Application of delay differential equations to foreign exchange reserves

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Declaration

I, Amos Soweto Kubeka, hereby certify that the thesis: "**Application of delay differential equations to foreign exchange reserves"**, submitted to the University of the Witwatersrand, Johannesburg, is my own work and has not in part or whole, been submitted to another institution for a similar qualification. All sources used have been properly acknowledged and referenced.

Signed :  **Date: 25 May 2021**

Abstract

First, we apply the theory of the delay differential equation to model the policy decisions of the central banks to deplete and accumulate the reserves during the global financial crisis. In the process we obtain a general first order monotonic delayed differential equation which we solve by transforming it to a recurrence relation using the theory of differential transform to generate the transformed recurrence equation. We then solve this recurrence equation to get a system of four linear equations which we solve to get general solutions of the reserve accumulations. Finally, we express the target reserves (i.e. gold or US dollars) as a function of the obtained reserve accumulation general solutions in a polynomial of time t to the fourth order.

To do the analysis, we then contract two realist economic scenarios of the obtained polynomial series solutions for the target reserves; 1) for the fast depletion and fast accumulation of the reserves during the time of the global financial crisis, and 2) for the slow depletion and slow accumulation of the reserves during the time of the global financial crisis.

For the two cases, we were able to model the policy decisions that can be made by the central banks of countries for the depletion and accumulation of the reserves with a fixed time delay of the response of the economy to the impact of these policy decisions.

In our findings, we found that the effects of the delay factor, if it is closer to 0 it prolongs the accumulation process of the reserves, and if is lager, it shorten the accumulation process. In addition, we have also observed that the same effects can be obtained by varying the other model variables while the delay factor is kept constant.

Finally, the inclusion of foreign exchange rate regime in our model will certainly require us to involve stochastic differential equations analysis. By doing this, we will then get a full understanding of the movement of the reserves in and out of a country in the form of the accumulation and depletion of the reserves.

1. Introduction

1.1. Background of the study

According to Wang (2011), the foreign exchange market is not a physical market where traders meets for discussing and closing deals. It is an Over The Counter (OTC) market responsible for; the transfer of the purchasing power between countries, the provision of credit, the provision of hedging facilities, and for transfers of the cheques and bank drafts. Foreign exchange reserves are; bank notes (United State (US) Dollars), bonds, treasury bills, deposits, government securities, special drawing rights, and most importantly gold.

Wang (2011) and Shin-ichi et. al. (2019) highlighted that these reserves are assets held by the central banks and other financial institutions of sovereign governments. The deposits are deposited by the governments and financial institutions to pay for sovereign liabilities. The reserves are held in one or more reserve currencies, mostly the US dollar and in gold as mentioned by Gevorkyan et. al. (2019) and John-Eun (2016).

The US Dollars and gold reserves held by central banks serves as a guarantee to pay depositors and traders on the stock markets. They also serves a store of value for emerging markets and developing economies, and also to support the value of the local currencies for developing counties Vermeiren (2013), Filardo et. al. (2016), and Jeanne et. al. (2011). They also help increase the confidence in the monetary and exchange rate policies of the central banks and fiscal policies of the government McKinnon (1988).

According to Shafiee et. al. (2010), the US Federal Reserve Bank is the biggest holder of gold reserves in the world and that makes the US dollar very strong in relation to other currencies of the world. As a result, the US economy is very strong also compared to other world economies. The US Dollar then became the reserve currency of the world by default and the Federal Reserve can print it when it is needed Wang (2011).

Calve (1998), Cespedes et. al. (2004), Lee (2007), and Dornbush (1976) have observed that the accumulation and depletion of foreign reserves is linked to the movement of foreign exchange rates. The movement of the foreign exchange rate determine the appreciation and depreciation of local currencies and also how much the local currency is worth relative to foreign currencies.

According to Nazeer et al. (2015), traders on the stock market determine the exchange rates of currencies. They also noted that the balance of trade among nations is a macroeconomic factor that influences the exchange rate with impact on the balance of payment.

The exchange rate is important because it indicates how strong the economy of country is relative to that of the United States (US). This then helps to determine the economic wealth of nations relative to the US Corrado et. el. (2009).

However, in this project we will not be looking at the full model incorporating the foreign exchange rates when modelling the movement of the reserves in and out of the open economy due to the policy decisions by the central banks to accumulate the reserves.

Including the exchange rate in our model would imply that we reformulate our model accordingly to take into account the inherent stochastics processes of the exchange rate. From Blanchard (2013) and Wang (2011), we note that foreign exchange rate is also a function of inflation, sovereign risk, and interest rate and this might complicate our project even further. We therefore, leave this part as a future recommendation for the extension our model which we shall formulate in section 4 and analyse in section 5 with conclusion and recommendations in section 6.

1.2. Problem statement

Understanding the movements of the foreign exchange reserves in and out in open economies is very important. It is evident from the literature that developing countries are the primary accumulator of foreign exchange reserves Steiner (2013). The developing countries must accumulate foreign reserves with the intention to use them to help mitigate the negative impact of the financial crisis. The crisis originate from developed and leading economies like the US Calve et. al. (1993) and Patnaik (2007).

Emerging markets must accumulate gold and US Dollar reserves and then manage them effectively manage to keep their economies healthy Filardo et al. (2016). This imply that the central banks and sovereign governments of these markets must develop sound and effective monetary and fiscal policies. These policies will guide then on how and when to decide to accumulate or to deplete their held reserves.

Therefore, we aim to develop a foreign exchange reserves model that is suitable for open economies of both emerging markets and developed countries for the depletion and accumulation and reserves. We shall opt to approach this problem differently from what normally has been done when modelling movement of the reserves. We shall do this in order to avoid dealing with many systems of equations that required extensive statistical analysis of advanced econometrics.

Thus, we shall utile the theory delay differential equations. This theory requires few equations to deal with and with less mathematical manipulations. The solutions are clear and concise and can be analysed easily as we shall see later.

1.3. Research objective

The problem of the foreign exchange reserve management is central to the functions of the central banks across the globe. The aim is for the central banks to formulate effective monetary policies decisions to help manage and mitigate the impact of adverse foreign risk

factors like the global financial crisis of 2007. This crisis began in the in the United State and lasted from 2007 to 2008. The effects were rippling and devastating for the world economy.

The consequence for some countries like South Africa were among others, slow economic recovery and growth, low foreign investments, high unemployment but with well-maintained inflation targeting, and still high foreign risk factors among others. The South African Reserve bank still maintain a good management of its foreign exchange reserves and protecting the value of Rand by implement effective monetary policy decisions.

For our project, the objective is to model the policy decisions by the central banks to manage the movement of the depletion and accumulation of the reserves during the global financial crisis like that of 2007.

