Enhancing teacher knowledge through an object-focused model of professional development

A thesis submitted to the Faculty of Humanities, University of the Witwatersrand, Johannesburg, in fulfillment of the requirements for the degree of Doctor of Philosophy

By

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DECLARATION

I declare that this research report is my own work, except as suggested in the acknowledgements, the text and references. It is being submitted to the faculty in fulfilment of the requirements for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other institution.

	December 2014
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Signed	Date

Shadrack Shadow Moalosi

DEDICATION

I dedicate this thesis to my family. My wife Kent, my sons Marcus and Calvin and my daughter Pauline. Your patience was worthwhile. Included are my mother Kegaisamang Moalosi and my sister Abueng Moalosi for their meaningful support.

ABSTRACT

This research was an exploratory study in which I was interested in the constitution of the enacted object of learning in an object-focused model of professional development and the lived object of learning for the participating mathematics teachers. The object-focused model of PD was a 5 year continous professional development which used secondary school mathematics content to strenghten mathematics teachers' knowledge of the content they taught in school.

To study the enacted object of learning I worked deductively with three theoretical resources. Firstly, I recriuted the notion of *patterns of variation* from Variation Theory to *visibilize* what was made possible for teachers to learn about functions. Secondly, to further explore the enacted object of learning I recruited the notions of the *duality of the object of learning* and *legitimating criteria* from Adler and Davis' (2006) methodology for studying mathematics teacher education to *illuminate* what was made possible to learn. Thirdly, I recruited Galileo's notions of *reality* and *appearance* to theorize the enacted object of learning. In my inductive analysis of the enacted object of learning I was interested in the nature of scaffold provided to teachers to discern the key features of functions. Two pedagogic strategies which were used to scaffold tasks emerged in the data and I described one as the *deductive* and the other as the *inductive pedagogic strategies*. While the former opened up scaffold, the latter closed it. To study the lived object of learning I recruited theoretical resources from phenomenology and Adler and Davis' (2006) notion of the *duality of the object of learning*.

Results from my analysis of the enacted object of learning revealed that the deductive pedagogic strategy resulted in moderate or high scaffold, and the inductive pedagogic strategy resulted in low scaffold. In addition, analysis with patterns of variation together with notions of duality of the object of learning and legitimating criteria revealed that the enacted object of learning was *mathematical reality* ($\boldsymbol{Mt_{re}}$) and *mathematical appearance* ($\boldsymbol{Mt_{ap}}$). Analysis of the lived object of learning revealed that the teachers' experiencing of an object focused PD was constituted by *personal learning of functions; professional learning of functions; personal learning of teaching functions* and *professional learning of teaching functions*. Methodologically, this research has contributed tools which can be used to study the enacted and lived object of learning. Theoretically, this research has extended Adler and Davis' notion of **Mt** at the level of enactment by subcategorising it into $\boldsymbol{Mt_{re}}$ and $\boldsymbol{Mt_{ap}}$. In addition, at the level of the lived object of learning, this research has extended both **Mt** and **Tm** into **Mt-pl; Mt-PL** and **Tm-pl; Tm-PL** respectively.

Keywords: object-focused, professional development, teacher knowledge, revisiting-mathematics, variation theory, phenomenology, legitimating criteria

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Conference proceedings article from Chapter Six was published as:

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CHAPTER 1: INTRODUCTION

1. 1. Introduction

The study reported in this thesis explores what and how the intended object of learning in an object-focused mathematics professional development was enacted in pedagogic practice, and became the lived object for the participating teachers. Object-focused professional development is a model where selected secondary school mathematical content in the school curriculum is deliberately in focus for teachers to re-learn or revisit.

In this chapter I discuss the rationale for the study, followed by a brief discussion of professional development project - the Wits Maths Connect Secondary (WMCS) project - in which it was situated. Together these locate my research questions. Suffice it to highlight here that the technical terms used in this chapter were derived from the theories which anchored the study. These are explained and/or operationalized in later chapters.

1.2. Rationale - Why a study of object focused professional development in South Africa?

The enduring problem of low achievement in mathematics has raised serious concerns in South Africa among stakeholders that include policy makers, practitioners, researchers, teachers and parents. All agree that the trend of low achievement in mathematics cannot remain unattended. Evidence abounds in local (Simkins, 2013; Spaull, 2013) and international (Mullis, Martin, Foy, & Arora, 2012; Mullis, Martin, Gonzalez, & Chrostowski, 2004) reports of a large proportion of South African learners performing far below their peers around the world. An instance of such low performance was captured recently by Spaull (2013) in his analysis of South Africa's TIMSS[1] results:

> … in 2011 a third of pupils (32 per cent) performed worse than guessing on the multiple choice items (i.e. no better than random). Furthermore, three quarters (76 per cent) of Grade Nine pupils in 2011 still had not acquired a basic understanding about whole numbers, decimals, operations or basic graphs… with the average South African Grade Nine child performing between two and three grade levels lower than the average Grade Eight child from other middle-income countries (p.6).

Low achievement in mathematics is indeed a serious concern considering the critical role mathematics plays in many spheres of human endeavor. Mathematics is a tool in people's everyday lives, a medium of communication, facilitates success in, and transition into the field of natural sciences such as medicine and engineering. Some go further to argue that mathematical knowledge, in particular algebra, is a civil right and thus important for all learners: it opens up opportunities to high rewarding high-tech jobs (Moses, 2001).

[1] TIMSS is an international comparative research focused on fifteen year old learners' performance on

It is believed (see McCarthy & Rebecca, 2013) that future economic success of each country will greatly rely on its skilled human resources. Needless to say, the shortage of skilled human resources is evident in recent reports (see Spaull, op cit). Mathematics is among the school subjects which are seriously affected by skills shortages, evident in the presence of un- and under-qualified mathematics teachers in some South African schools (Simkins, op cit). In less extreme cases, there are practicing teachers who took mathematics as a minor teaching subject during teacher preparation; in more extreme case we find teachers who are teaching at a level beyond which they were trained (e.g. some teachers were trained to teach up to Grade 9 and are teaching in Grades 10 – 12); as well as teachers who are teaching out-of-subject.

Having teachers who are untrained or inadequately trained to teach mathematics and yet placed in such positions, does not only compound the problem of poor teaching quality and low achievement in mathematics, but also undermines the teaching profession. The message conveyed is that it does not matter who we entrust with the delicate minds of our children. Offering mathematics teaching posts to unqualified teachers tends to suggest that anyone can teach. Simkins (op cit) and Spaull (op cit) agree that there are crucial weaknesses in some areas of the overall schooling system in South Africa, particularly in mathematics and they attribute these to un- and under-qualified teachers in the schooling system. The presence of un- and under-qualified teachers in any schooling system seems to vindicate George Bernard Shaw's assertion that ""[h]e who can does, [h]e who cannot teaches" (as cited by Shulman, 1986 p.1). The problem of poor training, shortage of mathematics teachers and the probable causes of such are well explained by Adler & Davis (2006):

> Majority of black secondary school teachers, trained under apartheid only had access to 3 year College of Education diploma, and the quality of training in general and in mathematics in particular was by and large poor. Hence, many current secondary mathematics teachers have not had an adequate opportunity to learn further mathematics (2006, p.277).

Indeed, there is wide agreement that the quality of an educational system is a function of the quality of its teachers and teaching, and teacher education and continuous professional development are central to such quality (Parker, 2009).

In the context of this study, it is interesting that in current and valued models of professional development that I discuss in detail in Chapter 3, the specific mathematics content of teaching appears backgrounded. Yet all models share a common agenda of enhancing teachers' professional knowledge and practice, and teachers' content knowledge for teaching – their subject matter knowledge (SMK) and pedagogic content knowledge (PCK) - is central to such (Shulman, 1987).

The issue of the location and focus on content knowledge for teaching in mathematics teacher professional development, and where the emphasis between SMK and PCK lies, was a key element of my study. It seems that in many professional development initiatives, teachers' SMK, or at least the basics of this, is assumed as unproblematic, and so emphasis is placed on teaching and learning, and so PCK. However, as discussed above, in South Africa, teachers' SMK cannot be taken for granted.

Hence the problem and focus of my study was mathematics teachers' content knowledge for teaching, with particular focus on SMK of Functions at Grades 9 – 11. Functions were a key focus in the Wits Maths Connect Secondary professional development (WMCS PD). I chose to focus my study on functions because they are an important integral part of the South African school mathematics curriculum. In South Africa functions are taught in Grades 9 – 12, with the focus ranging from linear to trigonometric functions and to inverse functions. Knowledge of functions is fundamental in the learning of advanced university mathematics such as Calculus and Mechanics. Furthermore, from what we already know from the national and international standardized learner tests, learners are not doing well in this domain of mathematics (e.g. Adler, November 2010).

Numerous educational reforms and efforts have been implemented in South Africa to improve educational outcomes. For instance, there have been several changes in the secondary school curriculum since independence. In spite of that, learner performance has continued to decline. Of course, as Young (2002) points out, changes in curriculum do not necessarily lead to improved understanding or knowledge acquisition, but instead, ensuring that all learners have access to powerful knowledge such as [mathematics] is critical. In South Africa, ensuring that all learners have access to powerful knowledge is not trivial, and as discussed above, a function of teachers' professional knowledge and practice. Evidence from literature of research on professional development in South Africa that has focused on PCK indicate that strengthening PCK levers up learning of SMK (Brodie, 2012; Chauraya & Brodie, 2011). Other research (Graven, 2005; Huillet, Adler, & Berger, 2011) has suggested that there was a horizon to the learning of SMK: where there were significant gaps this required deliberate focus. At the same time, pure content programs or content per se have not levered up improvements (e.g. Advanced Certificate in Education). This illuminates the significant tension around balance and the relationship between SMK and PCK in any professional development – so it was interesting to explore what was made possible to learn in an object focused PD where the object of learning was explicitly the mathematics content in the school curriculum, and thus aligned more towards SMK as defined by Shulman.

1.3. WMCS and SMK as part of professional knowledge

The study was located within the Wits Maths Connect-Secondary (WMCS) project, a project in South Africa focused on teachers' mathematical knowledge for teaching, with specific attention and so focus on SMK (see Adler, November 2010). In my research I focused on an aspect of the PD where SMK involved teachers *revisiting/(re)learning school mathematics* (see Pournara & Adler, 2014), and the particular school mathematics was Functions. Later in this research report I review the literature on professional development models and show there that little is known about professional development focused on mathematical objects of learning. So, what is the WMC project, its model, and why has it selected to focus on mathematical objects of learning?

Wits Maths Connect-Secondary project (WMCS) was a five year longitudinal professional development and research project targeting secondary mathematics teachers from 11 schools in one district in South Africa's Gauteng Province with two main development aims: (i) strengthening teachers' content knowledge for teaching so as to improve teaching quality and (ii) improving learner performance so as to increase the number of learners able to pursue pure mathematics in Further Training and Education (FET) level (i.e. Grade 10 – 12), and into tertiary study (Adler, November 2010). What is unique about the WMCS professional development initiative is its focus on what it refers to as the 'object of learning'.

Interestingly, or strangely, the WMCS project did not directly copy existing professional development models, despite, as I discuss in Chapter 3, their reported successes. However, aspects of those models were built into the WMCS model. For instance, the WMCS recognized and facilitated teacher collaboration, and the importance of situating teacher learning in teachers' own classroom practices was acknowledged. However, the choice of focus in any professional development is informed by the nature and context of the problem being addressed. The inequitable and poor quality of teacher preparation in apartheid South Africa is well known, as is the evidence, discussed above, that quality in the post- apartheid system is not improving.

I have already pointed to the well-known problem of the level and depth of SMK of many in South Africa's secondary teaching corps (e.g McCarthy & Rebecca, 2013). As a way to deal with this problem, Huillet, Adler & Berger (2011) argued strongly that SMK needed to be at the centre of mathematics teachers' professional learning particular where gaps in SMK exist. As noted earlier, these authors have shown that where there are gaps or limits in teachers' SMK, teachers need the opportunity to learn such directly.

It is thus understandable that the WMCS's goal was to strengthen participating teachers' mathematical knowledge for and in teaching through a deliberate focus on key mathematical contents. The underlying assumption at work here was that a focus by teachers on *what* is to be learned will create opportunities for teachers to strengthen their mathematics knowledge (for teaching) and so their professional knowledge. I return later to elaborate the way in which I am using professional knowledge in this study, which will further illuminate this assumption and the notion that even when mathematics per se is the focus of teacher education activity, and particularly professional development, these mathematical objects of learning will necessarily be oriented in some way to the activity and practice of teaching. This in turn, shapes how such content is enacted in professional development practice (Adler & Hulliet, 2008), and the opportunities opened up for teachers to strengthen their mathematical knowledge for teaching.

How SMK was enacted and what then became possible to learn was part of the key focus of my research. So how was the professional development enacted by the WMCS project? Detailed analysis of professional development tasks are the focus of Chapters 6 – 9. In this chapter, however, and for purposes of developing my research questions, I introduce the key activity in professional development which was the focus of this research.

In its second year, 2011, the WMCS project selected four topics or content areas (Integers, Functions, Geometry, and Probability) as the foci of its PD work. As I have already indicated, my study was focused on functions, and thus a part of the PD work with teachers conducted in 2011. In each of the four selected contents - or what the project referred to as themes - there were four sessions conducted over a four week period, one session per week after school, and in three of the project schools, to which the other teachers travelled for these sessions. In the functions theme, the first two sessions over the first two weeks, Functions 1 and 2, focused on Functions per se, and involved the teachers themselves only. The remaining two sessions, Functions 3 and 4, focused on teaching of functions, and included a lesson with learners, followed by joint reflection with teachers and project members. Thus, in Functions 1 and 2 teachers participated in classroom learning of Functions directly; in Functions 3 and 4 they observed and then reflected on learners being taught functions. In the latter sessions, teachers observed the lesson, paying attention to what and how aspects of ‘function’ as object of learning in that lesson were in focus, and engaged with by learners. These observations were followed by teachers and facilitators coming together to reflect on the lesson with particular attention to key features of functions. I wish to explain with the aid of variation theory, the difference (with respect to the object of learning) between the first two and the last two sessions of the Functions theme work. I do this from two perspectives.

From the perspective of the WMCS PD, the intended object of learning was Functions per se, and was the same across the four sessions irrespective of who the learner was. But from my perspective as the researcher, the enacted object of learning was different across the four sessions. Taking the teacher as the learner across the four sessions, it is clear that teachers were directly involved in the co-constitution of the enactment of functions in Functions 1 and 2 and this is evident in the legitimating criteria (see Chapters 6 – 9). In Functions 3 and 4, teachers were not learning functions directly even though the object of learning in the demonstration lesson was functions. Follow up discussions at the end of Functions 3 and 4 were not on teachers’, but on learners’ experiencing of Functions. So there was some ambiguity in what teachers were expected to learn, because whereas the goal of the WMCS PD was to strengthen teachers’ SMK in Functions 3 and 4, it was not clear what teachers were to learn. Hence data from Functions 3 and 4 were out of focus in my research.

WMCS PD has found resonance with the content focus in Variation Theory, and the work of Marton, Runesson and Tsui (2002). I briefly discuss this theory in Chapter 2, where I discuss the context of the study, and in more detail in Chapter 5, where I discuss the methodology which guided the research. Variation Theory was central in the design of tasks and examples, particularly in the first two years (2010 – 2011), when its use was deliberate. As a consequence, I recruited analytic tools from this theory to analyse video and lesson plan data (see Chapters 6 – 10).

1.4. Problem and research questions

It is evident from research that participating in continuous professional development has links with improved teaching quality. It is also evident that teachers’ professional knowledge has links with both teaching quality and learner achievement. This invaluable information suggests that continuous professional development that aims to enhance teacher professional knowledge could be a solution to the problem of low achievement in mathematics. In spite of the existing information about the importance of continuous professional development and teacher professional knowledge, how and what teachers learn in professional development (Buddin & Zamarro, 2009), and how they acquire professional knowledge is not clear (Ball, Bass, & Hill, 2004). In addition, it appears, from the review of literature that less is known about professional development models focused on key mathematical content as the object of learning. Hence the purpose of the proposed research was to understand what and how the object of learning was constituted in the WMCS project's professional development model, and whether and how this model mediated teachers' mathematical knowledge for teaching.

1.4.1. Research Question

The major question that guided my study was:

In what ways, and how, does a professional development model focused on key mathematical objects in the curriculum mediate mathematics teachers' professional knowledge, in particular their SMK? This question is informed by the theoretical framework adopted for this research

1.4.2. Critical Questions

1. What is constituted as the enacted object of learning in object focused professional development, and how is it constituted, i.e., what is the space of variation and what is the nature of the object of learning?

2. What becomes the lived object for teachers, i.e., what capabilities with respect to the mathematical object are reflected in teachers' lesson planning and in object-focused interviews ?

3. (a). How do these capabilities mediate teachers' professional knowledge?

(b). How might the mediation of teachers' professional knowledge be explained?

1.5. Conclusion

The significance of this study lies in its interrogation of the foregrounding of SMK in object focused mathematics professional development, confronting the question of whether and how, professional development (and not a formal course) where teachers learn mathematics directly (revisiting school mathematics content), might be a possible solution to the problem of un- and under-qualified teachers in the South African schooling system.

1.6. Outline of Chapters

CHAPTER 2: Context of the study and the intended object of learning. This chapter discusses the context of the study with focus on the work of the Wits Maths Connect Secondary project and so includes discussion of the intended object.

CHAPTER 3: Literature review

Here I review literature on professional development, teacher knowledge and functions. This is necessary because my research looks into professional development to study its enactments that aim to enhance teacher knowledge of functions.

CHAPTER 4: Theoretical Framework.

This chapter discusses epistemological, ontological and axiological issues in general and also particularizes these to this research. This chapter is central in my study because it provides a background on the origin of my analytical tools.

CHAPTER 5: Methodology. This chapter discusses the methodology underpinning the study, and so the research design and analytic tools

CHAPTER 6: The enacted object of learning: Analysis of Task 1 in Functions 1

This chapter presents the analysis focused on Task 1 of Functions 1. Enactment at each cluster is analyzed separately. The task was focused on Card Sorting.

CHAPTER 7: The enacted object of learning: Analysis of Task 2 of Functions 1.

This chapter presents analysis of enactment of Task 2 of Functions. The task focused on Card Matching of multiple representations of functions.

CHAPTER 8: The enacted object of learning: Analysis of Task 1 of Functions 2.

This chapter presents the analysis of the enactment of Task 1 of Functions 2. Its focus was on horizontal transformation of quadratic functions.

CHAPTER 9 The enacted object of learning: Summary of the Analysis of Task 2, 3, & 4 of Functions 2

This chapter summarises the analysis of the enactment of Tasks 2, 3, and 4 of Functions 2. Task 2 was on Card Matching involving two representations, domain and range. Task 3 involved finding functions which satisfied three stated conditions. Task 4 involved finding functions that satisfied given domains and ranges.

CHAPTER 10: The lived object of learning - Analysis and discussion of lesson planning. This Chapter discusses analysis done on nine pairs of Pre-PD and Post-PD lesson plans, and part of the interview where teachers talked about the difference between the two lesson plans.

CHAPTER 11: The lived object of learning- Analysis and discussion of teacher interviews with respect to learning from resources and PD activities

CHAPTER 12: Conclusion. This concluding chapter of the research contains answers to the research questions, contributions, implications and ideas for possible new research.

CHAPTER 2: WMCS PD – THE PROJECT CONTEXT OF THE STUDY

2.1. Introduction

As noted in Chapter 1, this study was located within the Wits Maths Connect Secondary (WMCS) project. In this chapter I describe the project, its PD work, and its deliberate use of Variation Theory in its planning and delivery. I also discuss how the model of professional development in the project was not static, but evolved over time. As with other professional development activity, WMCS faced tensions in its work, and the final part of this chapter discusses how the project managed those inherent tensions in PD that are pertinent to my study.

2.2. Wits Maths Connect Secondary (WMCS) project

The WMCS was a five year research and development project. The aim of the WMCS PD is to improve learner performance in mathematics by improving the quality of teaching through professional development (PD) of teachers. The immediate goal of the PD was to strenghten the teachers' subject matter knowledge of mathematics. The teachers came from 11 schools serving previously disadvantaged learners in one district in South Africa. While there were differences across schools[2], in general, performance in mathematics across all eleven schools was poor, with 8 of the schools labelled by the provincial department as 'underperforming'.

The relationship between learner performance, teaching quality and strong SMK has been described in the literature. Strong SMK is associated with improvements in teaching quality and improved teaching quality is associated with increased opportunities to learn (Hill, et al., 2008).

- **The core PD in WMCS**

The WMCS project PD was informed by previous research in the QUANTUM[3] project (e.g. Adler & Davis, 2006). From its empirical work, the QUANTUM project theorized *mathematics* and *teaching* as dual objects in mathematics teacher education, arguing that while both are inevitably present across teacher education courses and programmes, merging the two is fraught with challenges. Hence in the WMCS project, two strands were developed. The first - and core strand - was the Developing Mathematics Judgement project, which

[2] Of the 11 schools, 6 are fully state funded ("no fee") schools in townships, overcrowded, poorly resourced, with learners drawn largely from the township. The remaining five schools were previously served more advantaged communities, were better resourced, and differentially state funded. All were fee paying schools, though fees were relatively low, and learners drawn largely from nearby townships and sometimes those further away.

[3] QUANTUM stands for **qu**alifications for **t**eachers **u**nder qualified in **m**athematics and is also the name of a research and development project that focused on improving the qualifications of under qualified teachers in South Africa.

rested on the view that mathematical judgment was key to skillful mathematics teaching and depended on strong SMK and PCK, in Shulman's terms. Following QUANTUM, while SMK and PCK were both key to skillful teaching, in PD, attention on these needs to be separated. How this was done will become evident below.

The second, and non-core, but nevertheless part of WMCS work, was one short topic-focused workshop per term as requested by teachers, typically on sections of the curriculum that were, in mathematical terms, new to them (e.g. probablility), These workshops and their contents are not in focus in this study, and I only refer to them again below as I describe the evolving model of the core work of the project over time. I thus return now to detailed elaboration of the developing mathematical judgment (DMJ) core work in WMCS PD.

The Developing Mathematical Judgement project (DMJ) was at the centre of the WMCS project. This was a school-based professional development which incorporated development and research. By way of example, in my study I researched professional development that was conducted under the DMJ strand. The mandate of the DMJ project was to enhance mathematics teachers' professional knowledge, with particular focus on subject matter knowledge (SMK) pedagogical content knowledge (PCK) and curricular knowledge (CK) **(**Adler, July 2011).

The DMJ project was informed by the view that mathematical judgement was essential for skilful teaching. Since the development of mathematical judgement rested on strong mathematical knowledge for teaching, the aim of the DMJ project was to develop '... teachers' mathematical judgement through deliberate teaching focused on key mathematical objects of learning ...' (Adler, July 2011, p.1). In the next sub-headings I describe the professional development work under the DMJ project which was done in 2010 and 2011 separately.

The DMJ work was organised around selected contents, called 'themes' each of which became the focus of four consecutive sessions with teachers. Each theme first engaged teachers in *(re)learning* of the secondary school mathematics content. This was followed by (though with less intensity) what was entailed in making elements of this content available to learn in a particular class/lesson. Hence, while there was attention to both SMK and PCK these were deliberately separated in the PD sessions. In the Functions theme work the focus was on teachers learning the key features of functions (i.e. their properties, multiple representations and terminology) and then on how these might be exemplified and explained in a particular lesson with learners. To exemplify the duality in the DMJ I draw from the Functions theme work with particular focus in Functions 1.

In Functions 1 the major focus was on key features of functions, and hence on mathematical objects. There were two tasks for teachers to (re-) learn functions . Task 1 required them to sort cards bearing multiple representations into functions and non-functions; Task 2 required them to make piles of matching representations of functions. I return to elaborate on Functions 1 tasks in Section 2.3.

- **Planning and organising delivery of DMJ**

The project began its work in the second half of 2010, where work was, however, disrupted by a protracted public service strike. Many teachers across the project schools were on strike for over three weeks, and under considerable pressure when they returned to complete their year's work with learners. This impacted on participation in the project, and it became clear that work in school time with teachers, particularly if they were to collaborate, would not be effective. PD work in the project got under way in 2011, and themework was organised so that it occurred in the afternoon. Moreover, instead of a combination of central campus activity followed by activity in all eleven schools, the PD was planned to take place in clusters of schools that were named the N, E and I clusters. Schools in each cluster took turns in hosting this work.

This study revolved around functions as the object of learning in 2011. As noted in Chapter 1, of the four sessions, two were focused on functions themselves, with teachers from schools in each of the clusters. In the remaining two sessions, a key aspect of functions was taught (largely by project staff) to a class of learners in the host school. Project staff and teachers observed this 'demonstration' lesson, and this was followed by a reflection focused on the object of learning made available in the lesson. While the latter two sessions were in a teaching/learning setting, the project focus remained on the object of learning. Discussion was structured by what learners were to know, and whether this had been made available. The content of learning was deliberately kept in focus.

- **My interest and focus in this study**.

In this study I was interested on what was constituted as the enacted and the lived object of learning, particularly in sessions (i.e. Functions 1 and 2) where subject matter knowledge was directly in focus. The enacted object of learning was spread over three sites and it was valuable to study each since each site provided further insight into the relationship between the intended, enacted and lived objects of learning. Hence I conducted detailed analysis of each task in each site to study the enacted object, followed by in-depth interview of teachers from these sites to study the lived object of learning.

2. 2. Deliberate use of Variation Theory in task construction

In addition to QUANTUM, the DMJ was also informed by variation theory**.** Variation theory ensures that object of learning is always in focus, and while this has the learner in view, the focus is also on *what it is* they are to learn. Thus, the object of learning has both direct and indirect aspects. The direct is the content to be learnt, and the indirect object of learning is the capability to be developed in the learner (Lo, 2012). Variation Theory (or difference theory) is a theory of learning which asserts that learning is a function of discernment of variation and invariance on the *object of learning* (Marton, Runesson, & Tsui, 2004).

According to Lo (2012), learning is a function of discernment of difference, and discernment is a function of experienced difference. So there can never be learning unless there was discernment of difference and there can never be discernment of difference unless there has been some experienced difference. Hence variation theorists believe that discernment of difference makes it possible for the learner to know what something is and what it is not (Lo, 2012). Furthermore, to fully know what something is, one needs to *simultaneously* know what it is not.

Variation Theorists (Lo, 2012; Marton, et al., 2004) have defined three patterns of variation which are critical for awareness on the object of learning and another which is also important but does not bring about awareness. The first is brought about by experiencing difference between the critical features and is known as **contrast**. This pattern of variation informed the design and enactment of both Functions 1 and 2. For instance, Task 1 of Functions 1 required teachers to sort multiple representations into functions and non- functions. The task was intended to help teachers discern difference between Function and what a Function is not. Similarly, Task 1 of Functions 2 was designed to help teachers discern difference between one-to-one and many-to-one functions. There, both graphs of each type of function were foregrounded simultaneously, and a horizontal ruler test was brought to bear on them so that teachers could notice that the ruler cut a one-to-one graph once and a many-to-one graph more than once. In this example, the one-to-one and many-to-one functions were critical features and the dimension of variation (or critical aspect) was Functions. In contrast, the critical features are varying and are in focus and the dimension of variation is invariant. In other words, when attention is on a one-to-one function it does not change the fact that it is a *function* and when it is on many-to-one, we are still dealing with a *function*. Hence function is *invarian*t.

The notion of invariance refers to noticing those aspects of the object of learning that do not vary. Invariance is central to noticing similarity between the critical features. Awareness brought about by experiencing similarity between the critical features constitutes **generalisation** (another pattern of variation). In other words, we are enabled to generalise from discernment of similarity. In generalisation, what is in focus is that which is invariant. Here the critical features do vary, just as in contrast, but are out of focus. This pattern of variation was present in Task 1 of Functions 1. For instance, a vertical line test was used to explain that a parabola was a function. To do this the Facilitator introduced both the parabola and a linear graph simultaneously and applied the vertical line test to show that it cut each graph once. Now, what was invariant here was the fact that the vertical line cut each graph once, and this was what teachers were required to notice - it was critical for the realisation that for every x value (input) there was only one y value .

The third pattern of variation that brought about awareness was **Fusion**. **Fusion** is awareness brought about by simultaneous variation on two or more dimensions of variation. Its significance is that it facilitates the discernment of the relationship between the parts themselves and between the parts and the whole. Fusion informed the construction of Task 3 where both domain and range (two dimensions of variation) were designed to vary simultaneously. For instance, the task first required teachers to find a function with the domain and range both real numbers. Second, they then had to find a function where the domain was all real numbers and range not all real numbers. Third, they had to find a function with the domain not all real numbers and the range all real numbers. Fourth, they had to find a function with both the domain and range not all real numbers. The significance of fusion in this task was that it was meant to facilitate discernment of the relationship between the domain and range, and between these aspects and the function itself. **Separation** was one pattern of variation that does not directly bring about awareness but instead facilitates **generalisation**.

The **space of variation/learning** is another key aspect of Variation Theory. The space of variation/learning is constituted by several **dimensions of variation,** and so all that is possible to learn is located in this space. In a teaching sequence the teacher calls learners' attention to one critical aspect of the object of learning and moves from there to other aspects. Now, each aspect which has been attended to constitutes a dimension of variation. For instance, in the teaching of Linear Functions, the teacher might focus on each of these: (i) definition of function, (i) one-to-one function (iii) line test (iv) gradient, and (v) y and x intercepts. Each of these five aspects constitutes a dimension of variation. So the space of variation contains all the dimensions of variation, or aspects of the object of learning, which were made possible to learn – i.e. those aspects to which learners' attention was called.

As will be seen, later tasks in Functions 1 and Functions 2 moved on from the relationship between input and output, domain and range, and what was and was not a function, to multiple representations of functions, and their transformations. The space of variation can thus be considered both within and across sessions and tasks In general, I hope through discussion of Task 1 of Functions 1, that I have both illustrated aspects of the theme work in the WMCS PD in 2011, and at the same time, elaborated those aspects of Variation Theory that informed this work, and thus framed my interpretation of the PD in the study.

2.3. Evolving object focused model of professional development

2.3.1. The DMJ project theme work for 2010

As discussed, DMJ was the core of the PD in the WMCS project. The goal of DMJ project was to enhance mathematics teachers' professional knowledge with particular focus on subject matter knowledge (SMK), and pedagogical content knowledge (PCK) and curricular knowledge (CK) (Adler, July 2011)**.**

The DMJ project was informed by the view that mathematical judgement was essential for skilful teaching. Since the development of mathematical judgement rested on strong mathematical knowledge for teaching, the aim of the DMJ project was to develop '... teachers' mathematical judgement through deliberate teaching focused on key mathematical objects of learning ...' (Adler, July 2011, p.1). In the next sub-headings I describe the professional development work under the DMJ project which was done in 2010 and 2011 separately.

In 2010 the DMJ project focused on the teaching and learning of key mathematical objects of learning (Adler, July 2011). As noted above, the DMJ professional developemnt was conducted at the Wits School of Education and in the teachers' schools. DMJ project theme work took place across four weeks. In the first week all teachers from the three clusters converged at the Wits School of Education where they were introduced to the theme and the rationale for a particular selected mathematics object, its key features, some pedagogical content knowledge issues (like learner error) and where the topic was located in the curriculum. In the second week two facilitators from the WMCS project team members went to schools to demonstrate a short lesson where a key feature of a particular agreed object was in focus. The demonstrations took place in a classroom with learners during the school hours at a particular Grade level, and the intention was that teachers would organise their time so that at least one other could observe the lesson as well, followed by some collective discussion, when this was possible to arrange in the school time-table. Typically, this had to occur in the lunch break or after school. In the third week volunteer teachers taught the same demonstration lesson in the classroom, where it was intended that other teachers would observe, as well as WMCS project team members. In the fourth week all teachers converged at the Wits School of Education to share experiences and reflect on demonstration lessons.

In short, while there was much that was beneficial, particularly working with learners and in classroom conditions in the schools in which teachers worked, the model, depending on teachers being able to collaborate in schools during teaching hours, was fraught with challenges. Most significantly, overloaded timetables made it impossible for other teachers to observe demonstration lessons in their colleagues' classes, and reflection time between the teacher who taught the lesson, his/her colleagues who observed it and WMC-S project team members, was limited (see Adler, July 2011). Changes were thus made to how the themework was organised in 2011.

2.3.2. The DMJ project theme work for 2011

In 2011 the DMJ project maintained the principle of using mathematics topics as objects of learning in its theme work. In the period January – June 2011 the selected mathematics topics were Integers, Functions, Geometry and Probability. Theme 1 focused on Integers, Theme 2 on Functions at FET level, Theme 3 on Geometry and Theme 4 on Probability. Each theme had four sessions, and in my research I refer to the Functions theme work sessions as Functions 1, 2, 3 and 4.

As mentioned frequently already, it was the Functions theme work that became the focus of my study. Hence I used its tasks to illustrate the DMJ project theme work done in 2011. Suffice it to mention that though each theme work had its targeted level (i.e. GET or FET), all teachers were free to participate irrespective of the level (FET and GET teachers participated in both Integer and Functions Theme works). This is worth mentioning because participants in my study were GET and FET teachers, and the focus of my research was on Functions theme work (which focused on Functions at FET level). Clarity on this point as early as now and here, will obviate confusion in later chapters, particualarly in the methodology chapter when I discuss participants in my study. The confusion might raise the question of why I worked with GET teachers in my research when the Functions Theme work was focused at FET level. So I provide this clarification here in anticipation.

The DMJ project retained the four weeks format and demonstration lessons with learners. The major adjustment to the theme work saw the work in schools taking place for two hours after school and outside normal classroom teaching.

Below I describe examples from the Functions Theme work sessions to illustrate the content of PD activities. I use examples from the video data that I have analyzed for my research. Discussing them here will assist the reader to comprehend my analysis in later chapters of the enacted object of learning. In addition my data analysis assumes the reader has a general view of the WMCS project's professional development and a view of the planned tasks particularly for Functions 1 and Functions 2.

In my research I was concerned with teacher learning with respect to the mathematics content, which was the primary object of learning in the DMJ project. The DMJ project's immediate goal was to strenghten teachers' subject matter knowledge of functions, and so I was interested in what became the enacted object of learning when a mathematical object was in focus. Hence I did not work with video-recordings of Functions 3 and Functions 4 because, as noted ealier, it was not clear what teachers were to learn from these sessions.

2.3.3. Functions Theme work

The duration of this theme work was over four weeks. The central resource in the Functions Theme work was a set of card activities involving multiple representations (see **Table 2.1**.) These activites required teachers to work in groups.

- **First Session: - Functions 1**

The goal of Functions 1 was to help teachers see Functions as constituting a relationship, and not as a procedure that produces a graph. Functions 1 had two tasks, which are elaborated on below:

Task 1: Card Sorting

The task required teachers to sort cards containing the various multiple representations into two piles of function and non-function. Teachers worked in groups of three at the E-cluster and four at the other two clusters. Team members sat in the groups to provide assistance . Each group had a set of 24 cards consisting of 23 function representations and 1 non-function. Team members were the only ones who were aware of the composition of the cards. My analysis chapters show teachers' responses to Task 1.

Task 2: Card Matching

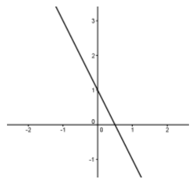
Task 2 required teachers, still working in groups, and with the same resource (see **Table 2.1**) to match multiple representations of the same linear or quadratic or hyperbolic function. In

Table 2.1 the first row contains multiple representation of the same Linear Function and the second row has multpile representations of the same quadratic function.

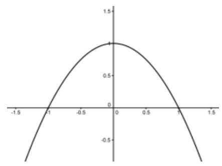
Table 2.1 shows some of the cards that were used in Functions 1 tasks. While the complete resource contained 24 cards, I have shown only 8 for brevity. The complete resource is appended (**see Appendix A**)

Table 2.1: Resource used in Functions 1 Tasks

<table>
<tr><td></td><td>Algebraic</td><td>Numeric</td><td>Graphical</td><td>Verbal</td></tr>
<tr><td>Linear</td><td>1

y = – 2x + 1</td><td>5
<table><tr><td>x</td><td>-3</td><td>0</td><td>1</td><td>5</td></tr><tr><td>y</td><td>7</td><td>1</td><td>-1</td><td>-9</td></tr></table></td><td>11
</td><td>16

To get y you double x, multiply by -1 and then add 1</td></tr>
<tr><td>Quadratic 1</td><td>4

f (x) = 1 – x²</td><td>23
<table><tr><td>X</td><td>f (x)</td></tr><tr><td>-2</td><td>-3</td></tr><tr><td>0</td><td>1</td></tr><tr><td>1</td><td>0</td></tr><tr><td>12</td><td>-143</td></tr></table></td><td>12
</td><td>3

To get f (x) you square x and subtract it from 1</td></tr>
</table>

Source: **WMC-S project archives**

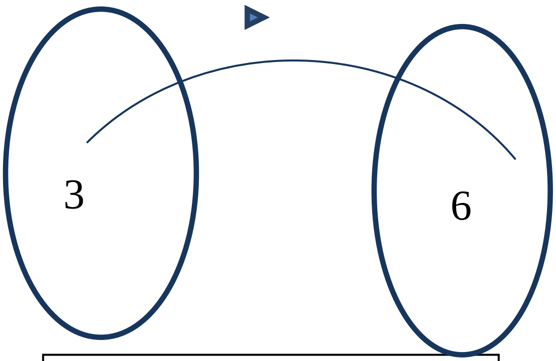
- **Second Session: Functions 2**

The goal of Functions 2 was to engage teachers in mathematical thinking tasks that challenged and extended their knowledge of Functions. The mathematical thinking tasks centred around transformation of functions. In these activities teachers had to find algebraic or graphical representations of transformed functions. Functions 2 had five tasks, and I show examples Task 1 and 4 below. In Task 1 teachers had to find Functions with a turning point at (5,4). In Task 4, teachers were given three conditions to work with to find eight Functions that satisfied one or more or all conditions.

Task 1 was used to introduce the notion of function as a relationship, which was central in Task 2. The introduction involved the use of four representations of a quadratic function to build on the work done in Functions 1. In that work the terms Input and Output were extensively used to help teachers see a function as a relationship between inputs and outputs. To build on that work the facilitator used Task 1 to show the connection between the terms

Input and Domain and Output and Range. **Figure 2.1** shows an example of a resource used in Task 1 of Functions 2. Teacher activity associated with this task involved teachers identifying the domain and range of $f(x)=(x-1)^2+2$ from its graphical representation. Building on teachers' responses the facilitator asked teachers to adapt $f(x)=(x-1)^2+2$ such that it had a maximum at its turning point. Teachers were also asked to find a function with a turning point at (5; 4). Geogebra was used to help teachers to experience compression and expansion on the graph of $f(x) = (x-1)^2 + 2$ by varying the coefficient of $(x-1)^2$ with the aid of a slider.

