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Contributions to simple random walks enumeration

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Chapter 1

General Introduction

In this research lattice paths (or simple random walks) with at most three directions are considered. In essence, we are going to use an elementary but elegant approach to enumerate Dyck, Knödel and Motzkin paths according to various parameters such as number visits to some nonnegative height and number of returns to the x-axis.

1.1 Basic concepts and notation

Definition 1 *A **Dyck path (or Dyck random walk)** is a nonnegative lattice path of $\mathbb{N}^2$ running from $(0,0)$ to $(0,n)$ and whose allowed steps are the up diagonal step $(1,1)$ and the down diagonal step $(1,-1)$.*

Definition 2 *A **Motzkin random walk** of length n is a nonnegative lattice path of $\mathbb{N}^2$ running from $(0,0)$ to $(n,0)$, whose allowed steps are the up diagonal step $(1,1)$, down diagonal step $(1,-1)$ and the horizontal step $(1,0)$.*

A Dyck path allowing horizontal steps only on the x-axis (or 0-level) is called **Knödel path.** Any Dyck walk which goes below the x-axis will be called an **unrestricted Dyck random walk.** In the same manner we shall define **unrestricted Motzkin random walk.**

Definition 3 *We define the **height** of a path to be the maximum y value attained and the **length** of the path by the total number of steps of size 1. A path is said to be*

- ***open*** *if it ends at a point* (n, i) *for* $i > 0$,
- ***closed*** *if it ends on the* x*-axis.*

All these types of random walks described above are very important in many disciplines such as Combinatorics, Probability Theory, Statistics, and Theoretical Computer Science. In the last century various researches, such as, [4], [13], [14], [15], [17], [18], [19], and [20] were done concerning enumeration of random walks with regard to:

- the average number of returns to the origin,
- the average nonnegative height.
- its maximum deviation from the origin.
- combinatorial connections between some of these random walks.

Definition 4 *[9] Given a sequence* $a_0, a_1, a_2, a_3, ..., a_k, ...,$ *the function*

$$f(z) = \sum_{k \geqslant 0} a_k z^k,$$

is called the ***ordinary generating function*** *(OGF) of the sequence. We use the notation* $\left[z^k\right] f(z)$ *to refer to the* ***coefficient*** a_k.

We want to emphasize the methodological point of view to be employed here, i.e., the use of combinatorial constructions as well as their associated generating functions. The advantage of this approach is that it makes these known results more extendable into a larger class of walks and new problems. For example, the correspondence or bijections between Motzkin and Catalan permutations studied in [2] will be done here using asymptotic lattice paths enumeration. The famous results [8] by Flajolet and Odlyzko play a key role here in transferring our results into asymptotic expansions.

Prodinger and Panny in [19] derived an asymptotic formula for the average height of unrestricted Dyck random walks which end at $(n, 0)$, which we call **closed ending**. In this research we will try to prove these results in the cases where both Knödel and Motzkin paths are unrestricted. In all these models we will try to generalize all our results even to walks which end anywhere at (n, i) for any $i \neq 0$ in $\mathbb{Z}$, which we call **open ending**.

1.2 Combinatorial Identities

Most combinatorial identities used in this thesis have already been used in many literatures such as [3], [7], [9], [10], [11], [22] and hence will not be proved again here.

Definition 5 *[9] Given a double indexed sequence* $\{a_{nk}\}$*, the function*

$$A(u,z)=\sum_{n\geq 0}\sum_{k\geq 0}a_{nk}u^k z^n,$$

is called a ***bivariate generating function*** *of the sequence.*

Binomial Theorem: Let n be a positive integer. Then for all x and y,

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

Some useful binomial coefficients:

1. $$\binom{n}{k}=\left[z^k\right](1+z)^n.$$

2. $$\binom{-n}{k}=(-1)^k\binom{n+k-1}{k}.$$

3. $$\binom{n+k-1}{k}=\left[z^k\right]\left(1+z+z^2+...\right)^n.$$

4. $$\binom{-\frac{1}{2}}{n}=\frac{(-1)^n}{4^n}\binom{2n}{n}.$$

5. $$\binom{\frac{1}{2}}{n}=\frac{(-1)^{n-1}}{2^{2n-1}n}\binom{2n-2}{n-1}.$$

6.
$$\binom{-\frac{3}{2}}{n} = \frac{(2n+2)(-1)^n}{2^{2n+1}}\binom{2n+1}{n+1}.$$

7.
$$\binom{\frac{3}{2}}{n} = \frac{3(-1)^n}{2^{2n-2}(n-2)(n-3)}\binom{2n-4}{n-4}.$$

8.
$$\binom{-\frac{5}{2}}{n} = \frac{(2n+3)(2n+4)(-1)^n}{2^{2n+2}.3}\binom{2n+2}{n+2}.$$

9.
$$\binom{\frac{5}{2}}{n} = \frac{15(-1)^{n-1}}{2^{2n-3}(n-3)(n-4)(n-5)}\binom{2n-6}{n-6}.$$

10. (*Symmetry relation*)
$$\binom{n}{k} = \binom{n}{n-k}.$$

The **composition of an integer** n :
$$\sum_{k=1}^{n}\binom{n-1}{k-1} = \sum_{k=0}^{n-1}\binom{n-1}{k} = 2^{n-1}.$$

Cauchy Integral Formula:
$$\begin{aligned}[z^n]\,g(z) &= \frac{1}{2\pi i}\oint \frac{(1-v^2)(1+v^2)^{n-1}}{v^{n+1}} g(z)\,dv \\ \text{where } z &= \frac{v}{1+v^2}.\end{aligned}$$

or
$$\begin{aligned}[z^n]\,f(z) &= \frac{1}{2\pi i}\oint \frac{(1-v^2)(1+v+v^2)^{n-1}}{v^{n+1}} f(z)\,dv \\ \text{where } z &= \frac{v}{1+v+v^2}.\end{aligned}$$

Catalan Numbers:
$$[z^n]\,\frac{1-\sqrt{1-4z}}{2z} = \frac{1}{n+1}\binom{2n}{n}.$$

Some useful trinomial coefficients:

$$\begin{aligned}\binom{n,3}{k} &= \left[v^k\right]\left(1+v+v^2\right)^n \\ &= \left[v^k\right]\left(1+v+v^2\right)\left(1+v+v^2\right)^{n-1} \\ &= \binom{n-1,3}{k}+\binom{n-1,3}{k-1}+\binom{n-1,3}{k-2}.\end{aligned}$$

Motzkin numbers:

$$[z^n]\,\frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}=\binom{n,3}{n}-\binom{n,3}{n-2}.$$

Asymptotic expansions:

1. [9]; Given a function $f(n)$, we write

$$f(n)\sim g(n) \text{ if and only if } \frac{f(n)}{g(n)}\to 1 \text{ as } n\to\infty.$$

2. [20]; An asymptotic expansion of trinomial coefficients formula:

$$\binom{n,3}{n-2k}\sim 3^n\sqrt{\frac{3}{4\pi n}}\exp\left(-\frac{3k^2}{n}\right)\left\{1-\frac{3}{16n}+\frac{9}{4n^2}k^2-\frac{9}{4n^3}k^4\right\}.$$

Chapter 2

Dyck paths

2.1 Introduction.

The purpose of this Chapter is to analyze how often an arbitrary Dyck path which starts at $(0, 0)$ and ends at (n, i),

(i) visits a nonnegative height r

(ii) returns to the 0-level

as shown in the following diagram:

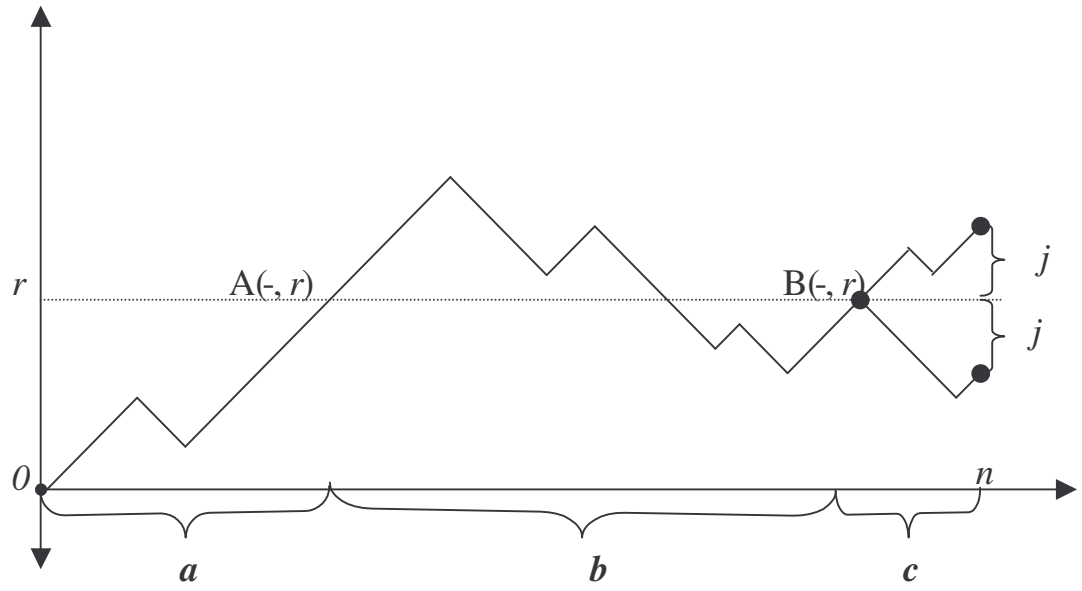


Figure 2.1: Open ended Dyck path of length n

These results will be achieved by first finding the generating function of a Dyck random walk in terms a nonnegative height r or its returns to the origin. Applying [8] and [16] onto this generating function we obtain corresponding moments through asymptotic expansion technique. This problems were researched in [14], [15], [17], and [18] by different methods. In this research we use the method of combinatorial constructions and their associated generating functions. A matrix approach will be employed and this will yields more contributions to Dyck paths enumeration which are easily extendable into newer results.

The generating function for the Dyck path of length n, ending at (n, i) where $i = 0, 1, 2, ..., h$ is denoted by $\varphi_{i,h}(z)$ where z counts the single steps is

$$\varphi_{i,h}(z) = \sum_{n\geq 0} \left[\begin{array}{l} \text{Number of paths of length } n, \text{ starting at } (0,0) \\ \text{ending at } (n,i) \text{ with height } \leq h \end{array} \right] z^n.$$

We describe the step from level i to the next level with the following recursions, which differ according to the value of i:

(i) For $1 \leq i < h$ we have that

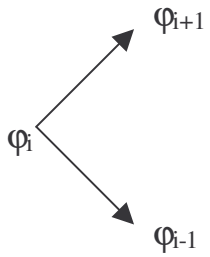


Fig. 2.2.

which gives the recursion

$$\varphi_{i,h}(z) = z\varphi_{i+1,h}(z) + z\varphi_{i-1,h}(z),$$

indicating that from level i we either go up to level $i+1$ or down to level $i-1$.

(ii) Next, for $i = 0$ we have a special case

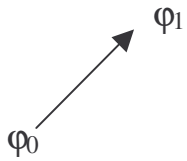


Fig. 2.3.

which gives the typical recursion

$$\varphi_{0,h}(z) = 1 + z\varphi_{1,h}(z)$$

indicating that we either stay at the origin or go up to level 1.

(iii) The final special case is for $i = h$ shown by the following diagram:

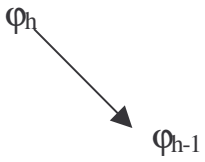


Fig. 2.4.

This is represented by the typical recursion

$$\varphi_{h,h}(z) = z\varphi_{h-1,h}(z).$$

These recursions, for $i = 0, 1, 2, ..., h$ are expressed in a single matrix equation

$$B_h \Phi_h = C_{h,}$$

where the coefficient matrix B_h is a square matrix with $h+1$ rows, Φ_h is the column matrix representing the $\varphi_{i,h}(z)$'s and C_h is the column matrix consisting of h zeros and a 1 in the first entry. Thus

$$\begin{bmatrix} 1 & -z & & & & & & \\ -z & 1 & -z & & & & & \\ & -z & 1 & -z & & 0 & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & 0 & & & -z & 1 & -z \\ & & & & & & -z & 1 \end{bmatrix} \begin{bmatrix} \varphi_{0,h}(z) \\ \varphi_{1,h}(z) \\ \varphi_{2,h}(z) \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{h-1,h}(z) \\ \varphi_{h,h}(z) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix} \tag{2.1}$$

We will calculate the $\varphi_{i,h}$'s using Cramer's rule. To do this we first determine the determinant of matrix B_h. For simplicity we shall first find the determinant of matrix A_h but with h rows only. We shall denote it by $\det A_h$ so that A_h has h rows whereas B_h has $h+1$ rows.

Lemma 6 *The determinant of the matrix B_h of the system of the $\varphi_{i,h}$'s with $h+1$ rows is given by*

$$\det B_h = \frac{1 - v^{2h+4}}{(1-v^2)(1+v^2)^{h+1}}.$$

Proof. By expanding the first row of A_h we obtain the recursion of order two:

$$\det A_h = \det A_{h-1} - z^2 \det A_{h-2}, \;\; \det A_0 = 1, \;\; \det A_1 = 1.$$

The characteristic equation is $\lambda^2 - \lambda + z^2 = 0$ with roots $\lambda_{1,2} = \frac{1 \pm \sqrt{1-4z^2}}{2}$.

Now using the substitution

$$z = \frac{v}{1+v^2},$$

we get

$$\lambda_1 = \frac{1}{1+v^2} \text{ and } \lambda_2 = \frac{v^2}{1+v^2}.$$

However the determinant of A_h can be written as a linear combination of λ_1^h and λ_2^h so that

$$\det A_h = \alpha\lambda_1^h + \beta\lambda_2^h.$$

Solving for α and β :

By letting $h = 0$ we obtain $1 = \alpha + \beta$; similarly $h = 1$ gives $1 = \alpha\lambda_1 + \beta\lambda_2$.

Combining the above two equations we finally get

$$1 = \alpha\left(\frac{1+\sqrt{1-4z^2}}{2}\right) + \beta\left(\frac{1-\sqrt{1-4z^2}}{2}\right),$$

so that

$$\alpha = \frac{\lambda_1}{\sqrt{1-4z^2}} \text{ and } \beta = \frac{-\lambda_2}{\sqrt{1-4z^2}}.$$

Hence

$$\begin{aligned}
\det A_h &= \frac{1}{\sqrt{1-4z^2}}\left[\lambda_1^{h+1} - \lambda_2^{h+1}\right] \\
&= \frac{1+v^2}{1-v^2}\left[\frac{1}{(1+v^2)^{h+1}} - \left(\frac{v^2}{1+v^2}\right)^{h+1}\right], \text{ since } z = \frac{v}{1+v^2} \\
&= \frac{1-v^{2h+2}}{(1+v^2)^h(1-v^2)}.
\end{aligned}$$

Hence the required determinant of the system with $h+1$ is

$$\begin{aligned}
\det B_h &= \det A_{h+1} \\
&= \frac{1-v^{2h+4}}{(1-v^2)(1+v^2)^{h+1}}.
\end{aligned}$$

■

Theorem 7 *The generating function of all Dyck paths bounded above by the line $h > 0$ which ends at (n, i) is given by*

$$\varphi_{i,h}(z) = \frac{v^i\left(1 - v^{2h-2i+2}\right)(1+v^2)}{1 - v^{2h+4}},$$

where $z = \frac{v}{1+v^2}$.

Proof. We shall now compose the new matrix called $B_{i,h}$ which is B_h where the i-th column is replaced by the column matrix C_h. Expanding by the i-th column we obtain the determinant

$$\det B_{i,h} = z^i \det B_{h-i-1}.$$

Hence using Lemma 6 and Cramer's rule we get

$$\varphi_{i,h}(z) = z^i \frac{\det B_{h-i-1}}{\det B_h} = \frac{v^i\left(1 - v^{2h-2i+2}\right)(1+v^2)}{1 - v^{2h+4}},$$

as required. ■

Remark 1 *If we remove the restriction on the height of the path and let $h \to \infty$ we get*

$$\varphi_{i,\infty}(z) = v^i(1+v^2).$$

Summing up over all $i \geq 0$ we get

$$\sum_{i\geq 0} \varphi_{i,\infty}(z) = \frac{(1+v^2)}{(1-v)},$$

so that

$$\begin{aligned}
D_{open}(n) &= [z^n] \sum_{i\geq 0} \varphi_{i,\infty}(z) \\
&= \frac{1}{2\pi i} \oint \frac{(1-v^2)(1+v^2)^{n-1}}{v^{n+1}} \frac{(1+v^2)\,dv}{(1-v)}, \text{ by Cauchy's Integral Formular} \\
&= [v^n](1+v)\left(1+v^2\right)^n \\
&= \begin{cases} \binom{n}{\frac{n}{2}} & , \text{for } n \text{ even} \\ \binom{n}{\frac{n-1}{2}} & , \text{for } n \text{ odd} \end{cases} \\
&\sim \begin{cases} 2^n\sqrt{\frac{2}{\pi n}}\left(1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{5}{128n^3} + ...\right) & , \text{for } n \text{ even} \\ 2^n\sqrt{\frac{2}{\pi n}}\left(1 - \frac{3}{4n} + \frac{25}{32n^2} - \frac{105}{128n^3} + ...\right) & , \text{for } n \text{ odd} \end{cases} \text{, using computer algebra,}
\end{aligned}$$

which is the asymptotic expansion of the total number of open Dyck paths of length n, as $n \to \infty$.

■

2.2 Generating functions for the number of visits to the r level.

We now need to find the bivariate generating function for open Dyck paths starting at $(0,0)$ and ending at (n,i) where z counts the single steps and u counts the number of returns to level r, $r > 0$.

We split the family of paths into three groups, as shown in Figure 2.1, and work out the generating function for each case:

(a) Lemma 8: Paths reaching r for the first time

(b) Lemma 9: Paths starting and ending on r

(c) Lemma 11: Paths starting on level r and not returning to it.

Lemma 8 *The generating function of the Dyck path starting from* $(0,0)$ *reaching* r *for the first time is given by*

$$z\varphi_{r-1,r-1}(z) = \frac{v^r(1-v^2)}{(1-v^{2r+2})},$$

where $z = \frac{v}{1+v^2}$.

Proof. This Dyck path starts at $(0,0)$ and ends at $\mathbf{A}(-,r)$. We can further decompose this Dyck path into two paths, namely;

(i) a path which starts at $(0,0)$, ends at $r-1$, and is consequently bounded below by 0 and above by the line $r-1$. Using the $\varphi_{i,h}(z)$ formula derived in Theorem 7 we get the generating function for this paths as $\varphi_{r-1,r-1}(z)$.

(ii) the last extra single z step, which makes the paths in (i) above to reach the r-level for the first time.

Combining the above two decompositions and using the substitution $z = \frac{v}{1+v^2}$ we get the generating function for part (a) of the Dyck path in Figure 2.1 which is given by

$$z\varphi_{r-1,r-1}(z) = \frac{v^r(1-v^2)}{(1-v^{2r+2})}.$$

■

Lemma 9 *The generating function of the Dyck path which starts and ends on r is given by*

$$F_r(z,u) = \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right)},$$

where $z = \frac{v}{1+v^2}$ and u is counting the number of returns to the origin.

Proof. This random walk starts at $A(-,r)$ and ends at $B(-,r)$ in Figure 2.1. By a simple shift we let level r be the 0-level. Let's denote the generating function for this random walk by $F_r(z,u)$, where u is counting the number of returns to the r-level. Furthermore, let $F_{r,+}(z)$ and $F_{r,-}(z)$ denote the generating functions of random walks that are above and below the 0-level respectively. Between any two returns on the 0-level, we have a sequence of paths above and below the x-axis, which we denote symbolically as $F_r(z,u) = (F_{r,+}(z) \cup F_{r,+}(z))^*$. Hence

$$F_r(z,u) = \frac{u}{1-u(F_{r,+}(z)+F_{r,+}(z))}.$$

The evaluation of $F_{r,+}(z)$ is done by treating the $(r+1)$- level as the 0-level and looking at paths starting at level 0 with no height limit and forcing the paths to touch the original r level at the beginning and at the end with two extra steps denoted by z^2. Hence

$$F_{r,+}(z) = z^2\varphi_{0,\infty}(z).$$

For $F_{r,-}(z)$, we let the 0-level be the $(r-1)$-level and use symmetry. We need paths starting and ending at the 0-level with a maximum height of $r-1$ and then force the ends to touch the r-level with two extra steps so that

$$F_{r,-}(z) = z^2\varphi_{0,r-1}(z).$$

Thus the decomposition of $F_r(z,u)$ is given by

$$F_r(z,u) = \frac{u}{1-u\left[z^2\varphi_{0,\infty}(z) + z^2\varphi_{0,r-1}(z)\right]}.$$

Using the substitutions

$$z = \frac{v}{1+v^2} \text{ and } \varphi_{i,h}(z) = \frac{v^i\left(1-v^{2h-2i+2}\right)\left(1+v^2\right)}{1-v^{2h+4}},$$

we get

$$\begin{aligned} 1-\left[z^2\varphi_{0,\infty}(z) + z^2\varphi_{0,r-1}(z)\right] &= 1 - \frac{v^2}{(1+v^2)} - \frac{v^2}{(1+v^2)}\frac{(1-v^{2r})}{(1-v^{2r+2})} \\ &= \frac{1-v^2}{(1+v^2)(1-v^{2r+2})}, \end{aligned}$$

so that

$$\begin{aligned} z^2\varphi_{0,\infty}(z) + z^2\varphi_{0,r-1}(z) &= 1 + z^2\varphi_{0,\infty}(z) + z^2\varphi_{0,r-1}(z) - 1 \\ &= 1 - \left[1 - \left(z^2\varphi_{0,\infty}(z) + z^2\varphi_{0,r-1}(z)\right)\right] \\ &= 1 - \frac{1-v^2}{(1+v^2)(1-v^{2r+2})}. \end{aligned}$$

It follows that

$$F_r(z,u) = \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right)},$$

as required. ■

Proposition 10 *The generating function of the s-factorial moments multiplied by the number of walks of length n in $F_r(z,u)$ is given by*

$$M_{(s)}(z) = s!\sum_{i=1}^{s-1}\binom{s-1}{i}(-1)^i\left(1-v^2\right)^{i-s-1}\left(1+v^2\right)^{s-i+1}\left(1-v^{2r+2}\right)^{s-i+1}.$$

Proof. Now

$$\begin{aligned}
M_{(s)}(z) &= \frac{\partial^s}{\partial u^s} F_r(z,u)\big|_{u=1} \\
&= s!\frac{\left[z^2\varphi_{0,\infty}(z)+z^2\varphi_{0,r-1}(z)\right]^s}{\left[1-z^2\varphi_{0,\infty}(z)-z^2\varphi_{0,r-1}(z)\right]^{s+1}} + s!\frac{\left[z^2\varphi_{0,\infty}(z)+z^2\varphi_{0,r-1}(z)\right]^{s-1}}{\left[1-z^2\varphi_{0,\infty}(z)-z^2\varphi_{0,r-1}(z)\right]^{s}} \\
&= \frac{s!\left[z^2\varphi_{0,\infty}(z)+z^2\varphi_{0,r-1}(z)\right]^{s-1}}{\left[1-z^2\varphi_{0,\infty}(z)-z^2\varphi_{0,r-1}(z)\right]^{s+1}} \\
&= \frac{s!\left[1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right]^{s-1}}{\left[\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right]^{s+1}} \\
&= s!\sum_{i=0}^{s-1}\binom{s-1}{i}(-1)^i\left(1-v^2\right)^{i-s-1}\left(1+v^2\right)^{s-i+1}\left(1-v^{2r+2}\right)^{s-i+1}.
\end{aligned}$$

■

Lemma 11 *The generating function for a Dyck path which starts at the r-level and does not return to the r-level is given by*

(a) $z\varphi_{j-1,\infty}(z) = v^j$, when j is above r.

(b) $z\varphi_{j-1,r-1}(z) = \frac{v^j\left(1-v^{2r-2i+2}\right)}{(1-v^{2r+2})}$, when j is below r.

Proof. This lattice path starts at $B(-,r)$, and either ends at $(n,r+j)$ or $(n,r-j)$ without returning to the r-level as shown on part (c) of Figure 2.1. So we have two unsymmetrical cases here:

Case a: *Random walk which ends at* $(n,r+j)$.

This random walk starts at $B(-,r)$ and ends above the r-level at $(n,r+j)$ without making any returns to the r-level. We make a first step and then treat the $(r+1)$-level as the zero level and therefore end at a point $(n,j-1)$ with no boundary for the height. Hence the generating function for this case is

$$z\varphi_{j-1,\infty}(z) = v^j.$$

Case b: *Random walk which ends at* $(n, r-j)$.

This lattice path starts at $B(-,r)$ and ends at $(n, r-j)$ without making any returns to the r-level. We make a first step and then treat the $r-1$ as the 0-level and therefore end up at a point $(n, j-1)$, but this time we are bounded above by $r-1$ as the path must not touch level r. Hence the generating function is

$$z\varphi_{j-1,r-1}(z) = \frac{v^j\left(1-v^{2r-2j+2}\right)}{\left(1-v^{2r+2}\right)}.$$

■

2.3 Random walks with closed endings

The random walk discussed in case (b) of Lemma 11 gives a random walk with a **closed ending** if $r-j=0$.

Theorem 12 *The generating function for a Dyck path which starts at the origin and has a closed ending at* $(n,0)$ *is given by*

$$G_{r,closed}(z,u) = \frac{\left(1-v^{2r}\right)\left(1+v^2\right)}{\left(1-v^{2r+2}\right)} + \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right)}\frac{v^{2r}\left(1-v^2\right)^2}{\left(1-v^{2r+2}\right)^2},$$

where $z=\frac{v}{1+v^2}$ *and* u *is counting the number returns to the origin.*

Proof. In order to have a closed ending walk we substitute $j=r$ into case (b) of Lemma 11 to get

$$z\varphi_{r-1,r-1}(z) = \frac{v^r\left(1-v^2\right)}{\left(1-v^{2r+2}\right)}.$$

Combining the two generating functions for parts (a) and (b) together with $z\varphi_{r-1,r-1}(z)$ we get the required generating function

$$\begin{aligned} G_{r,closed}(z,u) &= \varphi_{0,r-1}(z) + z\varphi_{r-1,r-1}(z)\,F_r(z,u)\,z\varphi_{r-1,r-1}(z) \\ &= \varphi_{0,r-1}(z) + F_r(z,u)\left(z\varphi_{r-1,r-1}(z)\right)^2, \end{aligned}$$

where $\varphi_{0,r-1}$ is the generating function for random walks not reaching level r. Using the substitutions

$$F_r(z,u) = \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right)} \text{ and } \varphi_{j,h}(z) = \frac{v^j\left(1-v^{2h-2j+2}\right)(1+v^2)}{1-v^{2h+4}},$$

we get

$$G_{r,closed}(z,u) = \frac{(1-v^{2r})(1+v^2)}{(1-v^{2r+2})} + \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right)}\frac{v^{2r}(1-v^2)^2}{(1-v^{2r+2})^2}.$$

■

Remark 2 *We make a simple test on $G_{r,closed}(z,u)$ by plugging in $u=1$:*

$$\begin{aligned} G_{r,closed}(z,1) &= \frac{(1-v^{2r})(1+v^2)}{(1-v^{2r+2})} + \frac{v^{2r}(1-v^2)(1+v^2)}{(1-v^{2r+2})} \\ &= 1+v^2 \\ &= \frac{1-\sqrt{1-4z^2}}{2z^2}, \end{aligned}$$

which is the famous generating function for ***Catalan numbers****. Now*

$$\begin{aligned} D_{closed}(n) &= [z^n]\,G_r(z,1) \\ &= \frac{1}{\left\lfloor\frac{n}{2}\right\rfloor+1}\binom{n}{\left\lfloor\frac{n}{2}\right\rfloor} \; for\ n\ even. \\ &\sim 2^n\sqrt{\frac{2}{\pi n}}\left(\frac{2}{n}-\frac{9}{2n^2}+\frac{145}{16n^3}\right). \end{aligned}$$

This $D_{closed}(n)$ is the required total number of Dyck paths of even lengths n, starting at $(0,0)$ and with a closed ending at $(n,0)$. ■

Proposition 13 *The generating function of the s-factorial moments multiplied by the number of walks of length n in $G_{r,closed}(z,u)$ is given by*

$$M_{i,s}(z) = v^{2r}s!\sum_{i=0}^{s-1}\binom{s-1}{i}(-1)^i\left(1-v^2\right)^{i-s+1}\left(1+v^2\right)^{s-i+1}\left(1-v^{2r+2}\right)^{s-i-1}.$$

Proof. Differentiating $G_{r,closed}(z,u)$ s times and applying Lemma 11 we get

$$\begin{aligned} M_{i,s}(z) &= \frac{\partial^s}{\partial u^s} G_{r,closed}(z,u)|_{u=1} \\ &= \frac{v^{2r}\left(1-v^2\right)^2}{(1-v^{2r+2})^2}\frac{\partial^s}{\partial u^s} F_r(z,u)|_{u=1} \\ &= \frac{v^{2r}\left(1-v^2\right)^2}{(1-v^{2r+2})^2} \times s! \sum_{i=0}^{s-1} \binom{s-1}{i} (-1)^i \left(1-v^2\right)^{i-s-1} \left(1+v^2\right)^{s-i+1} \left(1-v^{2r+2}\right)^{s-i+1} \\ &= v^{2r} s! \sum_{i=0}^{s-1} \binom{s-1}{i} (-1)^i \left(1-v^2\right)^{i-s+1} \left(1+v^2\right)^{s-i+1} \left(1-v^{2r+2}\right)^{s-i-1}. \end{aligned}$$

■

We are now ready to calculate the first and second moments of the number of returns to the r-level, where $r>0$, by a Dyck path with a closed ending.

Notation: We denote a combined asymptotic results from both the positive and negative contributions by $\asymp$.

Corollary 14 *The average number of visits to the r-level by Dyck paths with closed ending at* $(n,0)$ *is given by*

$$m_{(1)}(n) \asymp 4\left[r+1-\frac{(2r^3+6r^2+7r+3)}{n}+...\right].$$

Proof. Substituting $s=1$ into $M_{i,s}(z)$ in Proposition 13 gives

$$M_{(1)}(z) = v^{2r}\left(1+v^2\right)^2.$$

Using [8] we do series expansion on $M_{(1)}(z)$ around $z=\frac{1}{2}$ by using the substitution $z=\frac{v}{1+v^2}$ to get the contribution

$$\begin{aligned} M_{(1)}(z) \sim\ & 4-8\sqrt{2}(r+1)\sqrt{1-2z}+8\left(2r^2+4r+3\right)(1-2z) \\ & -\frac{2\sqrt{2}\left(16r^3+48r^2+77r+45\right)(1-2z)^{\frac{3}{2}}}{3} \\ & +\frac{4\left(8r^4+32r^3+82r^2+100r+51\right)(1-2z)^2}{3} \\ & -\frac{\sqrt{2}\left(256r^5+1280r^4+4640r^3+8800r^2+9489r+4305\right)(1-2z)^{\frac{5}{2}}}{60} \\ & +... \end{aligned}$$

Expanding $M_{(1)}(z)$ around $z=-\frac{1}{2}$ we obtain the contribution

$$\begin{aligned} M_{(1)}(z) &\sim 4-8\sqrt{2}(r+1)\sqrt{1+2z}+8\left(2r^2+4r+3\right)(1+2z) \\ &\quad -\frac{2\sqrt{2}\left(16r^3+48r^2+77r+45\right)(1+2z)^{\frac{3}{2}}}{3} \\ &\quad +\frac{4\left(8r^4+32r^3+82r^2+100r+51\right)(1+2z)^2}{3} \\ &\quad -\frac{\sqrt{2}\left(256r^5+1280r^4+4640r^3+8800r^2+9489r+4305\right)(1+2z)^{\frac{5}{2}}}{60} \\ &\quad +... \end{aligned}$$

Combining the above two contributions and extracting the coefficient of z^n we get

$$\begin{aligned} [z^n]\,M_{(1)}(z) &\asymp -(-2)^n\left(1+(-1)^n\right)\sqrt{2}\left[8(r+1)\binom{\frac{1}{2}}{n}+\frac{2\left(16r^3+48r^2+77r+45\right)}{3}\binom{\frac{3}{2}}{n}\right. \\ &\quad \left.+\frac{\left(256r^5+1280r^4+4640r^3+8800r^2+9489r+4305\right)}{60}\binom{\frac{5}{2}}{n}+...\right] \\ &= 2^n\left(1+(-1)^n\right)\sqrt{2}\left[-\frac{(r+1)}{2^{2n-4}n}\binom{2n-2}{n-1}\right. \\ &\quad +\frac{\left(256r^5+1280r^4+4640r^3+8800r^2+9489r+4305\right)}{2^{2n-1}(n-3)(n-4)(n-5)}\binom{2n-6}{n-6} \\ &\quad \left.+\frac{\left(16r^3+48r^2+77r+45\right)}{2^{2n-3}(n-2)(n-3)}\binom{2n-4}{n-4}+...\right] \\ &\sim 2^n\left(1+(-1)^n\right)\sqrt{\frac{2}{\pi n}}\left[\frac{(4r+4)}{n}-\frac{\left(8r^3+24r^2+37r+21\right)}{n^2}\right. \\ &\quad \left.+\frac{\left(8r^5+40r^4+130r^3+230r^2+\frac{1801r}{8}+\frac{745}{8}\right)}{n^3}+...\right], \end{aligned}$$

where the above asymptotic was done using computer algebra. Again using computer algebra and the total number of objects $D_{closed}(n)$ derived in Remark 2 we get

$$\begin{aligned} m_{(1)}(n) &= \frac{[z^n]\,M_{(1)}(z)}{D_{closed}(n)} \\ &\asymp 4\left[r+1-\frac{\left(2r^3+6r^2+7r+3\right)}{n}+...\right], \end{aligned}$$

which is the required average number of visits to height r. ■

Note: A Dyck path with an odd length cannot return to the 0-level, hence $m_{(1)}(n)=0$ for n odd.

Corollary 15 *The second moments of $G_{r,closed}(z,u)$ are given by*

$$m_{(2)}(n) \asymp 8\left[3r^2+5r+2-\frac{(15r^4+58r^3+90r^2+65r+18)}{n}+...\right].$$

Proof. Substituting $s=2$ into $M_{i,s}(z)$ in Proposition 13 gives

$$M_{(2)}(z)=\frac{2v^{2r+2}\left(1+v^2\right)^2\left(2-v^{2r}-v^{2r+2}\right)}{1-v^2}.$$

Expanding $M_{(2)}(z)$ around $z=\frac{1}{2}$ we obtain the contribution

$$\begin{aligned}
M_{(2)}(z) \sim\ & 8(1+2r)-16\sqrt{2}\left(3r^2+5r+2\right)\sqrt{1-2z} \\
&+\frac{16\left(28r^3+78r^2+80r+27\right)(1-2z)}{3} \\
&-\frac{4\sqrt{2}\left(120r^4+464r^3+783r^2+625r+186\right)(1-2z)^{\frac{3}{2}}}{3} \\
&+\frac{3968r^5+19520r^4+46240r^3+59920r^2+40512r+10920}{15}(1-2z)^2 \\
&-\frac{\sqrt{2}\left(5376r^6+32000r^5+98160r^4+178080r^3+193539r^2+115765r+28770\right)}{30}(1-2z)^{\frac{5}{2}} \\
&+....
\end{aligned}$$

Expansion around $z=-\frac{1}{2}$ gives the contribution

$$\begin{aligned}
M_{(2)}(z) \sim\ & 8(1+2r)-16\sqrt{2}\left(3r^2+5r+2\right)\sqrt{1+2z} \\
&+\frac{16\left(28r^3+78r^2+80r+27\right)(1+2z)}{3} \\
&-\frac{4\sqrt{2}\left(120r^4+464r^3+783r^2+625r+186\right)(1+2z)^{\frac{3}{2}}}{3} \\
&+\frac{3968r^5+19520r^4+46240r^3+59920r^2+40512r+10920}{15}(1+2z)^2 \\
&-\frac{\sqrt{2}\left(5376r^6+32000r^5+98160r^4+178080r^3+193539r^2+115765r+28770\right)}{30}(1+2z)^{\frac{5}{2}} \\
&+...
\end{aligned}$$

Combining these two contributions we get

$$\begin{aligned}
& [z^n]\,M_{(2)}(z) \\
\asymp\ & -(-2)^n\left(1+(-1)^n\right)\sqrt{2}\left[16\left(3r^2+5r+2\right)\binom{\frac{1}{2}}{n}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{4\left(120r^4+464r^3+783r^2+625r+186\right)}{3}\binom{\frac{3}{2}}{n}\\
&+\frac{\left(5376r^6+32000r^5+98160r^4+178080r^3+193539r^2+115765r+28770\right)}{30}\binom{\frac{5}{2}}{n}\\
&+...]\\
=\;& 2^n\left(1+(-1)^n\right)\sqrt{2}\left[-\frac{\left(120r^4+464r^3+783r^2+625r+186\right)}{2^{2n-4}n\left(n-2\right)\left(n-3\right)}\binom{2n-4}{n-4}\right.\\
&+\frac{\left(3r^2+5r+2\right)}{2^{2n-5}n}\binom{2n-2}{n-1}\\
&\left.+\frac{\left(5376r^6+32000r^5+98160r^4+178080r^3+193539r^2+115765r+28770\right)}{2^{2n-2}n\left(n-5\right)\left(n-4\right)\left(n-3\right)}\binom{2n-6}{n-6}\right.\\
&+...]\\
\sim\;& 2^n\left(1+(-1)^n\right)\sqrt{\frac{2}{\pi n}}\left[\frac{\left(24r^2+40r+16\right)}{n}-\frac{\left(120r^4+464r^3+774r^2+610r+180\right)}{n^2}+...\right]
\end{aligned}
$$

It follows that

$$m_{(2)}\left(n\right)\asymp 8\left[3r^2+5r+2-\frac{\left(15r^4+58r^3+90r^2+65r+18\right)}{n}+...\right].$$

This formula is only valid for n even as well. That is, $m_{(2)}\left(n\right)=0$ for n odd, as mentioned earlier on. ■

2.4 Observation

In this section we give an enumerative comparison between the asymptotic and exact average number of visits to height r, together with their corresponding variances, by a simple Dyck path with a closed ending at $(n,0)$. The exact results were found using Mathematica programming and series expansion; and the asymptotic results were calculated using the formula for $m_{(1)}\left(n\right)$ given in Corollary 14.

	Type of	**Average r-visits**			**Variances**		
n	**results**	$r=1$	$r=2$	$r=5$	$r=1$	$r=2$	$r=5$
400	exact	7.8232	11.4472	20.0229	21.8927	49.9097	135.9794
400	*asympt.*	7.8200	11.4302	19.6226	21.7462	48.5402	66.21.79
634	exact	7.8877	11.6473	21.4011	22.6449	53.4025	173.1360
634	*asympt.*	7.8864	11.6404	21.2372	22.5861	52.8473	143.7187
856	exact	7.9166	11.7375	22.0446	22.9878	55.0347	176.3885
856	*asympt.*	7.9159	11.7337	21.9535	22.9554	54.7276	171.8261
1000	exact	7.9285	11.7748	22.3152	23.1304	55.7214	201.8071
1000	*asympt.*	7.9159	11.7720	22.2481	23.1067	55.4957	189.5234

Table 2.1: Average r-level visits by a closed ended Dyck path.

The above Table shows that the asymptotic and the exact results are asymptotically the same for small values of r as n grows larger.

2.5 Random walks with open endings.

We use results from Lemma 11 where the open path can end either above or below the r-level, for $r > 0$.

Lemma 16 *The generating function for a random walk starting at* $(0,0)$ *and with an ending at* $r+j$ *is given by*

$$W_{r,above}(z,u) = \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right)} v^{j+r} \frac{(1-v^2)}{1-v^{2r+2}},$$

where $z = \frac{v}{1+v^2}$ *and* u *is counting the number returns to the origin.*

Proof. To get the generating function for a path ending at $(n, r+j)$, we combine results for Lemma 8, 9 and 11(a) to get

$$W_{r,above}(z,u) \;=\; z\varphi_{r-1,r-1}(z)\, F_r(z,u)\, z\varphi_{j-1,\infty}(z)$$

$$\begin{aligned} &= z\varphi_{r-1,r-1}(z)\frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right)}z\varphi_{j-1,\infty}(z) \\ &= \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right)}z^2\varphi_{r-1,r-1}(z)\,\varphi_{j-1,\infty}(z), \end{aligned}$$

However

$$\begin{aligned} \varphi_{j-1,\infty}(z) &= \lim_{t\to\infty}\varphi_{j,t}(z) \\ &= \lim_{t\to\infty}\frac{v^j(1+v^2)(1-v^{2t-2j+2})}{1-v^{2t+4}} \\ &= v^j\left(1+v^2\right), \end{aligned}$$

so that

$$\varphi_{j-1,\infty}(z) = v^{j-1}\left(1+v^2\right).$$

Using the substitution $z=\frac{v}{1+v^2}$ we get

$$\begin{aligned} z^2\varphi_{r-1,r-1}(z)\,\varphi_{j-1,\infty}(z) &= \frac{v^2}{(1+v^2)^2}\frac{v^{r-1}(1+v^2)(1-v^2)}{1-v^{2r+2}}v^{j-1}\left(1+v^2\right) \\ &= v^{j+r}\frac{(1-v^2)}{1-v^{2r+2}}. \end{aligned}$$

It follows that

$$W_{r,above}(z,u) = \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right)}v^{j+r}\frac{(1-v^2)}{1-v^{2r+2}}.$$

■

Lemma 17 *The generating function for a Dyck path starting at* $(0,0)$ *and with an ending at* $(n, r-j)$ *is given by*

$$W_{r,below}(z,u) = \varphi_{r-j,r-1}(z) + \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right)}\frac{v^{r+j}(1-v^2)(1-v^{2r-2i+2})}{(1-v^{2r+2})^2},$$

where $z=\frac{v}{1+v^2}$ and u is counting the number returns to the origin.