To model the problem, we apply the delay differential equations with a fixed time delay of the response of the economy to the impact of the policy decisions to depleting and accumulating the reserves. In the final analysis, we obtain the first order delay differential equation that we solve to obtain the series solutions that expresses the target reserves of the bank a as function of the reserve accumulation by the bank in time t.

1.4. Research question

What kind of policy (fiscal and monetary) decisions can be made to slowly or faster accumulate and deplete held foreign reserves by an open economy in times of global financial crisis? The economy can be either developing or developed.

1.5. Contribution to the literature

Our study will contribute to the understanding of the movement of the foreign exchange reserves subject to certain policy decisions by the central banks for the accumulation of the reserve during the global financial crisis. The problem of the foreign reserve accumulation is currently a major active area of research in international finance. Our results will be very important for all open economies including emerging markets Pontines et. al. (2011). We conduct our research by extending the applications of the continuous-time Kalecki's system Keller (2010). We do that by modifying the system and adapting it for foreign exchange reserves movement in and out of open economy.

For our purpose, we do not include foreign exchange rates movements in the economy. This is because if we do, we might have to reformulate our model further and use the theory of Delayed Stochastic Differential Equations which will complicate the problem even further because of the high volatility nature of the exchange rate movement. However, the problem can be done for future extensions the project. This will then give a comprehensive picture of the full movement of the foreign exchange reserves.

The field of study of foreign exchange rate movement is very extensive in the literature, and on the contrary, the DSDE has not been applied yet in this direction. Only the normal theory of stochastic differential equations have been applied Neisy et. al. (2011). We also note that the application of DDE to foreign exchange reserves is a new adventure for our project.

1.6. Road map structure

We shall apply the theory of delay differential equations to study the movement of the depletion and accumulation of foreign reserves in an open economy. Among others, we modify the aggregate demand equation of Kalecki (1935) and Keller (2011) (i.e. Eq. (5) in Keller) model. This equation was assumed to be a function of consumption and investment only. In its place, we shall now have the national income equation. This equation is assumed to be a function of a domestic absorption function, export function and import function. The domestic adoption function is taken to be a function of consumption and investment.

For our problem we model the policy decisions taken by the central banks to deplete the held reserves and also to compensate for the depleted reserved by accumulating more reserves during the global financial crisis like of 2007.

In the process we obtain a general first order delayed differential equation which solve by transforming it to a recurrence relation using the theory of differential transform to generate the transformed recurrence equation. We then solve this recurrence equation to get a system of four linear equations which we solve to get four set of the general solutions of the reserve accumulations.

Finally, we then express the target reserves (i.e. gold or US dollars) as a function of the obtained reserve accumulation general solutions in polynomial of time t to the fourth order. Time can be taken to be months days, weeks, months, or years.

To do the analysis, we then contract two realist economic scenarios of the obtained polynomial series solutions for the target reserves; 1) for the fast depletion and fast accumulation of the reserves during the time of the global financial crisis, and 2) for the slow depletion and slow accumulation of the reserves during the time of the global financial crisis.

We then discuss in detail the implications of the results for the two scenarios and we then conclude with recommendations for possible extension of our model for similar future project.

1.7. Conclusion

We have highlighted the significant of our research problem to foreign exchange markets and in particular the reserve accumulation and depletion process. We also outlined in brief, the road map for our research and how we will go about analysis the analytic results of the project.

In the next three sections we shall outline the problem statement, the research objective, and the research question for our project. These shall form the basis of our research problem later in the report.

2. Literature Review

The analytical efforts to modeling large bulk foreign reserve was done by Davis et. al. (2019) where they have looked at modeling large capital inflows and outflows in emerging markets. These large capital flows have a direct impact on the financial and macroeconomic risks of the country. This was a result of the argument put forward by the International Monetary Fung (IMF) in 2012. In that report, the IMF argued that, an active capital account must be maintained and be used as a mechanism to effectively manage these risks. Other researcher immediate followed suite in this direction, Jeanne et al. (2011).

In our case, we shall apply the theory of DDE to model the large accumulation and depletion of the foreign reserves for an open economy. As discussed by Ladas et. al. (1982b) and Stavroulakis (2014), the DDE have constant or variable lags and the time derivative is given by the previous solution. The delays or lags can simply lump complicated economic processes together, accounting only for the time required for these processes to occur. If the time delay is sufficiently small then it can be replaced by the Taylor series.

Such models have the advantage of combining a simple, intuitive derivation with a wide variety of possible behavior regimes for a large economic system. The analytic solutions to the governing DDE are obtained by applying the method of differential transform to the governing Arikoglu et. al. (1999). Also, the inverse scattering method McLeod et. al. (2006), the Jacobian elliptic function method Dai et. al. (2006), and the numerical methods Dormand (1996) can be used to obtain the solution to the DDE.

Jevons (1871) was among the first economics to note the important property that capital is concern with time. Hayek (1931) did a further study of this property based on the work of Böhm-Bawerk (1891). Then Kalecki (1935), confirm rigorously the work of Hayek by developing a macroeconomic model of the movement of capital stock in the continuous-time. We note that Kalecki's model is differential equation with a time lag i.e. DDE. Classically, the DDE with constant or flexible (varying) lags are also applied to the study of the neoclassical Solow (1956), Swan (1956), Ramsey (1928), Cass (1965), and Koopmans (1965) economic growth models. The flexibility of the time lags is as results of the heterogeneity of good and services.

Despite the original contribution by Keller (1935), Frisch et. al. (1935), and Belz et. al. (1938) to model the flow of capital using DDE, few economist have done research in this direction. They have rather focused their attention on using difference or differential equations to do the modeling. Kydland (1982) reignited the idea of applying DDE to the problem of capital dynamics. He systematically studied the time required for the capital goods to receive investment in order to be productive.

In addition, Keller (2010) recently revisited Kalecki's continuous-time and discrete-time economic models and applied DDE to develop the generalized DDE theory of the economic systems and control. In that work, he also applied the DDE to the Solow vintage growth model with heterogeneous capital i.e. variable delay.

The recent analytical models in the literature for modelling foreign exchange reserves in emerging markets and developing economies are discussed by Aizenman et. al. (2012) and Khemraj et al (2019) respectively. They employed econometric modelling techniques to do the modelling.

Also, Jeanne et al. (2011) has done a study to model the optimization of the reserve levels for emerging markets using pure optimization techniques. These economic models are based on the theory of differential equations and not DDE. The models sufferer the fact that they require larger systems of equations with many parameters to accurately model problem. The models cannot capture the natural richness of the variety of the dynamics inherent in financial economics of the problem.

Therefore, for our problem because the capital flows is concern with time, and that capital is another form of the reserve like gold, then we can use the DDE to model foreign exchange reserves. The DDE's are also suited to model economic problems with a time lag as a delay parameter Keller (2010).