$f(x)=(x-1)^2+2$



For this function, you can input any value of x but every output value is greater than or equal to 2.

x	-3	-2	-1	0	1	2	3	4
f(x)	18	11	6	3	2	3	6	11

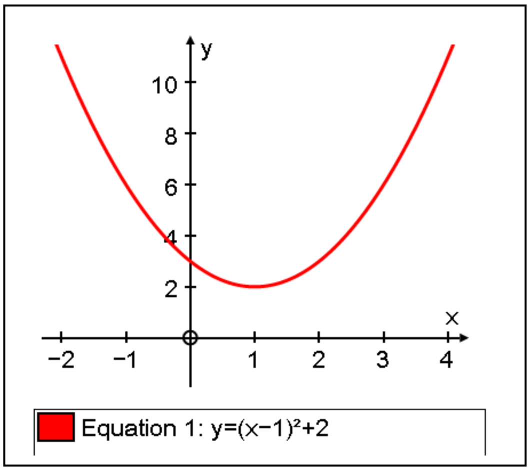
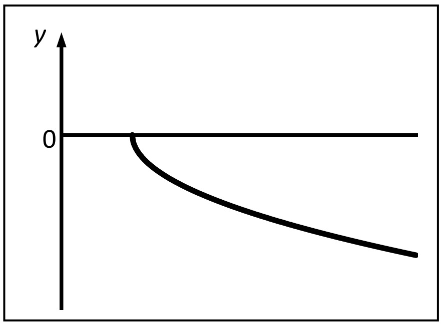


Figure 2.1: A resource used in Functions 2

Task 2 of Functions 2 was a card-matching activity involving an algebraic, graph, domain and range. Teachers worked in groups of three or four to find and match graphs to their equations, domains and ranges. The resource was 36 cards, with 9 cards for equation, 9 for graph, 9 for domain and 9 for range (see **Appendix B**). Some of the cards were incomplete and teachers were required to complete these such that they always had a set of four cards. The task was complete when a group had produced nine sets of four cards. **Figure 2.2** shows an example of four matching cards

$$y = -\sqrt{x-1}$$



$$x \in \mathbb{R}, \quad x \geq 1$$

$$y \in \mathbb{R}, \quad y \leq 0$$

Figure 2. 2: Example from Task 2 of Functions 2

Source: WMC-S project archives

- **Example of a resource used in Task 3 of Functions 2.**

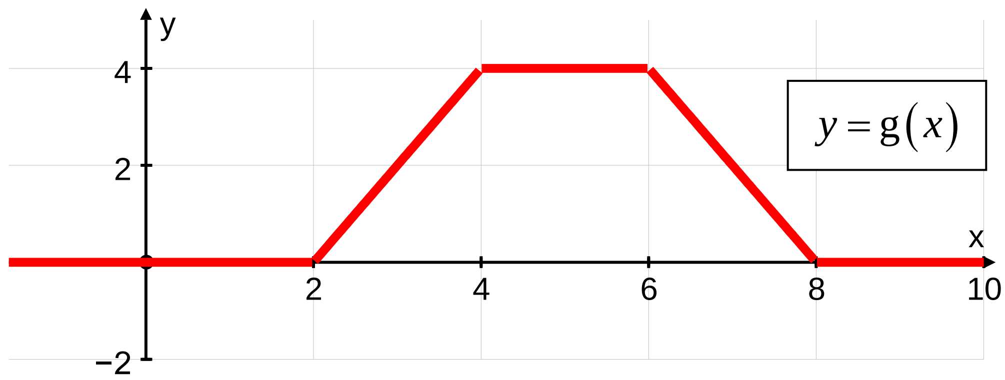
Task 3 was an extension of Task 2. It required teachers to find a function for each domain – range combination. This was done by filling the missing spaces with functions that satisfied the given domain – range combination. By way of illustration, in a space where I have placed the capital letter A, teachers needed to find a function with domain all real numbers and the range all real numbers. In the space where I placed B, teachers needed to find a function with the domain consisting of non-negative numbers and range all positive integers. In the last space of column 3 I have inserted an example of a function which teachers needed to find to fill that particular space. This function was the product of discussions between teachers and between teachers and the Facilitator at the E-Cluster.

Table 2.2: Resource used in Task 3 of Functions 2

Domain / Range	$x \in \mathbb{R}$	$x \in \mathbb{R}, x \geq 0$
$y \in \mathbb{R}$	A	
$y \in \mathbb{R}$, $y \geq 1$		B
$y \in \mathbb{R}$, $-1 \leq y \leq 1$		$f(x) = Sin(\sqrt{x}\ )$

- **Example of a resource used in Task 4 of Functions 2**

Task 4 was used at the I-Cluster, where it was used to explain why the graph of $g(x-1)$ shifts to the right. Teachers were required to complete the table and then plot the graph on the cartesian plane to observe its shift. The following figure was referred to as Bernard's table by one of the teachers. Hence I referred to it as Bernard's table in my analysis.



x	0	1	2	3	4	5	6	7	8	9	10
$g(x)$											
$g(x-1)$											
$g(2x)$											

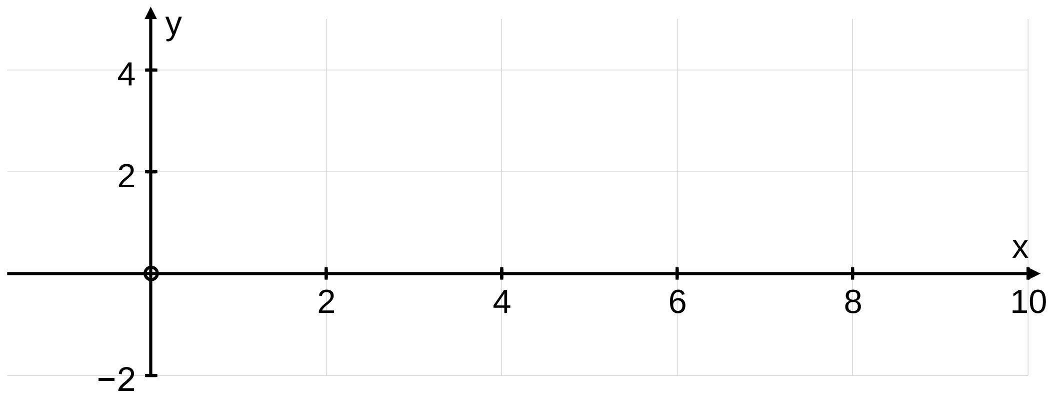


Figure 2.3: Resource used in Task 4 at I-Cluster only

- **Example of a resource used in Task 5 of Functions 2**

Task 5 required teachers to fill the regions constituted by the intersection of the three circles. In the task teachers had to think about all transformations of $\mathrm{f}(x)=x^2$ under these three properties

A: The range includes the value $y=-1$
B: The graph is of the form $y=\mathrm{f}(x-3)+a$
C: The graph passes through the fourth quadrant

In this task teachers had to find functions which were transformations of $\mathrm{f}(x)=x^2$ and which either satisfied all three properties, A, B, and C; any two properties; any one property; or none. If found, it had to be written in the appropriate region. A function that satisfied none of the properties was to be written outside the three circles. So altogether there were 8 regions, with 7 constituted by the circles, and the eight regions being the space outside the three circles.

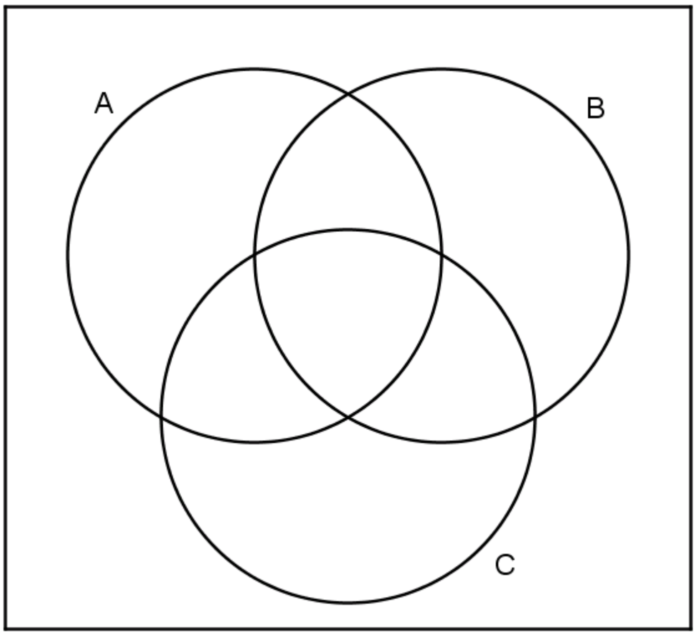


Figure 2.4: A resource used in Task 4 at E- & N-Clusters only

2.4. The WMCS project professional development in 2012 and 2013 – 20 day courses

As I will argue through this study, teachers learned a great deal in the functions themework. However, it became increasingly clear to the project that conducting PD weekly after school work was placing considerable strain on teachers. There was insufficient time in and across theme sessions to focus in detail on some key features of mathematics where teachers were less confident and knowledgeable; and whilst there were support materials for teachers to work on in between sessions this generally did not occur. Time for teachers to work on their mathematics, with facilitators and other teachers, and on their own, was critical, yet not optimal, hence the model evolved into a 20-day course in 2012, 2013 and 2014 (see Adler, July 2012, June 2013). While the time and place of the PD changed, its object focused orientation remained unchanged. Hence the continued relevance of my research on the enacted and lived object of learning.

2.5. Management of tensions/dilemmas in the WMCS PD

As has become clear in the discussion of the DMJ work, WMCS professional development encountered tensions before and during its implementation, and how these were managed shaped its constitution. Understanding this contributes a clearer understanding and description of what has come to be known as the WMCS project, and more importantly, its object focused professional development, and the context of the professional development in focus in this study.

Evident in the above sections are what Graven (2005) and Adler (June 2013), building on Adler's earlier work on teaching dilemmas, Adler (2001) and Graven's work described as inherent tensions in any professional development. They discuss 'where' (i.e., place), when (i.e., duration and time) and 'what' (selection of content in the PD), together with 'who' (selection of participants) 'sustainability' (scale, influence) and ethos' as six inherent tensions, which can only be managed, but cannot be resolved. Choices are inevitably made, as is evident in the discussion above. WMCS made choices about the content focus of the PD, as well as where and when this would occur. In both 2010 and 2011, the feasibility of working, even over time, with teachers in the afternoons after schools constrained activity, and so for 2012 (and continuing since then) the PD was organised in a 20-day programme, with 8 x 2 full day sessions where teachers are released from school on some of the days; other days are Saturdays; and 4 days in school work. In each of the 8 x 2 day sessions 75% of the time is on SMK (again, with key contents pertinent to the secondary school curriculum), and 25% of the time, on turning aspects of the content into objects of learning in school.

In her 2013 report (Adler, op cit) about the project, Adler illuminates how the five dilemmas played out in the project over time. In this research I focus on the 'what', together with brief discussion of 'ethos', as these are pertinent to the study, and drawn upon in reflecting and explaining findings in Chapter 10.

2.6. Managing tensions around 'what' in the WMC-S project

Adler (op cit) described the critical tension around 'what' as between SMK and PCK. Thus the 'what' tension was imposed by the duality of the object of learning, and was managed by privileging mathematics content and not its teaching. The privileged mathematics content in the DMJ and Content Focused Project (CFP) were Integers; Functions; Probability; Geometry; and Trigonometry. In the 20-day course the content was increased to include topics such as Differential Calculus. While school mathematics was the 'what' of the PD, Functions per se was the 'what' of the Functions theme work. In the PD, Functions per se, were constituted by key features. I elaborate on key features below.

As earlier noted, the work of the QUANTUM project has described the duality in the object of learning in teacher education and elaborated there - that the object of learning is either a mathematics or a teaching object, and when one is in the fore the other is implicitly present. Similarly, professional development as a form of teacher education is not free from this tension, and so choices have to be made in the design of the professional development with respect to what becomes the object of learning between these two contending objects. As noted earlier, while most professional development tends to take the teaching object route (Brodie, 2012; Chauraya & Brodie, 2011), the WMCS project took the mathematics object route.

As noted earlier in Chapter 1, the WMCS project chose not to copy other models of professional development because the choice of focus in any professional development is informed by the nature and context of the problem being addressed (Adler, November 2010).

In the WMC-S project the focus was to strengthen teachers' subject matter knowledge (SMK), and this was reinforced by earlier classroom observations by the WMCS project's team members which had revealed gaps in the teachers' SMK of the content they taught (Adler, November 2010). The presence of gaps in teachers' SMK of the mathematics they taught suggested a content focused model of professional development because of its capability to help teachers experience such content directly (Huillet, et al., 2011). Adler (op cit) reaffirmed this argument further in her report, where she asserted that teachers need opportunities to learn the mathematics directly rather than through an amalgam of mathematics and teaching objects, because this might impact on what is made possible to learn.

The **key features** of the functions were the intended object of learning in the functions theme work. Elsewhere I mentioned that in the planning of the functions sessions professional developers selected to privilege what they called key features of functions, and these included properties of Functions, multiple representations of Functions and Function terminology. The properties of Functions covered (i) definition; (ii) x and y intercepts; (iii) effects of parameters on Functions (e.g. compression) and (iv) transformation of functions (e.g. vertical and horizontal shifts), (v) arbitrariness; (vi) univalence, (vii) interpreting, (viii) translating between representations, (ix) treating functions as objects and (x) being able to act on the objects; and (xi) the ability to distinguish between the concept of function as a relation satisfying univalence and its instantiations.

I have shown examples of tasks from Functions 1 and 2 where the key features were present. I have worked with the same examples in my study to gain insight into both the enacted and lived object of learning. Hence the key features were the intended object of learning for the PD and my research.

2.7. Managing tensions around 'ethos' in the WMC-S project

As a member of the WMCS project's team members I noticed that it was relatively easy for us to design and implement professional development tasks with teachers. However, we were not clear to ourselves or to teachers what we expected them to be able to do at the end of their participation in these tasks. This was constrained by the view that teachers were not to be positioned as deficit of knowledge. This view made it difficult for us to tell teachers what we expected them to do with ideas learnt from experiencing professional development tasks.

Arising from our earlier classroom observations, teachers' requests to be resourced on certain topics and teacher responses on professional development tasks, showed that teachers' knowledge of mathematics per se was tenuous. However, we could not agree on what message needed to be communicated to teachers, because we did not want to give teachers the impression that what we were doing or demonstrating in professional development was the best practice which they needed to follow. There were divisions around this tension, with some team members feeling that teachers did not know, and so they should be encouraged to try professional development ideas in their classrooms, and others feeling that doing so would position teachers as lacking in knowledge. This tension is well summarised by Adler (June 2013) when she asked: '[h]ow does a programme attend to repair, without simultaneously constructing a 'lack' in teachers ...?' The tension about 'ethos' was managed by the introduction of the MFT tool which was able to shift attention from the teacher to exemplification, explanation and learner activity.

2.8. Summary

In this chapter I have discussed the focus of the WMCS PD with respect to its goal of strengthening mathematics teachers' SMK. In my discussion I used examples of tasks used in the PD, and moved on to foreground three tensions which were of particular interest in my research. Having discussed the WMCS PD and its focus here, it is interesting to know how this model is similar to, or different from other PD models. Hence in the next chapter I compare and contrast the WMCS PD model against other PD models. At the center of my comparison or contrast, is Graven's (op cit) three elements, which she believes are inherent in any PD model.

CHAPTER 3: LITERATURE REVIEW

3.1. Introduction

This research rests on the following key components:

(i) Professional development needed

(ii) Models of professional development

(iii) Mathematical knowledge for teaching (MKT)

(iv) Functions

In this review I open the chapter with a discussion of the importance of professional development. In the discussion I describe the features of effective professional development as emerging from research in the field which is identified in the literature, and show which features were inherent in the Wits Maths Connect project's professional development. My next discussion is on models of professional development. Here I describe general models and compare and contrast the focus of these with the focus of the WMC-S project's model. In my third discussion I focus on mathematical knowledge for teaching (MKT). It is essential that I discuss the constitution of the MKT because the development of Part 3 of my interview protocol was guided by earlier work done in this area. My fourth discussion focuses on functions because this mathematics content was central in the WMCS project's key features of Functions and pertinent literature on mediating Functions. I conclude the chapter with a summary of the reviewed literature.

3.2. Significance of Teacher professional development

Teacher professional development is considered key to unlocking learning barriers the majority of learners experience in schools (Fennema & Romberg, 1999; Kazemi & Franke, 2003), especially those emanating from mathematics (CTL, 2009). It suffices to note that most challenges teachers encounter in their teaching practice are novel to them. These challenges arise during the course of teaching as learners interact with new knowledge (Ball & Cohen, 1999). The nature of these challenges is sometimes in the form of questions learners ask, while at other times it is classroom records of practice that include errors, misconceptions, and learning difficulties. In most instances these challenges go unattended, even when teachers have noticed them (Ball & Cohen, 1999).

There are reasons why these challenges may go on unattended. Firstly, teachers may not mediate unanticipated classroom events because they do not recognize them as potential learning hazards. Second, in cases when they do recognize them, teachers may possess neither the required knowledge nor skill (Shulman, 2004). It is assumed teachers' professional knowledge grows in the field during the course of teaching practice (Shulman, 1987). This knowledge is thought to grow from teachers' continued interaction with classroom events, which are thought to provide teachers with learning opportunities (Shulman, 2004). It is believed that in the process of mediating unanticipated classroom events teachers develop unique ways designed to address such. Hence their knowledge grows. In spite of that, research (Blum & Krauss, 2008) on mathematics teaching and learning provides evidence that refutes this assumption.

Hill, Rowan, and Ball (2005) and Blum and Krauss (2008) recently found no relationship between teaching experience and teacher professional knowledge. This finding has since been interpreted by Blum and Krauss (2008) as indicating that teachers in their study drew from knowledge acquired during teacher preparation in their teaching, and not from experience. This finding help explain why unanticipated classroom records of practice are so persistent. Teachers in the mathematics classroom draw on knowledge acquired during teacher preparation. However, this knowledge does not provide them with ways of mediating unanticipated classroom records of practice, such as learner errors and misconceptions (Blum & Krauss, 2008; Hill, et al., 2005). Furthermore, teacher preparation courses equip teachers with theoretical knowledge that prepares them for general situations that may arise in the course of teaching. However, in practice teachers encounter unique and specific challenges which are often divorced from theoretical knowledge earlier acquire during teacher preparation (Shulman, 2004). In other words, more often than not, mathematics teachers encounter teaching and learning challenges for which they may never have been prepared by teacher preparation programs (Ball & Cohen, 1999).

Notwithstanding this fact, teachers are often blamed for low mathematical achievement, which in most cases is a consequence of unmediated learning constraints beyond their knowledge. This tendency has been questioned in the mathematics education fraternity (Ball, Hill, & Bass, 2005). For instance, Ball and Cohen (1999) wondered "how [could] teachers teach mathematics that they never learned, in ways they never experienced [themselves]"? (p1). In addition, Ball, et al., (2005) noted that teachers were products of defective education systems which were currently being revised. Ball and Cohen (1999) have also described the dilemma facing reforms that promote changes in instruction to deepen students' mathematical understanding. The authors describe the dilemma as follows:

> On the one hand, teachers are seen as the root of the problem: Their instruction is mechanical, often boring, and superficial. On the other hand, teachers are cast as the key agents of improvement because students will not learn the new mathematics that policymakers intend unless teachers learn that math and teach it (p1).

The evidence provided by Hill et. al (2005) and Blum and Kruass (2008) imply that professional development of teachers is indeed a necessity which cannot be left to chance. Teacher professional development which uses classroom records of practice as the basis of its content has potential to prepare teachers to mediate unanticipated challenges that usually arise in mathematics classrooms. At the WMCS project the content of professional development is adopted from the school curriculum and, more importantly, teachers engage with content that they will later teach in their own classrooms. Professional development is aligned with the education department teaching schedule such that content covered in professional development becomes useful in teachers' own classes.

In their recommendation, McCarthy and Oliphant (2013) point to teacher professional development as a possible solution to the problem of low learner achievement. It is interesting, but not surprising, that, while the problem is low learner achievement, the recommendation points to the need to improve teachers. This is so because these reports locate the problem of low learner achievement in teachers' subject matter knowledge. In the introduction chapter I provided quotes from these reports, where the authors were articulating the magnitude of low learner achievement and its associated gaps in teachers' subject matter knowledge. The perception that teacher professional development is a way to redress the problem suggests its significance.

The recognition that the nature of work teachers do does situate them as lifelong learners suggests continous professional development. The existence of a school suggests the intention of a society's desire to pass its culture to future generations, but culture is not static, as it keeps changing in order to remain relevant to the society's changing needs (Wood & Wood, 1996a). By extension, teachers, as agents of change, need to be continually learning to improve their knowledge and skills to facilitate the transfer of the society's culture to its future generations. In South Africa mathematics teachers are challenged to learn in order to meet the challenges of the continous change in the curriculum, and to also know the mathematics they teach. Providing teacher professional development is one way of ensuring that teachers continue to learn from practice.

Third, evidence from early research (Kaino & Moalosi, 2013; Moalosi, 2008; Moalosi & Kaino, 2010b; Shulman, 1987; Tirosh, Even, & Robinson, 1998) has shown that many teachers do not promote meaningful learning of mathematics in their classrooms. Meaningful learning resonates well with Artigue (2013) 's learner-friendly mathematics. For Artigue (2013, p. 27) , mathematics education is learner-friendly when it makes learners:

> confident in their mathematical abilities, cultivate their interest towards this sequence and their capacity for learning. It must open the world, eanbling learners to understand and experience the power of mathematics for understanding the world around them and addressing the challenges it raises, but it must also make them experience and appreciate the pleasure and beauty of mathematical practice and inquiry for its own sake; it must equip learner with the mathematical literacy skills for citizenship, ...

It is expected of teachers to change their practice on the basis of evidence accumulated over the years. Usually, this might be evidence of unattained educational goals, such as improved learner achievement. However, it appears experiences gained in the classroom are not sufficient to challenge and encourage teachers to re-examine their classroom practices (Even, 1999). Desforges (as cited by Even, 1999) claims that in practice teachers "tend to close down rather than open up to experience by means of [restructuring their knowledge]" (p.1) For example, according to Desforges, an encounter with unfamiliar events in the classroom does not usually encourage teachers to restructure their own conceptions and beliefs about teaching and learning. Even (1999) explains the status quo by citing Chinn and Brewer who say that “when encountering contradictory data in practice, teachers usually ignore them, reject, exclude as irrelevant, hold in abeyance, re-interpret them in terms of their existing personal theory, or make minor and peripheral changes to the theory. Consequently, practice alone is not enough to challenge educational practitioners' existing conceptions and beliefs. What teachers learn from experience does not appear to enhance their professional knowledge unless teachers are delibearately assisted to reflect on their practice (Chauraya & Brodie, 2011).

Since teachers do not change their practice unless they are challenged to do so, I chose to study teachers who participated in a professional development initiative which deliberately worked with them to help them improve their own subject matter knowledge of the mathematics content they teach.

3.3. Models of professional development

Graven (2005) has identified (i) professional community, (ii) artefacts of practice and (iii) specificity of content as three inherent elements in any professional development. I have used these elements to identify the foci of each model that I review below. Building on Pillay’s (2013) work with these elements, I used them to compare and contrast each model with the WMCS PD model. I have used shadings to show similarities and differences between each model and the WMCS PD model. Now, while the shading in the WMCS project’s model is invariant, in other models the shading varies according to what was foregrounded. There are several models that foregrounds specificity of content. However, among the models that I have discussed in this chapter, the model that foregrounds content other than the WMCS model is the Learning Study model. I return later to explain the source of the similarity between the two models in my summary of this review. In this review I focus my discussion on five models. Each model has a unique combination of the three elements and emphasis.

3.4. Professional learning communities

The professional learning community as a model of professional development brings professional developers and teachers together to co-constitute knowledge that is of relevance and immediate use in the teachers’ classroom. Borko (2004) has described the functioning of the professional learning community, and key among the requirements for an effective professional learning community is what she refers to as Teacher Leader. In a professional learning community a teacher leader is a more knowledgeable person who guides the learning community. The work of the professional learning community begins with the establishment of a community, and to do this is not trivial. For instance, Borko (2004) cited Carpenter and his colleagues, stating that ‘the development of teacher communities is difficult and time-consuming work’ and that ‘the norms that promote supportive yet challenging conversations about teaching are one of the most important features of successful learning communities’ (p.14). For the professional learning community to function effectively it needs to foster these conversations to help teachers to trust each other, so that they may talk openly about their own practices. The professional learning community also needs to develop communication norms that facilitate critical dialogue, respect and critical analysis of teaching-related issues (Borko, 2004).

The essence of the professional learning community is to help teachers to be accountable to their practice, and this approach to PD might be seen as a great shift from positioning teachers as coming to *get* knowledge, but as coming to *construct* knowledge which has relevance and is of immediate use in the classroom. Past PD initiatives are believed to have failed to bring about change in the classroom because they were far removed from what teachers did in the classroom on daily basis. These PD initiatives were often in the form of one-day-workshops or in-service seminars that positioned teachers as lacking in knowledge - and so were designed to give them that knowledge (CTL, 2009). Borko (2004) says they were 'fragmented, intellectually superficial,' and did not consider what was known about how teachers learn. These conventional teacher professional development initiatives did not consider that learning in 'professional development was a complex process which was not yet well understood' (Brodie & Shalem, 2011).

The function of a professional learning community is to work around a common teaching and learning task under the guidance of a knowledgeable other. This model of PD views learning as situated in the community, and draws from socio-cultural perspectives and also largely from Wenger (1998) and other scholars in this domain. Professional Learning Communities is believed to be the kind of PD which attempts to address the needs and demands of the new reforms in education. This model goes beyond helping teachers to acquire new knowledge and skills to helping them to: reflect on their own practice, and develop "new classroom roles and expectations about student outcomes, and to teach in ways they have never taught before'' (Darling-Hammond & McLaughlin, as cited by Vescio, Ross, & Adams, 2008, p. 80).

The Data Informed Practice Instruction Project (DIPIP) in South Africa is an example of a professional development initiative that follows the PLC model. I use the work of DIPIP to tease out the features of a professional learning community model of professional development. The DIPIP professional learning community was constituted by professional development team members and teachers from selected secondary and primary schools. According to Brodie and Shalem (2011), the number of teachers participating in this project ranged between 45 and 50, and the teachers were teaching at Grades 3 - 9 levels. In this particular professional learning community learner errors were in focus, and so the community worked together to help teachers to develop their understanding of these errors. The work of the DIPIP is briefly summarised by Brodie and Shalem (2011, p. 423) when they point out that the central focus of the DIPIP was to work with teachers to strengthen their understanding of learner errors, and this they did by adhering to the following process:

Analysis of learner results on an internationally standardized, multiple-choice test, with a particular focus on the reasoned errors implied by the distractors for each test item; Mapping of the test in relation to the South African mathematics curriculum; Reading and discussions of texts in relation to learner errors on a mathematical concept; Drawing on the above three analyses, developing lesson plans for between three and five lessons, which aim to engage with learner errors and misconceptions in relation to the concept; and reflections on videotaped lessons of some teachers teaching from the lesson plans.

Table 3. 1: Three elements inherent in any PLCs

		Teaching object	Mathematics object
Model	**Professional Community**	**Artefacts of practice**	**Specificity of Content**
DIPIP	Community of practice	Learner errors	Mathematical concepts
WMC-S project	teamwork	Readings, card activities, grids & geogebra	Functions per se

In Table 3.1 the DIPIP model (i.e., PLC) both community of practice and learner error are shaded grey. This suggests that the PLC model focused on developing a community of practice around learner error (i.e., teaching object). In summary, it is evident in the professional learning community model as exemplified by the DIPIP that, while the focus is on developing a community of practice around learner error, the specificity of content is backgrounded. In contrast, the specificity of content is in the fore in the WMCS project (see **Table 3.1**).

3.5. Lesson Study

A lesson study is a professional development model that has been used to strengthen teachers' mathematical knowledge in the classroom setting. This model has its origins in Japan, where it was used to facilitate scaffolding of teaching experiences from the more experienced to novice teachers. It is a professional development process that teachers engage in to systematically examine their practice. Its goal is to improve the effectiveness of the experiences that the teachers provide to their students. The essence of this model is teacher collaboration in planning, implementing and evaluation of taught lessons (Jita, Maree, & Ndalane, 2005).

Teacher collaboration in teaching of lessons provides opportunities for teachers to learn from each other by sharing experiences, ideas, teaching strategies and challenges. The use of this model brought about success in some teaching situations where it was applied. The focus of this model is on the lesson. The lesson study cycle is characterized by (i) Collaborative planning - where a team of teachers plan a lesson together (ii) Observing the lesson- teachers observe a lesson that they had planned earlier, (ii) Discussing the lesson- teachers reflect on the taught lesson, (iv) Revising the lesson - teachers make improvements to the lesson and (v) Re-teaching the lesson- the improved lesson is taught again. This series of activities is performed by a small group of teachers who are sometimes guided by the more experienced teacher or expert.

This form of professional development has recorded notable success in Japan (Ono & Ferreira, 2009). The importance of this model to my study is that this aspect – teacher collaboration - is built into the WMC professional development model. For instance, teachers in the WMC professional development are divided into clusters, and activities allow for teachers to work on tasks in teams and present their work to the rest of the cluster.

In summary, unlike the WMC-S model, this model focus on the teaching aspect by foregrounding the artefact of practice while remaining silent about the specific content of the lesson, but emphasized more on ways to improve the quality of the lesson. This does not suggest there is no content, but rather, proponents of this model improving the quality of lessons is important for improving teaching quality in the classroom (see **Table 3.2**)

Table 3. 2: Three elements inherent in Lesson Study

		Teaching object	Mathematics object
Model	**Professional Community**	**Artefacts of practice**	**Specificity of Content**
Lesson Study	Collaboration	Lesson planning	Mathematical concepts
WMC-S project	teamwork	Readings, card activities, grids & geogebra	Functions per se

3.6. Learning Study model

The learning study model of professional development is a recent development in the domain of professional development. It evolved from the Lesson Study model discussed above. The similarities between the two are that both models foreground the lesson plan. In the Learning Study about six teachers work together to design, implement, observe, reflect, improve and re-teach the lesson focused on a specific object of learning (Marton & Pang, 2006). A learning study draws its structure and focus from variation theory. How does the learning study cycle operate? A Learning Study begins with teachers identifying a particular object of learning which is critical for learners to learn. This might be an object of learning which is difficult for learners to discern, or difficult to teach in ways that makes it easy for learners to comprehend. The next stage involves teachers working together collaboratively to plan a lesson around this particular object of learning. In the third stage one teacher teaches the lesson in a real classroom with learners, while other teachers who planned the lesson are present as observers (Marton & Pang, 2006). In the fourth stage teachers come together to reflect on the taught lesson plan with suggestions on what needs to be improved. In the fifth stage the improved lesson is taught again.

Table 3. 3: Three elements inherent in Learning Study

		Teaching object	Mathematics object
Model	**Professional Community**	**Artefacts of practice**	**Specificity of Content**
Learning Study	Collaboration	Lesson plan	Mathematical content
WMC-S project	Teamwork	Readings, card activities, Geogebra, Grids,	Functions per se

In summary, the Learning Study model foregrounds both artefact and specific mathematics content. What is similar between this model and the WMCS project model in the context of my research is that both foreground the specific content as the object of learning. In the overall model of the WMCS project artefacts and content are both present.

3.7. Collaborative Apprenticeship Model

This model is concerned with teacher growth, and its focus is collaboration. Proponents of this model believe that professional growth can be pervasive when teacher learning is viewed as a collective enterprise, and can be stifled without continual interactions (Gallander and Ford, as cited by Glazer & Hannafin, 2006). This model suggests that professional growth is a result of teachers sharing successful experiences and learning from each other's mistakes (Glazer & Hannafin, 2006). Teachers participating in this professional development model work together, where they continually use each other's instructional resources and skills to support mutual growth for the purpose of reaching the shared instructional and curricular goals. Glazer and Hannafin (2006) believe that professional development is more effective when situated in the teacher's everyday experiences such as classroom settings – a similar statement to that advanced by other researchers (e.g. Ball & Cohen, 1999; CTL, 2009). All these researchers believe that teacher learning experiences are more enhanced when they are situated in the context in which they will be needed.

A key tenet of this model is the notion of Reciprocal Interactions. Reciprocal Interactions is defined by Glazer and Hannafin (2006) as "interactions demonstrating and influencing a mutual relationship supporting teacher learning and development" ,(p.180). These interactions are usually expressed in different communication strategies such as "in writing, verbal and non-verbal gestures, and physical movements" (p.180). An example of reciprocal interaction between teachers is where colleagues troubleshoot an instructional problem together to reach a common solution. Another example is where a community of teachers collaboratively designs a teaching curriculum or any relevant instructional material such as writing a textbook. It is believed that during the interactions learning is stimulated when individuals help one another maximize their learning potential (Glazer & Hannafin, 2006).

In their interactions teachers are given opportunities to share experiences through storytelling, back-scratching, discussing and resolving conflict, brainstorming, giving and seeking advice, modeling, sharing ideas, motivating and reinforcing, posing and responding to task-based questions. There are four progressive professional development phases in the model. As is the case with lesson study model, this model backgrounds content, as it puts more emphasis on collaboration. However, some of its components are present in the WMC-S model, and they include storytelling - where teachers share their own classroom experiences. These experiences are essential for planning professional development. In addition, the WMC-S model facilitates brainstorming and posing and responding to task-based questions through its thematic sequencing.

For instance, in 2010 the WMC-S PD worked in schools following a three-cycle model. In cycle 1 the teachers sat to observe professional developers teaching an aspect of functions to learners. In cycle 2, teachers and professional developers sat together to reflect on the teaching, and in Cycle 3 a teacher was observed by professional developers and colleagues teaching the same aspect to his learners. This model was revised in 2012, but this time teachers across the project schools came to one venue to observe professional developers teaching 40 learners from one of the ten schools. The teaching was followed by a reflection session. Teachers were later followed to their schools to observe how they recontextualized the teaching in their classrooms.

This model resonates well with the WMCS professional development activities of 2010 and 2011, which respectively took place in mathematics classrooms and schools. For instance, in 2010 professional developers taught selected mathematics content in the school curriculum where key features of that content were in focus. The teaching was conducted in the afternoons after formal school lessons had ended. All mathematics teachers in that particular cluster came in to observe the teaching and learning, and the lesson was followed by a feedback session where teachers and professional developers sat together to discuss the outcomes of the lesson.

The following paragraphs explain the three stages embedded in the Reciprocal Interactions model, The model explains the roles of teacher-leaders (professional developers) and peer-teachers in the process of collaborative apprenticeship. There are four progressive professional development phases in the model. These are introduction, development, proficient, and mastery.

Introduction phase:

In this phase the teacher-leader introduces an instructional resource or strategy. The role of teacher-leader in this phase is that of mentor. The teacher-leader may be interested in sharing a strategy that has been successful in his own class, and therefore wishes to share it with other teachers who may be experiencing difficulties in enacting a certain component of the curriculum (Glazer and Hannafin, 2006). The role of peer-teachers in this phase is that of learners. Teachers reflect on the lesson modeled by the teacher-leader and analyze how their learning experiences can be applied to their own classroom practices. This phase is characterized by posing and responding to task-based questions that seek to transfer knowledge and skills among members of a community. Peer-teachers later implement a model of the lesson to their own learners. In the next meeting peer-teachers reflect on their experiences in group discussions through story-telling as a means of sharing and constructing knowledge and strategies. The use of storytelling helps teachers relate experiences in order to obtain interpretations of implementation strategies and expectations (Orr, as cited by Glazer and Hannafin, *ibid*).

Development phase

In this phase the teacher-leader and peer-teachers collaborate in the design, development and implementation of new instructional strategies or resource. The phase is characterized by brainstorming, which is a constituent of reciprocal interactions. The discussions are facilitated by the teacher-leader. Participation of less experienced peers is legitimized by the teacher-leader engaging them collaboratively in the development of learning activities. Initially, most of the work is performed by teacher-leader, whose role gradually evolves to that of an ordinary peer, as he/she begins to scaffold and coach peers. Teachers later implement the lesson together and reflect on it as a collective, where they discuss and resolve conflict so as to refine their initiative.

Proficient phase

The phase involves peer-teachers developing learning activities on their own. The responsibility of designing, developing and implementations of learning activities shifts to them. Interactions are constituted in the form of peer-teachers giving and seeking advice from one another. They begin to explore different ways in which available resources may be used in the classroom. They also begin to evaluate these resources and methods to determine their efficiency in their teaching (Glazer and Hannafin, *ibid*)

Mastery phase

Glazer and Hannafin (***ibid***) explain that this stage is not always addressed in professional development, though it is the most critical. Ensuring that members of a community reach this stage facilitates the sustainability of the community's enterprise. As teacher-leaders assume different roles in their career and new members join the community there is need for new teacher-leaders to emerge and take the reins in steering the pursuit of the community. However, this is possible only when there are peer-teachers who have reached the mastery stage.

Table 3. 4: Three elements inherent in Collaborative Apprentice Model

		Teaching object	Mathematics object
Model	Professional Community	Artefacts of practice	Specificity of Content
Collaborative apprentice model	teacher interactions	Resource or strategy or instructional problem	Mathematics concepts
WMC-S project	Teamwork	Readings, card activities, Geogebra, Grids,	Functions per se

Table 3.4 shows that the collaborative apprentice model foregrounds interactions between teachers. Proponents of this model believe that teachers learn through interations with each other. There are similarities between this model and the PLC model shown ealier. In summary, the collaborative model foregrounds artefacts of practice. Here the focus is not necessarily on the lesson plan, but on a new resource or strategy, which teachers need to discern. Unlike the WMC-S project the content of specificity is a teaching object.