Proof. To get the generating function for a path ending at $(n, r-j)$, we combine results for Lemma 8, 9 and 11(b) to get

$$W_{r,below}(z,u) = \varphi_{r-j,r-1}(z) + z\varphi_{r-1,r-1}(z)\times F_r(z,u)\times z\varphi_{j-1,r-1}(z),$$

where $\varphi_{r-j,r-1}(z)$ is the generating function for the random walks not reaching r. However

$$\begin{aligned}\left(z\varphi_{r-1,r-1}(z)\right)\left(z\varphi_{j-1,r-1}(z)\right) &= z^2\varphi_{r-1,r-1}(z)\,\varphi_{j-1,r-1}(z)\\ &= \frac{v^2}{(1+v^2)^2}\frac{v^{r+j-2}(1+v^2)^2(1-v^2)(1-v^{2r-2j+2})}{(1-v^{2r+2})^2}\\ &= \frac{v^{r+j}(1-v^2)(1-v^{2r-2j+2})}{(1-v^{2r+2})^2},\end{aligned}$$

so that

$$W_{r,below}(z,u)=\varphi_{r-j,r-1}(z)+\frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right)}\frac{v^{r+j}(1-v^2)(1-v^{2r-2j+2})}{(1-v^{2r+2})^2},$$

as required. ■

We are now ready to get the bivariate generating function for the entire Dyck path with an open ending at $(n,-)$. This is achieved by summing up all the generating functions for the cases where j is above, below and equal to r.

Theorem 18 *The generating function of a Dyck path with an open ending at $(n,-)$ is given by*

$$W_{r,open}(z,u)=\sum_{k=0}^{r-1}\varphi_{k,r-1}(z)+\frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right)}\times\frac{v^r(1-v^2)(1-v^{r+1})(1+v)}{(1-v)(1-v^{2r+2})^2},$$

where $z=\frac{v}{1+v^2}$ and u is counting the number of returns to the origin.

Proof. Summing up both the generating functions for the two cases proved in Lemmas 16 and 17 together with the generating function $z\varphi_{r-1,r-1}(z)\,F_r(z,u)$ we get

$$W_{r,open}(z,u)=\sum_{k=0}^{r-1}\varphi_{k,r-1}(z)+z\varphi_{r-1,r-1}(z)\,F_r(z,u)\left(z\sum_{j\geq 1}\varphi_{j-1,\infty}(z)+z\sum_{j=1}^{r}\varphi_{j-1,r-1}(z)+1\right),$$

where $\sum_{k=0}^{r-1}\varphi_{k,r-1}(z)$ is the generating function random walks not reaching r. However

$$\begin{aligned}\left(z\sum_{j\geq 1}\varphi_{j-1,\infty}(z)+z\sum_{j=1}^{r}\varphi_{j-1,r-1}(z)+1\right) &= \frac{v}{1-v}+\frac{(v^r-1)(v-v^{r+2})}{(v-1)(1-v^{2r+2})}+1\\ &= \frac{(1-v^{r+1})(1+v)}{(1-v)(1-v^{2r+2})},\end{aligned}$$

and

$$z\varphi_{r-1,r-1} = \frac{v^r\left(1-v^2\right)}{\left(1-v^{2r+2}\right)},$$

so that

$$z\varphi_{r-1,r-1}\left(z\right)\left(z\sum_{j\geq 1}\varphi_{j-1,\infty}\left(z\right)+z\sum_{j=1}^{r}\varphi_{j-1,r-1}\left(z\right)+1\right) = \frac{v^r\left(1-v^2\right)\left(1-v^{r+1}\right)\left(1+v\right)}{\left(1-v\right)\left(1-v^{2r+2}\right)^2}.$$

Thus

$$\begin{aligned} W_{r,open}(z,u) &= \sum_{k=0}^{r-1}\varphi_{k,r-1}\left(z\right)+F_r\left(z,u\right)\frac{v^r\left(1-v^2\right)\left(1-v^{r+1}\right)\left(1+v\right)}{\left(1-v\right)\left(1-v^{2r+2}\right)^2} \\ &= \sum_{k=0}^{r-1}\varphi_{k,r-1}\left(z\right)+\frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1-v^{2r+2})}\right)}\times\frac{v^r\left(1-v^2\right)\left(1-v^{r+1}\right)\left(1+v\right)}{\left(1-v\right)\left(1-v^{2r+2}\right)^2}. \end{aligned}$$

■

Remark 3 *We make a simple test on $W_{r,open}(z,u)$ by plugging in $u=1$:*

$$\begin{aligned} W_{r,open}(z,1) &= \sum_{k=0}^{r-1}\varphi_{k,r-1}\left(z\right)+\frac{v^r\left(1+v^2\right)\left(1-v^{r+1}\right)\left(1+v\right)}{\left(1-v\right)\left(1-v^{2r+2}\right)} \\ &= \frac{\left(1+v^2\right)\left(1-v^{r+1}\right)\left(1-v^r\right)}{\left(1-v\right)\left(1-v^{2r+2}\right)}+\frac{v^r\left(1+v^2\right)\left(1-v^{r+1}\right)\left(1+v\right)}{\left(1-v\right)\left(1-v^{2r+2}\right)} \\ &= \frac{\left(1+v^2\right)}{\left(1-v\right)}, \end{aligned}$$

which matches with the results in Remark 1. ■

Proposition 19 *The generating function of the s-factorial moments multiplied by the number of walks of length n in $W_{r,open}(z,u)$ is given by*

$$\begin{aligned} M_{(s)}\left(z\right) &= s!\sum_{k=0}^{s-1}\binom{s-1}{k}\left(-1\right)^k\left(1-v^2\right)^{k-s-1}\left(1+v^2\right)^{s-k+1}\left(1-v^{2r+2}\right)^{s-k+1} \\ &\quad\times\frac{v^r\left(1-v^2\right)\left(1-v^{r+1}\right)\left(1+v\right)}{\left(1-v\right)\left(1-v^{2r+2}\right)^2}, \end{aligned}$$

where $z=\frac{v}{1+v^2}$.

Proof. Now

$$\begin{aligned} M_{(s)}(z) &= \left.\frac{\partial^s}{\partial u^s} W_{r,open}(z,u)\right|_{u=1} \\ &= \frac{v^r(1-v^2)(1-v^{r+1})(1+v)}{(1-v)(1-v^{2r+2})^2} \times \left.\frac{\partial^s}{\partial u^s} F_r(z,u)\right|_{u=1} \\ &= s!\sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^k\left(1-v^2\right)^{k-s-1}\left(1+v^2\right)^{s-k+1}\left(1-v^{2r+2}\right)^{s-k+1} \\ &\quad \times\frac{v^r(1-v^2)(1-v^{r+1})(1+v)}{(1-v)(1-v^{2r+2})^2}. \end{aligned}$$

■

Corollary 20 *The average number of visits to level r by a Dyck path with an open ending at $(n,-)$ is given by*

$$m_{(1)}(n) \asymp \begin{cases} 2r+2-\frac{\left(14r^3+42r^2+43r+15\right)}{6n}+(-1)^r\frac{r+1}{2n} & \text{, for } n \text{ even} \\ 2r+2-\frac{\left(14r^3+42r^2+49r+21\right)}{6n}-(-1)^r\frac{r+1}{2n} & \text{, for } n \text{ odd} \end{cases}.$$

Proof. Specializing on $s=1$ into Proposition 19 gives

$$M_{(1)}(z) = \frac{v^r(1+v^2)^2(1+v)(1-v^{r+1})}{(1-v^2)(1-v)}.$$

We then expand around the two singularities, $z=\pm\frac{1}{2}$.

$$\begin{aligned} M_{(1)}(z) \asymp\ & -6(1+r)^2+\frac{4(r+1)}{\sqrt{2}\sqrt{1-2z}}+\frac{\left(28r^3+84r^2+101r+45\right)\sqrt{2}\sqrt{1-2z}}{6} \\ & -\left(5r^4+20r^3+41r^2+42r+16\right)(1-2z) \\ & +\frac{\left(496r^5+2480r^4+7480r^3+12520r^2+10849r+3825\right)\sqrt{2}(1-2z)^{\frac{3}{2}}}{240} \\ & +1+(-1)^r-\sqrt{2}\left(2r+2+(-1)^r(r+1)\right)\sqrt{1+2z} \\ & +\frac{8r^2+16r+11+(-1)^r\left(2r^2+4r+5\right)}{2}(1+2z) \\ & -\frac{\sqrt{2}}{12}\left[32r^3+96r^2+142r+78+(-1)^r\left(4r^3+12r^2+35r+27\right)\right](1+2z)^{\frac{3}{2}}+.... \end{aligned}$$

Hence the coefficient of z^n $M_{(1)}(z)$ is

$$[z^n]M_{(1)}(z) \asymp (-2)^n\sqrt{2}\left[2(r+1)\binom{-\frac{1}{2}}{n}+\frac{\left(28r^3+84r^2+101r+45\right)}{6}\binom{\frac{1}{2}}{n}\right.$$

$$
\begin{aligned}
&+\frac{(496r^5+2480r^4+7480r^3+12520r^2+10849r+3825)}{240}\binom{\frac{3}{2}}{n}\\
&-(-1)^n\left(2r+2+(-1)^r(r+1)\right)\binom{\frac{1}{2}}{n}\\
&-\frac{(-1)^n\left[32r^3+96r^2+142r+78+(-1)^r\left(4r^3+12r^2+35r+27\right)\right]}{12}\binom{\frac{3}{2}}{n}+...\Bigg]\\
=\;& 2^n\left[\frac{(r+1)}{2^{2n-2}\sqrt{2}}\binom{2n}{n}-\frac{\left(28r^3+84r^2+101r+45\right)\sqrt{2}}{3n2^{2n}}\binom{2n-2}{n-1}\right.\\
&+\frac{(496r^5+2480r^4+7480r^3+12520r^2+10849r+3825)\sqrt{2}}{5(n-2)(n-3)2^{2n+2}}\binom{2n-4}{n-4}\\
&+\frac{\left(2r+2+(-1)^r(r+1)\right)}{2^{2n-1}n}\binom{2n-n}{n-1}\\
&\left.-\frac{\left[32r^3+96r^2+142r+78+(-1)^r\left(4r^3+12r^2+35r+27\right)\right]}{2^{2n}(n-2)(n-3)}\binom{2n-4}{n-4}+....\right]\\
\asymp\;& 2^n\sqrt{\frac{2}{\pi n}}\left\{2r+2+\frac{1}{n}\left[-\frac{\left(7r^3+21r^2+26r+12\right)}{3}+(-1)^n(r+1)+(-1)^{n+r}\frac{r+1}{2}\right]\right.\\
&+\frac{1}{n^2}\left[\frac{(124r^5+620r^4+1800r^3+2920r^2+2461r+845)}{80}\right.\\
&\left.\left.-(-1)^n\frac{4r^3+12r^2+17r+9}{2}-(-1)^{n+r}\frac{r^3+3r^2+8r+6}{4}\right]+....\right\}.
\end{aligned}
$$

Applying the total number of objects calculated in Remark 3 we get

$$
\begin{aligned}
m_{(1)}(n) &= \frac{[z^n]M_{(1)}(z)}{D_{open}(n)}\\
&\asymp \begin{cases} 2r+2-\frac{\left(14r^3+42r^2+43r+15\right)}{6n}+(-1)^r\frac{r+1}{2n}+... & \text{, for } n \text{ even}\\ 2r+2-\frac{\left(14r^3+42r^2+49r+21\right)}{6n}-(-1)^r\frac{r+1}{2n}+... & \text{, for } n \text{ odd}\end{cases},
\end{aligned}
$$

which is the required average number of visits to level r. ■

Corollary 21 *The second moments on $W_{r,open}(z,u)$ is given by*

$$
m_{(2)}(n)\asymp\begin{cases} 8r^2+12r+4-\frac{\left(80r^4+306r^3+442r^2+285r+69\right)}{3n}-(-1)^r\left(4r^2+7r+3\right)+... & \textit{, for } n \textit{ even}\\ 8r^2+12r+4-\frac{\left(80r^4+306r^3+466r^2+327r+87\right)}{3n}+(-1)^r\left(4r^2+7r+3\right)+... & \textit{, for } n \textit{ odd}\end{cases}.
$$

Proof. Substituting $s=2$ into Proposition 19 gives

$$
M_{(2)}(z)=\frac{2v^{r+2}\left(1-v^{r+1}\right)\left(1+v^2\right)^2\left(2-v^{2r}-v^{2r+2}\right)}{(1-v)^2\left(1-v^2\right)}.
$$

We expand $M_{(2)}(z)$ around the two singularities $z = \pm\frac{1}{2}$:

Around $z = \frac{1}{2}$ gives

$$M_{(2)}(z) \sim \frac{4\sqrt{2}\left(2r^2+3r+1\right)}{\sqrt{1-2z}} - \left(40r^3+108r^2+96r+28\right)$$
$$+\frac{\left(160r^4+612r^3+926r^2+639r+165\right)\sqrt{2}\sqrt{1-2z}}{3}$$
$$-2\left(50r^5+245r^4+530r^3+609r^2+358r+84\right)(1-2z)$$
$$\frac{\sqrt{2}\left(26432r^6+157104r^5+445040r^4+730200r^3+703838r^2+367881r+79875\right)(1-2z)^{\frac{3}{2}}}{360}$$
$$+....$$

Around $z = -\frac{1}{2}$ gives

$$M_{(2)}(z) \sim 4r+2+(-1)^r(4r+2) - \sqrt{2}\sqrt{1+2z}\left[12r^2+20r+8+(-1)^r\left(8r^2+14r+6\right)\right]$$
$$+\frac{112r^3+312r^2+314r+105+(-1)^r\left(52r^3+150r^2+170r+63\right)(1+2z)}{3}$$
$$-\frac{240r^4+928r^3+1530r^2+1190r+348+(-1)^r\left(80r^4+316r^3+592r^2+533r+177\right)(1+2z)^{\frac{3}{2}}}{3\sqrt{2}}$$
$$+....$$

The coefficient of z^n in the combined expansions is:

$$[z^n]\,M_{(2)}(z) \asymp (-2)^n\sqrt{2}\left\{4\left(2r^2+3r+1\right)\binom{-\frac{1}{2}}{n}\right.$$
$$-\frac{\left(160r^4+612r^3+926r^2+639r+165\right)}{3}\binom{\frac{1}{2}}{n}$$
$$+\frac{\left(26432r^6+157104r^5+445040r^4+730200r^3+703838r^2+367881r+79875\right)}{360}\binom{\frac{3}{2}}{n}$$
$$-(-1)^n\left[12r^2+20r+8+(-1)^r\left(8r^2+14r+6\right)\right]\binom{\frac{1}{2}}{n}$$
$$\left.-\frac{\left[240r^4+928r^3+1530r^2+1190r+348+(-1)^r\left(80r^4+316r^3+592r^2+533r+177\right)\right]}{6(-1)^{-n}}\binom{\frac{3}{2}}{n}\right.$$
$$\left.+...\right\}$$
$$= 2^n\left\{\frac{\left(2r^2+3r+1\right)}{2^{2n-3}\sqrt{2}}\binom{2n}{n}\right.$$
$$\left.-\frac{\left(160r^4+612r^3+926r^2+639r+165\right)\sqrt{2}}{3n2^{2n-1}}\binom{2n-2}{n-1}\right.$$

$$
\begin{aligned}
&+\frac{\sqrt{2}\left(26432r^6+157104r^5+445040r^4+730200r^3+703838r^2+367881r+79875\right)}{15\left(n-2\right)\left(n-3\right)2^{2n+1}}\binom{2n-4}{n-4}\\
&+\frac{\left(-1\right)^n\left[12r^2+20r+8+\left(-1\right)^r\left(8r^2+14r+6\right)\right]\sqrt{2}}{2^{2n-1}n}\binom{2n-2}{n-1}\\
&-\frac{\left(-1\right)^n\sqrt{2}}{2^{2n-1}\left(n-2\right)\left(n-3\right)}\binom{2n-4}{n-4}\left[240r^4+928r^3+1530r^2+1190r+348+\right.\\
&\left.+\left(-1\right)^r\left(80r^4+316r^3+592r^2+533r+177\right)\right]\Big\}\\
\asymp\ & 2^n\sqrt{\frac{2}{\pi n}}\left[8r^2+12r+4-\frac{\left(80r^4+306r^3+466r^2+324r+84\right)}{3n}+\frac{\left(-1\right)^n\left(6r^2+10r+4\right)}{n}\right.\\
&+\frac{\left(-1\right)^{n+r}\left(4r^2+7r+3\right)}{n}\\
&+\frac{6608r^6+39276r^5+110060r^4+177960r^3+169022r^2+87189r+18735}{120n^2}\\
&-\frac{\left(-1\right)^n\left(1920r^4+7424r^3+12222r^2+9490r+2772\right)}{8n^2}\\
&\left.-\frac{\left(-1\right)^{r+n}\left(640r^4+2528r^3+4724r^2+4243r+1407\right)}{8n^2}+....\right].
\end{aligned}
$$

Applying the total number of objects $D_{open}\left(n\right)$ calculated in Remark 3 we get the second moments given by

$$
\begin{aligned}
m_{(2)}\left(n\right) &= \frac{\left[z^n\right]M_{(2)}\left(z\right)}{D_{open}\left(n\right)}\\
&\asymp \begin{cases} 8r^2+12r+4-\frac{\left(80r^4+306r^3+442r^2+285r+69\right)}{3n}-\left(-1\right)^r\left(4r^2+7r+3\right) & \text{, for } n \text{ even}\\ 8r^2+12r+4-\frac{\left(80r^4+306r^3+466r^2+327r+87\right)}{3n}+\left(-1\right)^r\left(4r^2+7r+3\right) & \text{, for } n \text{ odd}\end{cases}.
\end{aligned}
$$

■

2.6 Observation

We are now ready to give a table listing an enumerative comparison between the asymptotic and exact average number of visits to height r, together with their corresponding variances, by a simple Dyck paths with an open ending at $(n,-)$. The exact results were found using Mathematica programming and from Taylor expansion on $M_{(1)}\left(z\right)$ using Maple software. Asymptotic results were calculated using Corollary 20.

	Type of	$m_{(1)}(n)$			**Variances**		
n	**results**	$r=1$	$r=3$	$r=6$	$r=1$	$r=3$	$r=6$
367	expected	3.9409	7.5902	11.9687	11.3036	45.5494	101.0745
367	*asympt.*	*3.9455*	*7.5858*	*11.7875*	*11.2668*	*44.0355*	*65.0284*
593	expected	3.9632	7.7428	12.6978	11.5625	49.2357	125.6367
593	*asympt.*	*3.9663*	*7.7437*	*12.6307*	*11.5469*	*48.6358*	*110.7624*
712	expected	3.9721	7.7902	12.9314	11.6307	50.2814	133.8064
712	*asympt.*	*3.9719*	*7.7865*	*12.8792*	*11.6228*	*49.8758*	*123.5057*
844	expected	3.9764	7.8225	13.0919	11.6875	51.1301	140.2719
844	*asympt.*	*3.9751*	*7.8175*	*13.0421*	*11.6819*	*50.8396*	*132.8193*
1001	expected	3.9781	7.8462	13.2100	11.7382	51.8697	145.7522
1001	*asympt.*	*3.9800*	*7.8482*	*13.1888*	*11.7319*	*51.6533*	*140.2511*

Table 2.2: Average r-level visits by an open ended Dyck path.

The above Table confirms that the asymptotic and the exact results are asymptotically the same for small values of r as n grows larger.

2.7 Return statistics

In this section, we look at the number of times an open ended path returns to the x-axis. In the generating function, the number of returns to the origin is counted by the variable w. Note, if the paths ends on the x-axis i.e., is a closed path, then that last visit is not counted. Like in the previous sections we apply the asymptotic expansions given in [8] to calculate our moments. If we count the number of returns to the origin by the variable w then the special system discussed in Section 2.1 is given by

$$\varphi_0(z) = 1 + zw\varphi_1(z).$$

The other two recursions $\varphi_{i,h}(z) = z\varphi_{i+1,h}(z) + z\varphi_{i-1,h}(z)$ and $\varphi_{h,h}(z) = z\varphi_{h-1,h}(z)$ remain the same as before. These recursions are expressed collectively in a matrix equation

$$\begin{bmatrix} 1 & -zw & & & & & & \\ -z & 1 & -z & & & & & \\ & -z & 1 & -z & & 0 & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & 0 & & & -z & 1 & -z \\ & & & & & & -z & 1 \end{bmatrix} \begin{bmatrix} \varphi_{0,h}(z) \\ \varphi_{1,h}(z) \\ \varphi_{2,h}(z) \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{h-1,h}(z) \\ \varphi_{h,h}(z) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}. \tag{2.2}$$

As in the previous case we calculate the $\varphi_{i,h}$'s using Cramer's rule by first finding the determinant of this system, denoted by $\det C_h$ with $h+1$ rows.

Lemma 22 *The determinant of the matrix of the system of the $\varphi_{i,h}$'s with $h+1$ rows is given by*

$$\det C_h = \frac{\left(1+v^2\right)\left(1-v^{2h+2}\right) - v^2 w\left(1-v^{2h}\right)}{\left(1-v^2\right)\left(1+v^2\right)^{h+1}},$$

where w is counting the number returns to the origin.

Proof. Applying the formula for $\det A_h$ derived in Lemma 6 we get

$$\begin{aligned} \det C_h &= \det A_h - z^2 w \det A_{h-1} \\ &= \frac{1-v^{2h+2}}{\left(1-v^2\right)\left(1+v^2\right)^h} - \frac{v^2 w\left(1-v^{2h}\right)}{\left(1-v^2\right)\left(1+v^2\right)^{h+1}} \\ &= \frac{\left(1+v^2\right)\left(1-v^{2h+2}\right) - v^2 w\left(1-v^{2h}\right)}{\left(1-v^2\right)\left(1+v^2\right)^{h+1}}. \end{aligned}$$

■

Theorem 23 *The generating function for the number of returns to the origin by a Dyck path starting at $(0,0)$ and ending at (n,i) is given by*

$$\varphi_{i,\infty}(z,w) = \frac{v^i}{1 - w\left(\frac{v^2}{1+v^2}\right)},$$

where $z = \frac{v}{1+v^2}$ and w is counting the number returns to the origin.

Proof. Applying Cramer's rule for solving system of linear equations in equation (2.2) we get

$$\begin{aligned}\varphi_{i,h}(z,w) &= \frac{z^i \det A_{h-i}}{\det C_h} \\ &= \frac{v^i\left(1-v^{2h-2i+2}\right)}{(1-v^2)(1+v^2)^h} \times \frac{(1-v^2)(1+v^2)^{h+1}}{(1+v^2)(1-v^{2h+2})-v^2w(1-v^{2h})} \\ &= \frac{v^i\left(1+v^{2h-2i+2}\right)(1+v^2)}{(1+v^2)(1-v^{2h+2})-v^2w(1-v^{2h})}.\end{aligned}$$

If we let $h \to \infty$, we obtain the generating function for the number of visits to the x-axis of an open path ending at (n,i) :

$$\varphi_{i,\infty}(z,w) = \frac{v^i}{1-w\left(\frac{v^2}{1+v^2}\right)}.$$

■

Remark 4 *We test the consistency of the generating function $\varphi_{i,\infty}(z,w)$ by plugging $w=1$ and summing over all $i \geq 0$ to get*

$$\sum_{i\geq 0}\varphi_{i,\infty}(z,1) = \frac{(1+v^2)}{(1-v)},$$

which is the generating function for the total number of all Dyck paths of length n with an open ending as seen in Remarks 1 and 3. ■

Proposition 24 *The generating function of the s-factorial moments multiplied by the number of walks of length n in $\varphi_{i,\infty}(z,w)$ is given by*

$$M_{(s)}(z) = \frac{s!v^{2s}(1+v^2)}{1-v},$$

where $z = \frac{v}{1+v^2}$.

Proof. Now

$$\begin{aligned}M_{(s,i)}(z) &= \frac{\partial^s}{\partial w^s}\varphi_{i,\infty}(z,w)\big|_{w=1} \\ &= s!\frac{\left(\frac{v^2}{1+v^2}\right)^s v^i}{\left(\frac{1}{1+v^2}\right)^{s+1}} \\ &= s!v^{2s+i}\left(1+v^2\right).\end{aligned}$$

Summing up the above generating function for all $i \geq 0$ we get

$$\begin{aligned} M_{(s)}(z) &= s!v^{2s}\left(1+v^2\right)\sum_{i\geq 0} v^i \\ &= \frac{s!v^{2s}\left(1+v^2\right)}{1-v}. \end{aligned}$$

■

Corollary 25 *The average number of returns to the origin by a Dyck path with an open ending is given by*

$$m_{(1)}(n) \asymp \begin{cases} 1-\frac{(13-5(-1)^n)}{4n}+\frac{149-85(-1)^n}{16n^2}+... & \text{, for } n \text{ even} \\ 1+\frac{(5(-1)^n)-11}{4n}+\frac{117-75(-1)^n}{16n^2}+... & \text{, for } n \text{ odd} \end{cases}.$$

Proof. Inserting $s=1$ into Proposition 24 gives

$$M_{(1)}(z) = \frac{v^2\left(1+v^2\right)}{1-v}.$$

Substituting $z=\frac{v}{1+v^2}$ and making series expansion on $M_{(1)}(z)$ around $z=\frac{1}{2}$ we get

$$M_{(1)}(z) \sim -5+\frac{2}{\sqrt{2}\sqrt{1-2z}}+\frac{27\sqrt{2}\sqrt{1-2z}}{4}-15(1-2z)+\frac{487\sqrt{2}(1-2z)^{\frac{3}{2}}}{32}+....$$

Furthermore, doing series expansion on $M_{(1)}(z)$ around $z=-\frac{1}{2}$ we obtain

$$M_{(1)}(z) \sim -\frac{5\sqrt{2}\sqrt{1+2z}}{2}+7(1+2z)-\frac{65\sqrt{2}(1+2z)^{\frac{3}{2}}}{8}+...,$$

so that

$$\begin{aligned} [z^n]M_{(1)}(z) &\asymp (-2)^n\left[\frac{2}{\sqrt{2}}\binom{-\frac{1}{2}}{n}+\frac{27\sqrt{2}}{4}\binom{\frac{1}{2}}{n}+\frac{487\sqrt{2}}{32}\binom{\frac{3}{2}}{n}+...\right] \\ &\quad +2^n\left[-\frac{5\sqrt{2}}{2}\binom{\frac{1}{2}}{n}-\frac{65\sqrt{2}}{8}\binom{\frac{3}{2}}{n}+...\right] \\ &= 2^n\left[\frac{2}{2^{2n}\sqrt{2}}\binom{2n}{n}-\frac{27\sqrt{2}}{2^{2n+1}n}\binom{2n-1}{n-1}+\frac{3(487)\sqrt{2}}{2^{2n+3}(n-2)(n-3)}\binom{2n-4}{n-4}\right. \\ &\quad \left.+\frac{5(-1)^n\sqrt{2}}{2^{2n}}\binom{2n-2}{n-1}-\frac{195\sqrt{2}(-1)^n}{2^{2n+1}(n-2)(n-3)}\binom{2n-4}{n-4}+...\right] \\ &\asymp 2^n\sqrt{\frac{2}{\pi n}}\left(1+\frac{5(-1)^n-14}{4n}+\frac{325-180(-1)^n}{32n^2}+...\right). \end{aligned}$$

Using computer algebra and the total number of objects derived in Remark 3 we get

$$\begin{aligned} m_{(1)}(n) &= \frac{[z^n] M_{(1)}(z)}{D_{open}(n)} \\ &\asymp \begin{cases} 1 - \frac{(13-5(-1)^n)}{4n} + \frac{149-85(-1)^n}{16n^2} + ... & , \text{ for } n \text{ even} \\ 1 + \frac{(5(-1)^n)-11}{4n} + \frac{117-75(-1)^n}{16n^2} + ... & , \text{ for } n \text{ odd} \end{cases}. \end{aligned}$$

■

Corollary 26 *The second moments are given by*

$$m_{(2)}(n) \asymp \begin{cases} 2 + \frac{(9(-1)^n-41)}{2n} + \frac{(1045-405(-1)^n)}{8n^2} + ... & , \textit{for } n \textit{ even} \\ 2 + \frac{(9(-1)^n-39)}{2n} + \frac{(957-387(-1)^n)}{8n^2} + ... & , \textit{for } n \textit{ odd} \end{cases}.$$

Proof. Substituting $s = 2$ into Proposition 24 gives

$$M_{(2)}(z) = \frac{2v^4(1+v^2)}{1-v}.$$

Making series expansion on $M_{(2)}(z)$ around $z = \frac{1}{2}$ we get

$$M_{(2)}(z) \sim \frac{4}{\sqrt{2}\sqrt{1-2z}} - 18(1-2z)^0 + \frac{83\sqrt{2}\sqrt{1-2z}}{2} - 138(1-2z) + \frac{3063\sqrt{3}(1-2z)^{\frac{3}{2}}}{16} + ...$$

Again, making series expansion on $M_{(2)}(z)$ around $z = -\frac{1}{2}$ we get

$$M_{(2)}(z) \sim -2(1+2z)^0 - 9\sqrt{2}\sqrt{1+2z} + 42(1+2z) - \frac{285\sqrt{2}(1+2z)^{\frac{3}{2}}}{4} + ...$$

We combine the two contributions to obtain

$$\begin{aligned} [z^n] M_{(2)}(z) &\asymp (-2)^n \left[\frac{4}{\sqrt{2}} \binom{-\frac{1}{2}}{n} + \frac{83\sqrt{2}}{2} \binom{\frac{1}{2}}{n} + \frac{3063\sqrt{3}}{16} \binom{\frac{3}{2}}{n} + ... \right] \\ &\quad + 2^n \left[-9\sqrt{2} \binom{\frac{1}{2}}{n} - \frac{285\sqrt{2}}{4} \binom{\frac{3}{2}}{n} + ... \right] \\ &= 2^n \left[\frac{1}{2^{2n-2}\sqrt{2}} \binom{2n}{n} - \frac{83\sqrt{2}}{2^{2n}n} \binom{2n-2}{n-1} + \frac{9189\sqrt{2}}{2^{2n+2}(n-2)(n-3)} \binom{2n-4}{n-4} \right. \\ &\quad \left. + \frac{9\sqrt{2}(-1)^n}{2^{2n-1}n} \binom{2n-2}{n-1} - \frac{855\sqrt{2}(-1)^n}{2^{2n}(n-2)(n-3)} \binom{2n-4}{n-4} + ... \right] \\ &\asymp 2^n \sqrt{\frac{2}{\pi n}} \left(2 + \frac{9(-1)^n - 42}{2n} + \frac{2173 - 828(-1)^n}{16n^2} + \frac{135765 - 50850(-1)^n}{512n^3} + ... \right). \end{aligned}$$

Using computer algebra and the total number of objects derived in Remarks 3 we get

$$\begin{aligned} m_{(2)}(n) &= \frac{[z^n] M_{(2)}(z)}{D_{open}(n)} \\ &\asymp \begin{cases} 2 + \frac{(9(-1)^n - 41)}{2n} + \frac{(1045 - 405(-1)^n)}{8n^2} + \ldots & \text{, for } n \text{ even} \\ 2 + \frac{(9(-1)^n - 39)}{2n} + \frac{(957 - 387(-1)^n)}{8n^2} + \ldots & \text{, for } n \text{ odd} \end{cases}. \end{aligned}$$

■

2.8 Observation

Like in the previous sections we give a Table which enumeratively compares asymptotic and exact average number of returns to the origin.

	Returns		**Variance**	
n	exact	*asympt.*	exact	*asympt.*
50	0.9615	0.9616	1.7064	1.7489
101	0.9615	0.9616	1.8149	1.8158
174	0.9886	0.9886	1.9219	1.9219
222	0.9911	0.9911	1.9384	1.9384
367	0.9892	0.9892	1.9465	1.9465
593	0.9933	0.9933	1.9667	1.9667
712	0.9972	0.9972	1.9210	1.9210
844	0.9976	0.9976	1.9835	1.9835
1001	0.9960	0.9960	1.9802	1.9802

Table 2.3: Average 0-level returns by an open ended Dyck path.

From the above Table 2.3 it is evident that the asymptotic and the exact results are asymptotically the same as n grows larger.

Chapter 3

Unrestricted Dyck random walks

3.1 Introduction.

We use the same procedure as in Chapter 2 to construct all our generating functions. However, this time around we *unrestrict* all our Dyck random walks from going below the x-axis. Throughout this Chapter, all **unrestricted Dyck random walks** will be called random walks unless otherwise stated. The work in this Chapter was highly influenced by [19] where the maximum height of a walk is calculated chiefly via Mellin transform approach.

In what follows we asymptotically analyze the average number of

(i) visits to height r where $r > 0$.

(ii) returns to the origin

by an arbitrary random walk which starts at the origin and has an ending at (n, i) as shown in the diagram below:

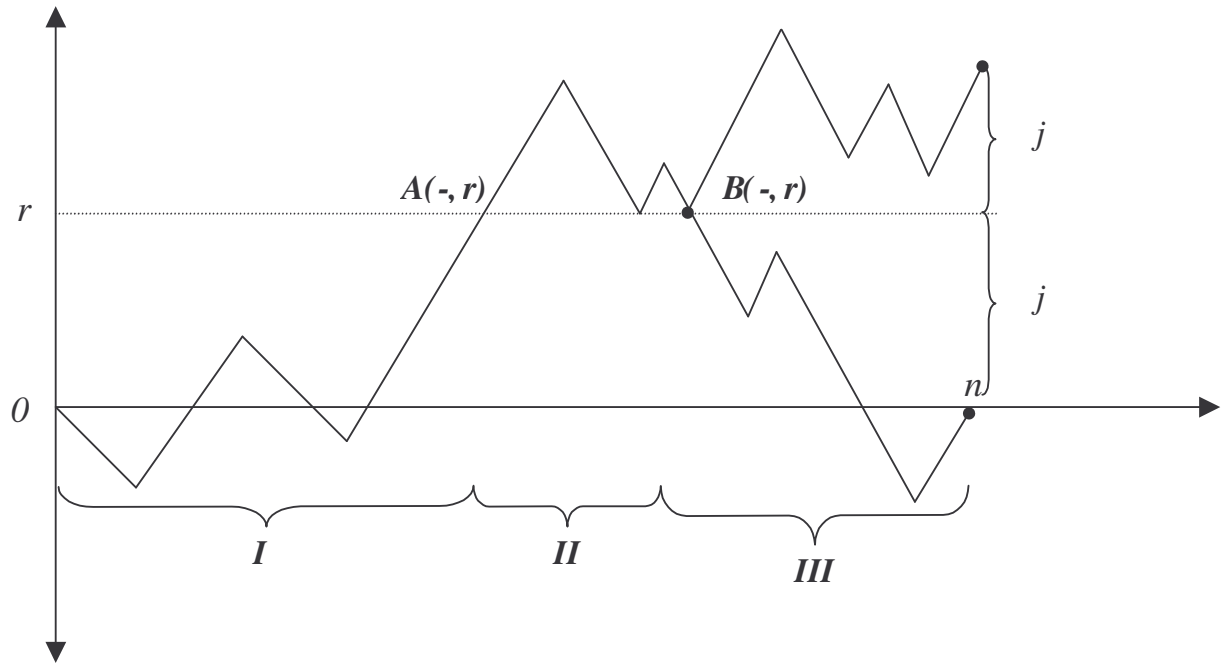


Figure 3.1: Open ended unrestricted Dyck path of length n.

The generating function for the random walk of length n for all i where $-k < i < h$ is given by:

$$\varphi_{i;\,h,k}\,(z) = \sum_{n\geq 0} \left[\begin{array}{l} \text{Number of paths of length } n, \text{ starting at } 0, \text{bounded above,} \\ \text{by the line } h \text{ below by } -k, \text{leading to } (n,i)\,. \end{array} \right] z^n,$$

where z counts the single steps. For all i between $-k$ and h we describe the step from level i to the next level by the following diagram

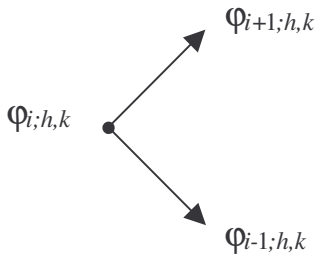


Figure 3.2

which gives recursions

$$\varphi_{i;\,h,k}(z) = z\varphi_{i-1;h,k}(z) + z\varphi_{i+1;h,k}(z)\,,\ \text{for } i \neq 0$$

and

$$\varphi_{i;\,h,k}(z) = z\varphi_{i-1;h,k}(z) + z\varphi_{i+1;h,k}(z) + 1,\ \text{for } i = 0.$$

The special systems are given by

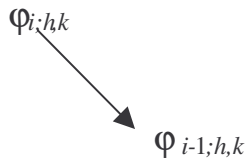


Figure 3.3

and

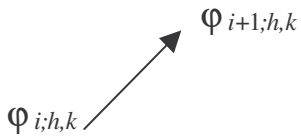


Figure 3.4

$$\varphi_{i;\,h,k}(z) = z\varphi_{i-1;h,k}(z) \text{ and } \varphi_{i;\,h,k}(z) = z\varphi_{i+1;h,k}(z)\,,$$

where $i = h$ and $i = -k$ respectively. These recursions, for all i where $-k \leq i \leq h$ are best expressed in a single matrix equation

$$B_{h+k}\Phi_{h+k} = C_{h+k},$$

where the coefficient matrix B_{h+k} is a square matrix with $h+k+1$ rows, Φ_{h+k} is the column matrix representing the $\varphi_{i;\,h,k}(z)$'s and C_{h+k} is the column matrix of $h+k$ zeros and a 1 in the middle (or the $h+1$) entry. Thus

$$\begin{bmatrix} 1 & -z & & & & & & & & & \\ -z & 1 & -z & & & & 0 & & & & \\ & & . & & & & & & & & \\ & & & . & & & & & & & \\ & & & -z & 1 & -z & & & & & \\ & & & & -z & 1 & -z & & & & \\ & & & & & -z & 1 & -z & & & \\ & & & & & & . & & & & \\ & & & & & & & . & & & \\ & & & 0 & & & & -z & 1 & -z & \\ & & & & & & & & -z & 1 & \end{bmatrix} \begin{bmatrix} \varphi_{h;h,k} \\ \varphi_{h-1;h,k} \\ . \\ . \\ \varphi_{1;h,k} \\ \varphi_{0;h,k} \\ \varphi_{-1;h,k} \\ . \\ . \\ \varphi_{-k+1;h,k} \\ \varphi_{-k;h,k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 1 \\ 0 \\ . \\ . \\ 0 \\ 0 \end{bmatrix} \tag{3.1}$$

The $\varphi_{i;h,k}$'s satisfy the system that has $h+k+1$ rows; this is only true for $h+k \geq 2$. We have special values $h+k=0$ and $h+k=1$ which would be considered separately. The calculations of the $\varphi_{i;h,k}$'s is done using Cramer's rule.

Lemma 27 *The determinant of the matrix of the system of the $\varphi_{i;h,k}$'s with $h+k+1$ rows is given by*

$$\det(B_{h+k}) = \frac{1-v^{2(h+k)+4}}{(1+v^2)^{h+k+1}(1-v^2)}.$$

Proof. Replacing h in the results of Lemma 6 of Chapter 2 by $h+k$ we get the desired results. ■

Theorem 28 *The generating function of any random walk bounded above and below by the lines $h>0$ and $-k<0$ respectively, and is ending at (n,i) is given by*

$$\varphi_{i;h,k}(z) = \begin{cases} v^i \frac{1+v^2}{1-v^2} \frac{\left(1-v^{2h-2i+2}\right)\left(1-v^{2k+2}\right)}{\left(1-v^{2h+2k+4}\right)} & \text{for } i \geq 0 \\ v^{|i|} \frac{1+v^2}{1-v^2} \frac{\left(1-v^{2h+2}\right)\left(1-v^{2k-2|i|+2}\right)}{\left(1-v^{2h+2k+4}\right)} & \text{for } i \leq 0 \end{cases},$$

with $z = \frac{v}{1+v^2}$.

Proof. We shall now compose the new matrix called $B_{i,h+k}$ which is B_{h+k} where the i-th column is replaced by the column matrix C_{h+k}. Expanding by the $(h-i+1)$-th column we obtain the determinant

$$\begin{aligned}\det\left(B_{i,h+k}\right) &= z^i \det\left(B_{h+k-i-1}\right) \\ &= \begin{cases} z^i \det\left(B_{h-i}\right)\det\left(B_k\right), \text{ for } i \geq 0 \\ z^i \det\left(B_k\right)\det\left(B_{k-i}\right), \text{ for } i \leq 0 \end{cases}.\end{aligned}$$

Hence by Lemma 29 and Cramer's rule we obtain

$$\begin{aligned}\varphi_{i;h,k}\left(z\right) &= \begin{cases} \frac{z^i \det(A_{h-i})\det(A_k)}{\det(A_{h+k})}, \text{ for } i \geq 0 \\ \frac{z^i \det(A_k)\det(A_{k-i})}{\det(A_{h+k})}, \text{ for } i \leq 0 \end{cases} \\ &= \begin{cases} v^i \frac{1+v^2}{1-v^2} \frac{\left(1-v^{2h-2i+2}\right)\left(1-v^{2k+2}\right)}{\left(1-v^{2h+2k+4}\right)} \text{ for } i \geq 0 \\ v^{|i|} \frac{1+v^2}{1-v^2} \frac{\left(1-v^{2h+2}\right)\left(1-v^{2k-2|i|+2}\right)}{\left(1-v^{2h+2k+4}\right)} \text{ for } i \leq 0 \end{cases}\end{aligned}$$

as required.

■

Remark 5 *If we unrestrict the generating function $\varphi_{i;h,k}\left(z\right)$, either from above or below we get*

$$\varphi_{i;\infty,k}\left(z\right) = \begin{cases} v^i \frac{1+v^2}{1-v^2}\left(1-v^{2k+2}\right) \ for \ i \geq 0 \\ v^{|i|} \frac{1+v^2}{1-v^2}\left(1-v^{2k-2|i|+2}\right) \ for \ i \leq 0 \end{cases},$$

and

$$\varphi_{i;h,\infty}\left(z\right) = \begin{cases} v^i \frac{1+v^2}{1-v^2}\left(1-v^{2h-2i+2}\right) \ for \ i \geq 0 \\ v^{|i|} \frac{1+v^2}{1-v^2}\left(1-v^{2h+2}\right) \ for \ i \leq 0 \end{cases},$$

so that

$$\begin{aligned}\sum_i \varphi_{i;\infty,\infty}\left(z\right) &= \frac{\left(1+v^2\right)}{1-v^2}\left(2\sum_{i\geq 1} v^i + 1\right) \\ &= \frac{\left(1+v^2\right)\left(1+v\right)}{\left(1-v^2\right)\left(1-v\right)} \\ &= \frac{\left(1+v^2\right)}{\left(1-v\right)^2}\end{aligned}$$

$$= \frac{1}{1-2z}, \text{ using the substitution } z = \frac{v}{1+v^2}$$
$$= \sum_{n\geq 0} 2^n z^n.$$

Thus

$$[z^n] \sum_i \varphi_{i;\infty,\infty}(z) = 2^n,$$

which is the required total number of random walks with an open ending at (n, i).