3. The Methodology framework

3.1 Accumulation of foreign exchange reserves

According to Rodrik (2006), the developing countries have experience the rapid rise in foreign reserve accumulation since the beginning of the 1990S. According to the International Monetary Fund (2019) and Rodrik (2006), the developing countries have experienced the rapid rise in foreign reserve accumulation sine the early 1990s with China ranking the second and India the fifth. Together these two countries hold a total of about 60% of developing countries foreign reserve equivalent of eight months of imports Rodrik (2006). China and India accumulated their foreign reserves by focusing on the fiscal policies and the dynamical choices of the exchange rates regime to drive the fiscal policies.

In India, the accumulation of foreign exchange reserve was a function of the volatility of the exchange rate and the degree of openness of the economy according to Romero (2005). According to Romero (2005), it remains unclear in the case of China which factors are driving the accumulation of foreign exchange reserves.

As noted by Kato et al. (2018), Chinas' huge reserve protect it against currency crisis as it was the case in the beginning of 2016 when there was significant capital flight out of the country which did not develop to a currency crisis.

Other developing countries with small reserves and with the fear of depleting all of their reserves may use the currency depreciation method or exchange rate depreciation to deal with huge exchange market pressure as it was the case with South Africa, Russia, Poland, Malaysia, Korea, Brazil, Colimbia, Indonesia, and India Aizenman et al. (2012).

It is clear from the IMF data (2019) and Aizenman et al. (2012) that developing countries have chosen the path of foreign reserve accumulation as a means to create liquidity to withstand outside shock and sudden outward capital lows during the time of financial market crisis around the world. The level of the foreign exchange reserves held by the developing countries amount to about 30 percent of their GDP Rodrik (2006).

Countries need to develop self-protection policy framework that will i them from outside financial markets shocks through the creation of more liquidity as reasoned by Feldstein (1999). Aizenman et al. (2012) noted the fact that about half of the emerging markets used their reserves as the adjusting mechanism to deal with the global financial crisis that began in 2008 to 2009 in the US by depleting not more than one- third of their reserve.

On the other hand, Rodrik (2006) pointed out that accumulating and holding foreign reserves has some associated costs. In that work, he further asked an important question; " why the developing countries accumulate more reserves as an economic way to create liquidity to deal with global crisis, than to create liquidity by reducing their short term-term foreign liabilities or by creating a collateralized credit facility?"

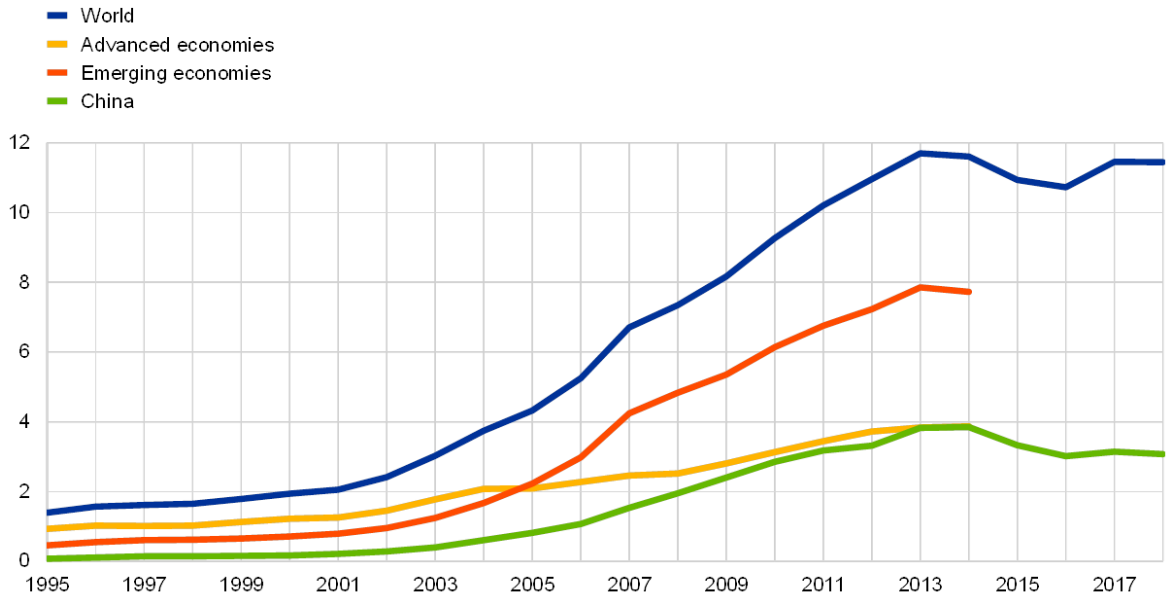
Furthermore, Aizenmen et al. (2007) pointed out that other developing countries particularly in East Asia prefer either the precautionary or mercantilist approach to accumulate foreign reserves depending on the fiscal policy they chose. The precautionary approach relates accumulating reserves to the exposure to sudden stops, capital flight, and volatility while the mercantilist approach relates the accumulation of the reserve to the residual of an industrial policy that have the effect of imposing negative external factors on other trading partners.

3.2 Evident trend analysis of the movement of foreign reserves

From Chiţu et al. (2019) we show as a background evidence for our model, the analysis and discussions will follow thereafter, the trend analysis of the movement of foreign reserves for advanced economies and emerging economies.

Below, is the trend analysis of the data for the accumulation and depletion of foreign reserves for advanced economies, emerging economies and the world for the years 1995 to 2017.

Figure 1Trend analysis of the accumulation and depletion of foreign reserves for advanced and emerging economies



Sources: IMF COFER, Haver Analytics and ECB calculations. Note: The breakdown between advanced and emerging economies is no longer available since China started reporting data for the IMF COFER.

From Figure 1. We observe that the both developed and emerging countries starts to accumulate reserves slowly from 1995 to 2001 and then starts to accumulate the reserve faster from 2001 to 2013. Due to the 2007 financial crisis, we observe from 2013 to 2016, countries started to deplete their reserves very fast, and again from 2016 to 2017 they started to accumulate and from 2017 to 2018 they then deplete the reserves again.

3.3 The methodology

International transactions in the form of international trade activities and the accompanied international financial settlements over a certain period are recorded in balanced of payment accounts. The balance of payment account is further subdivided in to the current account/ or trade account, capital account, and the financial account.

In the case where there is no government activities and for a closed economy, we record the following in the current account; the gross national product (GNP) which is a function of consumption (C), and investment (I). This model was the basis of the original work by Kalecki (1935) to model the depletion of the capital stock in time in a closed economy.

For an open economy, we include exports (X) and imports (Z) as the balance of goods and services in defining our national income (Y) for our model. The exports and imports are recorded in the nation's capital account.