3.8. The interconnected model

In the model the complex nature of teachers' professional growth as an inevitable and continuing process of learning is recognized. The interconnected model of teacher professional development is focused on teacher growth. According to Clarke and Hollingsworth (2002) the "model offers a powerful framework to support the analysis of those studying teacher change (or growth)" (p.1). They believed that understanding the process by which teachers grew professionally, and conditions that support and promote that growth, is essential for maximizing the outcomes of any professional development initiative. This model rejects previous models' views about the development of teachers' professional knowledge. The craft model saw teachers' professional knowledge as originating from the teacher's classroom experiences. This view is dismissed on the grounds that it does not make clear how teachers produce new meanings from their experiences, nor does it explain why some teachers only reproduce the same experience many times without learning from it (Clarke & Hollingsworth, 2002).

The expert model saw change in teachers' professional knowledge as a consequence of training. In that model teachers were positioned as passive recipients, because professional development was constituted as something that is done to teachers. It was also faulted for its failure to recognize professional development as constituting teacher learning where teachers do not only participate to improve their knowledge and skills, but also brought rich, useful experiences. In the case of the WMC-S professional development model teachers were positioned as active participants, who brought into the PD a wealth of experiences for professional developers to learn from as they went about conceptualizing PD activities and foci. This perception enabled trust and mutual respect between teachers and professional developers, and this is evident in teachers' interview utterances.

The interconnected model asserts that teacher professional development is non-linear. Its proponents argue that failure to see teacher change as a process led many professional developers to that changing teacher' beliefs and attitudes lead to changes in classroom practices and behaviors (Clarke & Hollingsworth, 2002). However, teachers' beliefs and attitudes are likely to change only after changes in student learning outcomes are evident (Guskey, as cited by Clarke & Hollingsworth, 2002). Guskey's assertion led to the development of a cyclic professional development model known as the interconnected model by Clarke and Hollingsworth (2002). Below is a diagrammatic illustration of the interconnected model.

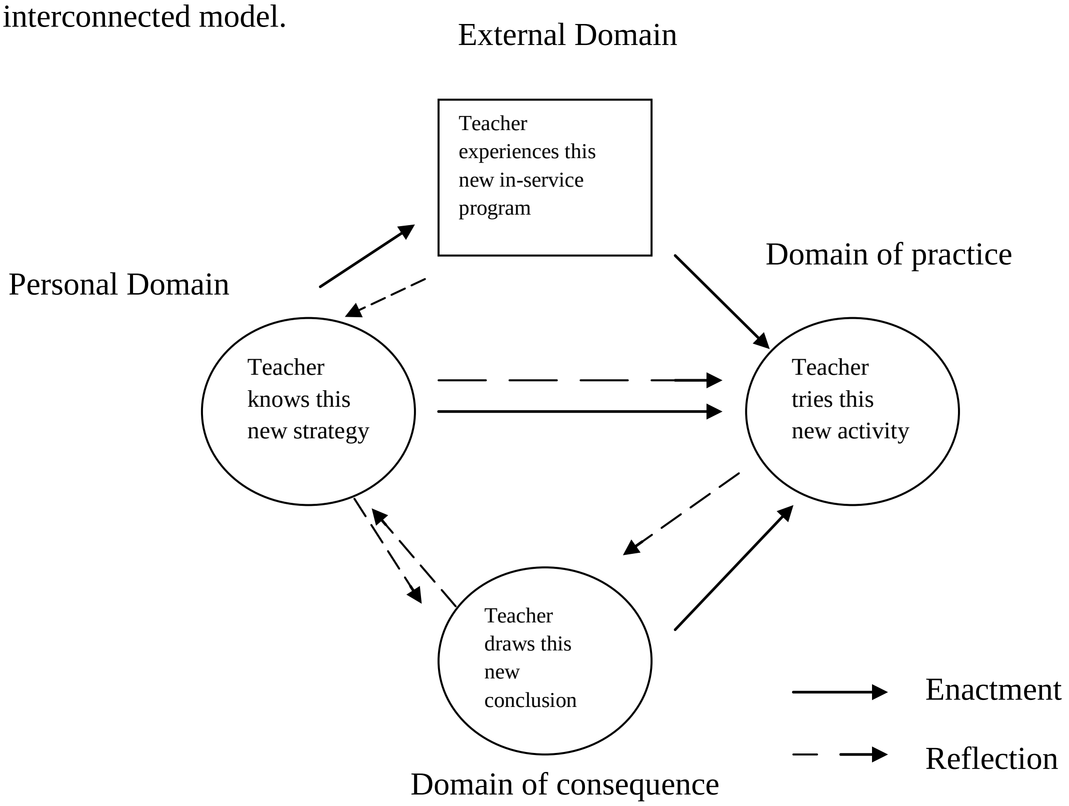


Figure 3.1: Interconnected model of professional development

This model suggests that change in teachers' practices are a result of the mediation processes which are enactment and reflection that must take place in each of the four domains shown below. The domains constitute the teacher's world (Clarke & Hollingsworth, 2002). In the diagram below the arrows linking the domains represent the mediating processes.

The multiple paths represent the interconnections between domains whilst arrows indicate that the model is no-linear. Change in one domain is thought to trigger corresponding changes in another domain. The mediating processes of enacting and reflection are necessary for change to occur (Clarke & Hollingsworth, 2002).

Enacting: refers to putting into action a new idea or new belief or a newly-encountered practice. Enacting is not simply acting, but instead refers to the teacher putting into practice something the teacher knows, believes or has experienced (Clarke & Hollingsworth, 2002).

Reflection: this is where the teacher actively, persistently and with careful consideration, critics his decisions and actions to improve enactment.

Among the four domains, personal, consequence and practice are said to be within the teachers' world and are therefore personally experienced by the teacher and the external domain is out of the teacher's control.

Table 3. 5: Three elements inherent in the interconnected model

		Teaching object	Mathematics object
Model	Professional Community	Artefacts of practice	Specificity of Content
Interconnected model	Teacher beliefs, knowledge & experience	New strategy	Mathematics concepts
WMC-S project	Teamwork	Readings, card activities, Geogebra, Grids,	Functions per se

In summary, unlike the WMCS model, this model foregrounds teacher beliefs. The specificity of content and artefact of practice are out of focus but of course present.

3.9. Summary of the models of professional development

In this summary I show the similarities and differences between the above models and the WMCS model. The main similarity is that the foci of these models are built into the WMCS' model. These models focus on teacher collaboration and achieving success as a collective. In the WMCS this aspect is considered important because all learning takes place in a context, and this is the context in which teachers will continue to work even after professional development. Teacher collaboration, or teachers as community of practice, provides a social context in which teachers participating in the WMCS activities learn from available opportunities. Learning Study is one model which foregrounds mathematics content like the WMCS model. The source of this similarity is that both draw from Variation Theory, which is known for its ability to bring the object of learning into focus.

What is different about the WMCS professional development is that, though teachers participate in activities where they collaborate and achieve set goals as a collective, this is not the focus. Instead, all teachers are challenged to focus on key mathematics concepts, and at the end the evaluation of the activities is not on the extent to which teachers collaborated, but rather the extent to which the selected content improved teachers' mathematical knowledge for teaching, and their ability to foreground this object of learning in their own teaching in a real classroom. Recently this has been done by working with variation theory to visibilize the object of learning, and the MFT tool to enact co-planned lessons in the context of learning study in schools. The models discussed above do not use content to improve teachers' professional knowledge - though content is present. It is evident from the models presented above that none uses key mathematical content as the object of learning. While some emphasize teachers learning as a community as key to improving teachers' professional knowledge and classroom practices, others focus on teacher change as key to improved learner achievement. Differences in the object of learning confirm the view that the location and focus of any professional developement is dependent on what is at interest (Adler, November 2010). In the WMCS the interest was on deepening teachers' SMK. Hence the PD was content focused, and took place in schools to locate the PD in the teachers' practice.

In Chapter 2 I discussed tensions/dilemmas in the WMCS PD. It is interesting that some of those tensions, particular the 'what' tension, are evident in other models. The above review on models of professional development has confirmed two critical aspects emerging from the work on tensions/dilemmas. Firstly, the review has confirmed that Graven's three elements are indeed inherent in any teacher professional development model, and that each element is a possible 'what' of any model. For instance, there are variations in the 'what' across the reviewed models, and which element is in focus is dependent on the model. Together, the three elements constitute the range of the possible 'what' for any model of professional development. Secondly, the presence of the three possible 'whats' (i.e. inherent elements), whether foregrounded or backgrounded, point to the inherent tension in the selection of the 'what'.

3.10. Content Knowledge for teaching

Following Shulman's Presidential address at the American Education Research Association conference in 1986, professional knowledge in teaching became a topical area of interest in mathematics education and research. This is exemplified by the extensive research that has been done so far. Shulman (1986) noted the absence of content knowledge, which he referred to as the missing paradigm in teaching in literature of research on teaching. To emphasize his observations from research in the study of teaching, he noted that "[n]o one focused on the subject matter content itself [and] ... no one asked how subject matter was transformed from knowledge of the teacher into content of instruction" (p. 6). He called for attention to this deficit in the literature in the study of teaching and further recommended the need to conceptualize teacher knowledge.

In his description of content knowledge Shulman began by categorizing teacher knowledge into seven categories and moving on to describe them. In this research two key components are elaborated on because of the role they play. These two are subject matter knowledge (SMK) and pedagogical content knowledge PCK). Shulman described SMK as "the amount and organization of knowledge per se in the mind of the teacher" (p. 9). For him, SMK has both the substantive structure - "the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts" (p. 9), and the syntactic structure – "the set of ways in which truth or falsehood, validity or invalidity, are established" (p. 9). Authors in the field of mathematics education have added to the description of SMK. Schifter (1998) has described SMK as the teacher's own mathematics or the teacher's mathematical knowledge and Bromme (1994) has described SMK as what the teacher learnt in his/her studies.

The importance of SMK in teaching has been articulated. For instance, it is generally believed that SMK influences the quality of explanations that teachers give to their students. In addition, the teachers' ability to integrate students' contributions into their teaching has been linked to an in-depth knowledge of the subject taught (Roehler, et al., 1987). It is further believed that teachers who possess good SMK are sufficiently sure of themselves to be able to direct classroom activities, even in very challenging situations (Dobey & Schafer, 1984). These beliefs were later supported empirically by Ma's (1999) study. In a comparative research of mathematics teachers Ma found that teachers who had in-depth knowledge of the subject matter were able to influence good classroom practices. According to Shulman, a deep and flexible understanding of the subject helps teachers to not only teach all students according to the desired expectations, but also to assist students in creating useful cognitive maps, relating ideas, and in addressing misconceptions that students bring to class (Shulman, 1987). Shulman believes that this kind of understanding provides a foundation from which pedagogical content knowledge (PCK) may grow (*ibid*).

PCK has been described in different ways by different authors. For Shulman, PCK is the subject matter knowledge for teaching which includes "an understanding of what makes the learning of specific topics easy or difficult" (p. 9). Veal and Macnister (1999) have descibed PCK as the knowledge possessed by the expert teachers. The constitution of PCK has also been articulated. For Tirosh, Even and Robinson (1998), PCK is constituted, among other things, by the teacher's understanding of learner thinking, the teacher's ability to identify sources of learner errors, and his/her knowledge of different ways of representing specific topics. In my research I define SMK and PCK in the same way as done by Shulman (1986).

3.11. Mathematical Knowledge for teaching

Building on Shulman's (1986) initial definition of content knowledge for teaching, several mathematics educators and researchers across the globe e.g., Ma (1999) in China and USA, the COACTIV team in Germany, Ball and her colleagues at the University of Michigan, Adler and her colleagues at University of the Witwatersrand and Cape Town University (see Ball and Bass (2003); Ball, et al, (2009) ; Blum and Krauss, (2008); Adler and Davis, (2006) and others, traversed this terrain to redress the problem of the missing paradigm in mathematics teaching. At the University of Michigan Ball and colleagues conducted extensive research in mathematics education where they, *inter alia*, developed a theory of mathematical work of teaching from studying teaching practice (Ball & Bass, 2003); conceptualized content knowledge for teaching (Ball, et al., 2009) and also developed measures of content knowledge for teaching. Parallel research was done in Germany by the COACTIV team to address the problem of the 'missing paradigm' in teaching.

In earlier research, Ma (1999) studied the mathematical knowledge for teaching of both Chinese and American elementary teachers. Her research revealed that though American teachers were on average more academically qualified to teach at elementary level, their content knowledge was flimsy and also fragmented. Some of them were not able to provide solutions to some items in the test which elementary Chinese learners were able to solve. Ma's (1999) research concluded that American teachers in her study lacked profound understanding of fundamental mathematics. This study, and many others, provided the impetus for the need to conceptualize teacher knowledge in mathematics education.

Both Ball and her colleagues and the COACTIV team subdivided Shulman's categories of SMK and PCK. Ball and her colleagues (Ball, et al., 2009) subdivided SMK into Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK) – where CCK is defined as the "mathematical knowledge and skill used in settings other than teaching" (p. 32), and is knowledge of the mathematics taught in the class, and which is found in student textbooks. Specialized content knowledge (SCK) is knowledge that is unique to teaching. It separates a mathematics teacher from a mathematician, and so is defined as "the mathematical knowledge and skill uniquely needed by teachers in the conduct of their work" (Ball, et al., 2009, p. 34). On the other hand, the COACTIV team subdivided SMK into knowledge of school mathematics and knowledge of mathematics content taught in the university.

Though there are some slight differences in how SMK is recontextualized between the COACTIV team and Ball and her colleagues, there seems to be little or no difference in how they subdivide PCK. Both research teams subdivided PCK into three. For Ball et al (2009) PCK consists of knowledge of content and teaching (KCT), knowledge of content and curriculum (KCC) and knowledge of content and students (KCS), while the COACTIV team's corresponding subdivisions are knowledge of mathematical tasks, knowledge of instruction, and knowledge of students. It suffices to note that both these researchers' work, though independent, were a response to the need to conceptualize professional knowledge in teaching. Hence, while Ball et al (2009) refer to teacher professional knowledge as mathematical knowledge for teaching (MKT), the COACTIV team simply refers to it as teachers' professional knowledge, and in both cases it is constituted by SMK and PCK. Both these researchers have also developed measures for measuring content knowledge for teaching.

Furthermore, these measures have been used to research teacher knowledge. In a study where MKT measures were used, Hill, Rowan & Ball (2005) found strong links between MKT and teaching quality, and between MKT and student achievement. In a related study Hill, et al. (2008) found strong links between MKT and mathematical quality of teaching (MQI). In this particular study Hill, et al (2008) reported the existence of "a powerful relationship between what a teacher knows, how s/he knows it, and what s/he can do in the context of instruction" (p. 496). Other important contributions related to conceptualization of content knowledge for teaching were made by Adler and her colleagues in the QUANTUM project. That project was research focused on understanding what and how mathematics for teaching was constituted in mathematics teacher education. A major contribution of this project was the development of a methodology for studying mathematics for teaching in teacher education, and their underlying assumption that the object of learning in mathematics teacher education would inevitably consist of both a mathematics and a teaching component, though one is more likely to be the primary object. Thus, whenever one was in focus the other was present, but in the background (Adler, 2009; Adler & Davis, 2006).

The above review of literature informs how professional knowledge was conceptualized in my study. For instance, professional knowledge as used here refers to Shulman's (1986) constructs of SMK and PCK. In this research I chose to background curriculum knowledge (CK) because my main interest was in SMK. I foregrounded PCK though it was not my main interest, because SMK and PCK are dual objects in mathematics teacher education (Adler & Davis, 2006). Since these concept are generic, to situate them in mathematics teacher education and in my research, I adopted Adler and Davis' (2006) constructs of mathematics objects (Mt) and Teaching objects (Tm) to mean SMK and PCK. Adler and Davis' (2006) Mt and Tm are elaborated on in Chapter 5, where I discuss the methodology and the analytical framework section below. Hence the object of learning was either focused on mathematics (Mt), or on teaching (Tm). In my research Mt has the same meaning as Ball, et al's., SCK. However, due to the connection between my analytical tools and Adler and Davis' methodology, I chose to work with **Mt** instead of SCK and **Tm** instead of KCT or KCS.

In my research I studied one aspect of the WMCS PD where in-service teachers were *revisiting school mathematics* (Pournara & Adler, 2014) in the secondary school mathematics which they taught. In a recent research by Pournara (2013) SMK is subcategorised into two components that include (i) *revisiting school mathematics* and (ii) *constructing new knowledge of mathematics*, which are important components of mathematics for teaching (MfT). In their recent paper to the British Congress for Mathematics Education, Pournara and Adler (2014) argued strongly for the need to provide pre-service teachers with the oportunity to revisit or (re)learn school mathematics in South Africa.

> ... the calls for revisiting school mathematics tend to assume pre-service teachers (PSTs) had adequate opportunity to learn the mathematical content at school, and now need to deepen and/or "top up" their knowledge with particular attention to conceptual aspects. In South Africa many PSTs have not had the kinds of opportunities to learn mathematics at school that might be assumed in other parts of the world. So for many students revisiting school mathematics is much more than "topping up". In the most extreme cases, it may involve learning for the first time the mathematics they will teach in schools (p. 1).

The above arguments are applicable for in-service teachers who had limited opportunites to learn mathematics under the Apartheid system (Adler & Davis, 2011). I return later to discuss the notion of *revisiting school mathematics* in my conclusion chapter. In particular, I discuss what gets enacted and lived when teachers (re)learn school matheamtics. The importance of teachers revisiting school mathematics has recently been articulated (see Goos, 2013). Drawing from examples in her research of prospective mathematics teachers Goos (2013) argued strongly that 'content knowledge apparently learned during secondary schooling is not necessarily secure, and it needs to be revisited during teacher preparation' (p 982). Her viewpoint seems to explain gaps that were noticed by the WMCS professional developers in their classroom observation of teachers prior to teachers participating in the PD. An interesting finding from the research was that while mathematical experience was critical for learning mathematics content knowledge, this was not the case with PCK. This led the researcher to conclude that university-level study of mathematics is inadequate, on its own, for preparing effective teachers of secondary school mathematics because it does not affect PCK. Hence this called for other alternatives, and this research might be seen as answering that call looking into what an object focused PD made possible for teachers to learn.

In the PD, teachers were (re)learning different areas of mathematics. However, in this research, I focused on Functions. As noted earlier, Functions were privileged in the PD because of low learner achievement in this area and the fact that teachers had identified it as one area of mathematics where they needed help, and their critical place in secondary school mathematics

3.11. Functions

The concept of Function, which is thought to have been a late development in the history of mathematics, has grown in importance such that it has become one of the cornerstones of mathematical thought (Thomas, 1969). Functions carry various definitions, some of which are recontexualisations of the formal definition. While the school curriculum and textbooks work with the recontexualised definition, for example "a correspondence between two sets which assigns to every element in the domain exactly one element in the range", university mathematics uses the formal definition of Functions. This definition, which is formulated in terms of the concept of Set, says that:

> If X and Y are sets, a function from (or on) X to (or on) Y is a a relation f such that dom f = X and such that for each $x \in X$ there is a unique $y \in Y$ with (x,y)$(x, y) \in f$. The uniqueness condition can be formulated explicitly as follows: If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$ (Thomas, 1969, p. 1)

In the WMC-S project's professional development the formal definition of Function was used in Task 1 of Functions 1 to explain why some of the cards teachers had identified as non-function were actually Functions. This was not the only definition which was availed to teachers. Several recontextualised definitions were present, and these were definitions which are normally used in secondary schools mathematics. While some definitions presented functions as relations, mappings or correspondence, others presented them as sets of ordered pairs. An example of a recontextualised definiton would be as follows:

> If with each element of set A there is associated in some way exctly one element of another set B, then this association constitutes a function from A to B. (DeMarois & Tall, 1996)

Literature on the concept of function has articulated several aspects which one must acquire in order to have a complete understanding of functions. For Even and Tirosh (1995) there must be an understanding of the arbitrary nature of functions. Arbitrariness is one property of functions. The meaning of the arbirary nature of functions is that functions do not have to be described by any known equation or expression, or be described by some known or familiar graph. There should also be an understanding of two conditions which are inherent in the definition of functions. The first condition is that a function must be defined on *every* element in the domain, and the second is that for each element in the domain there must be *one* element in the range. Together, the two conditions constitute another important property of functions, which is known as univalence. Even and Tirosh have also suggested the vertical line test as the quickest way to determine if a graph represents a function. They assert that a graph represents a function if and only if the vertical line cuts the graph in only one place.

O'Callaghan (as cited by Watson & Harel, 2013), has also listed five areas of expertise in functions which all learners need, irrespective of whether they will proceed to do advanced mathematics. These included (i) modeling, (ii) interpreting, (iii) translating between representations, (iv) treating functions as objects and (v) being able to act on the objects. Recently they added two further necesssary understandings of the concept of function. The first is the ability to differentiate between a function and its various representations. For instance, a function does not have to be representable on the coordinate system for it to be continous or differentiable (Watson & Harel, 2013). Second is the ability to differentiate between the concept of function as a relation satisfying univalence from its instantiations. Here Watson and Harel point that instantiations of functions do not define functions, but only exemplify them.

DeMarios and Tall (1996) have subcategorised the Function concept into facets and layers, and used these to investigate students' conceptual image of function. They defined facets of a mathematical object as various ways of thinking about the object and communicating to others in different ways such as verbal, numeric, symbolic and geometric or graphical, and the defined layer as constituting action, process and object. The ability to move between the layers and between the facets are thought to be critical for attainment of the Funciton concept. Connecting the multiple representations is essential because it is a property of high-order function (DeMarois & Tall, 1996). According to Cuoco (as cited by DeMarois & Tall, 1996), learners who percieve Functions as actions hold the view that a Function is a sequence of isolated calculations. Consequently, learners rely on procedures which are followed to obtain an output when given an input. Those who see functions as process percieve functions as dynamic tranformations. An object conception of Function is attained when learners are able to see Functions as "atomic structures that can be inputs and outputs to higher-order processess" (p.3).

Vinner (1983) has described the relationships between what he refers to as mental pictures, concept image and concept definition of functions. For this author, a mental picture of a concept consists of a collection of all pictures that one has ever associated with that particular concept. Pictures, as used here, refer to pictures as we know them, and may include any visual representations or symbols such as $y = f(x)$ (Vinner, 1983). In other, words, a mental picture, as defined here, refers to any object that one might associate with the concept. Vinner defines concept image as the sum of the mental picture and a set of properties associated with the concept. The example given is where one "might think that a function should always be defined by means of an algebraic expression" (p.293). Vinner is of the view that for one to successfully handle a concept, one needs a concept image and not concept definition because concept definition remains inactive or may soon be forgotten

In this research, the development of the notions of *mathematical appearance* and *mathematical reality* was informed theoretically by the notion of duality of the object of learning and Galileo's perspective about truth and also empirically by the data. Hence mathematical reality and appearance, as used in this study transcends Vinner's notions of mental picture and properties of a concept. Mathematical reality and appearance fill out the notion of mathematics with teaching backgrounded (Mt). They are not labels per se, but build on Adler & Davis' (2009) more general notion to theorize the enacted object of learning in object focused professional development. These notions are also grounded on Galileo's notion of truth. According to Galileo, truth is either permanent or temporary. Truth which is permanent is referred to as reality while truth which is temporary is referred to as appearance. Similarly, evidence from my data revealed that some legitimating criteria might appeal to permanently true mathematical objects and others might appeal to temporarily true mathematical objects. Hence bringing the notion of duality of object and Galileo's notions to bear on the data led to the emergence of the notions of mathematical reality and appearance.

Several researches focusing on teachers' understanding of specific mathematics topics in the school curriculum have been conducted (e.g Eisenhart, et al., 1993; Even & Tirosh, 1995; Leinhardt, Zaslavsky, & Stein, 1990; Tirosh, et al., 1998). By and large, most of the studies pointed to gaps in the teachers' understanding of the SMK they taught. Leinhardt et. al's (1990) investigation of mathematics teachers' understanding of the arbitrariness and univalence nature of Functions revealed gaps in their SMK. In particular, they found that teachers in their study believed that functions were only those that could be represented by a specific expression. Even and Tirosh (1995) found that a teacher in their study held contradictory conceptions of the univalence. While the teacher was able to apply this property of functions to classify mathematical objects into functions and non-functions, they found that when the teacher came across familiar graphs, such as circle and ellipse, the teacher said they were functions, even though he could see that they did not satisfy univalence. This prompted the authors to conclude that:

> teacher education should explicitly refer to topics included in the high-school curriculum, such as functions and undefined mathematical operations.These topics have already been studied by the teachers during their own high-school years. However, as we saw in this paper, one cannot assume that teachers' subject-matter knowledge with respect to the two aspects ("knowing what" and "knowing why") are sufficiently comprehensive and articulated for teaching (p.7)

The concept of function as described in the liteature contains aspects that relate to the WMCS's key features. These aspects, univalence, arbitrariness, translation between representations, functions as objects, and being able to act on functions objects were components of the key features, even though they were not expressed in similar terms. Univalence is the essence of the definition of function, and function was foregrounded. Arbitrariness was not articulated explicitly at the level of the intended object, but during enactment. Translation between representations was present at the level of the intended object, and enactement in card activities. The PD activities aimed at helping teachers percieve functions as a relationship, and also had to find the domain and range of various functions.

In research that investigated a student's function concept DeMarois and Tall (1996) found that the student was able to operate satisfactorily with algebraic representation, but encountered difficulties with table representation. These results are relevant to my research because in the WMCS project teachers were asked to reflect on the representations which were easy or difficult to work with, and they identified the table as the most difficult. In a related research Tall found that when students were given a formal definition of function their experience from examples of Functions made them develop a personal concept image of a function such that if they encountered functions mainly in algebraic form, this led to them believing that the existence of an algebriac representation was essential for a Function to exist (Tall, 1997). As if to counter this, in the WMCS PD functions were presented through multiple representations for teachers to see the connections between them, and the arbitrariness of functions was emphasized.

In the South African Mathematics curriculum, Functions are introduced formally at Grade 9, and this is restricted to Linear Functions. Later, at Grade 10, learners encounter more complex Functions that include quadratic and hyperbolic functions. Trigonometric functions are taught at Grade 11 and Inverse Functions are taught at Grade 12. As a consequence, in the WMC-S project teachers experienced trigonometric functions in 2010 when the focus was on the Sine and Cosine functions. In 2011 the Function theme work exposed teachers to Functions at Grades 9 – 10. The professional development tasks of 2011 were characterized by multiple representations of functions.

3.12. Conclusion

In this chapter I have surveyed literature on mathematics professional development, organising the review as a reflection on various models. I argued that all models have three components (i.e. professional community, artefacts and specificity of content) where what is in focus to learn differs. I showed that none has the content focus emphasis of the WMCS model, which makes it worth focused study. Though the Learning Study model does focus on content, it does so implicitly, and so teachers do not learn it directly, while in the WMCS model teachers learn content directly. In the next chapter I discuss the theoretical resources that I drew on to frame my research.

CHAPTER 4: THEORETICAL FRAMING OF THE STUDY

4.1. Introduction

The purpose of this chapter is to position the study theoretically. According to Bogdan and Biklen (1982), all research is guided by some theoretical orientation, whether stated or not. These authors consider good researchers to be those who are always aware of their theoretical base and use it to collect and analyze data.

My main theoretical resources are phenomenology and variation theory. I used the former to study aspects of the the lived object of learning through interviews, and the latter to analyze the enacted object of learning in video recordings of the PD sessions, and the lived object of learning in teachers' lesson plans. While variation theory provided useful and important handles for my study, it did not suffice with respect to fully describing the enacted object of learning in the PD in particular. I thus recruited additional resources, particularly the notions of scaffolding and legitimating to augment the main theories, all of which are detailed in this chapter. As noted earlier, one of the reasons I selected to work with variation theory was because the PD drew on this theoretical resource on its planning and work. As such I found it appropriate to extend the use of this resource into my research.

4.2. What theory framed my research?

4.2.1. Variation theory

This study, and its interest in the intended, enacted and lived object of learning, is framed by variation theory – that has been put extensively to work in a range of learning situations other than teacher education (e.g. Runesson, 1999). In the discussion of the theory below, and its key concepts, I will indicate how and why these do not easily extend to professional development where the object of learning – the 'what' of professional development - is complex. This has led to recruiting additional theoretical resources, firstly to fully describe the enacted object of learning in mathematics professional development, and secondly to illuminate what I describe as scaffolding strategies, that also emerged as pertinent to a description of the enacted object.

Following the brief introduction to variation theory in Chapter 2, I reintroduce it and go further to elaborate here on its major constructs, and at the same time open up discussion on what Adler & Davis (2006) have referred to as the dual objects of learning in mathematics teacher education. Based on a social epistemology (Adler & Hulliet, 2008), both mathematics and teaching will be present in some form in all teacher education activity. There is thus a complexity to 'objects of learning' in professional development, even when specific mathematical content is in focus.

According to Lo (2012) "*Variation Theory* takes the object of learning as the point of departure, and highlights some necessary conditions for learning that are related to how the object of learning should be dealt with" (p.17). The object of learning is the cornerstone of of Variation Theory in two main ways. Firstly, the focus of Variation Theory on the object of learning marks the difference between this theory and other learning theories. So while other theories of learning tend to previlege other aspects of learning and teaching, variation theory is concerned with that which is to be learnt. Secondly, the object of learning holds multiple meanings, and this has made possible the conceptualisation or theorisation of Variation Theory. Lo (*ibid*) states the importance of the object of learning as follows:

> ... all learning must be directed towards something. We cannot talk about learning without referring to what is to be learnt, i.e., the 'object of learning'. It is in this respect that Variation Theory differs from many other learning theories. In Variation Theory, the 'object of learning' is a special term encompassing multiple meanings (p.41).

Variation Theorists explain learning as: "a person is said to have learnt with respect to [the object of learning] when that person is capable of being simultaneously and focally aware of other aspects or more aspects of [the object of learning] than was previously the case (Marton & Booth, 1997, p. 142). There are several constructs in Variation Theory which, of course, relate to the object of learning in one way or another.

Major constructs in the theory of variation include (i) *object of learning, (ii) variation, (iii) discernment, (iv) space of variation, (iv) dimension of variation (*also known as *critical aspect)* and *(v) critical features (*also known as *values in the dimension of variation)*. In the following paragraphs these constructs are elaborated, and questions posed for the object of learning in mathematics teacher education and so my study.

Marton, et al, (2004) describe the *object of learning* in terms of three sub-constructs, i.e. intended, enacted and lived objects. The *intended object* is the object of learning at the level of the teacher. It is the capabilities that the teacher aims to develop in the learner, and is also described as what is intended to be learned. In mathematics teacher education, the intended object of learning could be either, or both, some mathematics, some teaching or an amalgam. In the WMCS PD, and so also in my research, the major focus was on the teachers' capabilities with respect to mathematics. In particular, the intended object of learning was constituted by key features of functions, and these included (i) properties of functions (intercepts, parameters and definitions) (ii) multiple representations and (iii) function terminology.

In this research the *enacted object of learning* is discussed from two orientations. Firstly, it is the object of learning from the point of view of the researcher, and it is through the enactment that the *space of variation* (opportunities to learn) is opened up. The object is normally co-constructed by the teacher and learners in the classroom (Runesson, 1999, 2005) in pedagogic practice. In my research opportunities to learn functions were co-constituted between the Facilitator and teachers and between teachers. Teachers were (re)learning school mathematics to deepen their knowledge of the mathematics they taught. As such, the teachers were learners.

Secondly, in this study, what is enacted in mathematics object focused PD sessions may have more or less inclusion of aspects of learning and teaching with respect to capabilities and the object. Discussion between participating teachers and the teacher educator might well shift into aspects of teaching and learning. It is careful analysis of the pedagogic discourse that illuminates not only the co-constitution of the object of learning between learners and the teacher, but also between mathematics and teaching as dual objects in mathematics teacher education.

The *lived object* is the object of learning at the level of the learner, in this case the teachers in the PD. Runesson (2005) describes the *lived object* as what is actually learned and in research it is usually studied from the point of view of the learner through interviews, post and delayed tests. In this study the lived object was studied from the point of view of mathematics teachers.

The object of learning also has both *direct* and *indirect* objects (Marton and Pang, 2006). The *direct object* is the *content* that teachers bring into focus for learners to engage with, while the *indirect object* is the *capability* that they intend to develop in the learner. For example, suppose a teacher decides to teach learners how to transform trigonometric functions. The trigonometric function being used is the direct object while the learner's ability to transform a trigonometric function is the capability that the teacher intends to develop in the learner. In teacher education, and following up on the trigonometric example, the direct object is functions and the indirect object (i.e. the teacher's capability), is firstly to functions themselves, and then possibly to the teaching of these.

The most critical component of the theory of variation is the notion of *variation/difference*. Variation theorists believe that learning is a consequence of discernment of variation. This is the essence of the theory. Variation with respect to the enacted object is usually accompanied by *discernment* of some *critical features* of the enacted object, and when that happens a *dimension of variation* is created. Critical features are those aspects of the object of learning upon which the teacher chooses to focus learners' attention, and this is of decisive importance in any learning situation because when learners focus on the critical features it becomes possible for them to learn what the teacher intended (Marton, et al., 2004). Lo (2012) explains the importance of foregrounding the critical features (i.e. that which learners need to know/learn) as follows:

> There is a limit to our capacity to focus, and we cannot focus on all of the features of [a phenomenon] simultaneously. We can only focus on a limited number of aspects of a phenomenon or object at a time. This results in some aspects coming to the forefront of our awareness and moving into focus while other aspects that are not in focus recede to the background. The understanding and meaning that we attach to a phenomenon depends on which aspects of the phenomenon come to our focal awareness (p 19).

However, it is possible that learners may fail to discern critical features because they may choose to focus on other aspects that are of interest to them, and this explains why in some instances the intended object and the lived object of learning vary. Though the two seldom align, the intention of any enactment is to bring the two as close as possible. I return to talk about the alignment between these objects in Chapter 12.

Hence it can be concluded that critical features of the object of learning are not trivial because they determine the extent to which the intended, enacted and lived objects of learning will vary from each other, and so minimising gaps between critical features of both the intended and enacted objects is critical for learning. As noted earlier, in this study I focused on the critical features of functions which were identified by WMCS professional developers to be key in the learning of functions.

Discernment is an integral part of the theory, especially its usefulness in explaining how the *space of learning* is opened up. What is learnt is dependent on, but not reducible to what is made available, and what is made available are *critical features* that learners might discern (Lo, 2012). The importance of discernment in theory of variation is emphasized by Runesson and Mok (2004) when they state that "[t]he way in which a situation is seen or experienced is dependent on the features of the object that were discerned" (p. 63). Series of variation and accompanying discernments create more dimensions of variation. The dimensions of variation on the same object open up a *space of learning*. When some aspects of the object vary while others remain invariant an awareness of those aspects creates a dimension of variation (Runesson, 1999). As the object continues to vary more aspects of the object are discerned, and more dimensions of variation are created. So these dimensions of variation on the same object open up a space for what is possible to learn. The space of learning is where opportunities to learn are made available and, as noted, in teacher education or teacher professional development the space of learning will consist of both the mathematical and the teaching objects, but with one primary and the other secondary (Adler & Davis, 2006).

For Marton et al the *enacted object* is seen in discourse in classrooms. Hence in my study I analyse the teaching task of the WMCS professional development activities focused on Functions using patterns of variation. The authors assert that one needs to pay particular attention to classroom discourse to understand what is happening to the enacted object. However, there are no explicit tools for analyzing pedagogic discourse. To address this problem I drew tools from Adler and Davis' (2006) methodology, the roots of which lie in Bernstein's social theory of pedagogic discourse (Bernstein, 2000), which is beyond the scope of my research. In addition to addressing the problem, Adler and Davis' tools do so in the context of teacher education. For instance, in my analysis of the enacted object of learning patterns of variation enabled visibilization of the object of learning, but not its illumination. Simply put, I was able to look into the visual component but not the verbal component of the enacted object of learning. This pointed to the first gap in my analysis which I closed by recruiting Adler and Davis' (2006) notion of legitimating criteria. The second gap was present in the space of variation. Analysis with patterns of variation revealed that there were opportunities to learn, but could not point to the nature of those opportunities. To close the gap I went back to Adler and Davis, but this time I recruited the notion of dual objects of learning to establish the identity of the enacted object of learning.

The challenge of studying teacher education using variation theory emerged further when conceptualizing the *lived object*. For Marton et al the lived object is made visible through learner tests and examinations. How does this transfer to teacher education, and in particular, professional development, as was the case in my study? What was the lived object at the level of the teacher in professional development? It was the challenge of my research to address these questions. While I worked productively with patterns of variation to analyze the enacted object of learning, I could not do the same to analyse the lived object of learning. This pointed back to the second gap, which I closed with resources from phenomenology.

Bringing the two perspectives together (the phenomenological underpinnings of Variation Theory, and the social epistemology of Adler and Davis (2006) to better understand the enacted and lived objects of learning, raised both epistemological and ontological confusion, and so it was the work of my thesis to integrate these two perspectives in a manner that was epistemologically and ontologically coherent. This was particularly true in my research where I was studying both the lived and the enacted objects of learning. While the lived object of learning was studied at the level of the individual teacher, the enacted object of learning was studied as a co-constitution by teachers and the facilitator in a social setting.

Working with these two orientations to meaning-making I was able to work well with both phenomenological and the social theory of pedagogic discourse. In particular, I worked with variation theory and the social epistemology of Adler and Davis' (2006) methodology for studying teacher education to examine both the lived and the enacted objects. For instance, I drew on Adler and Davis' notions of legitimating appeals and duality of object of learning to study the constitution of the enacted object in professional development tasks. To understand this further I worked with patterns of variation to see what was made possible to learn. For the lived object of learning, as elaborated below, I analysed teachers' interview responses first with phenomenological research tools (see **Chapter 5** for details) and thereafter with Adler and Davis' notion of duality of objects of learning in teacher education. I also studied the lived object of learning through teachers' lesson plans. Here I worked with variation theory's notion of the space of variation to analyse what teachers considered to be important to learn about functions, both before and after participation in professional development.

I must once more point out that it was not trivial putting phenomenological and Adler and Davis's theoretical tools together. However, it was part of this study to mould these two contrasting perspectives into tools that cohered with the overall orientation of the study. On the basis on the outcomes of my analysis, suffice it to mention that the study was able to work productively with these two perspectives.

4.2.2. Phenomenology

In one aspect of my research I was interested in constituting the lived object of learning for teachers in my study after they participated in an object focused PD. I called on phenomenoloy to engage with interview data after noticing that it was not going to be productive to work with variation theory. So while variation theory was my main theoretical resource in the analysis of video data and lesson plans, phenomenology was my main resource in my analysis of interview data. Phenomenology had shortcomings, just as did variation theory. For instance, while I was able to work with phenomenology to describe the meanings teachers ascribed to their experiencing the phenomena, I was not able to work productively with it to determine the nature of those meanings – i.e. were they more mathematical or more about teaching/pedagogy? Hence I recruited aspects of the methodology for studying teacher education to do so.