■

3.2 Generating functions for the number of visits to the r-level.

In order to get a suitable bivariate generating function for the entire walk from $(0,0)$ to $(n, r \pm j)$ we decompose the family of these random walks in Figure 3.1 using the symbolic method in [3] and [9] into three parts (or Lemmas given below). As in the previous Chapter z counts the single steps and u counts the number of returns to level r, $r > 0$.

Lemma 29 *The generating function of any random walk which starts from* $(0,0)$ *and reaching* r *for the first time is given by* $z\varphi_{r-1;r-1,\infty}(z)$.

Proof. This is the random walk for part I of Figure 3.1. It starts at $(0,0)$ and ends at $\mathrm{A}(-, r)$. Hence this walk is bounded above and below by r and $-\infty$ respectively. The generating function for this walk is therefore given by $\varphi_{r;r,\infty}(z)$ as $k \to \infty$. By decomposing this random walk, we can rewrite this generating function as $z\varphi_{r-1;r-1,\infty}(z)$ where the z in the product is the single step that separate the function $\varphi_{r-1;r-1,\infty}(z)$ and the line r. ■

Lemma 30 *The generating function of any random walk which starts and ends on the* r*-level is given by*

$$F_r(z,u) = \frac{u}{1 - u\left(\frac{2v^2}{1+v^2}\right)},$$

where $z = \frac{v}{1+v^2}$ *and* u *counts the number of returns to the origin.*

Proof. This random walk starts at $\mathrm{A}(r,-)$ and ends at $\mathrm{B}(r,-)$ in Figure 3.1. By a simple shift we let level r be the 0-level. Let's denote the generating function for this random walk by $F_r(z,u)$, where u is counting the number of returns to the r-level. Furthermore, let $F_{r,+}(z)$ and $F_{r,-}(z)$ denote the generating functions of random walks that are above and below the 0-level respectively. Between any two returns on the 0-level, we have a sequence of paths above and below the x-axis, which we denote symbolically as $F_r(z,u) = (F_{r,+}(z) \cup F_{r,+}(z))^*$ as shown in the following diagram:

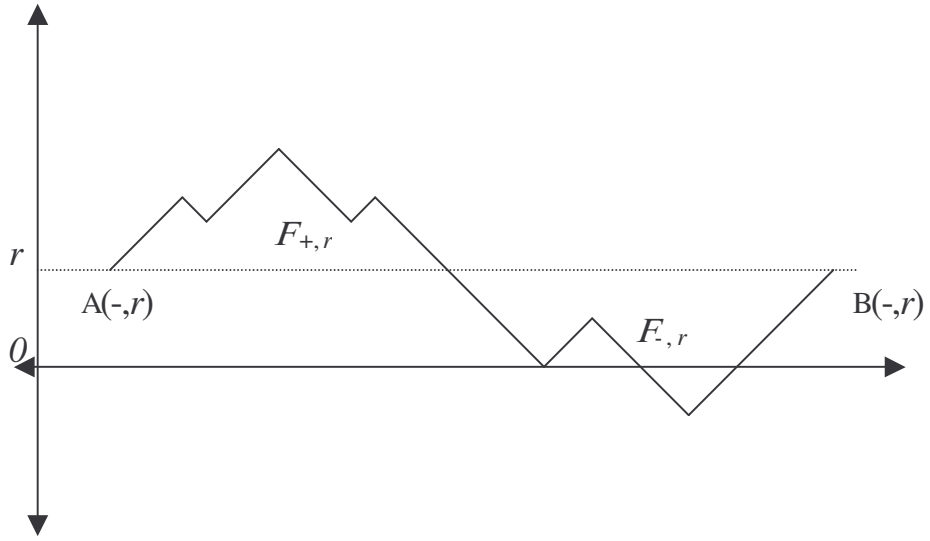


Figure 3.5

Hence

$$F_r(z,u) = \frac{u}{1-u\left(F_{r,+}(z)+F_{r,+}(z)\right)}.$$

Using the formula for $\varphi_{i;h,k}(z)$ derived in Theorem 28 we decompose $F_{r,+}(z)$ using the procedure used in Lemma 9 of Chapter 2 to get

$$F_{r,+}(z) = z^2\varphi_{0;\infty,0}(z).$$

However both $F_{r,-}(z)$ and $F_{r,+}(z)$ are symmetrical so that $F_{r,-}(z) = F_{r,+}(z)$. Therefore

$$\begin{aligned} F_r(z,u) &= \frac{u}{1-u\left[z^2(F_{r,+}(z))+z^2(F_{r,-}(z))\right]} \\ &= \frac{u}{1-u\left[z^2(\varphi_{0;\infty,0}(z))+z^2(\varphi_{0;0,\infty}(z))\right]} \\ &= \frac{u}{1-u\left(2z^2\varphi_{0;\infty,0}(z)\right)}, \text{ since } \varphi_{0;\infty,0}(z) = \varphi_{0;0,\infty}(z) \text{ by symmetry} \end{aligned}$$

$$= \frac{u}{1-u\left(\frac{2uv^2}{1+v^2}\right)}, \text{ since } \varphi_{0;\infty,0}(z) = \left(1+v^2\right)$$

■

Note: *The generating function $F_r(z,u)$ is unrestricted both above and below since*

$$(h,k) \to (\infty,\infty).$$

Lemma 31 *The generating function for any random walk that leaves r and never returns to the r-level is given by*

(a) $z\varphi_{j-1;\infty,0}(z)$ *where the end point is above r as $h \to \infty$*

(b) $z\varphi_{1-j;0,\infty}(z)$ *where the end point is below r as $-k \to -\infty$*

Proof. These walks start at $\mathrm{B}(r,-)$ and either end at $(n, r+j)$ or $(n, r-j)$ without making any return to the r-level in between these two ends. And so we have two symmetrical cases here:

Case a. *Random walks ending at* $(n, r+j)$ as $h \to \infty$.

If we treat $r+1$ as the 0-level, then these walks will end at $j-1$, and are consequently bounded below by 0 and above by ∞. Therefore the generating function for these walks in terms of Theorem 28 is given by $z\varphi_{j-1;\infty,0}(z)$ as $h \to \infty$. The z in this product is for the single z-step that separate $\varphi_{j-1;\infty,0}(z)$ and the line r.

Case b. *Random walks ending at* $(n, r-j)$ as $-k \to -\infty$.

If we treat $r-1$ as the 0-level, then these walks end at $1-j$, and are consequently bounded below by 0 and above by ∞. Symbolically, we have that the generating function for these walks is given by $z\varphi_{1-j;0,\infty}(z)$ as $k \to \infty$. The z in this product is for the single z-step that separate $z\varphi_{1-j;0,\infty}(z)$ and the line r. ■

Remark 6

$$\begin{aligned}\varphi_{1-j;0,\infty}(z) &= v^{|1-j|}\frac{(1+v^2)}{(1-v^2)}\left(1-v^2\right)\\ &= v^{j-1}\left(1+v^2\right)\\ &= \varphi_{j-1;\infty,0}(z),\end{aligned}$$

and this is not surprising because $\varphi_{1-j;0,\infty}(z)$ and $\varphi_{j-1;\infty,0}(z)$ are symmetrical. ■

3.3 Closed ending random walks.

Theorem 32 *The generating function for a random walk which starts at* $(0,0)$ *and ends at* $(n,0)$ *is given*

$$\Delta_{r,closed}(z,u) = \frac{(1+v^2)(1-v^{2r})}{(1-v^2)} + \frac{uv^{2r}}{1-u\left(\frac{2v^2}{1+v^2}\right)},$$

where $z = \frac{v}{1+v^2}$ *and* u *counts the number of visits to the* r*-level.*

Proof. If we substitute $j = r$ into Lemma 31(b) we get a closed ending which starts at height r and end at the x-axis. Treating $r-1$ as the 0-level we have that this closed ending walk is bounded above and below by $r-1$ and k respectively. The mathematical expression for this generating function is therefore given by $z\varphi_{r-1;r-1,\infty}(z)$ as $k \to \infty$, where the z in the product is the single step that separate the function $\varphi_{r-1;r-1,\infty}(z)$ and the line r. Hence the overall generating function for this type of random walk comprises of the generating functions for parts I, II and $z\varphi_{r-1;r-1,\infty}(z)$ and is consequently given by:

$$\begin{aligned}\Delta_{r,closed}(z,u) &= \varphi_{0;r-1,\infty}(z) + z\varphi_{r-1;r-1,\infty}(z) \times F_r(z,u) \times z\varphi_{r-1;r-1,\infty}(z)\\ &= \frac{(1+v^2)(1-v^{2r})}{(1-v^2)} + \frac{uv^{2r}}{1-u\left(\frac{2v^2}{1+v^2}\right)},\end{aligned}$$

where $\varphi_{0;r-1,\infty}(z)$ are the random walks not reaching r. ■

Remark 7 *We test the generating function* $\Delta_{r,closed}(z,u)$ *by plugging in* $u = 1$*:*

$$\Delta_{r,closed}(z,1) = \frac{(1+v^2)(1-v^{2r})}{1-v^2} + \frac{v^{2r}}{1-\left(\frac{2v^2}{1+v^2}\right)}$$

$$= \frac{(1+v^2)}{1-v^2}.$$

And so

$$\begin{aligned} B(n) &= [z^n]\,\Delta_{r,closed}(z,1) \\ &= \frac{1}{2\pi i}\oint \frac{(1-v^2)(1+v^2)^{n-1}}{v^{n+1}}dv \\ &= [v^n]\left(1+v^2\right)^n \\ &= \binom{n}{\frac{n}{2}},\ for\ n\ even, \end{aligned}$$

is the total number of a walk with a closed ending at $(n,0)$. ■

Proposition 33 *The generating function of the* s*-th factorial moments of* $\Delta_{r,closed}(z,u)$ *multiplied by the number of paths is given by*

$$M_{(s)}(z) = s!\left(2v^2\right)^{s-1}\frac{(1+v^2)^2}{(1-v^2)^{s+1}}v^{2r},$$

where $z = \frac{v}{1+v^2}$.

Proof. Now,

$$\begin{aligned} M_{(s)}(z) &= \frac{\partial^s}{\partial u^s}\Delta_{r,closed}(z,u)\Big|_{u=1} \\ &= \left[\frac{s!\left(\frac{2v^2}{1+v^2}\right)^s}{\left(1-\frac{2v^2}{1+v^2}\right)^{s+1}} + \frac{s!\left(\frac{2v^2}{1+v^2}\right)^{s-1}}{\left(1-\frac{2v^2}{1+v^2}\right)^{s}}\right]v^{2r} \\ &= s!\left(2v^2\right)^{s-1}\frac{(1+v^2)^2}{(1-v^2)^{s+1}}v^{2r}. \end{aligned}$$

■

We are now ready to asymptotically expand $\Delta_{r,closed}(z,u)$ around the dominant singularity $\pm\frac{1}{2}$ using our famous results [8] by Flajolet and Sedgewick.

Corollary 34 *The average number of visits to level* r *by a closed ended random walk of lengths* n *is given by*

$$m_{(1)}(n) \asymp \begin{cases} 2\left[\sqrt{2\pi n}\left(\frac{1}{4}+\frac{1}{16n}+\frac{1}{128n^2}+...\right)+\frac{r}{3n}+\frac{2r^3}{3n}-\frac{4r}{15n^2}-\frac{2r^5}{5n^2}-\frac{4r^3}{3n^2}-r\right] & ,\ n\ even \\ 0 & ,\ n\ odd \end{cases}.$$

Proof. Substituting $s = 1$ into Proposition 33 we get

$$M_{(1)}(z) = \frac{v^{2r}\left(1+v^2\right)^2}{\left(1-v^2\right)^2}.$$

Expanding $M_{(1)}(z)$ around $z = \frac{1}{2}$ gives the contribution

$$\begin{aligned} M_{(1)}(z) &\sim \frac{1}{2(1-2z)} - \frac{2r}{\sqrt{2}\sqrt{1-2z}} + \frac{(8r^2+1)}{4} - \frac{r\left(11+16r^2\right)\sqrt{2}\sqrt{1-2z}}{12} \\ &\quad + \frac{\left(32r^4+64r^2+3\right)(1-2z)}{24} - \frac{r\left(349+1120r^2+256r^4\right)\sqrt{2}\,(1-2z)^{\frac{3}{2}}}{480} + ... \end{aligned}$$

Expanding $M_{(1)}(z)$ around $z = -\frac{1}{2}$ gives the contribution

$$\begin{aligned} M_{(1)}(z) &\sim \frac{1}{2(1+2z)} - \frac{2r}{\sqrt{2}\sqrt{1+2z}} + \frac{(8r^2+1)}{4} - \frac{r\left(11+16r^2\right)\sqrt{2}\sqrt{1+2z}}{12} \\ &\quad + \frac{\left(32r^4+64r^2+3\right)(1+2z)}{24} - \frac{r\left(349+1120r^2+256r^4\right)\sqrt{2}\,(1+2z)^{\frac{3}{2}}}{480} + ... \end{aligned}$$

Now adding up the two contributions we get

$$\begin{aligned} [z^n]\, M_{(1)}(z) &\asymp 2^n \left[\frac{1}{2} - \frac{r\left(349+1120r^2+256r^4\right)\sqrt{2}}{5(n-2)(n-3)2^{2n+3}} \binom{2n-4}{n-4} - \frac{r}{2^{2n-1}\sqrt{2}} \binom{2n}{n} \right. \\ &\quad + \frac{r\left(11+16r^2\right)\sqrt{2}}{3(n-1)2^{2n+1}} \binom{2n-2}{n-2} - \frac{r\left(349+1120r^2+256r^4\right)(-1)^n\sqrt{2}}{5(n-2)(n-3)2^{2n+3}} \binom{2n-4}{n-4} \\ &\quad \left. + \frac{(-1)^n}{2} - \frac{r(-1)^n}{2^{2n-1}\sqrt{2}} \binom{2n}{n} + \frac{r\left(11+16r^2\right)(-1)^n\sqrt{2}}{3(n-1)2^{2n+1}} \binom{2n-2}{n-2} + ... \right]. \end{aligned}$$

However, for an even n the total number of objects for a closed ended walk is given by $\binom{n}{\frac{n}{2}}$ as calculated in Remark 7.

$$\begin{aligned} m_{(1)}(n) &= \frac{[z^n]\, M_{(1)}(z)}{\binom{n}{\frac{n}{2}}} \\ &\asymp \left((-1)^n+1\right)\left[\sqrt{2\pi n}\left(\frac{1}{16n}+\frac{1}{128n^2}+\frac{1}{4}+...\right)+\frac{2r^3}{3n}+\frac{r}{3n}-\frac{4r^3}{3n^2}-\frac{4r}{15n^2}-\frac{2r^5}{5n^2}-r\right] \\ &= \begin{cases} 2\left[\sqrt{2\pi n}\left(\frac{1}{4}+\frac{1}{16n}+\frac{1}{128n^2}+...\right)+\frac{r}{3n}+\frac{2r^3}{3n}-\frac{4r}{15n^2}-\frac{2r^5}{5n^2}-\frac{4r^3}{3n^2}-r\right] & , n \text{ even} \\ 0 & , n \text{ odd} \end{cases}, \end{aligned}$$

where the last asymptotic expansion was done using computer algebra. ■

Corollary 35 *The second moments on $\Delta_{r,closed}(z,u)$ are given by*

$$m_{(2)}(n) \asymp \begin{cases} 2\left[n+2r^2+2r+2-\frac{4\left(r+2r^2+2r^3+r^4\right)}{3n}-\sqrt{2\pi n}\left(r+\frac{1}{2}+\frac{r}{4n}+\frac{1}{8n}+...\right)\right] & , \text{ where } n \text{ is even} \\ 0 & , \text{ where } n \text{ is odd} \end{cases}$$

Proof. Substituting $s=2$ into Proposition 33 we get

$$M_{(2)}(z) = \frac{4v^{2r+2}\left(1+v^2\right)^2}{\left(1-v^2\right)^3}.$$

Expanding $M_{(2)}(z)$ around $z=\frac{1}{2}$ gives the contribution

$$M_{(2)}(z) \sim \frac{1}{\sqrt{2}\left(1-2z\right)^{\frac{3}{2}}}-\frac{(2r+1)}{(1-2z)}+\frac{\left(16r^2+16r+3\right)}{4\sqrt{2}\sqrt{1-2z}}+...$$

On the other hand, expanding around $z=-\frac{1}{2}$ gives the contribution

$$M_{(2)}(z) \sim \frac{1}{\sqrt{2}\left(1+2z\right)^{\frac{3}{2}}}-\frac{(2r+1)}{(1+2z)}+\frac{\left(16r^2+16r+3\right)}{4\sqrt{2}\sqrt{1+2z}}+...$$

Now, adding up the two contributions we obtain

$$\begin{aligned} [z^n]M_{(2)}(z) &\asymp 2^n\left[\frac{(2n+2)}{2^{2n+1}\sqrt{2}}\binom{2n+1}{n}-(2r+1)+\frac{\left(16r^2+16r+3\right)}{2^{2n+2}\sqrt{2}}\binom{2n}{n}\right. \\ &\quad -\frac{\sqrt{2}\left(256r^4+512r^3+608r^2+352r+45\right)}{3\left(n-1\right)2^{2n+5}\sqrt{2}}\binom{2n-2}{n-2}-(2r+1)(-1)^n \\ &\quad +\frac{\left(16r^2+16r+3\right)(-1)^n}{2^{2n+2}\sqrt{2}}\binom{2n}{n}+\frac{(2n+2)(-1)^n}{2^{2n+1}\sqrt{2}}\binom{2n+1}{n} \\ &\quad \left.-\frac{\sqrt{2}\left(256r^4+512r^3+608r^2+352r+45\right)(-1)^n}{3\left(n-1\right)2^{2n+5}\sqrt{2}}\binom{2n-2}{n-2}+...\right] \\ &\sim 2^n\left((-1)^n+1\right)\left[-2r-1\sqrt{\frac{2n}{\pi}}\left(1+\frac{3}{4n}+\frac{2r^2}{n}+\frac{2r}{n}+...\right)\right] \end{aligned}$$

where the last asymptotic expansion was done using computer algebra. And so

$$\begin{aligned} m_{(2)}(n) &= \frac{[z^n]\,M_{(2)}(z)}{\binom{n}{\frac{n}{2}}} \\ &\asymp \begin{cases} 2\left[\begin{array}{c} n+2r^2+2r+2-\frac{4\left(r+2r^2+2r^3+r^4\right)}{3n} \\ -\sqrt{2\pi n}\left(r+\frac{1}{2}+\frac{r}{4n}+\frac{1}{8n}+...\right) \end{array}\right], & \text{where } n \text{ is even} \\ 0 & \text{where } n \text{ is odd} \end{cases}, \end{aligned}$$

as required. ■

3.4 Observation

Table 3.1 given below helps us to make an enumerative comparison between the asymptotic and exact average number of visits to height r, together with their corresponding variances, by an unrestricted Dyck path with a closed ending at $(n, 0)$.

	Type of	**Average r-visits**			**Variances**		
n	**results**	$r=1$	$r=2$	$r=5$	$r=1$	$r=2$	$r=5$
400	exact	23.087	21.112	15.490	149.569	150.451	142.752
400	*asympt.*	23.087	21.112	15.489	149.569	150.451	142.752
634	exact	29.573	27.589	21.831	243.564	244.653	238.767
634	*asympt.*	29.573	27.589	21.831	243.564	244.653	238.767
856	exact	34.682	32.693	26.874	333.765	334.970	330.270
856	*asympt.*	34.682	32.693	26.874	333.765	334.970	330.270
1000	exact	37.645	35.655	29.810	392.619	393.879	389.778
1000	*asympt.*	37.645	35.655	29.810	392.619	393.879	389.778

Table 3.1: Average r-level visits by a closed ended unrestricted Dyck path.

From this table it is evident that exact and asymptotic results are approximately the same.

3.5 Random walks with open endings.

These walks end anywhere either above, on, or below the r-level.

Lemma 36 *The generating function for a random walk which starts at* $(0, 0)$ *and has an open ending at* $(n, r-j)$ *is given by*

$$\Omega_{r,below}(z,u) = \varphi_{r-j;r-1,\infty}(z) + \frac{uv^{r+j}}{1-u\left(\frac{2v^2}{1+v^2}\right)},$$

where $z = \frac{v}{1+v^2}$ and u counts the number of visits to the r-level.

Proof. If the end point is below the r-level, then the ending of this random walk has a generating function corresponding to part III of Fig. 3.1 given by $z\varphi_{1-j;\infty,0}(z)$ derived in

Lemma 31(b). Hence the overall generating function for the entire walk is given by

$$\Omega_{r,below} = \varphi_{r-j;r-1,\infty}(z) + z\varphi_{r-1;r-1,\infty}(z) \times F_r(z,u) \times z\varphi_{1-j;\infty,0}(z),$$

where $\varphi_{j-1;r-1,\infty}(z)$ are random walks not reaching r. Now

$$\begin{aligned} z\varphi_{r-1;r-1,\infty}(z) &= \left(\frac{v}{1+v^2}\right) v^{r-1}\left(1+v^2\right) \\ &= v^r, \end{aligned}$$

and

$$\begin{aligned} z\varphi_{j-1;\infty,0}(z) &= \left(\frac{v}{1+v^2}\right) v^{j-1}\left(1+v^2\right) \\ &= v^j, \end{aligned}$$

so that

$$\Omega_{r,below} = \varphi_{j-1;r-1,\infty}(z) + \frac{uv^{r+j}}{1-u\left(\frac{2v^2}{1+v^2}\right)}.$$

■

Lemma 37 *The generating function for a random walk starting at* $(0,0)$ *and has an open ending at* $(n, r+j)$ *is given by*

$$\Omega_{r,above}(z,u) = \frac{uv^{r+j}}{1-u\left(\frac{2v^2}{1+v^2}\right)},$$

where $z = \frac{v}{1+v^2}$ and j is above r.

Proof. By symmetry $\varphi_{1-j;0,\infty}(z) = \varphi_{j-1;\infty,0}(z)$, so that

$$\begin{aligned} \Omega_{r,above}(z,u) &= z\varphi_{r-1;r-1,\infty}(z) \times F_r(z,u) \times z\varphi_{1-i;0,\infty}(z) \\ &= \frac{uv^{r+j}}{1-u\left(\frac{2v^2}{1+v^2}\right)} \end{aligned}$$

as required. ■

We are now ready to get the bivariate generating function for the entire random walk with an open ending at $r \pm j$. To do so, we *sum* up the generating functions for the above two cases together with the generating function $z\varphi_{r-1;r-1,\infty}(z) \times F_r(z,u)$.

Theorem 38 *The generating function for all open ended paths is given by*

$$\Omega_{r,open}(z,u)=\frac{(1+v^2)(1-v^r)(1+v)}{(1-v^2)(1-v)}+\frac{uv^r}{1-u\left(\frac{2v^2}{1+v^2}\right)}\left(\frac{1+v}{1-v}\right),$$

where $z=\frac{v}{1+v^2}$ *and* u *counts the number of returns to the* r*-level.*

Proof. Summing up the two generating functions for the above two Lemmas together with the generating function $z\varphi_{r-1;r-1,\infty}(z)\times F_r(z,u)$ we get

$$\begin{aligned}\Omega_{r,open}(z,u) &= \sum_{k=0}^{r-1}\varphi_{k;r-1,\infty}(z)+\sum_{k\leq -1}\varphi_{k;r-1,\infty}(z)+\frac{uv^r}{1-u\left(\frac{2v^2}{1+v^2}\right)}\left(2\sum_{j\geq 1}v^j+1\right)\\ &= \frac{(1+v^2)(1-v^r)(1+v)}{(1-v^2)(1-v)}+\frac{uv^r}{1-u\left(\frac{2v^2}{1+v^2}\right)}\left(\frac{2v}{1-v}+\frac{1-v}{1-v}\right)\\ &= \frac{(1+v^2)(1-v^r)(1+v)}{(1-v^2)(1-v)}+\frac{uv^r}{1-u\left(\frac{2v^2}{1+v^2}\right)}\left(\frac{1+v}{1-v}\right).\end{aligned}$$

■

Remark 8 *We test the generating function* $\Omega_{r,open}(z,u)$ *by plugging in* $u=1$*:*

$$\begin{aligned}\Omega_{r,open}(z,1) &= \frac{(1+v^2)(1-v^r)(1+v)}{(1-v^2)(1-v)}+\frac{v^r(1+v^2)}{(1-v^2)}\left(\frac{1+v}{1-v}\right)\\ &= \frac{(1+v^2)}{(1-v)^2}\\ &= \frac{1}{1-2z}\\ &= \sum_{n\geq 0}2^n z^n,\end{aligned}$$

which is the required generating function for the total number of random walks with an open ending at $(n,-)$*.* ■

Proposition 39 *The generating function of the* s*-th factorial moments of* $\Omega_{r,open}(z,u)$ *multiplied by the number of paths is given by*

$$M_{(s)}(z)=s!\left[\left(2v^2\right)^{s-1}\frac{(1+v^2)^2}{(1-v^2)^{s+1}}\right]\left(\frac{1+v}{1-v}\right)v^r.$$

Proof. From the above Theorem 38 we have that

$$\begin{aligned} M_{(s)}(z) &= \frac{\partial^s}{\partial u^s}\Omega_{r,open}(z,u)\Big|_{u=1} \\ &= \left[\frac{s!\left(\frac{2v^2}{1+v^2}\right)^s}{\left(1-\frac{2v^2}{1+v^2}\right)^{s+1}} + \frac{s!\left(\frac{2v^2}{1+v^2}\right)^{s-1}}{\left(1-\frac{2v^2}{1+v^2}\right)^{s}}\right]\left(\frac{1+v}{1-v}\right)v^r \\ &= s!\left[\left(2v^2\right)^{s-1}\frac{\left(1+v^2\right)^2}{\left(1-v^2\right)^{s+1}}\right]\left(\frac{1+v}{1-v}\right)v^r. \end{aligned}$$

■

Corollary 40 *The average number of visits to level r by random walk with an open ending at $(n,-)$ is given by*

$$m_{(1)}(n) \asymp \sqrt{\frac{2n}{\pi}}\left(1+\frac{r^2+1}{n}-\frac{(-1)^{n+r}}{4n}-\frac{r^4}{24n^2}-\frac{r^2}{3n^2}-\frac{3}{32n^2}+\frac{(-1)^{n+r}\left(r^2+1\right)}{n^2}+...\right)-r.$$

Proof. Replacing s by 1 into Proposition 39 we get

$$M_{(1)}(z) = \frac{v^r\left(1+v^2\right)^2(1+v)}{\left(1-v^2\right)^2(1-v)}.$$

Now, by making the substitution that $z=\frac{v}{1+v^2}$ and upon expansion around $z=\frac{1}{2}$ we obtain

$$\begin{aligned} M_{(1)}(z) \sim\ & \frac{1}{\sqrt{2}(1-2z)^{\frac{3}{2}}}-\frac{r}{1-2z}+\frac{4r^2+1}{4\sqrt{2}\sqrt{1-2z}}-\frac{\left(r^3+2r\right)}{3} \\ & +\frac{\left(16r^4+104r^2+9\right)\sqrt{2}\sqrt{1-2z}}{192}-\frac{\left(r^5+15r^3+14r\right)(1-2z)}{30}+.... \end{aligned}$$

Again, by making the substitution that $z=\frac{v}{1+v^2}$ in $M_{(1)}(z)$ and upon expansion around $z=-\frac{1}{2}$ we obtain

$$\begin{aligned} M_{(1)}(z) \sim\ & \frac{(-1)^r}{2\sqrt{2}\left(\sqrt{1+2z}\right)}-\frac{(-1)^r r}{2}+\frac{(-1)^r\left(4r^2+3\right)\sqrt{2}\sqrt{1+2z}}{16}-\frac{(-1)^r\left(2r^3+7r\right)(1+2z)}{12} \\ & +\frac{(-1)^r\left(16r^4+152r^2+45\right)\sqrt{2}(1+2z)^{\frac{3}{2}}}{384}+.... \end{aligned}$$

Therefore

$$\begin{aligned}
[z^n] M_{(1)}(z) &\asymp 2^n (-1)^r \left[\frac{1}{2\sqrt{2}} \binom{-\frac{1}{2}}{n} + \frac{(4r^2+3)\sqrt{2}}{16} \binom{\frac{1}{2}}{n} + \frac{(16r^4+152r^2+45)\sqrt{2}}{384} \binom{\frac{3}{2}}{n} \right] \\
&\quad + (-2)^n \left[\frac{1}{\sqrt{2}} \binom{-\frac{3}{2}}{n} + \frac{(4r^2+1)}{4\sqrt{2}} \binom{-\frac{1}{2}}{n} + \frac{(16r^4+104r^2+9)\sqrt{2}}{192} \binom{\frac{1}{2}}{n} \right. \\
&\quad \left. -r \binom{-1}{n} + ... \right] \\
&= 2^n \left[\frac{(-1)^{r+n}}{2^{2n+1}\sqrt{2}} \binom{2n}{n} - \frac{(4r^2+3)(-1)^{r+n}\sqrt{2}}{(n-1)2^{2n+3}} \binom{2n-2}{n-2} + \frac{(4r^2+1)}{2^{2n+2}\sqrt{2}} \binom{2n}{n} \right. \\
&\quad + \frac{(2n+2)}{2^{n+1}\sqrt{2}} \binom{2n+1}{n} + \frac{(16r^4+152r^2+45)(-1)^{r+n}\sqrt{2}}{2^{2n+5}(n-2)(n-3)} \binom{2n-4}{n-4} \\
&\quad \left. -r - \frac{(16r^4+104r^2+9)}{3(n-1)2^{2n+5}} \binom{2n-2}{n-2} + ... \right].
\end{aligned}$$

Hence

$$\begin{aligned}
m_{(1)}(n) &= \frac{[z^n] M_{(1)}(z)}{2^n} \\
&\asymp \frac{(2n+2)}{2^{n+1}\sqrt{2}} \binom{2n+1}{n} - \frac{(16r^4+104r^2+9)\sqrt{2}}{3(n-1)2^{2n+5}} \binom{2n-2}{n-2} \\
&\quad -r + \frac{(4r^2+1)}{2^{2n+2}\sqrt{2}} \binom{2n}{n} + \frac{(-1)^{n+r}}{2^{n+1}\sqrt{2}} \binom{2n}{n} - \frac{(4r^2+3)(-1)^{n+r}\sqrt{2}}{(n-1)2^{2n+3}} \binom{2n-2}{n-2} \\
&\quad + \frac{(16r^4+152r^2+45)(-1)^{n+r}\sqrt{2}}{2^{2n+5}(n-2)(n-3)} \binom{2n-4}{n-4} + ..., \\
&\asymp \sqrt{\frac{2n}{\pi}} \left(1 + \frac{r^2+1}{n} - \frac{(-1)^{n+r}}{4n} - \frac{r^4}{24n^2} - \frac{r^2}{3n^2} - \frac{3}{32n^2} + \frac{(-1)^{n+r}(r^2+1)}{n^2} + ... \right) - r,
\end{aligned}$$

where last asymptotic results were done using computer algebra. ■

Corollary 41 *The second moments on $\Omega_{r,open}(z,u)$ are given by*

$$\begin{aligned}
m_{(2)}(n) &\asymp n + \frac{(-1)^{n+r}}{2} + r^2 + 2r + \frac{3}{2} \\
&\quad + \sqrt{\frac{2n}{\pi}} \left(-2r - 2 - \frac{5r+r^3}{3n} - \frac{(1+r^2)}{n} + \frac{(-1)^{n+r}(r+1)}{2n} + ... \right).
\end{aligned}$$

Proof. Inserting $s = 2$ into Proposition 39 we get

$$M_{(2)}(z) = \frac{4v^{r+2}(1+v^2)^2(1+v)}{(1-v^2)^3(1-v)}.$$

Now, by making the substitution that $z=\frac{v}{1+v^2}$ in $M_{(2)}(z)$ and upon expansion around $z=\frac{1}{2}$ we get that

$$\begin{aligned} M_{(2)}(z) &\sim \frac{1}{(1-2z)^2}-\frac{2(r+1)}{\sqrt{2}(1-2z)^{\frac{3}{2}}}+\frac{(2r^2+4r+1)}{2(1-2z)}-\frac{(4r^3+12r^2+11r+3)}{6\sqrt{2}\sqrt{1-2z}} \\ &\quad -\frac{(2r^4+8r^3+16r^2+16r+3)}{12} \\ &\quad +\frac{(16r^5+80r^4+280r^3+520r^2+349r+45)\sqrt{2}\sqrt{1-2z}}{480}+... \end{aligned}$$

On the other hand, expansion around $z=-\frac{1}{2}$ gives

$$\begin{aligned} M_{(2)}(z) &\sim -\frac{(-1)^r}{2(1+2z)}-\frac{(-1)^r(r+1)}{\sqrt{2}\sqrt{1+2z}}-\frac{(-1)^r(r+1)^2}{2} \\ &\quad -\frac{(-1)^r(4r^3+12r^2+17r+9)\sqrt{2}\sqrt{1+2z}}{24}+..., \end{aligned}$$

so that

$$\begin{aligned} [z^n]M_{(2)}(z) &\asymp (-2)^n\left[\binom{-2}{n}-\frac{2(r+1)}{\sqrt{2}}\binom{-\frac{3}{2}}{n}+\frac{2r^3+4r+1}{2}\binom{-1}{n}-\frac{(4r^3+12r^2+11r+3)}{3\sqrt{2}}\binom{-\frac{1}{2}}{n}\right. \\ &\quad \left.-\frac{(16r^5+80r^4+280r^3+520r^2+349r+45)\sqrt{2}}{480}\binom{\frac{1}{2}}{n}\right] \\ &\quad +2^n(-1)^r\left[-\frac{1}{2}\binom{-1}{n}-\frac{(4r^3+12r^2+17r+9)\sqrt{2}}{24}\binom{\frac{1}{2}}{n}-\frac{(r+1)}{\sqrt{2}}\binom{-\frac{1}{2}}{n}+...\right] \end{aligned}$$

Therefore

$$\begin{aligned} m_{(2)}(n) &= \frac{[z^n]M_{(2)}(z)}{2^n} \\ &\asymp \binom{n+1}{n}-\frac{(r+1)}{\sqrt{2}}\frac{(n+1)}{2^{n-1}}\binom{2n+1}{n}-\frac{(4r^3+12r^2+11r+3)}{3\sqrt{2}\,2^{2n+1}}\binom{2n}{n} \\ &\quad +\frac{(2r^2+4r+1)}{2}+\frac{(16r^5+80r^4+280r^3+520r^2+349r+45)\sqrt{2}}{15(n-1)2^{2n+4}}\binom{2n-2}{n-2} \\ &\quad -\frac{(-1)^{n+r}}{2}-\frac{(4r^3+12r^2+17r+9)(-1)^{n+r}\sqrt{2}}{3(n-1)2^{2n+2}}\binom{2n-2}{n-2}+\frac{(r+1)(-1)^{n+r}}{2^{2n}\sqrt{2}}\binom{2n}{n}+... \\ &\asymp n+\frac{(-1)^{n+r}}{2}+r^2+2r+\frac{3}{2} \\ &\quad +\sqrt{\frac{2n}{\pi}}\left(-2r-2-\frac{5r+r^3}{3n}-\frac{(1+r^2)}{n}+\frac{(-1)^{n+r}(r+1)}{2n}+...\right), \end{aligned}$$

where last asymptotic result was done using computer algebra. ■

3.6 Observation

We are now ready to give an enumerative comparison between the asymptotic and exact average number of visits to height :

	Type of	$m_{(1)}(n)$			**Variances**		
n	**results**	$r=1$	$r=3$	$r=6$	$r=1$	$r=3$	$r=6$
367	expected	14.337	12.503	10.038	119.432	116.815	106.498
367	*asympt.*	14.358	12.692	10.825	119.432	116.815	106.498
593	expected	18.471	16.601	14.024	197.422	194.708	183.558
593	*asympt.*	18.487	16.749	14.645	197.422	194.708	183.558
712	expected	20.313	18.432	15.848	238.459	235.714	224.980
712	*asympt.*	20.357	18.596	16.388	238.459	235.714	224.980
844	expected	22.201	20.310	17.693	284.537	281.765	270.795
844	*asympt.*	22.242	20.461	18.188	284.537	281.765	270.795
1001	expected	24.275	22.376	19.703	339.877	337.078	325.189
1001	*asympt.*	24.288	22.490	20.181	339.877	337.078	325.189

Table 3.2: Average r-level visits by an open ended Dyck path.

3.7 Return statistics.

In the sequel we analyse the number of returns to the origin by an arbitrary random walk starting at $(0,0)$ with an open ending at (n,i). Unlike in [14], [17] and [18], here we don't count the origin as a return in all our walks. If we count the number of returns to the origin by the variable w then the special system

$$\varphi_0 = 1 + z\varphi_1 + z\varphi_{-1},$$

from section 3.1 becomes

$$\varphi_0 = 1 + w\left[z\varphi_1 + z\varphi_{-1}\right].$$

Therefore the system of all the φ 's given in 3.1 is now rewritten as:

$$\begin{bmatrix} 1 & -z & & & & & & & \\ -z & 1 & -z & & & & 0 & & \\ & . & & & & & & & \\ & & . & & & & & & \\ & -z & 1 & -z & & & 0 & & \\ & & -zw & 1 & -zw & & & & \\ & & & -z & 1 & -z & & & \\ & & & & & . & & & \\ & & 0 & & & -z & 1 & -z & \\ & & & & & & -z & 1 & \end{bmatrix} \begin{bmatrix} \varphi_h \\ \varphi_{h-1} \\ . \\ . \\ \varphi_1 \\ \varphi_0 \\ \varphi_{-1} \\ . \\ \varphi_{-k+1} \\ \varphi_{-k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 1 \\ 0 \\ . \\ 0 \\ 0 \end{bmatrix}. \tag{3.2}$$

As in the previous case we calculate the $\varphi_{i;h,k}$'s using Cramer's rule by first finding the determinant of the matrix given in the System (3.2).

Lemma 42 *The determinant of the matrix of the system, denoted by C_{h+k} with $h+k+1$ rows is given by*

$$\begin{aligned} \det C_{h+k} &= \frac{\left(1-v^{2h+2}\right)\left(1-v^{2k+2}\right)\left(1+v^2\right)}{\left(1-v^2\right)^2\left(1+v^2\right)^{h+k+1}} \\ &\quad -\frac{v^2 w\left[\left(1-v^{2h+2}\right)\left(1-v^{2k}\right)+\left(1-v^{2h}\right)\left(1-v^{2k+2}\right)\right]}{\left(1-v^2\right)^2\left(1+v^2\right)^{h+k+1}}, \end{aligned}$$

where $z=\frac{v}{1+v^2}$.

Proof. Substituting h by $h+k$ into the formula for $\det\left(A_{h+k}\right)$ in Lemma 6 we get

$$\begin{aligned} \det C_{h+k} &= \det\left(C_{h+k}\right)-z^2\det\left(C_{h+k-1}\right) \\ &= \det(B_{h-1}B_{k-1})-z^2w\left[\det(B_{k-2}B_{h-1})+\det(B_{k-1}B_{h-2})\right], \text{ first row expansion} \\ &= \frac{\left(1-v^{2h+2}\right)\left(1-v^{2k+2}\right)\left(1+v^2\right)}{\left(1-v^2\right)^2\left(1+v^2\right)^{h+k+1}} \\ &\quad -\frac{v^2 w\left[\left(1-v^{2h+2}\right)\left(1-v^{2k}\right)+\left(1-v^{2h}\right)\left(1-v^{2k+2}\right)\right]}{\left(1-v^2\right)^2\left(1+v^2\right)^{h+k+1}}. \end{aligned}$$

■

Theorem 43 *The generating function for an unbounded random walk ending at (n,i) is given by*

$$\varphi_{i;h,k}(z,w)=\frac{v^{|i|}}{1-w\left(\frac{2v^2}{1+v^2}\right)},$$

where w is counting the number of returns to the origin and $z=\frac{v}{1+v^2}$.

Proof. Applying the formula for $\det(A_{h+k})$ derived in Lemma 6, Lemma 42 and Cramer's rule on the System (3.2) we get

$$\begin{aligned}
\varphi_{i;h,k}(z,w) &= \frac{\det A_{h+k-i}}{\det C_{h+k}}\\
&= \frac{z^i \det A_{h-i-1}\det A_{k-1}}{\det C_{h+k}}\\
&= \frac{(1-v^2)^2(1+v^2)^{h+k+1}}{(1-v^{2h+2})(1-v^{2k+2})(1+v^2)-v^2w\left[(1-v^{2h+2})(1-v^{2k})+(1-v^{2h})(1-v^{2k+2})\right]}\\
&\quad\times\frac{v^i\left(1-v^{2h-2i+2}\right)\left(1-v^{2k+2}\right)}{(1-v^2)^2(1+v^2)^{h+k}}\\
&= \frac{v^i\left(1-v^{2h-2i+2}\right)\left(1-v^{2k+2}\right)(1+v^2)}{(1-v^{2h+2})(1-v^{2k+2})(1+v^2)-v^2w\left[(1-v^{2h+2})(1-v^{2k})+(1-v^{2h})(1-v^{2k+2})\right]}.
\end{aligned}$$

If we let $(h,k)\to(\infty,\infty)$ we obtain

$$\begin{aligned}
\varphi_{i;\infty,\infty}(z,w) &= \frac{v^{|i|}(1+v^2)}{(1+v^2)-2v^2w}\\
&= \frac{v^{|i|}}{1-w\left(\frac{2v^2}{1+v^2}\right)}
\end{aligned}$$

■

Remark 9 *Plugging $w=1$ into the generating function $\varphi_{i;\infty,\infty}(z,w)$ we get*

$$\varphi_{i;\infty,\infty}(z,1)=\frac{v^{|i|}}{1-\left(\frac{2v^2}{1+v^2}\right)}=\frac{v^{|i|}(1+v^2)}{(1-v^2)}$$

Therefore

$$\begin{aligned}
\sum_i \varphi_{i;\infty,\infty}(z,1) &= \frac{(1+v^2)}{(1-v^2)}\left(2\sum_{i\geq 1}v^i+1\right)\ \ for\ all\ i\\
&= \frac{(1+v^2)}{(1-v^2)}\frac{(1+v)}{(1-v)}\\
&= \frac{(1+v^2)}{(1-v)^2},
\end{aligned}$$

which matches with the results in Remarks 5 and 8. ■

Proposition 44 *The generating function of the s-th factorial moments of* $\varphi_{i,h,k}(z,u)$ *multiplied by the number of paths of a random walk with an open ending at* (n,i) *is given by*

$$M_{(s)}(z)=\frac{2^s s!v^{2s}\left(1+v^2\right)}{\left(1-v^2\right)^{s+1}\left(1-v\right)},$$

where $z=\frac{v}{1+v^2}$.