The capital account balance is a function of the IS-gap (i.e. the difference between savings (S) and investments (I)) and the fiscal balance. In the case where X –Z is less than zero, then we have an economy being a net importer and has spent more than it has produced. Similarly, in

the case where X – Z is greater than zero, then we have an economy being a net exporter and it has produced more than it has spent.

In our case, we model the policy decision to accumulate and deplete foreign exchange reserves (gold or US dollars). These reserves are recorded in the financial account of nations. The national income is then assumed to be a function of the consumption (C), investment (I), exports (X), and imports (Z). Where the consumption is a function of the disposal income**,** interest burden of households, net worth of assets and liabilities, and labour confidence or expectations. Investment is a function of foreign growth, and real exchange rate.

Exports is a function of foreign growth, and real exchange rate. Imports is function of government spending, taxes rates, interest rates, net worth of assets and liabilities, and labour confidence. We note that investment is problematic function because it is highly volatile.

Thus, our model uses the current account, capital account and the financial account but for an open economy. The current account records the trade balance, nations net Income, and direct payments. On the other hand the trade balance is the balance between X and Z. The capital account is part of the balance of payments and it records the country's assets and liabilities.

In the case where current account deficit less than zero then we have the country consuming more than it can produce i.e. using more of its gold reserves to purchase imports to meet the demand for its citizen's consumption. Similarly, the case where current deficit is greater than zero means we have the country producing more than it consumes.

That is, it uses less of its gold reserves to purchase imported goods to meet the consumption demand of the citizen. Likewise, all the transactions are to be recorded in the balance of payments by the country's central bank for monitoring.

For the analysis, we shall vary the relevant model parameters. This will enable us to control the reserves movement in and out of the economy. By varying the parameters will in real time be synonymous to central banks and governments implementing various kinds of monetary and fiscal policy decisions.

We will do the analysis of our model for two realistic economic scenarios; a) for fast depletion and fast accumulation of the reserve by a country with fixed delay of the response of the economy to the policy decisions, b) for slow depletion and slow accumulation of foreign reserves by a country with a fixed delay of the response of the economy to the policy decisions. These scenarios will be elaborated more in detail in sections 5 and 6.

4. The Model framework

4.1. Introduction

In this section, we use apply the theory of delay differential equations to develop a model for an open economy for emerging markets and developing economies to model the decision to accumulate and to deplete foreign exchange reserves i,e, gold and US dollars. In the process, we manage to obtain the general polynomial series solutions for the model for the reserve accumulations as a function of time t. Time could be days, weeks, months or years.

4.2. The model

Understanding the dynamics of the foreign exchange reserve is very important for the country to manage effectively its foreign reserve effectively in the midst of global financial crises as it was the case in the early 1990s and recently in 2008 to 2009.

In our model below, we have modified the Kalecki (1935) model and have assumed that the aggregate demand of the gold reserve is a function of the country production and domestic consumption. Our model is given below,

$$Y(t) = A(t) + X(t) - Z(t), \tag{1}$$

$$A(t) = C(t) + I(t), \tag{2}$$

$$A(t) = \delta Y(t), \qquad \delta \in (0,1)\ , \tag{3}$$

$$Z(t) = zY(t), \qquad z \in (0,1)\ , \tag{4}$$

$$R^*(t) = \alpha Z(t), \qquad \alpha > 0, \tag{5}$$

$$B(t) = \lambda[R^*(t) - R(t)],\ \lambda \in (0,1), \tag{6}$$

$$X(t) = \frac{1}{\theta}\int_{t-\theta}^{t} B(\tau)d\tau, \tag{7}$$

$$\dot{R}(t) = B(t-\theta), \tag{8}$$

where

$Y(t)$ is the national income,

$A(t)$) is the domestic absorption,

$Z(t)$ is the imports,

$R^*(t)$ is the target reserve,

$B(t)$ is the decision to accumulate reserves,

λ is the response parameter ,

$X(t)$ is the exports, which are imports of the foreign economy and so are a function of past import orders of the foreign economy, which are equal to past foreign exchange payment to the national economy,

$R(t)$ is the level of existing/actual reserve, and

$\dot{R}(t)$ is the reserve accumulation.

θ is fixed time delay of the response of the economy to the policy decisions.

Eq. (1) express the national income as a function of the domestic absorption and the trade balance of the country i.e. exports, and imports. Eq. (2) defines the domestic absorption as a function of consumption C(t) and investments I(t) in the economy. Eq. (3) expresses the domestic absorption as a fraction of the national income.

The range of the parameter δ is from 0 to 1, with 0 implying that there is no domestic absorption and with 1 implying that the domestic absorption equals the national income. Eq. (4) also expresses imports as a fraction of the national income as defined in the International Monetary Fund (2019) guidelines of foreign exchange reserve management.

The range of the parameter z is from 0 to 1, with 0 implying that there are no imports and with 1 implying that imports equal the national income.

Eq. (5) expresses the target reserve as a fraction of the imports, with the parameter ranging from anything greater than 0 because the target reserve can never be zero. Eq. (6) expresses the decision to accumulate reserves as a fraction of the difference between the target reserves and the level of the existing/actual reserves.

The range of the fractional parameter λ is from 0 to 1 which implies the determination of the level of the decision to accumulate the reserves with 0 implying that there is not decision to accumulate the reserves. In Eq. (7) the export orders depends on the past decisions to accumulate the reserves with a fixed delay.

Then lastly, Eq. (8) expresses the reserve accumulation as function of the delayed decision to accumulate the reserve.

4.3. Solving the model

Substituting Eq. (3) and Eq. (4) into Eq.(1) we get,

$$Y(t) = \frac{1}{(1-\delta+z)} X(t). \tag{9}$$

Further substituting Eq. (4) into Eq. (5) we get

$$R^*(t) = \alpha z Y(t), \tag{10}$$

and also substituting Eq. (9) into Eq. (10), we get

$$R^*(t) = \frac{\alpha z}{(1-\delta+z)} X(t). \tag{11}$$

Continuing further, we substitute Eq. (11) into Eq. (6), and we get

$$B(t) = \lambda \left[\frac{\alpha z}{(1-\delta+z)} X(t) - R(t) \right], \tag{12}$$

And then substitute Eq. (7) into Eq. (12) to get

$$B(t) = \lambda \left[\frac{\alpha z}{(1-\delta+z)} \frac{1}{\theta} \int_{t-\theta}^{t} B(\tau) d\tau - R(t) \right],$$

which we then rewrite it as

$$R(t) = \frac{\alpha z}{\theta(1-\delta+z)} \int_{t-\theta}^{t} B(\tau) d\tau - \frac{1}{\lambda} B(t) . \tag{13}$$