Phenomenology is a perspective that acknowledges and values the meanings people ascribe to their own existence. Phenomenologists study the people's lived experiences of some phenomena in order to gain insight into the basis of their existence. Central to this perspective is the assumption that meaning is in the things that we want to research. Hence the prime intention of phenomenology is to identify (Bernstein, 2000), explore and describe phenomena as we find these in their natural settings. The type of phenomenology being described here is referred to as transcendental phenomenology, or phenomenology of experiences. This perspective was developed by Husserl, who believed that "experiences contain both the outward appearance and inward consciousness based on memory, image and meaning" (Creswell, 1998, p. 52). There are three main tenets of phenomenology: (i) phenomena, (ii) phenomenological epoche and (iii) intentionality of consciousness.

Phenomena - refers to the subjective experiences of individuals, whatever they are, and however they are interpreted by those individuals (Creswell, 1998). Phenomena may be such things as events, situations, experiences or concepts in which we lack understanding because the phenomenon has not been explicitly described and explained to us. In my research the phenomena was an object of focused PD, and I was interested in understanding what becomes the enacted and the lived object of learning is this type of PD.

Epoche – refers to the need to suspend jugdgement about what is real until the jugdements are founded on a more certain basis (Creswell, 1998). As a way to exemplify this, in the case of data analysis, the research should not jump to conclusions before the data has been saturated – i.e. there is no longer anything new to emerge from the data. Imposing one's judgment might obscure the emergence of new meanings. David (2007) explains this further when he asserts that:

> [In phenomenological research] the researcher must attempt to 'return to the things themselves', that is, to capture the 'essence' of a lived experience prior to the theorising about the experience. Returning to the phenomenon itself enables an 'uncovering' of what is taken for granted about the phenomenon. In this way, the researcher can describe a phenomenon that is presently 'covered' by our taken for granted assumptions.

Husserl summarises the notion of epoche by saying one must judge only by evidence as it avails itself in the data, and not by any preconcieved ideas, as is normally the case in natural sciences where hypothesis are put upfront (Stumpf, 1994).

Intentionality of consiousness – according to Hursserl, consciousness is always directed toward an object (Creswell, 1998). As a consequence, the ontology of an object is inseparable from one's consciousness. This has served as the basis for phenomenologists to dismiss the dichotomy between the subject and the object. For them, reality can never be separated into subjects and objects. Instead the two are essentially one, because the reality of an object is only percieved within the meaning of the experience of an individual (Stumpf, 1994)

4.2.3. Methodology for studying mathematics teacher education

The experiencing of phenomena/learning is not equal to possibilities made available. Hence, to study the possibilities made available required different tools and framing. For instance, while phenomenology facilitated the analysis of the teachers' experiences it was not sufficient in explaining either the nature of the experiences or what was made possible for teachers to learn. Hence I recruited the methodology for studying mathematics teacher education to further illuminate the enacted and the lived object of learning. As already noted, a methodology for studying teacher education has its roots in Bernstein's (2000) social theory of pedagogic discourse. According to him, "evaluation condenses meaning of the entire pedagogic device" (p.29). The device is constituted by both the distributive, recontextualising and evaluative rules. However, among the three, evaluative rules had a bearing on my research. For Bernstein and those who built on his theory:

> Evaluative rules regulate pedagogic practice at the classroom level, for they define the standards which must be reached. In as much as they do this, then evaluative rules act selectively on contents, the form of transmission and their distribution to different groups of pupils in different contexts (p.29).

Singh (2002) made an attempt to clarify the above quote when she pointed out that evaluation rules were embedded in pedagogic practices, where they regulate the recognition of what counted as valid acquisition of instructional and regulative texts. A similar attempt came from Maton (2006), who said that evaluative rules were manifested 'in assessment regimes which send strong messages about what counts as legitimate knowledge and knowing' (p 46). This aspect of the pedagogic device has been recontextualised into mathematics teacher education research, where it has been used to study the constitution of mathematics for teaching (MfT) (see Adler & Davis, 2006; Adler, Davis, Kazima, Parker, & Webb, op cit; Davis, 2001; Parker & Adler, 2012). These authors worked from the view that evaluation transmit criteria for the production of legitimate texts to develop the notion of legitimating criteria. Elaborating on this, Adler et.al., (op cit) pointed out that 'any act of evaluation has to appeal to some or other authorising ground in order to justify the selection of criteria' (p.8). In their research they found that the ground for appeal varied considerably within and across sites.

The methodology for studying mathematics teacher education is a sequel of a series of studies (e.g. Adler & Davis, 2006, 2011; Davis, 2001, 2005; Parker & Adler, 2012) coming out of the QUANTUM project. The goal of the project was to conceptualise the constitution of mathematical knowledge for teaching (MKT) in teacher education, and the outcome was the methodology for studying teacher education. In a proposal to the National Research Foundation (NRF) Adler explained the aim of the QUANTUM as follows:

> Our aim is to produce a principled description of what and how mathematics for teaching comes to be constituted across varying sites of mathematics teacher education practice. We contend that this description will contribute in significant and powerful ways to understanding variations in opportunities for teachers to learn mathematics for teaching, and thus its possible effects across diverse contexts. This, in turn, will provide the basis for a framework for intervention into the content preparation and development of mathematics teachers (p.4).

The methodology for teaching is the framework alluded to in the above quote and is the outcome of the QUANTUM project. While the entire methodology for studying teacher education is rooted in Bernstien's notion of pedagogic device, my discussion of the methodology is limited to the *duality in the object of learning* and Bernstien's notion of *evaluation* in pedagogic discourse – wherein lies the notion of *legitimating criteria* and *appeals.*

The *duality in the object of learning* is a major construct in the methodology for studying mathematics teacher education, and the essence of this notion "is that there are two interacting objects of acquisition ... [which] are deeply intertwined in any act of mathematics teaching" (Adler, 2008, p. 12). The objects of acquistition or learning referred to here are mathematics and teaching. The interaction of the dual objects has since been categorised into four components that included **M, T, m**, and **t.** While the capital letters indicated the primary object, the lower-case letters indicated the secondary object, such that **Mt** meant that mathematics was primary and mathematics teaching was secondary. Similarly with **Tm.** As a construct of the methodology, and also as I worked with it in my research, **Mt** is a construct that says that in a given mathematics teacher education pedagogic practice, mathematics objects are in focus, while teaching objects, though present, are backgrounded. Similarly, **Tm** is a construct which says that in a particular mathematics teacher education pedagogic practice, teaching objects are in fore and mathematics objects are backgrounded. Hence the two are not abbreviations of the words 'mathematics' and 'teaching', but two conceptions of mathematical knowledge for teaching.

(ii) The notion of *legitimating criteria* is based on the premise that what counts as legitimate text or knowledge rests on criteria transmitted through assessment. Furthermore, assessment has to *appeal* to some authorising ground to justify the selected criteria. Parker and Adler (2012) elaborate on appeals further by asserting that "it is accepted that all judgement, hence all evaluation, necessarily appeals to some or other locus of legitimation to ground itself, even if only implicitly" (p. 6). The proponents of the methodology have worked empirically from this position to develop six categories of appeals: (i) *mathematics itself*, (ii) *mathematics education*, (ii) *everyday knowledge,* (iv) *experiential knowledge*, (v) *curriculum knowledge*, and (ii) *authority of the adept*. In my research I worked with this notion, together with the notions of mathematical reality and appearance in my analysis of video data to study the nature of the facilitators and teachers' appeals, and so what was constituted as mathematics for teaching.

The purpose of working with the methodology for studying teacher education in my research was, first, to augment variation theory, and second, to study the nature of the enacted and the lived object of learning in an object focused professional development. The second reason has already been elaborated elsewhere. However, it rested on Adler, et. al.,'s (op cit) finding that suggested that in mathematics teacher education and/or professional development, the object of learning was either a mathematics object with its teaching backgrounded, or the teaching object with the mathematics backgrounded. This introduced a tension/dilemma into my research which I managed by working with the notion of the duality of the object of learning.

As noted earlier, I have worked with the methodology of studying teacher education as my additional theoretical resource to augment the shortcomings of variation theory. The methodology is constituted by two notions: (i) legitimating criteria and (ii) duality of the object of learning in mathematics teacher education. Analysis with these notions was useful and producive in theorizing the enacted object of learning into *mathematical appearance* ($\boldsymbol{Mt_{ap}}$) and *mathematical reality* ($\boldsymbol{Mt_{re}}$) and the lived object of learning into *personal mathematics* (**Mt-pl**), *professional mathematics* (**Mt-PL**), *personal teaching of mathematics* (**Tm-pl**) and *professional teaching of mathematics* (**Tm-PL**). In my analysis of the enacted object of learning the notion of the duality of the object of learning revealed that the mathematics object (**Mt**) was foregrounded and the legitimating criteria were also characterised by appeals to **Mt**. In order to understand the nature of the **Mt** I drew on Galileo's notions of *reality* and *appearance* (see Stumpf, 1994). While Galileo used these to describe truth in general, I have recontextualised them into matheamtics teacher education to further study the enacted object. Thus the notions of *mathematical reality* and *appearance* emerged as the study and the analysis unfolded. In my analysis I noticed that the *appeal to mathematics* was the most dominant category. My further analysis of appeals to mathematics was able to categorise **Mt** into $\boldsymbol{Mt_{ap}}$ and $\boldsymbol{Mt_{re}}$**.** Similarly, in my analysis of the lived object of learning through interviews I drew from the methodology for studying teacher education to study the nature of the lived object. The duality in the object of learning suggested that some aspects of the lived object of learning were **Mt** and others were **Tm**. Grounded analysis of these two revealed that the teachers' learning of **Mt** was both *personal* and *professional*. Similarly so with **Tm**.

Figure 4.1 is an illustration of the analysis of the enacted object of learning involving the methodology for studying teacher education. The analysis began with subdivisions of the ACTIVIY in which the intended object of learning was embedded into evaluative events and sub-events. Analysis with legitimating criteria showed that the enacted object was **Mt** and not **Tm**. Working with Galileo's notions showed that the **Mt** was constituted by *mathematical reality* and *mathematical appearance*.

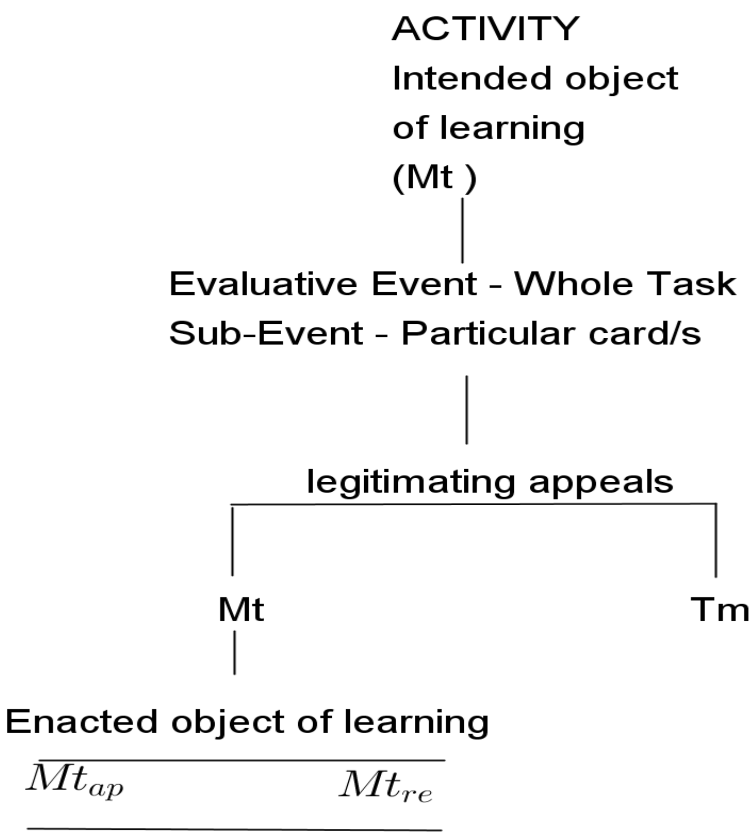


Figure 4.1: Illustration of the analysis done with the methodology on the enacted object

The analysis of the enacted object of learning with both variation theory and the methodology for studying teacher education were useful for studying the 'what' of the enacted object of learning, but not the 'how' part of the enactment. Facilitators' natural attitudes (i.e., tendency to take it for granted that the object of learning was easy for others to comprehend) led to different ways of enacting the sessions. To study how the sessions were enacted across the three clusters I worked with the notion of scaffolding which I discuss in the next subsection. This analytical tool also emerged during the study and as the analysis unfolded.

4.2.4. Scaffold

According to Wood and Wood (1996b) the term scaffold has its origins in the earlier work of Wood, Brunner and Ross . The term was used to describe the tutorial interaction between an adult and a child. The proponents of scaffold used this notion to gain insight into the nature of support given to a child on a task which the child could not do on its own. Working separately, Vygotsky developed a more general theory known as the zone of proximal development (ZPD) (Wood & Wood, 1996b). The ZPD is defined as "the gap between what a given child can achieve alone, their 'potential development as determined by independent problem solving', and what they can achieve 'through problem solving under adult guidance or in collaboration with more capable peers" (Wood & Wood, 1996b, p. 2).

In my research I worked with the metaphor of scaffold to study the nature of support that was provided to teachers under two distinct pedagogic strategies that emerged in the data. As I have elaborated more on this in later chapters, this theoretical resource was not part of my resources at the beginning of the study. Instead, it was a process and product of the study. Scaffold was recruited into my study during my analysis of the Functions 1 video data. I noticed the differentiated ways in which teachers were supported on tasks across three sites. This analysis led to the development of two pedagogic strategies: (i) inductive pedagogic strategy and (ii) deductive pedagogic strategy, which were used to close and open scaffold respectively. Further analysis focused on scaffold produced three categories of scaffold which I referred to as: (i) low scaffold, (ii) moderate scaffold and (iii) high scaffold. I then used these categories to analyze scaffold in the rest of the data.

4.3. Conclusion

In this chapter I discussed the theories underpinning my research. In the discussion I singled out *variation theory* together with *phenomenology* as my overarching theoretical lenses. I moved further to point out the limitations of these theories and how the limitations necessitated the recruitment of other theoretical resources such as notions of *duality in the object of learning* and *legitimating criteria*. Other resources emerged in the data and they included my notions of *inductive* and *deductive pedagogic strategies*, *low, moderate* and *high scaffold* as well as *mathematical reality* and *mathematical appearance*. These theoretical resources are operationalized in the analysis chapters and discussed further in my methodology and study design, to which I now turn in the next chapter.

CHAPTER 5: METHODOLOGY

5.1. Introduction

In this chapter I begin by articulating my world view about the nature of reality, truth/knowledge and values in research, and how these have informed my study. To do this, I discuss the interpretivist paradigm, and follow this with a discussion on fallibilism. While the former informed the conduct of the entire study, the latter informed my assumptions about the nature of mathematical truth. Building on this, I discuss both the positivist paradigm and absolutist perspective to illuminate the existing debates between the two paradigms. The main reason for bringing positivisim and absolutism into the dicussion was, in effect, to solidify my own understanding of the interpretivist paradigm.

This chapter provides a description of the overall methodology adopted for this research, since all other sections must resonate with it. In qualitative research the trustworthiness or believability of the research outcomes depends on the agreement between the aspects of the methodology (Crotty, 2009). In this research I bring together those elements of research that conform to a Social Constructionist view of knowledge. This view holds that knoweldge is personal, subjective and unique because there are multiple realities. Furthermore, knowledge is socially constructed and research is value-laden (Crotty, 2009).

All research is guided by the researchers' philosophical assumptions, and these assumptions are the bases for different ways of looking at social reality. As such, the ways in which we look at social reality give rise to different approaches in conducting social research. The assumptions I am referring to here are those that address the varying perspectives on the ontological (reality), epistemological (knowledge) and axiological (values) nature of research.

5.2. What paradigm informs my study?

5.2.1. The interpretivist paradigm

The role of a paradigm in social research is to frame 'the entire constellation of beliefs, values, techniques, and so on, [which] are shared by members of a given community' (Smith, 1975, p. 14). Paradigms are the assumptions or conceptualizations which underlie any data, theory or method used in the research. Most researchers and authors hold a particular paradigm or world view that guides their research process. In this research my underlying assumptions rested in the interpretivist paradigm. In this paradigm researchers do not focus on isolating and objectively measuring causes, or in developing generalizations (Cohen, Manion, & Morrison, 2001). Its focus is on attempts to gain empathic understanding of how people feel inside so that they can be able to interpret individuals' everyday experiences, their deeper meanings and feelings, and idiosyncratic reasons for their behaviors (Creswell, 1998). When conducting research, they tend to "hang out" with people and observe them in their natural settings. Hence it is often called the naturalistic paradigm (Rubin & Babbie, 1997). In my research I was interested in the teachers' experiences of the PD, and so working within this paradigm allowed me to get closer to the teachers and observe them in their natural settings. By way of example, I sat in PD sessions as a participant observer where they were experiencing PD activities.

According to Rubin and Babbie (1997) the interpretive researchers "hang out" with their participants so as to develop an in-depth subjective understanding of their lives and the researchers report their findings in such a manner that helps the reader to 'sense what it is like to walk in the shoes of the small number of people they study'. Rubin and Babbie argue that one cannot adequately learn about people by relying solely on objective measurement instruments that are used in the same standardized manner from person to person. They are opposed to the use of instruments that attempt to remove the observer from observed in order to pursue objectivity. In this research I analyzed video recordings of PD sesssions where teachers were (re-)learning school mathematics, and personally interviewed them following their participation in the PD. As such I was able to interact with the teachers.

For Cohen et. al (op cit), 'the interpretive researchers advocate for flexibility and subjectivity in one's approach so as to see the subject's world in the subject's eyes'. In this paradigm researchers are mainly interested in discovering and also understanding the way people perceive and experience the world on an internal, subjective basis (Cohen & Manion, 1980). These researchers believe that it is not possible to explain social reality without an in-depth understanding of how people's subjective interpretations of reality influence the creation of their social reality (Rubin & Babbie, 1997). This sentiment is echoed by Cohen and Manion (1980) who stated:

> The central endeavor in the context of the interpretive paradigm is to understand the subjective world of human experience. To retain integrity of the phenomena being investigated, efforts are made to get inside the person and understand from within. The imposition of external form and structure is resisted, since this reflects the viewpoint of the observer as opposed to that of the actor directly involved (p.40).

Therefore the purpose of social science is to understand social reality as different people see it and to demonstrate how their views shape the action which they take within that reality. The importance of working within this paradigm was that it served as a unifying factor. While they both subcribe to this paradigm, they stand in stark contrast to each other on the origin of meaning. While the former attributes the origin of meaning to the intentionality of the individual's consciousness, the latter attributes the origin of meaning to the society or context in which it is created. Hence in this research I worked from both views. Firstly, in my analysis of the enacted object of learning I embraced the view that meaning was socially constructed because what became the enacted object of learning happened in context and was co-constituted by facilitators and teachers, and this was evident in the legitimating criteria. There are other world views which are in stark contrast to the interpretivist paradigm.

Secondly, in my analysis of interviews I worked from the view that meaning was a consequence of an individual's intentionality of consciousness. It was challenging to work with these two views. However, I set out from my proposal stage with the resolve that one of the problems for my research would be to bringing the two views together to provide insight into the nature of the enacted and lived object of learning in an object focused PD. I could not work from one view, because no single view was sufficient to illuminate both objects of learning. The need to work with the two despite the differences rested on the fact that the two were each a variant perspective under the interpretivist paradigm. On the basis of the immense contribution of each perspective on my data analysis process, I believe that this research has been able to deal with the problem posed by working with the two.

The Positivist paradigm is the one that stands opposed to the interpretivist paradigm. Hence I discuss it later to illuminate the differences between the two rival paradigms.

5.2.2. What is the nature of mathematical knowledge within the Interpretivist paradigm?

While the interpretivists question the single nature of reality and the objective nature of knowledge, Fallibilists question the universality, absoluteness and perfection of mathematical knowledge. Fallibilism is a perspective within the interpretivist paradigm, and its central concern is to provide an alternative view to the nature of mathematical knowledge. As noted earlier that this study was located within the interpretivist paradigm, I must add that, similarly, I viewed mathematical objects (**Mt**) from the perspective of Fallibilists. Fallibilism is a perspective of those who stand opposed to the absolutist view of mathematical knowledge, which asserts that mathematical truths are incorrigible – i.e., they are beyond revision, and that they exist outside the human mind. I elaborate on the absolutist perspective later when I discuss the nature of mathematical knowledge within the positivist paradigm.

The emergence of the fallibilist perspective was seen as a great achievement by those who disputed the absolutist view (e.g. Enerst, 1999). For instance, Ernest (1999) stated that "an important background development has been the emergence of fallibilist perspectives in the philosophy of mathematics" (p.67). This view asserts that 'the status of mathematical truth is determined, to some extent, relative to its contexts and is dependent, at least in part, on historical contingency" (p.67). I use examples from Ernest (1999) to illustrate the source of fallibilists' dismissals of the absolutist views.

Machie (as cited by Enerst, 1999) proved by reason alone, that 2 + 2 = 4, and this he used to demonstrate that mathematical knowledge was the only true knowledge. In response, and to demonstrate that mathematical knowledge is born of human activity and so is susceptible to human error, fallibilists worked with the same example, 2 + 2 = 4 and showed that 2 + 2 is not always equal to 4. By way of example, they showed that in arithmetic modulo 4, 2 + 2 = 0 and in arithmetic modulo 3; 2 + 2 = 1. So for fallibilists, 2 + 2 = 4 is a social construction and not a discovery. Hence it is true by convention and is situational. It is by convention because it has been agreed upon, and so it is a case of *deciding* and not *discovering* that it is so. The decision is situational since there are other alternatives (such as I have illustrated with examples from Ernest (1999) earlier in this paragraph. Hence, multiple solutions to the same problem suggest that there are multiple realities and that mathematical knowledge does not exist outside human activity. In addition, that it is true by convention suggests that it is not value-free.

The purpose of this section was to exemplify the subjective view of reality by demonstrating that mathematical truths/objects are not beyond revision, and so are not objective. In my analysis I worked from the notion of **Mt** and moved further to subcategorise it into *mathematical reality* and *mathematical appearance*. The basis for further developments on **Mt** was my understanding and believe that, mathematical objects or truths are a result of one's interpretation of reality, and so are subjective.

5.2.3 The positivist paradigm

Positivists hold the belief that says that explanation proceeds by way of scientific description (Acton as city by Cohen, et al., 2001). Hence their doctrine that holds that all genuine knowledge is based on sense experience, and can only be advanced by means of observation and experiment (Ernest, 1991). Barrel and Morgan (as cited by Cohen, et al., 2001) add that the positivist perspective expresses itself most forcefully in a search for universal laws which explain and govern the reality which is being observed.

This paradigm has been criticized for trying to explain a person's behavior in mechanistic terms. This is where it suggests that people behave the way they do as a result of some internal and external causes. For Cohen and Manion (1980) explaining people/human agents in these terms tends to suggest to the social science researcher to adopt a perspective of 'a detached, outside observer intent upon classifying what he/she sees and hears in the light of some theory he/she holds about the way in which society is structured or some hunch he/she has about a range and variety of the needs that a person must have'. There are two properties of this paradigm, and these have been criticized. First is the mechanistic image of a person it poses. Its critics argue that it does not consider a person's unique ability to interpret his experiences and represent them him/herself. It is also accused of being ignorant of the profound differences between natural and social sciences. This paradigm is thought to be preoccupied with the belief that the best way to understand society is through the application of methods that are synonymous with natural sciences. Its findings are thought to be often banal and trivial, such that they are of little consequence to those for whom they are intended (Cohen & Manion, 1980).

The Positivist paradigm perceives reality as single and objective. It is tangible and can be observed. In other terms - there is one reality or truth. They see reality as the same across time, space and continents (Cohen, et al., 2001). For Positivists, knowledge is hard, objective and tangible. This paradigm asserts that the researcher does not have values. Instead, he is value-free since he/she does not carry his/her values into the research. For them research is value neutral because there is one truth and one objectivity (Cohen, et al., 2001).

5.2.4. What is the nature of mathematical knowledge within the positivist paradigm?

The absolutist paradigm claims that mathematical truth is infallible, incorrigible and above revision (Enerst, 1991; Hart, 1996). Ernest (1991) describes this view to mathematics knowledge as one that firmly believes '... that mathematical truth is absolutely certain, unquestionable and objective knowledge' (p.3). This belief is evident in Hart's (1999) assertion when he says that absolutism "... is an ancient and honorable view that mathematics is a body of truths, and indeed that the known truths of mathematics are the most absolute and unqualified truths known to us" (p.2). This view has its origins in the widely adopted approach to epistemology that assumes that knowledge in any field is represented by propositions and procedures. The procedures are thought to be essential for verifying or providing warrant for the asserted propositions.

Absolutists believe that mathematical knowledge stems from propositions and procedures. Hence it yields certain truths, and as such, is above correction and is the most certain of all knowledge. Absolutists included, among others, Euclid, Newton and Spinoza (Ernest, 1991). Euclid's contributions were his Elements, while Spinoza contributed Ethics. One of the most contested positions of the absolutists is the view that mathematical knowledge is the only one worthy of classification as true knowledge, since it is "asserted on the basis of reason, and reason alone" (Enerst, 1991). This view has been refuted by several scholars who hold the view that mathematics is fallible.

While my research is located in the interpretivist paradigm, I found it informative to discuss its rival paradigm, because I work from the view that to suffieciently know something, one must know what it is and what it is not. Hence by discussing the positivist paradigm I am essentially elaborating further on the interpretivist paradigm which anchored my research.

5.3. Philosophical assumptions in the study

Paradigms are different, and the distinguishing features between paradigms are the way they attend to ontology, epistemology and axiology. The *ontological assumption* is concerned with answering the question "what is the nature of reality?' Every paradigm has its own understanding or definition of reality. Cohen, Manion and Morrison (2001) view ontological assumptions in educational research as mainly concerned with the essence of the social phenomena which are being investigated. They explain that here we seek to know whether social reality is external to individuals. The question of interest is the one that seeks to find out if reality imposes itself on the individual's consciousness from outside, or if it is a product of individual consciousness. In other words, is it objective or subjective. So the crux of ontology lies in the answer to the question: Can one say reality is a given, or is it constructed? Is this construction individual or social? The design of my study suggested that I work from two perspectives. I worked with the view that reality was individually constructed at the level of the lived object of learning, and that it was socially constructed at the level of the enacted object of learning. As noted earlier, working with these two views was not only fraught with problems, but also introduced some contradiction.

In this research, and at the level of the enacted object of learning my *ontological assumtion* was that reality was both individually and socially constructed, because the experiences of phenomena were not only individual since an individual is a social subject whose learning occurs in a socially structured context. For instance, at the level of the enacted object of learning I worked with the view that meaning was socially constructed, and at the level of the lived object of learning I worked with the view that meaning was a consequence of the individual's intentionality of consciousness. However, at both levels I maintained that multiple realities existed – i.e. reality was either social or individual, and so differed between contexts and between individuals. This assumption was evident in my data collection and analysis methods. I analyzed video data where facilitators and teachers co-constituted the object of learning, and I engaged each teacher in an in-depth interview to capture each teacher's view of the nature of reality with respect to experiencing object focused PD.

The *epistemological assumption* is concerned with questions such as "how can we acquire knowledge?", and how, once acquired, can knowledge be communicated to other human beings (Barrel & Morgan as cited by Cohen, et al., 2001, p. 34)? Barrel and Morgan explain that:

> we seek to find whether it is possible to identify and communicate the nature of knowledge as being hard, real and capable of being transmitted in tangible form, or whether knowledge is of a softer, more subjective, spiritual or even transcendental kind, based on experience and insight of a unique and essentially personal nature

In this research my *epistemological assumption* was that knowledge was personal, subjective and unique (Creswell, 1998). Hence in my research I analyzed each participant's data set separately to gain insight into the meaning structure of each participant's experiencing of object focused PD. I later pulled these together to establish the essential invariant meaning structure of the experience. I also analyzed video data according to clusters to see what was made possible to learn at each cluster. Hence my epistemological assumptions shaped the nature of knowledge produced by my research.

The *axiological assumptions* are concerned with what values are held by the researcher. It is essential to know what values are held within a given paradigm when conducting research because the researcher's values shape the research and the derived meanings (Hatch, 2010). Understanding the nature of values in a particular paradigm helps the researcher to know if the research is value-neutral or not. According to (Creswell, 1998) knowing where the researcher stands with his values when conducting research is imperative.

In this research my *axiological assumption* was that research was value-laden because researchers have vested interests in the research they conduct - and so are likely to introduce bias, more so when they research their own practice - as was the case in my research. As an illustration, this research was located in a professional development project in which the researcher was a team member. My vested interests were manifested in the decisions that I made. As noted earlier (see **Chapter 2**), the decision-making process is fraught with tensions (see Adler, June 2013; Graven, 2005). Thus, decisions pertaining to content selection in my study were not value free. The grounds for working with SMK only are discussed in **Chapter 2,** and I have discussed how these decisions have limited my study in **Chapter 12**.

5.4. Overarching methodology -Social Constructionist

Some authors from the domain of educational research (e.g. Crotty, 2009) argue that the trustworthiness and credibility of any research, and qualitative research in particular, lie in the extent to which the four basic elements of research cohere. These elements include epistemological and ontological assumptions, theoretical perspectives, methodology and methods. While methodology holds the same meaning as research design, methods refer to ways of data collection and analysis. Social Constructionism, a variant of Constructionism, provided the ontological, epistemological and axiological assumptions undergirding the study. As noted earlier, my assumptions were located in the interpretive paradigm.

Constructionism, and in particular, Social Constructionist was the overarching methodology which was selected to guide my research. It informed my research questions, methods of data collection and analysis, methodologies and theoretical perspective which were selected to address the problem of my study. In particular, the problem of interest was to provide insight into two questions which were raised in teacher professional development literature (see Ball & Cohen, 1999; Buddin & Zamarro, 2009). The questions were: (i) What do teachers learn in professional development, and (ii) How do teachers learn in professional development?

Social Constructionism rejects both the objective and subjective orientations to knowledge, and instead asserts the social origin of meaning, and also propounds that all objects are made rather than found. In addition, the means by which the objects are made are said to be social and conventional (Crotty, 2009). This view opposes the Objectivist view that holds that meaning, and therefore meaningful reality, exists as it is outside and independent of the human mind; and the subjectivist view that says meaning is imposed on the object by the subject (Crotty, 2009). However, for Social Constructionists both the subject and the object are central in the making of meaning. By way of example, in my research teachers talked about their experiencing of professional development tasks. From their talk I was able to obtain the essential invariant composite meaning structure of their experiencing of the phenomena. In other words, teachers had to experience the object of learning in order to discern its critical features.

Crotty (2009) explains the relationship between the subject and object when he says that under Social Constructionism "meanings are constructed by human beings as they engage with the world they are interpreting", and in addition Social Constructionists assert that "[b]efore there was consciousness on earth capable of interpreting the world, the world held no meaning at all" (Crotty, 2009, p. 43). This epistemology resonates well with both phenomenology and variation theory, the two theoretical perspectives adopted for my research.

Social Constructionism, phenomenology, variation theory and the methodology for studying teacher education emerged through the development of the research problem, related questions, and empirical work. By way of example: on the one hand I was concerned with the constitution of the enacted object of learning in professional development activity, and viewed this constitution as a social process. As such I was clear that the constitution of the object of learning resulted in some forms of meanings originating in social settings (professional development) and professional development as a social setting had its own pedagogic practices which, according to the social theory of pedagogic discourse, made meaning-making possible. Hence here I worked in a grounded way with the notion of scaffolding to analyze the pedagogic practices inherent in the social setting. I also worked with a social epistemology to analyze what was appealed to, to ground the enacted object of learning. Further analysis was conducted with tools from variation theory to analyze what was made possible to learn.

On the other hand, I was concerned with the lived object, and thus the knowledge of the teachers in the professional development. I worked deductively to analyze the lesson plans and in a grounded way to analyze interviews. Thus, to understand the lived object of learning I drew from phenomenology to analyze interviews, and from variation theory to analyze the lesson plans. This theoretical perspective holds that culture impedes new meaning-making because it offers a comprehensive set of meanings which serve to close us from an abundance of untapped significant new meaning (Creswell, 1998). To deal with the interference from culture, this perspective emphasises phenomenological epoch, or bracketing, when new meaning is encountered, because meaning is embedded in the things themselves (Crotty, 2009; Stumpf, 1994). By way of example, in my analysis of the enacted object of learning I worked deductively with established analytical tools. To a greater extent, working with these tools shaped my interpretations, and the meanings I derived from my analysis. In phenomenology working with pre-established tools imposes meaning, and so obscures new meaning from emerging. Hence working in a grounded way is considered sufficient because here new meaning emerges from the data (things themselves) with limited influence.

As discussed in the theory chapter, Variation Theory and Phenomenology were my overarching theoretical perspectives and the social epistemology of Adler and Davis'(2006), scaffold, and mathematical reality and appearance were my methodologies. Each theoretical resource had links with a particular methodology. From variation theory I recruited the language of description to describe the object of learning at three levels. I drew on patterns of variation to describe the analysis process and to articulate evidence of the enacted object of learning in videos. I further called on the space of variation to articulate the lived object of learning from lesson plans. Phenomenology provided concepts which I used to describe the meaning structures from the interviews. I worked with the notion of legitimating appeals and the duality of the object of learning and evaluative event to analyze video data. My unit of analysis was evaluative events, and I have used this construct to frame my analysis of the enacted object of learning. In a teaching sequence an evaluative event is determined by a shift from one object to a new object. The methods of data collection and analysis followed known qualitative research practices (Cohen, et al., 2001; Creswell, 1998). As such, the trustworthiness of my research findings rested on the coherence between the four basic elements.

5.5. Research Methods

5.5.1. Participants and methods of data collection

The selection of participants is dependent on the tradition of inquiry underlying the research. Creswell (1998) explains a tradition of inquiry as "an approach to qualitative research that has a distinguished history in one of the disciplines ... [and has its own] distinct methodologies that characterizes its approach". Phenomenological research is one of the five traditions of inquiry which meets Creswell's criteria of being a tradition of inquiry. Application of Phenomenology requires that a participant must have experienced the phenomenon under investigation to be able to talk about his/her experiencing of it. In this research phenomenology was used to analyse teacher interviews after they had experienced PD.

The study had two types of participants, and they included nine mathematics teachers and six video recordings. I have discussed each type of participant separately. I do so by discussing sources of data; data collection and analysis for human participants first, before moving on to do the same for non-human participants. As I do this, I also show whether the data source is related to the lived or enacted object of learning.

5.5.2. Participating teachers – my human participants

The human participants were nine mathematics teachers. These teachers were selected after experiencing the WMCS professional development activities, particularly the Functions 1 and Functions 2 sessions. The participants were five male and four female mathematics teachers. The youngest teacher was 29 and the oldest was 50 years. The most experienced teacher had been teaching for 22 years and the least experienced teacher had taught for 7 years. The highest qualification held was a 4 year professional teaching degree, which was also the modal teaching qualification.

The nine teachers came from three school clusters which I refer to as the E-Cluster, I-Cluster and N-Cluster to maintain the anonymity of the participants. Since each cluster had both GET and FET teachers participating in the PD, the combination was inherent in the constitution of my participants. By way of example, the E-Cluster was represented by one GET and one FET teacher. I must add that these two teachers were not trained secondary school mathematics teachers. For instance, while the FET did not hold a teaching qualification, the GET teacher held a lower teaching qualification which required her to teach at Grades lower than Grade 8. The I-Cluster was represented by one GET, one FET and one teacher who taught across the levels. The N-Cluster was represented by three FET and one GET teachers. Altogether, there were two teachers from the E-Cluster, three from the I-Cluster and four from the N-Cluster. While the participants were drawn from the three clusters, not all schools were represented. Only six of the 11 project schools were represented. The two E-Cluster schools were each represented by a teacher. The three teachers from the I-Cluster were from the same school. So the other two schools in the cluster were not represented. The four teachers from the N-Cluster were from three of the six N-Cluster schools. A discussion of the demography of participants is important because it provides evidence to the problem I have noted in my discussion of the rationale in Chapter 1. There, I pointed to the presence of un-qualified and under-qualified teachers in the education system of South Africa.

Suffice it to note here that the nine teachers were purposefully selected, and the criterion was that each teacher needed to have developed two lesson plans and to have experienced Functions 1 and Functions 2 of the PD. While most teachers experienced the Functions 1 and Functions 2 tasks, only nine produced the required Pre- and Post-PD lesson plans. This number was 1 less than my intended sample of 10 participants. However, phenomenological research requires any number between five and 25 to be able to construct a reasonable composite meaning structure (Creswell, 1998).

- **Interview - source of data for the lived object**

The nine teachers who developed the Pre- and Post-PD also participated in in-depth individual interviews. As a consequence there were nine interviews, each averaging 2 hours in length. In phenomenological research, a 2-hour interview is considered to be sufficient to provide insight into the participant's experiencing of the phenomena. The interview protocol had three sections. In the first section teachers were asked to talk about the WMCS project's resources with respect to how they facilitated their learning in Functions 1 and Functions 2. The second section required teachers to talk about their Pre- and Post-PD lesson plans.

The third section contained seven scenarios which were designed to tap into teachers' subject matter knowledge of functions. However, in the end they were largely unhelpful in relating the teachers' lived object of learning to the enacted object of learning. However, there was one instance where the response to one scenario was useful because of the discussion that took place during the enactment of the object of learning and one teacher's involvement in that discussion. Hence I drew only from that scenario to illustrate the teachers' lived object of learning with respect to the enacted object of learning, and to make claims about the teachers' knowledge of functions before and after PD. Scenario seven was the only one that coincided with what happened in the PD during enactment. My illustration of the relationship between teachers' lived object of learning and the enacted object of learning can be found in **Chapter 10,** and the claim I make about the teachers' knowledge is discussed in **Chapter 12,** where I return to reflect on this element of the interview data and what might have improved its use in my research.

- **Pre- and Post PD lesson plans - source of data for the lived object**

All the nine participants developed the Pre- and Post-PD lesson plans, which were later analyzed to gain insight into the teachers' lived object of learning before and after participation in the Functions Theme work, where Functions 1 and 2 were in the fore, and where they experienced Functions directly. According to Variation Theorists, the lived object of learning is understood from the viewpoint of the learner, and is normally assessed through pencil and paper tests. However, when the learner is the teacher, such as in the case of the WMCS project's professional development, it is not always possible to get teachers to sit for pencil and paper tests. Hence, I asked each teacher to develop the two lesson plans to see what they privileged before and after professional development with respect to key features of functions.