Proof. Now,

$$\begin{aligned}M_{(s,i)}(z) &= \frac{\partial^s}{\partial u^s}\varphi_{i;\infty,\infty}(z,w)\Big|_{w=1}\\ &= \frac{s!v^i\left(\frac{2v^2}{1+v^2}\right)^s}{\left(\frac{1-v^2}{1+v^2}\right)^{s+1}}\\ &= \frac{2^s s!v^{2s}\left(1+v^2\right)}{\left(1-v^2\right)^{s+1}}v^i.\end{aligned}$$

Summing up the above generating function for all i we get

$$\begin{aligned}M_{(s)}(z) &= \frac{2^s s!v^{2s}\left(1+v^2\right)}{\left(1-v^2\right)^{s+1}}\left(2\sum_{i\geq 1}v^i+1\right)\\ &= \frac{2^s s!v^{2s}\left(1+v^2\right)\left(1+v\right)}{\left(1-v^2\right)^{s+1}\left(1-v\right)}.\end{aligned}$$

■

Corollary 45 *The average number of returns to the origin by an open ended walk of length* n *is given by*

$$m_{(1)}(n)\asymp\sqrt{\frac{2n}{\pi}}\left(1+\frac{1}{2n}+\frac{(-1)^n}{4n}-\frac{(-1)^n}{8n^2}-\frac{3}{32n^2}+\frac{7(-1)^n}{128n^3}+\frac{1}{64n^3}+...\right).$$

Proof. Specializing on $s=1$ on Proposition 44 we obtain

$$M_{(1)}(z)=\frac{2v^2\left(1+v^2\right)\left(1+v\right)}{\left(1-v^2\right)^2\left(1-v\right)}.$$

Expanding around $z=\frac{1}{2}$ we obtain the contribution

$$M_{(1)}(z) \sim \frac{1}{\sqrt{2}(1-2z)^{\frac{3}{2}}} - \frac{1}{1-2z} + \frac{1}{4\sqrt{2}\sqrt{1-2z}} + \frac{3\sqrt{2}\sqrt{1-2z}}{64} + \frac{5\sqrt{2}(1-2z)^{\frac{3}{2}}}{256} + ...$$

Similarly, expansion around $z = -\frac{1}{2}$ we obtain the contribution

$$M_{(1)}(z) \sim \frac{1}{2\sqrt{2}\sqrt{1+2z}} - \frac{1}{2} + \frac{3\sqrt{2}\sqrt{1+2z}}{16} - \frac{(1+2z)}{4} + \frac{15\sqrt{2}(1+2z)^{\frac{3}{2}}}{128} - \frac{(1+2z)^2}{8} + ...$$

Now adding up the two contributions gives

$$\begin{aligned}
[z^n] M_{(1)}(z) &\asymp 2^n \left[\frac{(2n+2)}{2^{2n+1}\sqrt{2}} \binom{2n+1}{n} - 1 + \frac{1}{2^{2n+2}\sqrt{2}} \binom{2n}{n} - \frac{3\sqrt{2}}{2^{2n+5}(n-1)} \binom{2n-2}{n-2} \right. \\
&\quad - \frac{3(-1)^n\sqrt{2}}{2^{2n+3}(n-1)} \binom{2n-2}{n-2} + \frac{45(-1)^n\sqrt{2}}{2^{2n+5}(n-2)(n-3)} \binom{2n-4}{n-4} \\
&\quad \left. + \frac{(-1)^n}{2^{2n+1}\sqrt{2}} \binom{2n}{n} - \frac{15\sqrt{2}}{2^{2n+6}(n-2)(n-3)} \binom{2n-4}{n-4} + ... \right] \\
&\asymp 2^n \left[\sqrt{\frac{2n}{\pi}} \left(1 + \frac{1}{2n} + \frac{(-1)^n}{4n} - \frac{(-1)^n}{8n^2} - \frac{3}{32n^2} + \frac{7(-1)^n}{128n^3} + \frac{1}{64n^3} + ... \right) \right],
\end{aligned}$$

where the above asymptotic expansion was done using computer algebra. Since the total number of objects for an open ended walk is 2^n, it follows that

$$\begin{aligned}
m_{(1)}(n) &= \frac{[z^n] M_{(1)}(z)}{2^n} \\
&\asymp \sqrt{\frac{2n}{\pi}} \left(1 + \frac{1}{2n} + \frac{(-1)^n}{4n} - \frac{(-1)^n}{8n^2} - \frac{3}{32n^2} + \frac{7(-1)^n}{128n^3} + \frac{1}{64n^3} + ... \right) - 1.
\end{aligned}$$

■

Corollary 46 *The second moments on $\varphi_{i;h,k}(z)$ are given by*

$$m_{(2)}(n) \asymp n + \frac{(-1)^n}{2} + \frac{7}{2} - \sqrt{\frac{2n}{\pi}} \left(4 + \frac{2}{n} + \frac{(-1)^n}{n} - \frac{3}{8n^2} - \frac{(-1)^n}{2n^2} + \frac{1}{16n^3} + \frac{7(-1)^n}{32n^3} + ... \right).$$

Proof. Specialising on $s = 2$ in Proposition 44 we get

$$M_{(2)}(z) = \frac{8v^4(1+v^2)(1+v)}{(1-v^2)^3(1-v)}.$$

Now expansion around $z = \frac{1}{2}$ gives the contribution

$$\begin{aligned} M_{(2)}(z) \;\sim\; & \frac{1}{(1-2z)^2} - \frac{4}{\sqrt{2}(1-2z)^{\frac{3}{2}}} + \frac{5}{2(1-2z)} - \frac{1}{\sqrt{2}\sqrt{1-2z}} + \frac{1}{4} \\ & - \frac{3\sqrt{2}\sqrt{1-2z}}{16} + \frac{(1-2z)}{8} - \frac{5\sqrt{2}(1-2z)^{\frac{3}{2}}}{64} + \ldots \end{aligned}$$

And again, expansion around $z = -\frac{1}{2}$ gives

$$M_{(2)}(z) \sim \frac{1}{2(1+2z)} - \frac{2}{\sqrt{2}\sqrt{1+2z}} + \frac{3}{2} - \frac{3\sqrt{2}\sqrt{1+2z}}{4} + \frac{7(1+2z)}{8} - \frac{15\sqrt{2}(1+2z)^{\frac{3}{2}}}{32} + \ldots$$

Now adding up the above two contributions gives

$$\begin{aligned} [z^n] M_{(2)}(z) \;\asymp\; & 2^n \left[\binom{n+1}{n} - \frac{(2n+2)}{2^{2n-1}\sqrt{2}} \binom{2n+1}{n} + \frac{3\sqrt{2}}{2^{2n+3}(n-1)} \binom{2n-2}{n-2} \right. \\ & + \frac{(-1)^n}{2} + \frac{3(-1)^n\sqrt{2}}{2^{2n+1}(n-1)} \binom{2n-2}{n-2} + \frac{15\sqrt{2}}{2^{2n+4}(n-2)(n-3)} \binom{2n-4}{n-4} \\ & \left. + \frac{5}{2} - \frac{(-1)^n}{2^{2n-1}\sqrt{2}} \binom{2n}{n} - \frac{1}{2^{2n}\sqrt{2}} \binom{2n}{n} - \frac{45(-1)^n\sqrt{2}}{2^{2n+3}(n-2)(n-3)} \binom{2n-4}{n-4} + \ldots \right] \\ \asymp\; & 2^n \left[n + \frac{7}{2} + \frac{(-1)^n}{2} - \sqrt{\frac{2n}{\pi}} \left(4 + \frac{2}{n} + \frac{(-1)^n}{n} - \frac{3}{8n^2} - \frac{(-1)^n}{2n^2} + \frac{1}{16n^3} \right. \right. \\ & \left. \left. + \frac{7(-1)^n}{32n^3} \right) + \ldots \right], \end{aligned}$$

so that

$$\begin{aligned} m_{(2)}(n) \;=\; & \frac{[z^n] M_{(2)}(z)}{2^n} \\ \asymp\; & n + \frac{7}{2} + \frac{(-1)^n}{2} \\ & - \sqrt{\frac{2n}{\pi}} \left(4 + \frac{2}{n} + \frac{(-1)^n}{n} - \frac{3}{8n^2} - \frac{(-1)^n}{2n^2} + \frac{1}{16n^3} + \frac{7(-1)^n}{32n^3} + \ldots \right) \end{aligned}$$

■

3.8 Observation

And so from the above two cases (i.e. $s = 1$ and $s = 2$) we get Table 3.3 given below, which compares the expected, asymptotic, and results from [18] and [14] for $m_s(n)$ where:

$$m_{(s)}(n) = s! \sum_{j=0}^{s} \binom{s}{j} (-1)^{s-j} \binom{\frac{j}{2} + \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor},$$

and

$$m_{(1)}(n) = \begin{cases} 2^{-n}(n+1)\binom{n}{\frac{n}{2}} & , n \text{ even} \\ 2^{-n} n \binom{n-2}{\frac{n-1}{2}} & , n \text{ odd}, n \geq 3. \end{cases},$$

are the explicit formulas for [18] and [14] respectively.

n	$m_{(1)}(n)$				$m_{(2)}(n)$			$Variance$
	Expected	asympt.	[18]	[14]	Expected	asympt.	[18]	
3	0.5	0.5	0.5	1.5	0	0	0	0.25
20	2.7	2.7	2.7	3.7	9.2	9.2	9.2	4.61
131	8.15	8.15	8.15	9.15	97.4	97.4	97.4	39.13
350	13.96	13.96	13.96	14.96	294.16	294.16	294.16	113.24
600	18.57	18.57	18.57	19.57	525.73	525.73	525.73	199.46
998	24.23	24.23	24.23	25.23	901.1	901.1	901.1	338.24

Table 3.3: Average 0-level returns by an open ended unrestricted Dyck walks.

The numerical values for $m_{(1)}(n)$ are more by 1 in [14] as compared to the corresponding one's in [18] and our asymtotic results, and this is because return statistics in [14] include the last return at $(n, 0)$.

Chapter 4

Unrestricted Dyck random walk-Alternative approach.

4.1 Introduction

We consider an unrestricted Dyck random walk

$$S_m = \sum_{k=1}^{m} X_m, \text{ with } S_0 = 0 \text{ and } S_n = 0,$$

where X_m, $k = 1, 2, ...$ are independent and identically distributed random variables with $P\{X_m = 1\} = P\{X_m = -1\} = \frac{1}{2}$. This random walk starts at $(0, 0)$ and leads to (n, i) after n steps that reaches a maximum level above a nonzero height r as shown in Figure 3.1 drawn in Chapter 3. This alternative approach in the study of unrestricted Dyck random walk was chiefly motivated by some recent results such as [17] and [18]. In this Chapter we explicitly analyze some of the problems researched in Chapter 3 using the symbolic method instead of using the Matrix method and its associated Cramer's rule. However return statistics have already been researched in [18] using the symbolic method, and hence will not be repeated here.

4.2 Closed ending random walk.

Theorem 47 *The generating function of any random walk starting at* $(0,0)$ *and leading to* 0 *after* n *steps that reaches a maximum level which is above a nonzero height* r *is given by*

$$W_{r,closed}(z,u) = \frac{u}{1-u+u\sqrt{1-4z^2}}\left(\frac{1-\sqrt{1-4z^2}}{2z}\right)^{2r},$$

where u *is counting the number of returns to the origin and* $z = \frac{v}{1+v^2}$.

Proof. In order to get a suitable expression for the required generating function we decompose the family $\mathcal{W}$ of random walks in question according to their returns, either to the origin or height r. Hence this decomposition consists of three cases, namely, random walks between:

(a) (0,0) and the first visit to height r

(b) the first visit to height r and the last visit to height r.

(c) last visit to height r and $(n,0)$.

Denoting the family of random walks corresponding to case (b) by $\mathcal{W}_0$, and those to case (a) and (c) by $\mathcal{N}_r$ we get

$$\mathcal{W} = (\mathcal{N}_r \times \mathcal{W}_0 \times \mathcal{N}_r) \approx \mathcal{W}_0 \times \mathcal{N}_r^2.$$

We now focus on $\mathcal{W}_0$: Let $\mathcal{W}_{0,+}$ denote the subset of walks that are strictly positive between the first and the last point in $\mathcal{W}_0$, and $\mathcal{W}_{0,-}$ denote the corresponding set with a strictly negative sojourn along the r-axis. Noting that between any two consecutive returns to the r-level a walk is either positive ($\in \mathcal{W}_{0,+}$) or negative ($\in \mathcal{W}_{0,-}$) we get that $\mathcal{W}_0$ can be decomposed as a sequence of elements in $\mathcal{W}_{0,+} \cup \mathcal{W}_{0,-}$, which is symbolically given by

$$\mathcal{W}_0 = (\mathcal{W}_{0,+} \cup \mathcal{W}_{0,-})^*,$$

where the asterisk denotes the combinatorial construction of forming finite sequences of elements of the concerned set of objects. Altogether we have

$$\mathcal{W}_r = (\mathcal{W}_{0,+} \cup \mathcal{W}_{0,-})^* \times \mathcal{N}_r^2.$$

If we count the steps by the variable z, the returns by the variable u and observe that disjoint unions, Cartesian products and stars translate into sums, Cauchy products and geometric series, respectively, we get

$$W_{r,closed}\,(z,u) = \frac{u}{1-[W_{0,+}(z,u)+W_{0,-}(z,u)]} N_r^2(z,u).$$

However, $W_{0,+}(z,u) = W_{0,-}(z,u)$ since the corresponding situations are symmetrical. Furthermore, we have

$$W_{0,+}(z,u) = uW_{0,+}(z,1),$$

since we have exactly one return. The term $W_{0,+}(z,1)$ in the above equation serves as a counting function of the positive random walks, and it is well known to be $C(z^2)$, where

$$C(z) = \frac{1-\sqrt{1-4z}}{2} = \sum_{n\geq 1} \frac{1}{n}\binom{2n-2}{n-1} z^n,$$

is one version of the generating function of the Catalan numbers. Substituting

$$W_{0,-}(z,u) = W_{0,+}(z,u) = uW_{0,+}(z,1) = uC(z^2)$$

into $W_0\,(z,u)$ gives

$$\begin{aligned} W_0\,(z,u) &= \frac{u}{1-2uC(z^2)} \\ &= \frac{u}{1-u+u\sqrt{1-4z^2}}, \end{aligned}$$

so that

$$W_{r,closed}\,(z,u) = \frac{u}{1-u+u\sqrt{1-4z^2}} N_r^2(z,u).$$

Since we have assumed $r > 0$, then $N_r(z,u)$ has no return to the r-level so that $N_r(z,u) = N_r(z,1)$. In addition, each walk in N_r can be decomposed according to its last points on the

levels $1, 2, 3, ..., r$ $(r > 0)$ into an r -tuple of walks that start at level 0, are strictly positive and end at level 1. It then follows that

$$N_r(z,1) = z^r \left(\frac{C(z^2)}{z^2}\right)^r = \left(\frac{C(z^2)}{z}\right)^r = \left(\frac{1-\sqrt{1-4z^2}}{2z}\right)^r.$$

Combining the above formulae we get

$$\begin{aligned} W_{r,closed}(z,u) &= \frac{u}{1-u+u\sqrt{1-4z^2}} \left(\frac{C(z^2)}{z}\right)^{2r} \\ &= \frac{u}{1-u+u\sqrt{1-4z^2}} \left(\frac{1-\sqrt{1-4z^2}}{2z}\right)^{2r}. \end{aligned}$$

■

Proposition 48 *The generating function of the s-th factorial moments multiplied by the number of walks of length n in $W_{r,closed}(z,u)$ is given by*

$$M_r^{(s)}(z) = \frac{s!}{(2z)^{2r}} \left[\frac{\left(1-\sqrt{1-4z^2}\right)^{s+2r-1}}{\left(\sqrt{1-4z^2}\right)^{s+1}}\right].$$

Proof. Now,

$$\begin{aligned} M_r^{(s)}(z) &= \frac{\partial^s}{\partial u^s} W_{r,closed}(z,u) \mid_{u=1} \\ &= \frac{s!\left(1-\sqrt{1-4z^2}\right)^{s+2r}}{(2z)^{2r}\left(\sqrt{1-4z^2}\right)^{s+1}} + \frac{s!\left(1-\sqrt{1-4z^2}\right)^{s+2r-1}}{(2z)^{2r}\left(\sqrt{1-4z^2}\right)^{s}} \\ &= \frac{\left(1-\sqrt{1-4z^2}\right)^{s+2r}\left[1+\left(\sqrt{1-4z^2}\right)\left(1-\sqrt{1-4z^2}\right)^{-1}\right]}{\left(\sqrt{1-4z^2}\right)^{s+1}} \\ &= \frac{s!}{(2z)^{2r}}\left[\frac{\left(1-\sqrt{1-4z^2}\right)^{s+2r-1}}{\left(\sqrt{1-4z^2}\right)^{s+1}}\right]. \end{aligned}$$

■

In order to get asymptotic results we specialize our s-th factorial moments for $s = 1$ and $s = 2$. We will use the above generating function and *singularity analysis of generating functions*, as described in [8].

Corollary 49 *The average number of visits to level r by a closed ended random walk of lengths n is given by*

$$m_{(1)}(n) \asymp 2\left[\sqrt{2\pi n}\left(\frac{1}{4}+\frac{1}{16n}+\frac{1}{128n^2}+...\right)+\frac{r}{3n}+\frac{2r^3}{3n}-\frac{2r^5}{5n^2}-\frac{4r^3}{3n^2}-\frac{4r}{15n^2}-r\right]$$

for all n even.

Proof. Specializing on $s=1$ in Proposition 48 get

$$M_{(1)}(z)=\frac{1}{(2z)^{2r}}\left[\frac{\left(1-\sqrt{1-4z^2}\right)^{2r}}{\left(\sqrt{1-4z^2}\right)^2}\right].$$

Expanding $M_{(1)}(z)$ around $z=\frac{1}{2}$ gives the contribution

$$\begin{aligned} M_{(1)}(z) &\sim \frac{1}{2(1-2z)}+\frac{2r}{\sqrt{2}\sqrt{1-2z}}+\left(8r^2+1\right)+\frac{r\left(11+16r^2\right)\sqrt{2}\sqrt{1-2z}}{12} \\ &+\frac{\left(32r^4+64r^2+3\right)(1-2z)}{24}-\frac{r\left(349+1120r^2+256r^4\right)\sqrt{2}(1-2z)^{\frac{3}{2}}}{480}+... \end{aligned}$$

Expanding $M_{(1)}(z)$ around $z=-\frac{1}{2}$ gives the contribution

$$\begin{aligned} M_{(1)}(z) &\sim \frac{1}{2(1+2z)}+\frac{2r}{\sqrt{2}\sqrt{1+2z}}+\left(8r^2+1\right)+\frac{r\left(11+16r^2\right)\sqrt{2}\sqrt{1+2z}}{12} \\ &+\frac{\left(32r^4+64r^2+3\right)(1+2z)}{24}-\frac{r\left(349+1120r^2+256r^4\right)\sqrt{2}(1+2z)^{\frac{3}{2}}}{480}+... \end{aligned}$$

Both these asymptotic expansions are identically the same as those derived using the Matrix method in Chapter 3. Hence

$$m_{(1)}(n) \asymp 2\left[\sqrt{2\pi n}\left(\frac{1}{4}+\frac{1}{16n}+\frac{1}{128n^2}+...\right)+\frac{r}{3n}+\frac{2r^3}{3n}-\frac{2r^5}{5n^2}-\frac{4r^3}{3n^2}-\frac{4r}{15n^2}-r\right],$$

and $m_{(1)}(n) \asymp 0$ when n even or odd respectively, as derived in Corollary 34.

■

Corollary 50 *: The second moments $m_{(2)}(n)$ of $W_{r,closed}(z,u)$ are given by*

$$\begin{aligned} m_{(2)}(n) &\asymp 2\left[n+2r^2+2r+2-\frac{2r^4}{3n}-\frac{4r^3}{3n}-\frac{4r^2}{3n}-\frac{2r}{3n}\right. \\ &\left.-\sqrt{2\pi n}\left(\frac{1}{2}+r+\frac{r}{4n}+\frac{1}{8n}+\frac{r}{32n^2}+\frac{1}{64n^2}+...\right)\right] \end{aligned}$$

for all n even.

Proof. Specializing on $s = 2$ in Proposition 48 we get

$$M_{(2)}(z) = \frac{2}{(2z)^{2r}} \left[\frac{\left(1-\sqrt{1-4z^2}\right)^{2r+1}}{\left(\sqrt{1-4z^2}\right)^3} \right].$$

Expanding $M_{(2)}(z)$ around $z = \frac{1}{2}$ gives the contribution

$$M_{(2)}(z) \sim \frac{1}{\sqrt{2}(1-2z)^{\frac{3}{2}}} - \frac{(2r+1)}{(1-2z)} - \frac{(16r^2+16r+3)}{4\sqrt{2}\sqrt{1-2z}} + ...$$

On the other hand, expanding around $z = -\frac{1}{2}$ gives the contribution

$$M_{(2)}(z) \sim \frac{1}{\sqrt{2}(1+2z)^{\frac{3}{2}}} - \frac{(2r+1)}{(1+2z)} - \frac{(16r^2+16r+3)}{4\sqrt{2}\sqrt{1+2z}} + ...$$

Both these asymptotic expansions are identically the same as those derived using the Matrix method in Chapter 3. Hence

$$\begin{aligned} m_{(2)}(n) \asymp\ & 2\left[n + 2r^2 + 2r + 2 - \frac{2r^4}{3n} - \frac{4r^3}{3n} - \frac{4r^2}{3n} - \frac{2r}{3n} - \sqrt{2\pi n}\left(\frac{1}{2} + r + \frac{r}{4n} \right.\right. \\ & \left.\left. + \frac{1}{8n} + \frac{r}{32n^2} + \frac{1}{64n^2} + ... \right) \right], \end{aligned}$$

and $m_{(2)}(n) \asymp 0$ when n even or odd respectively, as derived in Corollary 35.

■

4.3 Open ending random walks.

In what follows we explicitly analyze the number of visits to the nonzero height r for an arbitrary unrestricted random walk starting at zero, that reaches a maximum level which is above a nonzero height r and has an open ending which is either above or below r as shown in Figure 3.1.

Theorem 51 *The s-factorial moments of any random walk with an open ending at* $(n, -)$ *are given by*

$$m_{(s)}(n) = \begin{cases} s! \sum_{k=0}^{s+r-1} \binom{s+r-1}{k} \binom{\frac{k-s-2}{2}}{\frac{n+r}{2}} (-1)^{k+\frac{n+r}{2}}, & \text{where } n+r \text{ even} \\ \\ s! \sum_{k=0}^{s+r-1} \binom{s+r-1}{k} \binom{\frac{k-s-2}{2}}{\frac{n+r-1}{2}} (-1)^{k+\frac{n+r-1}{2}}, & \text{where } n+r \text{ odd} \end{cases}.$$

Proof. We consider the set

$$\mathcal{W}_{r,open} = \bigcup_{i\in\mathbb{Z}} \mathcal{W}_i.$$

The corresponding bivariate generating function is given by

$$\begin{aligned}
W_{r,open}(z,u) &= N_r(z,u) \times W_0(z,u) \times \left[1 + 2\sum_{i\geq 1}\left(\frac{C(z^2)}{z}\right)^i\right] \\
&= \left(\frac{C(z^2)}{z}\right)^r \left[\frac{u}{1-u+u\sqrt{1-4z^2}}\right]\left[1 + 2\sum_{i\geq 1}\left(\frac{C(z^2)}{z}\right)^i\right] \\
&= \left(\frac{1-\sqrt{1-4z^2}}{2z}\right)^r \left[\frac{u}{1-u+u\sqrt{1-4z^2}}\right]\left[-1 + 2\sum_{i\geq 0}\left(\frac{1-\sqrt{1-4z^2}}{2z}\right)^i\right] \\
&= \left[\frac{u}{1-u(1-\sqrt{1-4z^2})}\left(\frac{1-\sqrt{1-4z^2}}{2z}\right)^r\right]\left[\frac{1+2z}{\sqrt{1-4z^2}}\right],
\end{aligned}$$

and so we have that

$$\begin{aligned}
M_{(s)}(z) &= \frac{\partial^s}{\partial u^s} W_{r,open}(z,u)\mid_{u=1} \\
&= \left[\frac{s!(1-\sqrt{1-4z^2})^s}{[\sqrt{1-4z^2}]^{s+1}} + \frac{s!(1-\sqrt{1-4z^2})^{s-1}}{[\sqrt{1-4z^2}]^s}\right]\left(\frac{1-\sqrt{1-4z^2}}{2z}\right)^r\left[\frac{1+2z}{\sqrt{1-4z^2}}\right] \\
&= \frac{s!}{(2z)^r}\frac{\left(1-\sqrt{1-4z^2}\right)^{s+r-1}}{[\sqrt{1-4z^2}]^{s+2}}\,(1+2z)\,,
\end{aligned}$$

is the generating function of the s-th factorial moments multiplied by the number of paths. It then follows that

$$\begin{aligned}
[z^n]M_{(s)}(z) &= [z^n]\frac{s!}{(2z)^r}\frac{\left(1-\sqrt{1-4z^2}\right)^{s+r-1}}{\left(\sqrt{1-4z^2}\right)^{s+2}}(1+2z) \\
&= [z^{n+r}]\frac{s!}{2^r}\sum_{k=0}^{s+r-1}\binom{s+r-1}{k}(-1)^k\left(\sqrt{1-4z^2}\right)^{k-s-2}(1+2z).
\end{aligned}$$

Now the even and the odd powers of z are easily distinguished by the following two cases:

Case 1: $n+r$ is even.

$$[z^n]M_{(s)}(z) = [z^{n+r}]\frac{s!}{2^r}\sum_{k=0}^{s+r-1}\binom{s+r-1}{k}(-1)^k\left(\sqrt{1-4z^2}\right)^{k-s-2}$$

$$\begin{aligned} &= [x^{\frac{n+r}{2}}]\frac{s!}{2^r}\sum_{k=0}^{s+r-1}\binom{s+r-1}{k}(-1)^k\left(\sqrt{1-4x}\right)^{k-s-2}, \text{ where } z^2=x \\ &= \frac{s!}{2^r}\sum_{k=0}^{s+r-1}\binom{s+r-1}{k}(-1)^k\binom{\frac{k-s-2}{2}}{\frac{n+r}{2}}(-4)^{\frac{n+r}{2}}. \end{aligned}$$

Dividing the above formula by 2^n we get the s-factorial moments:

$$\begin{aligned} m_{(s)}(n) &= 2^{-n}\left[\frac{s!}{2^r}\sum_{k=0}^{s+r-1}\binom{s+r-1}{k}(-1)^k\binom{\frac{k-s-2}{2}}{\frac{n+r}{2}}(-4)^{\frac{n+r}{2}}\right] \\ &= 2^{-n-r}s!\sum_{k=0}^{s+r-1}\binom{s+r-1}{k}(-1)^k\binom{\frac{k-s-2}{2}}{\frac{n+r}{2}}(-4)^{\frac{n+r}{2}} \\ &= s!\sum_{k=0}^{s+r-1}\binom{s+r-1}{k}\binom{\frac{k-s-2}{2}}{\frac{n+r}{2}}(-1)^{k+\frac{n+r}{2}}. \end{aligned}$$

Case 2: $n+r$ is odd.

$$\begin{aligned} [z^n]M_{(s)}(z) &= [z^{n+r}]\frac{s!}{2^r}\sum_{k=0}^{s+r-1}(-1)^k\left(\sqrt{1-4z^2}\right)^{k-s-2}(2z) \\ &= [z^{n+r-1}]\frac{s!}{2^{r-1}}\sum_{k=0}^{s+r-1}\binom{s+r-1}{k}(-1)^k\left(\sqrt{1-4z^2}\right)^{k-s-2} \\ &= [x^{\frac{n+r-1}{2}}]\frac{s!}{2^{r-1}}\sum_{k=0}^{s+r-1}\binom{s+r-1}{k}(-1)^k\left(\sqrt{1-4x}\right)^{k-s-2}, \text{ where } z^2=x \\ &= \frac{s!}{2^{r-1}}\sum_{k=0}^{s+r-1}\binom{s+r-1}{k}(-1)^k\binom{\frac{k-s-2}{2}}{\frac{n+r-1}{2}}(-4)^{\frac{n+r-1}{2}}. \end{aligned}$$

Dividing the above formula by 2^n we can similarly get the s-factorial moment:

$$\begin{aligned} m_{(s)}(n) &= 2^{-n-r-1}s!\sum_{k=0}^{s+r-1}\binom{s+r-1}{k}(-1)^k\binom{\frac{k-s-2}{2}}{\frac{n+r-1}{2}}(-4)^{\frac{n+r-1}{2}} \\ &= s!\sum_{k=0}^{s+r-1}\binom{s+r-1}{k}\binom{\frac{k-s-2}{2}}{\frac{n+r-1}{2}}(-1)^{k+\frac{n+r-1}{2}}. \end{aligned}$$

■

Corollary 52 *The average number of visits to level r by $W_{r,open}(z,u)$ is given by:*

$$m_1(n)=\begin{cases}\sum\limits_{k=0}^{r}\binom{r}{k}\binom{\frac{k-3}{2}}{\frac{n+r}{2}}(-1)^{\frac{n+r}{2}+k}, & \text{where } n+r \text{ is even}\\ \sum\limits_{k=0}^{r}\binom{r}{k}\binom{\frac{k-3}{2}}{\frac{n+r-1}{2}}(-1)^{\frac{n+r-1}{2}+k}, & \text{where } n+r \text{ is odd}\end{cases}.$$

Proof. By substituting $s=1$ into $M_{(s)}(z)$ in Theorem 51 the result follows at once. ■

Corollary 53 *The specialized 2 -factorial moments of $W_{r,open}(z,u)$ are given by:*

$$m_2(n)=\begin{cases}2\sum\limits_{k=0}^{r+1}\binom{r+1}{k}\binom{\frac{k-4}{2}}{\frac{n+r}{2}}(-1)^{\frac{n+r}{2}+k} & \text{where } n+r \text{ is even}\\ 2\sum\limits_{k=0}^{r+1}\binom{r+1}{k}\binom{\frac{k-4}{2}}{\frac{n+r-1}{2}}(-1)^{\frac{n+r-1}{2}+k} & \text{where } n+r \text{ is odd.}\end{cases}.$$

Proof. If we substitute $s=1$ into $M_{(s)}(z)$ in Theorem 51 we immediately get the results.

Remark 10 *From the above two Corollaries it follows that the* ***variance*** *of a random variable $m_s(n)$ is given by:*

$$\begin{aligned}&m_2(n)+m_1(n)-(m_1(n))^2\\ &=\begin{cases}2\sum\limits_{k=0}^{r+1}\binom{r+1}{k}\binom{\frac{k-4}{2}}{\frac{n+r}{2}}(-1)^{\frac{n+r}{2}+k}\\ \quad+\sum\limits_{k=0}^{r}\binom{r}{k}\binom{\frac{k-3}{2}}{\frac{n+r}{2}}(-1)^{\frac{n+r}{2}+k}\left(1-\sum\limits_{k=0}^{r}\binom{r}{k}\binom{\frac{k-3}{2}}{\frac{n+r}{2}}(-1)^{\frac{n+r}{2}+k}\right), & n+r \textit{ is even}\\ 2\sum\limits_{k=0}^{r+1}\binom{r+1}{k}\binom{\frac{k-4}{2}}{\frac{n+r-1}{2}}(-1)^{\frac{n+r-1}{2}+k}\\ \quad+\sum\limits_{k=0}^{r}\binom{r}{k}\binom{\frac{k-3}{2}}{\frac{n+r-1}{2}}(-1)^{\frac{n+r-1}{2}+k}\left(1-\sum\limits_{k=0}^{r}\binom{r}{k}\binom{\frac{k-3}{2}}{\frac{n+r-1}{2}}(-1)^{\frac{n+r-1}{2}+k}\right), & n+r \textit{ is odd}\end{cases}.\end{aligned}$$

Remark 11 ■

The explicit formula for $m_{(1)}(n)$ researched by [2] is given by

$$m_{(1)}(n)=\begin{cases}2^{-n}\left[\binom{n}{\frac{n-r}{n}}4\sum_{j\geq 1}j\binom{n}{\frac{n-r}{n}-j}\right] & ,(n-r) \text{ even}\\ 2^{-n}4\sum_{j\geq 1}(4j+2)\binom{n}{\frac{n-r-1}{n}-j} & ,(n-r) \text{ odd}\end{cases}.$$

The formulas for $m_{(1)}(n)$ both from here and [14] gives the same numerical values as the expected ones given in Table 3.2 of Chapter 3.

Chapter 5

Knödel random walks

5.1 Introduction

We recall that a Dyck path allowing horizontal steps only on the x-axis (or 0-level) is called **Knödel path.** In this Chapter we extend a study on Knödel random walks done by [21]. We will however avoid using Mellin transform method in order to make our results easily extendable to newer results. As usual we will be analyzing the average number of :

(i) visits to the nonnegative height r

(ii) and returns to the 0-level

by an arbitrary Knödel random walk which starts at the origin and has an ending at (n, i) as shown by the following diagram:

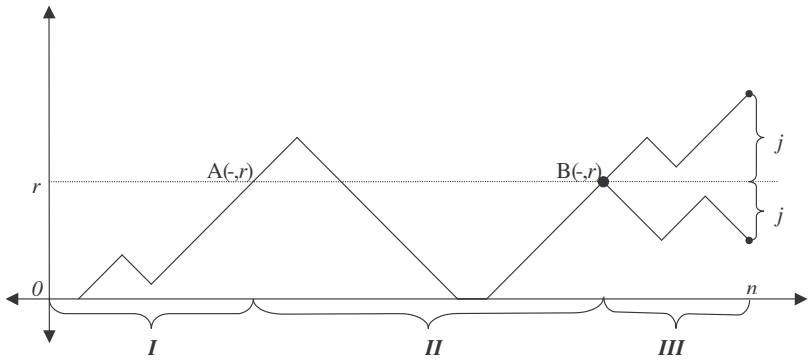


Figure 5.1: Open ended Knödel path of length n

path of length n for all i where $0 \leq i \leq h$ is given by:

$$\xi_{i;\,h}(z) = \sum_{n\geq 0} \left[\begin{array}{l} \text{Number of paths of length } n, \text{ starting at } 0, \text{bounded above} \\ \text{by the line } h, \text{ allows horizontal steps at level 0, and leading to } (n,i)\,. \end{array} \right] z^n.$$

For all i between $1 \leq i < h$ we have that

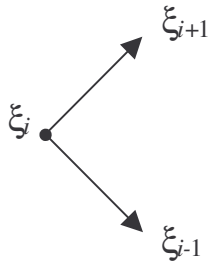


Figure 5.2

which gives the recursion

$$\xi_{i,h} = z\xi_{i-1,h} + z\xi_{i+1,h},$$

indicating that from level i we either go diagonally up to level $i+1$ or diagonally down to level $i-1$. The special system occurs at $i = 0$:

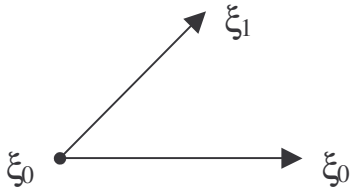


Figure 5.3

and $i = h$:

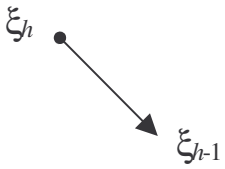


Figure 5.4

and have the recursions

$$\xi_0 = 1 + z\xi_1 \; + z\xi_0 \text{ and } \xi_h = z\xi_{h-1},$$

respectively. Thus the recursions, for $i = 0, 1, 2, ..., h$ of all these ξ_i's is best described by a single matrix equation

$$B_h \Phi_h = C_h,$$

where the coefficient matrix B_h is a square matrix with $h+1$ rows, Φ_h is the column matrix representing the $\xi_{i,h}(z)$'s and C_h is the column matrix consisting of h zeros and a 1 in the first entry:

$$\begin{bmatrix} 1-z & -z & & & & & & & \\ -z & 1 & -z & & & & & & \\ & & & & & 0 & & & \\ & & & -z & 1 & -z & & & \\ & & & & -z & 1 & -z & & \\ & & 0 & & & . & . & . & \\ & & & & & & . & . & \\ & & & & & & -z & 1 & -z \\ & & & & & & & -z & 1 \end{bmatrix} \begin{bmatrix} \xi_{0,h} \\ \xi_{1,h} \\ . \\ \\ \\ \xi_{i,h} \\ \xi_{i-1,h} \\ . \\ . \\ \xi_{h-1,h} \\ \xi_{h,h} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \\ \\ \\ 0 \\ 0 \\ \\ \\ 0 \\ 0 \end{bmatrix}. \qquad (5.1)$$

The $\xi_{i,h}$'s satisfy the system which has $h+1$ rows; this is only true for $h \geq 2$. The special values $h = 0$ and $h = 1$ are treated separately. As usual we calculate the $\xi_{i,h}$'s using Cramer's rule.

Lemma 54 *The determinant of the system of $\xi_{i,h}$'s with $h+1$ rows is given by*

$$\det B_h = \frac{(1-v)\left(1+v^{2h+3}\right)}{(1-v^2)\left(1+v^2\right)^{h+1}}.$$

Proof. By expanding the first row of our actual matrix in equation (5.1) we obtain the following recursion of order two:

$$\det B_h = (1-z)\det A_h - z^2 \det A_{h-1}, \ \det A_0 = 1, \text{ and } \det A_1 = 1-z,$$

where the quantities $\det A_h$ and $\det A_{h-1}$ are as derived in Lemma 6 of Chapter 2:

$$\det A_h = \frac{1}{(1-v^2)}\frac{1-v^{2h+2}}{(1+v^2)^h} \text{ and } \det A_{h-1} = \frac{1}{(1-v^2)}\frac{1-v^{2h}}{(1+v^2)^{h-1}}.$$

Using the substitution

$$z = \frac{v}{1+v^2},$$

we get

$$\begin{aligned}\det B_h &= \left(1-\frac{v}{1+v^2}\right)\frac{1}{(1-v^2)}\frac{1-v^{2h+2}}{(1+v^2)^h} - \frac{v^2}{(1+v^2)^2}\frac{1}{(1-v^2)}\frac{1-v^{2h}}{(1+v^2)^{h-1}} \\ &= \frac{(1-v)\left(1+v^{2h+3}\right)}{(1-v^2)(1+v^2)^{h+1}},\end{aligned}$$

as required. ■

5.2 Fundamental generating functions.

We are now ready to calculate all the $\xi_{i,h}$'s.

Theorem 55 *The generating function for all Knödel random walks which start at* $(0,0)$*, end at* $(n.i)$*, and bounded above by the line* $h>0$ *is given by*

$$\xi_{i,h}(z) = \frac{v^i(1+v^2)\left(1-v^{2h-2i+2}\right)}{(1-v)(1+v^{2h+3})}$$

where $z = \frac{v}{1+v^2}$.

Proof. Replacing the i-th column of the matrix B_h by the column matrix C_h we get a new matrix which we shall denote by $B_{i,h}$. Expanding by the i-th column of $B_{i,h}$ we obtain the determinant

$$\det B_{i,h} = z^i \det B_{h-i-1}.$$

Hence using Lemma 6 and Cramer's rule we get We apply Cramer's rule we get:

$$\begin{aligned}\xi_{i,h}(z) &= z^i\frac{\det B_{h-i-1}}{\det B_h} \\ &= \left(\frac{v}{(1+v^2)}\right)^i\left(\frac{1}{(1-v^2)}\frac{1-v^{2h-2i+2}}{(1+v^2)^{h-i}}\right)\times\left(\frac{(1-v^2)(1+v^2)^{h+1}}{(1-v)(1+v^{2h+3})}\right) \\ &= \frac{v^i(1+v^2)\left(1-v^{2h-2i+2}\right)}{(1-v)(1+v^{2h+3})}.\end{aligned}$$

■

Remark 12 *We observe that the overall number of random walks from* $(0,0)$ *to* (n,i) *is given by*

$$\sum_{i\geq 0}\xi_{i,\infty}(z)=\frac{1+v^2}{(1-v)^2}=\frac{1}{1-2z}=\sum_{n\geq 0}2^n z^n.$$

That is, the total number of objects for any Knödel random walk starting at $(0,0)$ *and ending at* (n,i) *is* 2^n*. This remark validates the correctness of the generating function* $\xi_{i,h}(z)$*.* ■

In order to get a suitable expression for the generating function for the entire random walk we decompose the family of these random walks into three parts (or three Lemmas) corresponding to three main parts of Figure 5.1.

Lemma 56 *The generating function of any Knödel path which starts at* $(0,0)$ *and is reaching level* r *for the first time is given by* $z\xi_{r-1,r-1}(z)$*.*

Proof. This random walk starts at the origin and ends at $\mathbf{A}(-,r)$ as shown in Figure 5.1. We further decompose this walk into two parts, namely:

(i) the random walk which starts at $(0,0)$, ends at $r-1$, and is consequently bounded above and below by $r-1$ and 0 respectively. Hence this generating function is given by $\xi_{r-1,r-1}(z)$.

(ii) the single z-step generating function which separate the above generating function in (i) with the r-level.

