35 is covered with a tiling of pattern ϕ_{17}.

Once we are able to colour Layer 47 we may then utilise the ordered sequence of patterns (from the Central Layer up to and including Layer 47) to cover every layer of $\mathbb{Z}^3_{3b}$. Unfortunately, our current analysis has failed to yield a feasible repeatable colour pattern to this end, leading to the following open question.

Open Question 1. *Is it true that* $\chi_\rho(\mathbb{Z}^3_{3b}) < p$ *for some* $p \in \mathbb{Z}^+$*?*

Open Question 2. *Are* $\mathbb{Z}^3_{3a}$ *and* $\mathbb{Z}^3_{3b}$*, respectively, minimal and maximal three regular connected spanning subgraphs of* $\mathbb{Z}^3$*?*

Open Question 3. *Is it possible to find* $a, b \in \mathbb{Z}^+$ *so that* $a \leq \chi_\rho(\mathbb{Z}^3_3) \leq b$*, where* $\mathbb{Z}^3_3$ *is an arbitrary three regular connected spanning subgraph of* $\mathbb{Z}^3$*? Also, is* $a = 23$*?*

To conclude this chapter, notice that the proportion of edges of an arbitrary three regular connected spanning subgraph of $\mathbb{Z}^3$ relative to $\mathbb{Z}^3$ is $\frac{1}{2}$.

Chapter 2

Four and five regular connected spanning subgraphs of $\mathbb{Z}^3$

2.1 Preamble

Let $\mathbb{Z}_i^3$ be an i-regular connected spanning subgraph of $\mathbb{Z}^3$, where $i \in \{4, 5\}$. For the cases $\mathbb{Z}_4^3$ and $\mathbb{Z}_5^3$, our intuition suggests that χ_ρ is infinite. In order to show this, we will utilise the technique introduced by Finbow and Rall in [14], which requires estimating the order and size of the induced subgraph of $N_i(x)$, $i \in \mathbb{Z}^+$, where $x \in V(\mathbb{Z}_4^3)$ or $x \in V(\mathbb{Z}_5^3)$ respectively, and $\mathbb{Z}_4^3$ or $\mathbb{Z}_5^3$ arbitrary (respectively).

Generation of Subgraphs and Estimation of Properties: We begin by exploring the feasibility of generating all possible subgraphs of the induced subgraph of $N_k(x)$, where $k \in \{1, 2, .., 5\}$, and $x \in V(\mathbb{Z}^3)$, after which we can then filter the generated subgraphs to identify those that are either four or five regular, which are relevant to our study. These specific subgraphs will then serve as the foundation for applying the least squares method to estimate the order and size of the induced subgraph of $N_i(x)$, $i \in \mathbb{Z}^+$.

It is crucial to note that (in general) for any graph G with size m, the number of spanning subgraphs is 2^m. Define G_i as the induced subgraph of $N_i(x)$ with associated size m_i, s_i as the number of spanning subgraphs of G_i, c_i as the computational time (in days) to generate s_i, and d_i as the size (in gigabytes) on disk of s_i. From Lemma 10 in [14], we have $m_i = 4i^3 + 2i$, which implies that $s_i = 2^{4i^3+2i}$.

distance i	m_i	s_i	c_i	d_i
1	6	2^{6}	$\approx \frac{1}{86400}$	$\approx \frac{4}{10^6}$
2	36	2^{36}	$\approx \frac{143165576533333}{10^{13}}$	$\approx \frac{3607772}{1000}$
3	114	2^{114}	$\approx \frac{18938500465}{159279}$	$\approx \frac{1864345029634599}{895813}$
4	264	2^{264}	$\approx \frac{35259911806526}{413779}$	$\approx \frac{4193497528590885}{128}$
5	510	2^{510}	$\approx \frac{7744408264633035}{524288}$	$\approx 64024301125204784$

Table 2.1: A summary of the observations and approximations for each induced subgraph G_i, size m_i, number of spanning subgraphs s_i, computational time c_i in days, and size on disk d_i in gigabytes.

The values of c_1, c_2, d_1, and d_2 were obtained by executing the algorithms represented in Python script ([8]) on a MacBook Pro (Processor: 1.4 GHz Quad-Core Intel Core i5, Memory: 8 GB 2133 MHz LPDDR3). The algorithms utilised are provided below for reference.

Algorithm 1 Main Program

```
1: Set graph, start_vertex, distance
2: max_subgraphs_per_file ← k
3: output_directory ← path
4: subgraph ← FIND_SUBGRAPH(graph, start_vertex, distance)
5: GEN_ALL_SUBGRAPHS(subgraph, max_subgraphs_per_file,output_directory)
```

Algorithm 2 Function to find subgraph

```
function FIND_SUBGRAPH(graph, start_vertex, distance)
    Initialize visited as an empty list, current_level as the start_vertex
    # visited stores all unique vertices in the subgraph.
    for i ← 0 to distance − 1 do
        Initialize next_level as an empty list
        for each vertex in current_level do
            Fetch neighbours of vertex from the graph
            for each neighbour in neighbours do
                if neighbour is not in visited then
                    Add neighbour to next_level
                    Add neighbour to visited
                end if
            end for
        end for
        current_level ← next_level
    end for
    return visited
end function
```

Algorithm 3 Function to generate all subgraphs

```
function GEN_ALL_SUBGRAPHS(graph, max_subgraphs_per_file, output_dir)
    Initialize edges as a set containing all unique edges in graph
    Initialize file_count to 1, subgraph_count to 0, and total_count to 0
    for each possible edge_count from 0 to the total number of edges do
        for sub_edges in combination(edges, edge_count) do
            Write sub_edges to the open file
            Increment subgraph_count and total_count
            if subgraph_count equals max_subgraphs_per_file then
                Close the current file
                Increment file_count
                Reset subgraph_count to 0
            end if
        end for
    end for
    Close the final file
end function
```

For practical considerations, we estimated c_3, c_4, c_5, d_3, d_4, and d_5 using an exponential function fitted by linear regression on the logarithmically transformed data. Despite potential improvements with more powerful computing resources, we surmise that the conclusions regarding the infeasibility of generating all possible subgraphs of the induced subgraph of $N_k(x)$ would remain consistent.

Sampling Approach: Exhaustively generating all possible subgraphs provides a complete view of a graph's structural properties but becomes computationally infeasible for large graphs. To address this, we adopt a sampling approach that selects a representative subset of subgraphs, maintaining key structural properties while significantly reducing computational complexity.

Our approach offers a balanced trade-off between computational efficiency and depth of exploration without the exhaustive (and infeasible) enumeration of all subgraphs.

Algorithm 4 Sampling Approach to Generate Subgraphs

```
function GenerateSubgraphs(graph, no_subgr, degree)
    Initialize sampled_subgraphs as an empty list
    for i ← 1 to no_subgr do
        Initialize subgraph_sample
        for each vertex in graph do
            Determine all configurations of degree edges around vertex
            Randomly select one of these configurations
            Update subgraph_sample with the selected configuration
        end for
        Append subgraph_sample to sampled_subgraphs
    end for
    return sampled_subgraphs
end function
```

Algorithm 5 Extracting a Neighbourhood Subgraph

```
function ValidSubgraph(graph, start_vertex, dist)
    Initialize: step ← start_vertex, step_tmp as an empty lst
    Initialize: valid_graph as an empty list, degree to 0
    for i = 0 to dist − 1 do
        for all j in step do
            discard ← True
            checklist ← getNeighbours(graph, j)
            if i = 0 then
                degree ← length(checklist)
            end if
            if length(checklist) ≠ degree then
                valid_graph ← empty
                break
            end if
            for all k in checklist do
                if k ∉ V(valid_graph) then
                    discard ← False
                end if
            end for
            if discard then
                valid_graph ← empty
                break
            end if
            Append {j, getNeighbours(graph, j)} to valid_graph
            Append {getNeighbours(graph, j)} to step_tmp
        end for
        step ← step_tmp
    end for
    return valid_graph
end function
```

Our sampling approach tests directly for connectedness. However, the question of whether these induced subgraphs are spanning remains more ambiguous. In our investigation of specific (observable) i-regular, $i \in \{1, 2, 3, 4, 5\}$ connected spanning subgraphs of $\mathbb{Z}^3$, we noticed, considering some $N_k(x)$, $k \in \mathbb{Z}^+$, $x \in V(\mathbb{Z}^3)$, that it is necessary to extend the closed neighbourhood $N_k(x)$ well beyond k to ensure that every vertex of $N_k(x)$ is indeed included in the vertex set of the resulting i-regular connected (spanning) subgraph of $\mathbb{Z}^3$. Recognizing the challenge of demonstrating this directly, we opted for an indirect and more efficient approach. Let $\mathcal{F}$ be a set of subgraphs of a graph G. A subgraph $X \in \mathcal{F}$ is said to be a *maximal subgraph* of $\mathcal{F}$ if there exists no $H \in \mathcal{F}$ such that $X \subset H$ and $X \neq H$.

Let $\mathcal{A}$ be a set of k-regular, $k \in \mathbb{Z}^+$, connected subgraphs of the induced subgraph of $N_M(x)$, where $x \in V(\mathbb{Z}^3)$ and $M \in \mathbb{Z}^+$ is sufficiently large. Let $\mathbb{Z}^3_{k'}$ be a k-regular connected maximal subgraph of $\mathcal{A}$. It is clear that $\mathbb{Z}^3_{k'}$ is also a subgraph of $\mathbb{Z}^3$.

We will demonstrate that for arbitrary four and five regular connected maximal subgraphs of $\mathbb{Z}^3$, denoted by $\mathbb{Z}^3_{4'}$ and $\mathbb{Z}^3_{5'}$ respectively, that $\chi_\rho(\mathbb{Z}^3_{4'})$ and $\chi_\rho(\mathbb{Z}^3_{5'})$ are infinite. It will then follow directly that $\chi_\rho(\mathbb{Z}^3_4)$ and $\chi_\rho(\mathbb{Z}^3_5)$ are also infinite. Using our sampling approach, we obtain the following approximations for the maximum order, o_m, and the maximum size s_m of an arbitrary four or five regular connected subgraph of an induced subgraph of $N_k(x)$, $x \in V(\mathbb{Z}^3)$.

distance i	No. of generated subgraphs	No. of connected subgraphs	% Connected subgraphs	o_m	s_m
1	N/A	N/A	N/A	5	4
2	600 000	600 000	100%	17	16
3	1 196 000	968 354	80.97%	49	52
4	1 117 000	201 234	18.02%	105	144
5	1 163 000	7 034	0.60%	185	276

Table 2.2: Four regular subgraphs of the induced subgraph of $N_k(x)$, $x \in V(\mathbb{Z}^3)$.

From Table 2.2, we approximate the order and size of the induced subgraph $N_i(x)$, $x \in V(\mathbb{Z}^3_{4'})$. We do this by making use of least squares regression applied to the values of o_m and s_m for known distances i.

For every $i \in \mathbb{Z}^+$

$$|N_i(x)| \approx a_1 i^2 + a_2 i + a_3, \tag{2.1}$$

where $a_1 = \frac{80}{7}$, $a_2 = \frac{-832}{35}$, $a_3 = \frac{89}{5}$ and $x \in V(\mathbb{Z}^3_{4'})$.

Similarly, the size of the induced subgraph $N_i(x)$ is approximately equal to

$$b_1 i^2 + b_2 i + b_3, \tag{2.2}$$

where $b_1 = \frac{148}{7}$, $b_2 = \frac{-2088}{35}$ and $b_3 = \frac{224}{5}$.

distance i	No. of generated subgraphs	No. of connected subgraphs	% Connected subgraphs	o_m	s_m
1	N/A	N/A	N/A	6	5
2	1 000 000	1 000 000	100%	23	25
3	1 000 000	648 535	64.85%	61	90
4	1 000 000	38 514	3.85%	127	215
5	1 000 000	145	0.01%	218	405

Table 2.3: Five regular subgraphs of the induced subgraph of $N_k(x)$, $x \in V(\mathbb{Z}^3)$

Furthermore, from Table 2.3, we again approximate the order and size of the induced subgraph $N_i(x)$, $x \in V(\mathbb{Z}^3_{5'})$ by making use of least squares regression applied to the values of o_m and s_m for known distances i.

For every $i \in \mathbb{Z}^+$

$$|N_i(x)| \approx a_1 i^2 + a_2 i + a_3, \tag{2.3}$$

where $a_1 = \frac{88}{7}$, $a_2 = -\frac{792}{35}$ and $a_3 = \frac{83}{5}$, $x \in V(\mathbb{Z}^3_{5'})$.

Finally, the size of the induced subgraph $N_i(x)$ is approximately equal to

$$b_1 i^2 + b_2 i + b_3, \tag{2.4}$$

where $b_1 = \frac{200}{7}$, $b_2 = -\frac{507}{7}$ and $b_3 = 51$.

Since these bounds were obtained from sampled data, extrapolation to larger i requires validation. However, the observed quadratic trend aligns with expectations from combinatorial analysis, supporting its use as an asymptotic estimate.

2.2 Four regular connected spanning subgraphs of $\mathbb{Z}^3$

It follows from Equation 2.1, Section 2.1 that for every $i \in \mathbb{Z}^+$

$$\begin{aligned} |N_i(x)| &\approx a_1 i^2 + a_2 i + a_3 \\ &= \frac{400i^2 - 832i + 623}{35}, \end{aligned}$$

where $a_1 = \frac{80}{7}$, $a_2 = \frac{-832}{35}$, $a_3 = \frac{89}{5}$ and $x \in V(\mathbb{Z}^3_{4'})$.

In addition, from Equation 2.2, the size of the induced subgraph $N_i(x)$ is approximately equal to

$$b_1 i^2 + b_2 i + b_3 = \frac{740i^2 - 2088i + 1568}{35},$$

where $b_1 = \frac{148}{7}$, $b_2 = \frac{-2088}{35}$ and $b_3 = \frac{224}{5}$.