Altogether, there were 9 Pre-PD and 9 Post-PD lesson plans making a total of 18. All the 18 lesson plans were focused on Functions, with each teacher planning for the Grade s/he taught. Teachers were asked to focus the Pre and Post-PD lesson plans on the same aspect or topic under Functions. By way of example, Teacher 2's two lesson plans were focused on the Exponential Function, with focus on the effect of the base when the base was between 0 and 1 and when it was greater than 1 (i.e. $y = b^x; when\ b > 1\ and\ when\ 0 < b < 1$) (see **Chapter 10** for Teacher 2's two lesson plans).

5.5.3. Data collection and procedures for human participants

- **Pre- and Post-PD lesson plans**

Two weeks before the beginning of the Functions 1, 25 teachers who had participated in the Integer Theme work were each requested to develop a lesson plan focused on any topic under Functions at the Grade level each taught. Teachers were instructed to show what they considered to be important for learners to know in the chosen topic. By the time Functions 1 began only 14 teachers had responded. I then requested them to ensure that they participate in the Functions Theme work in order to be eligible to develop the second lesson plan. By the end of the Functions Theme work the 14 teachers were eligible to develop the Post-PD lesson plan because they had experienced the Functions Theme work, particularly Functions 1 and Functions 2. Two weeks after Functions 4, I asked each of the 14 teachers to develop a second lesson plan and the instruction was that the lesson plan should have the same focus and target as the first one. In response, only 9 teachers developed the required second lesson plan, and became my participants.

In Part 2 of the interview each teacher was required to talk about inherent similarities and differences between the two lessons plans the 5 teachers who did not develop the second lesson plan could not participate in my research. **Table 5.2** shows the dates on which lesson plans were obtained from the 9 teachers.

Table 5. 1: Dates on which lesson plans were developed

Teacher	Pre-PD Date	Post-PD Date
Teacher 1	04 May 2011	25 July 2011
Teacher 2	04 May 2011	05 July 2011
Teacher 3	29 April 2011	25 July 2011
Teacher 4	04 May 2011	02 August 2011
Teacher 5	04 May 2011	10 August 2011
Teacher 6	11 April 2011	26 August 2011
Teacher 7	29 April 2011	08 November 2011
Teacher 8	04 May 2011	02 August 2011
Teacher 9	03 May 2011	23 August 2011

Column 1 of Table 5.1 shows that all the 9 teachers submitted the Pre-PD lesson plan, and this took place between 11^{th} April and 4^{th} May 2011. At the E-Cluster I collected the last Pre-PD lesson plan of on 3^{rd} May 2011. The Functions 1 sub-theme was held on this date at this cluster. The last Pre-PD lesson plans were collected on 4^{th} May 2011 at the I-Cluster and the N-Cluster and this was the date of Functions 1 in the two clusters. As a consequence, all Pre-PD lesson plans were collected before the start of the Functions Theme work. Similarly, all Post-PD lesson plans were collected at the end of the Functions Theme work. For instance, while the Functions theme work ended on 01-June (see **Table 5.3**), the first Post-PD lesson plan was collected a month later on 05-July, and the last of these was collected on 08 November 2011.

What procedures did I follow to get teachers to develop the Pre-PD lesson plans? In April 2011 teachers were participating in the Integer Theme work. Each participating teacher was requested to develop a lesson plan which he/she was to hand in to the researcher by the first contact day of the Functions Theme work. **Excerpts 5.1** and **Excerpt 5.2** contain instructions I gave teachers to develop the two lesson plans. The excerpts were taken from the two letters I wrote requesting teachers to develop the Pre-PD and the Post-PD lesson plan.

Instructions for developing the Pre-PD lesson plan

I humbly request you to develop ***one Lesson Plan on Grade 10 Function before the Wits Maths Cluster Functions Theme work***

I request that you bring your plan to the first cluster workshop.

You might not be teaching Grade 10. If so, you can write a plan for a lesson on functions at a Grade 9 or Grade 11 level.

Could you please include in your plan

1. The topic and purpose of the lesson
2. The grade it is aimed at
3. The steps you plan to follow in the lesson and what you expect learners to be doing. You might find the template attached helpful.

Excerpt 5.1: Instructions for developing Pre-PD lesson plan

Instructions for developing the Post-PD lesson plan I wish to request that you develop a second Lesson Plan on the same topic of Functions. I have attached details of the first lesson plan to remind you how you had planned. Information from your first and second lesson plans will be used to research the Wits Maths Connect project's professional development. My desire is that you plan for the same topic, with the same objectives, for the same Grade. After your participation in the just ended WMC theme on Functions I seek to understand how you might plan for the same lesson. In particular, I are interested in how your second lesson plan might resemble or differ from the first one.

Excerpt 5.2: Instructions for developing Post-PD lesson plan

- **Delayed Interview data collection**

Prior to conducting interviews with teachers I wrote each of the 9 teachers a letter inviting him/her to participate in the interview. The letters were written and sent to teachers at the beginning of November. As a consequence, the first interview was conducted on 22 November 2011 (see **Table 5.2**).

I wish to invite you to an interview in which I seek to understand your experiencing of the Wits Maths Connect professional development activities focused on functions. The interview is divided into three Parts. Part 1 focuses on resources that were used in Theme 2 (Functions). Part 2 focuses on the two lesson plans that you have developed. Part 3 focuses on functions themselves. Each part is expected to last for 20 minutes but may take longer. Please note that you are not the focus of this interview. Instead I want you to assist me to obtain information about the Wits Maths Connect model of professional development.

Excerpt 5.3: Information about the interview

The interviews were conducted 5 months after the Functions Theme work. Each teacher sat for a 2 hour in-depth interview to talk about his/her experiencing of Functions 1 and Functions 2. The interviews were delayed deliberately since I was interested in the lived object of learning. Variation Theorists believe that a longer period between experiencing the phenomena and the interview gives the learner sufficient time to make sense of the experiences and construct knowledge about the phenomena. Delaying the interview also resonated well with Mason's (2013) assumption about teaching, which says that while teaching takes place in time, "learning takes place over time, as a maturation process" (p.3).

The 9 interviews were conducted in the period 22nd November – 1st December 2011, and took 6 days (see **Table 5.2**). Each teacher was interviewed at his/her school during school hours when the teacher was not teaching. Teachers were emailed the interview protocol a day before the interview to familiarize and to prepare for the interview. I brought resources which were used in Functions 1 and Functions 2 to the interview. Bringing along resources to the interview served two purposes. First, it revitalized the teachers' memory since the interview was taking place 5 months after the Functions Theme work, and so they were likely to have forgotten the tasks or activities in which were used. The second purpose was to shift attention from teachers onto the resources. In **Table 5.3** the names of teachers and their schools have been replaced with pseudonyms to protect their identity. In the pseudonyms of schools the E-C, N-C, and I-C are abbreviations of the three clusters and the numbers in front of these are used to distinguish schools in the same cluster. The arrangement of teachers in the table rests on the date on which the teacher was interviewed. For instance, Teacher 8 was the first to be interviewed and this took place on the 22nd November 2011, and the last interview was conducted with Teacher 6 on the 1st December 2011.

Table 5. 2: Interview data collection time table

Name	Interview Date	School	Status
Teacher 8	22 Tuesday November, 2011 1230 -1400hrs	I-1	Interviewed
Teacher 5	24 Thursday November, 2011 1230-1400hrs	N-1	Interviewed
Teacher 2	25 Friday November, 2011 0800-0930hrs	I-1	Interviewed
Teacher 4.	25 Friday November, 2011 1000-1130hrs	I-1	Interviewed
Teacher 7	28 Monday November, 2011 1130hrs-1300hrs	N-2	Interviewed
Teacher 3	29 Tuesday November, 2011 0800 – 0930hrs	N-3	Interviewed
Teacher 1	29 Tuesday November, 2011 1000-1130hrs	N-3	Interviewed
Teacher 9	30 Wednesday November, 2011 0800-0930hrs	E-1	Interviewed
Teacher 6.	01 Thursday December, 2011 0900 – 1030hrs	E-2	Interviewed

The purpose of the interview data was to provide insights into the lived object of learning from the point of view of teachers following their participation in the PD, where they were revisiting school mathematics with Functions in the fore. To gain insight into the primary enacted object of learning in the PD I turned to video recordings of the WMCS PD, in which teachers experienced Functions 1 and 2. As such the next section details how the video recordings were obtained.

5.5.4. Video Recordings – my non-human participants

The non-human participants were six videos selected from 12 video recordings of the functions theme work from three sites (i.e. E-Cluster, I-Cluster and N-Cluster). These video-recordings of classroom learning were obtained from the WMCS project archives. As noted earlier, six videos were analysed to answer questions about the enacted object of learning and these contained data from Functions 1 and Functions 2. In these sessions teachers were learning functions directly.

As noted (see **Chapter 1 section 1.3** for details) I did not work with video recordings of Functions 3 and 4 because in those sessions it was not clear what teachers were expected to learn. I explained in **Chapter 1** that in Functions 3 and 4 teachers were not learning functions directly. In these sessions teachers observed demonstration lessons in which the facilitator taught functions to 40 learners from one of the 11 project schools. At the end of each lesson teachers sat together with the facilitator to reflect on the lesson with particular focus on learners' experiencing of functions. Therefore, data from Functions 3 and 4 were beyond the scope of my research, since the enacted object of learning was not co-constituted by the facilitator and teachers. Of course, I have noted in **Chapter 12** how this decision introduced a limitation into my research.

5.5.5. Data collection and procedures for non-human participants

- **Video – source of data for the enacted object of learning**

The video data was obtained with permission from the WMCS project. The video recordings of Functions 1 and Functions 2 across the three clusters were obtained and later analyzed. This particular data was used to answer my research question, which inquired about the enacted object of learning in the Functions Theme work. The video-recordings of Functions 1 and 2 were conducted in the period May – June 2011 (see **Table 5.3**).

Obtaining video data was less demanding because the WMCS project's archives were located on the same premises as the researcher. In addition, this research was covered by the WMCS project's ethics clearance application. As such, accessing this data was less challenging. However, I followed due processes to obtain permission from the WMCS project to use its video data which was obtained from schools with permission from the Gauteng Department of Education under reference number **D2011/30**, and the University of the Witwatersrand's ethics clearance protocol number **2010ECE60C**. Table 5.3 shows the dates on which the Functions Theme work was conducted. All the 12 video recordings were conducted on these dates by WMCS project team members.

Table 5. 3: Dates on which video recordings were conducted

Date	Event	No. of teachers attending
3, 4 May	Functions 1	32
10, 11 May	Functions 2	33
24, 25 May	Functions 3	25
31 May, 1 June	Functions 4	16

Source: *(Adler, July 2011)*

Column 1 of Table 5.3 indicates that the duration of the Functions Theme work was four weeks (i.e. 3rd May – 1st June). There are two dates in each row of this column. For instance, 3, 4 May in Row 1 of Column 1 stands for 3rd May and 4th May. Similarly, 10, 11 May in Row 2 of Column 1 stands for 10th May and 11th May. Now, the first date (e.g. 3rd May) is the date on which the Functions Theme work was conducted at the E-Cluster, and the second date is the date on which it was conducted concurrently at the N-Cluster and I-Cluster. The first dates were on Tuesdays, while the second dates were on Wednesdays. Later in my inductive analysis of pedagogic strategies I mention there that the E-Cluster teachers' responses on Functions 1 and 2 tasks informed the scaffolding of these tasks at the other two clusters. The responses revealed gaps in their subject matter knowledge of the secondary school Functions content. As a consequence, Facilitators provided higher scaffolding on the following day at the other two clusters.

Column 2 shows the sub-theme which was conducted on a particular date and Column 3 shows the total number of teachers who attended a particular sub-theme. For instance, between the 3rd and 4th May, 32 teachers attended Functions 1 across the three clusters. Earlier I mentioned that I worked with 6 video recordings out of a population of 12, and that the 6 were focused on Functions 1 and Functions 2 across the three clusters. The table shows that Functions 1 and Functions 2 were conducted on the first and second weeks of May 2011. Suffice it to mention that the researcher was present at the enactment of both Functions 1 and Functions 2 at the E-Cluster and the I-Cluster as a WMC-S project team member and he participated in the Functions 1 and Functions 2 tasks.

5.6. Data Analysis

5.6.1. Unit of analysis

- **Lesson plan and interview utterance**

As noted earlier, the first type of participant was 9 teachers, and two sources of data from these teachers were lesson plans and interviews. The unit of analysis in the lesson plans was individual lesson plans. In the interviews my unit of analysis was teacher utterances about their experiencing of Functions 1 and Functions 2 tasks. In the interviews teachers talked about resources, lesson planning and scenarios.

- **Evaluative event and sub-events**

As noted earlier, the second type of participants were 6 video recordings of the WMC-S project professional development where Functions 1 and Functions 2 were in the fore. Data from the videos sought to answer the research question in which I was exploring the constitution of the enacted object of learning in those video recordings. Here the unit of analysis was professional development tasks. I have discussed these tasks, with illustrations, in Chapter 2, when I discussed the context of this study. There, I showed examples of tasks used in Functions 1 and Functions 2. While Functions 1 had two tasks (i.e. Card Sorting and Card Matching), Functions 2 had five tasks. Decisions about making tasks units of analysis were informed by Davis (2005). An evaluative event as described by Davis (2005) and then Adler and Davis (2006) is some interaction that invites evaluation/judgment. In my data, the most coherent breaking-up of the PD sessions was in terms of tasks – the sessions were clearly punctuated by the ending of one task and the beginning of the next. Davis, Adler and Parker (2007) have elaborated on the constitution of an evaluative event.

> We accept as axiomatic that pedagogic practice entails continuous evaluation, the purpose of which is to transmit criteria for the production of legitimate texts (Bernstein, 1996). Further, any evaluative act, implicitly or explicitly, has to appeal to some or other authorising ground in order to justify the selection of criteria. Our unit of analysis is what we call an *evaluative event*, that is, a teaching-learning sequence that can be recognised as focused on the pedagogising of particular mathematics and/or teaching content. In other words, an evaluative event is an evaluative sequence aimed at the constitution of a particular mathematics/teaching object. The shift from one event to the next is taken as marked by a change in the object of acquisition (p.5).

The usefulness of this notion to their work was that it constituted their unit of analysis. In this research I also work with this notion, particularly when I work with professional development tasks as my units of analysis. As a consequence, my tasks are my evaluative events. I exemplify this with illustrations from Functions 1 later, with **Excerpts 5.4** and **Excerpt 5.5**.

Working from Davis' et. al., (2007) explanation that an evaluative event has a beginning and an end, and so the shift from one event to the next is marked by a change in the enacted object of learning, I was able to mark the beginning and the end of each task. While doing this, I noticed that the end of one task marked the beginning of the next. By way of example, Functions 1 had two tasks and during enactments the shift from Task 1 to Task 2 was marked by both the facilitator's announcements and task instructions (see **Excerpt 5.4**). The announcements were made to introduce the resources which teachers were to work with and task instructions were used to explain the task. By way of example, in **Excerpt 5.4** the facilitator introduces the cards which teachers are to work with on Task 1. Simultaneously, task instructions explain what teachers are expected to do in the task.

While **Excerpt 5.4** portrays the beginning and ending of Task 1, **Excerpt 5.5** portrays the same for Task 2. In Task 1 the first announcement was made 4 minutes 19 seconds into the

1hour time slot allocated to both Task 1 and Task 2. The second announcement came after 39 minutes of the allocated 1 hour. As such, the time spent on Task 1 was 35 minutes 32 seconds.

Beginning of Task 1: 0:04:19

F: Ok, so on your table you have an envelope that has got several cards in it, about 25 of them.

[on the overhead:
Card sorting

- Task 1
  Sort the cards into 2 piles.
  Those that represent functions
  Those that DO NOT represent functions *]*

F: And I would like you to sort your cards into 2 piles – those that represent functions and those that don't. Ok. So functions and, if you want, non-functions, just separate the 2 piles. You'll see there are some cards with words on them, some with pictures, some with tables of values. Take a look at them and decide, is this a function or is it not? And make your 2 piles. And I'll give you a few minutes to work on that.

End of Task 1: 0:39:32

F: So please will you take 24 out of your pack and now what I'd like you to do is to go back to your cards, and to put them in piles. I have an instruction. Here it is. *[puts an overhead on the screen:*

Excerpt 5.4: The beginning and ending of Task 1

It is discernible from the two excerpts that the last announcement in **Excerpt 54** is the first in **Excerpt 5.5**. This occurrence resonates well with Davis and his colleagues' assertion that a shift is marked by the change in the enacted object. In Task 1, teachers worked with all the cards, including card 24, to sort them into two piles of functions and non-functions. In Task 2 the Facilitator tells teachers to put aside card 24 which will not be needed in this task. While this announcement marks the end of Task 1, it simultaneously marks the beginning of Task 2. Task 2 commenced after 41 minutes into the total allocated time for Functions 1. The task ended after 57 minutes 22 seconds of the allocated 60 minutes which were shared by both tasks. The time spent on Task 2 was approximately 16 minutes.

An evaluative event might have sub-events depending on the total length of the event. Due to the length of Task 1 I subdivided my analysis to into sub-events, where each was constituted by focus on a particular card or a collection of related cards. So Task 1 had three sub-events

(Card 24; Cards 3,4,12 and19; Card 7). While Card 24 was a representation of non-function, Cards 3,4,12 and 19 were multiples of the same quadratic function, and Card 7 was a representation of a mapping. Similarly, Task 2 was sub-divided into sub-events and these were three. The first focused on matching multiple representations of a linear, the second on quadratic and the third on hyperbolic function.

Beginning of Task 2: 0:41:01

F: So please will you take 24 out of your pack and now what I'd like you to do is to go back to your cards, and to put them in piles. I have an instruction. Here it is.
[puts an overhead on the screen:

More card sorting

- Task 2

Take only the cards representing functions.
Make piles of cards that belong to the same function
Give reasons why you think the cards belong together. *]*

F: Ok, so we are taking only the cards representing functions. You might not be sure about 19. You can decide whether you want it in or out. I want you to make piles of cards for cards that belong to the same function. And obviously then be able to give reasons. So when I'm saying the same function, I'm not saying these are all straight lines or these are all parabolas – it would be a specific parabola or specific straight line that would be in the pile, ok. So if it were… If it's a different equation then it's not the same function. If it's a different graph, then it can't be the same function. Ok. So can you put them in piles of, or in rows or whatever, the ones that you think belong together.

End of Task 2: 0:57:22

[on the overhead:
Time to reflect:

- Which cards did you find the most difficult to match?
- Which representation/s did you find
  easy to deal with?
  difficult to deal with?
- How do you think Gr 10 learners will cope with a task like this? *]*

F: Alrighty. I would like us to take a moment to reflect for yourself – which of the cards did you find the most difficult to place? And which representations did you find the easiest and the most difficult to work with? And then thirdly how would you think a Grade 10 would cope with a task like this? I'm not saying this particular set of cards but a task like this? Maybe you just want to chat briefly with the people at your table?
Can you pinpoint ones that you found harder than others? And that might be different for everybody, because we have different things that speak to us.

Excerpt 5.5: Beginning and ending of Task 2

I have used the above two excerpts to illustrate why I used tasks as my evaluative events, and so as my unit of analysis. To do this, and to do it as I have done, I drew from research coming out of the QUANTUM project.

5.6.2. How was each data set analyzed?

- **Analysis of Video data**

In my analysis of tasks I worked both inductively and deductively with three analytic tools. As discussed earlier in Chapter 4, I analyzed video data with the notions of scaffold, legitimating criteria, patterns of variation and Galileo's notions of reality and appearance. My **Analytical tool A** was all inductive analysis, done with the notion of *scaffold***.** In my **Analytical tool B** I worked with the notion of *legitimating criteria* together with the notions *of mathematical reality* and *appearance* to gain insight into the constitution of the enacted of learning. In my **Analytical tool C** I worked with patterns of variation, together with *mathematical reality* and *appearance.* While I used tool A to gain insight into how tasks were made easy or difficult to discern, I also used tools B and C to gain insight into the primary enacted object of learning in the PD

Analytical tool A

I used my analytical tool A to analyze the classroom pedagogy in the videos in terms of how tasks were scaffolded. I worked with phenomenologists' notion of natural attitude to discern the Facilitator's attitude towards teachers' prior subject matter knowledge of secondary school Functions. Here I was interested in whether teachers were positioned as knowing or not knowing. To discern this, I paid attention to how tasks were scaffolded and the teachers' consequent responses to tasks. It was from this grounded analysis that three themes emerged: (i) *low scaffolding*, (ii) *moderate scaffolding* and *high scaffolding*.

Table 5. 4: Analytical tool A

Natural attitude:	- refers to the assumption that the object of learning is not shrouded with learning difficulties. In variation theory terms, a natural attitude is a tendency to take it for granted that the object of learning is easy to learn and so the teacher provides minimum scaffolding. *Tendency to take for granted that the task will be easy to others as it is easy to you*
Inductive pedagogic strategy:	- refers to an instructional approach where the Facilitator does not bracket his/her natural attitude. Facilitator assumes the object of learning will not present challenges, and provides low scaffolding of the task. *Absence of systematic provision of scaffold and involves provision of task instructions only*
Deductive pedagogic strategy:	-refers to an instructional approach where the Facilitator brackets his/her natural attitude. Here the Facilitator does not assume that the object of learning is easy to learn, and so provides sufficient scaffolding to enable progress on the task. *Prevalence of systematic step by step provision of scaffold*
Scaffold:	-refers to the help or assistance and the availing of resources to learners before, during and after work on the given task. **Help or assistance** refers to *Facilitator's cues and probes:* and *lowering of the cognitive demand of task* to facilitate progress. **Resources** as used here refers to provision of: *task instructions* **-** usually provided before work on the task, and spelling out what teachers were expected to do on the task; *definition/s* – provided before, during and after work on the task and *familiarisation with multiple representations (MR)* prior to work on the task.
Low Scaffold:	- refers to provision of *task instruction only* – before work on the task. There is no help provided during work on the task. *Scaffold is constituted by task instructions only*.
Moderate scaffold:	- refers to provision of: *task instruction*; *definition* - before work on the task; and *familiarization with multiple representations* – before work on the task. There is no help provided during work on the task.
High Scaffold:	- refers to provision of all available **resources**: *task instruction*; *definition* - before and during work on the task; and *familiarisation with multiple representations* before work on the task. Provision of **help** include**:** *Facilitator's cues and probes*, and the l*owering of the cognitive demand of the task. Scaffold is constituted as in moderate scaffold, but here Facilitator moves between the groups providing assistance to groups that are making slow progress, and this help is constituted by prompts and probes*

Analytical tool B – Legitimating criteria and Galileo's notions

In my **Analytical tool B** I worked with the notions of legitimating appeals and duality of the object of learning which I recruited from Adler and Davis (2006). In particular, I used this tool to analyze the way tasks were continuously evaluated to transmit criteria for the production of legitimate texts (Davis, et al., 2007). Here I was interested in the authority to which facilitators and teachers were appealing, because "any evaluative act, implicitly or explicitly, has to appeal to some or other authorising ground in order to justify the selection of criteria" (Davis, et al., 2007, p. 5). Now working with this tool I first colour-coded all the appeals in each statement. In my first level of analysis of appeals I worked with the notion of duality of objects to determine whether the appeals were related to mathematics or teaching objects. This analysis revealed that all appeals were grounded in mathematics. This realisation prompted my second analysis, whereby I used Galileo's notions of *reality* and *appearance* (Stumpf, 1994) to further subcategorise earlier appeals to mathematics. For illustrative purposes, I used *appeal to mathematical reality*, with the abbreviation, $\boldsymbol{Mt_{re}}$ and "*appeal to mathematical appearance*", with the abbreviation, $\boldsymbol{Mt_{ap}}$.

According to Galileo, the distinction between reality and appearance was that reality consisted of primary qualities, while appearance was constituted by secondary qualities. His examples of primary qualities of a body were size, position and motion. These he believed were truly real because they could be dealt with mathematically. By contrast, secondary qualities of the same body included colour, tastes and sounds. These were qualities which were imposed by a human mind and so were provisional. Galileo's notions are comparable to Bernstein's classification of linguistic acquisition of middle- and working-class children into universal and particular. For him, universal linguistic acquisition is free from context, while the particular is context-bound. So here I used reality in a similar manner as universal and appearance as particular, because in my opinion reality is not context-bound, while appearance is.

In my analysis the primary qualities included properties of functions, such as definition, vertical line test, and horizontal line test (also referred to as horizontal ruler test). My constitution of the properties of functions resonated with views from the reviewed literature and the WMCS PD's key features of functions. On the other hand, secondary qualities included reference to Function terminology and symbols (e.g. input, output, f(x) and Cartesian plane). As a consequence, appeals involving primary qualities of Functions were subcategorised as $\boldsymbol{Mt_{re}}$ and appeals to secondary qualities were subcategorised as $\boldsymbol{Mt_{ap}}$**.** In the third level of analysis I was interested in the correctness or otherwise, of the appeal. Hence I further subcategorised *appeal* to *mathematical reality* ($\boldsymbol{Mt_{re}}$) into *correct appeal to mathematical reality* ($\boldsymbol{Mt_{re+}}$) and *incorrect appeal to mathematical reality* **(**$\boldsymbol{Mt_{re-}}$**).** Similarly, I subcategorised *appeal to mathematical appearance* into *correct appeal to mathematical appearance* ($\boldsymbol{Mt_{ap+}}$) and *incorrect appeal to mathematical appearance* ($\boldsymbol{Mt_{ap-}}$**).**

Correct appeal to mathematical reality ($\boldsymbol{Mt_{re+}}$)

F:	<u>And a function by definition says that you can't have one x value that is mapped onto two different y values</u>. <u>Any vertical line wherever it is, it's only going to cut the graph once.</u>

Incorrect appeal to mathematical reality $(\boldsymbol{Mt_{re-}})$

F:	Alright. So you're saying this is the graph. But now tell us why it's not a function.
T3:	Right. I used the horizontal test. The horizontal test, ok
F:	Which says what?
T3:	A function should have one input, one output.
	Ja. So if we take our horizontal test we discover that two inputs versus one output. A function should be a one-to-one for it to be called a function.

Correct appeal to mathematical appearance ($\boldsymbol{Mt_{ap+}}$)

F:	Teacher J was saying earlier that if you've got f of x that already says it's a function, am I right or am I misquoting you? Because he said that if we can write it like that *[points to* y = f(x) *on the flipchart]* then it is a function

Incorrect appeal to mathematical appearance $(\boldsymbol{Mt_{ap-}})$

F:	Let's go to number 7. So let's just pass that one for a sec. [*puts a dot next to 19*] Ok. Teacher 6 were you … Did you suggest this one? [*looks at Teacher 6*]. Ok, tell us why you put it up? *[Two sets: one consists of A, E, I, O U, the other 1, 5, 10, 20, 30 with lines linking the letters and numbers*]
T6:	It's like 10 and I know that A is the first letter and it represents 1, there's no x and y [*Teacher 6 says Card 7 is not a function because it is not expressed in algebraic form]*

In **Table 5.5** I demonstrate the analysis process of appeals to mathematics objects. The table shows how I have analyzed legitimating criteria which were initially characterised by appeals to mathematics objects. Now, further analysis revealed that, essentially, the appeals to mathematics objects were not homogeneous. So drawing from Galileo's distinctions between appearance and reality, I was able to subcategorise Mt into $\boldsymbol{Mt_{re}}$ and $\boldsymbol{Mt_{ap}}$**.** Analysis of these subcategories revealed that while some $\boldsymbol{Mt_{re}}$ were correct, others were incorrect**.** Similarly with $\boldsymbol{Mt_{ap}}$. Consequently, my analysis produced four sub-subcategories of appeals to **Mt**. suffice it to mention that while the $\boldsymbol{Mt_{re+}}$ sub-subcategory was a characteristic of Facilitators' appeals to **Mt**, the other three ($\boldsymbol{Mt_{re-}}$; $\boldsymbol{Mt_{ap+}}$; $\boldsymbol{Mt_{ap-}}$) were inherent in teachers' appeals to **Mt**.

Table 5. 5: Illustration of four subcategories of appeals to Mt

	Legitimating criteria	Appeal	Enacted object
1	for every one x there must be one y	Definition of function	Mt_{re+}
2	Ja. So if we take our horizontal test we discover that two inputs versus one output. A function should be a one-to-one for it to be called a function.	Horizontal line test	Mt_{re-}
4	Teacher J was saying earlier that if you've got f of x that already says it's a function, am I right or am I misquoting you? Because he said that if we can write it like that *[points to* y = f(x) *on the flipchart]* then it is a function	Mathematical symbol	Mt_{ap+}
5	Card 24 is not a function because it had dots	Graphical representation	Mt_{ap-}

Analytical tool C – Patterns of variation and Galileo's notions

In my second deductive analysis I worked with patterns of variation (Lo, 2012; Marton, et al., 2004).

Contrast:- awareness brought about by experiencing difference between the critical features

Generalisation:- awareness brought about by experiencing invariance in the focused critical feature and variance in the other feature

Fusion:- awareness brought about by experiencing simultaneous variation between two or more dimensions of variation

I used these patterns to analyse what was made possible to learn. In particular, I used *contrast, generalisation* and *fusion*. I used the relationship between these two constructs: invariance and variation to decide whether the foregrounded awareness was contrast or generalisation or fusion. By way of example, when that which was varying (i.e. critical features) was foregrounded, then learning was made possible through **contrast**. Here it was possible to discern difference between the critical features. In **generalisation** the invariant aspect was foregrounded (i.e. critical aspect/dimension of variation), and so what was possible to discern was similarity between the critical features. Below I exemplify **contrast** from the data.

Figure 4.1 is used here to show an instance of contrast in video data. Below the figure I show enactment involving one-to-one and many-to-one. Facilitator is using the figure to explain the constitution of a one-to-one function, and he does this by simultaneously foregrounding what is not a one-to-one function.

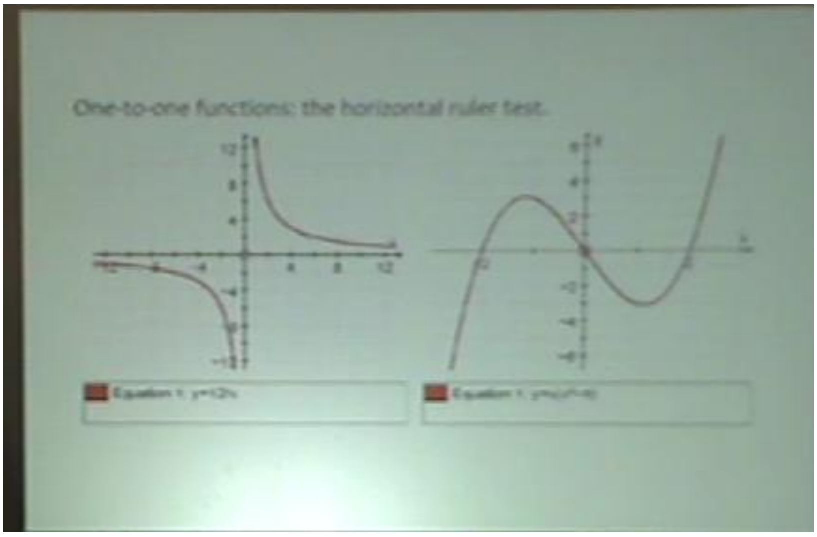


Figure 5.1: Example of contrast from the data

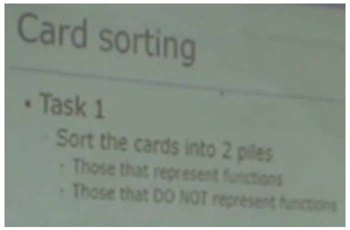
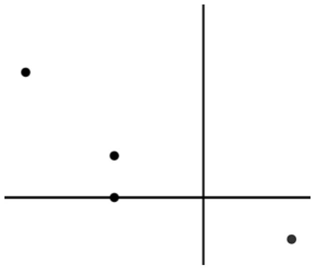
F:	A one-to-one function is important in mathematics because it allows you to work backwards. Now look at this one. *[points to a graph on the overhead]* I've put in a number into this function 12 over x, and out has come the number 2. What number did I put in?
Trs:	6
F:	So it's 6. Ja, Out came the number 2 and the number *[looks at the graph]* 6. What about this one? *[points to another graph on the overhead]* This one I've got a number and out came 2. What number did I put in?
T41:	Negative a half
F:	No
T1:	Or positive 1
F:	There's all sorts of things. If 2 comes out …
T1:	Sorry
F:	… it could have been that one or that one. You'd better go for all three then *[chuckles]* Well, the point is that you can't tell.
T1:	Ja
F:	You can't be sure. If you'd been lucky and I'd said 'Well out comes the number 4' then sure enough, there's only 1 possible input. So if we've got a function like this and it's got the horizontal ruler test, as the horizontal ruler goes up the page it never crosses the curve more than once. Sometimes horizontal will miss it altogether. It doesn't matter. But here, wherever it goes, the horizontal ruler cuts it only once. So it allows me to undo the function, to think backwards. You tell me an output, I will tell you your input. Here though *[shakes his head]* horizontal ruler here cuts it in 3 places
T42:	Ja
F:	That's not a one-to-one function. So if as you're working today I'm gonna ask you, you've got function and I ask you, 'Is it one-to-one?' that's what I'm asking about. *[points to the overhead]* Yeah? Ok

Fusion was present whenever two critical aspects of the object of learning varied simultaneously. For instance, this pattern of variation was dominant in my analysis of Task 4 of Functions 2. There, teachers were required to find functions which met some specified criteria. For instance they were asked to find a Function which had the domain and range not all real numbers. Now, both domain and range were dimensions of variation and were so, in this way – each had values or critical features which made it a dimension of variation. For example, domain can be all real numbers or not all real numbers. Similarly with the range. So fusion was present whenever a function was found, because both domain and range simultaneously came into focus. This then made it possible for the discernment of the relationship between the domain and range of that particular Function and between these and the Function itself.

The following is an illustration from my analysis of Task 1 of Functions 1 at the E-Cluster. I use this section of that analysis to show an enactment involving the use of **contrast**. In this occurrence the Facilitator was explaining why Card 24 was not a representation of a Function. To do this, he called on the definition of Function and also demonstrated diagrammatically a vertical line test on a cubic and a square root Function.

Key	
Coding	**Evidence of:**
Underlined text	Legitimating criteria
Highlighted text	Scaffold
Bold text	Patterns of variation

Table 5. 6: illustration of my analysis process using Sub-Event 2 of Event 1 (Task 1 of Functions 1)

Line numbering	What is said and done	Time	Event 1: Task 1	Description of what is said and done?	Examples	Analysis
127.	Um, 24. Why can't I find it? Oh there. *[puts a graph on the overhead]* Ok. Nelson(?) (T7) I think you gave us this one.	11:00	Sub-Event 2: Card 24 (see **Figure 5.3)**		 **Figure 5.2: Task instructions**  **Figure 5.3. The graph in Card 24**	**Analysis of Scaffold** At the E-Cluster the Facilitator did not withhold his natural attitude (i.e. he took it for granted that teachers would not encounter difficulties). Hence he adopted an inductive pedagogic strategy which closed scaffold. Teachers were provided with task instructions
128.	T7: Mmm					

129.	F: Do you want to talk to it?					only (see **Figure 5.2**) and this amounted to *low scaffold* (compared to the other two clusters where *moderate* and *high scaffold* were provided). Consequently, teachers' incorrectly placed functions in the pile of non-functions. Nonetheless, this pedagogic strategy was very productive in revealing gaps in teachers' knowledge of functions. The gaps were evident in the legitimating criteria that teachers called to fix meaning to their
130.	T7: No					
131.	F: No?					
132.	*[everyone chuckles]*					
133.	F: Ok. Nice when your colleagues say no on camera.					
134.	T7: I don't think there's any reason…					
135.	F: So T7 claims it's not a function.					
136.	T4: Before I talk…					
137.	F: Before he talks, would you agree, other people?					
138.	T4: I can see it's a function <u>because it's a non-linear graph</u>.			Facilitator put the transparency for Card 24 up for discussion. He asked what made it a non function and one teacher said it was a function because it was non-linear and had x and y. (see line number 138 – 168)		
139.	F: A non-linear graph? Ok.					