Combining these two generating functions we get $z\xi_{r-1,r-1}(z)$ as required. ■

Lemma 57 *The generating for any Knödel path which starts and ends on the* r*-level is given by*

$$F_r(z,u)=\frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right)},$$

where $z=\frac{v}{1+v^2}$ *and* u *counts the number of returns to the origin.*

Proof. This random walk starts at $\mathbf{A}(-,r)$ and ends at $\mathbf{B}(\ -,r)$ as shown in Figure 5.1. Using the same contstruction used in the proof of Lemma 9 we get

$$F_r(z) = \frac{u}{1 - u\left(F_{r,+}(z) + F_{r,-}(z)\right)}$$

The generating function for each walk $F_{r,+}(z)$ has already been derived using the formula for $\varphi_{i,\infty}(z)$ in Chapter 2 and is given by

$$F_{r,+}(z) = z^2 \varphi_{0,\infty}(z).$$

However, because of the speciality of the 0-level we cannot apply the $\varphi_{i,h}$ formula derived in Chapter 2 to get the generating function for $F_{r,-}(z)$ as shown in the diagram below

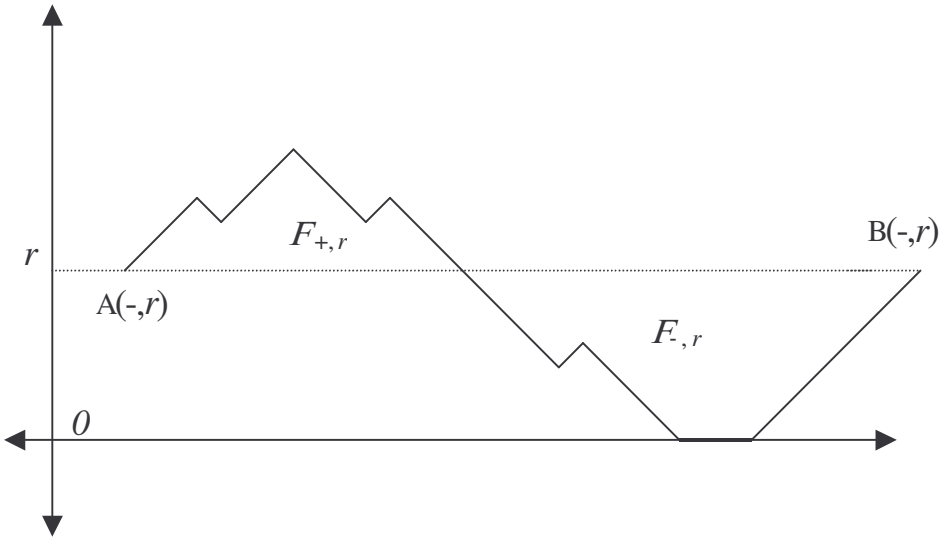


Figure 5.5

And so we will derive a new generating function for $F_{r,-}(z)$ by considering the following random walks living within the strip $0 \leq i \leq r-1$:

$$\theta_{i,h}(z) = \sum_{n \geq 0} \left[\begin{array}{l} \text{Random walks starting at } (0, r-1), \text{ living inside the strip} \\ 0 \leq i \leq r-1, \text{ allows horizontal steps at the 0-level and} \\ \text{leading to } (n,i) \end{array} \right] z^n.$$

The system of all these θ_i 's is described by the following matrix:

$$\begin{bmatrix} 1-z & -z & & & & & & & & \\ -z & 1 & -z & & & & & & & \\ & -z & 1 & -z & & & & & & \\ & & -z & 1 & -z & & & 0 & & \\ & & & -z & 1 & -z & & & & \\ & & & & -z & 1 & -z & & & \\ & & & & & . & . & . & & \\ & & 0 & & & & . & . & . & \\ & & & & & & & . & . & \\ & & & & & & & -z & 1 & -z \\ & & & & & & & & -z & 1 \end{bmatrix} \begin{bmatrix} \theta_{0,r-1} \\ \theta_{1,r-1} \\ . \\ . \\ . \\ \theta_{i,r-1} \\ \theta_{i-1,r-1} \\ . \\ . \\ . \\ \theta_{r-1,r-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ 0 \\ 0 \\ 1 \end{bmatrix}. \quad (5.2)$$

However the determinant of the above $r \times r$ matrix has already been calculated in the ξ_i situation in Section 5.1 and is given by

$$\det B_r = \frac{(1-v)\left(1+v^{2r+1}\right)}{\left(1-v^2\right)\left(1+v^2\right)^r}.$$

Using Cramer's rule we solve the system of linear equations of the $r \times r$ matrix in equation (5.2) to get the generating function

$$\begin{aligned} \theta_{i,r-1}(z) &= \frac{z^{r-1-i} c_i(z)}{c_r(z)} \\ &= \frac{v^{r-1-i}\left(1+v^{2i+1}\right)\left(1+v^2\right)}{\left(1+v^{2r+1}\right)}, \end{aligned}$$

where the substitutions

$$z = \frac{v}{1+v^2} \text{ and } \det B_r = \frac{(1-v)\left(1+v^{2i+1}\right)}{\left(1-v^2\right)\left(1+v^2\right)^i},$$

were made. Applying the formula for $\theta_{i,r-1}(z)$ in the decomposition of $F_{r,-}(z)$ we get

$$F_{r,-}(z) = z^2 \theta_{r-1,r-1}(z).$$

So using the substitutions

$$z = \frac{v}{1+v^2},\ z^2 \varphi_{0,\infty}(z) = \frac{v^2}{1+v^2},$$

and

$$z^2\theta_{r-1,r-1}(z) = \frac{v^2\left(1+v^{2r-1}\right)}{\left(1+v^2\right)\left(1+v^{2r+1}\right)},$$

we get

$$z^2\varphi_{0,\infty}(z) + z^2\theta_{r-1,r-1}(z) = \frac{v^2\left(2+v^{2r+1}+v^{2r-1}\right)}{\left(1+v^2\right)\left(1+v^{2r+1}\right)},$$

or

$$1 - z^2\varphi_{0,\infty}(z) - z^2\theta_{r-1,r-1}(z) = \frac{1-v^2}{\left(1+v^2\right)\left(1+v^{2r+1}\right)}.$$

It follows that

$$\begin{aligned} z^2\varphi_{0,\infty}(z) + z^2\theta_{r-1,r-1}(z) &= 1 + z^2\varphi_{0,\infty}(z) + z^2\theta_{r-1,r-1}(z) - 1 \\ &= 1 - \left[1 - z^2\varphi_{0,\infty}(z) - z^2\theta_{0,r-1}(z)\right] \\ &= 1 - \frac{1-v^2}{\left(1+v^2\right)\left(1+v^{2r+1}\right)}, \end{aligned}$$

so that

$$\begin{aligned} F_r(z) &= \frac{u}{1 - u\left(F_{r,+}(z) + F_{r,-}(z)\right)} \\ &= \frac{u}{1 - u\left(z^2\varphi_{0,\infty}(z) + z^2\theta_{r-1,r-1}(z)\right)} \\ &= 1 - \frac{1-v^2}{\left(1+v^2\right)\left(1+v^{2r+1}\right)}, \end{aligned}$$

■

Proposition 58 *The generating function of the s-factorial moments of $F_r(z,u)$ multiplied by the number of paths of length n in $F_r(z,u)$ is given by*

$$M_s(z) = s!\sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^k\left(1+v^2\right)^{s-k+1}\left(1+v^{2r+1}\right)^{s-k+1}\left(1-v^2\right)^{k-s-1}.$$

Proof. Now

$$\begin{aligned} M_s(z) &= \frac{\partial^s}{\partial u^s}F_r(z,u)|_{u=1} \\ &= \frac{s!\left[z^2\varphi_{0,\infty}(z) + z^2\theta_{r-1,r-1}(z)\right]^s}{\left[1 - z^2\varphi_{0,\infty}(z) - z^2\theta_{0,r-1}(z)\right]^{s+1}} + \frac{s!\left[z^2\varphi_{0,\infty}(z) + z^2\theta_{r-1,r-1}(z)\right]^{s-1}}{\left[1 - z^2\varphi_{0,\infty}(z) - z^2\theta_{0,r-1}(z)\right]^{s}} \end{aligned}$$

$$\begin{aligned} &= \frac{s!\left[z^2\varphi_{0,\infty}(z)+z^2\theta_{r-1,r-1}(z)\right]^{s-1}}{\left[1-z^2\varphi_{0,\infty}(z)-z^2\theta_{0,r-1}(z)\right]^{s+1}} \\ &= \frac{s!\left[1-\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right]^{s-1}}{\left[\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right]^{s+1}} \\ &= s!\sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^k\left(1+v^2\right)^{s-k+1}\left(1+v^{2r+1}\right)^{s-k+1}\left(1-v^2\right)^{k-s-1}. \end{aligned}$$

■

5.3 Random walks with closed ending.

Theorem 59 *The generating function for a Knödel random walk starting at* $(0,0)$ *and with a closed ending at* $(n,0)$ *is given by*

$$K_r(z,u)=\xi_{0,r-1}(z)+\frac{uv^{2r}(1+v)^2}{\left[1-u\left(1-\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right)\right](1+v^{2r+1})^2}$$

where $z=\frac{v}{1+v^2}$ *and* u *is counting the number of returns to the origin.*

Proof. A closed ending random walk for part III starts at B(-,r) and ends at $(n,0)$. If we treat $r-1$ as the 0-level then this walk is bounded above and below by 0 and $r-1$ respectively. It follows that the generating function for this closed ending walk is given by $z\theta_{0,r-1}(z)$. Thus the overall generating function is given by

$$K_{r,closed}(z,u)=\xi_{0,r-1}(z)+z\xi_{r-1,r-1}(z)\times F_r(z,u)\times z\theta_{0,r-1}(z),$$

where $\xi_{0,r-1}(z)$ is the generating function for all random walks not reaching r. However

$$z\xi_{r-1,r-1}(z)\times z\theta_{0,r-1}(z)=\frac{v^{2r}(1+v)^2}{(1+v^{2r+1})^2},$$

so that

$$\begin{aligned} K_{r,closed}(z,u) &= \xi_{0,r-1}(z)+\frac{uv^{2r}(1+v)^2}{1-u\left(z^2\varphi_{0,\infty}(z)+z^2\theta_{r-1,r-1}(z)\right)}\frac{v^{2r}(1+v)^2}{(1+v^{2r+1})^2} \\ &= \xi_{0,r-1}(z)+\frac{u}{\left[1-u\left(1-\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right)\right](1+v^{2r+1})^2}. \end{aligned}$$

■

Remark 13 *We test the correctness of* $K_{r,closed}(z,u)$ *by plugging in* $u=1$ *to get:*

$$\begin{aligned} K_{r,closed}(z,1) &= \frac{(1+v^2)(1-v^{2r})}{(1-v)(1+v^{2r+1})}+\frac{v^{2r}(1+v^2)(1+v)^2}{(1-v^2)(1+v^{2r+1})}, \\ &= \frac{1+v^2}{1-v}, \end{aligned}$$

so that

$$\begin{aligned} B(n) &= [z^n]\,K_{r,closed}(z,1) \\ &= \frac{1}{2\pi i}\oint \frac{(1-v^2)(1+v^2)^{n-1}}{v^{n+1}}K_r(z,1)\,dv \\ &= [v^n](1+v)\left(1+v^2\right)^n \\ &= \begin{cases} \binom{n}{\frac{n}{2}} & for\ n\ even \\ \binom{n}{\frac{n-1}{2}} & for\ n\ odd \end{cases} \\ &\sim \begin{cases} 2^n\sqrt{\frac{2}{\pi n}}\left(1-\frac{1}{4n}+\frac{1}{32n^2}+\frac{5}{128n^3}\right) & for\ n\ even \\ 2^n\sqrt{\frac{2}{\pi n}}\left(1-\frac{3}{4n}+\frac{25}{32n^2}-\frac{105}{128n^3}\right) & for\ n\ odd \end{cases}, \end{aligned}$$

which is the generating function for the total number of Knödel random walks with a ***closed ending*** *at* $(n,0)$. ■

Proposition 60 *The generating function of the s-factorial moments multiplied by the number of paths of length* n *in* $F(z,u)$ *is given by*

$$M_s(z)=v^{2r}s!\sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^k\left(1+v^2\right)^{s-k+1}\left(1+v^{2r+1}\right)^{s-k-1}\left(1-v^2\right)^{k-s-1}(1+v)^2.$$

Proof. Applying Theorem 59 we get

$$\begin{aligned} M_s(z) &= \frac{\partial^s}{\partial u^s}K_{r,closed}(z,u)\big|_{u=1} \\ &= \frac{v^{2r}(1+v)^2}{(1+v^{2r+1})^2}\frac{\partial^s}{\partial u^s}F_r(z,u)\big|_{u=1} \\ &= v^{2r}s!\sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^k\left(1+v^2\right)^{s-k+1}\left(1+v^{2r+1}\right)^{s-k-1}\left(1-v^2\right)^{k-s-1}(1+v)^2. \end{aligned}$$

■

We use the method of singularity analysis of generating function given by [8] to prove the following two Corollaries:

Corollary 61 *The average number of visits to the r-level by a Knödel random walk with a closed ending at* $(n,0)$ *is given by*

$$m_1(n) \asymp \begin{cases} \sqrt{2\pi n} - 4r - 2 + \frac{1}{4}\sqrt{\frac{2\pi}{n}} + \frac{16r^3+24r^2+26r-9(-1)^n+9}{6n}, & \text{for } n \text{ even} \\ \sqrt{2\pi n} - 2r - 2 + \frac{3}{4}\sqrt{\frac{2\pi}{n}} + \frac{16r^3+24r^2+14r-9(-1)^n+3}{6n}, & \text{for } n \text{ odd} \end{cases}.$$

Proof. specializing on $s=1$ into Proposition 60 we have that

$$M_1(z) = \frac{v^{2r}\left(1+v^2\right)^2(1+v)^2}{\left(1-v^2\right)^2}.$$

Now, using the substitution that $z = \frac{v}{(1+v^2)}$and upon expansion around $z = \frac{1}{2}$ we obtain

$$\begin{aligned} M_1(z) \sim\ & \frac{2}{1-2z} - \frac{4(2r+1)}{\sqrt{2}\sqrt{1-2z}} + \left(8r^2+8r+4\right) - \frac{\left(32r^3+48r^2+58r+21\right)\sqrt{2}\sqrt{1-2z}}{6} \\ & + \frac{2\left(8r^4+16r^3+34r^2+26r+9\right)(1-2z)}{3} \\ & + \frac{\left(512r^5+1280r^4+4160r^3+4960r^2+3938r+1185\right)\sqrt{2}\,(1-2z)^{\frac{3}{2}}}{240} + ... \end{aligned}$$

Again, by making the substitution that $z = \frac{v}{(1+v^2)}$and upon expansion around $z = -\frac{1}{2}$ we get

$$\begin{aligned} M_1(z) \sim\ & 1 + (2r+1)\sqrt{2}\sqrt{1+2z} - \frac{\left(8r^2+8r+5\right)(1-2z)}{2} \\ & - \frac{\left(32r^3+48r^2+70r+27\right)\sqrt{2}\,(1+2z)^{\frac{3}{2}}}{12}. \end{aligned}$$

Adding up the above two contributions gives

$$\begin{aligned} [z^n]\,M_1(z) \asymp\ & (-2)^n\left[2\binom{-1}{n} - \frac{4(2r+1)}{\sqrt{2}}\binom{-\frac{1}{2}}{n} - \frac{\left(32r^3+48r^2+58r+21\right)\sqrt{2}}{6}\binom{\frac{1}{2}}{n}\right. \\ & \left. + \frac{\left(512r^5+1280r^4+4160r^3+4960r^2+3938r+1185\right)\sqrt{2}}{240}\binom{\frac{3}{2}}{n}\right] \\ & + 2^n\left[+(2r+1)\sqrt{2}\binom{\frac{1}{2}}{n} - \frac{\left(32r^3+48r^2+70r+27\right)\sqrt{2}}{12}\binom{\frac{3}{2}}{n}\right] \\ =\ & 2^n\left[2 - \frac{(2r+1)}{2^{2n-2}\sqrt{2}}\binom{2n}{n} - \frac{\left(32r^3+48r^2+58r+21\right)\sqrt{2}}{3n2^{2n}}\binom{2n-2}{n-1}\right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{\left(512r^5+1280r^4+4160r^3+4960r^2+3938r+1185\right)\sqrt{2}}{5\left(n-2\right)\left(n-3\right)2^{2n+2}}\binom{2n-4}{n-4}\\
&-\frac{\left(-1\right)^n\left(32r^3+48r^2+70r+27\right)\sqrt{2}}{2^{2n}\left(n-2\right)\left(n-3\right)}\binom{2n-4}{n-4}\\
&-\frac{\left(2r+1\right)\left(-1\right)^n\sqrt{2}}{2^{2n-1}n}\binom{2n-2}{n-1}\Bigg]\\
\asymp\ & 2^n\left\{2+\sqrt{\frac{2}{\pi n}}\left(-4r-2+\frac{16r^3+24r^2+26r+6r\left(-1\right)^n+3\left(-1\right)^n+9}{6n}\right.\right.\\
&+\frac{1}{160n^2}(256r^5+640r^4+(1920-320(-1)^n)r^3+(2240-3(-1)^n)r^2\\
&+(1674-760(-1)^n)r+485-300(-1)^n)+...\Big\},
\end{aligned}
$$

where the above asymptotic results were done using computer algebra. From Remark 12 we have that $B(n)$ is the total number of objects for a closed ended walk so that

$$
\begin{aligned}
m_1\left(n\right) &= \frac{\left[z^n\right]M_1\left(z\right)}{B\left(n\right)}\\
&\asymp \begin{cases} \sqrt{2\pi n}-4r-2+\frac{1}{4}\sqrt{\frac{2\pi}{n}}+\frac{16r^3+24r^2+26r-9(-1)^n+9}{6n}, & \text{for } n \text{ even}\\ \sqrt{2\pi n}-2r-2+\frac{3}{4}\sqrt{\frac{2\pi}{n}}+\frac{16r^3+24r^2+14r-9(-1)^n+3}{6n}, & \text{for } n \text{ odd.}\end{cases}.
\end{aligned}
$$

■

Corollary 62 *The second moments are given by*

$$
m_2\left(n\right)\asymp\begin{cases} 8n-\sqrt{2\pi n}\left(12r+8+\frac{3r+2}{n}+\frac{3r+2}{8n^2}\right)+40r^2+48r+24 & \\ -\frac{\left(272r^4+576r^3+676r^2+408\bar{r}+99\right)}{6n}-\frac{\left(-1\right)^n\left(12r^2+8r+1\right)}{2n}, & \textit{for } n \textit{ even}\\ 8n-\sqrt{2\pi n}\left(12r+8+\frac{9r+6}{n}+\frac{21r+14}{8n^2}\right)+40r^2+48r+28 & \\ -\frac{\left(272r^4+576r^3+556r^2+264\bar{r}+90\right)}{6n}-\frac{\left(-1\right)^n\left(12r^2+8r+2\right)}{2n}, & \textit{for } n \textit{ odd}\end{cases}
$$

Proof. Specializing in $s=2$ onto Proposition 60 we get

$$
\begin{aligned}
M_2\left(z\right) &= 2\left[\frac{v^{2r}\left(1+v^2\right)^3\left(1+v^{2r+1}\right)\left(1+v\right)^2}{\left(1-v^2\right)^3}-M_1\left(z\right)\right]\\
&= 2\frac{\left(1+v^2\right)\left(v^{4r+1}+2v^{2r+2}+2v^{4r+3}+v^{2r+4}+v^{4r+5}\right)}{\left(1-v\right)^2}.
\end{aligned}
$$

By making the substitution that $z=\frac{v}{1+v^2}$ and upon expansion around $z=\frac{1}{2}$ we get

$$\begin{aligned} M_2\left(z\right) &\sim \frac{8}{\sqrt{2}\left(1-2z\right)^{\frac{3}{2}}}-\frac{8\left(3r+2\right)}{1-2z}+\frac{2\left(40r^2+48r+19\right)}{\sqrt{2}\sqrt{1-2z}}-\left(96r^3+160r^2+128r+40\right) \\ &+\frac{\left(2176r^4+4608r^3+5648r^2+3552r+933\right)\sqrt{2}\sqrt{1-2z}}{24}+... \end{aligned}$$

Again, by making the substitution that $z=\frac{v}{1+v^2}$ and upon expansion around $z=-\frac{1}{2}$ we get

$$\begin{aligned} M_2\left(z\right) &\sim -1+\left(10r+4\right)\sqrt{2}\sqrt{1+2z}-\frac{\left(104r^2+128r+41\right)\left(1+2z\right)}{2} \\ &+\frac{\sqrt{2}\left(128r^2+192r+139\right)\left(1+2z\right)^{\frac{3}{2}}}{6}+... \end{aligned}$$

Combining the above two contributions we obtain

$$\begin{aligned} [z^n]\,M_2\left(z\right) &\asymp \left(-2\right)^n\left[\frac{8}{\sqrt{2}}\binom{-\frac{3}{2}}{n}-8\left(3r+2\right)\binom{-1}{n}+\frac{2\left(40r^2+48r+19\right)}{\sqrt{2}}\binom{-\frac{1}{2}}{n}\right. \\ &\left.+\frac{\left(2176r^4+4608r^3+5648r^2+3552r+933\right)\sqrt{2}}{24}\binom{\frac{1}{2}}{n}\right] \\ &+2^n\left[\left(10r+4\right)\sqrt{2}\binom{\frac{1}{2}}{n}+\frac{\sqrt{2}\left(128r^2+192r+139\right)}{6}\binom{\frac{3}{2}}{n}\right] \\ &= 2^n\left[\frac{\left(n+1\right)}{2^{2n-3}\sqrt{2}}\binom{2n+1}{n+1}+\frac{\left(40r^2+48r+19\right)}{2^{2n-1}\sqrt{2}}\binom{2n}{n}-24r-16\right. \\ &-\frac{\left(2176r^4+4608r^3+5648r^2+3552r+933\right)\sqrt{2}}{3n2^{2n+2}}\binom{2n-2}{n-1} \\ &\left.+\frac{\left(-1\right)^n\left(10r+4\right)\sqrt{2}}{2^{2n-1}n}\binom{2n-2}{n-1}+\frac{\sqrt{2}\left(-1\right)^n\left(128r^2+192r+139\right)}{2^{2n-1}\left(n-2\right)\left(n-3\right)}\binom{2n-4}{n-4}+...\right] \\ &\asymp 2^n\left[-24r-16+\sqrt{\frac{2n}{\pi}}\left(8+\frac{\left(40r^2+48r+22\right)}{n}\right.\right. \\ &\left.\left.-\frac{\left(544r^4+1152r^3+1472r^2+960r+267\right)}{12n^2}-\frac{\left(-1\right)^n\left(12r^2+8r+1\right)}{2n^2}\right)\right] \\ &\asymp 2^n\left[-24r-16+\sqrt{\frac{2n}{\pi}}\left(8+\frac{\left(40r^2+48r+22\right)}{n}\right.\right. \\ &\left.\left.-\frac{\left(544r^4+1152r^3+1472r^2+960r+267\right)}{12n^2}-\frac{\left(-1\right)^n\left(5r+2\right)}{n^2}+...\right)\right], \end{aligned}$$

where the above asymptotic results were done using computer algebra. From Remark 12 we have that $B(n)$ is the total number of objects for a closed ended walk so that

$$\begin{aligned} m_2(n) &= \frac{[z^n]\, M_2(z)}{B(n)} \\ &\asymp \begin{array}{l} 8n - \sqrt{2\pi n}\left(12r + 8 + \frac{3r+2}{n}\right) + 40r^2 + 48r + 24, \quad \text{for } n \text{ even} \\ 8n - \sqrt{2\pi n}\left(12r + 8 + \frac{9r+6}{n}\right) + 40r^2 + 48r + 28, \quad \text{for } n \text{ odd} \end{array} . \end{aligned}$$

■

5.4 Observation

We construct Table 5.1 to give an enumerative comparison between the asymptotic and exact average number of visits to height r, together with their corresponding variances, by a simple Knödel path with a closed ending at $(n, 0)$. The exact results were found using computer algebra and the asymptotic results were calculated using the formula for $m_{(1)}(n)$ given in Corollary 61.

		$m_{(1)}(n)$			**Variances**		
n	**Results type**	$r=1$	$r=3$	$r=6$	$r=1$	$r=3$	$r=6$
222	expected	1.682	1.280	0.770	59.764	33.243	13.372
222	*asympt.*	1.682	1.282	0.812	59.759	33.056	8.554
367	expected	1.752	1.431	0.996	85.505	54.748	27.05
367	*asympt.*	1.752	1.432	1.010	85.506	54.713	25.61
593	expected	1.804	1.548	1.188	117.571	83.414	48.352
593	*asympt.*	1.804	1.549	1.192	117.572	83.405	47.900
712	expected	1.821	1.587	1.253	132.100	96.810	59.100
712	*asympt.*	1.821	1.587	1.255	132.100	96.796	58.775
844	expected	1.836	1.620	1.310	146.915	110.641	70.556
844	*asympt.*	1.836	1.620	1.311	146.914	110.641	70.339
1001	expected	1.849	1.650	1.363	163.146	125.954	83.591
1001	*asympt.*	1.849	1.650	1.364	163.147	125.953	83.468

Table 5.1: Average r-level returns by closed ended Knödel paths.

The above Table shows that the asymptotic and the exact results are almost the same for small values of r as n approaches infinity.

5.5 Random walks with open ending.

Lemma 63 *The generating function for a Knödel random walk starting at* $(0,0)$ *and ending at* $(n, r+j)$ *is given by*

$$L_{r+j,above}(z,u) = \frac{uv^{r+j}(1+v)}{\left[1-u\left(1-\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right)\right](1+v^{2r+1})},$$

where $z = \frac{v}{1+v^2}$ *and* u *counts the number of returns to the origin.*

Proof. Combining generating functions for parts I and II together with the generating function for part III ending at $(n, r+j)$ in Figure 5.1 we get the generating function for

the entire walk starting at $(0,0)$ and ending at $(n, r+j)$. The random walk for part III starts at $B(-,r)$ and ends at $(n, r+j)$. If we treat $r+1$ as the 0-level, we realize that the generating function for part III random walk is unbounded from above and below by $r-1$ and 0 respectively. Hence the generating function for part III is similar to the one derived in the Dyck paths situation in Lemma 6(a) given by $z\varphi_{j-1,\infty}(z)$. Therefore the overall generating function for this walk is given by

$$L_{r+j,above}(z,u) = z\xi_{r-1,r-1}(z) \times F_r(z,u) \times z\varphi_{j-1,\infty}(z).$$

However

$$F_r(z,u) = \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right)}, \quad z\xi_{r-1,r-1}(z) = \frac{v^r(1+v)}{(1+v^{2r+1})},$$

and

$$z\varphi_{j-1,\infty}(z) = v^j,$$

so that

$$L_{r+j,above}(z,u) = \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right)} \frac{v^{r+j}(1+v)}{(1+v^{2r+1})}.$$

■

Lemma 64 *The generating function for a Knödel random walk starting at* $(0,0)$ *and ending at* $(n, r-j)$ *is given by*

$$L_{r-j,below}(z,u) = \xi_{r-j,r-1}(z) + \frac{uv^{2r-j}(1+v)(1+v^{2j+1})}{\left[1-u\left(1-\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right)\right](1+v^{2r+1})^2},$$

where $z = \frac{v}{1+v^2}$ *and* u *counts the number of returns to the origin.*

Proof. Combining generating functions for parts I and II together with the generating function for part III ending at $(n, r+j)$ in Figure 5.1 we get the generating function for the entire walk starting at $(0,0)$ and ending at $(n, r-j)$. The random walk for part III starts at $B(-,r)$ and ends at $(n, r-j)$. If we treat $r-1$ as the 0-level we realize that part III random is bounded above and below by $r-1$ and 0 respectively. Because of the speciality of

the $(r-1)$-level discussed in Lemma 51 we have that the generating function for part III of Figure 5.1 is unsymmetrical to the one ending at $(n, r+j)$ proved in the above Lemma 63; and so we will denote it by $z\theta_{j,r-1}(z)$ and not $z\varphi_{j-1,\infty}(z)$. Therefore the overall generating function for this case is given by

$$L_{r-j,below}(z,u) = \xi_{j,r-1}(z) + z\xi_{r-1,r-1}(z) \times F_r(z,u) \times z\theta_{j,r-1}(z),$$

where $\xi_{j,r-1}(z)$ is the generating function for all random walks starting at $(0,0)$ and not reaching r. However

$$F_r(z,u) = \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right)}, \quad z\xi_{r-1,r-1}(z) = \frac{v^r(1+v)}{(1+v^{2r+1})},$$

and

$$z\theta_{j,r-1}(z) = \frac{v^{r-j}(1+v^{2j+1})}{(1+v^{2r+1})},$$

so that

$$L_{r-j,below}(z,u) = \xi_{r-j,r-1}(z) + \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right)} \frac{v^{2r-j}(1+v^{2j+1})(1+v)}{(1+v^{2r+1})^2}.$$

■

Theorem 65 *The generating function for a Knödel random walk starting at the origin with an open ending at $(n,-)$ is given by*

$$L_{r,open}(z,u) = \sum_{k=0}^{r-1} \xi_{k,r-1}(z) + \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right)} \frac{v^r(1+v)^2}{(1-v)(1+v^{2r+1})^2},$$

where $z = \frac{v}{1+v^2}$ and u counts the number of returns to the origin.

Proof. If we sum up the two unsymmetrical generating functions derived in Lemmas 63, and 64, for all $j \geq 0$ together with the generating function

$$z\theta_{r-1,r-1}(z) \times F_r(z,u),$$

we get a bivariate generating function for the entire random walk with an open ending at $(n,-)$:

$$L_{r,open}(z,u)=\sum_{k=0}^{r-1}\xi_{k,r-1}(z)+z\xi_{r-1,r-1}(z)\times F_r(z,u)\left(z\sum_{j\geq 1}\varphi_{j-1,\infty}(z)+z\sum_{j=0}^{r-1}\theta_{j,r-1}(z)+1\right),$$

Since

$$z\sum_{j\geq 1}\varphi_{j-1,\infty}(z)=\frac{v}{1-v}\ ,\quad z\sum_{j=0}^{r-1}\theta_{j,r-1}(z)=\frac{v\left(1+v^r\right)\left(1-v^r\right)}{\left(1+v^{2r+1}\right)(1-v)},$$

we have that

$$\left(z\sum_{j\geq 1}\varphi_{j-1,\infty}(z)+z\sum_{j\geq 1}\theta_{j,r-1}(z)+1\right)=\frac{(1+v)}{(1-v)\left(1+v^{2r+1}\right)}.$$

It follows that

$$\begin{aligned}L_{r,open}(z,u) &= \sum_{k=0}^{r-1}\xi_{k,r-1}(z)+\frac{u}{1-u\left(z^2\varphi_{0,\infty}(z)+z^2\theta_{r-1,r-1}(z)\right)}\frac{v^r(1+v)^2}{(1-v)\left(1+v^{2r+1}\right)^2}\\ &= \sum_{k=0}^{r-1}\xi_{k,r-1}(z)+\frac{u}{1-u\left(1-\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right)}\frac{v^r(1+v)^2}{(1-v)\left(1+v^{2r+1}\right)^2},\end{aligned}$$

as required. ■

Remark 14 *We test the correctness of the generating function $L_{r,open}(z,u)$:*

The generating function for all random walks starting at $(0,0)$ and not reaching r is given by

$$\sum_{k=0}^{r-1}\xi_{k,r-1}(z)=\frac{\left(1+v^2\right)\left(1-v^r\right)\left(1-v^{r+1}\right)}{(1-v)^2\left(1+v^{2r+1}\right)},$$

so that

$$\begin{aligned}L_{r,open}(z,1) &= \sum_{k=0}^{r-1}\xi_{k,r-1}(z)+\frac{\left(1+v^2\right)\left(1+v^{2r+1}\right)}{1-v^2}\frac{v^r(1+v)^2}{(1-v)\left(1+v^{2r+1}\right)^2}\\ &= \frac{\left(1+v^2\right)\left(1-v^r\right)\left(1-v^{r+1}\right)}{(1-v)^2\left(1+v^{2r+1}\right)}+\frac{v^r\left(1+v^2\right)(1+v)}{(1-v)^2\left(1+v^{2r+1}\right)}\\ &= \frac{\left(1+v^2\right)}{(1-v)^2}\\ &= \frac{1}{1-2z}\\ &= \sum_{n\geq 0}2^n z^n,\end{aligned}$$

which is generating function the total number of objects for any Knödel random walk starting at $(0,0)$ and ending at $(n,-)$. ■

Proposition 66 *The generating function of the s-factorial moments of $L_{r,open}(z,u)$ multiplied by the number of paths in this case is given by*

$$\begin{aligned} M_s(z) &= v^r s! \sum_{k=0}^{s-1} \binom{s-1}{k} (-1)^k \left(1-v^2\right)^{k-s-1} \left(1+v^2\right)^{s-k+1} (1-v)^{-1} \\ &\quad \times \left(1+v^{2r+1}\right)^{s-k-1} (1+v)^2. \end{aligned}$$

Proof. Now

$$\begin{aligned} M_s(z) &= \frac{\partial^s}{\partial u^s} L_{r,open}(z,u)\Big|_{u=1} \\ &= \frac{s!\left[1-\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right]^{s-1}}{\left[\frac{1-v^2}{(1+v^2)(1+v^{2r+1})}\right]^{s+1}} \frac{v^r(1+v)^2}{(1-v)\left(1+v^{2r+1}\right)^2} \\ &= v^r s! \sum_{k=0}^{s-1} \binom{s-1}{k} (-1)^k \left(1-v^2\right)^{k-s-1} \left(1+v^2\right)^{s-k+1} (1-v)^{-1} \\ &\quad \times \left(1+v^{2r+1}\right)^{s-k-1} (1+v)^2. \end{aligned}$$

■

Like in the previous Section we apply the method of singularity analysis for generating function given by [8] to get formular for the average number of visits to level r by a nonnegative height r.

Corollary 67 *The average number of visits to level r by a Knödel random walk with an open ending at $(n,-)$ is given by*

$$\begin{aligned} m_1(n) \asymp & \left[\sqrt{\frac{2n}{\pi}}\left(2+\frac{2r^2+2r+3}{2n}\right.\right. \\ & \left.\left.-\frac{\left(4r^4+8r^3+44r^2+40r+12(-1)^n r+6(-1)^n+27\right)}{48n^2}\right)-2r-1\right]. \end{aligned}$$

Proof. Specializing on $s=1$ onto Proposition 66 gives

$$M_1(z) = \frac{v^r\left(1+v^2\right)^2(1+v)^2}{\left(1-v^2\right)^2(1-v)}.$$

By making the substitution that $z=\frac{v}{1+v^2}$ and upon expansion around $z=\frac{1}{2}$ we obtain

$$\begin{aligned} M_1(z) &\sim \frac{2}{\sqrt{2}(1-2z)^{\frac{3}{2}}}-\frac{2r+1}{(1-2z)}+\frac{4r^2+4r+3}{2\sqrt{2}\sqrt{1-2z}}-\frac{(2r^3+3r^2+7r+3)}{3} \\ &\quad +\frac{\sqrt{2}(16r^4+32r^3+152r^2+136r+69)\sqrt{1-2z}}{96}+... \end{aligned}$$

Again, by making the substitution that $z=\frac{v}{1+v^2}$ and upon expansion around $z=-\frac{1}{2}$ we get

$$\begin{aligned} M_1(z) &\sim \frac{(-1)^2}{2}-\frac{(-1)^r(2r+1)\sqrt{2}\sqrt{1+2z}}{4}+\frac{(-1)^r(2r^2+2r+3)(1+2z)}{4} \\ &\quad -\frac{\sqrt{2}(-1)^r 2(1+2z)^{\frac{3}{2}}}{48}+... \end{aligned}$$

Adding up the above two contributions we get

$$\begin{aligned} [z^n]M_1(z) &\asymp (-2)^n\left[\frac{2}{\sqrt{2}}\binom{-\frac{3}{2}}{n}-(2r+1)\binom{-1}{n}+\frac{4r^2+4r+3}{2\sqrt{2}}\binom{-\frac{1}{2}}{n}\right. \\ &\quad \left.+\frac{\sqrt{2}(16r^4+32r^3+152r^2+136r+69)}{96}\binom{\frac{1}{2}}{n}+...\right] \\ &\quad +2^n\left[\frac{(-1)^r(2r+1)\sqrt{2}}{4}\binom{\frac{1}{2}}{n}-\frac{\sqrt{2}(-1)^r(8r^3+12r^2+46r+21)}{48}\binom{\frac{3}{2}}{n}+..\right] \\ &= 2^n\left[\frac{(2n+2)}{2^{2n}\sqrt{2}}\binom{2n+1}{n}-\frac{\sqrt{2}(16r^4+32r^3+152r^2+136r+69)}{96n2^{2n-1}}\binom{2n-2}{n-1}\right. \\ &\quad +\frac{(4r^2+4r+3)}{2^{2n+1}\sqrt{2}}\binom{2n}{n}-(2r+1)-\frac{(-1)^{n+r}(2r+1)\sqrt{2}}{2^{2n+1}n}\binom{2n-2}{n-1} \\ &\quad \left.+\frac{\sqrt{2}(-1)^{n+r}(8r^3+12r^2+46r+21)}{2^{2n+2}(n-2)(n-3)}\binom{2n-4}{n-4}+...\right] \\ &\asymp 2^n\left[\sqrt{\frac{2n}{\pi}}\left(2+\frac{2r^2+2r+3}{2n}\right.\right. \\ &\quad \left.\left.-\frac{(4r^4+8r^3+44r^2+40r+12(-1)^n r+6(-1)^n+27)}{48n^2}\right)-2r-1\right]. \end{aligned}$$

where the above asymptotic results were done using computer algebra. From Remark 10 we have that 2^n is the total number of objects for a closed ended walk so that

$$m_{(1)}(n) = \frac{[z^n]M_1(z)}{2^n}$$

$$\asymp \left[\sqrt{\frac{2n}{\pi}}\left(2+\frac{2r^2+2r+3}{2n}\right.\right.$$
$$\left.\left.-\frac{\left(4r^4+8r^3+44r^2+40r+12\left(-1\right)^n r+6\left(-1\right)^n+27\right)}{48n^2}+...\right)-2r-1\right],$$

as required. ■

Corollary 68 *The second moments on $L_{r,open}(z,u)$ are given by*

$$m_{(2)}(n) \asymp 4n+\sqrt{\frac{2n}{\pi}}\left[-16r-12-\frac{\left(56r^3+90r^2+94r+39\right)}{3n}+...\right]+20r^2+24r+14.$$

Proof. Specializing on $s=2$ into Proposition 66 we get

$$\begin{aligned} M_2(z) &= 2\left[\frac{v^r\left(1+v^2\right)^3\left(1+v^{2r+1}\right)\left(1+v\right)^2}{\left(1-v^2\right)^3\left(1-v\right)}-M_1(z)\right] \\ &= \frac{2\left(v^{3r+1}+2v^{r+2}+v^{3r+3}\right)\left(1+v^2\right)^2}{\left(1-v\right)^3\left(1-v^2\right)}. \end{aligned}$$

By making the substitution that $z=\frac{v}{1+v^2}$ and upon expansion around $z=-\frac{1}{2}$ we get

$$\begin{aligned} M_2(z) \sim\ & \frac{4}{\left(1-2z\right)^2}-\frac{4\left(4r+3\right)}{\sqrt{2}\left(1-2z\right)^{\frac{3}{2}}}+\frac{2\left(10r^2+12r+5\right)}{1-2z}-\frac{\left(112r^3+180r^2152r+51\right)}{3\sqrt{2}\sqrt{1-2z}} \\ &+\left(82r^4+168r^3+224r^2+150r+42\right) \\ &+\frac{\left(3904r^5+9840r^4+18880r^3+19080r^2+10876r+2715\right)\sqrt{2}\sqrt{1-2z}}{240} \\ &+\left(140032r^7+490560r^6+16665664r^5+2951760r^4+3735088r^3+2949660r^2\right. \\ &\left.+1364016r+285075\right)\sqrt{2}\frac{\left(1-2z\right)^{\frac{3}{2}}}{20160}+... \end{aligned}$$

Upon expansion around $z=\frac{1}{2}$ we get the contribution

$$\begin{aligned} M_2(z) \sim\ & \frac{\sqrt{2}\left(-1\right)^r\left(8r^2+2r+1\right)\sqrt{1+2z}}{4} \\ &+\frac{\sqrt{2}\left(-1\right)^r\left(160r^4+104r^3+164r^2+46r+9\right)\left(1+2z\right)^{\frac{3}{2}}}{48}+... \end{aligned}$$

Therefore

$$[z^n]M_2(z) \asymp \left(-2\right)^n\left[4\binom{-2}{n}-\frac{4\left(4r+3\right)}{\sqrt{2}}\binom{-\frac{3}{2}}{n}+\left(20r^2+24r+10\right)\binom{-1}{n}\right.$$

$$
\begin{aligned}
&-\frac{(112r^3+180r^2152r+51)}{3\sqrt{2}}\binom{-\frac{1}{2}}{n}\\
&+\frac{(3904r^5+9840r^4+18880r^3+19080r^2+10876r+2715)\sqrt{2}}{240}\binom{\frac{1}{2}}{n}\\
&+\left(140032r^7+490560r^6+16665664r^5+2951760r^4+3735088r^3+2949660r^2\right.\\
&\left.+1364016r+285075\frac{\sqrt{2}}{20160}\binom{\frac{3}{2}}{n}+...\right]+2^n\sqrt{2}\,(-1)^r\left[\frac{(1+2r+8r^2)}{4}\binom{\frac{1}{2}}{n}\right.\\
&\left.+\frac{(160r^4+104r^3+164r^2+46r+9)}{48}\binom{\frac{3}{2}}{n}\right]\\
=\;& 2^n\left[4\binom{n+1}{n}-\frac{(4r+3)(n+1)}{2^{2n-2}\sqrt{2}}\binom{2n+1}{n+1}\right.\\
&+2\left(10r^2+12r+5\right)-\frac{(112r^3+180r^2152r+51)}{2^{2n}3\sqrt{2}}\binom{2n}{n}\\
&\frac{(3904r^5+9840r^4+18880r^3+19080r^2+10876r+2715)\sqrt{2}}{15n2^{2n+3}}\binom{2n-2}{n-1}\\
&+\left(140032r^7+490560r^6+16665664r^5+2951760r^4+3735088r^3+2949660r^2\right.\\
&\left.+1364016r+285075\right)\frac{\sqrt{2}}{2^{2n+4}(n-2)(n-3)105}\binom{2n-4}{n-4}\\
&\frac{(-1)^{n+r}(1+2r+8r^2)}{2^{2n}n}\binom{2n-2}{n-1}\\
&\left.+\frac{(-1)^{n+r}(160r^4+104r^3+164r^2+46r+9)}{2^{2n+2}(n-2)(n-3)}\binom{2n-4}{n-4}\right]\\
\asymp\;& 2^n\left\{4n+20r^2+24r+14-\sqrt{\frac{2n}{\pi}}\left[\left(16r+12-\frac{(56r^3+90r^2+94r+39)}{3n}\right)\right.\right.\\
&\left.\left.-\frac{\left(976r^5+2460r^4+5000r^3+60r^2\left(87+2(-1)^{n+r}\right)+3204r+15(-1)^{n+r}(2r+1)+8\right.}{120n^2}\right.\right.\\
&\left.\left.+...\right]\right\}
\end{aligned}
$$

where the above asymptotic results were done using computer algebra. From Remark 14 we have that 2^n is the total number of objects for a closed ended walk so that so that

$$
\begin{aligned}
m_{(2)}(n) &= \frac{[z^n]\,M_2(z)}{2^n}\\
&\asymp \left\{4n+20r^2+24r+14-\sqrt{\frac{2n}{\pi}}\left[\left(16r+12-\frac{(56r^3+90r^2+94r+39)}{3n}\right)\right.\right.\\
&\left.\left.-\frac{\left(976r^5+2460r^4+5000r^3+60r^2\left(87+2(-1)^{n+r}\right)+3204r+15(-1)^{n+r}(2r+1)+885\right)}{120n^2}\right.\right.
\end{aligned}
$$

$+...]\}$

■

5.6 Observation

The following Table gives an enumerative comparison between the asymptotic and exact average number of visits to height r, together with their corresponding variances, by a simple Knödel path with an open ending at $(n, -)$.