We now substitute Eq. (8) into Eq. (13) to get

$$R(t) = \frac{\alpha z}{\theta(1-\delta+z)} \int_{t-\theta}^{t} \dot{R}\ (\tau+\theta) d\tau - \frac{1}{\lambda} \dot{R}(t+\theta) ,$$

which then simply to

$$R(t)\ = \frac{\alpha z}{\theta(1-\delta+z)} R(t+\theta) - \frac{\alpha z}{\theta(1-\delta+z)} R(t) - \frac{1}{\lambda} \dot{R}(t+\theta) . \tag{14}$$

We rewrite Eq. (14) above as

$$\dot{R}(t+\theta) = \frac{\alpha z \lambda}{\theta(1-\delta+z)} R(t+\theta) - \lambda(1 + \frac{\alpha z}{\theta(1-\delta+z)}) R(t) \tag{15}$$

But this is the same as rewriting it as

$$\dot{R}(t) - aR(t) + bR(t-\theta) = 0, \qquad R(0) = 1, \tag{16}$$

with

$$a = \frac{\alpha z \lambda}{\theta(1-\delta+z)} \qquad \text{and } b = \lambda(1 + \frac{\alpha z}{\theta(1-\delta+z)}) \tag{17}$$

Which is the deduced first-order monotonic Delay Differential Equation (DDE) for our forex accumulation model. In the next section we solve it as systems of DDE. From Eq. we can interpret the parameter $-a$ as a reserve deletion and b as a reserve accumulation after some fixed time delay θ of the response of the economy to the policy decision to mitigate the crisis shock.

Similar to theorem 1.3 of Ladas et al. (1982a), we formulate the following theorem for Eq. (16)

Theorem 1. Given the central bank response parameter λ, the fraction of the national income parameter z, the fraction of the import parameter α, the fraction of the national income parameter δ, and the fixed time delay parameter θ of the response of the economy to the policy decisions, then we have the condition

$$b\theta e^{-a\theta} > \frac{1}{e}, \qquad \text{for} \qquad a \geq 0 \text{ and } b > 0,$$

where e is a mathematical constant written explicitly as,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \ldots$$

which implies that every solutions of Eq. (20) oscillates.

4.4 Transforming the deduced delay differential equation and solving it in general terms

Proof of Theorem 1:

To transform the delay differential equation Eq. (16), we use the differential transform method and convert it into recurrence equation or recurrence relations Karakoç et al. (2009) , and Keller (2010). This method is very efficient in obtaining analytical and numerical solutions for both differential and integral equations in a form of convent series solutions Zhou (1986, 2010).

In transforming Eq. (16), we note the error in Eq. (10) of Keller's' paper that occurred when he performs a similar transformation of that Equation. The error is in the last term and here we write that term correctly and we then continue with transformation process.

Eq. (16) then transforms into a recurrence equation

$$(k+1)\bar{R}(k+1) - a\bar{R}(k) + b \lim_{T\to\infty} \sum_{h_1=k}^{T} \binom{h_1}{k} \theta^{h_1-k} \bar{R}\,(h_1) = 0, \tag{18}$$

where $\dot{R}(t)$ and $R(t-\theta)$ transformed as

$$\dot{R}(t) = (k+1)\bar{R}(k+1), \text{ and } R(t+\theta) = \lim_{T\to\infty} \sum_{h_1=k}^{T} \binom{h_1}{k} \theta^{h_1-k} \bar{R}\,(h_1). \tag{19}$$

k is the new recurrence time parameter of the transformed function $\bar{R}$ of the reserve accumulation $\dot{R}$, T is the summation order of the finite series in the limit.

We expand Eq. (18) explicitly taking T=4, and k=0,..3 and we get the following four systems of equations

For k=0, we get

$$1\bar{R}(1) - a\bar{R}(0) + b \sum_{h_1=0}^{T=4} \binom{h_1}{0} \theta^{h_1-0} \bar{R}(h_1) = 0. \tag{20}$$

For k=1, we get

$$2\bar{R}(2) - a\bar{R}(1) + b \sum_{h_1=1}^{T=4} \binom{h_1}{1} \theta^{h_1-1} \bar{R}(h_1) = 0. \tag{21}$$

For k=2, we get

$$3\bar{R}(3) - a\bar{R}(2) + b \sum_{h_1=2}^{T=4} \binom{h_1}{2} \theta^{h_1-2} \bar{R}(h_1) = 0. \tag{22}$$

For k=3, we get

$$4\bar{R}(4) - a\bar{R}(3) + b\sum_{h_1=3}^{T=4}\binom{h_1}{3}\theta^{h_1-3}\bar{R}(h_1) = 0. \quad (23)$$

Expanding further the binomial coefficients in the above system of equations we get

For k=0

$$1\bar{R}(1) - a\bar{R}(0) + b\left\{\binom{0}{0}\theta^{0-0}\bar{R}(0) + \binom{1}{0}\theta^{1-0}\bar{R}(1) + \binom{2}{0}\theta^{2-0}\bar{R}(2) + \binom{3}{0}\theta^{3-0}\bar{R}(3) + \binom{4}{0}\theta^{4-0}\bar{R}(4)\right\} = 0, \quad (24)$$

For k=1,

$$2\bar{R}(2) - a\bar{R}(1) + b\left\{\binom{1}{1}\theta^{1-1}\bar{R}(1) + \binom{2}{1}\theta^{2-1}\bar{R}(2) + \binom{3}{1}\theta^{3-1}\bar{R}(3) + \binom{4}{1}\theta^{4-1}\bar{R}(4)\right\} = 0, \quad (25)$$

For k=2,

$$3\bar{R}(3) - a\bar{R}(2) + b\left\{\binom{2}{2}\theta^{2-2}\bar{R}(2) + \binom{3}{2}\theta^{3-2}\bar{R}(3) + \binom{4}{2}\theta^{4-2}\bar{R}(4)\right\} = 0 \quad (26)$$

For k=3,

$$4\bar{R}(4) - a\bar{R}(3) + b\left\{\binom{3}{3}\theta^{3-3}\bar{R}(3) + \binom{4}{3}\theta^{4-3}\bar{R}(4)\right\} = 0 \quad (27)$$

Using a symbolic manipulation software called Maple, we simplified Eq. (14) into the following systems of equations which we are then able to solve,

$$\bar{R}(1) - a\bar{R}(0) + b(\bar{R}(0) + \theta\bar{R}(1) + \theta^2\bar{R}(2) + \theta^3\bar{R}(3) + \theta^4\bar{R}(4)) = 0; \quad (28)$$

$$2\bar{R}(2) - a\bar{R}(1) + b(\bar{R}(1) + 2\theta\bar{R}(2) + 3\theta^2\bar{R}(3) + 4\theta^3\,\bar{R}(4)\,) = 0; \quad (29)$$

$$3\bar{R}(3) - a\bar{R}(2) + b(\bar{R}(2) + 3\theta\bar{R}(3) + 6\theta^2\bar{R}(4)) = 0; \quad (30)$$

$$4\bar{R}(4) - a\bar{R}(3) + b\big(\bar{R}(3) + 4\theta\bar{R}(4)\big) = 0. \quad (31)$$