140.	T4: So it's a function because if you look at the graph maybe you're looking at the plotting of the graph, and you get your x, because you are plotting on y and x.					explanations. **Analysis of legitimating criteria** The discussions around Card 24 produced several appeals from the Facilitator and teachers. The first legitimating criterion was from a teacher (T4) who said that Card 24 was a function because the graph was non-linear and was on the x and y-axis. The teacher appealed to the structure of the representation. Second legitimating criteria came from another teacher (T7) who said it was not a
141.	0:13:22					
142.	F: So you are assuming this is x *[points to the horizontal axis]*, this is y *[points to the vertical axis]* right?					
143.	T4: Yes					
144.	F: Ok					
145.	T4: And it's a non-linear. So I don't know why is it...					
146.	F: When you said non-linear, you did this with your hands *[moves his hands in a wave movement]*.					
147.	T4: It's not straight – the linear graph moves.					
148.	F: Ok					
149.	T4: Yes					
150.	F: So are you saying… Are you going to draw something to join these? *[points to the*					

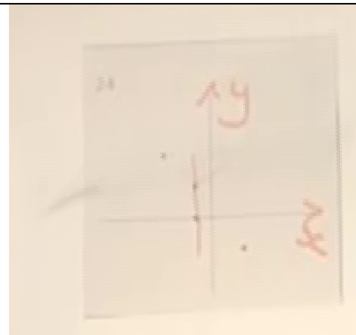
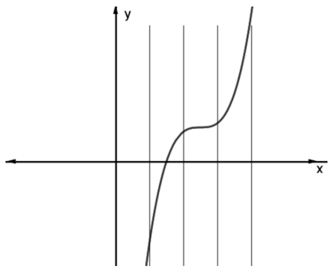
	graph on the overhead]					function because one value mapped onto two y values and this was not in agreement with the Grade 10 definition of function. This teacher was appealing to the definition of function. The third, fourth and fifth legitimating criteria came from the Facilitator. The legitimating criteria were verbal and visual. The third legitimating criterion was -to the definition of function and it was elaborated verbally. He pointed out that
151.	T4: I'm going to link it up. I think it is a graph.					
152.	F: *[holds out the overhead pen]* Do you want to come draw?					
153.	T4: No I'm not drawing now					
154.	*[everyone laughs]*					
155.	T4: What I'm saying is that I think it is a graph.					
156.	F: Ok					
157.	T4: But they are not joining it.					
158.	F: Ok					
159.	T4: Like I can see a parabola and this kind of ? there. So I think if you want to draw this it must be on a graph. And we're talking about linear graph and non-linear graph. Like a linear must be straight so it is going to be a function.					
160.	F: So you're saying it is a linear function but it's not linear.					
161.	T4: It's a non-linear function. Like there is one here. What I was thinking,					

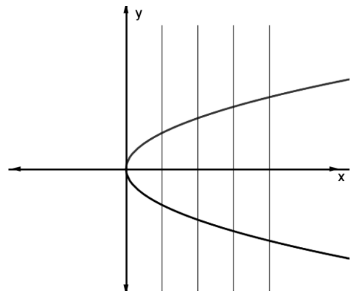
	a kind of y equals x over 2. It will give us various values for x. It's not very ? when you want to plot the graph. Maybe it's going to be non-linear now.					Card 24 was not a unique correspondence because one element in one set should map onto one element in another set and he pointed out that this was not the case in Card 24 because one x value mapped onto two y values. In the first criterion the In the fourth legitimating criteria Facilitator appealed to the vertical line test and was elaborated pictorially. He used it on the graph of Card 24. In his demonstration he showed that it cut the graph of Card 24 in two places.
162.	F: For x over 2?					
163.	T4: Yes					
164.	0:14:37					
165.	F: Ok. Let's stick with this one first. So you're saying it's not linear but it's a function?					
166.	T4: *[nods]*					
167.	F: Ok. And maybe they wanted to join them both but they haven't?					
168.	T4: Yes					
169.	F: Alright. How do other people feel about that? *[silence]* Nelson(?) do you want to tell us why you put it?					
170.	T7: So I see it as a set of 4 coordinates, and when I look at each x value of each of those 4 coordinates, I find that for one x value it's got 2 y values. <u>And if I look at</u>			T7 said it was not a function because one x value was mapped to two y values and so according to Grade 10 definition it is not a unique a		

<table>
<tr><td></td><td><u>the definition from grade 10 we look at the unique correspondence of elements from one set mapped onto another set. And then that is not a unique correspondence from a set x to a set y.</u></td><td></td><td></td><td>correspondence because one element in one set should map onto one element in another set</td><td></td><td rowspan="9">The fifth legitimating criterion was present when a further demonstration of the vertical line test was performed simultaneously on the cubic and a square root graphs to show that it cut the former once and the latter twice. So the appeal associated with the use of vertical line test was to contrast - a pattern of variation.

Analysis with patterns of variation

Sub-Event 2 was characterised by one pattern of</td></tr>
<tr><td>171.</td><td>F: So we’ve got… So you’re saying if you look over here, is it this part here that you’re worried about? [writes on the overhead, labelling the y and the x axis and joining two points in red]</td><td></td><td></td><td></td><td></td></tr>
<tr><td>172.</td><td>T7: Ja</td><td></td><td></td><td></td><td></td></tr>
<tr><td>173.</td><td>F: There is one x value, so we are assuming, right, that this is the x axis and this is the y axis</td><td></td><td></td><td></td><td></td></tr>
<tr><td>174.</td><td>T7: Yes</td><td></td><td></td><td></td><td></td></tr>
<tr><td>175.</td><td>F: Right? And you’re saying for one x value there’s two y values?</td><td></td><td></td><td></td><td></td></tr>
<tr><td>176.</td><td>T7: Mmm. One of which is zero – that’s on the x axis.</td><td></td><td></td><td></td><td></td></tr>
<tr><td>177.</td><td>F: Ok</td><td></td><td></td><td></td><td></td></tr>
<tr><td>178.</td><td>T7: Some kind of</td><td></td><td></td><td></td><td></td></tr>
</table>

	positive value.					variation. This
179.	F: And that one we don’t know. But if we assume they are in the same vertical line *[moves his left hand down, indicating a vertical line]*, then they have both got the same x value. And a function by definition says that you can’t have one x value that is mapped onto two different y values. So **when we talk about the vertical line test, if you can draw a vertical line *[moves his hand down again, indicating a vertical line]* and it cuts the graph more than once, then we say the graph is not a function**. Is that familiar?			Facilitator drew a vertical line to demonstrate that it cuts the graph on Card 24 in two places. Thus failing the vertical line test. In addition to this, the Facilitator pointed that Card 24 did not satisfy the definition of a function because one x-value cannot map onto two different y values	 **Figure 5.4: A demonstration of vertical line test on Card 24**	pattern of variation was contrast and was present in three instances. The first **contrast** was present when the definition of function was applied on the graph of Card 24. The dimension of variation was definition and the critical features were the number of y values that one x value may correspond with. While the definition called
180.	**So you might have other examples where, if you have a graph that looks like that, if you draw any vertical *[drops the pen: Whoops. Picks the pen up again]* any vertical line wherever it is, it’s only going to cut the**			Facilitator drew on the flipchart a cubic graph and some vertical lines cutting across it to illustrate an example of a function. He pointed that each vertical line cut the graph once and so this showed that the	 **Figure 5.4: Cubic graph**	for each value of x to map onto a unique y value, the graph showed that one -x value mapped onto two y values. *Here it was made possible to*

	graph once. *[demonstrates this by drawing a graph on the flipchart]*			graph represented a function		*discern that in a function, one value of x may not correspond to more than one y values but in a non function one x value may correspond to more than one y values* The second **contrast** was present when the dimension of variation was the vertical line test. The critical features in this dimension were the number of times a vertical line cut a graph of a function and a non-function. *Here it was made possible for teachers to discern that a vertical line cut a graph of a*
181.	**But if you have something, for example something that looks like that *[using another colour, F draws a graph over the one he has just drawn]*, which is like a parabola on its side, then you can see there are several places where the vertical line is going to cut the graph twice**, ok? And according to the grade 10 definition of function, that can't happen. If that happens then the thing is not a function. So that would mean this one because there are two points where they have same x value and different y value, that means that this one would not be a function. Ok. So 24 can stay on our list. *[puts a tick next to* 24 *on the flipchart]*			Next the Facilitator drew on the flipchart what he referred to as a parabola on its side with each vertical line shown cutting the graph in two places. Here the facilitator was demonstrating an example of a non-function. Facilitator pointed that a situation where the vertical line cuts the graph in two places goes against the definition of a function. He pointed that according to the Grade 10 definition this could not happen if the graph was representation of a function. Hence he said Card 24 could remain in the list of non-functions	 **Figure 5.6: Square root graph**	
182.	The next one is number 3.	17:00	End of SE -2			

						function once and a non-function more than once. The third **contrast** was present when the cubic and square root functions were in the fore. The dimension of variation was relationship. Here the critical features were a function relationship and a non-function relationship. *The use of the vertical line test on their graphs made it possible for teachers to discern the difference between a function and a non-function relationship*.

- **Analysis of Interviews**

In my analysis of interviews I recruited the following tools from Phenomenology and from the methodology for studying teacher education. While phenomenological tools were responsible for putting together the composite meaning structure of the experiencing of the PD and so the lived object of learning, the duality of the objects of acquisition illuminated the nature of the experiencing by explaining whether the lived object was a mathematics object (i.e. *personal learning of functions* (**Mt-pl); ***professional learning of functions* **(Mt-PL**) or teaching object (i.e. *personal learning of teaching functions* (**Tm-pl); ***professional learning of teaching functions* **(Tm-PL**)). These four themes of the lived object of learning are elaborated in **Chapter 10**.

Horizonalization: refers to statements in the interviews about how individuals talk about their experiencing of the phenomena

Clusters of Meanings: refers to grouping the statements into meaning units or themes

Themes: refers to a collection of related meaning units

Meaning structure: refers to what it means for an individual to experience the phenomenon.

Essential invariant structure: is the essence or goal of phenomenological research, which is to reduce meanings of experiences to a brief description that typifies the experiences of all the participants in the study. This is constituted by grouping all the **themes** together. In my analysis the essential invariant structure was constituted by four themes (i.e. **Mt-pl; Mt-PL; Tm-pl**; and **Tm-PL**) (see **Chapter 10** for details).

In the first stage of my analysis I **horizonalized** the data (captured significant statements) to capture utterances which were my unit of analysis. These contained the participants' meanings of their experiencing of the Functions 1 and Functions 2. In the second stage I created **meaning units** from the significant statements and in the third stage I collapsed related meaning units to create **themes.** By the end of my analysis I had four themes which together constituted the composite meaning structure, also known as the **essential invariant composite meaning structure**. In other words, the four themes constituted what it meant for participants to experience an object focused professional development model which worked with secondary school mathematics content to strengthen teachers' subject matter knowledge.

Table 5.7: Illustration of my analysis of Teacher 1's utterances about resources used before PD

Horizonalization (Unit of analysis)	**Primary object (a Mt or a Tm)**	**Meaning units**	**Meaning structure**	**Themes**
There were things that one found rather interesting which were not defined like in the same way as maybe Classroom Mathematics would put it or Maths Today would put it, you know.	***Tm:*** Teacher noticed that learning material was presented differently in PD compared to prescribed texts	***Functions were presented differently in PD compared to prescribed texts***	Teaching of functions in the PD was unique	Tm-pl
. Like for instance when I talk about the functions, né, there were examples, specific examples that were given of functions that you don't quite find in let's say Classroom Mathematics, you know.	***Tm:*** Teachers says in PD examples of functions were quite different from those in prescribed text	***Examples of functions used in PD were different from prescribed textbook***	Teaching of functions in the PD was unique	Tm-pl
So somehow you know… Ok, sometimes we sort of like go by the book, I mean, so to speak. So there are certain things that you may not bring out, like I think it's important from the beginning when you introduce functions that learners get to know what functions really are, and maybe even give some examples of functions and stuff like that	***Tm:*** Teacher says teaching that follows prescribed text does not bring out important things. He thinks learners need to know what are functions and there should be examples of functions.	***The teacher tend to follow the textbook but it does not bring out key features of functions***	Teaching that follows text books does not bring out key features of functions for learners to discern	Tm-PL

But then going by their textbook, that doesn't come like that. Do you understand?	***Tm:*** Teacher says following prescribed text does not help bring out what are functions nor examples of functions	***Following prescribed text does not bring out key features of Functions***	Teaching that follows text books does not bring out key features of functions for learners to discern	Tm-pl
It's from the word go you have to, they have to plot this and that and so on. And they are not even sure what it is that they are doing.	***Tm:*** Teacher says prescribed text goes straight into asking learners to plot and join points to produce graphs	***Prescribed text focuses on plotting points***	Teaching that emphasis plotting of points, and joining them to produce graph is not productive for learners	Tm-PL
Exactly, you know. So I as an educator, if I'm just like limiting myself to a textbook, I also lose the plot somehow.	***Tm:*** Teachers says he limits himself and miss the point when he chooses to follow the text	***Following prescribed text limits one's capability to teach functions***	Teaching in a way that follows the textbook is not productive	Tm-pl

- **Analysis of Lesson Plans**

To gain further insight into the lived object of learning I turned to the Pre- and Post-PD lesson plans and I introduced other theoretical resources. In my analysis of the two lesson plans I recruited the notion of *space of variation* and *dimensions of variation* from Variation Theory and also worked with the notions of *privileged features* and *key features*. A detailed discussion on the notions of space of variation and dimension of variation were described in Chapter 4. However, I provide a brief discussion on all tools below.

Space of variation/learning (Marton, et al., 2004) is constituted by critical aspects of the object of learning. It is the location of all that which is possible to learn.

Dimensions of variation are also known as critical aspects of the object of learning. Critical aspects are constituted by critical features which are vital for discernment

Teacher's privileged features: These are aspects of functions which are the focus of the teacher's lesson plan, and may be construed as what the teacher believes is important to learn about functions. Teachers choose what they think and believe is essential to learn about the object of learning, and this is what they include in their planning and attempt to focus learners' attention on during the lesson. In my analysis I derived these features empirically from teachers' lesson plan objectives and from the body of the lesson plan.

WMC's key features: these are features of functions which professional developers considered to be important to learn, and so should form part of the lesson whenever functions were taught. These are derived from the perspective of school mathematics. These features are distinct from critical features as propounded by variation theorists. While *critical features* are what is difficult to learners, *key features* are what is important to teach about a particular concept - whether this is difficult or not.

In my analysis of lesson plans I was interested in shifts in the teachers' lived object of learning as depicted in the lesson plans. First I looked at shifts in what was privileged by the teacher. I also looked at the space of variation to extract the WMC-S project's key features whenever they were present. In my final analysis I compared the presence or absence of the key features across the two lesson plans for evidence of shifts in the lived object of learning. I have used my analysis of Teacher 3's Pre-PD and Post-PD lesson plans to illustrate how I conducted my analysis of all the lesson plans (see **Table 5.8** and **Table 5.9**).

Column 1 shows what was privileged after participation in the WMC-S project's Functions Theme work. My analysis of this column produced Column 2. This column contains what would be made possible to learn during enactment. My analysis of Column 2 involved teasing out the WMC-S project's key features to produce Column 3. In my analysis of shifts I compared and contrasted Column 3 of both the Pre-PD and Post-PD to see the extent to which key features were present or absent in each lesson plan.

Table 5. 8: Illustration of my analysis of Teacher 3's Pre-PD lesson plan

Intended Object of Learning for the teacher **The Linear Function: $y = mx + c$** Teacher 3's Pre Plan		
Teacher 3' s privileged features	**Space of variation**	**Intended object of learning for WMCS PD (key Features/ Predetermined typologies)** **(Domain/Range; Multiple Representations; Parameters) Use of definition**
<u>Three approaches to Sketching a linear graph</u> -Dual intercept -Table method -Gradient –intercept Sketch of each method is provided in the lesson plan	**Dimension of Variation** Example -Dual intercept method: $y = 2x + 3; y = 0; x = \frac{-3}{2}$ -Gradient-intercept $y = -2x - 2$ -Table method $y = x - 2$	Intercepts To be used in the sketching of linear graphs

Shape of linear graph when gradient is positive & negative	**Dimension of Variation** Example: If m is (+ve) the graph will slant to the right If the m is (–ve) the graph will slant to the left	Parameters: Negative & positive gradients used to determine the shape of the graph
Learners will be asked the meaning of m, and c in y = mx + c	**Dimension of Variation** - Meaning of m, and c -M is gradient -C is y-intercept	Parameters Meaning of 'm' and 'c'

Table 5.9: Illustration of my analysis of Teacher 3's Post-PD lesson plan

Intended Object of Learning for the teacher **Topic: Linear Function $f(x) = ax + q$** Teacher 3's Post Plan		
Teacher 3' s privileged features	**Space of variation**	**Intended object of learning for the WMCS PD (keyFeatures/ Predetermined typologies)** **(Domain/Range; Multiple Representations; Parameters; Intercepts; Definition)**
Definition of a function Multiple Representations	**Dimension of Variation** Teacher defines a function Representing functions in several ways: (i) Mapping diagram (ii) Table; (iii) Graphically; (iv) verbal Learners give examples of functions	**Definition** **Multiple Representations**
Defining Domain and Range	**Dimension of Variation** Teacher define the Domain Range as follows: $Domain: x \in R/(-\infty; \infty)$ $Range: y \in R/(-\infty; \infty)$	**Domain and Range**
Domain and Range from different representations	**Dimension of Variation** Teacher produces different representations of functions showing domain and range	**Multiple representations /Domain and Range**
Finding domain and Range from given representation	**Dimension of Variation** From a given graphical representation learners respond to questions: (i) what is the domain from the graph? What is range from the graph?	**Domain and Range**

A summary of the analysis of Teacher 3's two lesson plans indicate that the Pre-PD lesson plan has two key features that include *intercepts* and *parameters*. On the other hand, the Post-PD lesson plan has three key features that include *definition*, *domain and range* and *multiple representations* of functions. These key features were privileged in both Functions 1 and Functions 2. The analysis indicates that while the teacher considered intercepts and parameters important to teach before participation in professional development, after professional development there was a shift in what the teacher considered to be important to learn about Functions.

5.7. Summary of the analytic framework underpinning the study

Table 5.6 contains a summary of the constitution of my analytic framework. The framework was made up of tools which I drew from the theoretical resources which guided my research. The table shows that while some tools were orientated towards phenomenology, others were towards social theory of pedagogic discourse. It is interesting that both sets of tools were useful in the analysis of both the enacted and lived objects of learning irrespective of orientation. I have noted ealier that while analytical tools aligning to phenomenology visibilized the object of learning, those orientated to social theory of discourse were productive in illuminating the object of learning. I discuss this further in **Chapter 12**.

Table 5.10: An illustration of my analytical framework

SOCIAL CONSTRUCTIONISM

Phenomenology

Variation Theory

Patterns of Variation

Video Data

Enacted object of learning

(Tasks as unit of analysis)

Space of variation

Lesson plan Data

Lived object of learning

(Lesson plans as unit of analysis)

Inductive data analysis

Analysis of Pedagogic Strategies

Video Data

Enacted object of learning

(Tasks as unit of analysis)

Analysis of Meaning Structure

Interview Data

Lived object of learning

(utterance as unit of analysis)

Social theory of discourse

Adler & Davis' Methodology

Notion of duality of object of learning (Mt or Tm)

Video Data

Enacted object of learning

(Tasks as unit of analysis)

Interview Data

Lived object of learning (utterance as unit of analysis)

Notion of legitimating criteria

Video Data

Enacted object of learning

(Tasks as unit of analysis)

The analytical framework shows how I have conducted my analysis. I have ensured that I create coherence between my assumptions, theoretical perspectives, methodology and methods to enhance the credibility of my findings (Crotty, 2009). While both phenomenology and the social theory speak to the social constructionist assumptions, variation theory and the methodology for studying matheamtics teacher education speak to phenomenology and the social theory respectively. Though the perspectives hold divergent views pertaining to whether knowledge is a social or individual construction, the two were able to complement each other. For instance, I worked with the notions of legitinmating criteria and patterns of variation simultaneously to analyze the enacted object of learning. I have also worked with phenomenology and Adler and Davis' methodology in my analysis of the lived object of learning. I return to show how I have connected these theoretical resources in the conclusion chapter.

5.6. Ethics Issues

Any research that involves people has the potential to cause unintentional damage (Opie, 2004), hence the need for a careful elaboration on how the moral aspect of the research was addressed to minimize the damage. The proposal, on which this research rests, was reviewed and approved by the Ethics committee at the Wits School of Education to ensure protection of the participants. Subsequently, I obtained an ethics clearance with protocol number 2011ECE2013 (see Appendix) which authorised me to conduct this research.

This research was conducted in six secondary schools in Gauteng Province. Access into these schools was granted through the Gauteng Department of Education's permission which was issued under reference number D2011/30 to the WMC-S project In addition to this, the WMC-S project applied to individual schools, parents, learners and teachers, seeking permission to conduct research in schools and also consent from these groups of people. Since my research was embedded in the WMC-S project's wider research aims, I used the same permission to access the six schools. Another permission which I did not apply for directly but still benefited my research was the WMC-S project's ethics clearance, obtained under protocol number 2010ECE60C. The WMC-S project video recorded the Functions Theme work under this protocol and I later sought their permission to use the video data in my research.

In order to collect Pre- and Post-PD and interview data I applied for a separate ethics clearance (protocol number 2011ECE2013). To gain access to teachers I first wrote each teacher a letter inviting him/her to participate in my research. In that letter I explained the purpose of the study; how I was going to use the information and the teachers' rights. The purpose of my research was to study what was taught and what was learnt in the Functions Theme work. I explained that the information was going to be used in this research. Under the teachers' rights I mentioned three key things. First I mentioned that their names were not going to appear in either my thesis nor research presentations. Second, I mentioned that their involvement in my research would not affect their professional responsibilities in the school. By way of example, that at no point would information for research purposes affect their appraisal. Third, I mentioned that if they agreed to participate in my research and later decide to withdraw, they were free to withdraw any time they so wished. After reading through the letter, 9 teachers accepted the invitation to participate in my research.

Following the teachers' consent to participate I wrote three separate letters sent at different intervals, were to each teacher. The first requested each teacher to develop a Pre-PD lesson plan before participating in the Functions Theme work (see **Appendix D**). The second requested teachers to develop a Post-PD lesson plan after participating in the Functions Theme work (see Appendix). The third letter invited each teacher to an in-depth interview (see **Appendix D**) I have appended the both the WMC-S project and my clearance certificate.

5.7. Trustworthiness and credibility

The trustworthiness of the findings of my research will largely depend on the coherence of the four basic elements of research (see Crotty, 2009), which have been discussed in the research design section of this proposal. It is when these four elements are in resonance with each other that the trustworthiness of research can be ascertained. In this research, therefore, I assembled the methods, methodologies, theoretical perspectives and epistemology that were compatible; hence the trustworthiness of the findings rested on the coherence of these four elements. The value of coherence between the four was acknowledge by Maxwell (Maxwell, 2005) when he asserted that "although methods and procedures do not guarantee validity, they are nonetheless essential to the process of ruling out validity threats and increasing the credibility of the our conclusions" (p.109).

5.7.1. Validity Issues

In my research I ensured descriptive validity (Maxwell, 2005) by working with video recordings of classroom observations and audio recording my interviews with teachers. Hence my data analysis was focused on what was actually seen and heard from participants. Maxwell locates the main threat to valid description of what was seen and heard in inaccuracy or incompleteness of the data. However, these, he says, are solved by audio or video recording the observations and interviews, and following this with verbatim transcriptions.

Researcher bias and reactivity are believed to be the two main validity threats. Researcher bias might involve the researcher's selection of data that fits his/her existing theory or preconceptions and data that stands out. Since subjectivity, or researcher bias, is inherently present in qualitative research, the researcher needs to articulate his/her biases to make it know how his/her values might influence the conduct and conclusions of the study. Hence in Chapter 2 of this research I made mention of the fact that the researcher was a team member in the WMCS project. So it can be expected that the researcher came into the study with vested interest. Another bias was inherent in the selection of content. As noted earlier, though teachers might have experienced both SMK and PCK in the PD, I chose to focus my research on SMK only, and did so by working with data from Functions 1 and Functions 2.

Reactivity is the effect of the research on the context or participants (Maxwell, 2005). Reactivity is believed to be less serious in participant observation studies because in natural settings an observer is generally much less of an influence on participants' behaviour. In my research I participated in the Functions Theme work, and so was able to interact with participants in their working groups. I met similar challenges on tasks just as the groups I worked with and we searched for solutions together. In a sense I was learning as much as the participants.

I have worked hard to minimize validity threat in my research right from participant selection to data analysis. Triangulation is one recommended strategy to address validity threat in qualitative research, which I used. I drew participants from across the three clusters to participate in my research and I used video, lesson plan and interview to capture data from each participant. So I captured information from a range of individuals and settings, using a variety of methods. In so doing I was consistent with the notion of triangulation. The use of triangulation was meant to minimize the risk of chance associations and of systematic biases (Maxwell, 2005) which might have resulted from the use of one particular method of data collection.

Another way to reduce validity threat was to continuously present my data analysis to colleagues in the WMC-S project and colleagues in the School of Education during PhD Weekends. Colleagues from the WMC-S project had all experienced the phenomena and the context of my research. Some participated in the Functions Theme work as Facilitators while others were team members. My presentations to this group were meant to solicit feedback which I then used to identify validity threats, as well as my own biases and assumptions, including inconsistencies in my analysis. Soliciting feedback is a productive strategy for identifying validity threats, especially when feedback comes from both those who are familiar with the phenomena being studied and those who are not familiar (Maxwell, 2005).

Obtaining 'rich' data is the third strategy which I used in my research to reduce the risk of validity threat. Maxwell describes 'rich' data as detailed and complete enough to provide a full and revealing picture of what transpired. In my research I used 9 in-depth interviews which were each about 2 hours long and 6 sets of video data of classroom learning. Both these sets of data required verbatim transcriptions. The advantages of obtaining 'rich' data in my research were that it reduced respondent duplicity and my own bias.

5.8. Generalization of research findings

There are two types of genralization in research, and these are internal genralisation and external generalisation (Maxwell, 2005). While the former is synonymous with qualitative research, the latter is associated with quantitative research. Internal generalisation is the generalisability of results to the contex or group studied while external generalisation is whereby results are generalized beyond the group studied – usually to the population from which the sample was obtained. In this research the findings were generalised to the nine participants in the study.

However, the theory developing from this research had potential for generalization beyond this research, particularly to the rest of the teachers who particiapted in the Functions Theme work. This potential was drawn from the view that the generalisability of qualitative research, unlike that of quantitative research, “is based on the development of a theory that can be extended to other cases” (Maxwell, 2005, p. 115). In addition, face-generalisability was also probable in my research. By this I am suggesting that “there [was] no obvious reason not to believe that” (p.115) the lived object of learning for other teachers who experienced the Functions Theme work but were not among the 9 participants, was characterised by one or more of these: Mt-pl; Mt-PL; Tm-pl or Tm-PL.

5.9. Limitations

Limitations are what the researcher knows might impact negatively on the research but does not have control over (Gay & Airasian, 2003). In my research the major limitation rested on the fact that the interviews were focused on teacher learning in Functions 1 and Functions 2, where mathematics objects were foregrounded, even though they had experienced Functions 3 and 4 where teaching objects (**Tm**) were in the fore. As a consequence, the presence of teacher utterances about teaching functions might have been influenced by their experiencing Functions 3 and Functions 4. This is particularly so since the interview was conducted at the end of Functions Theme work. I return to elaborate on this in the conclusion chapter.

CHAPTER 6: ANALYSIS OF TASK 1 OF FUNCTIONS 1

6.1. Introduction

This chapter presents the analysis of the enacted object of learning in Functions 1 across three sites. Functions 1 privileged discernment of connections between the multiple representations, definition of function - with particular focus on univalence and arbitrariness and function as constituting a relationship.

The specific content of Functions 1 is exemplified in Chapter 2, where I discuss the context of the study. The urgency underpinning the packaging and organization of both sessions rested on earlier classroom observations and low learner achievement on the WMC-S project's baseline test. Details of classroom observations and evidence of low learner achievement are also discussed in Chapter 1. Just to remind the reader, during classroom observations the teachers tended to begin with the algebraic representation, move on to the numeric representation (table of values) and end with the graphical representation. It suffices to point out that even as they were presented in this manner, there was never a time where these were referred to as representations of Functions. Rather, they were presented in ways that portrayed some representations, particularly the numeric, as a step in a procedure that produces a graph. Consequently, a decision to work with teachers to discern each representation as a separate object that constitutes a functional relationship, and so strengthen their own relationship to mathematics, was conceived by professional developers at the WMCS project.

6.2. Description of Task 1 at E-cluster

Evidence that supports my description of tasks is provided in both my description and analysis sections where I show evidence of scaffolding, legitimation and patterns of variation. While evidence of scaffold is shown by highlights in the excerpts, legitimation is underlined and patterns of variation are bolded. In the extracts, the letter **F** is an abreviation for *facilitator* and **T1**, **T2, T3**, ..., stand for *Teacher 1, Teacher 2, Teacher 3* etc. In Chapters 6 – 9, T1, T2, T3, ..., T48, do not necessarily refer to the nine teachers who were participants in my research, but to all teachers who participated in the functions themework. However, in Chapters 10 and 11, I use T1, T2, T3, ...T9 to label the nine teachers in my research.

- **Task 1 - Event 1: Identifying non-functions and functions**

In the E-Cluster, the facilitator introduced Task 1 by putting up a slide containing the first task for teachers to do (see **Figure 6.1**). The task required teachers to sort cards into two piles: functions and non-functions. Each teacher was provided with a set of 24 cards. Cards 1 to 23 contained multiple representations (including mappings) of linear, quadratic, hyperbolic functions. Card 24 contained a graphical representation of a non-function, and was the only card among the 24 that was not a function.

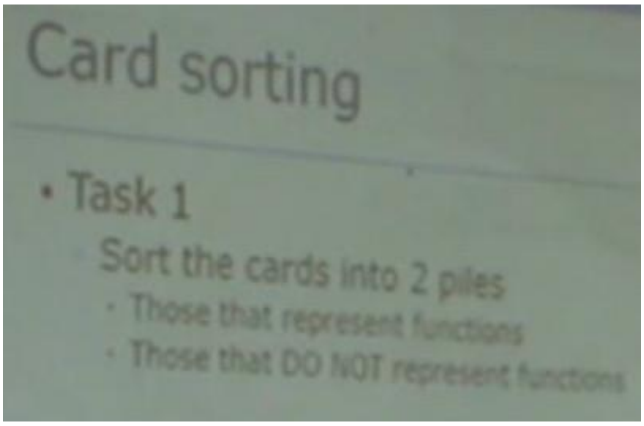


Figure 6.1. Task 1 instruction

Teachers sat in groups but worked individually on Task 1 for a minute at which point the Facilitator asked teachers to name the cards that they had each classified as non-functions. He called individual teachers by name to provide cards from their lists and as they did this he wrote the card number on the flipchart (**see Figure 6.2**). Altogether, six cards were suggested as non-functions. Figure 6.2 shows a list of teachers' non-functions which the Facilitator wrote down on the flipchart as teachers were listing the cards. Card 17 was cancelled after Teacher 4 (T4) had mentioned that it was not a non-function.

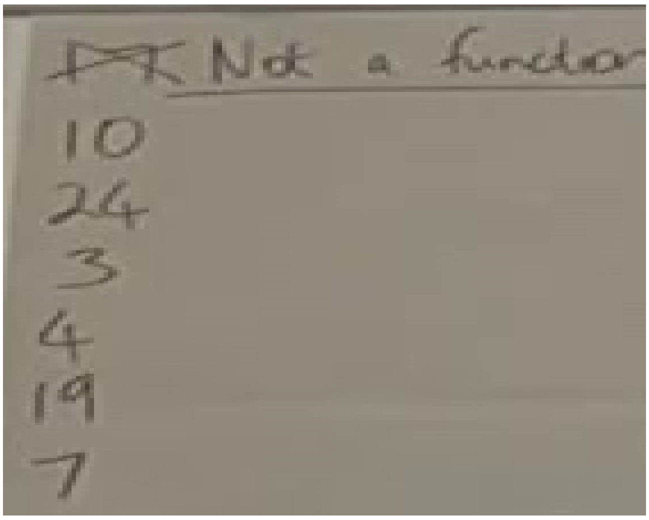


Figure 6.2: Cards listed as non-functions

Source: WMCS project archives

After writing all the cards on the flipchart the Facilitator moved to the overhead projector where he put up a transparency of each card for teachers to explain why they thought it was a non-function.

- **Task 1- Event 2: Discussion focused on Card 10**

Figure 6.3 is Card 10 which was the first to be put up for discussion. The card contains the verbal representation of a quadratic function. However, teachers have classified this as a non-function.

10

To get the output, take the input, square it, then double, then decrease it by two

Figure 6.3: Card 10

The first transparency put up contained Card 10. Teacher 6 had suggested this as non-function and so was asked to explain (**see Excerpt 6.1**). While Teacher 6 could not explain his selection, Teacher 12 suggested that card 10 was a function. In the excerpt I have underlined Teacher 12's explanation which according to my Analytical tool B constitutes a legitimating criteria. In the legitimation the teacher correctly appealed to the terms input and output. In terms of my recontextualisation of Galileo's terms, the appeal constituted *correct mathematical appearance*.

181 182 183	F:	Ok. So number 10 says to get the output you take the input, square it, then double it, then decrease it by 2. Alright. So we're thinking that some people are saying we don't think that's a function. I can't remember who gave us that one. Was it you?
185	T6:	It was me.
187	F:	Was it you? Ok, tell us. Sorry?
189	T6:	To get the output you take the input…
191	F:	*[F waits for T6 to think]* You think it is not a function?
193	T6:	Ja
195 196	F:	Tell us what you're… Tell us what else you're thinking because clearly there's other stuff going on.
202	F:	Anyone else want to comment?
204	T12:	I guess, I think it's a function because it's having input and output.
206	F:	Ok. So it's got an input and an output?

Excerpt 6.1: Teacher explain why Card 10 is a function

After some discussions teachers agree that it is a function because it has the words input and output. At the end the facilitator points that the aspect of input and output is very important. One teacher asked if Card 10 was now classified as a function because of the words input and output. Facilitator asked teachers to respond to this and building on **T12**'s response (**see**

Excerpt 6.1), they said the words input and output made them change their intial view in which they said Card 10 was a non-function.

- **Task 1- Event 3: Discussion focused on Card 24**

The next Card for discussion was card 24. Facilitator put the transparency for Card 24 up for discussion. He asked what made it a non-function. Teacher 4 suggested that it was a function because it was non-linear. Teacher 4's legitimation contained an incorrect appeal to the shape of the graph and so this was an appeal to *incorrect mathematical appearance*. When asked to explain why Card 24 was a non-function Teacher 7's legitimation was that one x-value was mapped onto two y-values. Teacher 7 appealed to the definition of a function and this was a correct appeal. In terms of Galileo's notions of reality and appearance, the teacher appealed to *correct mathematical reality*.

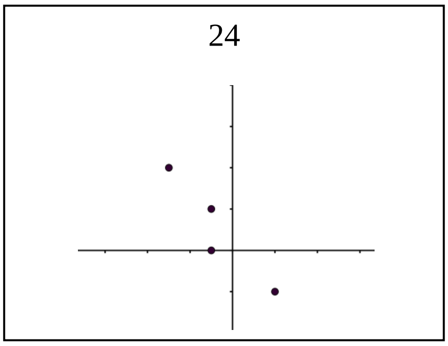


Figure 6.4: Card 24

Another said it was not a function because one x value was mapped to two y values and so according to Grade 10 definition it is not a unique correspondence because one element in one set should map onto one element in another set (**see Excerpt 6.2**).

325 326	F:	Um, 24. Why can't I find it? Oh there. *[puts a graph on the overhead]* Ok. Nelson (T7) I think you gave us this one
349	T4:	I can see it's a function because it's a non-linear graph. So it's a function because if you look at the graph maybe you're looking at the plotting of the graph, and you get your x, because you are plotting on y and x.
351	F:	F: A non-linear graph? Ok. Alright, how do other people feel about that? *[silence]* Nelson do you want to tell us why you put it?
423 424 425 426 427	T7:	So I see it as a set of 4 coordinates, and when I look at each x value of each of those 4 coordinates, I find that for one x value it's got 2 y values. And if I look at the definition from grade 10 we look at the unique correspondence of elements from one set mapped onto another set. And then that is not a unique correspondence from a set x to a set y.

Excerpt 6.2: Teachers' explanations involving Card 24

Building on Teacher 7's explanation, the Facilitator drew a vertical line to demonstrate that it cuts the graph on Card 24 in two places (**see Figure 6.5**). Thus failing the vertical line test. In addition to this, the Facilitator pointed that Card 24 did not satisfy the definition of a function because one x-value cannot map onto two different y values. Facilitator drew on the flipchart a cubic graph and some vertical lines cutting across it to illustrate an example of a function. He pointed that each vertical line cut the graph once and so this showed that the graph represented a function. Next the Facilitator drew on the flipchart what he referred to as a parabola on its side with each vertical line shown cutting the graph in two places. Here the Facilitator was demonstrating an example of a non function. **Figure 6.5** shows the graph of function on the left and graph of non- function on the right hand side. The use of the vertical line test on the two graphs opened a pattern of variation known as **contrast.** It was contrast because the facilitator wanted to illustrate what constituted a function and what it was not.

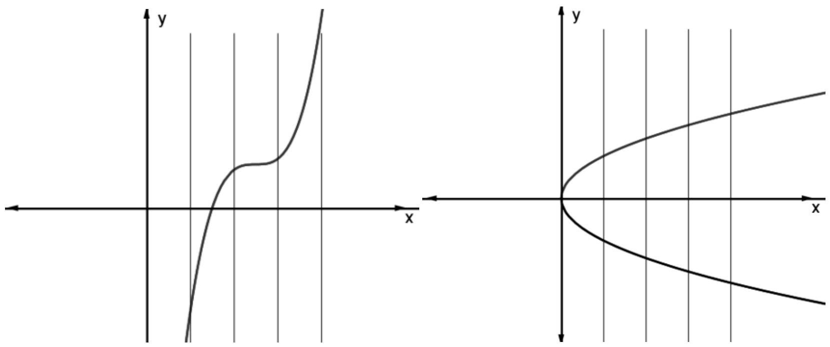


Figure 6.5: Graphs of function and non-function

After the two graphs were constructed the Facilitator pointed out that a situation where the vertical line cuts the graph in two places goes against the definition of a function. He explained that according to the Grade 10 definition this could not happen if the graph was a representation of a function. Hence he said Card 24 could remain in the list of non-functions because the vertical line cut its graph in two places.

- **Task - Event 4: Discussion focused on Card 3**

Facilitator put up Card 3 (see **Figure 6.6**) on the overhead projector for discussion. He read the verbal statement on Card 3 and thereafter invited teachers to comment on why they thought it was not a function.

3

To get f (x) you square x and subtract it from 1

Figure 6.6: Card 3

When teachers were taking time to comment he scaffolded the task by pointing that one teacher had earlier said that **f(x)** meant it was a function and if it could be written as **y = f(x)** then it was a function. The appeal associatied with this legitimation was correct and it constituted *correct mathematical appearance*. Facilitator did further scaffolding by translating the verbal statement algebraically to $1 - x^2$. More scaffolding was done when the facilitator asked teachers to think about the graph of Card 3 and visualise how the vertical line test would cut it. In spite of the scaffold on Card 3 teachers were not sure that it was a function (**see Excerpt 6.3**). Reference to the use of the vertical line test on the graph of Card 3 was a way to scaffold the task and also a legitimation. Appealing to the vertical line test constituted *correct mathematical reality*.

	F	The next one is number 3.To get f of x you square x and subtract from 1. Comments on that one? Teacher 2 was saying earlier that if you've got f of x that already says it's a function, am I right or am I misquoting you? Because he said that if we can write it like that *[points to* y = f(x) *on the flipchart]* then it is a function. And if we look at the story does the story still… So if we wrote it algebraically, to get f of x you would square x. So we would take x and square it and subtract it from 1. So how would we write that? *[on the flipchart F writes:* $1 - x^2$ *]* 1 minus x squared. Ok. Can you picture what this thing looks like? What would the graph of this thing look like? Because that might also help you in terms of the vertical line test to think about whether it's a function or not. We're saying it is a function so it must come off this list?
	T4	Yes
	F	Is that what we're saying? Or are we not sure?
	T5	Not sure.