Lemma 2.2.1. *Let $x \in V(\mathbb{Z}^3_{4'})$, $r \in \mathbb{Z}^+$ and $\epsilon \in \mathbb{R}$ be positive. Then there exists an $M_r \in \mathbb{Z}^+$ such that whenever $M \geq M_r$, $M \in \mathbb{Z}^+$, if $A \subseteq V(N_M(x))$ has the property that $\{N_r(a) : a \in A\}$ is a collection of pairwise disjoint subsets of $V(\mathbb{Z}^3_{4'})$, then*

$$\frac{|A|}{|N_M(x)|} < \frac{35}{400r^2 - 832r + 623} + \frac{\epsilon}{2^{r+1}}.$$

Proof. Let $r, k \in \mathbb{Z}^+$ with $A \subseteq V(N_k(x))$ such that $\{N_r(a) : a \in A\}$ is a collection of pairwise disjoint subsets of $V(\mathbb{Z}^3_{4'})$. Consider Figure 2.4 as a visual representation of our selection of r, k, $N_k(x)$ and A.

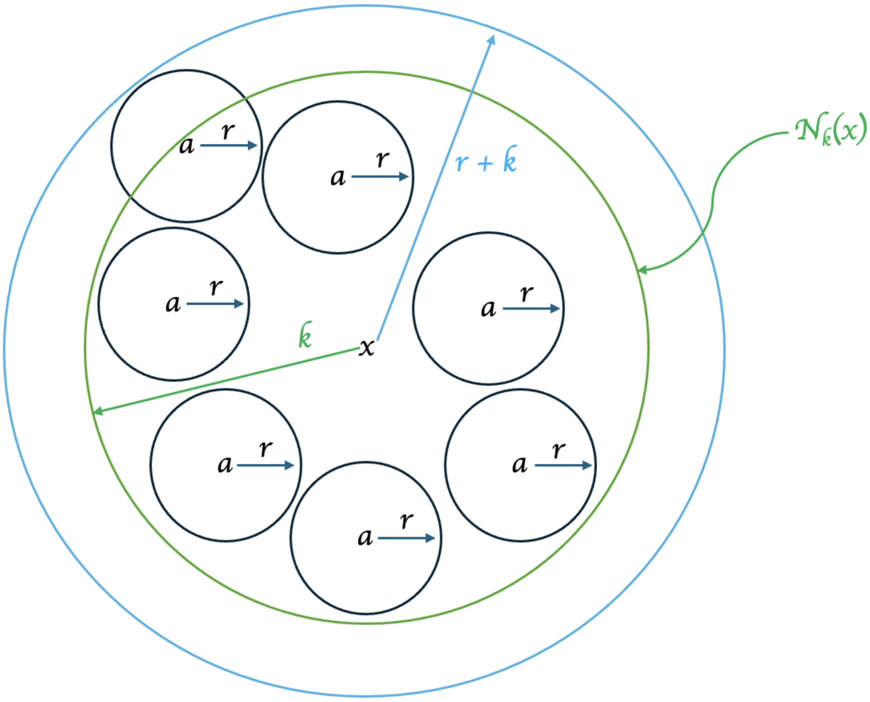

Table 2.4: A collection of pairwise disjoint subsets of $V(\mathbb{Z}^3_{4'})$.

Observe that

$$\bigcup_{a\in A} N_r(a) \subseteq N_{r+k}(x).$$

We obtain then that

$$|\bigcup_{a\in A} N_r(a)| \leq |N_{r+k}(x)|.$$

It follows that

$$|A| \cdot \frac{400r^2 - 832r + 623}{35} \leq \frac{400(r+k)^2 - 832(r+k) + 623}{35}.$$

Hence

$$|A| \leq \frac{400(r+k)^2 - 832(r+k) + 623}{400r^2 - 832r + 623}.$$

Dividing both sides of the inequality by $|N_k(x)|$, we obtain that

$$\frac{|A|}{|N_k(x)|} \leq \frac{35(400(r+k)^2 - 832(r+k) + 623)}{(400k^2 - 832k + 623)(400r^2 - 832r + 623)}.$$

Now, taking the limit of the right-hand side of the inequality as $k \to \infty$, we obtain that

$$\begin{aligned}
&\lim_{k\to\infty} \frac{35(400(r+k)^2 - 832(r+k) + 623)}{(400k^2 - 832k + 623)(400r^2 - 832r + 623)} \\
&= \frac{35}{400r^2 - 832r + 623} \cdot \lim_{k\to\infty} \frac{400(r+k)^2 - 832(r+k) + 623}{400k^2 - 832k + 623}, \ r \text{ fixed} \\
&= \frac{35}{400r^2 - 832r + 623} \cdot 1.
\end{aligned}$$

It follows then that there exists and $M_r \in \mathbb{Z}^+$ such that whenever $M \geq M_r$ with $M \in \mathbb{Z}^+$, then

$$\frac{|A|}{|N_M(x)|} < \frac{35}{400r^2 - 832r + 623} + \frac{\epsilon}{2^{r+1}},$$

as desired. $\square$

Lemma 2.2.2. *Let $x \in V(\mathbb{Z}^3_{4'})$, $r \in \mathbb{Z}^+$ and $\epsilon \in \mathbb{R}$ be positive. Then there exists an $M_r \in \mathbb{Z}^+$ such that whenever $M \geq M_r$, $M \in \mathbb{Z}^+$, if $A \subseteq V(N_M(x))$ has the property that $\{E(N_r(a)) : a \in A\}$ is a collection of pairwise disjoint subsets of $E(\mathbb{Z}^3_{4'})$, then*

$$\frac{|A|}{|N_M(x)|} < \frac{35}{\frac{20}{37} \cdot (740r^2 - 2088r + 1568)} + \frac{\epsilon}{2^{r+1}}.$$

Proof. Let $r, k \in \mathbb{Z}^+$ with $A \subseteq V(N_k(x))$ such that $\{E(N_r(a)) : a \in A\}$ is a collection of pairwise disjoint subsets of $E(\mathbb{Z}^3_{4'})$. Observe that

$$\bigcup_{a\in A} E(N_r(a)) \subseteq E(N_{r+k}(x)).$$

We obtain then that

$$\Big|\bigcup_{a\in A} E(N_r(a))\Big| \leq |E(N_{r+k}(x))|.$$

It follows that

$$|A| \cdot \frac{740r^2 - 2088r + 1568}{35} \leq \frac{740(r+k)^2 - 2088(r+k) + 1568}{35}.$$

Hence

$$|A| \leq \frac{740(r+k)^2 - 2088(r+k) + 1568}{740r^2 - 2088r + 1568}.$$

Dividing both sides of the inequality by $|N_k(x)|$, we obtain that

$$\begin{aligned}\frac{|A|}{|N_k(x)|} &\leq \frac{35(740(r+k)^2-2088(r+k)+1568)}{(740r^2-2088r+1568)(400k^2-832k+623)}\\ &= \frac{\frac{20}{37}\cdot 35(740(r+k)^2-2088(r+k)+1568)}{\frac{20}{37}\cdot(740r^2-2088r+1568)(400k^2-832k+623)}.\end{aligned}$$

Now, taking the limit of the right-hand side of the inequality as $k \to \infty$, we obtain that

$$\begin{aligned}&\lim_{k\to\infty}\frac{\frac{20}{37}\cdot 35(740(r+k)^2-2088(r+k)+1568)}{\frac{20}{37}\cdot(740r^2-2088r+1568)(400k^2-832k+623)}\\ &= \frac{35}{\frac{20}{37}\cdot(740r^2-2088r+1568)}\cdot\lim_{k\to\infty}\frac{400(r+k)^2-\frac{20}{37}\cdot 2088(r+k)+\frac{20}{37}\cdot 1568}{400k^2-832k+623},\ r \text{ fixed}\\ &= \frac{35}{\frac{20}{37}\cdot(740r^2-2088r+1568)}\cdot 1.\end{aligned}$$

It follows then that there exists and $M_r \in \mathbb{Z}^+$ such that whenever $M \geq M_r$ with $M \in \mathbb{Z}^+$, then

$$\frac{|A|}{|N_M(x)|} < \frac{35}{\frac{20}{37}\cdot(740r^2-2088r+1568)} + \frac{\epsilon}{2^{r+1}},$$

as desired. $\square$

Theorem 2.2.3. *The packing chromatic number of $\mathbb{Z}^3_{4'}$ is infinite.*

Proof. We fix an arbitrary $x \in V(\mathbb{Z}^3_{4'})$. Suppose to the contrary that $\chi_\rho(\mathbb{Z}^3_{4'}) \leq p$, $p \in \mathbb{Z}^+$. We select $k \in \mathbb{Z}^+$ such that $p \leq 2k$. Since $\chi_\rho(\mathbb{Z}^3_{4'}) \leq p$, it follows that there exists a packing colouring $f : V(\mathbb{Z}^3_{4'}) \to 1, 2, ..., p$ with corresponding i-packings $A_i, 1 \leq i \leq p, i \in \mathbb{Z}^+$. We define $I = \{1, 2, ..., 2k\}$. It is easily observed that for any even integer $q \in I$, it follows that $N_{r_q}(y) \cap N_{r_q}(z) = \emptyset$, where $y, z \in A_q, y \neq z$ and $r_q = \frac{q}{2}$.

It follows from Lemma 2.2.1 that there exists $M_q \in \mathbb{Z}^+$ such that whenever $M \geq M_q$, with $M \in \mathbb{Z}^+$, then

$$\frac{|A_q \cap N_M(x)|}{|N_M(x)|} < \frac{35}{400r_q^2-832r_q+623} + \frac{\epsilon}{2^{r_q+1}}. \tag{2.1}$$

Furthermore, for any odd integer $j \in I$, it follows by a known result (see Lemma 9 in

[14]) that $E(N_{r_j}(y)) \cap E(N_{r_j}(z)) = \emptyset$, where $y, z \in A_j, y \neq z, r_j = \frac{j+1}{2}$. We obtain then from Lemma 2.2.2 that

$$\frac{|A_j \cap N_M(x)|}{|N_M(x)|} < \frac{35}{\frac{20}{37} \cdot (740r_j^2 - 2088r_j + 1568)} + \frac{\epsilon}{2^{r_j+1}}. \tag{2.2}$$

We set $\epsilon = \frac{1}{200}$ and $K = \max_{i \in I}\{M_i\}$. Since by our assumption $\chi_\rho(\mathbb{Z}^3_{4'}) \leq p$, we deduce that the proportion ρ of vertices of $N_K(x)$ which are covered by f is exactly 1. On the other hand, the explicit calculation of ρ is given by

$$\begin{aligned}
\rho &= \frac{\sum\limits_{i=1}^{2k} |A_i \cap N_K(x)|}{|N_K(x)|} \\
&= \frac{\sum\limits_{i=1}^{k} |A_{2i} \cap N_K(x)|}{|N_K(x)|} + \frac{\sum\limits_{i=1}^{k} |A_{2i-1} \cap N_K(x)|}{|N_K(x)|}.
\end{aligned}$$

It follows from Inequalities 2.1 and 2.2 that

$$\begin{aligned}
&\frac{\sum\limits_{i=1}^{k} |A_{2i} \cap N_K(x)|}{|N_K(x)|} + \frac{\sum\limits_{i=1}^{k} |A_{2i-1} \cap N_K(x)|}{|N_K(x)|} \\
&< \sum_{i=1}^{k} \left(\frac{35}{400i^2 - 832i + 623} + \frac{\epsilon}{2^{i+1}}\right) + \sum_{i=1}^{k} \left(\frac{35}{\frac{20}{37} \cdot (740i^2 - 2088i + 1568)} + \frac{\epsilon}{2^{i+1}}\right) \\
&< \sum_{i=1}^{2} \frac{35}{400i^2 - 832i + 623} + \sum_{i=1}^{2} \frac{35}{\frac{20}{37} \cdot (740i^2 - 2088i + 1568)} + 2\sum_{i=3}^{\infty} \frac{1000}{2950i^2} + 2\sum_{i=1}^{\infty} \frac{\epsilon}{2^{i+1}} \\
&\leq \frac{35}{191} + \frac{35}{559} + \frac{35}{\frac{20}{37} \cdot 220} + \frac{35}{\frac{20}{37} \cdot 352} + 2\sum_{i=3}^{10000} \frac{1000}{2950i^2} + \frac{2000}{2950} \int_{i=10000}^{\infty} \frac{1}{i^2} di + \epsilon \sum_{i=1}^{\infty} \frac{1}{2^i} \\
&= \frac{35}{191} + \frac{35}{559} + \frac{35}{\frac{20}{37} \cdot 220} + \frac{35}{\frac{20}{37} \cdot 352} + 0.2677 + \frac{1}{200} \\
&= 0.9968 < 1, \text{ a contradiction.}
\end{aligned}$$

□

The following result follows immediately from Theorem 2.2.3.

Theorem 2.2.4. *The packing chromatic number of $\mathbb{Z}^3_4$ is infinite.*

Notice that the proportion of edges of an arbitrary four regular connected spanning subgraph of $\mathbb{Z}^3$ relative to $\mathbb{Z}^3$ is $\frac{2}{3}$.

2.3 Five regular connected spanning subgraphs of $\mathbb{Z}^3$

It follows from Equation 2.3, Section 2.1, that for every $i \in \mathbb{Z}^+$

$$\begin{aligned} |N_i(x)| &\approx a_1 i^2 + a_2 i + a_3 \\ &= \frac{440i^2 - 792i + 581}{35}, \end{aligned}$$

where $a_1 = \frac{88}{7}$, $a_2 = -\frac{792}{35}$ and $a_3 = \frac{83}{5}$, $x \in V(\mathbb{Z}^3_{5'})$.