The boundary condition $\bar{R}(0) = 1$ is given which then imply that the above systems of equations has the following general solutions

$$\bar{R}(1) = \frac{4(a-b)(4b^2\theta^3a+6ab\theta^2+a^2b\theta^3+b^3\theta^3+12b^2\theta^2+6+18b\theta)}{D}, \quad (32)$$

$$\bar{R}(2) = \frac{6(a^2-2ab+b^2)(b^2\theta^2+2+ab\theta^2+4b\theta)}{D}, \quad (33)$$

$$\bar{R}(3) = \frac{4(a^3-3a^2b+3ab^2-b^3)(1+b\theta)}{D}, \quad (34)$$

$$\bar{R}(4) = \frac{(a^4-4a^3b-4ab^3+6a^2b^2+b^4)}{D}, \quad (35)$$

Where

$$D = (a^3b\theta^4 + 11b^2\theta^4a^2 + 8a^2b\theta^3 + 11b^3\theta^4a + 56b^2\theta^3a + 36ab\theta^2 + b^4\theta^4 + 32b^3\theta^3 + 108b^2\theta^2 + 24 + 96b\theta) \quad (36)$$

Our model then has the following general series solution which is oscillatory and this proves theorem 1.

$$R(t) = 1 + \bar{R}(1)t + \bar{R}(2)t^2 + \bar{R}(3)t^3 + \bar{R}(4)t^4 + \mathcal{O}t^5 \, . \quad (37)$$

Eq. (37) then expresses the level of the target reserve $R(t)$ as a function of the series solutions of the reserve accumulation $\bar{R}$ to fourth order in time t. We note that, the more terms we have in the series i.e high order terms in t, the more closer we are to getting convergent series solution.

In this case, taking the series solution to fourth order is sufficient to guarantee converging solution. In doing the analysis for our model Eq (37), we shall only consider the real case scenarios. That is, we will not consider the cases where $z = 0, \delta = 0,$ and $\lambda = 0$ as they are unrealistic.

4.5 Conclusion

In this section, we have used the theory of delay differential equations to model the decision to accumulate foreign exchange reserves. We then solved the problem and we obtained the general polynomial series solution for the reserve accumulation as function of time t.

In the next section, we shall vary the coefficient of the model to get two cases for two different policy decisions to accumulate the reserve and we then analysis the accumulation solution graphs and we make the conclusions about the nature of the kind of decisions made.

5. Analysis and Discussions of the Model

5.1 Introduction

In this section, we look at two cases to analysis the validity of our model introduced in section 4. In the first case, we construct an economic scenario where the foreign exchange reserves are depleted and accumulated fast due to sudden external financial risk factors, like devastating global financial crisis of 2007 that originated in the US. In real time, this kind of economic scenario is observed by countries after the impact of the global financial crisis shocks.

To that effect, some countries then have to recover quickly to avoid prolonged economic stagnations that leads to higher unemployment, lack of foreign investments, lower national

income, higher inflation, and ultimately risk of financial instability in the economy. This problem is prevalent in weak economies, International Monetary Fund (2019).

For the second case, we construct a complementary economic scenario where foreign exchange reserves are slowly depleted and slowly accumulated. In real time, this kind of economic scenario is observed by countries that carefully implement policies to slowly accumulate foreign reserves over a time, and when the crisis materialise, they carefully use their reserves to mitigate the impact shocks. They then implement policies to carefully and slowly accumulate the reserves over time. This problem is prevalent in strong economies, International Monetary Fund (2019).

5.2 The management of foreign exchange reserves

The central banks are the primary custodian of accumulating, managing, and holding the foreign exchange reserves as mentioned by Steiner (2013). Zhang et al. 2013 highlight the fact that among others with regard to reserve management, the central banks on a daily basis must maintain and manage the foreign currency exchange payments or trades and must avoid liquidity crises. According to the IMF Guidelines for Foreign Exchange Reserve Management (2001), central banks entails the following functions;

● support and maintain confidence in the policies for monetary and exchange rate management including the capacity to intervene in support of the national or union currency,

● limit external vulnerability by maintaining foreign currency liquidity to absorb shocks during times of crises or when access to borrowing is curtailed,

● provide a level of confidence to markets that s country can meet its external obligations,

●demonstrate the backing of domestic currency by external assets,

●assist the government in meeting its foreign exchange needs and external debt obligations, and

●maintain a reserve for national disasters or emergencies.

Through its interaction with financial markets, the central banks are able to get the valuable information they need that they can use to develop monetary policies that they can use to respond effectively to the global financial crises. In the next section we will analyse the results of our model in section 4 and see how the country's central banks can help manage the accumulated reserves in times of global financial crisis.

5.3.1 Case 1: fast depletion of foreign exchange reserve and the accumulation thereafter

In the case of Keller (2010), the values $\bar{R}(1)$, $\bar{R}(2)$, $\bar{R}(3)$, and $\bar{R}(4)$ are found explicitly when assuming the following numerical values, $\lambda = 2/5, \theta = 1, z = 3/4, \delta = 1/2,\ \alpha = 1$, and then a, b simplify to

$a = 0.24\ ,\ b = 0.64.$ Therefore, from the recurrence relations Eqs. (28) – (31), we obtained the explicit series solutions of the reserve accumulations, $\bar{R}(1)$, $\bar{R}(2)$, $\bar{R}(3)$, and $\bar{R}(4)$, i.e. Eqs. (32) – (36) as,

$$\bar{R}(1) = -0.256, \bar{R}(2) = 0.0327, \bar{R}(3) = -0.0028, \bar{R}(4) = 0.00017, \quad (38)$$

The target reserve Eq. (37) then simplifies to

$$R(t) = 1 - 0.256t + 0.0327t\text{^}2 - 0.0028t\text{^}3 + 0.00017t\text{^}4 + \mathcal{O}(t^5). \quad (39)$$