Excerpt 6.3: Dicussion of Card 3

In the excerpt the facilitator began the discussion by converting the verbal representation of card 3 into algebraic hoping that teachers will recognise that it was a quadratic function which was familiar. The card was set aside for later discussion after they could not realise that it was a function.

- **Task 1 – Event 5: Discussion focused on Card 4**

The next discussion focused on Card 4 (see **Figure 6.7**). Facilitator put up Card 4 on the overheard projector. Card 4 was an algebraic representation of Card 3.

4

$$\mathbf{f\,(x) = 1 - x^2}$$

Figure 6.7: Card 4

Facilitator pointed out that since teachers were not sure about Card 3, then they may also be not sure about Card 4. Teacher 18 said it was not a function because it did not have y. The teacher's legitmation contained an incorrect appeal to mathematical notation and this costituted *incorrect mathematical appearance*. Facilitator suggested card 4 be set aside together with Card 3 (**see Excerpt 6.4**)

509 510 511 512	F:	So let's go to number 4. That's… I mean we don't have to decide on everything right now. Number 4 is? Number 4 is that itself, right? Ok, so if we weren't sure about 3, then presumably we are not sure about 4.
514	T18	There's no y there
516	F:	Theres no y?
518	T18:	ja
520 522 523	F:	Ok *[on flipchart puts a dot next to 4]* So we are unsure about that, alright. Um, so that's fine. Let's leave those two as question marks for now

Excerpt 6.4: Discussion of Card 4

In the discussion involving card 4 Teacher 18 said it was a non-function because it did not have a y value. Perhaps the teacher was not familiar with the function notation. At the end of the discussion the facilitator decided that this card, together with card 3, be put aside for later discussion. Card 4 was the algebraic representation of Card 3 and this was show earlier in the discussion of Card 3 when the facilitator converted the verbal representation of card 3 to algebraic form (**see Excerpt 6.3**).

- **Task 1-Event 6: Discussion focused on Card 19**

Following Card 4 was Card 19. Facilitator put the card up and invited teachers to comment about why it was not a function.

19

Halve the input to get the output

Figure 6.8: Card 19

Teacher 13 said it was a function because of the magic words 'input and output'. The teacher drew from the fact that earlier, Card 10 was categorised as a function because its verbal statement had the same words. Facilitator asked if the words input and output always suggest a function and the teacher affirmed. The legitimation contained the appeal to *correct mathematical appearance*.

525	F:	And then number 19.
527		*[on the overhead:*
528		*19*
529		*Halve the input to get the output]*
531	F:	Halve the input to get the output.
533		*[teachers work this out]*
535	F:	Yes, it's a function?
537	T13:	*[hesitantly]* Ja
539	F:	Convince me?
541	T13:	It's the magic words input and output *[chuckles]*
543		*[other teachers and F chuckle]*
545	F:	The magic words input and output. Input and output are gonna help us always?
547	T13:	Mmm
549	F:	Ok, I'm gonna suggest to you that it doesn't, but I'll pick that up now.

Excerpt 6.5: Discussion of Card 19

Facilitator then cancelled Card 19 from the list of non functions after pointing out that these words do not always suggest a function.

- **Task 1- Event 7: Discussion focused on Card 7**

The last card on the list of the teachers' non-functions was Card 7. This card was put up for teachers to explain why it was a non-function.

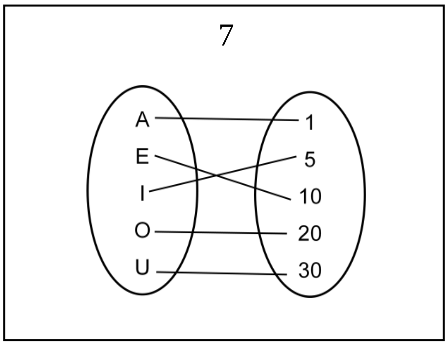


Figure 6.9: Card 7

He asked the teacher who said it was not a function to explain and she said it was not a function because it did not have x and y and that letters of the alphabet did not correspond to the numbers. That is, A mapped onto 1 but E did not map onto 5. Teacher 18's legitimation had an appeal to which was grounded in *incorrect mathematical appearance*. The teacher's legitimation seem to suggest that a function is one which is expressible in algebraic notation (**see Excerpt 6.6**).

551 552 564 565	F:	F: Let’s go to number 7. So let’s just pass that one for a sec. *[puts a dot next to* 19 *on the flipchart]* Number 7. Ok. T11, were you… Did you suggest this one? *[Looks at T11?]* Ok, tell us why you put it up?
569	T11:	It’s like 10 and I know that A is the first letter and it represents 1, there’s no x y
571	F:	Sorry? There’s no?
573	T11:	x y *[chuckles]*
575	F:	There’s no x’s and y’s?

Excerpt 6.6: Discussion focused on card 7

In another form of scaffolding what was and was not a function the facilitator referred teachers back to where one teacher pointed that the words input and output were magic words that could be used to identify a function. He pointed that in Card 7 these words are not there. So he asked if inputs and outputs were there in Card 7 but in another form. A teacher said they were present in the form of sets. She said the set of alphabets was inputs and numbers were outputs. Facilitator confirmed this by saying yes the inputs and outputs were indeed present but in another form. He then asked if Card 7 was a function and the teachers agreed. Facilitator added the letter J to the inputs set and mapped it to 30 (see **Figure 6.10**).

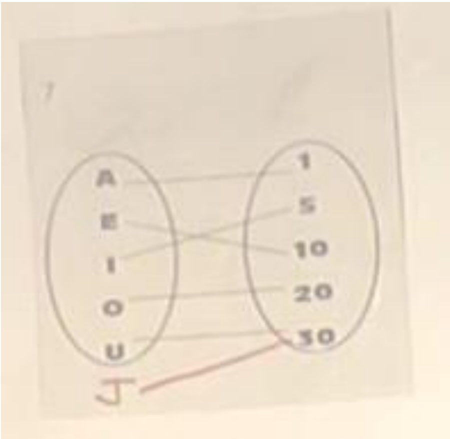


Figure 6.10: A mapping that makes Card 7 a function

He then asked if Card 7 was still a function and Teacher 24 said it was not a problem, meaning that it was still a function (**see Excerpt 6.8**). The addition of the letter J was a way to help teachers realise that both one-to-one and many-to-one mappings were functions. Hence the introduction of the letter J into the input set of card 7 was a way to scaffold this realisation. The scaffold moves through probes (see highlighted text in **Excerpt 6.7**).

638	F:	So if I add J for Jewels and I map it onto 30, is that ok?
640	T24:	Which one?
642	F:	U and J are both mapped onto 30, is that a problem?
644	T24:	No

Excerpt 6.7: Discussion focused on mapping U and J to 30

He then asked for a situation that would make it a non-function. In response, Teacher 25 suggested mapping J to 5 but later changed her mind and suggested mapping J to 20 such that both 20 and 30 were mapped to J (see **Excerpt 6.8** and **Figure 6.11**).

646	F:	Can you give me an example here that would cause a problem?
648	T25:	Let's have J going to 5
650	F	J goes to 5
652 658 666	T25	Ja. Oh sorry, sorry, sorry I'm not thinking. Let's have 20 going to J.
668	F:	20 goes to J like that? *[now makes the line go from J to 20 as well as to 30]*
670	T25:	No. I mean it has to change. It is going to change... because the ... is changing.
672	T19:	What are we trying to do? Sorry, I don't…
674	F:	Ok. So I said if I add J and map it onto 30 which…
676	T25:	That's not a function. It's not a function.
678	F:	That's not?
680	T25:	Yes
682	F:	<u>Because we have taken one input and mapped it onto two outputs.</u>
684	T25:	Ja, onto two.

Excerpt 6.8: Discussion focused on mapping J to 20 and 30

Building on this the Facilitator explained that mapping J to both 20 and 30 made Card 7 a non-function because one input was mapped onto two outputs (see **Figure 6.11**). The facilitator's appeal was to one-to-many mapping (**see Excerpt 6.8 line 682**). As such the appeal was to *correct mathematical reality*. While the mapping of J to both 20 and 30 constituted scaffolding it also opened possibilities for legitimation and contrast. I have highlighted the probe which constitutes scaffold, and I have underlined the legitimation from the facilitator in **Excerpt 6.8**. **Contrast** is evident in **Figures 6.10** and **11** when the two are simaltaneously discerned. In fact the two are the same figure showing what is, and what is not a function.

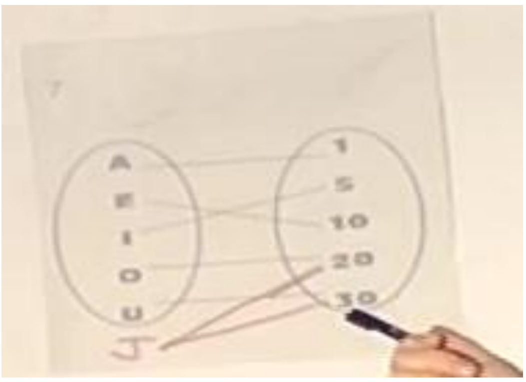


Figure 6.11: A mapping that makes Card 7 a non-function

Following on the discussion around the mapping of J to 20 and 30, facilitator asked for another example that would make Card 7 a non-function and none was given. Facilitator related the situation where J was mapped onto 20 and 30 to Card 24 where one value of x mapped onto two values of y. Facilitator demonstrated on Card 24 by replacing an x value with J and y values with 20 and 30 such that (J; 20) and (J; 30) were two pairs of coordinates on Card 24. Facilitator stressed that what made Card 7 a non-function was mapping one input onto two outputs. It can be said that the introduction of the letter J into the input set of Card 7 provided opportunites to scaffold, legitimate and contrast what is and is not a function. The legitimation contained appeal to the definition of function and this constituted an appeal to *correct mathematical reality*. It is interesting to notice that while the discussion on Card 7 began with Teacher 11 appealing to *incorrect mathematical appearance*, it ended with the facilitator's appeal to *correct mathematical reality*. It can be said that the use of Figure 6.10 and 11 were means of redressing Teacher 11's appeal to *incorrect mathematical appearance.*

- **Task 1- Event 8: Facilitator's explanation of 'what is a function'**

At the end of the discussions on why the six listed cards were non-functions the facilitator put up 3 definitions of functions on the power point slide to explain what was a function to redress the incorrect listing of functions as non-functions. To do this the facilitator began by pointing out that input and output were the language of function. He used the two terms to illustrate the definition of function in an example involving a one-to-one and many-to-one mappings (see **Figures 6.12** and **13**).

Facilitator explained the one-to one mapping by pointing out that no two elements in the input set map onto the same element in the output set. He pointed out that while all elements in the input set must have a partner in the output set, not all output elements need to have a partner in the output set. He demonstrated this by placing two unmapped elements in the output set (**see Figure 6.12**).

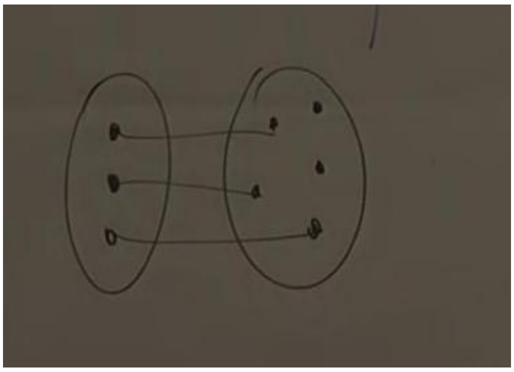


Figure 6.12: Illustration of a one-to-one mapping

For a many-to-one mapping two or more elements in the input set can map onto the same element in the output set (see **Figure 6.13**). Similarly, all elements in the input set must have a partner in the output set while the reverse is not necessary.

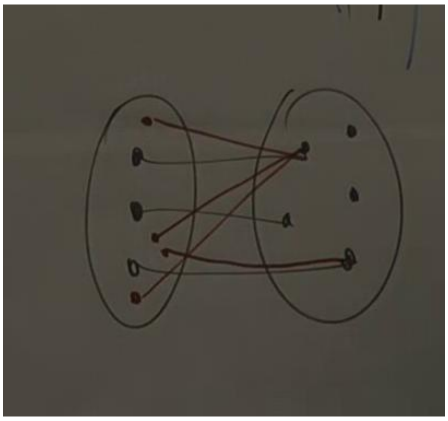


Figure 6.13: Illustration of a many-to-one mapping

In **Figure 6.13**, more than one element in the input set maps onto the same element in the output set- and this makes it a many-to-one mapping. In **Excerpt 6.9** (line **742** – **747**) facilitator unpacked the modern definition using a mapping diagram (**see Figure 6.14**) which he used before when he was explaining the previous definition.

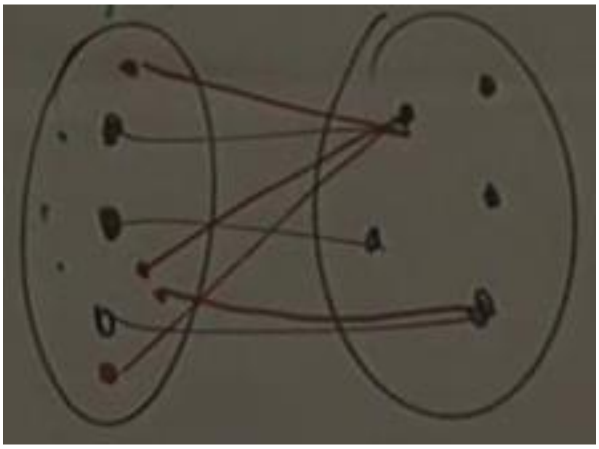


Figure 6.14: Unpacking the modern definition of function

Facilitator used an example on the flipchart to explain the definition further. In that example he added ummapped elelments in the input set (see **Figure 6.14)**

744 745 746 747	F	But there's a modern definition of function and I'm not going to go into the history of stuff that actually just says a function is a bunch of ordered pairs where the first element comes from one set and the other, the second element of the ordered pair comes from another set.
749 750 751		So two sets A and B is a collection of ordered pairs where the first element or ordered pairs, a comes from set A – ah, sorry – x comes from set A and y comes from set B. y is an element of B. And every element in A is associated with a unique element in B
753 754 755		So every element in, if this is set A *[above the first set on the flipchart F writes A]* we can't have a few extras hanging out there, ok. *[puts a few extra dots in set A]* It's like every element here *[points to set A]* is mapped to something
753 754 755 756 757		Alright. And it can't go to more than one; that's what we mean by unique. Unique as in this element goes to that element in that set – it can't go to another one as well, alright It doesn't say anything about, what kind of animals these sets are. The set can contain, one set can contain letters, the other can contain numbers. So this example that we have here with the vowels, can you see it, hiding in the background?
765 766 767 768		That one is still a function provided you ignore the J I put in. These things can be anything you like. *[points to the slide showing the two sets]* Ok. You can have letters in one and numbers in the other. You can have body parts in one (or the words of body parts) and…

Excerpt 6.9: Facilitator unpacking the modern definition of function

Facilitator is working from an example (**see Figure 6.14**) to explain and unpack the definition with focus on the two terms *every* and *unique*. To explain the term *every*, he added three elements into the input set (**see lines 753, 754** and **755** in **Excerpt 6.9**) which were not mapped onto any element in the output set and said this could not happen in a function because *every* element in the input set should have a partner in the output set. Hence **Figure 6.14** was an example of non-function. Moving on to the term *unique*, facilitator pointed out that one element in the input set cannot map onto more than one element in the output set. In addition to explaining the two terms he pointed in a function anything thing can be an element and he gave names of flowers, colours, body parts, numbers and letters as examples.

Connecting the explantion to card 7, he pointed out that it was a function because one set had vowels and another had numbers and that the condition that said *every* element in the input set must map onto a *unique* element in the output set was satisfied. Having said that the facilitator called attention to earlier cards (i.e. 3, 4 and 19) which were earlier put aside because teachers were not sure that they were functions. The purpose of unpacking the 3 definitions of function was a way to scaffold the realisation that cards 3, 4 and 19 were functions. Hence subsequent discussions on the cards hinged on the definition of function.

Returning to Cards 3, 4 and 19 facilitator began by scaffolding card 3 using mutiple representations (see **Figure 6.15**). The text related to scaffold has been highlighted. In **Excerpt 6.10** the facilitator is showing that card 3 was a many-to-one function and he did that by moving between representations to show that -2 and +2 produced -3 across the three representations. By showing that both negative and positive two produce negative 3, he wanted teachers to notice that it was a many-to-one mapping which was a function. Hence in the excerpt I underlined the text that draws attention to this because it constituted a legitimation.

F:	So think about… If we think about that representation *[on the flipchart F draws two sets: one has 2 and -2]* This one represents ... negative 2 and positive 2. And they both map onto negative 3. *[in the second set F puts -3. A line coming from both 2 and -2 in the first set link to -3 in the second set]* Was that a problem? No. **If we think in the Cartesian plane do you notice what I'm doing? I'm going from table to a different representation** *[points to the sets]* **and I'm going to another representation** *[points to the graph]*.

Excerpt 6.10: Facilitator scaffolding Card 3

While working with the three representations was a way to scaffold card 3, it was also implicitly a legitimation as well as openning a pattern of variation known as **generalisation**. The appeal in the legitimation was to many-to-one mapping and so this constituted an appeal to *correct mathematical reality*. In the above excerpt the facilitator called attention to similarity between the representations and this opened opportunity for **Generalisation**.

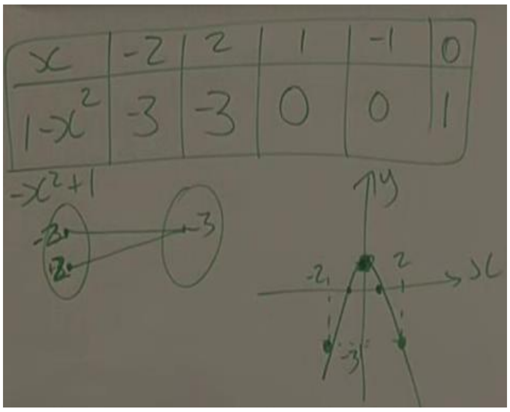


Figure 6.15: Multiple representations of Card 3

In a further scaffold of card 3 the facilitator explained the meaning of the term unique, as used in the definition of a function. He pointed that ‘unique value of y’ meant that one x value cannot have more than one y values and not that two x values cannot have the same y value. He concluded that since each x value on the table yielded one value of y, then Card 3 was a function. Facilitator extended this conclusion to Card 4 which was basically the equation for card 3. **Figure 6.16** below is card 20 which was an algebraic representation of card 19.

20

$$y = \frac{x}{2}$$

Figure 6.16: An algebraic representation of Card 19

The facilitator converted the verbal description on Card 19 to algebraic representation to show that it was a verbal representation of Card 20 and was a linear function (**see Figure 6.16**). Relating card 19 to card 20 was a way to scaffold card 19 because teachers were familiar with algebraic linear functions. Converting the verbal description on Card 19 to algebraic opened an opportunity for **generalisation**.

In section 6.2 I described the enactment of Task 1 and I did this by evidencing instances of scaffold, legitimation and patterns of variation. In the following sections I discuss scaffold, legitmating criteria and patterns of variation separately and in more detail. My discussions of these three aspects of my analysis are characterised by references to figures and excerpts in the description section.

6.3. Discussion of the pedagogic strategy used at the E- Cluster

At the E-Cluster the Facilitator did not bracket his **natural attittude**. This might be due to the fact that the subject matter knowledge of the task consisted of **revisiting school mathematics** which was familiar to teachers. Hence he adopted an inductive pedagogic strategy and this was evident in the **low scaffolding** provided on the task (**see Figure 6.1**). Here teachers were provided task instructions only. Teachers worked in groups of three on the task with task instructions as the only scaffold. **Excerpt 6.11** below contains evidence of low scaffolding which was provided by the Facilitator. Here the scaffold was limited to explaining task instructions to the teachers.

F:	Ok, so on your table you have an envelope that has got several cards in it, about 25 of them. *And I would like you to sort your cards into 2 piles – those that represent functions and those that don't. Ok. So functions and, if you want, non-functions, just separate the 2 piles. You'll see there are some cards with words on them, some with pictures, some with tables of values. Take a look at them and decide, is this a function or is it not? And make your 2 piles.* And I'll give you a few minutes to work on that.

Excerpt 6.11: Facilitator's scaffold on task instructions

At the end of the task teachers' incorrectly placed functions in the pile of non-functions (see **Figure 6.2**). As such, the inductive pedagogic strategy revealed gaps in teachers' knowledge of functions. The gaps were evident in the utterances in which teachers made when explaining their list of non-functions. Teachers were asked to explain why the cards were non-functions and so they appealed to different criteria to legitimize their explanations. By way of example, Card 10 was the first on the list and might have been included in the list because teachers are not familiar with verbal representations of functions. However, rather than justify why it is a non-function, teachers decided that they recognised that it was a function because it contained the words input and output. So here teachers appealed to mathematics language to discern that Card 10 was no longer a non-function as they had initially thought **(see Excerpt 6.1**). As such, the Facilitator cancelled Card 10 from the list of non-functions. The next card for discussion was Card 24 and this process went on up to Card 7, the last on the list of cards identified as non-functions by teachers (see **Figure 6.2**). I return to talk about the teachers' explanations when I discuss the legitimating criteria in the next section.

6.4. Discussion of legitimating criteria in Task 1 at the E-Cluster

The legitimating criteria were co-constituted by the Facilitators and teachers. Teachers explained why the cards in the pile of non-functions were not functions while the Facilitator had to explian why they were actually functions. The first legitimating criteria came into existence when attention was on Card 10. This card had earlier been classified as non-function but the group that put it there later decided it was a function. When explaining why it was a function, a teacher from the the group said it was a function <u>because its verbal statement had the words input and output</u> (see **Excerpt 6.1**). Here the teacher was appealing to function terminology.

The second legitimating criteria was present when attention was on Card 24. One teacher explained that Card 24 was a function <u>because its graph was non-linear</u>. This teacher was appealing to the shape of the graph. The third legitimating criteria was also present when attaention was on Card 24 when another teacher responded that it was a function <u>because one x value was mapped to two y values and so it was not a unique correspondence.</u> The appeal associated with this was to definition of a function. The fourth legitimating criteria emerged when the Facilitator was explaining why Card 24 was correctly listed as a non-fucntion. To do this he <u>drew a vertical line to show that it cut the graph of Card 24 in two places</u> and so this made it a non-function. Here the appeal was to vertical line test. The fifth legitimating criteria came into being when the Facilitator pointed that <u>Card 24 did not satisfy the definition of a function because one x-value cannot map onto two different y values</u>. His appeal was to definition of a function. The sixth appeal was present when the Facilitator used the vertical line test to show that it cut a cubic function once and a square root graph twice. In **Figure 6.4** in the section where I describe the enactment shows the cubic and square root graphs, each with vertical lines. **Excerpt 6.12** below contain utterances which evidence the fourth, fifth, and sixth legitimating criteria which were uttered by the Facilitator.

448 449 450 451 452 453 454 455 456 457 458 459 460 461 462 463 464 465 466 467	F:	And that one we don't know. But if we assume they are in the same vertical line *[moves his left hand down, indicating a vertical line]*, then they have both got the same x value. And a function by definition says that you can't have one x value that is mapped onto two different y values. So when we talk about the vertical line test, if you can draw a vertical line *[moves his hand down again, indicating a vertical line]* and it cuts the graph more than once, then we say the graph is not a function. Is that familiar? **So you might have other examples where, if you have a graph that looks like that, if you draw any vertical** *[drops the pen:* Whoops. *Picks the pen up again]* **any vertical line wherever it is, it's only going to cut the graph once**. *[demonstrates this by drawing a graph on the flipchart]* **But if you have something, for example something that looks like that *[using another colour, F draws a graph over the one he has just drawn]*, which is like a parabola on its side, then you can see there are several places where the vertical line is going to cut the graph twice,** ok? And according to the grade 10 definition of function, that can't happen. If that happens then the thing is not a function. So that would mean this one because there are two points where they have same x value and different y value, that means that this one would not be a function. Ok. So 24 can stay on our list. *[puts a tick next to* 24 *on the flipchart*

Excerpt 6.12: Facilitator's legitimating criteria related to Card 24

The seventh legitimating criteria was present when attention was Card 3. Here the Facilitator reminded teachers that one teacher had earlier said that if it can be written in the form **y = f(x)** then it is a function. That teacher was appealing to algebraic represenation. The eighth legitimating criteria appeared when attention was on Card 4. Teacher 18 said it was not a function because its algebraic representation did not contain y (see **Excerpt 6.13**).

509 510 511 512	F:	So let's go to number 4. That's… I mean we don't have to decide on everything right now. Number 4 is? *[looks for a slide to put on the overhead:* 4. $F(x) = 1 – x^2$ *]* Number 4 is that itself, right? Ok, so if we weren't sure about 3, then presumably we are not sure about 4. *[chuckles]*
514	T18:	There's no y there
516	F:	There's no y
518	T18:	Ja

Excerpt 6.13: Teacher 18's explanation on Card 4

The ninth legitimating criteria was present when attention was on Card 19. This card was earlier in the list of functions but a teacher from another group said it was a function because its verbal description had the words input and output. This teacher was appealing to functions terminology.

The tenth legitimating criteria came into being when attention was on Card 7. Here a teacher explained that this was not a function because there was no x and y. The appeal was to mathematical notation. The eleventh legitimating criteria was present when the same teacher said that there was no relationship between the alphabets and numbers. Here the appeal was to algebraic representation. The twelfth appeal was also related to Card 7 and it arose after the Facilitator had included the letter J in the set of alphabets such that J mapped onto both 20 and 30 in the output sets (**see Figure 6.11**). He pointed that mapping J to these two numbers made Card 7 a non-function and his legitimating criteria was that one input was mapped onto two outputs. As such he was appealing to a mathematics definition. The thirteenth legitimating criteria related to Card 7 arose when the Facilitator explained that when J is removed from the set of alphabets Card 7 was a function because its mapping satisfied the definition of a function. This was another appeal to a mathematics definition. The forteenth legitimating criteria was present when after realising that teachers failed to recognise that Card 3 and 4 were functions the Facilitator decided to explain the meaning of a function and he did this by working with definitions (**see Excerpt 6.7**). His explanations were largely communicated through examples in the form of mapping diagrams which constituted his legitimating criteria (**see Figures 6.12 and 6.13**). His appeals were to mathematics diagrams.

Table 6.1: Summary of legitimating criteria in Task 1 at E-Cluster

Legitimating criteria	**Appeals**	**Enacted object**
Because its verbal statement had the words input and output. Here the teacher was appealing to function terminology	function terminology	Mt_{ap+}
One teacher said Card 24 was a Function because its graph had x and y axes	Cartesian plane	Mt_{ap-}
Another teacher said Card 24 was not a function because one x value was mapped to two y values and so it was not a unique correspondence.	Definition of a function	Mt_{re+}
Facilitator drew a vertical line to show that it cut the graph of Card 24 in two places	Vertical line test	Mt_{re+}
Facilitator said Card 24 did not satisfy the definition because one x-value map onto two different y-values	Definition	Mt_{re+}
Facilitator used the vertical line test to show that it cut a cubic graph once and a square root graph twice	Vertical line test	Mt_{re+}
One teacher said a function is that which has y=f(x)	Algebraic representation	Mt_{ap+}
A teacher said Card 19 was a Function because it had the words input and output	Function terminology	Mt_{ap+}
A teacher said Card 7 was not a function because there is no x and y	algebraic representation	Mt_{ap-}
A teacher said Card 7 was not a function because there was no relationship between the alphabets and numbers	Algebraic representation	Mt_{ap-}
Facilitator said Card 7 was not a function because one input was mapped onto two outputs. This he said after adding the letter J into the input set and mapping this to 20 and 30.	Mathematical definition	Mt_{re+}

After removing the letter J, the Facilitator said Card 7 was a function because its mapping satisfied the definition of a function.	Mathematica definition	Mt_{re+}
Facilitator's use of mapping diagrams to explain the definition of function	Mathematical diagrams	Mt_{re+}
Facilitator's use of examples constituted by mapping to explain that Card 3 and 4 were functions	Mathematics examples	Mt_{re+}

6.5. Discussion of what was made possible to learn at the E-Cluster

Working on Task 1 opened 6 dimensions of variation. The first dimension of variation was the ***types of relationships*** and the two critical features within this dimension were ***function and non-function*** relationships. These critical features were simultaneously in focus in the card sorting task. Hence the task of sorting cards into functions and no-functions made it possible for teachers to experience difference between these critical features through. In the first dimension of variation the Facilitator used ***contrast*** - a pattern of variation where awareness was brought about by experiencing difference between functions and no-functions.

This pattern of variation helped teachers to discern that a relationship *might be a function or a non-function* and this was possible because in Task 1 teachers experienced difference between functions and non-functions. According to variation theory, it is not sufficient to only know what something is without also knowing what it is not – because for one to truly know the object of learning, one should learn about both its *internal* (structure and meaning) and *external horizons* (related or unrelated context in which it is embedded).This is indeed important because it was evident from teachers' responses to Task 1 that prior to engaging in they did not have a deeper understanding of what constitutes a function and this was evident when they listed some functions as non-functions. The teachers responded in this way in spite of the fact that they have all taught functions for 7 or more years. So their responses seemed to confirm the view that experiencing sameness only and learning from experience do not guarantee a full understanding of the phenomena (Lo, 2012).

In **Excerpt 6.12** above the bold utterances show related **contrast** - a pattern of variation. The excerpt contain utterances which were related to **Figure 6.5** in the description section above. The demonstration with the cubic and the square root graphs opened up several dimensions of variation and so introduced patterns of variation that opened possibilities for teachers to experience difference and the discernment of difference between subsequent critical features. In particular, the discussion around card 24 resulted in three dimensions being opened up and I must point that the first dimension (**relationship**) was explicit and the other two (**definition** and **vertical line tes**t) were implicit. The first was explicit because the Facilitator was deliberate in trying to help teachers to discern difference between functions and non-functions and the other two were implicit because they were never deliberately foregrounded but were used to explain the first dimension of variation. In the first dimension of variation the Facilitator put up Card 24 on the flipchart and proceeded from there to explain by means of the definition to point out that the graph on Card 24 had one x value mapped onto two y values and that made it a non-function because that was contrary to the definition of a function. In a follow up explanation the Facilitator simultaneously put up a cubic graph and a square root graph on the flipchart. He then applied the vertical line test on each of the two graphs.

The simultaneous presence of the two graphs on the flipchart opened up opportunities for discernment of difference between functions and non-functions (the two critical features) which are two types of relationships. Now, the fact that each of the two constituted a relationship meant that together they were values or critical features within a relationship. So this made the relationship a dimension of variation. The Facilitator's explanations brought about contrast in two ways. First it was awareness brought about by experiencing difference in the number of outputs that one input should map onto to produce a function or a non-function. Second was awareness brought about by experiencing difference in how many times a vertical line cut a cubic and a square root graphs. **Here it was possible for teachers to notice that a graph represented a function if and only if the vertical line cut it in one place and not a function if it cut it in more than one place.**

In **Table 6.2**, relationship and vertical line test are invariant because functions and non-functions are each a type of relationship. Whether the discussion is around the cubic or square root graph the fact that each is a relationship does not vary. What varied in the explanation was the number of outputs needed to map onto one input to constitute a function or a non-function. In a function one output mapped onto one input and in non-functions more than one outputs mapped onto one input. Similarly, the vertical line test is also invariant because it is applicable in functions and non functions. What varied was how many times the vertical line cut each graph. ***Consequently, teachers had opportunities to discern that in a function one input must map onto one output and in a non-function it should map onto more than one outputs.*** **They also had the opportunity to discern that the vertical line cut the graphs of a function and non-function once and more than once respectively.**

The Facilitator's addition of the letter J into the set of letters and then proceeding from there to map it to 30 and later to 20 opened a pattern of variation. Card 7 was a one to one mapping before J was added. So its addition made Card 7 a many to one mapping. So the presence of J made it possible for teachers to experience difference between a one to one and a many to one mapping. In Variation theory terms, it can be pointed out that adding J opened a dimension of variation where one to one and many to one mapping are critical features in the dimension. This then suggests that since the two critical features are the types of mappings then we have mappings as the dimension of variation. Now, when J was later mapped onto twenty such that J was mapped onto two outputs (i.e. 30 and 20) such that we had a one to many mapping, a third critical feature (one to many mapping) was introduced and the initial pattern of variation was extended. The dimension of variation remained the same as mappings but the number of critical features increased from two to three such that there were one-to-one, many-to-one and one-to-many mappings. The pattern of variation associated with this was contrast. **It was possible for teachers to discern difference between the three types of mappings. In particular, it was possible to discern that a one-to-one and a many-to-one mapping constituted a function but a one-to-many did not.**

In his explanation of the definition that says "Mappings that are one-to-one or many-to-one are called functions" the Facilitator drew two diagrams showing a one-to-one and a many-to-one mapping to illustrate the meaning of the definition (**see Figures 6.12-6.13**) and **Exerpt 6.13**. In the following excerpt the Facilitator explains what constitutes a one-to-one and many-to-one mapping. The verbal explanation was based on **Figure 6.12 and 6.13** which were one-to-one and many-to-one mappings. Facilitator began with **Figure 6.12** and moved further to convert it into **Figure 6.13**. So converting **Figure 6.12** to **Figure 6.13** opened opportunity for **contrast** between the two mappings. The bolded utterances are evidences of **contrast** in the Facilitator's explanation.

709 710 711 712 713 714 715 716 717	F:	Looking at the definition of a function, we can talk about it as a mapping. So we map two sets of items. In the language we have been using there's an input set and an output set, ok. So if you want we've got input over here and there's some mapping that's happening that's associating elements in this set to elements on the set on the right hand side**. Mappings that are one-to-one or many-to-one are called functions**. **So one-to-one would mean it takes one from the input set and maps it onto one in the output set. It can't go to two things – that's what we're saying. You can't take something from the input and go to two different things in the output.** Ok. **Or many-to-one would be several things in the input going to the same output.** If we draw them, those diagram kind of things
718		
719 720		*[On the flipchart F draws two sets. The first set has 3 dots, the second set has 5 dots. Lines link the 3 dots in the first set to 3 dots in the second set]*
721 723 724		So if they all go like this they don't have to use all of them. **If they go like that there's one-to-one, there's a one-to-one. That's what we are referring to as one-to-one. Many-to-one would be that many in the input set go to one in the output set.**
724		
725		So this for example *[Facilitator referring to **Figure 6.9**]*
726 727		*[draws another dot in the first set and the line links to the first dot in the second set (there are now 2 dots linked to the dot in the second set)]*
728 729		**If I add another one here and automatically this whole thing becomes many-to-one because there are many (as in 2) that go to 1. Ok. If I add some more here and say**
731 732		*[draws two more dots in the first set which both get linked to the first dot in the second set (now there are 4 dots linked to the dot in the second set)]*
733		That's still many-to-one.
735 736 737 738		And if I add another one over here and go over there that's still many-to-one. *[Another dot is drawn in the first set and is linked to the third dot in the second set. (This third dot now has 2 dots from the first set linked to it)]*. Alright. Many basically means more than one in this context.
740 741 742		And so that's, that's one of the key elements of a definition of functions – **there's a mapping that goes from one set to the other and that it's either one-to-one or many-to-one.**

Excerpt 6.14: Facilitor's explanation of one-to-one and many-to-one mappings

The simultaneous presence of the two diagrams helped teachers to experience difference between the one-to-one and many-to-one mappings and a pattern of variation was opened where the dimension of variation was the definition and the critical features were the two types of mapping. The pattern of variation was contrast and so **it was possible for teachers to discern difference between a one-to-one and a many-to-one function**.

In his explanation of the modern definition of function the Facilitator continued with his use of mathematics diagrams when he used the mapping diagram to explain the meaning of unique. He used the diagram to unpack the statement “every element in A is associated with a unique element in B by the function f”. To do this he used a counter example by adding more elements to the first set and leaving them without partners in the second set (**see Figure 6.10**). He pointed that the presence of unmapped elements in the input set shows that not every element in this set is associated to an element in the output set and so this is contrary to the definition. He also pointed that the meaning of unique was that one element in the input set must map onto one element in the output set **(see line 753- 755 in Excerpt 6.9)**. The Facilitator also explained that in the modern definition an element in the set can be any object.

The explanation of the second definition of function helped teachers to experience difference between a mapping and a non-mapping. A mapping was represented by the many to one and when the Facilitator added elements in Set A that did not have partners in Set B the many-to-one mapping became a non-mapping. As such, a pattern of variation was opened where relationship was the dimension of variation and the critical features were mapping and non-mapping. Another pattern of variation (*generalization*) opened when the Facilitator explained that an element can be any object. Here the dimension of variation was elements and its critical features were types of elements.

Table 6.2 shown below suggests that the explanation of the modern definition of function made it possible for teachers to learn that for a mapping to exist, every element in the input set should have a partner in the output set and if not, then there is no mapping (see **Figure 6.10**). It was also possible to discern that in a function an element can be any object. Teachers have had the opportunity to learn that one-to-many and many-to-many mappings were not functions. Now the explanation of the modern definition increased the opportunities by adding that for mapping to exist, every element in the input set must have a partner in the output set. As such this suggests that Task 1 helped teachers to experience three (i.e. one to many; many to many and when there is no mapping) conditions of non functions.

Since teachers did not realise that card 3 and 4 were functions after the Facilitator’s initial explanation using representations the Facilitator put them aside until he had introduced three definitions of a function. The introduction of the definitions was a pedagogic strategy the facilitator used to help teachers understand that most of the cards that they had earlier listed as non-functions were actually functions. Now in the second discussion of Card 3 the Facilitator asked teachers to use the definition to determine if this card was still a non-function. When teachers could not do so, he scaffolded the task further by transforming the verbal representation on Card 3 into a table of values, algebraic, graph, and as a mapping. He used the table, mapping and graph to demonstrate that any one value of x mapped onto one y value. By way of example, he demonstrated on each of the three representations that -2 mapped onto a unique y value which was -3. Similarly with 2.

The Facilitator's choice of example to explain the meaning of unique was productive pedagogic strategy that had the potential to scaffold the meaning of unique as used in the definition of a function because teachers held the impression that unique meant that two inputs cannot map onto the same output. Hence they believed that a many-to-one was not a function. As a matter of fact, one of the teachers had at the beginning of the task, pointed that he did not consider many-to-one mapping as functions and his reasons rested on his understanding of unique. For him, the term unique meant that each input should have its own output and so no two inputs should map onto the same output. As such, the use of a many-to-one mapping to explain the meaning of unique provided the Facilitator with the opportunity to help teachers redress the misconception.