		$m_{(1)}(n)$			**Variances**		
n	**Results type**	$r=1$	$r=3$	$r=6$	$r=1$	$r=3$	$r=6$
222	expected	20.963	17.494	13.063	189.696	106.197	61.431
222	*asympt.*	*20.963*	*17.494*	*13.061*	*189.661*	*104.993*	*40.019*
367	expected	27.716	24.130	19.362	353.690	222.629	128.960
367	*asympt.*	*27.716*	*24.130*	*19.362*	*353.673*	*222.048*	*118.393*
593	expected	35.974	32.301	27.275	624.766	432.603	266.160
593	*asympt.*	*35.974*	*32.301*	*27.275*	*624.758*	*432.317*	*260.864*
712	expected	39.685	35.983	30.874	771.879	551.657	349.540
712	*asympt.*	*39.685*	*35.983*	*30.873*	*771.885*	*551.441*	*345.492*
844	expected	43.456	39.730	34.549	937.463	688.453	448.761
844	*asympt.*	*43.456*	*39.730*	*34.549*	*937.458*	*688.285*	*445.606*
1001	expected	47.576	43.928	38.581	1136.919	856.231	574.270
1001	*asympt.*	*47.576*	*43.928*	*38.581*	*1136.917*	*856.099*	*571.806*

Table 5.1: Average r-level returns by open ended Knödel paths.

The above Table confirms that

$$\frac{expected\ (r)}{asympt\ (r)} \to 1$$

as n grows larger.

5.7 Return statistics

In what follows we analyze the average number of returns to the origin for an arbitrary Knödel random walk starting at $(0,0)$ with an open ending at $(n,-)$. The recursion for the special system at 0-level, unlike in the case of visits to the r-level, will be given by

$$\xi_0 = 1 + zw\xi_1 \ + zw\xi_0,$$

where the variable w is counting the number of returns to the origin. Therefore the system of all the ξ_i 's given in Section 5.1 will now be given by

$$\begin{bmatrix} 1-zw & -zw & & & & & & & & \\ -z & 1 & -z & & & & & & & \\ & -z & 1 & -z & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & -z & 1 & -z & & & \\ & & & & & -z & 1 & -z & & \\ & & & & & & . & . & . & \\ & & & & & & & . & . & . \\ & & & & & & & & -z & 1 & -z \\ & & & & & & & & & -z & 1 \end{bmatrix} \begin{bmatrix} \xi_{0,h} \\ \xi_{1,h} \\ . \\ . \\ . \\ \xi_{i,h} \\ \xi_{i-1,h} \\ . \\ . \\ . \\ \xi_{h,h} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{bmatrix}. \quad (5.3)$$

As before we calculate the $\xi_{i,h}$'s using Cramer's rule by first finding the determinant of this system with $h+1$ rows.

Lemma 69 *The determinant of the system of the $\xi_{i,h}$'s with $h+1$ rows is given by*

$$\det D_h = \frac{(1+v^2-wv)\left(1-v^{2h+2}\right) - wv^2\left(1-v^{2h}\right)}{(1+v^2)^{h+1}(1-v^2)},$$

where w counts the number of returns to the origin.

Proof. The determinant of the system in equation (5.3) is given by

$$\det D_h = \begin{vmatrix} 1-zu & -zu & & & & & & & \\ -z & 1 & -z & & & & & & \\ & -z & 1 & -z & & & 0 & & \\ & & . & . & . & & & & \\ & & & . & . & . & & & \\ & & & & . & . & . & & \\ & & 0 & & & . & . & . & \\ & & & & & & . & . & . \\ & & & & & & & -z & 1 & -z \\ & & & & & & & & -z & 1 \end{vmatrix}$$

$$\begin{aligned} &= (1-zw)\det A_h - z^2 w \det A_{h-1} \\ &= \left(1-\frac{vw}{1+v^2}\right)\left(\frac{1}{(1-v^2)}\frac{1-v^{2h+2}}{(1+v^2)^h}\right) - w\left(\frac{v}{1+v^2}\right)^2\left(\frac{1}{(1-v^2)}\frac{1-v^{2h}}{(1+v^2)^{h-1}}\right) \\ &= \frac{(1+v^2-wv)\left(1-v^{2h+2}\right)-wv^2\left(1-v^{2h}\right)}{(1+v^2)^{h+1}(1-v^2)}, \end{aligned}$$

where the substitutions for $\det A_h$ and $\det A_{h-1}$ are as in Lemma 6 of Chapter 2. ■

The $\xi_{i,h}$'s calculated in Section 5.1 will now change slightly because of the incorporation of the return counting variable u into the matrix of the system.

Theorem 70 *The generating function for the number of returns by a Knödel random walk with an open ending at $(n,-)$ is given by*

$$\xi_{i,\infty}(z,u) = \frac{v^i}{1-u\left(\frac{v+v^2}{1+v^2}\right)},$$

where $z = \frac{v}{1+v^2}$ and u counts the number of returns to the origin.

Proof. Applying Cramer's rule onto the system of linear equations given in equation (5.3) we get

$$\xi_{i,\infty}(z,u) = z^i \frac{\det A_{h-i}}{\det D_{h+1}}$$

$$\begin{aligned}
&= \left(\frac{v}{1+v^2}\right)^i \frac{1}{(1-v^2)} \frac{1-v^{2h-2i+2}}{(1+v^2)^{h-i}} \frac{(1+v^2)^{h+1}(1-v^2)}{(1+v^2-wv)(1-v^{2h+2})-wv^2(1-v^{2h})} \\
&= \frac{v^i\left(1-v^{2h-2i+2}\right)(1+v^2)}{(1+v^2-wv)(1-v^{2h+2})-wv^2(1-v^{2h})} \\
&= \frac{v^i(1+v^2)}{(1+v^2)-w(v+v^2)} \text{ as } h\to\infty,
\end{aligned}$$

and so

$$\xi_{i,\infty}(z,w) = \frac{v^i}{1-w\left(\frac{v+v^2}{1+v^2}\right)}.$$

■

Remark 15 *Plugging* $w=1$ *into* $\sum_{i\geq 0}\xi_{i,\infty}(z,w)$ *gives*

$$\sum_{i\geq 0}\xi_{i,\infty}(z,1) = \frac{1+v^2}{(1-v)^2}$$

which is the generating function for the total number of Knödel random walks with an open ending already derived in Remark 14.

Proposition 71 *The generating function of the* s*-factorial moments of* $\xi_{i,\infty}(z,w)$ *multiplied by the number of paths of length* n *is given by*

$$M_s(z) = \frac{v^s s!(1+v)^s(1+v^2)}{(1-v)^{s+2}}.$$

Proof. Now

$$\begin{aligned}
M_{(s)}(z) &= \frac{\partial^s}{\partial w^s}\xi_{i,\infty}(z,w)\Big|_{w=1} \\
&= \frac{v^i s!\left(\frac{v+v^2}{1+v^2}\right)^s}{\left(1-\frac{v+v^2}{1+v^2}\right)^{s+1}} \\
&= \frac{v^{i+s}s!(1+v)^s(1+v^2)}{(1-v)^{s+1}}.
\end{aligned}$$

Summing up the above generating function for all $i\geq 0$ we get the s-moments generating function for an open ending at $(n,-)$ given by

$$M_{(s)}(z) = \frac{v^s s!(1+v)^s(1+v^2)}{(1-v)^{s+2}}.$$

■

Corollary 72 *The average number of returns to the origin by a Knödel random walk with an open ending at* $(n,-)$ *is given by*

$$m_{(1)}(n) \asymp \sqrt{\frac{2n}{\pi}}\left(2+\frac{3}{2n}-\frac{9}{16n^2}\right)-2.$$

Proof. Specializing on $s=1$ into Proposition 71 we get

$$M_{(1)}(z)=\frac{v(1+v)(1+v^2)}{(1-v)^3}.$$

Expanding $M_{(1)}(z)$ around $z=\frac{1}{2}$ and using the substitution $z=\frac{v}{1+v^2}$ we get

$$M_{(1)}(z)\sim\frac{2}{\sqrt{2}(1-2z)^{\frac{3}{2}}}+\frac{2}{(1-2z)}+\frac{3}{2\sqrt{2}\sqrt{1-2z}}.$$

However expanding $M_{(2)}(z)$ around $z=-\frac{1}{2}$ gives a negligibly small contribution. Combining these two contributions gives

$$\begin{aligned}
[z^n]M_{(1)}(z) &\asymp (-2)^n\left[\frac{2}{\sqrt{2}}\binom{-\frac{3}{2}}{n}-2\binom{-1}{n}+\frac{3}{2}\binom{-\frac{1}{2}}{n}+\frac{23\sqrt{2}}{32}\binom{\frac{1}{2}}{n}\right]\\
&= 2^n\left[\frac{(2n+2)}{2^{2n}\sqrt{2}}\binom{2n+1}{n+1}+\frac{3}{2^{2n+1}\sqrt{2}}\binom{2n}{n}-2-\frac{23\sqrt{2}}{2^{2n+4}n}\binom{2n-2}{n-1}\right]\\
&\asymp 2^n\left[\sqrt{\frac{2n}{\pi}}\left(2+\frac{3}{2n}-\frac{9}{16n^2}\right)-2\right],
\end{aligned}$$

where the above asymptotic expansion was done using computer algebra. It follows that

$$\begin{aligned}
m_{(1)}(n) &= \frac{[z^n]M_1(z)}{2^n}\\
&\asymp \sqrt{\frac{2n}{\pi}}\left(2+\frac{3}{2n}-\frac{9}{16n^2}\right)-2,
\end{aligned}$$

as required. ■

Corollary 73 *The second moments of* $\xi_{i,\infty}(z)$ *are given by*

$$m_{(2)}(n)\sim 4n+18+\sqrt{\frac{2n}{\pi}}\left[-16-\frac{16}{n}+\frac{17}{2n^2}\right].$$

Proof. Specializing on $s = 2$ into Proposition 71 gives

$$M_{(2)}(z) = \frac{v^2(1+v)^2(1+v^2)}{(1-v)^4}.$$

Expanding $M_{(2)}(z)$ around $z = \frac{1}{2}$ and using the substitution $z = \frac{v}{1+v^2}$ we get

$$M_{(2)}(z) \sim \frac{4}{(1-2z)^2} - \frac{16}{\sqrt{2}(1-2z)^{\frac{3}{2}}} + \frac{14}{1-2z} - \frac{20}{\sqrt{2}\sqrt{1-2z}} - \frac{51\sqrt{2}\sqrt{1-2z}}{2}.$$

However expanding $M_{(2)}(z)$ around $z = -\frac{1}{2}$ gives a negligibly small contribution. Therefore

$$\begin{aligned}
[z^n] M_{(2)}(z) &\sim (-2)^n \left[4\binom{-2}{n} - \frac{16}{\sqrt{2}}\binom{-\frac{3}{2}}{n} + 14\binom{-1}{n} - \frac{20}{\sqrt{2}}\binom{-\frac{1}{2}}{n} - \frac{51\sqrt{2}}{4}\binom{\frac{1}{2}}{n} \right] \\
&= 2^n \left[4\binom{n+1}{n} - \frac{(n+1)}{2^{2n-4}\sqrt{2}}\binom{2n+1}{n+1} + 14 - \frac{5}{2^{2n-2}\sqrt{2}}\binom{2n}{n} \right. \\
&\quad \left. - \frac{51\sqrt{2}}{2^{2n+3}n}\binom{2n-2}{n-1} \right] \\
&\sim 2^n \left[4n + 18 + \sqrt{\frac{2n}{\pi}} \left(-16 - \frac{16}{n} + \frac{17}{2n^2} \right) \right].
\end{aligned}$$

Hence

$$\begin{aligned}
m_{(2)}(n) &= \frac{[z^n] M_2(z)}{2^n} \\
&\sim 4n + 18 + \sqrt{\frac{2n}{\pi}} \left[-16 - \frac{16}{n} + \frac{17}{2n^2} \right].
\end{aligned}$$

■

The two moments calculated in the above Corollaries do agree with moments of the random vaiable for the number of returns of a simple random walk given by Kemp in [16].

5.8 Observation

In this Section we give an enumerative comparison between the asymptotic and exact average number of returns to the 0-level, together with their corresponding variances, by a simple open ended Knödel path. The exact results were found using computer algebra and the asymptotic results were calculated using the formula for $m_{(1)}(n)$ given in Corollary 72.

n	**Returns**		**Variance**	
	expected	*asympt.*	*expected*	*asympt.*
50	9.4521	*9.4521*	46.0529	*46.0533*
101	14.1558	*14.1560*	106.2072	*106.2018*
174	19.1402	*19.1402*	197.4310	*197.4310*
222	21.8566	*21.8566*	259.0795	*259.0794*
367	28.6329	*28.6330*	449.5602	*449.5602*
593	36.9086	*36.9086*	753.2639	*753.2639*
712	40.6253	*40.6253*	915.0888	*915.0888*
844	44.4010	*44.4010*	1095.6349	*1095.6349*
1001	48.5257	*48.5257*	1311.4759	*1311.4759*

Table 5.3: Average 0-level returns by open ended Knödel paths.

The above Table confirms that the asymptotic and the exact results are the same for small values of r as n grows larger.

Chapter 6

Motzkin random walks

6.1 Introduction

First we recall that a **Motzkin random walk** of length n is a closed nonnegative lattice path of N^2 running from $(0,0)$ to $(n,0)$, whose allowed steps are the up diagonal step $(1,1)$, down diagonal step $(1,-1)$ and the horizontal step $(1,0)$. In this Chapter we enumerate Motzkin random walks of length n in terms of their visit to the nonnegative height r and returns to the origin as shown in the following diagram:

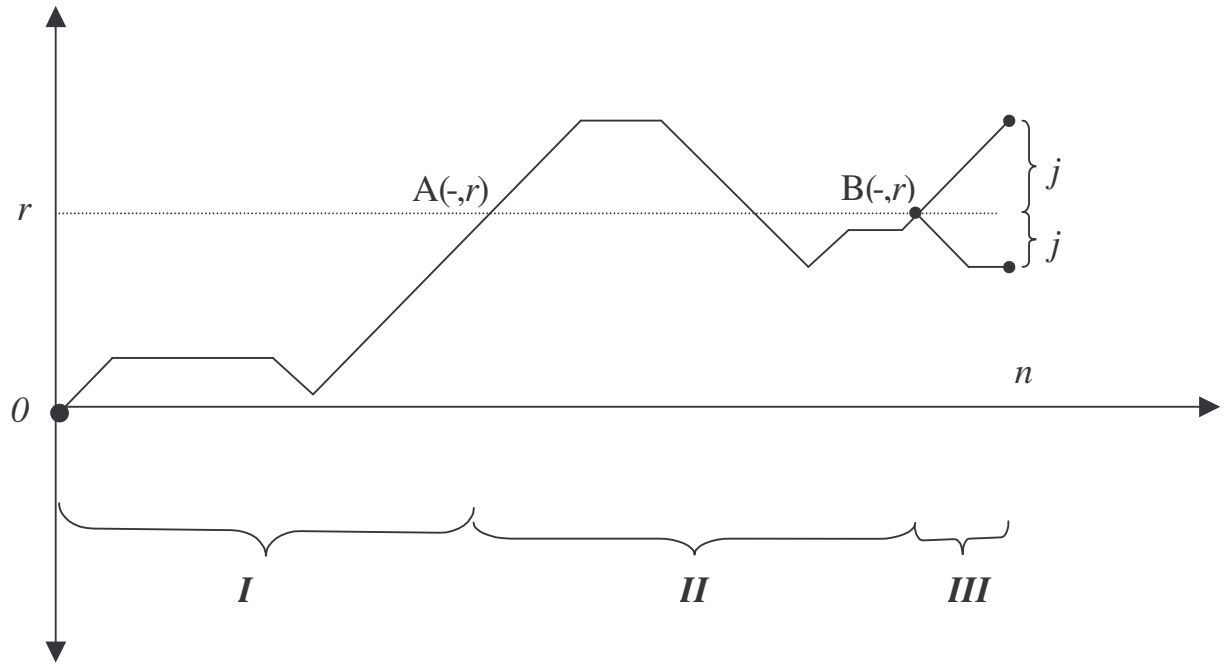


Figure 6.1: Open ended Motzkin path of length n

Further results on the enumeration of Motzkin random walk in terms of other parameters such as peaks and valleys will be dealt with in [23]. A wide range of articles dealing with the enumeration of Motzkin paths appears in many literatures such as [1], [2], [6], [7], [12], [23], [24], and [25]. As usual we apply the matrix approach and its associated Cramer's rule to construct our generating functions.

The generating function for any Motzkin random walk of length n for all $i = 0, 1, 2, 3, ..., h$, which we denote by $\delta_{i,h}$, is given by:

$$\delta_{i,h} = \sum_{i \geq 0} \left[\begin{array}{l} \text{Number of path of length } n, \text{ starting at 0 leading to } (n,\ i) \text{ and} \\ \text{allows horizontal steps at any level with height } \leq h \end{array} \right] z^n,$$

where z counts the single steps. For $0 < i < h$ we have that

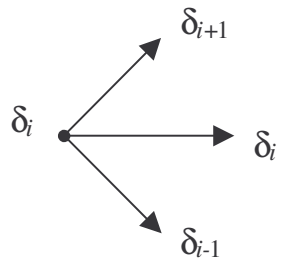


Figure 6.2

which gives the recursion

$$\delta_i(z) = z\delta_i(z) + z\delta_{i-1}(z) + z\delta_{i+1}(z),$$

indicating that from level i we either go up to level $i+1$, down to level $i-1$, or stay horizontal at level i. For $i = 0$ we have a special case:

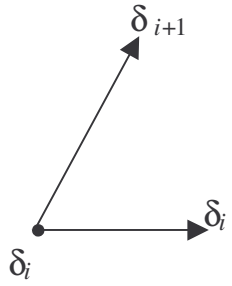


Figure 6.3

which gives the recursion

$$\delta_0(z) = 1 + z\delta_0(z) + z\delta_1(z),$$

indicating that we either stay at the origin, 0-level, or go to up to 1-level. The final special case is for $i = h$:

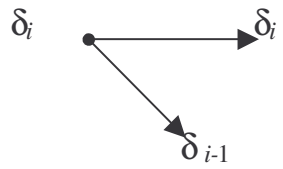


Figure 6.4

which gives the recursion

$$\delta_h(z) = z\delta_{h-1}(z) + z\delta_h(z),$$

indicating that from level h we either go down to level $h-1$ or stay horizontal at level h. We express all these recursions, for $i = 0, 1, 2, ..., h$ in a single matrix equation

$$B_h \Phi_h = C_h,$$

where the coefficient matrix B_h is a square matrix with $h+1$ rows, Φ_h is the column matrix representing the $\delta_{i,h}(z)$'s and C_h is the column matrix consisting of h zeros and a 1 in the first entry. It follows that

$$\begin{bmatrix} 1-z & -z & & & & & & \\ -z & 1-z & -z & & & 0 & & \\ & -z & 1-z & -z & & & & \\ & & . & . & . & & & \\ & & & . & . & . & & \\ & 0 & & & . & . & . & \\ & & & & & -z & 1-z & -z \\ & & & & & & -z & 1-z \end{bmatrix} \begin{bmatrix} \delta_{0,h} \\ \delta_{1,h} \\ \delta_{2,h} \\ . \\ . \\ . \\ \delta_{h-1,h} \\ \delta_{h,h} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{bmatrix}$$

As usual we calculate the $\delta_{i,h}$'s using Cramer's rule. But first, we determine the determinant of matrix B_h.

Lemma 74 *The determinant of the system of the $\delta_{i,h}$'s, denoted by $\det B_h$, with $h+1$ rows is given by*

$$\det B_h = \frac{\left(1 - v^{2h+4}\right)}{(1-v^2)\left(1+v+v^2\right)^{h+1}}.$$

Proof. First we determine the determinant of the matrix A_h but we h rows only. Expansion of the first row of A_h we obtain the following recursion of order two:

$$\det A_h = (1-z)\det A_{h-1} - z^2 \det A_{h-2}, \ \ \det A_0 = 1, \text{ and } \ \det A_1 = 1 - z.$$

The characteristic equation is $\lambda^2 - (1-z)\lambda + z^2 = 0$ with roots $\lambda_{1,2} = \frac{1-z \pm \sqrt{1-2z-3z^2}}{2}$. Now using the substitution

$$z = \frac{v}{1+v+v^2},$$

we get

$$\lambda_1 = \frac{1}{1+v+v^2} \text{ and } \lambda_2 = \frac{v^2}{1+v+v^2},$$

so that

$$\det A_h = \alpha\lambda_1^h + \beta\lambda_2^h.$$

Solving for α and β :

$$h = 0 \text{ gives } 1 = \alpha + \beta \text{ and } h = 1 \text{ gives } 1 - z = \alpha\lambda_1 + \beta\lambda_2.$$

Combining the above two equations we finally get

$$1 = \alpha\left(\frac{1-z+\sqrt{1-2z-3z^2}}{2}\right) + \beta\left(\frac{1-z-\sqrt{1-2z-3z^2}}{2}\right),$$

so that

$$\alpha = \frac{\lambda_1}{\sqrt{1-2z-3z^2}} \text{ and } \beta = \frac{-\lambda_2}{\sqrt{1-2z-3z^2}}.$$

Hence

$$\begin{aligned}
\det A_h &= \frac{1}{\sqrt{1-2z-3z^2}}\left[\lambda_1^{h+1} - \lambda_2^{h+1}\right] \\
&= \frac{1+v+v^2}{1-v^2}\left[\frac{1}{(1+v+v^2)^{h+1}} - \left(\frac{v^2}{1+v+v^2}\right)^{h+1}\right] \\
&= \frac{1-v^{2h+2}}{(1+v+v^2)^h(1-v^2)}.
\end{aligned}$$

Now expanding the first row of our actual matrix in equation (**??**) with $h+1$ rows we obtain,

$$\begin{aligned}
\det B_h &= \det A_{h+1} \\
&= \frac{\left(1-v^{2h+4}\right)}{(1-v^2)(1+v+v^2)^{h+1}}.
\end{aligned}$$

■

Theorem 75 *The generating function of the a Motzkin random walk which start at* $(0,0)$*, end at* $(n.i)$*, and bounded above by the line* $h>0$ *is given by*

$$\delta_{i,h}(z)=\frac{v^i\left(1+v+v^2\right)\left(1-v^{2h-2i+2}\right)}{\left(1-v^{2h+4}\right)}$$

with $z=\frac{v}{1+v+v^2}$.

Proof. The proof of this Theorem is an exact analogue of the proof of Theorem 7 in Chapter 2, hence will not be repeated here. ■

Remark 16 *If we remove the restriction on the height of the path and let* $h\to\infty$ *we get*

$$\delta_{i,\infty}(z)=v^i\left(1+v+v^2\right).$$

Summing up for all $i\geq 0$ we get

$$\sum_{i\geq 0}\delta_{i,\infty}(z)=\frac{\left(1+v+v^2\right)}{(1-v)}.$$

Therefore

$$\begin{aligned}
D_{open}(n) &= [z^n]\sum_{i\geq 0}\delta_{i,\infty}(z)\\
&= [v^n]\left(1-v^2\right)\left(1+v+v^2\right)^{n-1}\frac{\left(1+v+v^2\right)}{(1-v)}\\
&= [v^n]\left(1+v\right)\left(1+v+v^2\right)^{n}\\
&= \binom{n,3}{n}+\binom{n,3}{n-1}\\
&\sim 3^n\sqrt{\frac{3}{\pi n}}\left(1-\frac{9}{16n}+\frac{63}{128n^2}-\frac{351}{1024n^3}\right),\ \textit{using}\ [23]
\end{aligned}$$

which is the total number of Motzkin paths with an open ending at $(n,-)$. ■

6.2 Generating functions for the number of visits to the r-level.

In order to get a suitable expression for the generating function for the entire random walk we decompose it into three *parts,* which are *I, II* and *III* as shown in Figure 6.1.

Lemma 76 *The generating function of a Motzkin random walk which starts at* $(0,0)$ *and is reaching* r *for the first time is given by*

$$z\delta_{r-1,r-1}(z)=\frac{v^r(1-v^2)}{(1-v^{2r+2})},$$

where $z=\frac{v}{1+v+v^2}$.

Proof. The proof of this is the generating function is similar to the one done for the Dyck case in Lemma 8 and hence will not be repeated here. ■

Lemma 77 *The generating function for a Motzkin random walk which starts and ends on* r *is given by*

$$F_r(z,u)=\frac{u}{1-u\left(1-\frac{1-v^2}{(1+v+v^2)(1-v^{2r+2})}\right)},$$

where $z=\frac{v}{1+v+v^2}$ *and* u *counts the number of returns to the* r*-level.*

Proof. This random walk starts at $\mathbf{A}(r,-)$ and ends at $\mathbf{B}(\ r,-)$ as shown by part II of Fig. 6.1. By a simple shift we let level r be the 0-level. We denote the generating function for this random walk by $F_r(z,u)$, where u is counting the number of returns to the r-level. Furthermore, let $F_{r,+}(z)$ and $F_{r,-}(z)$ denote the generating functions of random walks that are above and below the 0-level respectively. Finally, let $F_{r,0}(z)$ denote the generating function of random walks which are before $F_{r,+}(z)$, between $F_{r,+}(z)$ and $F_{r,0}(z)$ and after $F_{r,0}(z)$. Therefore between any two returns on the 0-level, we have a sequence of paths above, below and on the r-level, which we denote symbolically as $F_r(z,u)=(F_{r,+}(z)\cup F_{r,-}(z)\cup F_{r,0}(z))^*$. Hence

$$F_r(z)=\frac{u}{1-u\left(F_{r,+}(z)+F_{r,-}(z)+F_{r,0}\right)}.$$

We decompose $F_{r,+}(z)$ and $F_{r,-}(z)$ using the formula for $\delta_{i,h}(z)$ as in the proof Lemma 9 so that

$$F_{r,+}(z)=z^2\delta_{0,\infty}(z)\ \text{ and }\ F_{r,-}(z)=z^2\delta_{0,r-1}(z).$$

The horizontal random walks $F_{r,0}(z)$ will be denoted by a single z-step. It follows that

$$F_r(z) = \frac{u}{1 - u\left(z^2\delta_{0,\infty}(z) + z^2\delta_{0,r-1}(z) + z\right)}.$$

Using the substitutions

$$z = \frac{v}{1+v+v^2},\ z^2\delta_{0,\infty}(z) = \frac{v^2}{1+v+v^2},\ \text{and}\ z^2\delta_{0,r-1}(z) = \frac{v^2\left(1-v^{2r}\right)}{\left(1+v+v^2\right)\left(1-v^{2r+2}\right)},$$

we have that

$$z^2\delta_{0,\infty}(z) + z^2\delta_{0,r-1}(z) = \frac{v^2\left(2-v^{2r+2}-v^{2r}\right)}{\left(1+v^2\right)\left(1-v^{2r+2}\right)},$$

or

$$1 - z^2\delta_{0,\infty}(z) - z^2\delta_{r-1,r-1}(z) - z = \frac{1-v^2}{\left(1+v^2\right)\left(1-v^{2r+2}\right)}.$$

Therefore

$$\begin{aligned} z^2\delta_{0,\infty}(z) + z^2\delta_{0,r-1}(z) + z &= 1 + z^2\delta_{0,\infty}(z) + z^2\delta_{0,r-1}(z) + z - 1 \\ &= 1 - \left[1 - z^2\delta_{0,\infty}(z) - z^2\delta_{0,r-1}(z) - z\right] \\ &= 1 - \frac{1-v^2}{\left(1+v+v^2\right)\left(1-v^{2r+2}\right)}, \end{aligned}$$

and hence the results. ■

Proposition 78 *The generating function of the s-factorial moments multiplied by the number of paths of length n $F_r(z,u)$ in is given by*

$$M_{(s)}(z) = s!\sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^k\left(1+v+v^2\right)^{s-k+1}\left(1-v^{2r+2}\right)^{s-k+1}\left(1-v^2\right)^{k-s-1},$$

where $z = \frac{v}{1+v+v^2}$.

Proof. Applying Lemma 77 we get

$$\begin{aligned} M_{(s)}(z) &= \frac{\partial^s}{\partial u^s}F_r(z,u)|_{u=1} \\ &= \frac{s!\left[z^2\varphi_{0,\infty}(z) + z^2\delta_{0,r-1}(z) + z\right]^s}{\left[1 - z^2\varphi_{0,\infty}(z) - z^2\delta_{0,r-1}(z) - z\right]^{s+1}} \end{aligned}$$

$$
\begin{aligned}
&+\frac{s!\left[z^2\varphi_{0,\infty}(z)+z^2\delta_{0,r-1}(z)+z\right]^{s-1}}{\left[1-z^2\varphi_{0,\infty}(z)-z^2\delta_{0,r-1}(z)-z\right]^{s}}\\
=\;&\frac{s!\left[z^2\varphi_{0,\infty}(z)+z^2\delta_{0,r-1}(z)+z\right]^{s-1}}{\left[1-z^2\varphi_{0,\infty}(z)-z^2\delta_{0,r-1}(z)-z\right]^{s+1}}\\
=\;&\frac{s!\left[1-\frac{1-v^2}{(1+v+v^2)(1+v^{2r+2})}\right]^{s-1}}{\left[\frac{1-v^2}{(1+v+v^2)(1+v^{2r+2})}\right]^{s+1}}\\
=\;&s!\sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^k\left(1+v+v^2\right)^{s-k+1}\left(1-v^{2r+2}\right)^{s-k+1}\left(1-v^2\right)^{k-s-1}.
\end{aligned}
$$

■

The generating function for *part III* of Figure 6.1 helps us to determine if the generating function for the entire walk has a **closed** or **open ending.**

6.3 Random walks with a closed ending.

These are random walks ending at $(n,0)$.

Theorem 79 *The generating function for a Motzkin random walk starting at* $(0,0)$ *and ending at* $(n,0)$ *is given by*

$$G_{r,closed}(z,u)=\frac{(1+v+v^2)(1-v^{2r})}{(1-v^{2r+2})}+\frac{uv^{2r}(1-v^2)^2}{\left[1-u\left(1-\frac{1-v^2}{(1+v+v^2)(1+v^{2r+2})}\right)\right](1-v^{2r+2})^2},$$

where $z=\frac{v}{1+v+v^2}$ *and* u *counts the number of returns to the* r*-level.*

Proof. We decompose a Motzkin random which starts at $(0,0)$ and ends at $(n,0)$ into three generating functions corresponding to walks in parts I, II and III of Fig. 6.1. The generating functions for part I and II are already known from Lemma 76 and Lemma 77 respectively. Treating r as the 0-level and realizing that the generating function for part III leaves the r-level at $B(-,r)$, without making no returns to r, and ending at $(n,0)$, we eventually see that this generating function is symmetrical to the one for part I. Thus the generating function for this *closed ending* (or *part III*) is given by $z\delta_{r-1,r-1}(z)$. If we combine the generating

functions for *part I, II,* and *III* we get the generating function for the entire walk given by

$$G_{r,closed}(z,u) = \delta_{0,r-1}(z) + z\delta_{r-1,r-1}(z) \times F_r(z,u) \times z\delta_{r-1,r-1}(z),$$

where $\delta_{0,r-1}(z)$ is the generating function for random walks not reaching level r. But then

$$(z\delta_{r-1,r-1}(z))^2 = \frac{v^{2r}(1-v^2)^2}{(1-v^{2r+2})^2},$$

so that

$$\begin{aligned} G_{r,closed}(z,u) &= \delta_{0,r-1}(z) + \frac{u}{\left[1-u\left(z^2\varphi_{0,\infty}(z) + z^2\delta_{0,r-1}(z) + z\right)\right]} \frac{v^{2r}(1-v^2)^2}{(1-v^{2r+2})^2}, \\ &= \frac{(1+v+v^2)(1-v^{2r})}{(1-v^{2r+2})} + \frac{uv^{2r}(1-v^2)^2}{\left[1-u\left(1-\frac{1-v^2}{(1+v+v^2)(1-v^{2r+2})}\right)\right](1-v^{2r+2})^2}. \end{aligned}$$

■

Remark 17 *Substituting $u=1$ into $G_{r,closed}(z,u)$ we get*

$$\begin{aligned} G_{r,closed}(z,1) &= \frac{(1+v+v^2)(1-v^{2r})}{(1-v^{2r+2})} + \frac{v^{2r}(1+v+v^2)(1-v^2)}{(1-v^{2r+2})} \\ &= 1+v+v^2, \end{aligned}$$

And so

$$\begin{aligned} B(n) &= [z^n] G_{r,closed}(z,1) \\ &= \frac{1}{2\pi i}\oint \frac{(1-v^2)(1+v+v^2)^{n-1}}{v^{n+1}}\left(1+v+v^2\right)dv \\ &= [v^n]\left(1-v^2\right)\left(1+v+v^2\right)^n \\ &= \binom{n,3}{n} - \binom{n,3}{n-2} \\ &\sim 3^n\sqrt{\frac{3}{\pi n}}\left(\frac{3}{2n} - \frac{117}{32n^2} + \frac{459}{64n^3}\right) \textit{using } [1], \end{aligned}$$

which is the ***total number of Motzkin paths of length*** n ***with closed endings****. So this remark serves as a good test for the consistency of* $G_{r,closed}(z,u)$. ■

Proposition 80 *The generating function of the s-factorial moments multiplied by the number of paths of length n in $G_{r,closed}(z,u)$ is given by*

$$M_s(z) = v^{2r} s! \sum_{k=0}^{s-1} \binom{s-1}{k} (-1)^k \left(1+v+v^2\right)^{s-k+1} \left(1-v^{2r+2}\right)^{s-k-1} \left(1-v^2\right)^{k-s+1},$$

where $z = \frac{v}{1+v+v^2}$.

Proof. Applying the generating function $G_{r,closed}(z,u)$ we get

$$\begin{aligned} M_s(z) &= \frac{\partial^s}{\partial u^s} G_{r,closed}(z,u)|_{u=1} \\ &= \frac{v^{2r}(1-v^2)^2}{(1-v^{2r+2})^2} \frac{\partial^s}{\partial u^s} F_r(z,u)|_{u=1} \\ &= v^{2r} s! \sum_{k=0}^{s-1} \binom{s-1}{k} (-1)^k \left(1+v+v^2\right)^{s-k+1} \left(1-v^{2r+2}\right)^{s-k-1} \left(1-v^2\right)^{k-s+1}, \end{aligned}$$

as required. ■

We are now ready to use asymptotic expansions given in [8] and [20] to find moments of Motzkin random walks.

Corollary 81 *The average number of visits to level r by a Motzkin random walk is given by*

$$\begin{aligned} m_{(1)}(n) \sim\ & 6r+6+\frac{36r^3+108r^2+117r+45}{2n} \\ & +\frac{432r^5+2160r^4+5328r^3+7344r^2+5319r+1575}{16n^2}+... \end{aligned}$$

Proof. Specializing on $s=1$ into Proposition 82 we get

$$M_1(z) = v^{2r}(1+v+v^2)^2.$$

To compute $[z^n] M_1(z)$ using Cauchy Integral Formula is a little bit tricky here, so we use the substitution

$$\binom{n+1,3}{k} = \binom{n,3}{k} + \binom{n,3}{k-1} + \binom{n,3}{k-2},$$

from [20] so that

$$[z^n] M_1(z) = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} v^{2r}(1+v+v^2)^2$$

$$\begin{aligned}
&= \frac{1}{2\pi i}\oint \frac{\left(1-v^2\right)\left(1+v+v^2\right)^{n-1}}{v^{n+1}} v^{2r}(1+v+v^2)^2 dv \\
&= \left[v^{n-2r}\right]\left(1-v^2\right)\left(1+v+v^2\right)^{n+1} \\
&= \binom{n+1,3}{n-2r} - \binom{n+1,3}{n-2r-2} \\
&= \binom{n,3}{n-2r} + \binom{n,3}{n-2r-1} - \binom{n,3}{n-2r-3} - \binom{n+1,3}{n-2r-4} \\
&\sim 3^n\sqrt{\frac{3}{\pi n}}\left[\frac{9\left(r+1\right)}{n} - \frac{9\left(195r+99\right)}{16n^2} - \frac{27\left(r^3+3r^2\right)}{n^2} + \frac{9(1005r^3+1575r^2)}{16n^3}\right. \\
&\quad \left. + \frac{9(5373r+1941)}{64n^3} + \frac{9(9r^5+45r^4)}{2n^3} + ...\right],
\end{aligned}$$

where the last asymptotic expansion was done using computer algebra, and applying [20]. Using the total number of objects derived in Remark 17 we get

$$\begin{aligned}
m_1\left(z\right) &= \frac{\left[z^n\right]M_1\left(z\right)}{B\left(n\right)} \\
&\sim 6r+6+\frac{36r^3+108r^2+117r+45}{2n} \\
&\quad + \frac{432r^5+2160r^4+5328r^3+7344r^2+5319r+1575}{16n^2} + ...
\end{aligned}$$

■

Corollary 82 *The second moments on $G_{r,closed}\left(z,u\right)$ are given by*

$$\begin{aligned}
m_{(2)}\left(n\right) &\sim 54r^2+96r+42-\frac{\left(405r^4+1584r^3+2403r^2+1665r+441\right)}{n} \\
&\quad + \frac{40824r^6+243648r^5+652050r^4+985176r^3+872208r^2+423387r+87129}{24n^2} + ...
\end{aligned}$$

Proof. Substituting $s=2$ in Proposition 80 gives

$$M_2\left(z\right) = 2v^{2r}\left[\frac{(1+v+v^2)^3(1-v^{2r+2})}{(1-v^2)} - (1+v+v^2)^2\right].$$

And so

$$\begin{aligned}
\left[z^n\right]M_2\left(z\right) &= \frac{1}{2\pi i}\oint \frac{dz}{z^{n+1}} M_2\left(z\right) \\
&= \frac{1}{2\pi i}\oint \frac{2v^{2r}\left[\left(v+2v^2\right)-v^{2r+2}\left(1+v+v^2\right)\right]\left(1+v+v^2\right)^n}{v^{n+1}} dv
\end{aligned}$$

$$
\begin{aligned}
&= 2\left[v^{n-2r-1}\right](1+2v)(1+v+v^2)^n - 2\left[v^{n-4r-2}\right](1+v+v^2)^{n+1} \\
&= 2\left[\binom{n,3}{n-2r-1} + \binom{n,3}{n-2r-2} - \binom{n+1,3}{n-4r-2}\right] \\
&\sim 3^n\sqrt{\frac{3}{\pi n}}\left(\frac{81r^2+144r+63}{n} - \frac{1215r^4+5697r}{2n^2} - \frac{60831r^2+13041}{16n^2}\right. \\
&\quad -\frac{2376r^3}{n^2} + \frac{15228r^5+67365r^3}{n^3} + \frac{254745r^2}{4n^3} + \frac{5887}{8n^3} \\
&\quad \left. +\frac{1063611r+1351485r^4}{32n^3} + \frac{5103r^6}{2n^3} + ...\right),
\end{aligned}
$$

where the last asymptotic expansion was done using computer algebra, and applying [20]. Using the total number of objects derived in Remark 17 we get

$$
\begin{aligned}
m_2(n) &= \frac{[z^n]\,M_2(z)}{B(n)} \\
&\sim 54r^2+96r+42 - \frac{(405r^4+1584r^3+2403r^2+1665r+441)}{n} \\
&\quad +\frac{40824r^6+243648r^5+652050r^4+985176r^3+872208r^2+423387r+87129}{24n^2} + ...
\end{aligned}
$$

■

6.4 Observation

With the help of computer algebra and Corollary 81 we construct a Table 5.1 to give an enumerative comparison between the asymptotic and exact average number of visits to height r, together with their corresponding variances, by a simple Motzkin path with a closed ending at $(n, 0)$.

	Type of	$m_{(1)}(n)$			**Variances**		
n	**results**	r=1	r=3	r=5	r=1	r=3	r=5
593	expected	11.746	22.114	29.993	55.203	195.265	318.485
593	*asympt.*	11.746	22.117	30.034	54.821	175.868	123.158
712	expected	11.788	22.418	30.919	55.9741	205.277	354.423
712	*asympt.*	11.788	22.419	30.943	55.708	191.642	215.213
844	expected	11.821	22.657	31.661	56.583	213.460	385.496
844	*asympt.*	11.821	22.658	31.675	56.393	203.654	284.277
1001	expected	11.849	22.862	32.303	57.105	220.654	414.089
1001	*asympt.*	11.849	22.863	32.312	56.969	213.620	340.794

Table 5.3: Average r-level visits by a closed ended Motzkin path.