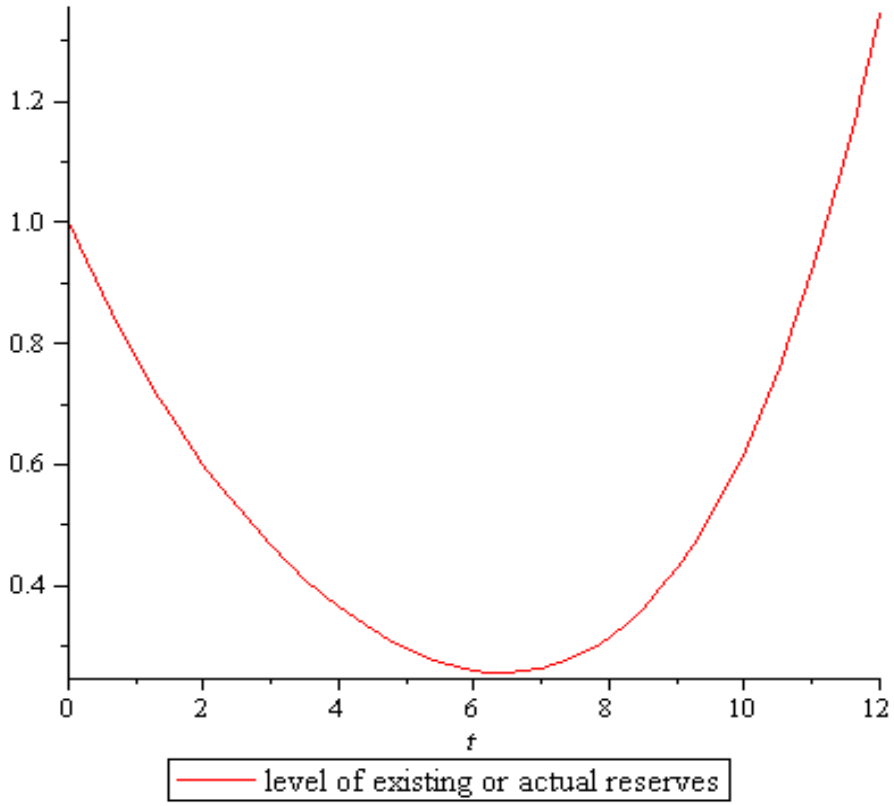
The above equation is a polynomial series solution of the target reserve R in time t to the fourth order.

In this section, we shall discuss the realistic scenario where the reserves are depleted very fast during the global financial crisis, and are then accumulated very fast as a result of the policy decisions of the central banks.

In the next section we shall discuss the complementary situation where the reserves are depleted very slowly during the global financial crisis, and are then accumulated very slowly as a result of the policy decisions of the central banks.

In the case where the reserves are depleted and accumulated very fast, is when we have a situation where a country is fast using its reserves to mitigate the crippling economic problems caused by the global financial crisis but the central bank management implement drastic policy measures to accumulate the reserves to boost its ability to withstand such crisis in the future. Below is the graph that depicts this scenario for Eq. (43) with time raging from $t = 0\ to\ 12$.

Figure 2Fast depletion and accumulation of foreign exchange reserves



5.3.2. Discussions

The model parameter δ if it is 0 in Eq. (3), then it imply that the domestic absorption is 0, that is, it is not a fraction of the national income. This then means that the national income is only determined by the trade balance between exports and imports in the economy.

On the other hand, if δ is 1, then this imply that the nation income will be determined by the domestic absorption only and that the trade balance will be zero because imports will equal exports and cancel out in Eq. (1).

For our case, we have chosen δ to be 0.5, which then imply that the domestic absorption function is half a fraction of the national income of the country which then imply that the national income will be determined by twice the trade balance in Eq. (1).

The parameter z if it is 0 in Eq. (4), then it implies that the imports in Eq. (4) are zero, which means that the imports are not a fraction of the national income because the country does noes not import goods and services.

This then imply that the trade balance is positive and it dependent only on the exports of goods and services. This means we have an inflow of capital and thus boosting the financial account of the country. This will then enable the country to have the capacity to accumulate gold reserves for example.

If z is 1, then this means imports equal the national income in Eq. (4). This then mean that all the national income is spend on buying imported goods and services and this means that we have outflow of capital and in this kind of economic situation the country does not have the means to accumulate.

But for our case, we have taken z to be 0.75 which means that imports are a substantial fraction (more than half) of the national income. That is, more than half of the national income is spend on buying imported goods and services. This means that we have a negative trade balance. Thus we have outflow of capital and also that the reserves are being depleted as can be observed in figure 1. above between $t = 0$ and $t = 6$. Furthermore, it means we have a negative balance of payments.

The parameter α in Eq. (5) can be taken to be any value greater than zero but never zero because if it is zero, then that means that the country does not have the intention to accumulate the reserve. That is, its target reserves to be accumulated is zero. The correct economic fact is that all countries must accumulate foreign exchange reserves to help them mitigate unforeseen global financial risks.

Therefore nations must always seek to have the desired or targeted level of the reserves to be accumulated at any given moment as long as the targeted level is within the financial abilities of these nations accumulate.

The model response parameter λ in Eq. (6) influence the decision by the central banks to accumulate the reserves. For example, if it is taken to be 0, then it implies that there won't be the decision to accumulate the reserves at all, but if it is taken as 1, then it implies that there will be a prompt decision to accumulate the reserves.

But now we have chosen it to be 0.4, which then imply that the decision to accumulate the reserve will not be hasty but taken slowly. That is the reserve will be accumulated in a moderate or adequate speed as can be observed in figure 1. above just after time $t = 6.$ This then imply that since the national income is determined by twice the trade balance as discussed above and that the reserves will be accumulated in a more steady speed, then we are more liked to have a positive balance of payment in in this case.

5.4.1. Case 2: slow depletion and accumulation of foreign exchange reserve

In the case of Keller (2010), the values $\bar{R}(1)$, $\bar{R}(2)$, $\bar{R}(3)$, and $\bar{R}(4)$ are found explicitly when assuming the following numerical values, $\lambda = 2/3, \theta = 1, z = 1/2, \delta = 1/3,\ \alpha = 0.1$, and then a, b simplify to $a = 0.029$, $b = 0.695.$

Therefore, from the recurrence relations Eqs. (28) – (31), we obtained the explicit series solutions of the reserve accumulations, $\bar{R}(1)$, $\bar{R}(2)$, $\bar{R}(3)$, and $\bar{R}(4)$, i.e. Eqs. (32) – (36) as,

$$\bar{R}1 = -0.426,\ \bar{R}2 = 0.091\ ,\ \bar{R}3 = -0.013, \bar{R}4 = 0.0013 \quad (40)$$