In addition, it follows from Equation 2.4 that the size of the induced subgraph of $N_i(x)$ is approximately equal to

$$b_1 i^2 + b_2 i + b_3 = \frac{200i^2 - 507i + 357}{7},$$

where $b_1 = \frac{200}{7}$, $b_2 = -\frac{507}{7}$ and $b_3 = 51$.

Lemma 2.3.1. *Let $x \in V(\mathbb{Z}^3_{5'})$, $r \in \mathbb{Z}^+$ and $\epsilon \in \mathbb{R}$ be positive. Then there exists an $M_r \in \mathbb{Z}^+$ such that whenever $M \geq M_r$, $M \in \mathbb{Z}^+$, if $A \subseteq V(N_M(x))$ has the property that $\{N_r(a) : a \in A\}$ is a collection of pairwise disjoint subsets of $V(\mathbb{Z}^3_{5'})$, then*

$$\frac{|A|}{|N_M(x)|} < \frac{35}{440r^2 - 792r + 581} + \frac{\epsilon}{2^{r+1}}.$$

Proof. Let $r, k \in \mathbb{Z}^+$ with $A \subseteq V(N_k(x))$ such that $\{N_r(a) : a \in A\}$ is a collection of pairwise disjoint subsets of $V(\mathbb{Z}^3_{5'})$. Observe that

$$\bigcup_{a \in A} N_r(a) \subseteq N_{r+k}(x).$$

We obtain then that

$$|\bigcup_{a \in A} N_r(a)| \leq |N_{r+k}(x)|.$$

It follows that

$$|A| \cdot \frac{440r^2 - 792r + 581}{35} \leq \frac{440(r+k)^2 - 792(r+k) + 581}{35}.$$

Hence

$$|A| \leq \frac{440(r+k)^2 - 792(r+k) + 581}{440r^2 - 792r + 581}.$$

Dividing both sides of the inequality by $|N_k(x)|$, we obtain that

$$\frac{|A|}{|N_k(x)|} \leq \frac{35(440(r+k)^2 - 792(r+k) + 581)}{(440r^2 - 792r + 581)(440k^2 - 792k + 581)}.$$

Now, taking the limit of the right-hand side of the inequality as $k \to \infty$, we obtain that

$$\begin{aligned}
&\lim_{k\to\infty} \frac{35(440(r+k)^2 - 792(r+k) + 581)}{(440r^2 - 792r + 581)(440k^2 - 792k + 581)} \\
&= \frac{35}{440r^2 - 792r + 581} \cdot \lim_{k\to\infty} \frac{440(r+k)^2 - 792(r+k) + 581}{440k^2 - 792k + 581}, \; r \text{ fixed} \\
&= \frac{35}{440r^2 - 792r + 581} \cdot 1.
\end{aligned}$$

It follows then that there exists an $M_r \in \mathbb{Z}^+$ such that whenever $M \geq M_r$ with $M \in \mathbb{Z}^+$, then

$$\frac{|A|}{|N_M(x)|} < \frac{35}{440r^2 - 792r + 581} + \frac{\epsilon}{2^{r+1}},$$

as desired. $\square$

Lemma 2.3.2. *Let $x \in V(\mathbb{Z}^3_{5'})$, $r \in \mathbb{Z}^+$ and $\epsilon \in \mathbb{R}$ be positive. Then there exists an $M_r \in \mathbb{Z}^+$ such that whenever $M \geq M_r$, $M \in \mathbb{Z}^+$, if $A \subseteq V(N_M(x))$ has the property that $\{E(N_r(a)) : a \in A\}$ is a collection of pairwise disjoint subsets of $E(\mathbb{Z}^3_{5'})$, then*

$$\frac{|A|}{|N_M(x)|} < \frac{35 \cdot \frac{5}{11}}{200r^2 - 507r + 357} + \frac{\epsilon}{2^{r+1}}.$$

Proof. Let $r, k \in \mathbb{Z}^+$ with $A \subseteq V(N_k(x))$ such that $\{E(N_r(a)) : a \in A\}$ is a collection of pairwise disjoint subsets of $E(\mathbb{Z}^3_{5'})$. Observe that

$$\bigcup_{a\in A} E(N_r(a)) \subseteq E(N_{r+k}(x)).$$

We obtain then that

$$|\bigcup_{a\in A} E(N_r(a))| \leq |E(N_{r+k}(x))|.$$

It follows that

$$|A| \cdot \frac{200r^2 - 507r + 357}{7} \leq \frac{200(r+k)^2 - 507(r+k) + 357}{7}.$$

Hence

$$|A| \leq \frac{200(r+k)^2 - 507(r+k) + 357}{200r^2 - 507r + 357}.$$

Dividing both sides of the inequality by $|N_k(x)|$, we obtain that

$$\begin{aligned}\frac{|A|}{|N_k(x)|} &\leq \frac{35(200(r+k)^2 - 507(r+k) + 357)}{(200r^2 - 507r + 357)(440k^2 - 792k + 581)} \\ &= \frac{35(200(r+k)^2 - 507(r+k) + 357)}{(200r^2 - 507r + 357)\frac{11}{5}(200k^2 - \frac{5}{11}792k + \frac{5}{11}581)}.\end{aligned}$$

Now, taking the limit of the right-hand side of the inequality as $k \to \infty$, we obtain that

$$\begin{aligned}&\lim_{k\to\infty} \frac{35(200(r+k)^2 - 507(r+k) + 357)}{(200r^2 - 507r + 357)\frac{11}{5}(200k^2 - \frac{5}{11}792k + \frac{5}{11}581)} \\ &= \frac{35 \cdot \frac{5}{11}}{200r^2 - 507r + 357} \cdot \lim_{k\to\infty} \frac{200(r+k)^2 - 507(r+k) + 357}{200k^2 - \frac{5}{11} \cdot 792k + \frac{5}{11} \cdot 581}, \ r \text{ fixed} \\ &= \frac{35 \cdot \frac{5}{11}}{200r^2 - 507r + 357} \cdot 1.\end{aligned}$$

It follows then that there exists an $M_r \in \mathbb{Z}^+$ such that whenever $M \geq M_r$ with $M \in \mathbb{Z}^+$, then

$$\frac{|A|}{|N_M(x)|} < \frac{35 \cdot \frac{5}{11}}{200r^2 - 507r + 357} + \frac{\epsilon}{2^{r+1}},$$

as desired. $\square$

Theorem 2.3.3. *The packing chromatic number of $\mathbb{Z}^3_{5'}$ is infinite.*

Proof. We fix an arbitrary $x \in V(\mathbb{Z}^3_{5'})$. Suppose to the contrary that $\chi_\rho(\mathbb{Z}^3_{5'}) \leq p$, $p \in \mathbb{Z}^+$. We select $k \in \mathbb{Z}^+$ such that $p \leq 2k$. Since $\chi_\rho(\mathbb{Z}^3_{5'}) \leq p$, it follows that there exists a packing colouring $f : V(\mathbb{Z}^3_{5'}) \to 1, 2, ..., p$ with corresponding i-packings $A_i, 1 \leq i \leq p, i \in \mathbb{Z}^+$. We define $I = \{1, 2, ..., 2k\}$. It is easily observed that for any even integer $q \in I$, it follows that $N_{r_q}(y) \cap N_{r_q}(z) = \emptyset$, where $y, z \in A_q, y \neq z$ and $r_q = \frac{q}{2}$.

It follows from Lemma 2.3.1 that there exists $M_q \in \mathbb{Z}^+$ such that whenever $M \geq M_q$, with $M \in \mathbb{Z}^+$, then

$$\frac{|A_q \cap N_M(x)|}{|N_M(x)|} < \frac{35}{440r_q^2 - 792r_q + 581} + \frac{\epsilon}{2^{r_q+1}}. \tag{2.1}$$

Furthermore, for any odd integer $j \in I$, it follows by a known result (see Lemma 9 in [14]) that $E(N_{r_j}(y)) \cap E(N_{r_j}(z)) = \emptyset$, where $y, z \in A_j, y \neq z, r_j = \frac{j+1}{2}$. We obtain then from Lemma 2.3.2 that

$$\frac{|A_j \cap N_M(x)|}{|N_M(x)|} < \frac{35 \cdot \frac{5}{11}}{200r_j^2 - 507r_j + 357} + \frac{\epsilon}{2^{r_j+1}}. \tag{2.2}$$

We set $\epsilon = \frac{1}{19}$ and $K = \max\limits_{i \in I}\{M_i\}$.

Since by our assumption $\chi_\rho(\mathbb{Z}^3_{5'}) \leq p$, we deduce that the proportion ρ of vertices of $N_K(x)$ which are covered by f is exactly 1. On the other hand, the explicit calculation of ρ is given by

$$\begin{aligned}
\rho &= \frac{\sum\limits_{i=1}^{2k} |A_i \cap N_K(x)|}{|N_K(x)|} \\
&= \frac{\sum\limits_{i=1}^{k} |A_{2i} \cap N_K(x)|}{|N_K(x)|} + \frac{\sum\limits_{i=1}^{k} |A_{2i-1} \cap N_K(x)|}{|N_K(x)|}.
\end{aligned}$$

It follows from Inequalities 2.1 and 2.2 that

$$\begin{aligned}
&\frac{\sum\limits_{i=1}^{k}|A_{2i}\cap N_K(x)|}{|N_K(x)|}+\frac{\sum\limits_{i=1}^{k}|A_{2i-1}\cap N_K(x)|}{|N_K(x)|}\\
&<\sum_{i=1}^{k}\left(\frac{35}{440i^2-792i+581}+\frac{\epsilon}{2^{i+1}}\right)+\sum_{i=1}^{k}\left(\frac{35\cdot\frac{5}{11}}{200i^2-507i+357}+\frac{\epsilon}{2^{i+1}}\right)\\
&<\sum_{i=1}^{2}\frac{35}{440i^2-792i+581}+\sum_{i=1}^{2}\frac{35\cdot\frac{5}{11}}{200i^2-507i+357}+2\sum_{i=3}^{\infty}\frac{1000}{2551i^2}+2\sum_{i=1}^{\infty}\frac{\epsilon}{2^{i+1}}\\
&\leq\frac{35}{229}+\frac{35}{757}+\frac{35\cdot\frac{5}{11}}{50}+\frac{35\cdot\frac{5}{11}}{143}+2\sum_{i=3}^{10000}\frac{1000}{2551i^2}+\frac{2000}{2551}\int_{i=10000}^{\infty}\frac{1}{i^2}+\epsilon\sum_{i=1}^{\infty}\frac{1}{2^i}\\
&=\frac{35}{229}+\frac{35}{757}+\frac{35\cdot\frac{5}{11}}{50}+\frac{35\cdot\frac{5}{11}}{143}+0.3096+\frac{1}{19}\\
&=0.9907<1,\ \text{a contradiction.}
\end{aligned}$$

□

The following result follows immediately from Theorem 2.3.3.

Theorem 2.3.4. *The packing chromatic number of $\mathbb{Z}_5^3$ is infinite.*

Notice that the proportion of edges of an arbitrary five regular connected spanning subgraph of $\mathbb{Z}^3$ relative to $\mathbb{Z}^3$ is $\frac{5}{6}$.

Chapter 3

δ-subgraphs of $\mathbb{Z}^3$

In Chapter 1, we investigated whether $\mathbb{Z}_3^3$ is finitely packing colourable, where $\mathbb{Z}_3^3$ denotes an arbitrary three regular connected spanning subgraph of $\mathbb{Z}^3$.

In Chapter 2, we demonstrated that $\mathbb{Z}_4^3$ is not finitely packing colourable (refer to Section 2.2, Theorem 2.2.4), where $\mathbb{Z}_4^3$ denotes an arbitrary four regular connected spanning subgraph of $\mathbb{Z}^3$.

The primary question addressed in this thesis is as follows: *What is the minimum proportion of edges that must be removed from $\mathbb{Z}^3$ to produce a connected spanning subgraph for which a finite packing colouring exists?*

To further refine our analysis, we focus on connected spanning subgraphs of $\mathbb{Z}^3$ where every vertex has degree 3 or 4. To formalize this, we define the following:

Definition 8. Let $\mathcal{F} = \{F \mid F$ is a connected spanning subgraph of $\mathbb{Z}^3$ with $3 \leq \deg(x) \leq 4\}$ for all $x \in V(F)$.
A graph H is called a *δ-subgraph* of $\mathbb{Z}^3$, denoted by H^δ, if $H \in \mathcal{F}$.

In Chapter 2, we noted that generating all possible subgraphs of the induced subgraph $N_k(x)$, $k \in \mathbb{Z}^+$ and $x \in V(\mathbb{Z}^3)$, is impractical for large k. Instead, we adopt an approach that selects meaningful δ-subgraphs of $\mathbb{Z}^3$ to address the primary question.

The graphs that we select should allow sufficient travel from an arbitrary vertex as to meet the requirement of being a connected spanning subgraph of $\mathbb{Z}^3$, however, the movement should also be sufficiently restrictive as to allow us to apply a finite packing colouring to the said subgraph. As the reader will find, this is an extremely challenging balance to strike.

We again make use of patterns ϕ_1, ϕ_2 and ϕ_3 as introduced in Section 1.2. We detail these patterns here again for ease of reference.

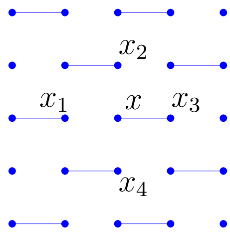

Figure 3.1: A representative tile of pattern ϕ_1

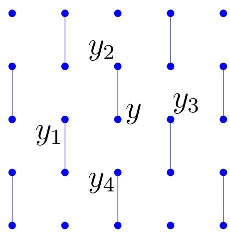

Figure 3.2: A representative tile of pattern ϕ_2.