The above Table shows that the asymptotic and the exact results are almost the same for small values of r as n grows larger.

6.5 Random walks with an open ending.

The generating function for *part III* of Fig. 6.1 has three choices, depending on whether it will end on $j+r$ (i.e. above r), r or on $r-i$ (i.e. below r) as proved in the following Lemma:

Lemma 83 *The generating function $L_r(z,u)$ for the entire Motzkin random walk starting at $(0,0)$ and ending either at $(n, r-j)$, r or $(n, r+j)$ is given by*

$$L_r(z,u) = \begin{cases} \frac{uv^{r+j}(1-v^2)}{\left[1-u\left(1-\frac{1-v^2}{(1+v+v^2)(1-v^{2r+2})}\right)\right](1-v^{2r+2})}, & \text{ending at } (n, r+j) \\ \delta_{j-1,r-1}(z) + \frac{uv^{r+j}(1-v^2)(1-v^{2r-2j+2})}{\left[1-u\left(1-\frac{1-v^2}{(1+v+v^2)(1-v^{2r+2})}\right)\right](1-v^{2r+2})^2}, & \text{ending at } (n, r-j) \\ \frac{uv^{r}(1-v^2)}{\left[1-u\left(1-\frac{1-v^2}{(1+v+v^2)(1-v^{2r+2})}\right)\right](1-v^{2r+2})}, & \text{ending at } (n, r) \end{cases}$$

where $z = \frac{v}{1+v+v^2}$ and u is counting the number of visits to the r-level.

Proof.

Case 1: *Ending at* $(n, r+j)$.

This random walk starts at $(0,0)$ and end at $(n, r+j)$, and is mainly composed of three generating functions; $z\delta_{r-1,r-1}(z)$ for *part I* , $F_r(z,u)$ for *part II* and the generating function for *part III* of Fig. 6.1. If we treat r as the 0-level we can easily see that it is bounded below and above by 0 and ∞ respectively, and it leaves the 0-level at B and end at i without making returns to the 0-level. Applying Theorem 75 we write the generating for *part III* of Fig. 6.1 as $\delta_{j,\infty}(z)$ or we can rewrite it as $z\delta_{j-1,\infty}(z)$ which is a product of its decompositions, namely, z and $z\delta_{j-1,\infty}(z)$. Combining the generating functions for all the three parts we get

$$L_{r,above}(z,u) = z\delta_{r-1,r-1}(z) \times F_r(z,u) \times z\delta_{j-1,\infty}(z),$$

Since

$$F_r(z,u) = \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v+v^2)(1-v^{2r+2})}\right)},$$

and

$$z^2\delta_{r-1,r-1}(z)\,\delta_{j-1,\infty}(z) = \frac{v^{r+j}(1-v^2)}{(1-v^{2r+2})},$$

it follows that

$$L_{r,above}(z,u) = \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v+v^2)(1-v^{2r+2})}\right)}\frac{v^{r+j}(1-v^2)}{(1-v^{2r+2})}.$$

Case 2 : *Ending at* $(n, r-j)$.

In this case the walk starts at $(0,0)$ and end at $(n, r-j)$. This walk consists of three generating functions, namely; $z\delta_{r-1,r-1}(z)$ for *part I*, $F_r(z,u)$ for *part II* and the generating function for *part III* of Figure 6.1. If we treat $r-1$ as the 0-level we see that the generating function for *part III* is bounded below and above by 0 and $r-1$ respectively, and it starts and end at 0 and $j-1$ respectively without making no returns to the 0-level. Hence the

generating function for *part III* in Fig. 6.1 is mathematically written as $z\delta_{j-1,r-1}(z)$, which is a decomposition of $\delta_{i,r-1}(z)$ into z and the generating function which starts at $r-1$ and end at $j-1$. Now combining these three generating functions we get

$$\begin{aligned} L_{r,below}(z,u) &= \delta_{j-1,r-1}(z) + z\delta_{r-1,r-1}(z) \times F_r(z,u) \times z\delta_{j-1,r-1}(z) \\ &= \delta_{j-1,r-1}(z) + \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v+v^2)(1-v^{2r+2})}\right)} \frac{v^{r+j}\left(1-v^{2r+2j+2}\right)\left(1-v^2\right)}{\left(1-v^{2r+2}\right)^2}, \end{aligned}$$

where $\delta_{j-1,r-1}(z)$ is the generating function for random walks not reaching r and

$$z^2\delta_{r-1,r-1}(z)\,\delta_{i-1,r-1}(z) = \frac{v^{r+j}\left(1-v^2\right)\left(1-v^{2r-2j+2}\right)}{\left(1-v^{2r+2}\right)^2}.$$

■

Case 3 : *Ending at* (n,r)

In this case the walk starts at $(0,0)$ and end at $B(-,r)$ as shown in Fig. 6.1 Unlike in the two previous cases, this walk consists of two generating functions, namely, $z\delta_{r-1,r-1}(z)$, and $F_r(z,u)$. Now combining these two generating functions we get

$$\begin{aligned} L_{r,0}(z,u) &= z\delta_{r-1,r-1}(z) \times F_r(z,u) \\ &= \frac{uv^r(1-v^2)}{\left[1-u\left(1-\frac{1-v^2}{(1+v+v^2)(1-v^{2r+2})}\right)\right]\left(1-v^{2r+2}\right)}. \end{aligned}$$

■

Theorem 84 *The generating function for a Motzkin random walk with open ending at* $(n,-)$ *is given by*

$$\begin{aligned} L_{r,open}(z,u) &= \frac{\left(1+v+v^2\right)\left(1-v^{r+1}\right)\left(1-v^r\right)}{(1-v)\left(1-v^{2r+2}\right)} \\ &\quad + \frac{u}{1-u\left(1-\frac{1-v^2}{(1+v+v^2)(1-v^{2r+2})}\right)} \frac{v^r\left(1-v^2\right)\left(1-v^{r+1}\right)(1+v)}{(1-v)\left(1-v^{2r+2}\right)^2}, \end{aligned}$$

where $z=\frac{v}{1+v+v^2}$ and u counts the number of returns to the r-level.

Proof. If we sum up the three generating functions derived in Lemma 83 above for all $j \geq 0$ we get a bivariate generating function for the entire random walk given by

$$\begin{aligned} L_{r,open}(z,u) &= \sum_{j\geq 0}\left(L_{r,open}(z,u)+L_{r,below}(z,u)+L_{r,0}(z,u)\right) \\ &= \sum_{k=0}^{r-1}\delta_{k,r-1}(z)+z\delta_{r-1,r-1}(z)\times F_r(z,u) \\ &\quad \times\left(z\sum_{j\geq 1}\delta_{j-1,\infty}(z)+z\sum_{j=1}^{r}\delta_{j-1,r-1}(z)+1\right). \end{aligned}$$

where $\sum_{k=0}^{r-1}\delta_{k,r-1}(z)$ is the generating function for random walks not reaching r. However

$$z\delta_{r-1,r-1}(z)\left(z\sum_{j\geq 1}\delta_{j-1,\infty}(z)+z\sum_{j=1}^{r}\delta_{j-1,r-1}(z)+1\right)=\frac{v^r\left(1-v^2\right)\left(1-v^{r+1}\right)(1+v)}{(1-v)\left(1-v^{2r+2}\right)^2},$$

so that

$$\begin{aligned} L_{r,open}(z,u) &= \sum_{k=0}^{r-1}\delta_{k,r-1}(z)+F_r(z,u)\frac{v^r\left(1-v^2\right)\left(1-v^{r+1}\right)(1+v)}{(1-v)\left(1-v^{2r+2}\right)^2} \\ &= \frac{\left(1+v+v^2\right)\left(1-v^{r+1}\right)\left(1-v^r\right)}{(1-v)\left(1-v^{2r+2}\right)} \\ &\quad +\frac{u}{1-u\left(1-\frac{1-v^2}{(1+v+v^2)(1-v^{2r+2})}\right)}\frac{v^r\left(1-v^2\right)\left(1-v^{r+1}\right)(1+v)}{(1-v)\left(1-v^{2r+2}\right)^2} \end{aligned}$$

as required. ■

Remark 18 *We test $L_{r,open}(z,u)$ by pugging in $u=1$:*

$$\begin{aligned} L_{r,open}(z,1) &= \sum_{k=0}^{r-1}\delta_{k,r-1}(z)+\frac{v^r\left(1+v+v^2\right)\left(1-v^{r+1}\right)(1+v)}{(1-v)\left(1-v^{2r+2}\right)} \\ &= \frac{\left(1+v+v^2\right)\left(1-v^{r+1}\right)\left(1-v^r\right)}{(1-v)\left(1-v^{2r+2}\right)}+\frac{v^r\left(1+v+v^2\right)\left(1-v^{r+1}\right)(1+v)}{(1-v)\left(1-v^{2r+2}\right)} \\ &= \frac{\left(1+v+v^2\right)}{(1-v)}. \end{aligned}$$

which corresponds to the generating function for the total number of Motzkin paths with an open ending at $(n,-)$ previously calculated in Remark 17. ■

Proposition 85 *The generating function of the s-factorial moments multiplied by the number of paths of length n in $L_{r,open}(z,u)$ is given by*

$$\begin{aligned} M_{(s)}(z) &= s!\sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^k\left(1-v^2\right)^{k-s-1}\left(1+v+v^2\right)^{s-k+1}\left(1-v^{2r+2}\right)^{s-k+1} \\ &\quad\times\frac{v^r\left(1-v^2\right)\left(1-v^{r+1}\right)(1+v)}{(1-v)\left(1-v^{2r+2}\right)^2}, \end{aligned}$$

where $z=\frac{v}{1+v+v^2}$.

Proof. Applying Theorem 84

$$\begin{aligned} M_{(s)}(z) &= \frac{\partial^s}{\partial u^s}L_{r,open}(z,u)\Big|_{u=1} \\ &= \frac{v^r\left(1-v^2\right)\left(1-v^{r+1}\right)(1+v)}{(1-v)\left(1-v^{2r+2}\right)^2}\times\frac{\partial^s}{\partial u^s}F_r(z,u)\Big|_{u=1} \\ &= s!\sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^k\left(1-v^2\right)^{k-s-1}\left(1+v+v^2\right)^{s-k+1}\left(1-v^{2r+2}\right)^{s-k+1} \\ &\quad\times\frac{v^r\left(1-v^2\right)\left(1-v^{r+1}\right)(1+v)}{(1-v)\left(1-v^{2r+2}\right)^2}. \end{aligned}$$

■

We now use the famous results by [8] and [20] to asymptotically expand $L_{r,open}(z,u)$ around the dominant singularity $\frac{1}{3}$.

Corollary 86 *The average number of visits to level r by an open ended Motzkin random walk is given by*

$$\begin{aligned} m_1(n) &\sim 3r+3-\frac{\left(84r^3+252r^2+264r+96\right)}{16n} \\ &\quad+\frac{\left(13392r^5+66960r^4+164160r^3+224640r^2+160143r+46095\right)}{2560n^2}. \end{aligned}$$

Proof. Substituting $s=1$ into Proposition 85 we get

$$M_1(z)=\frac{v^r\left(1-v^{r+1}\right)(1+v)\left(1+v+v^2\right)^2}{\left(1-v^2\right)(1-v)}.$$

Using the substitution $z = \frac{v}{1+v+v^2}$ and expanding $M_1(z)$ around $z = \frac{1}{3}$ gives

$$\begin{aligned} M_1(z) &\sim \frac{9(r+1)}{\sqrt{3}\sqrt{1-3z}} - \frac{(27r^2+54r+27)}{2} + \frac{3(28r^3+84r^2+95r+39)\sqrt{3}\sqrt{1-3z}}{8} \\ &\quad - \frac{27(5r^4+20r^3+37r^2+34r+12)(1-3z)}{8} \\ &\quad + \frac{3(1488r^5+7440r^4+19640r^3+29160r^2+22747r+7275)\sqrt{3}(1-3z)^{\frac{3}{2}}}{640}. \end{aligned}$$

Therefore

$$\begin{aligned} [z^n] M_1(z) &\sim (-3)^n \left[\frac{9(r+1)}{\sqrt{3}} \binom{-\frac{1}{2}}{n} + \frac{3(28r^3+84r^2+95r+39)\sqrt{3}}{8} \binom{\frac{1}{2}}{n} \right. \\ &\quad \left. + \frac{3(1488r^5+7440r^4+19640r^3+29160r^2+22747r+7275)\sqrt{3}}{640} \binom{\frac{3}{2}}{n} + ... \right] \\ &= 3^n \left[\frac{9(r+1)}{2^{2n}\sqrt{3}} \binom{2n}{n} - \frac{3(28r^3+84r^2+95r+39)\sqrt{3}}{2^{2n+2}n} \binom{2n-2}{n-1} \right. \\ &\quad \left. + \frac{9(1488r^5+7440r^4+19640r^3+29160r^2+22747r+7275)\sqrt{3}}{5(n-2)(n-3)2^{2n+5}} \binom{2n-4}{n-4} + ... \right] \\ &= 3^n \left[\frac{3\sqrt{3}(80r^4+18r^3-239r^2-273r-78)}{2^{2n+3}n} \binom{2n-2}{n-1} \right. \\ &\quad \left. - \frac{9(r^2-3r-2)}{2^{2n+1}\sqrt{3}} \binom{2n}{n} + ... \right] \\ &\sim 3^n \sqrt{\frac{3}{\pi n}} \left[3r+3 - \frac{(84r^3+252r^2+292r+123)}{16n} \right. \\ &\quad \left. + \frac{(13392r^5+66960r^4+171720r^3+247320r^2+187683r+58515)}{2560n^2} + ... \right] \end{aligned}$$

Applying the total number of objects, $D_{open}(n)$, calculated in Remark 16 we get

$$\begin{aligned} m_1(n) &= \frac{[z^n] M_1(z)}{T(n)} \\ &\sim 3r+3 - \frac{(84r^3+252r^2+264r+96)}{16n} \\ &\quad + \frac{(13392r^5+66960r^4+164160r^3+224640r^2+160143r+46095)}{2560n^2}, \end{aligned}$$

is the required average number of visits to level r for this case. ■

Corollary 87 *The second moments are given by*

$$\begin{aligned} m_{(2)}(n) &\sim 18r^2+30r+12-\frac{(180r^4+699r^3+1035r^2+174)}{2n} \\ &+\frac{(356832r^6+2127600r^5+5631120r^4+8354880r^3+7220493r^2+3401355r+675390)}{1280n^2} \\ &+... \end{aligned}$$

Proof. Substituting $s=2$ into Proposition 85 gives

$$M_2(z)=\frac{2v^{r+1}\left(1-v^{r+1}\right)\left(1+v+v^2\right)^2\left[1+2v-v^{2r+1}\left(1+v+v^2\right)\right]}{\left(1-v^2\right)(1-v)^2}.$$

By expanding $M_2(z)$ around $z=\frac{1}{3}$ and by making the substitution $z=\frac{v}{1+v+v^2}$ we get

$$\begin{aligned} &M_2(z) \\ \sim\ &\frac{6\sqrt{3}\left(3r^2+5r+2\right)}{\sqrt{1-3z}}-27(1+r)^2(4+5r)+\frac{3\left(240r^4+932r^3+1401r^2+955r+246\right)\sqrt{3}\sqrt{1-3z}}{4} \\ &-\frac{27\left(75r^5+370r^4+775r^3+848r^2+476r+108\right)(1-3z)}{4} \\ &+\frac{3\sqrt{3}\left(39648r^6+236400r^5+637680r^4+974920r^3+872937r^2+426695r+87750\right)(1-3z)^{\frac{3}{2}}}{320}. \\ &+... \end{aligned}$$

Therefore

$$\begin{aligned} &[z^n]\,M_2(z) \\ =\ &(-3)^n\left[-\frac{18\left(3r^2+5r+2\right)}{\sqrt{3}}\binom{-\frac{1}{2}}{n}+\frac{3\left(240r^4+932r^3+1401r^2+955r+246\right)\sqrt{3}}{4}\binom{\frac{1}{2}}{n}\right. \\ &\left.\frac{3\left(39648r^6+236400r^5+637680r^4+974920r^3+872937r^2+426695r+87750\right)\sqrt{3}}{320}\binom{\frac{3}{2}}{n}+...\right] \\ =\ &3^n\left[\frac{9\left(3r^2+5r+2\right)}{2^{2n-1}\sqrt{3}}\binom{2n}{n}-\frac{3\sqrt{3}\left(240r^4+932r^3+1401r^2+955r+246\right)}{2^{2n+1}n}\binom{2n-2}{n-1}\right. \\ &\left.\frac{9\left(39648r^6+236400r^5+637680r^4+974920r^3+872937r^2+426695r+87750\right)\sqrt{3}}{5(n-2)(n-3)2^{2n+4}}\binom{2n-4}{n-4}+...\right. \\ \sim\ &3^n\sqrt{\frac{3}{\pi n}}\left[18r^2+30+12-\frac{740r^4+2796r^3+4221r^2+2895r+750}{8n}\right. \\ &\left.+\frac{(356832r^6+2127600r^5+5631120r^4+8354880r^3+7220493r^2+3401355r+675390)}{1280n^2}+...\right]. \end{aligned}$$

Since the total number of objects for this case is $D_{open}(n)$, calculated in Remark 16, we get

$$\begin{aligned} m_{(2)}(n) &= \frac{[z^n] M_2(z)}{D_{open}(n)} \\ &\sim 18r^2+30r+12-\frac{(180r^4+699r^3+1035r^2+690r+174)}{2n} \\ &\quad +\frac{(356832r^6+2127600r^5+5631120r^4+8354880r^3+7220493r^2+3401355r+675390)}{1280n^2} \\ &\quad +... \end{aligned}$$

■

6.6 Observation

In this Section we give an enumerative comparison between the asymptotic and exact average number of visits to height r, together with their corresponding variances, by a simple Motzin path with an open ending at $(n,-)$. The exact results were found using computer algebra and the asymptotic results were calculated using the formula for $m_{(1)}(n)$ given in Corollary 86.

		$m_{(1)}(n)$			**Variances**		
n	**Results type**	$r=1$	$r=3$	$r=6$	$r=1$	$r=3$	$r=6$
222	expected	5.809	10.589	14.447	26.220	81.453	112.858
222	*asympt.*	5.809	10.596	14.729	26.247	85.266	353.594
367	expected	5.883	11.120	16.761	27.640	98.117	181.443
367	*asympt.*	5.883	11.121	16.121	27.647	99.018	242.631
593	expected	5.927	11.445	18.198	78.512	109.680	246.091
593	*asympt.*	5.927	11.445	18.215	28.514	109.905	261.951
712	expected	5.939	11.536	18.635	28.754	113.086	268.200
712	*asympt.*	5.939	11.536	18.644	28.756	113.219	277.601
844	expected	5.949	11.607	18.984	28.945	115.826	287.070
844	*asympt.*	5.949	11.607	18.989	28.946	115.908	292.833
1001	expected	5.957	11.667	19.284	29.107	118.200	305.063
1001	*asympt.*	5.957	11.667	19.287	29.108	118.256	307.780

Table 6.2: Average r-visits by an open ended Motzkin walk.

The above Table shows that the asymptotic and the exact results are the same for small values of r as n grows larger.

6.7 Return statistics

Here we analyze the number of times an open ended walk returns to the x-axis by an arbitrary Motzkin random starting at 0 and has an open ending at $(n,-)$. We use a different approach from [14] to return statistics. The recursion for the special system when $i=0$ for the system of the δ_i's described in **??** can be rewritten as

$$\delta_0 = 1 + zw\delta_1 + zw\delta_0,$$

where the variable w is counting the number of returns to the origin. The other two recursions $\delta_{i,h}(z)$ and $\delta_{h,h}(z)$ remain unchanged. These recursions are expressed collectively in a single

matrix equation

$$\begin{bmatrix} 1-zw & -zw & & & & & & & \\ -z & 1-z & -z & & & & 0 & & \\ & -z & 1-z & -z & & & & & \\ & & -z & 1-z & -z & & & & \\ & & & . & . & . & & & \\ & & & & . & . & . & & \\ & & & & & . & . & . & \\ & & 0 & & & . & . & . & \\ & & & & & & . & . & . \\ & & & & & & -z & 1-z & z \\ & & & & & & & -z & 1-z \end{bmatrix} \begin{bmatrix} \delta_{0,h} \\ \delta_{1,h} \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ \delta_{h-1,h} \\ \delta_{h,h} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ 0 \\ 0 \end{bmatrix} \tag{6.1}$$

As before we calculate the $\delta_{i,h}$'s using Cramer's rule by first finding the determinant of the above matrix system, denoted by $\det C_h$ with $h+1$ rows.

Lemma 88 *The determinant of the system of the $\delta_{i,h}$'s with $h+1$ rows is given by*

$$\det C_h = \frac{\left(1+v+v^2\right)\left(1-v^{2h+2}\right) - vw\left(1+v\right)\left(1-v^{2h+1}\right)}{\left(1-v^2\right)\left(1+v+v^2\right)^{h+1}},$$

where w is counting the number of returns to the origin.

Proof. Applying the formula for $\det A_h$ derived in the proof of Lemma 74 we get

$$\begin{aligned} \det C_h &= (1-zw)\det A_h - z^2 w \det A_{h-1} \\ &= \left(1 - \frac{vw}{1+v+v^2}\right)\frac{1-v^{2h+2}}{\left(1-v^2\right)\left(1+v+v^2\right)^h} - \frac{v^2 w\left(1-v^{2h}\right)}{\left(1-v^2\right)\left(1+v+v^2\right)^{h+1}} \\ &= \frac{\left(1+v+v^2\right)\left(1-v^{2h+2}\right) - vw\left(1+v\right)\left(1-v^{2h+1}\right)}{\left(1-v^2\right)\left(1+v+v^2\right)^{h+1}}. \end{aligned}$$

■

Theorem 89 *The generating function for Motzkin paths returns to the origin, which is bounded above by the line $h > 0$, and has an open ending at $(n, -)$ is given by*

$$\delta_{i,\infty}(z,w) = \frac{v^i}{1 - w\left(\frac{v+v^2}{1+v+v^2}\right)},$$

where $z = \frac{v}{1+v+v^2}$ and w is counting the number of returns to the origin.

Proof. Applying Cramer's rule onto the linear system given in (6.1) for all $i \geq 0$ we get the generating function

$$\begin{aligned}
\delta_{i,h}(z,w) &= z^i \frac{\det A_{h-i}}{\det C_h} \\
&= \left(\frac{v}{1+v+v^2}\right)^i \frac{1}{(1-v^2)} \frac{1-v^{2h-2i+2}}{(1+v+v^2)^{h-i}} \\
&\quad \times \frac{(1+v+v^2)^{h+1}(1-v^2)}{(1+v+v^2)(1-v^{2h+2}) - wv(1+v)(1-v^{2h+1})} \\
&= \frac{v^i\left(1-v^{2h-2i+2}\right)(1+v+v^2)}{(1+v+v^2)(1-v^{2h+2}) - wv(1-v^{2h+1})},
\end{aligned}$$

so that

$$\begin{aligned}
\delta_{i,\infty}(z,w) &= \frac{v^i(1+v+v^2)}{(1+v+v^2) - w(v+v^2)} \text{ as } h \to \infty \\
&= \frac{v^i}{1 - w\left(\frac{v+v^2}{1+v+v^2}\right)}.
\end{aligned}$$

■

Remark 19 *We test $\delta_{i,\infty}(z,w)$ by plugging in $w = 1$ into $\sum_{i\geq 0} \delta_{i,\infty}(z,1)$:*

$$\sum_{i\geq 0} \delta_{i,\infty}(z,1) = \frac{(1+v+v^2)}{(1-v)}$$

which is the generating function for the total number of all Motzkin paths with open ending already derived in Remarks 15.

Proposition 90 *The generating function of the s-factorial moments multiplied by the number of paths of length n in $\delta_{i,\infty}(z,w)$ is given by*

$$M_s(z) = \frac{s! v^s (1+v)^s (1+v+v^2)}{(1-v)},$$

where $z = \frac{v}{1+v+v^2}$.

Proof. Now

$$\begin{aligned} M_{(s)}(z) &= \frac{\partial^s}{\partial w^s}\delta_{i,\infty}(z)\Big|_{w=1} \\ &= \frac{s!v^i\left(\frac{v+v^2}{1+v+v^2}\right)^s}{\left(1-\frac{v+v^2}{1+v+v^2}\right)^{s+1}} \\ &= s!v^{i+s}(1+v)^s\left(1+v+v^2\right). \end{aligned}$$

Now summing up the above generating function for all $i \geq 0$ we get the s-moments generating function for an open ending at (n,i) given by

$$M_{(s)}(z) = \frac{s!v^s(1+v)^s(1+v+v^2)}{(1-v)}.$$

■

Since our return statistics to the origin are asymptotic ones, we apply [8] and [20] to expand our generating functions around the dominant singularity $\frac{1}{3}$.

Corollary 91 *The average number of returns to the origin by a Motzkin random walk with an open ending is given by*

$$m_{(1)}(n) \sim 2 - \frac{6}{n} + \frac{4609}{256n^2} - \frac{111195}{2048n^3} + ...$$

Proof. Specializing on $s=1$ into Proposition 90 we

$$M_{(1)}(z) = \frac{v(1+v)(1+v+v^2)}{(1-v)}.$$

By making the substitution $z = \frac{v}{1+v+v^2}$ and expanding $M_{(1)}(z)$ around $z = \frac{1}{3}$ we get

$$\begin{aligned} M_{(1)}(z) &\sim \frac{6}{\sqrt{3}\sqrt{1-3z}} - 12 + \frac{55\sqrt{3}\sqrt{1-3z}}{4} - \frac{117(1-3z)}{3} + \frac{2127\sqrt{3}(1-3z)^{\frac{3}{2}}}{64} \\ &\quad + \frac{30983\sqrt{3}(1-3z)^{\frac{5}{2}}}{512} + ..., \end{aligned}$$

so that

$$\begin{aligned} [z^n]M_{(1)}(z) &\sim (-3)^n\left[\frac{6}{\sqrt{3}}\binom{-\frac{1}{2}}{n} + \frac{55\sqrt{3}}{4}\binom{\frac{1}{2}}{n} + \frac{2127\sqrt{3}}{64}\binom{\frac{3}{2}}{n} + ...\right] \\ &= 3^n\left[\frac{3}{\sqrt{3}2^{2n-1}}\binom{2n}{n} + \frac{55\sqrt{3}}{2^{2n+1}n}\binom{2n-2}{n-1} + \frac{6381\sqrt{3}}{2^{2n+4}(n-2)(n-3)}\binom{2n-4}{n-4} + ...\right] \\ &\sim 3^n\sqrt{\frac{3}{\pi n}}\left(2 - \frac{57}{8n} + \frac{5725}{256n^2} + \frac{92985}{2048n^3} + ...\right). \end{aligned}$$

Therefore

$$\begin{aligned} m_{(1)}(n) &= \frac{[z^n] M_{(1)}(z)}{T(n)} \\ &\sim 2 - \frac{6}{n} + \frac{4609}{256n^2} - \frac{111195}{2048n^3} + ..., \end{aligned}$$

which is the average number of returns to the 0-level. ■

Corollary 92 *The second moments on $\delta_{i,\infty}(z)$ are given by*

$$m_{(2)}(n) \sim 8 - \frac{75}{n} + \frac{31465}{64n^2} - \frac{731325}{256n^3} + ...$$

Proof. Specializing on $s = 2$ into Proposition 90 we get

$$M_{(2)}(z) = \frac{2v^2(1+v)^2(1+v+v^2)}{(1-v)}.$$

By making the usual substitution $z = \frac{v}{1+v+v^2}$ and expanding $M_{(2)}$ around $z = \frac{1}{3}$ we get

$$\begin{aligned} M_{(2)}(z) \sim\ & \frac{24}{\sqrt{3}\sqrt{1-3z}} - 84 + 157\sqrt{3}\sqrt{1-3z} - 651(1-3z) + \frac{12099\sqrt{3}(1-3z)^{\frac{3}{2}}}{16} \\ & -2352(1-3z)^2 + \frac{287969\sqrt{3}(1-3z)^{\frac{5}{2}}}{128} + ..., \end{aligned}$$

so that

$$\begin{aligned} [z^n] M_{(2)}(z) &= (-3)^n \left[\frac{12}{\sqrt{3}} \binom{-\frac{1}{2}}{n} + \frac{157\sqrt{3}}{2} \binom{\frac{1}{2}}{n} + \frac{12099\sqrt{3}}{32} \binom{\frac{3}{2}}{n} \right] \\ &= 3^n \left[\frac{3}{2^{2n-2}\sqrt{3}} \binom{2n}{n} - \frac{157\sqrt{3}}{2^{2n}n} \binom{2n-2}{n-1} + \frac{36297\sqrt{3}}{2^{2n+3}(n-2)(n-3)} \binom{2n-4}{n-4} + ... \right] \\ &\sim 3^n \sqrt{\frac{3}{\pi n}} \left(8 - \frac{159}{2n} + \frac{34417}{64n^2} + \frac{3246285}{1024n^3} + ... \right). \end{aligned}$$

Hence

$$\begin{aligned} m_{(2)}(n) &= \frac{[z^n] M_{(2)}(z)}{T(n)} \\ &\sim 8 - \frac{75}{n} + \frac{31465}{64n^2} - \frac{731325}{256n^3} + ... \end{aligned}$$

■

6.8 Observation

Table 6.3 serves to validate that

$$\frac{expected\ (n)}{asympt\ (n)} \to 1$$

where $expected\ (n)$ and $asympt\ (n)$ are exact and asymptotic results respectively. The exact results were found using computer algebra and the asymptotic results were calculated using the formula for $m_{(1)}(n)$ given in Corollary 91.

n	$m_1(n)$		**Variance**	
	expected	*asympt.*	*expected*	*asympt.*
50	1.887	1.888	1.665	1.668
101	1.942	1.942	1.821	1.823
174	1.966	1.966	1.893	1.893
222	1.973	1.973	1.915	1.915
367	1.984	1.984	1.948	1.948
593	1.990	1.990	1.968	1.968
712	1.992	1.992	1.973	1.973
844	1.993	1.993	1.977	1.977
1001	1.994	1.994	1.981	1.981

Table 6.3: Average 0-level returns by an open ended Motzkin walk.

The above Table shows that the asymptotic and the exact results are the same for small values of r as n grows larger.

Chapter 7

Unrestricted Motzkin random walks

7.1 Introduction

Katzenbeisser and Panny in [14] derived a formular for calculating average number of visits to the nonzero height r by an arbitrary unrestricted Motzkin random walk where the approch was purely probabilistic. It is the aim of this paper to re-do this problem using an approach which will allow an extension these results into a larger class of walks.

Throughout this Chapter, all unrestricted Motzkin random walks will be called random walks unless otherwise stated. The average number of returns to the origin by an arbitrary walk will be researched into as well. The following figure shows an arbitrary random walk which starts at the origin, ends at n and has an open ending at (n, i):

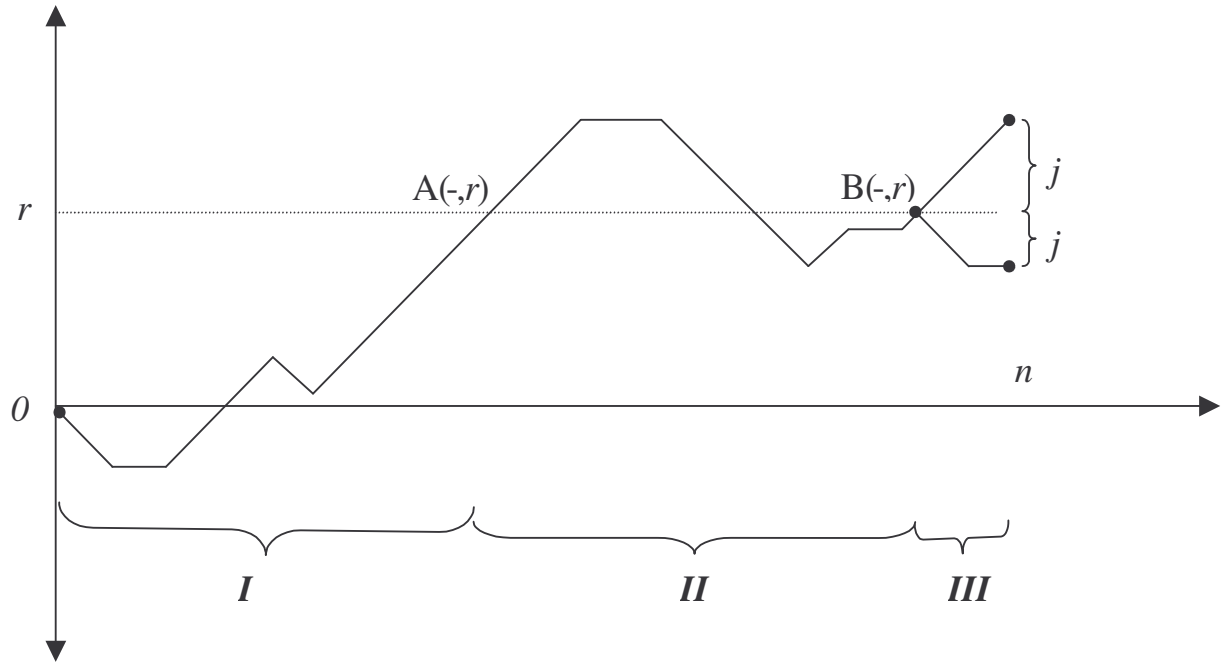


Figure 7.1: Open ended unrestricted Motzkin path of length n

Now the number of paths of length n for all i where $-k \leq i < h$ is given by:

$$\varphi_{i;\,h,k}(z) = \sum_{n\geq 0} \left[\begin{array}{l} \text{Number of paths of length } n, \text{ starting at } 0, \text{bounded above,} \\ \text{by the line } h \text{ below by } -k, \text{leading to } (n,i)\,. \end{array} \right] z^n,$$

where z counts the single steps. For all i between $-k$ and h we describe the step from level i to the next level by the following diagram:

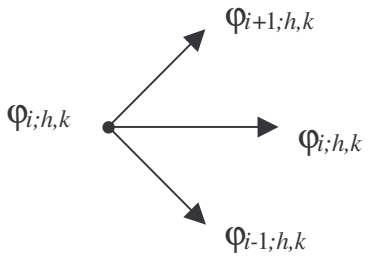


Figure 7.2

which gives recursions

$$\varphi_{i;\,h,k}(z) = z\varphi_{i-1;h,k}(z) + z\varphi_{i;\,h,k}(z) + z\varphi_{i+1;h,k}(z)\,, \text{ for } -k < i < h \text{ and } i \neq 0$$

and

$$\varphi_{i;\,h,k}(z) = z\varphi_{i-1;h,k}(z) + z\varphi_{i;\,h,k}(z) + z\varphi_{i+1;h,k}(z) + 1, \text{ for } i = 0.$$

Recursion for special systems are given by

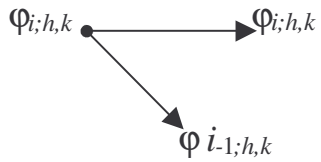


Figure 7.3

so that

$$\varphi_{i;\,h,k}(z) = z\varphi_{i-1;h,k}(z) + z\varphi_{i;\,h,k}(z)$$

where $i = h$; and

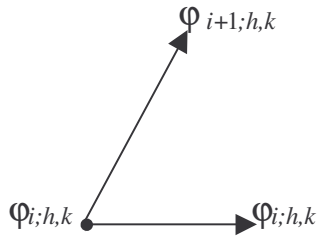


Figure 7.4

so that

$$\varphi_{i;\,h,k}(z) = z\varphi_{i+1;h,k}(z) + z\varphi_{i;\,h,k}(z),$$

where $i = k$. These recursions, for all i where $-k \leq i \leq h$ are best described by a single matrix equation

$$B_{h+k}\Phi_{h+k} = C_{h+k},$$

where the coefficient matrix B_{h+k} is a square matrix with $h+k+1$ rows, Φ_{h+k} is the column matrix representing the $\varphi_{i;\,h,k}(z)$'s and C_{h+k} is the column matrix of $h+k$ zeros and a 1 in the middle (or the $h+1$) entry. Thus we have the following matrix system:

$$\begin{bmatrix} 1-z & -z & & & & & & & \\ -z & 1-z & -z & & & & 0 & & \\ & . & . & . & & & & & \\ & & . & . & . & & & & \\ & & & -z & 1-z & -z & & & \\ & & & & -z & 1-z & -z & & \\ & & & & & & & & \\ & & & & . & . & . & & \\ & & 0 & & & . & . & . & \\ & & & & & & -z & 1-z & -z \\ & & & & & & & -z & 1-z \end{bmatrix} \begin{bmatrix} \varphi_{h;h,k} \\ \varphi_{h-1;h,k} \\ . \\ . \\ \varphi_{1;h,k} \\ \varphi_{0;h,k} \\ \varphi_{-1;h,k} \\ . \\ . \\ . \\ \varphi_{-k;h,k} \end{bmatrix} = \begin{bmatrix} 0 \\ . \\ . \\ . \\ 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} \tag{7.1}$$

The $\varphi_{i;h,k}$'s satisfy the system that has $h+k+1$ rows; this is only true for $h+k \geq 2$. The special values $h+k=0$ and $h+k=1$ are treated separately. The calculation of the $\varphi_{i;h,k}$'s is done using Cramer's rule.

Lemma 93 *The determinant of the system of the $\varphi_{i;h,k}$'s, denoted by $\det A_{h+k+1}$, with $h+k+1$ rows is given by*

$$\det A_{h+k+1} = \frac{1}{(1-v^2)} \frac{1-v^{2(h+k)+4}}{(1+v+v^2)^{h+k+1}}.$$

Proof. Replacing h by $h+k+1$ in the proof Lemma 74 of Chapter 6 we get

$$\det A_{h+k+1} = \frac{1}{(1-v^2)} \frac{1-v^{2(h+k)+4}}{(1+v+v^2)^{h+k+1}}.$$

as required. ■

Theorem 94 *The generating function a random walk bounded above and below by the lines*

$h > 0$ and $-k < 0$ respectively, and is ending at (n, i) is given by

$$\varphi_{i;h,k}(z) = \begin{cases} v^i \frac{\left(1+v+v^2\right)}{\left(1-v^2\right)} \frac{\left(1-v^{2h-2i+2}\right)\left(1-v^{2k+2}\right)}{\left(1-v^{2h+2k+4}\right)} \text{ for } i \geq 0 \\ \\ v^{|i|} \frac{\left(1+v+v^2\right)}{\left(1-v^2\right)} \frac{\left(1-v^{2h+2}\right)\left(1-v^{2k-2|i|+2}\right)}{\left(1-v^{2h+2k+4}\right)} \text{ for } i \leq 0 \end{cases},$$

where $z = \frac{v}{1+v+v^2}$.

Proof. The proof of this Theorem is an exact analogue of the proof of Theorem 28. Therefore it will not be repeated here. ■

Remark 20 *If we unrestrict the generating function $\varphi_{i;h,k}(z)$, either from above or below, then Theorem 94 can be simplified as*

$$\varphi_{i;\infty,k}(z) = \begin{cases} v^i \frac{\left(1+v+v^2\right)}{1-v^2} \left(1 - v^{2k+2}\right) \text{ for } i \geq 0 \\ \\ v^{|i|} \frac{\left(1+v+v^2\right)}{1-v^2} \left(1 - v^{2k-2|i|+2}\right) \text{ for } i \leq 0 \end{cases},$$

and

$$\varphi_{i;h,\infty}(z) = \begin{cases} v^i \frac{\left(1+v+v^2\right)}{1-v^2} \left(1 - v^{2h-2i+2}\right) \text{ for } i \geq 0 \\ \\ v^{|i|} \frac{\left(1+v+v^2\right)}{1-v^2} \left(1 - v^{2h+2}\right) \text{ for } i \leq 0 \end{cases},$$

so that

$$\begin{aligned} \sum_i \varphi_{i;\infty,-\infty}(z) &= \frac{(1+v+v^2)}{1-v^2} \left(2 \sum_{i \geq 1} v^i + 1\right) \\ &= \frac{(1+v+v^2)(1+v)}{(1-v^2)(1-v)} \\ &= \frac{(1+v+v^2)}{(1-v)^2} \\ &= \frac{1}{1-3z}, \text{ using the substitution } z = \frac{v}{1+v+v^2} \\ &= \sum_{n \geq 0} 3^n z^n. \end{aligned}$$

Thus

$$[z^n] \sum_i \varphi_{i;\infty,\infty}(z) = 3^n,$$

which is the required total number of random walks with an open ending at (n, i). ■

7.2 Generating functions for the number of visits to the r-level.

In order to get a suitable generating function for the entire walk from $(0,0)$ to (n,i) we decompose this random walk in Figure 7.1 using symbolic method in [9] into three parts corresponding to the following three Lemmas:

Lemma 95 *The generating function of any random walk starting from* $(0,0)$ *and is reaching* r *for the first time is given by* $z\varphi_{r-1;r-1,-\infty}(z)$.