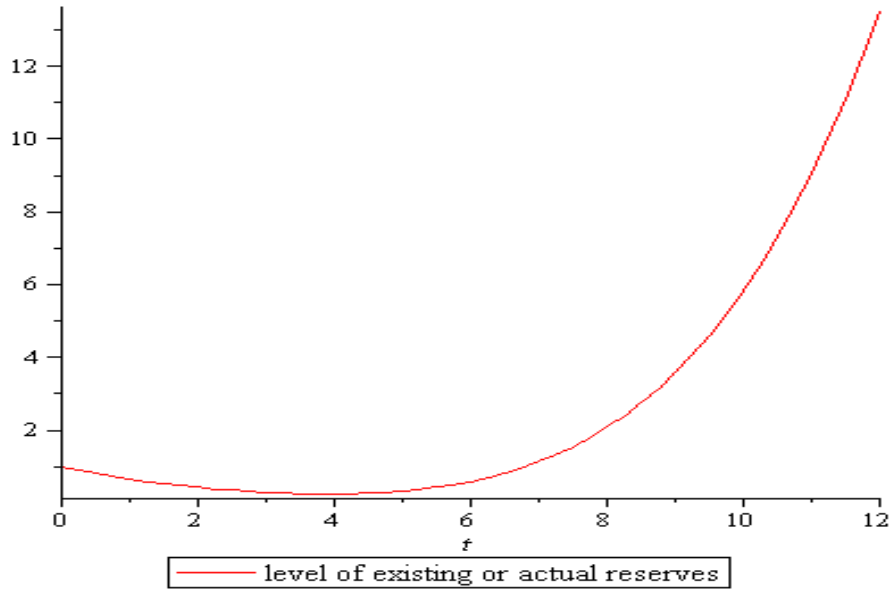
Similarly to Eq. (39), the target reserve Eq. (37) then simplifies to a polynomial series solution in time t,

$$R(t) = 1 - 0.426t + 0.091t\text{^}2 - 0.013t\text{^}3 + 0.0013t\text{^}4 + \mathcal{O}(t^5). \quad (41)$$

In the case where the reserve are depleted and accumulated very slowly, is when we have a situation where a country is slowly using its reserves to mitigate the impact of the global financial crisis over time, and the central bank implement policy decisions to slowly accumulate the reserves. We also note that the delay in the response of the economy to the decisions also has some impact to the effectives of the decisions between $t = 3$ and 5.

Below is the graph that depicts this scenario for Eq. (41). In the graph, we can also see that in later times, the accumulation rate is slightly increasing between $t = 8$ and 9. We can interpret this as being that the effects of the delay have decreased significantly in this time interval, and that the policies to accumulate are much more effective here. The equation have been plotted from t=0 to 12 .

Figure 3Slow depletion and accumulation of foreign exchange reserves



5.4.2. Discussions

In the case, we have chosen δ to be 0.33, which then imply that the domestic absorption is a less than half the fraction of the national income of the country in Eq. (3). This means that consumption and investment are not contributing substantially to the national income than

exports and imports in Eq. (1). Therefore, the balance of trade contribution to the GDP is substantially above that of the domestic consumption and investment.

Further, we chose the parameter z to be 0.5 in Eq. (4), which then imply that imports cost of goods and services to the national income in Eq. (5) is about half a fraction. This imply that the trade balance is negative because exports together with the domestic absorption contribute the other half of the fraction to the national income.

Therefore, because of the negative trade balance, we have an outflow of capital and this also affects the health of the financial account of the country because it be forced to used its foreign reserves either cash or gold to sustain the economic situation.

We can observe this phenomena in figure 2 above. In that graph, we see that the reserves are depleted gradually between $t = 0$ and $t = 4$. Thereafter, the country is then able to slowly accumulate the reserve between $t = 4$ and $t = 8{,}4$ (say) and from $t = 8.4$ it is accumulating the reserve fast.

Also, we have chosen the parameter λ in Eq. (6) to be 0.67, which then imply that the impact of the response to decide to accumulate the reserve will not be instantly strong as evident in figure 2 above. As discussed in the above paragraph, among others, this is because of the economic conditions we have chosen in this case.

It is only later that the impact of the policy decision is strong because of the resulting positive balance of payments due to the decision. That's is, the decision to accumulate inevitably results in the positive balance of payments and the inflow of capital either through investments (i.e. foreign direct investments or local invest) or exports in this case.

5.5. Conclusions

We have discussed and analysed the results of our model introduced in section 4 formulated using delay differential equations, and solved using differential transform method to obtained the polynomial series solutions of the target reserves.

We constructed two realistic scenarios for the depletion and accumulation of the reserves by countries during times of the global financial crisis. The one scenario was for the fast depletion and fact accumulation of the reserves, and the one scenario was for the slow depletion and slow accumulation during the global financial crisis.

For the two cases, we were able to model the policy decisions that can be made by the central banks of countries for the depletion and accumulation of the reserves with a fixed time delay of the response of the economy to the impact of these policy decisions.

To that effect, we varied the model parameters to formulate the scenarios. Intuitively, we found that the results of our model are in line with what can be expected about the accumulation of the reserves given the kind of cases we formulated, International Monetary Fund (2019).

The results of our modele show a very strong relevance to the empirical and practical results of figure 1 for all countries. The next step for our model is to use the empirical data as in figure 1 to do the robust check of model to the data by estimating the relevant model variables and do the comparisons.

In the next section, we conclude our research report and we also give recommendations for future work in this direction.

6. Conclusions and Recommendations

In conclusion the effects of the delay factor θ is that, if it is closer to 0 it prolongs the accumulation process of the reserves, and if is lager, it shorten the accumulation process. In addition, we have also observed that the same effects can be obtained by varying the other model variables (α, λ, and δ) while the delay factor is kept constant.

This is to say many permutations of varying the model variables (α, λ, and δ) for various economic scenarios can be obtain while the nature of the delay process from the source is constant either within the countries or outside the countries. This is true in real economic situations where various macroeconomic factors have a direct dynamical effects on the reserves Calve et. al. (1993).

Therefore, more research on the macroeconomic nature of the delay need to be done. Keller (2010) did not dwell much about explaining the fundamental implications of his mathematical results to economics. For example in real economic scenarios, the delay could be a function of other macroeconomic variables, for example the impact of foreign growth and foreign direct investment on the local economy during and after the financial crisis.

Furthermore, we did not include in our model, the foreign exchange rate to fully model the problem, but this can be done by modifying Eq. (1) and include it. This will then require us to include the elasticities of the exports and imports in the trade balance in Eq. (1), and then modifying other equations of the model (i.e. Eqs 2 - 8) accordingly.

We will then have a new single master equation Eq. (37) for the new model that must be solved along similar lines of arguments as we did, but perhaps much more involved and expansionary considering other relevant macroeconomic policy factors.

Finally, the inclusion of foreign exchange rate regime in our model will certainly require us to involve stochastic differential equations analysis. By doing this, we will then get a full understanding of the movement of the reserves in and out of a country in the form of the accumulation and depletion of the reserves.

Approaching the problem to study the movement of the reserve depletion and accumulation for an open economy using the delay differential equations has not been done before. This is the first time that an attempt has been made to this effect in this report.

As stated before, this approach is concise, yet it summaries all the characteristic of the problem in a single master equations as opposed to other well know econometrics methods.

The methods involve a large number of equations and variables that must be solved and interpreted at the end in the analysis of the results. So, indeed more future work needs to be done in this direction within the context of foreign exchange reserve analysis using delay differential equations.

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