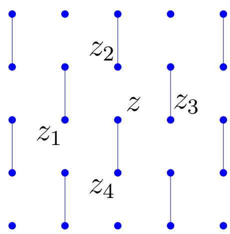

Figure 3.3: A representative tile of pattern ϕ_3.

In addition, we introduce the following patterns.

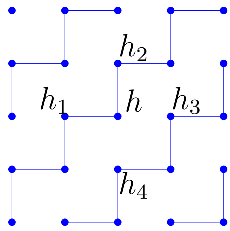

Figure 3.4: A representative tile of pattern α_1

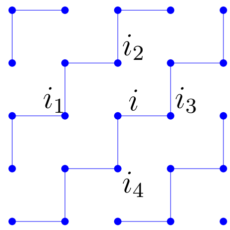

Figure 3.5: A representative tile of pattern α_2

We summarize the process of the construction and attempted colouring of specific δ-subgraphs of $\mathbb{Z}^3$ which guided our approach in constructing a finitely packing colourable δ-subgraph of $\mathbb{Z}^3$.

Let H be a graph. We define θ_H as the proportion of vertices of $V(H)$ having degree 4. For simplicty sake, we shall write $\theta_{H_a^\delta}$ as θ_a and so on.

Building on the algorithm used to construct $\mathbb{Z}^3_{3_a}$ and $\mathbb{Z}^3_{3_b}$, we begin by selecting an arbitrary vertex $x \in V(\mathbb{Z}^3)$ as an anchor. The naming conventions established in $\mathbb{Z}^3_{3_a}$ and $\mathbb{Z}^3_{3_b}$ are preserved:

i) Layers above the Central Layer are denoted as *Layer 1*, *Layer 2*, etc.

ii) Layers below the Central Layer are labeled *Layer 1′*, *Layer 2′*, and so forth.

The *Central Layer* contains the vertex x, with *Layer 1* containing the vertex y, and *Layer 1′* containing y'. Subsequent layers follow this pattern.

We also define a reference path (as we had in Section 1.2), $P = (m, l, k, j, \ldots, y, x, y', \ldots, m')$, which will be used in the construction of the δ-subgraphs discussed below.

The construction of H_a^δ: Select an arbitrary vertex $x \in V(\mathbb{Z}^3)$. Label its neighbours on the same layer as x by x_1, x_2, x_3, and x_4, as illustrated in Figure 3.6.

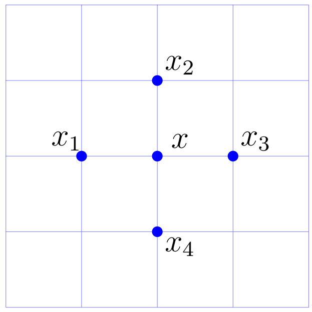

Figure 3.6: An arbitrary vertex x and its neighbours x_1, x_2, x_3, x_4 within $\mathbb{Z}^3$.

By aligning pattern ϕ_1 (from Figure 3.1) such that its vertices match those shown in Figure 3.6, we form a tiling of ϕ_1 around x. This tiling covers every vertex on x's layer (the Central Layer) with ϕ_1.

For the layer above the Central Layer (Layer 1), we create a tiling of pattern ϕ_2. We follow the same procedure used with ϕ_1, ensuring that x is incident with y, x_1 is incident with y_1, etc. Proceeding similarly, we apply the ordered sequence of patterns

$$\beta_1 = (\phi_2,\ \phi_3,\ \alpha_1,\ \phi_3,\ \phi_2,\ \alpha_2),$$

to Layers 1 through 6. We then repeat the tiling of β_1 for all subsequent layers above Layer 6.

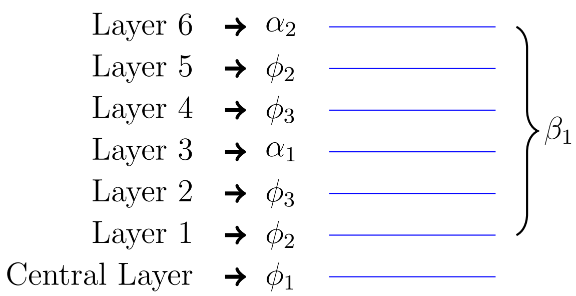

Figure 3.7: Side view illustration of selected layers of H_a^δ and their respective tiling patterns. The Central Layer is covered by ϕ_1, and each layer above (the Central Layer) is covered by the sequence of patterns in β_1.

Finally, we mirror this procedure for all layers below the Central Layer, applying the same tiling sequence as for the layers above. In this way, every layer of $\mathbb{Z}^3$ (except the Central Layer, which is covered by a tiling of ϕ_1) is covered by patterns chosen (sequentially) from β_1, completing the construction of H_a^δ.

We highlight the following observations:

i) Layers covered by tilings of patterns ϕ_1, α_1, and α_2 enable horizontal movement (left to right and vice versa).

ii) Layers covered by tilings of patterns ϕ_2, ϕ_3, α_1, and α_2 enable vertical movement (up and down).

iii) Every vertex belonging to a ϕ_1, ϕ_2 or ϕ_3 layer has degree 3.

iv) Every vertex belonging to a α_1 or α_2 layer has degree 4.

It follows that layers tiled with ϕ_1 restrict movement to horizontal directions, while layers tiled with ϕ_2 or ϕ_3 restrict movement to vertical directions. Although this may appear straightforward, these tiling constraints are crucial to limiting movement in our construction. At the same time, we preserve connectivity to ensure that H_a^δ remains a spanning subgraph of $\mathbb{Z}^3$.

Finally, note that layers tiled with patterns α_1 and α_2 allow diagonal movement from left to right but significantly limit diagonal movement from right to left. This selective restriction contributes to our goal of constraining traversal while maintaining the necessary connectivity. We observe that

$$\theta_a \approx \frac{1}{3}.$$

We proceed as follows in our attempt to apply a finite packing colouring to $V(H_a^\delta)$. All layers covered by tilings of ϕ_2 or ϕ_3 are coloured using only the colours 1, 2, and 3. Refer to Figure 3.8 for an illustration.

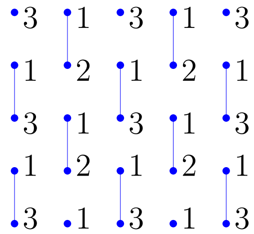

Figure 3.8: A representative colouring of those layers tiled with ϕ_2 or ϕ_3.

Because these layers represent approximately $\frac{2}{3}$ of $V(H_a^\delta)$, only the remaining (approximately) $\frac{1}{3}$ of the layers (those covered by tilings of ϕ_1, α_1, or α_2) remain to be coloured. Next, consider Layer 3. It can be coloured by using a tiling of pattern ϕ_{20}, depicted in Figure 3.9 below.

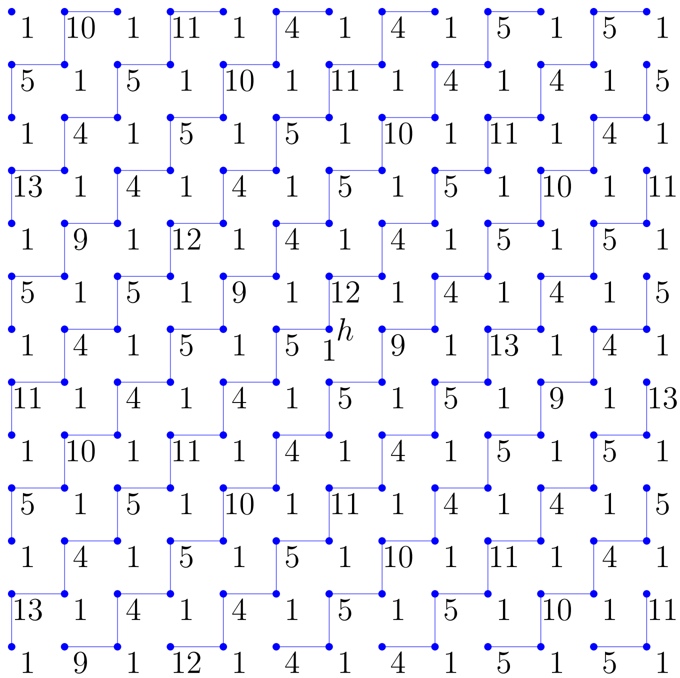

Figure 3.9: A representative tile of colour pattern ϕ_{20}

However, after colouring Layer 3, we observed that the number of colours required to colour subsequent uncoloured layers grows at an untenable rate (one may test the feasibility of colouring Layer 6 while minimizing the total number of colours used.) Consequently, we surmise that horizontal movement must be restricted even further to keep the colouring tractable. Notice that for H_a^δ, the average degree is

$$\overline{d} \;=\; \frac{2}{3} \times 3 \;+\; \frac{1}{3} \times 4 \;=\; \frac{10}{3} \;\approx\; 3.3333.$$

To find the *proportion* of edges in H_a^δ relative to those in $\mathbb{Z}^3$, we form the ratio

$$\frac{\text{average degree of a vertex in } H_a^\delta}{\text{average degree of a vertex in } \mathbb{Z}^3} \;=\; \frac{\overline{d}}{6}.$$

Plugging in $\overline{d} = \frac{10}{3}$ gives

$$\frac{\overline{d}}{6} \;=\; \frac{\frac{10}{3}}{6} \;=\; \frac{5}{9} \;\approx\; 0.5555.$$

The construction of $\boldsymbol{H_b^\delta}$: We follow a similar algorithm in the construction of H_b^δ as we had for H_a^δ. Consider now Figure 3.10 and Figure 3.11.

$$\left.\begin{array}{rcl}
\text{Layer } 6 & \rightarrow & \phi_2 \\
\text{Layer } 5 & \rightarrow & \phi_3 \\
\text{Layer } 4 & \rightarrow & \phi_2 \\
\text{Layer } 3 & \rightarrow & \phi_3 \\
\text{Layer } 2 & \rightarrow & \phi_2 \\
\text{Layer } 1 & \rightarrow & \phi_3 \\
\text{Central Layer} & \rightarrow & \alpha_1
\end{array}\right\}\beta_2$$

Figure 3.10: Side view illustration of selected layers of H_b^δ and their respective tiling patterns.

$$\left.\begin{array}{rcl}
\text{Layer } 13 & \rightarrow & \phi_3 \\
\text{Layer } 12 & \rightarrow & \phi_2 \\
\text{Layer } 11 & \rightarrow & \phi_3 \\
\text{Layer } 10 & \rightarrow & \phi_2 \\
\text{Layer } 9 & \rightarrow & \phi_3 \\
\text{Layer } 8 & \rightarrow & \phi_2 \\
\text{Layer } 7 & \rightarrow & \alpha_2
\end{array}\right\}\beta_3$$

Figure 3.11: Side view illustration of selected layers of H_b^δ and their respective tiling patterns.

We cover every layer above the Central Layer by applying β_2 and β_3 sequentially. All layers below the Central Layer is a mirror image of those layers above the Central Layer, completing the construction of H_b^δ. The reader will notice that horizontal movement has been severely restricted in H_b^δ as compared to H_a^δ (achieved only via those layers tiled with either α_1 or α_2). We observe that

$$\theta_b = \frac{1}{7}.$$

We proceed slightly differently in our attempt to apply a finite packing colouring to $V(H_b^\delta)$ as compared to $V(H_a^\delta)$. We start by targeting the α_1 and α_2 layers first before moving on to the ϕ_2 and ϕ_3 layers. We reserve colours 2 and 3 for ϕ_2 and ϕ_3 layers. Consider now Figure 3.12.

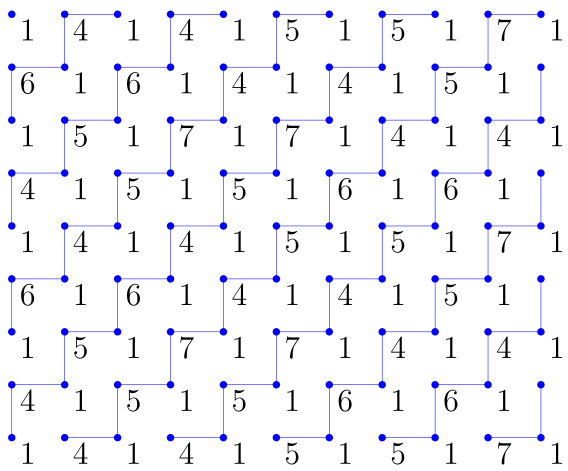

Figure 3.12: A representative tile of colour pattern ϕ_{21}

We cover the α_1 and α_2 layers by making use of tilings of pattern ϕ_{21}, representing $\frac{1}{7}$ of $V(H_b^\delta)$. Consider now Figure 3.13.

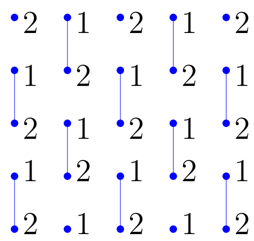

Figure 3.13: A representative tile of colour pattern ϕ_{22}.

We colour those Layers $T(i)$ and $T'(i)$, where

$$T(i) = \left\lfloor \frac{7i-1}{2} \right\rfloor = T'(i),$$

$i \in \mathbb{Z}^+$, with a tiling of colour pattern ϕ_{22}. Notice that

$$T(1) = 3, \quad T(2) = 6, \quad T(3) = 10, \quad T(4) = 13, \quad T(5) = 17, \quad T(6) = 20, \ldots \text{ and}$$

$$T'(1) = 3', \quad T'(2) = 6', \quad T'(3) = 10', \quad T'(4) = 13', \quad T'(5) = 17', \quad T'(6) = 20', \ldots$$

Next, consider Figure 3.14 and Figure 3.15.

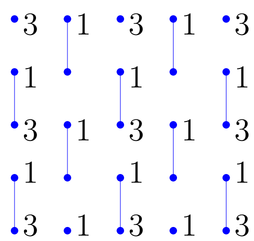

Figure 3.14: A representative tile of colour pattern ϕ_{23}.