Proof. This random walk starts at $(0,0)$ and end at $\mathrm{A}(-,r)$. This walk can be decomposed into a single z-step and $\varphi_{r-1;r-1,-\infty}(z)$; and hence the results. ■

Lemma 96 *The generating function of a random walk which starts and ends on* r *is given by*

$$F_r(z,u) = \frac{u}{1-u\left(\frac{v+2v^2}{1+v+v^2}\right)},$$

where $z = \frac{v}{1+v+v^2}$ *and* u *counts the number of visits to the* r*-level.*

Proof. This is the random walk that starts at $\mathrm{A}(r,-)$ and ends at $\mathrm{B}(r,-)$ as shown in part III of Fig.7.1. We shall treat height r as the 0-level to easy our proof . Let's denote the generating function of this random walk by $F_r(z,u)$, where u is counting the number of returns to the r-level. Furthermore, let $F_{r,+}(z)$, $F_{r,0}(z)$ and $F_{r,-}(z)$ denote the generating functions of random walks which are above the 0-level, strictly horizontal on 0-level and below the 0-level respectively. Between any two returns on the 0-level, we have a sequence of paths above, below and on the r-axis, which we denote symbolically as get

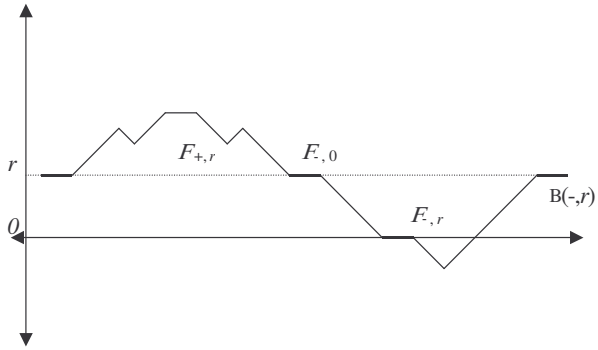


Figure 7.1: Figure 7.5

$F_r(z,u) = (F_{r,+}(z) \cup F_{r,-}(z) \cup F_{r,0}(z))^*$ as shown in the following diagram:

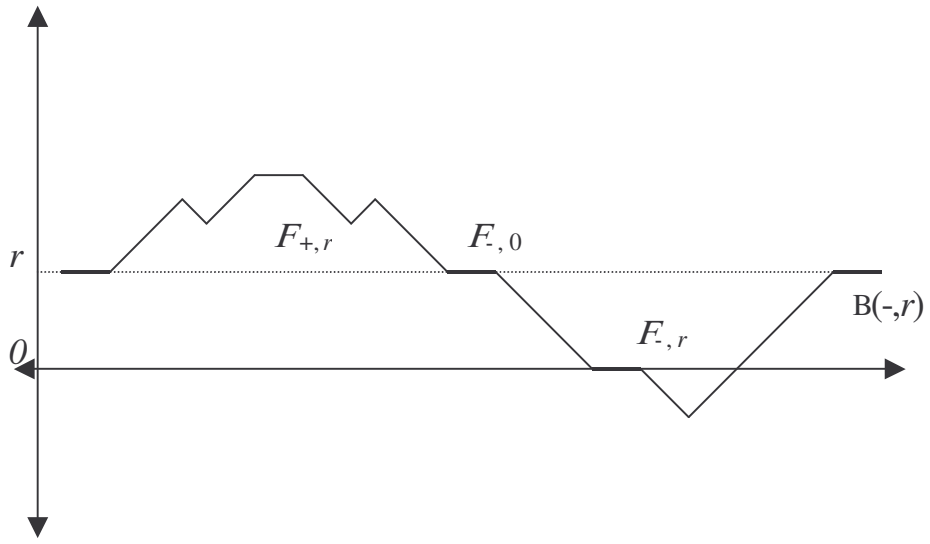


Figure 7.5

Hence

$$F_r(z,u) = \frac{u}{1 - u\left(F_{r,+}(z) + F_{r,-}(z) + F_{r,0}(z)\right)}.$$

Using the formula for $\varphi_{i;h,k}(z)$ we decompose $F_{r,+}(z)$ using the procedure used in Lemma 30 of Chapter 3 to get

$$F_{r,+}(z) = z^2 \varphi_{0;\infty,0}(z).$$

By symmetry, we have that $F_{r,-}(z) = F_{r,+}(z)$ so that

$$F_r(z,u) = \frac{u}{1 - u\left[z^2(\varphi_{0;\infty,0}(z)) + z^2(\varphi_{0;0,\infty}(z)) + z\right]}, \text{ where } z = F_{r,0}(z)$$

$$\begin{aligned} &= \frac{u}{1-u\left(2z^2\varphi_{0;\infty,0}\left(z,u\right)+z\right)}, \text{ since } \varphi_{0;\infty,0}=\varphi_{0;0,\infty} \text{ by symmetry} \\ &= \frac{u}{1-u\left(\frac{v+2v^2}{1+v+v^2}\right)}, \text{ since } \varphi_{0;\infty,0}\left(z\right)=\left(1+v+v^2\right). \end{aligned}$$

■

7.3 Closed ending random walks.

Theorem 97 *The generating function of any random walk starting at* $(0,0)$ *and ending at* $(n,0)$ *is given by*

$$\Delta_{r,closed}\left(z,u\right)=\frac{\left(1+v+v^2\right)\left(1-v^{2r}\right)}{1-v^2}+\frac{uv^{2r}}{1-u\left(\frac{v+2v^2}{1+v+v^2}\right)},$$

where $z=\frac{v}{1+v+v^2}$ *and* u *is counting the number of visits to the* r*-level.*

Proof. We decompose a random walk starting from $(0,0)$ and ending at $(n,0)$ into three parts corresponding to the three parts in Fig. 7.1. The generating functions for parts I and II were already derived in Lemmas 95 and 96 respectively. However the generating functions for part I and III are symmetrical (because $r+j=0$) so that the overall generating function for the entire random walk is given by:

$$\begin{aligned} \Delta_{r,closed}\left(z,u\right) &= \varphi_{0;r-1,\infty}(z)+z\varphi_{r-1;r-1,\infty}(z)\times F_r(z,u)\times z\varphi_{r-1;r-1,\infty}(z) \\ &= \frac{\left(1+v+v^2\right)\left(1-v^{2r}\right)}{1-v^2}+\frac{uv^{2r}}{1-u\left(\frac{v+2v^2}{1+v+v^2}\right)}, \end{aligned}$$

where $\varphi_{0;r-1,\infty}(z)$ are the random walk not reaching r. ■

Remark 21 *We test the consistancy of* $\Delta_{r,closed}\left(z,u\right)$ *by plugging in* $u=1$ *to get*

$$\begin{aligned} \Delta_{r,closed}\left(z,1\right) &= \frac{\left(1+v+v^2\right)\left(1-v^{2r}\right)}{1-v^2}+\frac{v^{2r}}{1-\left(\frac{v+2v^2}{1+v+v^2}\right)} \\ &= \frac{\left(1+v+v^2\right)}{1-v^2}. \end{aligned}$$

And so

$$B\left(n\right) = \left[z^n\right]\Delta_{r,closed}\left(z,1\right)$$

$$\begin{aligned} &= \frac{1}{2\pi i}\oint \frac{(1-v^2)\left(1+v+v^2\right)^{n-1}}{v^{n+1}}\Delta_{r,closed}(z,1)\,dv \\ &= [v^n]\left(1+v+v^2\right)^n \\ &= \binom{n,3}{n} \\ &\sim 3^n\sqrt{\frac{3}{\pi n}}\left(\frac{1}{2}-\frac{3}{32n}+...\right) using\ \ [20], \end{aligned}$$

which is the required total number of random walks with a closed ending at $(n,0)$. ■

Proposition 98 *The generating function of the s-factorial moments multiplied by the number of walks of length n in* $\Delta_{r,closed}(z,u)$ *is given by*

$$M_{(s)}(z) = s!\left(v+2v^2\right)^{s-1}\frac{\left(1+v+v^2\right)^2}{\left(1-v^2\right)^{s+1}}v^{2r},$$

where $z=\frac{v}{1+v+v^2}$.

Proof. Now

$$\begin{aligned} M_{(s)} &= \frac{\partial^s}{\partial u^s}\Delta_{r,closed}(z,u)\Big|_{u=1} \\ &= \left[\frac{s!\left(\frac{v+2v^2}{1+v+v^2}\right)^s}{\left(\frac{1-v^2}{1+v+v^2}\right)^{s+1}}+\frac{s!\left(\frac{v+2v^2}{1+v+v^2}\right)^{s-1}}{\left(\frac{1-v^2}{1+v+v^2}\right)^{s}}\right]v^{2r} \\ &= s!\left(v+2v^2\right)^{s-1}\frac{\left(1+v+v^2\right)^2}{\left(1-v^2\right)^{s+1}}v^{2r}. \end{aligned}$$

■

We are now ready to calculate $m_1(n)$ and $m_2(n)$ with the help of [8] and [20].

Corollary 99 *The average number of visits to level r by an arbitrary random walk with closed ending is given by*

$$\begin{aligned} m_1(n) \sim\ & \sqrt{3\pi n}\left(\frac{1}{2}+\frac{3}{32n}+\frac{9}{512n^2}+...\right)-3r+\frac{(3r+12r^3)}{4n} \\ & +\frac{(3423r+4320r^3+6912r^6+15840r^4)}{2560n^2}, \end{aligned}$$

Proof. Substituting $s=1$ into Proposition 98 we obtain

$$M_{(1)}(z)=\frac{v^{2r}\left(1+v+v^2\right)^2}{\left(1-v^2\right)^2}.$$

Substituting that $z=\frac{v}{1+v+v^2}$ into $M_{(1)}(z)$ and expanding the series around $z=\frac{1}{3}$ we get

$$\begin{aligned}M_{(1)}(z) &\sim \frac{3}{4(1-3z)}-\frac{9r}{2\sqrt{3}\sqrt{1-3z}}+\frac{(72r^2+3)}{16}-\frac{3r\left(16r^2+5\right)\sqrt{3}\sqrt{1-3z}}{16}\\ &+\frac{3\left(96r^4+96r^2+1\right)}{64}+\frac{3\left(768r^5+1760r^3+247r\right)\sqrt{3}\left(1-3z\right)^{\frac{3}{2}}}{1280}+...,\end{aligned}$$

so that

$$\begin{aligned}[z^n]M_{(1)}(z) &\sim (-3)^n\left[\frac{3}{4}\binom{-1}{n}-\frac{9r}{2\sqrt{3}}\binom{-\frac{1}{2}}{n}-\frac{3\sqrt{3}\left(16r^2+5\right)r}{16}\binom{\frac{1}{2}}{n}\right.\\ &\left.+\frac{3\sqrt{3}\left(768r^5+1760r^3+247r\right)}{1280}\binom{\frac{3}{2}}{n}+...\right]\\ &= 3^n\left[\frac{3}{4}-\frac{9r}{2^{2n+1}\sqrt{3}}\binom{2n}{n}+\frac{3\sqrt{3}\left(16r^2+5\right)r}{2^{2n+3}n}\binom{2n-2}{n-1}\right.\\ &\left.+\frac{9\sqrt{3}\left(768r^5+1760r^3+247r\right)}{5\left(n-2\right)\left(n-3\right)2^{2n+6}}\binom{2n-4}{n-4}+...\right]\\ &\sim 3^n\left[\frac{3}{4}-\sqrt{\frac{3}{\pi n}}\left(\frac{3r}{2}-\frac{3r^3}{2n}-\frac{21r}{32n}-\frac{3063r}{5120n^2}\right.\right.\\ &\left.\left.-\frac{9r^3}{16n^2}-\frac{27r^6}{20n^2}-\frac{99r^4}{32n^2}+...\right)\right],\end{aligned}$$

where the last asymptotic results was done using computer algebra. Using the total number of objects $B(n)$ calculated in Remark 21 we get

$$\begin{aligned}m_1(n) &\sim \sqrt{3\pi n}\left(\frac{1}{2}+\frac{3}{32n}+\frac{9}{512n^2}+...\right)-3r+\frac{(3r+12r^3)}{4n}\\ &+\frac{(3423r+4320r^3+6912r^6+15840r^4)}{2560n^2},\end{aligned}$$

is the required average number of visits to level r. ■

Corollary 100 *The second moments on $\Delta_r(z,u)$ are given by*

$$m_2(n)\sim 3n-\sqrt{3\pi n}\left(1+3r+\frac{9r+3}{16n}+...\right)+9r^2+6r+\frac{9}{4}.$$

Proof. Substituting $s = 2$ into Proposition 98 we get

$$M_{(2)}(z) = \frac{2v^{2r}(v+2v^2)(1+v+v^2)^2}{(1-v^2)^3}.$$

Substituting $z = \frac{v}{1+v+v^2}$ into $M_{(2)}(z)$ and expanding the series around $z = \frac{1}{3}$:

$$\begin{aligned} M_{(2)}(z) \sim\ & \frac{9}{4\sqrt{3}(1-3z)^{\frac{3}{2}}} - \frac{3(3r+1)}{2(1-3z)} + \frac{9(48r^2+32r+3)}{32\sqrt{3}\sqrt{1-3z}} - \frac{(72r^3+72r^2+27r+3)}{8} \\ & - \frac{3(288r^5+480r^4+720r^3+480r^2+117r+5)(1-3z)}{160} \\ & + \frac{3\sqrt{3}(768r^4+1024r^3+864r^2+320r+15)\sqrt{1-3z}}{512} + ... \end{aligned}$$

Therefore

$$\begin{aligned} [z^n] M_{(2)}(z) \sim\ & (-3)^n \left[\frac{9}{4\sqrt{3}} \binom{-\frac{3}{2}}{n} - \frac{3(3r+1)}{2} \binom{-1}{n} + \frac{9(48r^2+32r+3)}{32\sqrt{3}} \binom{-\frac{1}{2}}{n} \right. \\ & \left. - \frac{3\sqrt{3}(768r^4+1024r^3+864r^2+320r+15)}{512} \binom{\frac{1}{2}}{n} + ... \right] \\ =\ & 3^n \left[\frac{9(2n+2)}{2^{2n+3}\sqrt{3}} \binom{2n+1}{n+1} - \frac{3(3r+1)}{2} + \frac{9(48r^2+32r+3)}{2^{2n+5}\sqrt{3}} \binom{2n}{n} \right. \\ & \left. + \frac{3\sqrt{3}(768r^4+1024r^3+864r^2+320r+15)}{2^{2n+8}n} \binom{2n-2}{n-1} + ... \right] \\ \sim\ & 3^n \left[\sqrt{\frac{3n}{\pi}} \left(\frac{3}{2} + \frac{27}{32n} + \frac{9r^2}{2n} + \frac{3r}{n} + ... \right) - \frac{3(3r+1)}{2} \right], \end{aligned}$$

where the last asymptotic results was done using computer algebra. Using the total number of objects for a random walk with closed ending calculated in Remark 21 we get

$$\begin{aligned} m_2(n) =\ & \frac{[z^n] M_{(2)}(z)}{B(n)} \\ \sim\ & 3n - \sqrt{3\pi n} \left(1 + 3r + \frac{3}{16n} + \frac{9r}{16n} + ... \right) + 9r^2 + 6r + \frac{9}{4}. \end{aligned}$$

■

7.4 Observation

As usual we now give an enumerative comparison between the asymptotic and exact average number of visits to height r, together with their corresponding variances, by an unrestricted Motzkin path with a closed ending at $(n, 0)$. The exact results were found using computer algebra and the asymptotic results were calculated using the formula for $m_{(1)}(n)$ given in Corollary 99.

n	**Type of**	$m_{(1)}(n)$			**Variances**		
	results	r=1	r=3	r=6	r=1	r=3	r=6
222	expected	19.9069	14.2499	7.4338	123.6776	118.1490	81.0813
222	*asympt.*	19.9072	14.0275	5.8059	123.7538	126.7889	130.8269
367	expected	26.4314	20.6424	13.0476	210.6483	206.8344	169.4427
367	*asympt.*	26.4315	20.5009	11.8723	210.6964	214.0915	216.3499
593	expected	34.3976	28.5295	20.4319	348.2950	346.0704	311.5438
593	*asympt.*	34.3976	28.4394	19.6333	348.3280	352.0965	354.6630
712	expected	37.9747	32.0848	23.8442	421.3684	419.7178	386.8086
712	*asympt.*	37.9747	32.0092	23.1627	421.3959	425.3083	428.1590

Table 7.1: Average r-level visits by closed ended unrestricted Motzkin walks.

Both enumerative and asymptotic results shows that a closed ended unrestricted Dyck walk frequent the r-level less times than the unrestricted Motzkin walk. The expected and the asymptotic values are almost the same particularly when $n \to \infty$.

7.5 Random walks with open endings.

Lemma 101 *The generating function of any random walk which starts at the r-level and does not return to the r-level is given by*

$$\begin{cases} z\varphi_{j-1;\infty,0}(z) & \text{when } j \text{ is above the } r\text{-level as } h \to \infty \\ z\varphi_{1-j;0,\infty}(z) & \text{when } j \text{ is below the } r\text{-level as } k \to \infty \end{cases}.$$

Proof. This is the generating function of a random walk which starts at $\mathrm{B}(-,r)$, never returns to the r-level, and it either ends at $(n,j+r)$ or $(n,r-j)$ as shown in Fig. 7.1. Treating height r as the 0-level we prove these two cases separately:

Case 1. *The random walk that ends at* $(n,r+j)$

If we let this random walk starts at $r+1$, then it will end at $j-1$. Thus assuming that height $r+1$ is the 0-level we see that this random walk is bounded below by 0 and above by ∞. Symbolically, we can write the generating function for this case as $z\varphi_{j-1;\infty,0}(z)$ as $h\to\infty$ after decomposition. The z in this product is for the single step that separate $\varphi_{j-1;\infty,0}(z)$ and the line r.

Case 2. *Random walk ending at* $(n,r-j)$:

If we let this random walk starts at $r-1$, then it will end at $1-j$. Therefore assuming that height $r-1$ is the 0-level we see that this random walk is bounded below by 0 and above by ∞. Symbolically, we can write the generating function for this case as $z\varphi_{1-j;0,\infty}(z)$ as $k\to\infty$. The z in this product is for the single step that separate $\varphi_{1-j;0,\infty}(z)$ and the line r. ■

Remark 22

$$\begin{aligned}\varphi_{1-j;0,\infty}(z) &= v^{|1-j|}\frac{(1+v+v^2)}{(1-v^2)}\left(1-v^2\right)\\ &= v^{j-1}\left(1+v+v^2\right)\\ &= \varphi_{1-j;\infty,0}(z),\end{aligned}$$

and this is not surprising because $\varphi_{1-j;0,\infty}(z)$ and $\varphi_{1-j;\infty,0}(z)$ are symmetrical. ■

Theorem 102 *The overall generating function for the entire random walk that starts at* $(0,0)$ *and ends at* $(n,r\pm j)$ *is given by*

$$\Omega_{r,open}(z,u)=\frac{(1+v+v^2)(1-v^r)(1+v)}{(1-v^2)(1-v)}+v^r\frac{u}{1-u\left(\frac{v+2v^2}{1+v+v^2}\right)}\left(\frac{1+v}{1-v}\right),$$

where u is counting the number of visits to the r-level and $z = \frac{v}{1+v+v^2}$.

Proof. If j is above the r-level, then the generating function for ending of this random walk is $z\varphi_{j-1;\infty,0}(z)$ (calculated in case 1 of Lemma 101) so that the overall generating function for this case is given by

$$\Omega_{r,above}(z,u) = z\varphi_{r-1;r-1,\infty}(z) \times F_r(z,u) \times z\varphi_{j-1;\infty,0}(z)$$

However

$$\begin{aligned} z\varphi_{r-1;r-1,\infty}(z) &= \left(\frac{v}{1+v+v^2}\right) v^{r-1}\left(1+v+v^2\right) \\ &= v^r, \end{aligned}$$

and

$$\begin{aligned} z\varphi_{j-1;\infty,0}(z) &= \left(\frac{v}{1+v+v^2}\right) v^{i-1}\left(1+v+v^2\right) \\ &= v^j, \end{aligned}$$

so that

$$\begin{aligned} \Omega_{r,above}(z,u) &= v^r \times F_r(z,u) \times v^j \\ &= \frac{uv^{r+j}}{1-u\left(\frac{v+2v^2}{1+v+v^2}\right)}. \end{aligned}$$

By symmetry $\varphi_{1-j;0,\infty}(z) = \varphi_{1-j;\infty,0}(z)$, so that when j is below the r-level we have

$$\Omega_{r,below}(z,u) = \varphi_{r-j;r-1,\infty}(z) + \frac{uv^{r+j}}{1-u\left(\frac{v+2v^2}{1+v+v^2}\right)},$$

where $\varphi_{r-j;r-1,\infty}(z)$ are random walks not reaching r. Combining the generating functions $\Omega_{r,above}(z,u)$, $\Omega_{r,below}(z,u)$ and $v^r F_r(z,u)$; and summing up the overall generating function for all j we get

$$\begin{aligned} \Omega_{r,open}(z,u) &= \sum_j \left(\Omega_{r,above}(z,u) + \Omega_{r,below}(z,u) + v^r F_r(z,u)\right) \\ &= \sum_{k=0}^{r-1} \varphi_{k;r-1,\infty}(z) + \sum_{k\leqslant -1} \varphi_{k;r-1,\infty}(z) \end{aligned}$$

$$
\begin{aligned}
& +z\varphi_{r-1;r-1,\infty}(z)\times F_r(z,u)\left(2z\sum_{j\geq 1}\varphi_{j-1;\infty,0}+1\right)\\
= & \sum_{k=0}^{r-1}\varphi_{k;r-1,\infty}(z)+\sum_{k\leqslant -1}\varphi_{k;r-1,\infty}(z)\\
& +z\varphi_{r-1;r-1,\infty}(z)\times F_r(z,u)\left[\frac{2v}{1-v}+\frac{1-v}{1-v}\right]\\
= & \frac{(1+v+v^2)(1-v^r)(1-v^{r+1})}{(1-v^2)(1-v)}+\frac{(1+v+v^2)(v-v^{2r+1})}{(1-v^2)(1-v)}\\
& +v^r\frac{u}{1-u\left(\frac{v+2v^2}{1+v+v^2}\right)}\left(\frac{1+v}{1-v}\right)\\
= & \frac{(1+v+v^2)(1-v^r)(1+v)}{(1-v^2)(1-v)}+v^r\frac{u}{1-u\left(\frac{v+2v^2}{1+v+v^2}\right)}\left(\frac{1+v}{1-v}\right).
\end{aligned}
$$

■

Remark 23 *We test the correctness of* $\Omega_{r,open}(z,u)$ *by plugging in* $u=1$ *to get*

$$
\begin{aligned}
\Omega_{r,open}(z,u) &= \frac{(1+v+v^2)(1-v^r)(1+v)}{(1-v^2)(1-v)}+\frac{v^r}{1-\left(\frac{v+2v^2}{1+v+v^2}\right)}\left(\frac{1+v}{1-v}\right)\\
&= \frac{(1+v+v^2)}{(1-v)^2}\\
&= \frac{1}{1-3z}\\
&= \sum_{n\geq 0}3^n z^n
\end{aligned}
$$

which is the required generating function for the total number of random walks with an open ending at $(n,-)$. ■

Proposition 103 *The generating function of the* s*-th factorial moments of* $\Omega_{r,open}(z,u)$ *multiplied by the number of paths of length* n *is given by*

$$
M_{(s)}(z)=\frac{s!v^r(v+2v^2)^{s-1}(1+v)(1+v+v^2)^2}{(1-v^2)^{s+1}(1-v)},
$$

$z=\frac{v}{1+v+v^2}$.

Proof. From Theorem 102 we have that

$$\begin{aligned} M_{(s)}(z) &= \frac{\partial^s}{\partial u^s}\left(1-v^2\right)\Big|_{u=1} \\ &= \frac{\partial^s}{\partial u^s}F_r(z,u)\Big|_{u=1}\left(\frac{1+v}{1-v}\right)v^r \\ &= s!\left(v+2v^2\right)^{s-1}\frac{\left(1+v+v^2\right)^2}{\left(1-v^2\right)^{s+1}}\left(\frac{1+v}{1-v}\right)v^r \\ &= \frac{s!v^r\left(v+2v^2\right)^{s-1}(1+v)\left(1+v+v^2\right)^2}{\left(1-v^2\right)^{s+1}(1-v)}, \end{aligned}$$

as required. ■

As usual we employ [8] and [20] to calculate our moments:

Corollary 104 *The average number of visits to the r-level by an arbitrary random walk with an open ending is given by*

$$m_1(n) \sim \sqrt{\frac{3n}{\pi}}\left(1+\frac{7}{16n}+\frac{3r^2}{4n}+...\right)-\frac{3r}{2}.$$

Proof. Substituting $s=1$ into the above Proposition 103 we get

$$M_{(1)}(z)=\frac{v^r\left(1+v+v^2\right)^2}{\left(1-v^2\right)(1-v)^2}.$$

Now, by making the substitution that $z=\frac{v}{1+v+v^2}$ and upon expansion around $z=\frac{1}{3}$ we get

$$\begin{aligned} M_{(1)}(z) \sim\ & \frac{3}{2\sqrt{3}(1-3z)^{\frac{3}{2}}}-\frac{3r}{2(1-3z)}+\frac{3\left(12r^2+1\right)}{16\sqrt{3}\sqrt{1-3z}} \\ & +\frac{3\left(16r^4+56r^2+1\right)\sqrt{3}\sqrt{1-3z}}{256}+... \end{aligned}$$

Therefore

$$\begin{aligned} [z^n]M_{(1)}(z) &\sim (-3)^n\left[\frac{3}{2\sqrt{3}}\binom{-\frac{3}{2}}{n}-\frac{3r}{2}\binom{-1}{n}+\frac{3\left(12r^2+1\right)}{16\sqrt{3}}\binom{-\frac{1}{2}}{n}\right. \\ &\quad \left.+\frac{3\sqrt{3}\left(16r^4+56r+1\right)}{256}\binom{\frac{1}{2}}{n}+...\right] \\ &= 3^n\left[\frac{3(2n+2)}{2^{2n+2}\sqrt{3}}\binom{2n+1}{n+1}-\frac{3r}{2}+\frac{3\left(12r^2+1\right)}{2^{2n+4}\sqrt{3}}\binom{2n}{n}\right. \end{aligned}$$

$$+\frac{3\sqrt{3}\left(16r^4+56r+1\right)}{2^{2n+7}n}\binom{2n-n}{n-1}+...\Bigg]$$
$$\sim 3^n\left[\sqrt{\frac{3n}{\pi}}\left(1+\frac{7}{16n}+\frac{3r^2}{4n}+...\right)-\frac{3r}{2}\right],$$

where the last asymptotic expansion was done using computer algebra. Hence

$$\begin{aligned} m_1(n) &= \frac{[z^n]M_{(1)}(z)}{3^n} \\ &\sim \sqrt{\frac{3n}{\pi}}\left(1+\frac{7}{16n}+\frac{3r^2}{4n}+...\right)-\frac{3r}{2}, \end{aligned}$$

is the required average number of visits to level r. ■

Corollary 105 *The second moments are given by*

$$m_2(n)\sim\frac{3n}{2}-(3r+2)\sqrt{\frac{3n}{\pi}}+\frac{9r^2}{4}+3r+\frac{15}{8}+...$$

Proof. Substituting $s=2$ into Proposition 103 we get

$$M_{(2)}(z)=\frac{2v^r\left(v+2v^2\right)\left(1+v+v^2\right)^2(1+v)}{\left(1-v^2\right)^3(1-v)}. \tag{7.2}$$

Now, by making the substitution that $z=\frac{v}{1+v+v^2}$ into $M_{(2)}(z)$ and upon expansion around $z=\frac{1}{3}$ we obtain

$$\begin{aligned} M_{(2)}(z) \sim\ & \frac{3}{2(1-3z)^2}-\frac{3(3r+2)}{2\sqrt{3}(1-3z)^{\frac{3}{2}}}+\frac{3\left(6r^2+8r+1\right)}{8(1-3z)}-\frac{3\left(12r^3+24r^2+15r+2\right)}{16\sqrt{3}\sqrt{1-3z}} \\ & -\frac{3\sqrt{3}\sqrt{1-3z}\left(48r^5+160r^4+440r^3+560r^2+247r+10\right)}{1280}+..., \end{aligned}$$

so that

$$\begin{aligned} [z^n]M_{(2)}(z) \sim\ & (-3)^n\left[\frac{3}{2}\binom{-2}{n}-\frac{3(3r+2)}{2\sqrt{3}}\binom{-\frac{3}{2}}{n}+\frac{3\left(6r^2+8r+1\right)}{8}\binom{-1}{n}\right. \\ & -\frac{3\left(12r^3+24r^2+15r+2\right)}{16\sqrt{3}}\binom{-\frac{1}{2}}{n} \\ & \left.-\frac{3\sqrt{3}\left(48r^5+160r^4+440r^3+560r^2+247r+10\right)}{1280}\binom{\frac{1}{2}}{n}+...\right] \end{aligned}$$

$$\begin{aligned} &= 3^n \left[\frac{12n+15}{8} - \frac{3\left(2n+2\right)\left(3r+2\right)}{2^{2n+2}\sqrt{3}} \binom{2n+1}{n+1} + \frac{9r^2+12r}{4} \right. \\ &\quad + \frac{3\left(12r^3+24r^2+15r+2\right)}{2^{2n+4}\sqrt{3}} \binom{2n}{n} \\ &\quad \left. - \frac{3\sqrt{3}\left(48r^5+160r^4+440r^3+560r^2+247r+10\right)}{5n2^{2n+7}} \binom{2n-2}{n-1} + ... \right] \\ &\sim 3^n \left[\frac{3n}{2} - \left(3r+2\right)\sqrt{\frac{3n}{\pi}} + \frac{9r^2}{4} + 3r + \frac{15}{8} + ... \right], \end{aligned}$$

where the last asymptotic expansion was done using computer algebra. Therefore

$$\begin{aligned} m_2\left(n\right) &= \frac{\left[z^n\right] M_{(2)}\left(z\right)}{3^n} \\ &\sim \frac{3n}{2} - \left(3r+2\right)\sqrt{\frac{3n}{\pi}} + \frac{9r^2}{4} + 3r + \frac{15}{8} + ..., \end{aligned}$$

as required. ■

7.6 Observation

We give a table of enumerative results to compare asymptotic and exact average number of visits to height r, together with their corresponding variances:

n	**Type of**	$m_{(1)}\left(n\right)$			**Variances**		
	results	r=1	r=3	r=6	r=1	r=3	r=6
222	*expected*	13.1377	10.5314	7.3595	107.5239	100.925	81.0798
222	asympt.	13.1379	10.5314	7.3595	107.8581	103.5852	95.8712
367	*expected*	17.2810	14.5856	11.1201	182.4060	175.4705	152.8492
367	asympt.	17.2811	14.5872	11.1201	182.6659	177.5	164.4267
593	*expected*	22.3441	19.5842	15.8884	300.5198	293.2728	268.7493
593	asympt.	22.3441	19.5849	15.8975	300.7256	294.8792	277.6109
712	*expected*	24.6185	21.8377	18.0729	355.7598	355.7598	330.4641
712	asympt.	24.6186	21.8383	18.0799	357.2258	357.2258	338.5702

Table 7.2: Average r-level visits by open ended unrestricted Motzkin walks.

Both enumerative and asymptotic results shows that an open ended unrestricted Dyck walk has less average visits the r-level than the unrestricted Motzkin walk.

7.7 Return statistics

In the sequel we analyze the average number of returns to the origin by an arbitrary random walk starting at $(0,0)$. Unlike in [14], [17] and [18] we don't count the first step at the origin as a return. If we count the number of returns to the 0-level by the variable w then the special system

$$\varphi_0(z) = 1 + z\varphi_1 + z\varphi_{-1},$$

discussed in Section 7.1 becomes

$$\varphi_0(z,u) = 1 + w\left[z\varphi_1 + z\varphi_{-1} + z\varphi_0\right].$$

Therefore the system of all the φ 's given in Section 7.1 is now rewritten as:

$$\begin{bmatrix} 1-z & -z & & & & & & & \\ -z & 1-z & -z & & & & & & \\ & . & . & . & & & 0 & & \\ & & -zu & 1-zu & -zu & & & & \\ & & & . & . & . & & & \\ & & 0 & & . & . & . & & \\ & & & & & -z & 1-z & -z & \\ & & & & & & -z & 1-z \end{bmatrix} \begin{bmatrix} \varphi_h \\ \varphi_{h-1} \\ . \\ \varphi_0 \\ \varphi_{-1} \\ . \\ . \\ \varphi_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ 1 \\ . \\ . \\ . \\ 0 \end{bmatrix}. \qquad (7.3)$$

As in the previous case we calculate the $\varphi_{i;h,k}$'s using Cramer's rule by first finding $\det D_{h+k+1}$,the determinant of the above matrix system in (7.3) :

Lemma 106 *The determinant of this system, denoted by* $\det D_{h+k+1}$*, with* $h+k+1$ *rows is given by*

$$\det D_{h+k+1} = \frac{\left(1-v^{2h+2}\right)\left(1-v^{2k+2}\right)\left(1+v+v^2-vw\right)}{\left(1-v^2\right)^2\left(1+v+v^2\right)^{h+k+1}}$$

$$-\frac{v^2 w\left[\left(1-v^{2h+2}\right)\left(1-v^{2k}\right)+\left(1-v^{2h}\right)\left(1-v^{2k+2}\right)\right]}{\left(1-v^2\right)^2\left(1+v+v^2\right)^{h+k+1}}.$$

Proof. This proof is an exact analogue of the proof in Lemma 42 where the substitutions

$$\det A_h = \frac{1}{\left(1-v^2\right)}\frac{1-v^{2h+4}}{\left(1+v+v^2\right)^{h+1}} \text{ , and } z=\frac{v}{1+v+v^2}$$

were made. Hence we will not repeat the proof here. ■

Theorem 107 *The generating function for the number of returns by an arbitrary random walk starting at the origin, bounded above and below by the lines $h>0$ and $k<0$ respectively, and is ending at (n,i) is given by*

$$\varphi_{i,h,k}(z,w)=\frac{v^i}{1-w\left(\frac{v+2v^2}{1+v+v^2}\right)},$$

where w is counting the number of returns to the origin and $z=\frac{v}{1+v+v^2}$.

Proof. The generating function for $\varphi_{i,h,k}(z,w)$ calculated in the Section 7.2 using Cramer's rule will now be rewritten as

$$\begin{aligned}
&\varphi_{i,h,k}(z,w)\\
=\;&\frac{z^i \det A_{h-i-1}\det A_{k-1}}{\det D_{h+k+1}}\\
=\;&\frac{\left(1-v^2\right)^2\left(1+v+v^2\right)^{h+k+1}}{\left(1-v^{2h+2}\right)\left(1-v^{2k+2}\right)\left(1+v+v^2-vw\right)-v^2w\left[\left(1-v^{2h+2}\right)\left(1-v^{2k}\right)+\left(1-v^{2h}\right)\left(1-v^{2k+2}\right)\right]}\\
&\times\frac{v^i\left(1-v^{2h-2i+2}\right)\left(1-v^{2k+2}\right)}{\left(1-v^2\right)^2\left(1+v+v^2\right)^{h+k}}\\
=\;&\frac{v^i\left(1-v^{2h-2i+2}\right)\left(1-v^{2k+2}\right)\left(1+v+v^2\right)}{\left(1-v^{2h+2}\right)\left(1-v^{2k+2}\right)\left(1+v+v^2-vw\right)-v^2w\left[\left(1-v^{2h+2}\right)\left(1-v^{2k}\right)+\left(1-v^{2h}\right)\left(1-v^{2k+2}\right)\right]}\\
=\;&\frac{v^i\left(1+v+v^2\right)}{1+v+v^2-vw-2v^2w}\text{ , as } h,k\to\infty,\\
=\;&\frac{v^i}{1-w\left(\frac{v+2v^2}{1+v+v^2}\right)}.
\end{aligned}$$

■

Remark 24 *We test the correctness of the generating function $\varphi_{i,h,k}(z,w)$:*

$$\varphi_{i;\infty,\infty}(z,1)=\frac{v^i}{1-\left(\frac{v+2v^2}{1+v+v^2}\right)}=\frac{v^i\left(1+v+v^2\right)}{1-v^2},$$

so that

$$\begin{aligned}\sum_i \varphi_{i;\infty,\infty}(z,1) &= \frac{(1+v+v^2)}{1-v^2}\left(2\sum_{i\geq 1} v^i + 1\right) \text{ for all } i.\\ &= \frac{(1+v+v^2)}{1-v^2}\frac{(1+v)}{(1-v)}\\ &= \frac{(1+v+v^2)}{(1-v)^2},\end{aligned}$$

which is the generating function for the total number of random walks with open ending at $(n,-)$. ■

Proposition 108 *The generating function of the s-th factorial moments of* $\varphi_{i,h,k}(z,u)$ *multiplied by the number of paths of length n is given by*

$$M_{(s)}(z) = \frac{s!v^s(1+2v)^s(1+v+v^2)(1+v)}{(1-v^2)^{s+1}(1-v)},$$

where $z = \frac{v}{1+v+v^2}$.

Proof. Now,

$$\begin{aligned}M_{(s)}(z) &= \frac{\partial^s}{\partial w^s}\varphi_{i,\infty,\infty}(z,w)\Big|_{w=1}\\ &= \frac{s!v^i\left(\frac{v+2v^2}{1+v^2}\right)^s}{\left(\frac{1-v^2}{1+v+v^2}\right)^{s+1}}\\ &= \frac{s!v^{i+s}(1+2v)^s(1+v+v^2)}{(1-v^2)^{s+1}}.\end{aligned}$$

For an open ending we sum up for all i to get

$$\begin{aligned}M_{(s)}(z) &= \frac{s!v^s(1+2v)^s(1+v+v^2)}{(1-v^2)^{s+1}}\left(2\sum_{i\geq 1} v^i + 1\right)\\ &= \frac{s!v^s(1+2v)^s(1+v+v^2)(1+v)}{(1-v^2)^{s+1}(1-v)}.\end{aligned}$$

■

Corollary 109 *The average number of returns to the origin by a random walk of length n with an open ending is given by*

$$m_{(1)}(n) \sim \sqrt{\frac{3n}{\pi}}\left(1+\frac{7}{16n}+\frac{1}{n^2}+...\right)-1.$$

Proof. Specializing on $s=1$ into Proposition 107 we get

$$M_1(z)=\frac{v(1+2v)(1+v+v^2)(1+v)}{(1-v^2)^2(1-v)}.$$

Using the substitution $z=\frac{v}{1+v+v^2}$ into $M_{(1)}(z)$ we get the following series expansion around $z=\frac{1}{3}$:

$$M_{(1)}(z)\sim\frac{3}{2\sqrt{3}(1-3z)^{\frac{3}{2}}}-\frac{1}{(1-3z)}+\frac{3}{16\sqrt{3}\sqrt{1-3z}}+\frac{3\sqrt{3}\sqrt{1-3z}}{256}+...$$

so that

$$\begin{aligned}[z^n]M_{(1)}(z) &\sim (-3)^n\left[\frac{3}{2\sqrt{3}}\binom{-\frac{3}{2}}{n}-\binom{-1}{n}+\frac{3}{16\sqrt{3}}\binom{-\frac{1}{2}}{n}-\frac{3\sqrt{3}}{256}\binom{\frac{1}{2}}{n}+...\right],\\ &= 3^n\left[\frac{3(2n+2)}{2^{2n+2}\sqrt{3}}\binom{2n+1}{n+1}-1+\frac{3}{2^{2n+4}\sqrt{3}}\binom{2n}{n}-\frac{3}{2^{2n+7}n}\binom{2n-2}{n-1}+...\right]\\ &\sim 3^n\left[\sqrt{\frac{3n}{\pi}}\left(1+\frac{7}{16n}+\frac{1}{n^2}+...\right)-1\right],\end{aligned}$$

where the above asymptotic expansion was done using computer algebra. Therefore

$$\begin{aligned}m_1(n) &\sim \frac{[z^n]M_{(1)}(z)}{3^n}\\ &= \sqrt{\frac{3n}{\pi}}\left(1+\frac{7}{16n}+\frac{1}{n^2}+...\right)-1,\end{aligned}$$

as required. ■

Corollary 110 *The second moments on $\varphi_{i,h,k}(z,u)$ are given by*

$$m_{(2)}(n)\sim\frac{3n}{2}-\sqrt{\frac{3n}{\pi}}\left(4-\frac{7}{4n}+...\right)+\frac{31}{8}.$$

Proof. Specializing on $s=2$ into Proposition 107 we get

$$M_2(z)=\frac{2v^2(1+2v)^2(1+v+v^2)(1+v)}{(1-v^2)^3(1-v)}.$$

Using the substitution $z=\frac{v}{1+v+v^2}$ into $M_{(1)}(z)$ we get the following series expansion around $z=\frac{1}{3}$:

$$M_{(2)}(z)\sim\frac{3}{2(1-3z)^2}-\frac{6}{\sqrt{3}(1-3z)^{\frac{3}{2}}}+\frac{19}{8(1-3z)}-\frac{3}{4\sqrt{3}\sqrt{1-3z}}-\frac{3\sqrt{3}\sqrt{1-3z}}{64}+...,$$

so that

$$\begin{aligned}
[z^n]\, M_2(z) &= (-3)^n \left[\binom{-2}{n} + \frac{19}{8}\binom{-1}{n} - \frac{6}{\sqrt{3}}\binom{-\frac{3}{2}}{n} - \frac{3}{4\sqrt{3}}\binom{-\frac{1}{2}}{n} - \frac{3\sqrt{3}}{64}\binom{\frac{1}{2}}{n} + ... \right] \\
&= 3^n \left[\frac{3n}{2} + \frac{31}{8} - \frac{3(2n+2)}{2^{2n}\sqrt{3}}\binom{2n+1}{n+1} - \frac{3\sqrt{3}}{2^{2n+7}n} - \frac{3}{2^{2n+2}\sqrt{3}}\binom{2n}{n} + ... \right] \\
&\sim 3^n \left[\frac{3n}{2} - \sqrt{\frac{3n}{\pi}}\left(4 + \frac{7}{4n} + ...\right) + \frac{31}{8} \right].
\end{aligned}$$

Therefore

$$\begin{aligned}
m_{(2)}(n) &= \frac{[z^n]\, M_2(z)}{3^n} \\
&\sim \frac{3n}{2} - \sqrt{\frac{3n}{\pi}}\left(4 + \frac{7}{4n} + ...\right) + \frac{31}{8}.
\end{aligned}$$

■

7.8 Observation

As usual we give a table of enumerative results to compare asymptotic and exact average number of visits to height r, together with their corresponding variances:

n	**Returns**		**Variance**	
	expected	asympt.	*expected*	asympt.
50	5.9703	5.9702	21.3202	21.3194
101	8.8633	8.8633	46.2269	46.2269
174	11.9226	11.9226	82.9587	82.9587
222	13.5887	13.5887	107.4561	107.4561
367	17.7429	17.7429	182.336	182.3360
593	22.814	2.814	300.4542	300.4542
712	25.0911	25.0911	363.0363	363.0363
844	27.4042	27.4042	432.6724	432.6724
1001	29.9309	29.9309	515.7235	515.7235

Table 7.3: Average 0-level returns by open ended unrestricted Motzkin walks.

Both enumerative and asymptotic results shows that an open ended unrestricted Dyck walk has fewer average returns to the 0-level than the unrestricted Motzkin walk.

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