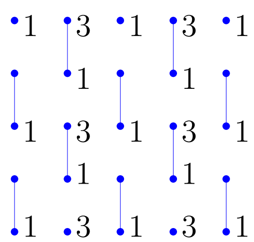

Figure 3.15: A representative tile of colour pattern ϕ_{24}.

We partially colour Layers $T(i)$ and $T'(i)$, where

$$T(i) = T'(i) = \begin{cases} 7\left\lfloor \frac{i}{4} \right\rfloor - 2, & \text{if } i \equiv 0 \pmod 4, \\ 7\left\lfloor \frac{i}{4} \right\rfloor + 1, & \text{if } i \equiv 1 \pmod 4, \\ 7\left\lfloor \frac{i}{4} \right\rfloor + 2, & \text{if } i \equiv 2 \pmod 4, \\ 7\left\lfloor \frac{i}{4} \right\rfloor + 4, & \text{if } i \equiv 3 \pmod 4, \end{cases}$$

$i \in \mathbb{Z}^+$, using tilings of patterns ϕ_{23} and ϕ_{24}. Above the Central Layer, we apply colour patterns ϕ_{23} and ϕ_{24} sequentially. Below the Central Layer, we reverse this order and apply ϕ_{24} followed by ϕ_{23}. Notice that

$$T(1) = 1, \quad T(2) = 2, \quad T(3) = 4, \quad T(4) = 5, \quad T(5) = 8, \quad T(6) = 9, \ldots \text{ and}$$

$$T'(1) = 1', \quad T'(2) = 2', \quad T'(3) = 4', \quad T'(4) = 5', \quad T'(5) = 8', \quad T'(6) = 9', \ldots$$

We proceed by focussing on completing the colouring of Layer 1. Consider Figure 3.16.

10	1	8	1	11	1	9	1	12	1	10
1	3	1	3	1	3	1	3	1	3	1
11	1	9	1	12	1	10	1	8	1	11
1	3	1	3	1	3	1	3	1	3	1
12	1	10	1	8	1	11	1	9	1	12
1	3	1	3	1	3	1	3	1	3	1
8	1	11	1	9	1	12	1	10	1	8
1	3	1	3	1	3	1	3	1	3	1
9	1	12	1	10	1	8	1	11	1	9
1	3	1	3	1	3	1	3	1	3	1
10	1	8	1	11	1	9	1	12	1	10

Figure 3.16: A representative tile of colour pattern ϕ_{25}.

Notice that colour pattern ϕ_{25} contains ϕ_{23} as a subpattern. We cover Layer 1 with a tiling of ϕ_{25}. Next, we consider Figure 3.17.

Figure 3.17: A representative tile of colour pattern ϕ_{26}.

Observe that the colour pattern ϕ_{26} contains ϕ_{24} as a subpattern. We cover Layer 2 with a tiling of ϕ_{26}. At this point, it becomes impractical to continue attempting a finite packing colouring of $V(H_b^\delta)$ (The reader may simply consider the feasibility of colouring Layers $1'$ or $2'$).

The key difficulty thus far arises from the relatively easy traversal within ϕ_2 and ϕ_3 layers when they lie adjacent to α_1 or α_2 layers. For example, Layers 1 and $1'$ adjoin the Central Layer, permitting movement that complicates efforts to restrict paths sufficiently for a finite packing colouring. Moreover, because every seventh layer is an α_1 or α_2 layer, the required number of colours escalates quickly when attempting to apply a packing colouring to $V(H_b^\delta)$.

We proceed now by introducing additional ϕ_2 and ϕ_3 layers between α_1 and α_2 layers in an attempt to arrest rapid escalation of colours required to apply a finite packing colouring to the graph in question. Notice that for H_b^δ, the average degree is

$$\overline{d} = \frac{6}{7} \times 3 + \frac{1}{7} \times 4 = \frac{22}{7} \approx 3.1429.$$

To find the *proportion* of edges in H_b^δ relative to those in $\mathbb{Z}^3$, we form the ratio

$$\frac{\text{average degree of a vertex in } H_b^\delta}{\text{average degree of a vertex in } \mathbb{Z}^3} = \frac{\overline{d}}{6}.$$

Plugging in $\overline{d} = \frac{22}{7}$ gives

$$\frac{\overline{d}}{6} = \frac{\frac{22}{7}}{6} = \frac{11}{21} \approx 0.5238.$$

The construction of $\boldsymbol{H_c^\delta}$: We follow a similar algorithm in the construction of H_c^δ as we had for H_b^δ. Consider now Figure 3.18 and Figure 3.19.

$$\left.\begin{array}{rcl}
\text{Layer 14} & \rightarrow & \phi_2 \\
\text{Layer 13} & \rightarrow & \phi_3 \\
\text{Layer 12} & \rightarrow & \phi_2 \\
\text{Layer 11} & \rightarrow & \phi_3 \\
\text{Layer 10} & \rightarrow & \phi_2 \\
\text{Layer 9} & \rightarrow & \phi_3 \\
\text{Layer 8} & \rightarrow & \phi_2 \\
\text{Layer 7} & \rightarrow & \phi_3 \\
\text{Layer 6} & \rightarrow & \phi_2 \\
\text{Layer 5} & \rightarrow & \phi_3 \\
\text{Layer 4} & \rightarrow & \phi_2 \\
\text{Layer 3} & \rightarrow & \phi_3 \\
\text{Layer 2} & \rightarrow & \phi_2 \\
\text{Layer 1} & \rightarrow & \phi_3 \\
\text{Central Layer} & \rightarrow & \alpha_1
\end{array}\right\}\beta_4$$

Figure 3.18: Side view illustration of selected layers of H_c^δ and their respective tiling patterns.

$$\left.\begin{array}{lcl}
\text{Layer 29} & \rightarrow & \phi_3 \\
\text{Layer 28} & \rightarrow & \phi_2 \\
\text{Layer 27} & \rightarrow & \phi_3 \\
\text{Layer 26} & \rightarrow & \phi_2 \\
\text{Layer 25} & \rightarrow & \phi_3 \\
\text{Layer 24} & \rightarrow & \phi_2 \\
\text{Layer 23} & \rightarrow & \phi_3 \\
\text{Layer 22} & \rightarrow & \phi_2 \\
\text{Layer 21} & \rightarrow & \phi_3 \\
\text{Layer 20} & \rightarrow & \phi_2 \\
\text{Layer 19} & \rightarrow & \phi_3 \\
\text{Layer 18} & \rightarrow & \phi_2 \\
\text{Layer 17} & \rightarrow & \phi_3 \\
\text{Layer 16} & \rightarrow & \phi_2 \\
\text{Layer 15} & \rightarrow & \alpha_2
\end{array}\right\}\beta_5$$

Figure 3.19: Side view illustration of selected layers of H_c^δ and their respective tiling patterns.

We cover every layer above the Central Layer by applying β_4 and β_5 sequentially. All layers below the Central Layer is a mirror image of those layers above the Central Layer, completing the construction of H_c^δ. Observe that horizontal movement has been further restricted in H_c^δ as compared to H_b^δ (achieved only via those layers tiled with either α_1 or α_2). We observe that

$$\theta_c = \frac{1}{15}.$$

We adjust our approach yet again in our attempt to apply a finite packing colouring to $V(H_c^\delta)$. This time we start by targeting the ϕ_2 and ϕ_3 layers. We partially colour Layers $T(i)$ and $T'(i)$, where

$$T(i) = \left\lfloor \frac{3i-1}{2} \right\rfloor = T'(i),$$

$i \in \mathbb{Z}^+$, using tilings of patterns ϕ_{23} and ϕ_{24}. Above the Central Layer, we apply colour patterns ϕ_{23} and ϕ_{24} sequentially. Below the Central Layer, we reverse this order and apply ϕ_{24} followed by ϕ_{23}. Notice that

$$T(1) = 1, \quad T(2) = 2, \quad T(3) = 4, \quad T(4) = 5, \quad T(5) = 7, \quad T(6) = 8, \ldots \text{ and}$$

$$T'(1) = 1', \quad T'(2) = 2', \quad T'(3) = 4', \quad T'(4) = 5', \quad T'(5) = 7', \quad T'(6) = 8', \ldots$$

We colour those Layers $T(i)$ and $T'(i)$, where

$$T(i) = 3i = T'(i), \quad i \in \mathbb{Z}^+, \quad 5 \nmid i,$$

using a tiling of colour pattern ϕ_{22}; see Figure 3.13 for reference.

Consider Figure 3.20.

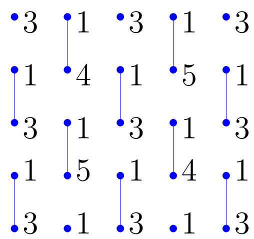

Figure 3.20: A representative tile of colour pattern ϕ_{27}.

Notice that colour pattern ϕ_{27} contains ϕ_{23} as a subpattern. We complete the colouring of those Layers $T(i)$, that is, the layers above the Central Layer, where

$$T(i) = 6i - 5,$$

$i \in \mathbb{Z}^+$, with a tiling of colour pattern ϕ_{27}. Similarly, we complete the colouring of those Layers $T'(i)$, that is, below the Central Layer, where

$$T'(i) = 6i - 1,$$

with a tiling of colour pattern ϕ_{27}.

Next, consider Figure 3.21.

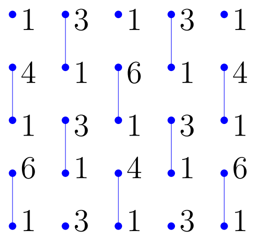

Figure 3.21: A representative tile of colour pattern ϕ_{28}.

Notice that colour pattern ϕ_{28} contains ϕ_{24} as a subpattern. We complete the colouring of those Layers $T(i)$, where

$$T(i) = 6i - 4,$$

$i \in \mathbb{Z}^+$, with a tiling of colour pattern ϕ_{28}. Similarly, we complete the colouring of those Layers $T'(i)$, where

$$T'(i) = 6i - 2,$$

with a tiling of colour pattern ϕ_{28}.

It is important to note that slight adjustments to the arrangement of subpatterns (for example, using 4 followed by 6 rather than 6 followed by 4) are necessary when transferring colour patterns (such as ϕ_{27} and ϕ_{28}) from one layer to the next. These adjustments ensure that the distance constraints required for a valid packing colouring are maintained. We have chosen not to elaborate on these distinctions further, as they should be self-evident to the reader. Consider Figure 3.22.

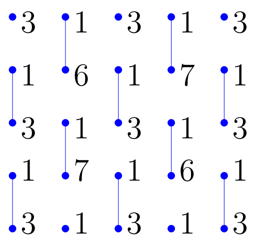

Figure 3.22: A representative tile of colour pattern ϕ_{29}.

The reader will again notice that colour pattern ϕ_{29} contains ϕ_{23} as a subpattern.

We complete the colouring of those Layers $T(i)$, that is, the layers above the Central Layer, where

$$T(i) = \begin{cases} 4 + 30k, & \text{if } i = 3k+1,\ k \geq 0, \\ 10 + 30k, & \text{if } i = 3k+2,\ k \geq 0, \\ 30k - 8, & \text{if } i = 3k, \quad k \geq 1, \end{cases}$$

$i \in \mathbb{Z}^+$ and $k \in \mathbb{Z}_{\geq 0}$, with a tiling of colour pattern ϕ_{29}. Notice that

$$T(1) = 4, \quad T(2) = 10, \quad T(3) = 22, \quad T(4) = 34, \quad T(5) = 40, \quad T(6) = 52, \ldots$$

Similarly, we complete the colouring of those Layers $T'(i)$, that is, those layers below the Central Layer, where

$$T'(i) = \begin{cases} 8 + 60\left\lfloor \frac{i-1}{6} \right\rfloor, & \text{if } i \equiv 1 \pmod 6, \\ 20 + 60\left\lfloor \frac{i-2}{6} \right\rfloor, & \text{if } i \equiv 2 \pmod 6, \\ 26 + 60\left\lfloor \frac{i-3}{6} \right\rfloor, & \text{if } i \equiv 3 \pmod 6, \\ 38 + 60\left\lfloor \frac{i-4}{6} \right\rfloor, & \text{if } i \equiv 4 \pmod 6, \\ 50 + 60\left\lfloor \frac{i-5}{6} \right\rfloor, & \text{if } i \equiv 5 \pmod 6, \\ 56 + 60\left\lfloor \frac{i-6}{6} \right\rfloor, & \text{if } i \equiv 0 \pmod 6, \end{cases}$$

with a tiling of colour pattern ϕ_{29}. Notice that

$$T'(1) = 8', \quad T'(2) = 20', \quad T'(3) = 26', \quad T'(4) = 38', \quad T'(5) = 50', \quad T'(6) = 56', \ldots$$

Next, consider Figure 3.23.

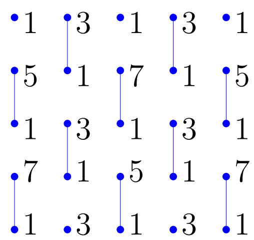

Figure 3.23: A representative tile of colour pattern ϕ_{30}.

We complete the colouring of those Layers $T(i)$, above the Central Layer, where

$$T(i) \;=\; \begin{cases} 5 \;+\; 30\left\lfloor \dfrac{i-1}{3} \right\rfloor, & \text{if } i \equiv 1 \pmod 3, \\ 11 \;+\; 30\left\lfloor \dfrac{i-2}{3} \right\rfloor, & \text{if } i \equiv 2 \pmod 3, \\ 30\left\lfloor \dfrac{i}{3} \right\rfloor \;-\; 7, & \text{if } i \equiv 0 \pmod 3, \end{cases}$$

$i \in \mathbb{Z}^+$, with a tiling of colour pattern ϕ_{30}. Notice that

$$T(1) = 5, \quad T(2) = 11, \quad T(3) = 23, \quad T(4) = 35, \quad T(5) = 41, \quad T(6) = 53, \ldots$$

Similarly, we complete the colouring of those Layers $T'(i)$, that is, those layers below the Central Layer, where

$$T'(i) \;=\; \begin{cases} 7 \;+\; 60\left\lfloor \frac{i-1}{6} \right\rfloor, & \text{if } i \equiv 1 \pmod 6, \\ 19 \;+\; 60\left\lfloor \frac{i-2}{6} \right\rfloor, & \text{if } i \equiv 2 \pmod 6, \\ 25 \;+\; 60\left\lfloor \frac{i-3}{6} \right\rfloor, & \text{if } i \equiv 3 \pmod 6, \\ 37 \;+\; 60\left\lfloor \frac{i-4}{6} \right\rfloor, & \text{if } i \equiv 4 \pmod 6, \\ 49 \;+\; 60\left\lfloor \frac{i-5}{6} \right\rfloor, & \text{if } i \equiv 5 \pmod 6, \\ 55 \;+\; 60\left\lfloor \frac{i-6}{6} \right\rfloor, & \text{if } i \equiv 0 \pmod 6, \end{cases}$$

with a tiling of colour pattern ϕ_{30}. Notice that

$$T'(1) = 7', \quad T'(2) = 19', \quad T'(3) = 25', \quad T'(4) = 37', \quad T'(5) = 49', \quad T'(6) = 55', \ldots$$

We make the following observation at this stage of our attempt to apply a finite packing colouring to $V(H_c^\delta)$: Considering the α_1 and α_2 layers, either the two layers above or the two layers below every such layer is only partially coloured by a ϕ_{23} or a ϕ_{24} colour pattern.

Specifically, we observe that the two layers below the Central Layer, that is, Layer $1'$ and Layer $2'$, are only partially coloured by colour patterns ϕ_{23} and ϕ_{24}. Consider Figure 3.24.

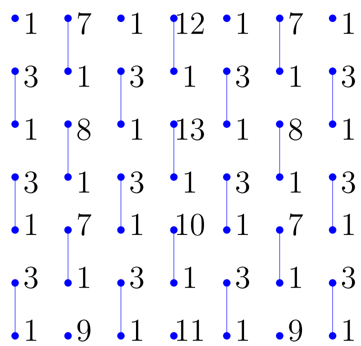

Figure 3.24: A representative tile of colour pattern ϕ_{31}.

We complete the colouring of Layer $1'$ with a tiling of colour pattern ϕ_{31}. In fact, we may extend this colouring to those Layers below the Central Layer, $T'(i)$, where

$$T'(i) = \begin{cases} 1 + 30\left\lfloor \frac{i-1}{2} \right\rfloor, & \text{if } i \equiv 1 \pmod 2, \\ -16 + 30\left\lfloor \frac{i}{2} \right\rfloor, & \text{if } i \equiv 0 \pmod 2. \end{cases}$$

Hence,

$$T'(1) = 1', \quad T'(2) = 14', \quad T'(3) = 31', \quad T'(4) = 44', \quad T'(5) = 61', \quad T'(6) = 74', \ldots$$

Similary, we extend a tiling of colour pattern ϕ_{31} to the Layers above the Central Layer,

$T(i)$, where

$$T(i) \;=\; \begin{cases} 16 \;+\; 30\left\lfloor \frac{i-1}{2} \right\rfloor, & \text{if } i \equiv 1 \pmod 2, \\ \\ -1 \;+\; 30\left\lfloor \frac{i}{2} \right\rfloor, & \text{if } i \equiv 0 \pmod 2. \end{cases}$$

Hence,

$$T(1) = 16, \quad T(2) = 29, \quad T(3) = 46, \quad T(4) = 59, \quad T(5) = 76, \quad T(6) = 89, \ldots$$

Next, we focus on completing the colouring of Layer $2'$. Table 3.1 presents a summarized representation of pattern ϕ_{32}, where the trivial uses of colours 1 and 3 have been omitted. The first column indicates the order in which this pattern should be applied. For an earlier example of a summarized pattern, see Table 1.1.

1	18	19	28	36	18	19	28	51	18	19	28	36	18	19	28	51
2	14	20	119	37	14	20	128	52	14	20	136	37	14	20	144	52
3	15	21	29	38	15	21	29	53	15	21	29	38	15	21	29	53
4	16	26	30	39	16	26	30	54	16	26	30	39	16	26	30	54
5	17	22	31	40	17	22	31	55	17	22	31	40	17	22	31	55
6	18	19	32	42	18	19	47	56	18	19	32	42	18	19	47	56
7	14	23	120	43	14	23	129	57	14	23	137	43	14	23	145	57
8	15	24	33	44	15	24	48	58	15	24	33	44	15	24	48	58
9	16	27	34	45	16	27	49	59	16	27	34	45	16	27	49	59
10	17	25	35	46	17	25	50	60	17	25	35	46	17	25	50	60
11	18	19	28	36	18	19	28	79	18	19	28	36	18	19	28	99
12	14	20	121	37	14	20	130	80	14	20	138	37	14	20	146	100
13	15	21	29	38	15	21	29	81	15	21	29	38	15	21	29	101
14	16	26	30	39	16	26	30	82	16	26	30	39	16	26	30	102
15	17	22	31	41	17	22	31	83	17	22	31	41	17	22	31	103
16	18	19	32	61	18	19	66	84	18	19	32	70	18	19	75	104
17	14	23	122	62	14	23	131	85	14	23	139	71	14	23	147	105
18	15	24	33	63	15	24	67	86	15	24	33	72	15	24	76	106
19	16	27	34	64	16	27	68	87	16	27	34	73	16	27	77	107
20	17	25	35	65	17	25	69	88	17	25	35	74	17	25	78	108
21	18	19	28	36	18	19	28	51	18	19	28	36	18	19	28	51
22	14	20	123	37	14	20	132	52	14	20	140	37	14	20	148	52
23	15	21	29	38	15	21	29	53	15	21	29	38	15	21	29	53
24	16	26	30	39	16	26	30	54	16	26	30	39	16	26	30	54
25	17	22	31	40	17	22	31	55	17	22	31	40	17	22	31	55
26	18	19	32	42	18	19	47	56	18	19	32	42	18	19	47	56
27	14	23	124	43	14	23	133	57	14	23	141	43	14	23	149	57
28	15	24	33	44	15	24	48	58	15	24	33	44	15	24	48	58
29	16	27	34	45	16	27	49	59	16	27	34	45	16	27	49	59
30	17	25	35	46	17	25	50	60	17	25	35	46	17	25	50	60
31	18	19	28	36	18	19	28	89	18	19	28	36	18	19	28	109
32	14	20	125	37	14	20	134	90	14	20	142	37	14	20	150	110
33	15	21	29	38	15	21	29	91	15	21	29	38	15	21	29	111
34	16	26	30	39	16	26	30	92	16	26	30	39	16	26	30	112
35	17	22	31	41	17	22	31	93	17	22	31	41	17	22	31	113
36	18	19	32	61	18	19	66	94	18	19	32	70	18	19	75	114
37	14	23	126	62	14	23	135	95	14	23	143	71	14	23	151	115
38	15	24	33	63	15	24	67	96	15	24	33	72	15	24	76	116
39	16	27	34	64	16	27	68	97	16	27	34	73	16	27	77	117
40	17	25	35	65	17	25	69	98	17	25	35	74	17	25	78	118

Table 3.1: A representative tile of colour pattern ϕ_{32}.

Next, Tables 3.2 and 3.3 provide summarized representations of the colour patterns ϕ_{33} and ϕ_{34}, respectively.

1	18	19	28	36	18	19	28	51	18	19	28	36	18	19	28	51
2	14	20	119	37	14	20	152	52	14	20	160	37	14	20	168	52
3	15	21	29	38	15	21	29	53	15	21	29	38	15	21	29	53
4	16	26	30	39	16	26	30	54	16	26	30	39	16	26	30	54
5	17	22	31	40	17	22	31	55	17	22	31	40	17	22	31	55
6	18	19	32	42	18	19	47	56	18	19	32	42	18	19	47	56
7	14	23	120	43	14	23	153	57	14	23	161	43	14	23	169	57
8	15	24	33	44	15	24	48	58	15	24	33	44	15	24	48	58
9	16	27	34	45	16	27	49	59	16	27	34	45	16	27	49	59
10	17	25	35	46	17	25	50	60	17	25	35	46	17	25	50	60
11	18	19	28	36	18	19	28	79	18	19	28	36	18	19	28	99
12	14	20	121	37	14	20	154	80	14	20	162	37	14	20	170	100
13	15	21	29	38	15	21	29	81	15	21	29	38	15	21	29	101
14	16	26	30	39	16	26	30	82	16	26	30	39	16	26	30	102
15	17	22	31	41	17	22	31	83	17	22	31	41	17	22	31	103
16	18	19	32	61	18	19	66	84	18	19	32	70	18	19	75	104
17	14	23	122	62	14	23	155	85	14	23	163	71	14	23	171	105
18	15	24	33	63	15	24	67	86	15	24	33	72	15	24	76	106
19	16	27	34	64	16	27	68	87	16	27	34	73	16	27	77	107
20	17	25	35	65	17	25	69	88	17	25	35	74	17	25	78	108
21	18	19	28	36	18	19	28	51	18	19	28	36	18	19	28	51
22	14	20	123	37	14	20	156	52	14	20	164	37	14	20	172	52
23	15	21	29	38	15	21	29	53	15	21	29	38	15	21	29	53
24	16	26	30	39	16	26	30	54	16	26	30	39	16	26	30	54
25	17	22	31	40	17	22	31	55	17	22	31	40	17	22	31	55
26	18	19	32	42	18	19	47	56	18	19	32	42	18	19	47	56
27	14	23	124	43	14	23	157	57	14	23	165	43	14	23	173	57
28	15	24	33	44	15	24	48	58	15	24	33	44	15	24	48	58
29	16	27	34	45	16	27	49	59	16	27	34	45	16	27	49	59
30	17	25	35	46	17	25	50	60	17	25	35	46	17	25	50	60
31	18	19	28	36	18	19	28	89	18	19	28	36	18	19	28	109
32	14	20	125	37	14	20	158	90	14	20	166	37	14	20	174	110
33	15	21	29	38	15	21	29	91	15	21	29	38	15	21	29	111
34	16	26	30	39	16	26	30	92	16	26	30	39	16	26	30	112
35	17	22	31	41	17	22	31	93	17	22	31	41	17	22	31	113
36	18	19	32	61	18	19	66	94	18	19	32	70	18	19	75	114
37	14	23	127	62	14	23	159	95	14	23	167	71	14	23	175	115
38	15	24	33	63	15	24	67	96	15	24	33	72	15	24	76	116
39	16	27	34	64	16	27	68	97	16	27	34	73	16	27	77	117
40	17	25	35	65	17	25	69	98	17	25	35	74	17	25	78	118

Table 3.2: A representative tile of colour pattern ϕ_{33}.

1	18	19	28	36	18	19	28	51	18	19	28	36	18	19	28	51
2	14	20	119	37	14	20	152	52	14	20	176	37	14	20	184	52
3	15	21	29	38	15	21	29	53	15	21	29	38	15	21	29	53
4	16	26	30	39	16	26	30	54	16	26	30	39	16	26	30	54
5	17	22	31	40	17	22	31	55	17	22	31	40	17	22	31	55
6	18	19	32	42	18	19	47	56	18	19	32	42	18	19	47	56
7	14	23	120	43	14	23	153	57	14	23	177	43	14	23	185	57
8	15	24	33	44	15	24	48	58	15	24	33	44	15	24	48	58
9	16	27	34	45	16	27	49	59	16	27	34	45	16	27	49	59
10	17	25	35	46	17	25	50	60	17	25	35	46	17	25	50	60
11	18	19	28	36	18	19	28	79	18	19	28	36	18	19	28	99
12	14	20	121	37	14	20	154	80	14	20	178	37	14	20	186	100
13	15	21	29	38	15	21	29	81	15	21	29	38	15	21	29	101
14	16	26	30	39	16	26	30	82	16	26	30	39	16	26	30	102
15	17	22	31	41	17	22	31	83	17	22	31	41	17	22	31	103
16	18	19	32	61	18	19	66	84	18	19	32	70	18	19	75	104
17	14	23	122	62	14	23	155	85	14	23	179	71	14	23	187	105
18	15	24	33	63	15	24	67	86	15	24	33	72	15	24	76	106
19	16	27	34	64	16	27	68	87	16	27	34	73	16	27	77	107
20	17	25	35	65	17	25	69	88	17	25	35	74	17	25	78	108
21	18	19	28	36	18	19	28	51	18	19	28	36	18	19	28	51
22	14	20	123	37	14	20	156	52	14	20	180	37	14	20	188	52
23	15	21	29	38	15	21	29	53	15	21	29	38	15	21	29	53
24	16	26	30	39	16	26	30	54	16	26	30	39	16	26	30	54
25	17	22	31	40	17	22	31	55	17	22	31	40	17	22	31	55
26	18	19	32	42	18	19	47	56	18	19	32	42	18	19	47	56
27	14	23	124	43	14	23	157	57	14	23	181	43	14	23	189	57
28	15	24	33	44	15	24	48	58	15	24	33	44	15	24	48	58
29	16	27	34	45	16	27	49	59	16	27	34	45	16	27	49	59
30	17	25	35	46	17	25	50	60	17	25	35	46	17	25	50	60
31	18	19	28	36	18	19	28	89	18	19	28	36	18	19	28	109
32	14	20	125	37	14	20	158	90	14	20	182	37	14	20	190	110
33	15	21	29	38	15	21	29	91	15	21	29	38	15	21	29	111
34	16	26	30	39	16	26	30	92	16	26	30	39	16	26	30	112
35	17	22	31	41	17	22	31	93	17	22	31	41	17	22	31	113
36	18	19	32	61	18	19	66	94	18	19	32	70	18	19	75	114
37	14	23	127	62	14	23	159	95	14	23	183	71	14	23	191	115
38	15	24	33	63	15	24	67	96	15	24	33	72	15	24	76	116
39	16	27	34	64	16	27	68	97	16	27	34	73	16	27	77	117
40	17	25	35	65	17	25	69	98	17	25	35	74	17	25	78	118

Table 3.3: A representative tile of colour pattern ϕ_{34}.

Finally, we use a tiling of the colour patterns ϕ_{32}, ϕ_{33}, and ϕ_{34} to complete the colouring of Layer $2'$.

φ_{32}	φ_{33}	φ_{32}	φ_{33}
φ_{32}	φ_{34}	φ_{32}	φ_{34}
φ_{32}	φ_{33}	φ_{32}	φ_{33}
φ_{32}	φ_{34}	φ_{32}	φ_{34}

Figure 3.25: A representative tiling of colour patterns ϕ_{32}, ϕ_{33}, and ϕ_{34} to complete the colouring of Layer $2'$.

Once Layer $2'$ is fully coloured, it becomes clear that continuing the packing colouring of $V(H_c^\delta)$ is no longer practical. Notice that for H_c^δ, the average degree is

$$\overline{d} \;=\; \frac{14}{15} \times 3 \;+\; \frac{1}{15} \times 4 \;=\; \frac{46}{15} \;\approx\; 3.0667.$$

To find the *proportion* of edges in H_c^δ relative to those in $\mathbb{Z}^3$, we form the ratio

$$\frac{\text{average degree of a vertex in } H_c^\delta}{\text{average degree of a vertex in } \mathbb{Z}^3} \;=\; \frac{\overline{d}}{6}.$$

Plugging in $\overline{d} = \frac{46}{15}$ gives

$$\frac{\overline{d}}{6} \;=\; \frac{\frac{46}{15}}{6} \;=\; \frac{23}{45} \;\approx\; 0.5111.$$

Construction of $\boldsymbol{H_d^\delta}$. We follow the same algorithm used for H_c^δ, applying it to construct H_d^δ. See Figure 3.26 for a side-view illustration of selected layers.

$$\begin{array}{rcl}
\text{Layer } 8 & \rightarrow & \left.\begin{array}{l} \phi_3 \\ \phi_3 \\ \phi_3 \\ \phi_3 \\ \phi_2 \\ \phi_2 \\ \phi_2 \\ \phi_2 \end{array}\right\} \beta_6 \\
\text{Layer } 7 & \rightarrow & \\
\text{Layer } 6 & \rightarrow & \\
\text{Layer } 5 & \rightarrow & \\
\text{Layer } 4 & \rightarrow & \\
\text{Layer } 3 & \rightarrow & \\
\text{Layer } 2 & \rightarrow & \\
\text{Layer } 1 & \rightarrow & \\
\text{Central Layer} & \rightarrow & \alpha_1
\end{array}$$

Figure 3.26: Side view illustration of selected layers of H_d^δ and their respective tiling patterns.

All layers above the Central Layer are covered by the pattern β_6, and all layers below the Central Layer mirror those above, thus completing the construction of H_d^δ. In contrast to H_c^δ, horizontal movement here is severely restricted, being possible only via the Central Layer.

Remark. Every layer in H_d^δ, except one, is 3-regular. There is exactly one layer where each vertex has degree 4. Since H_d^δ has infinitely many layers in total, the proportion of vertices with degree 4 tends to 0 in the infinite limit.

Finally, we note that

$$0 \; < \; \theta_d \; < \; \frac{1}{\ell},$$

for arbitrarily large $\ell \in \mathbb{Z}^+$.

Consider Figure 3.27.

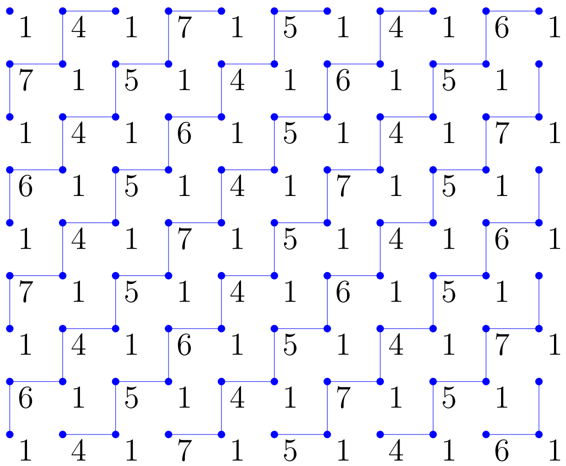

Figure 3.27: A representative tile of colour pattern ϕ_{35}

We colour the Central Layer with a tiling of colour pattern ϕ_{35}. Next, consider Figure 3.28.

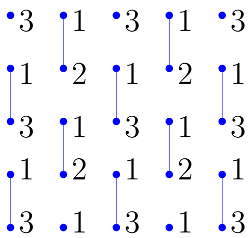

Figure 3.28: A representative tile of colour pattern ϕ_{36}.

We colour the Layers $T(i)$ and $T'(i)$, where

$$T(i) \;=\; \left\lfloor \frac{3i-1}{2} \right\rfloor \;=\; T'(i), \quad i \in \mathbb{Z}^+,$$

using a tiling of colour pattern ϕ_{36}, making any necessary adjustments to satisfy the distance constraints of a packing colouring. Note that

$$T(1) = 1,\ T(2) = 2,\ T(3) = 4,\ T(4) = 5,\ T(5) = 7,\ T(6) = 8, \ldots$$

and

$$T'(1) = 1',\ T'(2) = 2',\ T'(3) = 4',\ T'(4) = 5',\ T'(5) = 7',\ T'(6) = 8', \ldots$$

We now address Layers $j = 3k = j'$, where $k \in \mathbb{Z}^+$. We postpone though colouring Layers 3 and $3'$ for reasons that will soon become clear. Consider Figure 3.29.

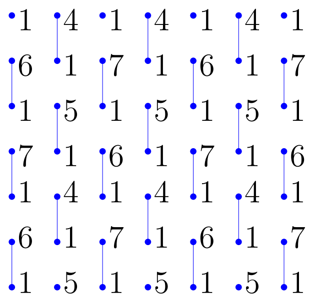

Figure 3.29: A representative tile of colour pattern ϕ_{37}.

We colour all Layers $j = 3k = j'$, $k \in \mathbb{Z}^+$ (excluding Layers 3 and $3'$) with a tiling of colour pattern ϕ_{37}, again making any necessary adjustments required for the packing distance constraint. Consider Figure 3.30.

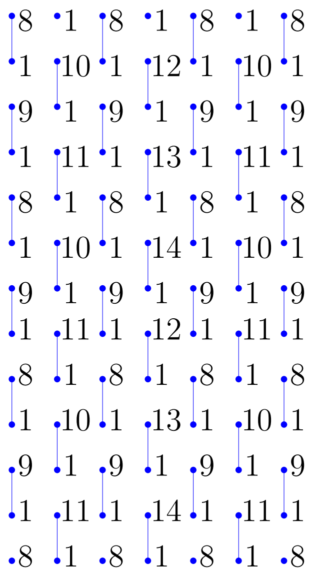

Figure 3.30: A representative tile of colour pattern ϕ_{38}.

We colour Layer 3 using a tiling of colour pattern ϕ_{38}. The only layer remaining is $3'$, which we colour last. Table 3.4 provides a summarized view of pattern ϕ_{39}, omitting the trivial use of colour 1 (The first row and column indicate the order in which this pattern is applied).

index	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	4	15	18	4	22	32	4	15	18	4	28	52	4	15	18	4	22	32	4	15	18	4	28	68
2	4	16	19	4	23	33	4	16	19	4	29	53	4	16	19	4	23	33	4	16	19	4	29	69
3	4	17	20	4	24	34	4	17	20	4	22	54	4	17	20	4	24	34	4	17	20	4	22	70
4	4	15	21	4	25	35	4	15	21	4	23	55	4	15	21	4	25	35	4	15	21	4	23	71
5	4	16	18	4	22	36	4	16	18	4	30	56	4	16	18	4	22	36	4	16	18	4	30	72
6	4	17	19	4	23	37	4	17	19	4	31	57	4	17	19	4	23	37	4	17	19	4	31	73
7	4	15	20	4	26	38	4	15	20	4	22	32	4	15	20	4	26	38	4	15	20	4	22	32
8	4	16	21	4	27	39	4	16	21	4	23	33	4	16	21	4	27	39	4	16	21	4	23	33
9	4	17	18	4	22	40	4	17	18	4	28	34	4	17	18	4	22	40	4	17	18	4	28	34
10	4	15	19	4	23	41	4	15	19	4	29	35	4	15	19	4	23	41	4	15	19	4	29	35
11	4	16	20	4	24	42	4	16	20	4	22	58	4	16	20	4	24	42	4	16	20	4	22	74
12	4	17	21	4	25	43	4	17	21	4	23	59	4	17	21	4	25	43	4	17	21	4	23	75
13	4	15	18	4	22	32	4	15	18	4	30	60	4	15	18	4	22	32	4	15	18	4	30	76
14	4	16	19	4	23	33	4	16	19	4	31	61	4	16	19	4	23	33	4	16	19	4	31	77
15	4	17	20	4	26	34	4	17	20	4	22	62	4	17	20	4	26	34	4	17	20	4	22	78
16	4	15	21	4	27	35	4	15	21	4	23	63	4	15	21	4	27	35	4	15	21	4	23	79
17	4	16	18	4	22	44	4	16	18	4	28	64	4	16	18	4	22	44	4	16	18	4	28	80
18	4	17	19	4	23	45	4	17	19	4	29	65	4	17	19	4	23	45	4	17	19	4	29	81
19	4	15	20	4	24	46	4	15	20	4	22	32	4	15	20	4	24	46	4	15	20	4	22	32
20	4	16	21	4	25	47	4	16	21	4	23	33	4	16	21	4	25	47	4	16	21	4	23	33
21	4	17	18	4	22	48	4	17	18	4	30	34	4	17	18	4	22	48	4	17	18	4	30	34
22	4	15	19	4	23	49	4	15	19	4	31	35	4	15	19	4	23	49	4	15	19	4	31	35
23	4	16	20	4	26	50	4	16	20	4	22	66	4	16	20	4	26	50	4	16	20	4	22	82
24	4	17	21	4	27	51	4	17	21	4	23	67	4	17	21	4	27	51	4	17	21	4	23	83

Table 3.4: A representative tile of colour pattern ϕ_{39}.

Finally, we colour Layer $3'$ with a tiling of colour pattern ϕ_{39}, completing the packing colouring of $V(H_d^\delta)$. It follows immediately that

$$\chi_\rho(H_d^\delta) \ \leq \ 83.$$

To extend this result to a new graph H_e^δ, we repeat the pattern α_1 and colour pattern ϕ_{35} every ninetieth layer (above and below the Central Layer). Additionally, we repeat colour pattern ϕ_{38} every ninetieth layer (above and below the Central Layer) and colour pattern ϕ_{39} every ninetieth layer (above and below the Central Layer). It follows then that

$$\chi_\rho(H_e^\delta) \ \leq \ 83 \quad \text{and} \quad \theta_e \ = \ \frac{1}{90}.$$

Additionally, consider H_e^δ, whose average degree is

$$\overline{d} \ = \ \frac{89}{90} \times 3 \ + \ \frac{1}{90} \times 4 \ = \ \frac{267 + 4}{90} \ = \ \frac{271}{90} \ \approx \ 3.0111.$$

To find the *proportion* of edges in H_e^δ relative to those in $\mathbb{Z}^3$, we form the ratio

$$\frac{\text{average degree of a vertex in } H_e^\delta}{\text{average degree of a vertex in } \mathbb{Z}^3} = \frac{\overline{d}}{6}.$$

Plugging in $\overline{d} = \frac{271}{90}$ gives

$$\frac{\overline{d}}{6} = \frac{\frac{271}{90}}{6} = \frac{271}{540} \approx 0.5019.$$

Thus, H_e^δ retains approximately 50.19% of the edges of $\mathbb{Z}^3$. Consequently, we obtain the following observation.

Observation 3.0.1. *Let κ be the proportion of edges that was removed from $\mathbb{Z}^3$ to produce the connected spanning subgraph H_e^δ for which a finite packing colouring exists. Then $\kappa = \frac{269}{540}$.*

Chapter 4

Conclusion

In this thesis, we investigated how modifications to the size of $\mathbb{Z}^3$ impacts the packing chromatic number of the resulting graph. Specifically, we sought to answer the following central question: What is the minimum proportion of edges that must be removed from $\mathbb{Z}^3$ to obtain a connected spanning subgraph that admits a finite packing colouring?

For practical reasons, we focused on specific classes of subgraphs of $\mathbb{Z}^3$, namely, two, three, four, and five regular connected spanning subgraphs.

Our study began by investigating whether a two regular connected spanning subgraph of $\mathbb{Z}^3$ could be constructed. We observed that such a subgraph exists if and only if $\mathbb{Z}^3$ is Hamiltonian, leading us to introduce the following conjecture:

Conjecture 4.0.1 (Restated, 1.1.1)**.** $\mathbb{Z}^3$ *is non-Hamiltonian.*

As an alternative, we constructed a two-way infinite path H, which serves as a connected spanning subgraph of $\mathbb{Z}^3$, and demonstrated that $\chi_\rho(H) = 3$. Observe that H is a Hamiltonian path of $\mathbb{Z}^3$.

Next, we examined three regular connected spanning subgraphs of $\mathbb{Z}^3$ by constructing two distinct subgraphs, denoted as $\mathbb{Z}^3_{3a}$ and $\mathbb{Z}^3_{3b}$. The structure of $\mathbb{Z}^3_{3a}$ was designed such that horizontal movement was restricted to a Central Layer, while only vertical movement was permitted in all other layers. This approach served two objectives: maintaining connectivity while reducing the number of colours required for a packing colouring. We established the following upper bound:

$$\chi_\rho(\mathbb{Z}^3_{3a}) \leq 23.$$

In contrast, $\mathbb{Z}^3_{3b}$ was constructed with minimal constraints on movement. We observed that $\mathbb{Z}^3_{3b}$ is a maximal three regular connected spanning subgraph of $\mathbb{Z}^3$. However, we were only able to partially colour $\mathbb{Z}^3_{3b}$, leading to the following open questions:

Open Question 4 (Restated, 1)**.** *Is it true that $\chi_\rho(\mathbb{Z}^3_{3b}) < p$ for some $p \in \mathbb{Z}^+$?*

Open Question 5 (Restated, 2)**.** *Are $\mathbb{Z}^3_{3a}$ and $\mathbb{Z}^3_{3b}$, respectively, minimal and maximal three regular connected spanning subgraphs of $\mathbb{Z}^3$?*

Open Question 6 (Restated, 3)**.** *Is it possible to find $a, b \in \mathbb{Z}^+$ so that $a \leq \chi_\rho(\mathbb{Z}^3_3) \leq b$, where $\mathbb{Z}^3_3$ is an arbitrary three regular connected spanning subgraph of $\mathbb{Z}^3$? Also, is $a = 23$?*

Our objective is to revisit the packing colouring of $\mathbb{Z}^3_{3b}$ using computational techniques to determine whether a finite packing colouring can be achieved.

In Chapter 1 we investigated whether arbitrary three regular connected spanning subgraphs of $\mathbb{Z}^3$ are finitely packing colourable. We started Chapter 2 by noting that our inuition suggested that $\chi_\rho(\mathbb{Z}^3_4)$ and $\chi_\rho(\mathbb{Z}^3_5)$ are infinite, where $\mathbb{Z}^3_4$ and $\mathbb{Z}^3_5$ are arbitrary four and five regular connected spanning subgraphs of $\mathbb{Z}^3$.

In order to show this, we used the technique introduced by Finbow and Rall in [14], which requires estimating the order and size of the induced subgraph of $N_i(x)$, $i \in \mathbb{Z}^+$, where $x \in V(\mathbb{Z}^3_4)$ or $x \in V(\mathbb{Z}^3_5)$ respectively. We considered two approaches in estimating the order and size of the induced subgraph of $N_i(x)$.

i) Generation of subgraphs and estimation of properties

ii) Sampling approach

We concluded that the generation of all possible subgraphs of the induced subgraph of $N_i(x)$, and then filtering the generated subgraphs to identify those that are either four or five regular is not feasible, which prompted us to make use of a sampling approach instead.

Based on our sampling approach, we were able to produce estimates of the maximum order and size of arbitrary induced subraphs of $N_i(x)$, $i \in \mathbb{Z}^+$, where $x \in V(\mathbb{Z}^3_{4'})$ or $x \in V(\mathbb{Z}^3_{5'})$ respectively.

4-regular connected spanning subgraphs of $\mathbb{Z}^3$:
Let $x \in V(\mathbb{Z}^3_{4'})$. We showed that for every $i \in \mathbb{Z}^+$

$$|N_i(x)| \approx a_1 i^2 + a_2 i + a_3, \tag{4.1}$$

where $a_1 = \frac{80}{7}$, $a_2 = \frac{-832}{35}$, and $a_3 = \frac{89}{5}$ for arbitrary $x \in V(\mathbb{Z}^3_{4'})$. Similarly, we showed that the size of the induced subgraph of $N_i(x)$ is approximately equal to

$$b_1 i^2 + b_2 i + b_3, \tag{4.2}$$

where $b_1 = \frac{148}{7}$, $b_2 = \frac{-2088}{35}$, and $b_3 = \frac{224}{5}$. Notice that Equations 4.1and 4.2 serve as approximate upper bounds of the order and size of arbitrary induced subgraphs of $N_i(x)$. Using these bounds, that is 4.1 and 4.2, we derived the following key lemmas:

Lemma 4.0.2 (Restated, 2.2.1)**.** *Let $x \in V(\mathbb{Z}^3_{4'})$, $r \in \mathbb{Z}^+$ and $\epsilon \in \mathbb{R}$ be positive. Then there exists an $M_r \in \mathbb{Z}^+$ such that whenever $M \geq M_r$, $M \in \mathbb{Z}^+$, if $A \subseteq V(N_M(x))$ has the property that $\{N_r(a) : a \in A\}$ is a collection of pairwise disjoint subsets of $V(\mathbb{Z}^3_{4'})$, then*

$$\frac{|A|}{|N_M(x)|} < \frac{35}{400r^2 - 832r + 623} + \frac{\epsilon}{2^{r+1}}.$$

Lemma 4.0.3 (Restated, 2.2.2)**.** *Let $x \in V(\mathbb{Z}^3_{4'})$, $r \in \mathbb{Z}^+$ and $\epsilon \in \mathbb{R}$ be positive. Then there exists an $M_r \in \mathbb{Z}^+$ such that whenever $M \geq M_r$, $M \in \mathbb{Z}^+$, if $A \subseteq V(N_M(x))$ has the property that $\{E(N_r(a)) : a \in A\}$ is a collection of pairwise disjoint subsets of $E(\mathbb{Z}^3_{4'})$,* *then*

$$\frac{|A|}{|N_M(x)|} < \frac{35}{\frac{20}{37} \cdot (740r^2 - 2088r + 1568)} + \frac{\epsilon}{2^{r+1}}.$$

From these lemmas, we derived the following important result.

Theorem 1 (Restated, 2.2.3)**.** The packing chromatic number of $\mathbb{Z}^3_{4'}$ is infinite.

This result directly led to the following conclusion considering arbitrary four regular connected spanning subgraphs of $\mathbb{Z}^3$.

Theorem 2 (Restated, 2.2.4)**.** The packing chromatic number of $\mathbb{Z}^3_4$ is infinite.

5-regular connected spanning subgraphs of $\mathbb{Z}^3$:

Let $x \in V(\mathbb{Z}^3_{5'})$. We showed that for every $i \in \mathbb{Z}^+$

$$|N_i(x)| \approx a_1 i^2 + a_2 i + a_3, \tag{4.3}$$

where $a_1 = \frac{88}{7}$, $a_2 = -\frac{792}{35}$ and $a_3 = \frac{83}{5}$, $x \in V(\mathbb{Z}^3_{5'})$.

Lastly, we showed that the size of the induced subgraph $N_i(x)$ is approximately equal to

$$b_1 i^2 + b_2 i + b_3, \tag{4.4}$$

where $b_1 = \frac{200}{7}$, $b_2 = -\frac{507}{7}$ and $b_3 = 51$.

Notice again that Equations 4.3and 4.4 serve as approximate upper bounds of the order and size of arbitrary induced subgraphs of $N_i(x)$. Using these bounds, that is 4.3 and 4.4, we derived the following key lemmas:

Lemma 4.0.4 (Restated, 2.3.1)**.** *Let $x \in V(\mathbb{Z}^3_{5'})$, $r \in \mathbb{Z}^+$ and $\epsilon \in \mathbb{R}$ be positive. Then there exists an $M_r \in \mathbb{Z}^+$ such that whenever $M \geq M_r$, $M \in \mathbb{Z}^+$, if $A \subseteq V(N_M(x))$ has the property that $\{N_r(a) : a \in A\}$ is a collection of pairwise disjoint subsets of $V(\mathbb{Z}^3_{5'})$, then*

$$\frac{|A|}{|N_M(x)|} < \frac{35}{440r^2 - 792r + 581} + \frac{\epsilon}{2^{r+1}}.$$

Lemma 4.0.5 (Restated, 2.3.2)**.** *Let $x \in V(\mathbb{Z}^3_{5'})$, $r \in \mathbb{Z}^+$ and $\epsilon \in \mathbb{R}$ be positive. Then there exists an $M_r \in \mathbb{Z}^+$ such that whenever $M \geq M_r$, $M \in \mathbb{Z}^+$, if $A \subseteq V(N_M(x))$ has the property that $\{E(N_r(a)) : a \in A\}$ is a collection of pairwise disjoint subsets of $E(\mathbb{Z}^3_{5'})$, then*

$$\frac{|A|}{|N_M(x)|} < \frac{35 \cdot \frac{5}{11}}{200r^2 - 507r + 357} + \frac{\epsilon}{2^{r+1}}.$$

From these lemmas, we derived the following important result.

Theorem 4.0.6 (Restated, 2.3.3)**.** *The packing chromatic number of $\mathbb{Z}^3_{5'}$ is infinite.*

This result directly led to the following conclusion considering arbitrary five regular connected spanning subgraphs of $\mathbb{Z}^3$.

Theorem 4.0.7 (Restated, 2.3.4)**.** *The packing chromatic number of $\mathbb{Z}^3_5$ is infinite.*

At this stage in our analysis of connected spanning subgraphs of $\mathbb{Z}^3$, we have established a two-way infinite path H as a connected spanning subgraph of $\mathbb{Z}^3$ which trivially is finitely packing colourable. Moreover, assuming our open question is resolved affirmatively, three regular connected spanning subgraphs of $\mathbb{Z}^3$ are also finitely packing colourable. In contrast, arbitrary four and five regular connected spanning subgraphs are not finitely packing colourable.

In Chapter 3 we then shifted our focus to connected spanning subgraphs of $\mathbb{Z}^3$ in which every vertex has either degree 3 or 4. To formalize this investigation, we introduced the following definition:

Definition 9 (Restated, 8)**.** Let $\mathcal{F} = \{F \mid F$ is a connected spanning subgraph of $\mathbb{Z}^3$ with $3 \leq \deg(x) \leq 4\}$ for all $x \in V(F)$.
A graph H is called a *δ-subgraph* of $\mathbb{Z}^3$, denoted by H^δ, if $H \in \mathcal{F}$.

We proceeded to construct specific δ-subgraphs of $\mathbb{Z}^3$, labeled H^δ_a, H^δ_b, and H^δ_c, which we attempted to colour using a finite packing colouring. In each case, we primarily adjusted those layers that permitted horizontal travel - specifically, the layers where each vertex has degree 4. We refer to such a layer as an α *layer*.

We observed that the more frequently an α layer is encountered along a path from an arbitrary vertex, the more challenging it becomes to apply a finite packing colouring to the corresponding graph. Indeed, as the reader will recall, in each of the cases of H_a^δ, H_b^δ, and H_c^δ, we concluded that obtaining a finite packing colouring was not feasible.
We then proceeded to construct H_d^δ, in which only a single layer - the Central Layer - was an α layer. We demonstrated that $\chi_\rho(H_d^\delta) \leq 83$. We then extended H_d^δ to H_e^δ by expanding the number of α layers, and obtained $\chi_\rho(H_e^\delta) \leq 83$.

Finally, we obtained the following observation, addressing in part the primary question of this thesis.

Observation 4.0.8 (Restated, 3.0.1)**.** *Let κ be the proportion of edges that was removed from $\mathbb{Z}^3$ to produce the connected spanning subgraph H_e^δ for which a finite packing colouring exists. Then $\kappa = \frac{269}{540}$.*

In our study of the r-regular (where $r \in \{2,3,4,5\}$) connected spanning subgraphs of $\mathbb{Z}^3$ we made use of computers as well as manual constructions of specific connected spanning subgraphs of $\mathbb{Z}^3$ in order to determine a lower bound for the proportion of edges that must be removed to produce a connected spanning subgraph which is finitely packing colourable. We would like to continue this study and assess further techniques which can be used to effectively and efficiently leverage the available computional power, ideally going even further to utilize available supercomputers.

Specifically, we would like to address the following two conjectures.

Conjecture 4.0.9. *Let κ denote the proportion of edges removed from $\mathbb{Z}^3$ to obtain a connected spanning subgraph H. If $\kappa \leq \frac{1}{3}$ then $\chi_\rho(H)$ is infinite.*

Conjecture 4.0.10. *Let H be an arbitrary connected spanning subgraph of $\mathbb{Z}^3$ which is fintely packing colourable. Let κ denote the proportion of edges that was removed from $\mathbb{Z}^3$ to construct H, then $\kappa \geq \frac{1}{2}$.*

Additionally, we found the work on the packing chromatic number of $\mathbb{Z}^2$ particularly intriguing, especially the analysis conducted using SAT-solvers (see [25]). We aim to explore alternative techniques while still utilising SAT-solvers to independently verify that $\chi_\rho(\mathbb{Z}^2) = 15$.

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Appendix A

Supplementary Materials

This appendix provides references for the supplementary datasets used in this study. To request copies, please contact my supervisor, Professor Elizabeth Jonck, at Betsie.Jonck@wits.ac.za.

A.1 Contents of the Supplementary Files

- **Connected three regular spanning subgraph 3a** [Workings related to the packing colouring of graph $\mathbb{Z}^3_{3a}$]

- **Connected three regular spanning subgraph 3b** [Workings related to the attempted packing colouring of graph $\mathbb{Z}^3_{3b}$]

- **Connected deg 3 and 4 spanning subgraph Ha** [Workings related to the attempted packing colouring of graph H^δ_a]

- **Connected deg 3 and 4 spanning subgraph Hb** [Workings related to the attempted packing colouring of graph H^δ_b]

- **Connected deg 3 and 4 spanning subgraph Hc** [Workings related to the attempted packing colouring of graph H^δ_c]

- **Connected deg 3 and 4 spanning subgraph Hd** [Workings related to the packing colouring of graph H^δ_d]