In an attempt to help teachers begin to see verbal descriptions as representations of functions the Facilitator called a pedagogic strategy in which he worked directly with the verbal description to complete a table of values. It is interesting that the Facilitator opted to work directly with the verbal representation rather than transforming it into algebraic form and working from there because teachers were familiar with working with algebraic representations to complete table of values. So this was a pedagogic strategy which was used to help teachers discern that the properties of a function are embedded into all its representations. In addition, teachers are not familiar with verbal representations and so this might have influenced them to list it as a non-function. **So working with the verbal representation directly to do what a familiar representation normal does helped teachers to recognise that verbal descriptions can also be representations of functions just as algebraic representations.**

When the table of values was completed the Facilitator asked teachers to see if each x value on the table had one y value. Teachers were also asked to do the same on the mapping diagram and the parabola and when teachers were satisfied that no x value yielded two y values the Facilitator concluded the discussion on Card 3 by pointing that since each x value on the table yielded one value of y, then Card 3 was a function. Similarly, it was concluded that Card 4 was a function because it was an algebraic representation of Card 3.

Working to explain that Card 3 and 4 were functions opened two patterns of variation that included generalisation and contrast. The pattern of variation related to generalisation emerged when the verbal representation of Card 3 was transformed into four different representations that included table of values; mapping diagram; graph and algebraic expression. During the transformation process, the fact that these were the same function did never changed and so in Variation theory terms it meant that function was invariant when the representations were varying. Moving between the representations of the same function helped teachers to discern similarities between the representations. According to variation theory discernment of sameness makes generalisations about the phenomena possible. The pattern of variation related to contrast opened when the Facilitator was explaining the meaning of the term 'unique' in the definition of function. In his explanation the Facilitator point that unique meant that one input should map onto one output and that this did not mean that two inputs could not map to the same output. **Table 6.2** shows a summary of what was possible to learn at the E-Cluster

Table 6.2: Summary of what was made possible to learn at the E-Cluster

Invariant	Varied	Discernment	Enacted object
Relationship	Kinds of relationships	A relationship can be non-function or function	$\boldsymbol{Mt_{re+}}$
Relationship	How many outputs should be mapped onto one input	In a function one input maps onto one output and in a non-function one input maps onto more than one outputs	$\boldsymbol{Mt_{re+}}$
Vertical line	How many times should a vertical line cut a graph	A vertical line cuts the graph of a function once and a non-function more than once	$\boldsymbol{Mt_{re+}}$
Mapping	Type of mapping	A one-to-one and many-to-one mappings are functions and one –to- many is not a function	$\boldsymbol{Mt_{re+}}$
Relationships	type of relationship	There is a mapping when every element in the input set has a partner in the output set and no mapping when some do not have partners.	$\boldsymbol{Mt_{re+}}$
Elements	Types of elements	In a function an element can be any object and does not need to be expressible in numeric or algebraic form.	$\boldsymbol{Mt_{re+}}$

6.6. Summary of the analysis of Task 1 of Functions 1 at the E-Cluster

The analysis of the enactment at the E-Cluster indicates that teachers experienced low scaffolding due to the inductive pedagogic strategy that was adopted there. Analysis of legitimating criteria suggests that the Facilitator and teachers were appealing to mathematics objects to fix meaning for the emerging enacted object of learning. Similarly, analysis with patterns of variation indicates that what was made possoble to learn was a mathematics object and this is evident in Column 4 of **Table 6.1** and **Table 6.2**. When the **Mt** of Column 4 for both Tables 6.1 and 6.2 are put together we notice that ten (10) were $\boldsymbol{Mt_{re+}}$; three (3) were $\boldsymbol{Mt_{ap+}}$ **;** two (2) were $\boldsymbol{Mt_{ap-}}$; and there were no $\boldsymbol{Mt_{re-}}$**.** The presence of appeals to categories of **Mt** other than $\boldsymbol{Mt_{re+}}$ were from teachers. The presence of $\boldsymbol{Mt_{ap-}}$ showed that teachers' subject matter knowledge of Functions was fraught with gaps prior to participation in professional development. The dominance of $\boldsymbol{Mt_{re+}}$**,** which were largely from the Facilitator was a way to redress the gaps particularly when the Facilitator deliberately worked with patterns of variation to open opportunities for teachers to discern $\boldsymbol{Mt_{re+}}$**.** Hence $\boldsymbol{Mt_{re+}}$ was present in larger proportions in **Table 6.2.** Consequently, when the two tables are combined it is noticeable that the primary enacted object of learning (**Mt**) at the E-Cluster was dominated by $\boldsymbol{Mt_{re+}}$.

6.7. Description of Task 1 of Functions 1 at the I-Cluster and N-Cluster

The enactement of Functions 1 was conducted concurrently at the I-Cluster and the N-Cluster. The two clusters experienced Functions 1 a day after it was conducted at the E-Cluster. Following the E-Cluster presentation, Facilitators met to reflect and modify the presentation, particularly the scaffolding of Task 1 at the outset. As a consequence, the enactment at these two clusters while similar to the E-cluster in content, were different in scaffolding.

In this analysis I describe what was similar across the I and N clusters and I use the I-Cluster as illustration. Where there are differences I use illustrations from both clusters. I must mention here that teachers at the N-Cluster did not produce an incorrect list of cards because of the high scaffolding experienced there. However, the emergent enactment of Task 1 was similar in content and sequencing.

- **Task 1-Event 0: Scaffolding Task 1 at I-and N-Clusters**

These two sessions began differently from the E-Cluster. Hence I refer to the this episode as Episode 0. Prior to teachers working on Task 1, facilitators at the I- and N-Clusters put up a slide in which they asked each teacher to write his/her own view of "a function" on a sheet of paper.

What is a function?

- Write your own ideas on the sheet provided
- Definition from Classroom Maths Gr 10:

A function is a special relationship between input and output values where every input value has only one output value

Figure 6.17: Information used to scaffold task

When teachers had written their ideas down the sheets were collected and then the facilitators put up the definition of function from a Grade 10 mathematics book onto the same slide. Facilitators read the definition to teachers and then left the slide showing the definition for the time teachers were working on Task 1. In the following excerpt the facilitator instructs teachers to write down what they thought constituted a function. The introduction of multiple representations together with teachers writing their thoughts about functions and the provision of the Grade 10 definition of function prior to teachers working on the task were ways of scaffolding the task for teachers. **Excerpt 6.15** captures the I-Cluster facilitator instructing teachers to write down their thoughts about functions and also reading aloud the definition of function. **Figure 6.18** shows the multiple representations which were introduced to teachers prior to work on Task 1.

F:	Ok. So before we start our task, on the… on the desk there's another sheet that looks like this. *[F holds up the sheet]* And what we would like to do is take this in at the end of today, after you've filled some of that stuff in. We're going to give it back to you at the end of the theme and check how you're thinking about some stuff at the beginning of the theme and whether it's changed by the end of the theme. *So when you hear the word 'function'… So in other words if someone said to you, 'What is a function?' what are the things that come to mind when you think of a function in a mathematical sense?* Won't you just write some stuff down in that space? Um, and then we'll move on. So I'm going to give you a minute or two. Whatever comes to mind when you hear the word 'function' in relation to mathematics.
	So we're saying a function is a special relationship between an input and an output where every input value has only one output value. Ok? [pauses] Are you happy? Ok. *Every input has only one output value.* So that's the definition that Classroom Maths is working with at grade 10 level.

Excerpt 6.15: Part of the facilitator's scaffold on Task 1

The decision to put up the definition before teachers began work on the task was meant to scaffold the task for teachers. In the E- Cluster the definition of function was introduced at the end of the task to explain why some of the cards they had identified as non functions were actually functions. So by putting the definition upfront in the I- and N-Clusters the facilitators sought to help teachers work from the definition to identify the correct non-functions and functions cards. As will become apparent, putting up the definition before Task 1, that is, this additional scaffolding, did not seem to make Task 1 less challenging. Indeed, more cards that were actually functions were listed as non-functions in the I-Cluster than in the E-Cluster.

In another form of scaffolding, the Facilitator put up a slide showing four types of representation (see **Figure 6.18**) of a linear function and explained that the envelope he was going to issue to each pair contained similar types of representations.

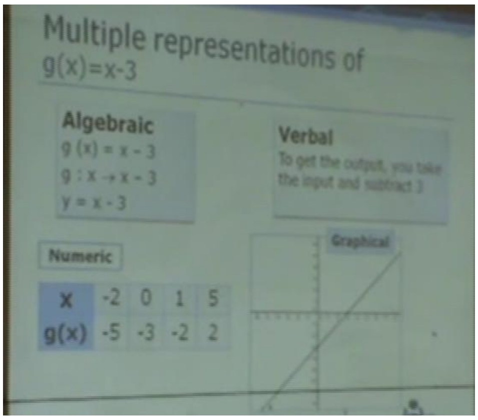


Figure 6.18: Slide showing four representations of a linear function

Following this, another slide containing a set of instructions (see **Figure 6.19**) for Task 1 was put up to mark the beginning of the task. The facilitator pointed that he wanted teachers to sort the cards into one pile of functions and another pile of non-functions. Each pair of teachers was provided with an envelope containing the same 25 cards to sort

Card Sorting: Task 1

- Sort the cards into 2 piles
- Those that represent functions
- Those that DO NOT represent functions

Figure 6. 19: Task 1 instruction

- **Task 1-Event 1: Identifying functions and non functions cards**

Teachers worked on the task for 1 minute before the Facilitator asked them to give him the numbers of the cards on the pile of non-functions (see **Excerpt 6. 16** and **Figure 6.15**)

F	Can you give me the numbers of the cards which you think, which you say are not functions. So if you look at the pile you have which are not functions, what are the numbers on those cards? I'd like to write them down. Let's start over there. The cards which are not functions, can you give me their numbers? Well just give me one. You don't have to give me all of them.
T1	Mine is 24
	[F writes 24 *on a sheet of paper on the easel]* 24. Ok, I'm not going to ask you why yet, I'm just going to put them down.
T2	25
F	25 *[writes* 25 *underneath* 24*]*
	[some teachers are talking amongst themselves]
F	*[gestures towards a teacher]*
T3	8 of them
F	8 of them?
T3	Ja
F	Ok, give it to me. Give me the numbers from those cards?
T3	Those that are not functions?
F	Not functions.
T3	Yes, it's still 8.
F	Ok, so tell me which numbers they are.
T3	I have number 3, number 21, number 12, number 24
F	*[writes down the numbers (underneath each other) as T3 calls them out. F crosses out 21]* I've got that, ja.
T3	T3: Number 14
F	F: 14?
T3	T3: I'm going… Put a star with 14
T4	14? 14 is a function?
T6	18
T3	T3: 9, 10, 4 and I have 16.

Excerpt 6.16: Teachers' list of non-function cards

Facilitator wrote the number of each card on the flipchart. One teacher who listed Card 14 suggested that an asterisk be placed beside the number 14 because he was not certain that it was a non-function. When the Facilitator had completed writing the numbers (see **Figure 6.20**) he called attention to each card.

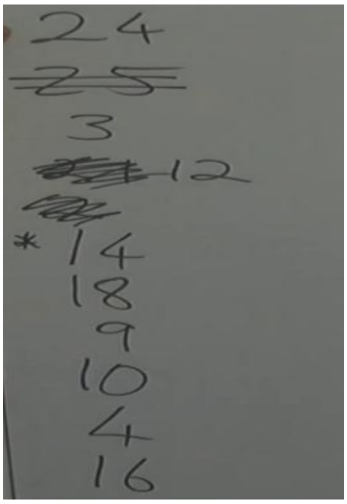


Figure 6.20: Cards listed as non-functions

- **Task 1-Event 2: Discussion focused on Card 24**

First card put up for discussion was Card 24. He asked teachers to explain why it was a non-function? He drew the graph on Card 24 on the flipchart so that the whole class could see and talk about it. Three legitimating criteria emerged in the discussion of card 24. First Teacher 6 said it was not a function because there were two y values for one x value (see **Excerpt 6.17 line 269**). The second was made by the facilitator when he made reference to the definition of function (see **line 278 – 279**). The facilitator called attention to the slide containing the definition of function and read the part that said 'for every input there is only one output value' and having read that he pointed out that Card 24 was not a function.

259 260	F	Alright let's start with 24. Ok, so we have… let's assume it's x and y axis. *[F draws an x and y axis]* Unfortunately I left my transparency stuff at home.
261	T4	I'll buy everybody a coffee or a cup of tea if we're wrong.
262	F	What with this? *[F points to* 24*]* You reckon this is not a function?
264	T4	I'll say it's not.
265		
266	F	Ok. And this table agree? *[F points to a table]*
267	T6	Ja
268	F	You both agree. Ok. Convince us.
269	T6	You have 2 y values for 1 value of x.
270	F	Two y values over here for the same x value? *[F points to the axes he has drawn]*
271	T6	Ja. One input.
272	F	Sorry?
273	T6	One input
274	F	One input
275	T6	Two outputs
276	F	Two outputs
277	T6	Ja

F	Ok. And the rule, the thing we're working with was saying for every input value there's only one output value. So this one fails on that thing that we've got an x value over here but it's linked to two y values, hey? *[F demonstrates this on the graph he has drawn]* And so you know this thing of the vertical line test? You familiar with that?
T6	Ja

Excerpt 6.17: Discussion on why Card 24 was a non-function

The third was also made by the facilitator when he made reference to the vertical line test which he said could be used to show that Card 24 was a non-function (**see line 280 -281**). Facilitator conducted a vertical line test on the graph of Card 24 to show that it was indeed not a function (see **Figure 6.21**).

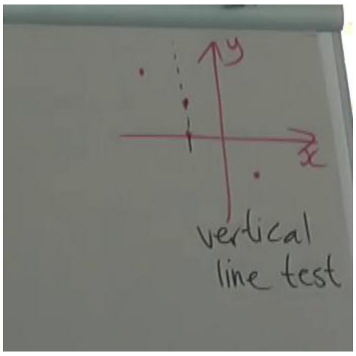


Figure 6.21: Graphical Representation of Card 24

He pointed out that since the vertical line cut the graph in two places it showed that it was not a function. He emphasised that failing the vertical line test did not suggest that Card 24 was not a graph. He said it was still a graph but only that it was not a graph of a function. He then concluded that Card 24 should remain in the list because it was a non-function. The three instances of legitimation associated with card 24 contained appeals to one-to-many mapping, definition of function and vertical line test. As such the three appeals were to *correct mathematical reality.*

- **Task 1-Event 3: Discussion focused on Card 3**

Facilitator then called teachers' attention to Card 3. He read the verbal statement on Card 3 aloud. As a way to scaffold the task, he wrote this algebraically on the flipchart and then drew its graph after one teacher said its graph would be a parabola facing down (see **Figure 6.22**). Facilitator drew the parabola on the flipchart after pointing that it will cut the y axis at positive 1. He also rewrote the function in an algebraic form as shown here; $f(x) = -x^2 + 1$.

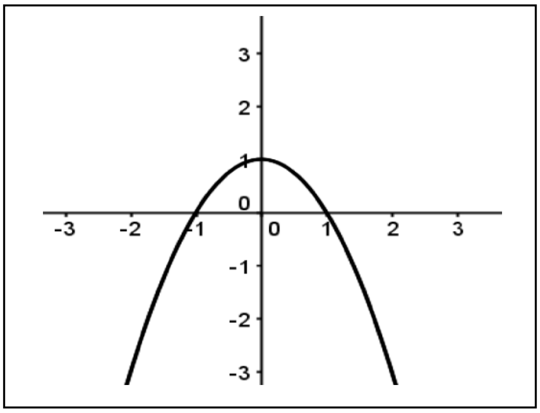


Figure 6.22: Graphical representation of the verbal description on Card 3

Facilitator asked one teacher to explain why Card 3 was not a function. The teacher pointed out that a horizontal line test would show that it is not a function.

328 329	F:	Number 3, we're just not used to seeing it and so it's not as easy. Alright. So you're saying this is the graph. But now tell us why it's not a function.
330	T3:	Right. <u>I used the horizontal test. The horizontal test</u>, ok
331	F:	Which says what?
332	T3:	<u>A function should have one input, one output</u>.
333	F:	Ok
334	T3:	Ja. <u>So if we take our horizontal test we discover that one input</u>…
335	F:	I'm going to just do it there.
336	T3:	Exactly, ja. <u>It will have two inputs versus one output.</u>
337	T8:	2 inputs, 1 output
338	T3:	One output
339	F:	Do you want to come and show us what you mean?

Excerpt 6.17: Dicussion on why card 3 was non-function

In the above excerpt Teacher 3 explains the horizontal line test which he used to realise that Card 3 was a non-function. He said the horizontal line test says that 'a function should have one input and one output' but when applied on the graph it would show that two inputs have one output. Now, Teacher 3's legitimation contained an incorrect appeal to the horizontal line test. This line test is only used when one wants to determine whether a graph was a one-to-one or many-to-one and not for checking if a graph was a function or non-function. So the appeal to the horizontal line test constituted an appeal to *incorrect mathematical reality*.

Following on Teacher 3's legitimation the facilitator drew a horizontal line on the graph to exemplify the teacher's view point (see **Figure 6.23**).

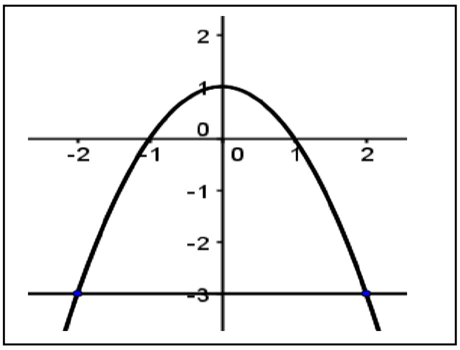


Figure 6.23: A horizontal line test on Card 3

Facilitator asked the teacher to come and explain his viewpoint on the flipchart. The teacher drew two sets showing a many-to-one mapping. In one set were 2 and -2 and were each mapped to -3 in another set (see **Figure 6.24**). In the following excerpt Teacher 3 speak to **Figure 6.24** to explain why card 3 was a non-function.

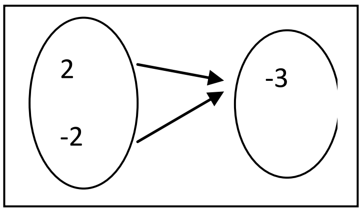


Figure 6.24: A mapping diagram of Card 3

Teacher 3's explanation was based on his interpretation of a part of the definition which said that one input should have one output. For this teacher that part of the definition meant that two inputs could not map onto the same output.

359 360 396 397	T3:	So it's now two inputs versus what? One output. And here we are saying it must be? One input, one output. But my argument is the input you are dealing with it is unique in the sense that this is negative 2 and this is 2.
398	T11:	But they are different inputs
399	T3:	They are different inputs?
400	T11:	Ja
401 402 409	T3:	So we preferably have different outputs. This one should have a different output… This input negative 2 is a unique input which should have a unique output.
410	T4:	Uh huh
411 412	T3:	And this 2 is a unique input which won't have the same as that. Which should have a unique output

Excerpt 6.18: Teacher 3 explaining that two inputs cannot map onto one output

In his explantion Teacher 3 argued that -2 an +2 were not supposed to share -3 since each input has to map onto a different output. Hence he concluded his explanation by pointing out that for him, a many-to-one mapping was not a function.

432	T6:	So are you saying, are you saying we can't have let's say many-to-one…
434	T3:	Ja. Many-to-one
435	T6:	Many-to-one
436	T8:	So it's still a function?
437	T3:	In my case I take it not to be that.
443 444	F:	What we have to do here is accept the mathematical definition that might not, you know, fit with what we want to accept.

Excerpt 6.19: Teacher 3 explaning that many-to-one was a non-function

Facilitator pointed that the teacher had to accept the mathematical definition of function even if it did not meet his expectations. In **Excerpt 6.20** the facilitator explained that it was acceptable for two inputs to have the same output. Following on Teacher 3's explanation, he spoke from **Figure 6.24** to argue that by definition a many-to-one mapping was a function.

457 458 459 460 461 462 463	F:	Ok here we've got two inputs having the same output *[points to the board where T3 has written -2 and 2 both leading to -3]* but it's still only one output each. And by definition that mathematically has been accepted as that's ok for a function. What you're starting to do is subdivide inside this category of functions between one-to-one and many-to-one. And so this *[F points to the -2 and 2 leading to -3]* would be considered a many-to-one function because there are many xs to the same y or many from this set to that set.

Excerpt 6.20: Facilitator's explanation focused on card 3

Facilitator pointed out that the teacher was only pointing at subdivisions of functions. Further in his explanation he cited linear functions with gradient greater or less than zero as examples of one-to-one functions and a linear function ($g(x) = 5$) as an example of a many-to-one function. The following graph of $g(x) = 5$ was used as an example of a many-to-one function (see **Figure 6.25**).

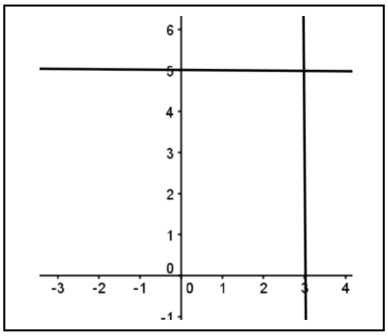


Figure 6.25: Graph of g(x) = 5

He pointed that in the graph of g(x) = 5 all x values mapped onto the same y value and yet that was still a function. Furthermore, he drew the graph of x = 3 (**see Figure 6.25**) and pointed out that on that graph one x value was mapped onto all possible y values and this was not a function. Following the facilitator's explanations it was agreed that Card 3 be removed from the list of non-functions.

- **Task 1- Event 4: Deciding on remaining cards**

The next card to consider was Card 12. This card was immediately removed from the list of non functions because it was the actual graph of Card 3. So agreeing that Card 3 was a function simultaneously made Card 12 a function. Facilitator mentioned that by seeing the definition of function differently might lead to more cards being removed from the list. One teacher pointed that a vertical line test would also be useful to determine whether a graph was a function or not. Card 14 was also removed from the list on similar grounds as Card 12. Facilitator decided that Card 18 be set aside because there were differing interpretations of the statement: 'divide into 2'. Card 9 was a function and so it moved off the list due to the discussion around Card 3 which was also a function. Card 10 was also removed because it was a parabola and this prompted the facilitator to point that teachers have noticed that a parabola is a function (**see Excerpt 6.21**)

537 539 542 543 552	F:	Number 10? We've all got number 10? If you change it to something we're more familiar with it starts… it's better. y equals? *[F writes on the board]* Is this a function or isn't it?
553	Trs:	Ja, it is a function
554 555	F:	Because we recognise that it's a parabola, right. And straight away we go, based on our definition, parabolas are functions.

Excerpt 6.21: Discussion focused on card 10

Card 4 was the equation for Card 3 which was a verbal statement. This card was removed because it was a function. Card 16 was removed after the facilitator wrote it algebraically to produce a linear function and teachers pointing out that a linear function was a function (see **Excerpt 6.22**)

563	F:	And number 16?
566	T3:	It's a linear function.
567	F:	A linear function?
568	T3:	Yes, a linear function
576	T20:	It's a function
577	F:	It's a function and we recognise it as a linear function

Excerpt 6.22: Discussion focused on card 16

Converting the verbal representation of card 16 to algebraic representation scaffolded card 16 and so helped teachers recognise that it was a function

- **Task 1 - Event 5: Focus on Card 7**

When all cards were discussed, the facilitator pointed that he was surprised that Card 7 was not listed among non-functions (see **Figure 6.26**). In spite of it not being in the list, Facilitator pointed that it met the definition of a function because each input was mapped onto one output. He said it did not matter what elements were in each set. Facilitator mentioned that in school teachers tend to work largely with inputs and outputs that are numbers.

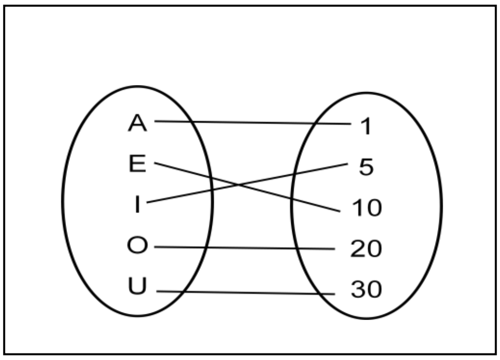


Figure 6.26: Representation of Card 7

- **Task 1-Event 6: Unpacking the definition of function**

He put up a slide containing the modern definition of function (see **Figure 6.27**). The slide also explained what a mapping is and that a one to one and many to one mapping were functions. He introduced the modern definition of function and went on to unpack its meaning.

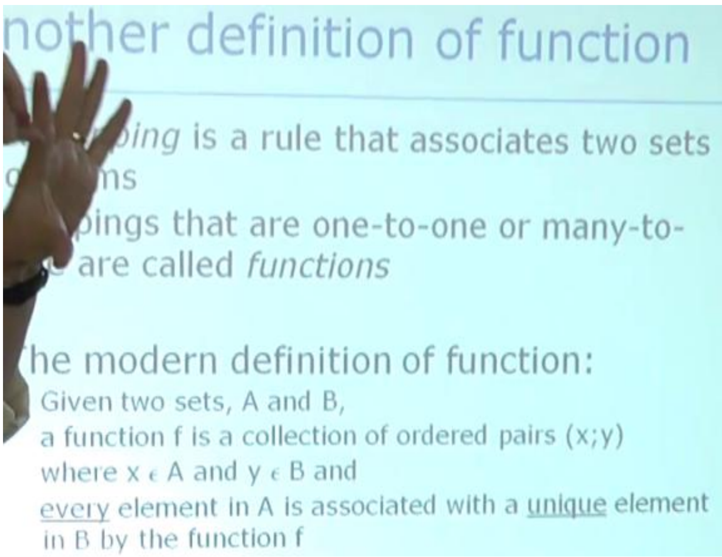


Figure 6.27: Slide showing three definitions of a function

Here he explained that unique as used in the definition does not mean one to one mapping but that each element in one set has one partner in another set. He said this did not mean that two elements in Set A cannot go to the same element in Set B. He said the definition does not specify what should go into the sets. He elaborated this by saying that Set A can have body parts and Set B can have shapes. Following this Facilitator put up a slide containing summary points (see **Figure 6.28**)

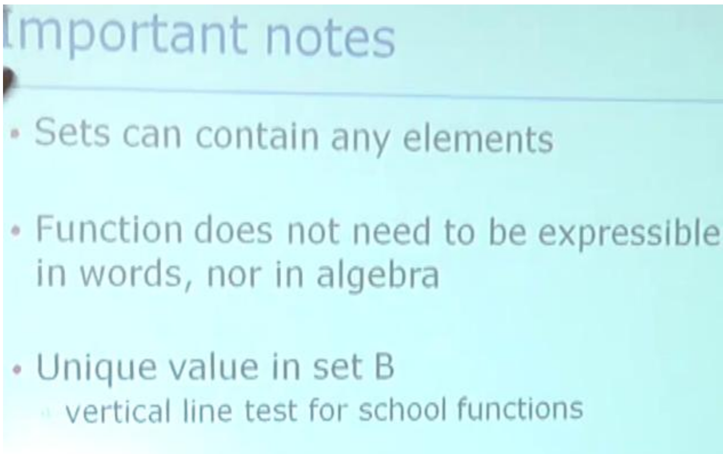


Figure 6.28: Slide used to summarise the task

6.8. Discussion of the pedagogic strategies used at the I-cluster and N-Cluster

The pedagogic strategy used at the I-Cluster and N-Cluster was a consequence of the Facilitators' reflection on the outcomes of Functions 1 at the E-Cluster. Here the Facilitators adopted the deductive pedagogic strategy which opened the scaffolding. While the task was moderately scaffolded at the I-Cluster, it was highly scaffolded at the N-Cluster (**see Excerpt 6.23**). At both clusters the first scaffold was when teachers were asked to write down their thoughts and ideas about what constituted a function. This move created opportunities for teachers to recall their knowledge of functions. The second scaffolding was done by familiarising teachers with unfamiliar representations of function. To do this, the Facilitator had a slide containing the four multiple representations of the same linear function. This move made it possible for teachers to familiarise and extend their knowledge of multiple representations. In the third scaffold, the Facilitator introduced and explained key words in the definition of function.

In spite of this amount of scaffolding the teachers at the I-Cluster incorrectly listed functions as non-functions. It was surprising that the deductive pedagogic strategy at the I-Cluster and its consequent scaffold did not produce a qualitatively different list of non-functions. As a matter of fact, I-Cluster list was longer than the E-Cluster list. This scenario raised the question "Which resources or help were teachers working with on the task since they did not appear to have drawn from the availed scaffold"? On the basis of their incorrect responses, It is probable that the teachers drew from an implicit resource and that resource was incorrect and was probably, their prior knowledge of functions.

The teachers at the N-Cluster experienced a high scaffolding after they were introduced to multiple representations and definition of function before working on the task. The facilitator took teachers through the definition of function by reading through it, unpacking its meaning and making it available on the slide as teachers were working on the task.

412 413 414 415 416 417	F:	You've got a lot of cards in front of you, some of them are empty. *They are different representations, they're equations, formulae, there are words, there's verbal, algebraic, diagrammatical, graphical and tabular. Look at these carefully, think about that definition, that for every x there is only one y*. Are any of the cards in front of you not a function? Have you found one?
418	T13:	Yes
441 442 446 447 448 449 450 454 460 465	F:	*And I'm going to say that now, I'm quite hesi… There's one card that is not a function, can you identify it?* *Ok, remember, we are working with the definition 'for every x value' in other words if we look across the X axis, for every X axis we should only be available to find one Y. If you look at the x values in the table, for each X value there should only be one Y. So if you've got naught in the, or zero in the… X's in the table and you see that 0 maps to more than value, it is not a function*. *So every X does it map onto one Y*? Ok, so I think most people have found… how are you doing? Have you found it? *[goes to group 2]* That's it. Right, so I think most people have identified that it is card number 24.

Excerpt 6.23: Facilitator providing scaffold on Task 1

The facilitator constantly reminded teachers to work from the definition of function and when progress was slow, moved from one group to another repeatedly stating that: "one x goes to one y" This statement was a simplified version of the definition which was present on the slide. Facilitator had noticed that inspite of the scaffolding already done the teachers were making slow progress. The facilitator was concerned that the time taken on Task 1 was likely to impact on the time allocated to Task 2. It suffices to point out that teachers in the N-Cluster worked on Task 1 which had been greatly scaffolded. For instance, the task required teachers to sort 24 cards into two piles of functions and non-functions. At the beginning the Facilitator asked teachers to find cards that were not functions and by so saying facilitator made the task easier because teachers did not have to produce two piles but one pile. Later on when facilitator noticed that less progress was made, the Facilitaotar made further scafolding by revealing to teachers that out of the 24 cards only one card was not a function (see **Excerpt 6.23**).

One would have expected teachers to immediately complete the task but this did not happen as they continued working on it trying to identify the card that was not a function. While teachers were working on this task the facilitator kept reminding them the definition of function. When the amount of scaffolding that went in to help teachers progress through Task 1 and the amount of time they took to complete the task are considered, it might be concluded that perhaps teachers were not working with the provided scaffoldresources but with their own prior knowledge of functions which was fraught with gaps. As a consequence of high scaffolding teachers at the N-Cluster did not produce an incorrect list but took long to identify Card 24 – the only card which was a non-function.

6.9. Discussion of legitimating criteria in Task at the I-Cluster and N-Cluster

Work on Task 1 produced nine Cards that were listed as non-functions at the I-Cluster while no cards were produced at the N-Cluster due to high sacffolding. As such at the I-Cluster legitimating criteria emerged when the Facilitator and teachers had to talk about the nine cards. At the N-Cluster the legitimating criteria emerged when the Facilitator went about scaffolding the task as teachers were working on it. As such all legitimating criteria from the N-Cluster were from the Facilitator.

At the I-Cluster, among the nine cards, eight were incorrectly identified as functions. The teachers were asked to explain, beginning with Card 24, why each of the listed cards were not functions. The teachers' explanations carried the legitimating criteria which was characterised by appeals to mathematics objects. There were two legitimating criteria advanced by teachers. The first legitimating criteria were that <u>Card 24 was not a function because there were two y values for one x value</u>. The appeal was to the <u>definition of function</u>. The second legitimating criteria <u>was that Card 24 had one input and two outputs</u>. The appeal associated with this criterion is an appeal to <u>mapping of non-function</u> (one to many). The third legitimating criterion was present when attention was on Card 3. Here a <u>teacher said Card 3 was not a function because it will fail the horizontal line test</u>. The appeal associated with this criterion was to the <u>horizontal line test,</u> It is discernible here that the recruited criterion was incorrect because the horizontal line test is recruited to show whether a function is one to one or many to one. In this case, the vertical line test would have been appropriate. The fourth legitimating criteria appeared when a teacher was asked to state the horizontal test. He said that a function should have one input mapped onto one output <u>but the horizontal test would show that in Card 3, two inputs mapped onto one output</u>. The teacher was appealing to the <u>horizontal line test</u>. However his appeal was betrothed with misconceptions.

The teacher's argument that two inputs cannot map onto the same output revealed a gap in his prior knowledge of functions and this takes the form of a misconception. For him, a function is one where there is a one to one mapping. This is evident when he said that he did not consider a many to one mapping as constituting a function. The fifth legitimating criteria came into being when the <u>Facilitator pointed that many to one was in fact a function</u> by <u>definition of function</u>. The Facilitator appealed to the <u>definition of function</u> to attend to the teacher's misconception. The sixth legitimating criteria was present when the <u>Facilitator used g(x) = 5 to exemplify a many to one function and a vertical line graph to exemplify a one-to-many function</u>. The appeal here was to <u>contrast</u>. The seventh legitimating criteria appeared when a teacher pointed that a vertical line test can be used to determine if Card 12 was a function or not.

Card 12 was removed from the list and the reason was that since the class were in agreement that Card 3 was a function that made Card 12 a function because it was a graphical representation of Card 3. This legitimating criterion carried an appeal to multiple representations of the same function. The eighth legitimating criteria was present when attention was on Cards 9, 10 and 4. These were representations of quadratic functions and it had earlier been agreed that since Card 3 was a parabola and the class had agree that it was a function then all parabola constituted functions. In these criteria the appeal was to parabola. The ninth legitimating criteria came into being when Card 16 was removed from the list because teachers recognised that it was a linear graph. The appeal associated with this criterion was to teachers' knowledge of linear graphs. The tenth legitimating criteria were present when Card 7 was in focus. The Facilitator had to explain that this card represented a function because each input was mapped onto one output. The appeal here was to mapping.

More legitimating criteria emerged when the focus shifted to a slide with three definitions. Providing explanations of the three definitions was scaffolding at the end of the task which was meant to close the gaps tin teachers' conceptions of the definition of function. To explain that one to one and many to one mapping were functions he appealed to the meaning of a mapping. He used the modern definition of function to unpack key words in the definition of function. He explained the meaning of the word every as used in 'every element' and unique as used in 'unique element'. Every element meant that all elements in the input set must each have a partner in the output set but not all elements in the output set need to have a partner. Unique element did not suggest one to one mapping but that each input is not mapped onto two or more outputs. To explain these keys words the Facilitator was appealing to the definition of function. There is another appeal to definition of function when the Facilitator explained that the definition does not specify what can be used to represent a function. He said in a Function an element can be anything one chooses and he gave flowers, body parts, and colours as examples. He pointed that a Function does not need to have an algebriaic representation to be a Function. So here the appeal was to definition of Function.

Table 6.3: Summary of legitimating criteria in Task 1 at the I-Cluster

Legitimating criteria	Appeal	Enacted object
Card 24 was not a function because there were two y values for one x value	definition of function	Mt_{re+}
Card 24 had one input and two outputs	mapping of non-function	Mt_{re+}
Card 3 was not a function because it will fail the horizontal line test	Horizontal line test	Mt_{re-}
but the horizontal test would show that in Card 3, two inputs mapped onto one output.	Horizontal line test	Mt_{re-}
Facilitator pointed that many to one was in fact a function by definition of function	Defintion of function	Mt_{re+}
Facilitator used g(x) = 5 to exemplify a many to one function and a vertical line graph to exemplify a one-to-many function	Discernment of contrast	Mt_{re+}
A teacher pointed that a vertical line test can be used to determine if Card 12 was a function or not	Vertical line test	Mt_{re+}
Card 12 was a function because it was a graphical representation of Card 3	Multiple representation	Mt_{re+}
since Card 3 was a parabola and the class had agree that aparabola was was a function then all parabola constituted functions	Parabola	Mt_{re+}
Card 16 was removed from the list because teachers recognised that it was a linear graph	Linear graphs	Mt_{ap+}
Card 7 was a Function because each input was mapped onto one output	Mapping (one-to-one)	Mt_{re+}
Every element meant that all elements in the input set must each have a partner in the output set but not all elements in the output set need to have a partner	Definition of Function (Meaning of the expression every element)	Mt_{re+}
Unique element did not suggest one to one mapping but that each input is not mapped onto two or more outputs	Definition of Function (Meaning of the term unique as used in the defintion of function)	Mt_{re+}
In a Function an element can be anything one chooses	Formal definiton of function	Mt_{re+}

At the N-C;uster, the first appeal made by the Facilitator at the N-Cluster was to definition of function and this was present on the slide (see **Figure 6.21**) and in a simplified form where the Facilitator kept saying that 'for every one x there must be one y". A second appeal was also to a definition of function and was present in the Facilitator's explanation of what constitutes a non-function. She explained that if zero was an x value and had two y values then that would not be a function. A third appeal was to definition of function and was evident when the Facilitator explained that Card 24 was not a function because one x value was associated with more than one y". In the fourth appeal the Facilitator pointed that learners might identify Card 24 as not function because it had dots and the appeal was to graphical representation. She articulated her fifth appeal after a teacher had pointed that a graph was easier to identify than the other representations. The Facilitator added to this by saying this was so because a vertical line test can be used to determine if a graph represented a function. Implying that it would cut a graph of a Function only once. This appeal was to the vertical line test. The sixth appeal was evident in the Facilitator's legitimating criteria which she used to fix the meaning of her explanation when she explained that Card 24 was not a function because one value of x was associated with more than one y values. Here the appeal was to definition of function. Her seventh appeal was to the vertical line test and this was present when she mentioned that the graphical representation was easier to identify because it was possible to apply the vertical line test on it.

The eigth appeal emerged during the explanation after she asked if a one to many was a function. The response was that it was a function and she then went on to point that one to many meant that one x value mapped onto many y values. Here the appeal was to a mapping. The nineth appeal was also to mapping and was evident when she mentioned that even many to many mapping was not a function. The last appeal was made when the focus was on Card 7. The facilitator was surprised that no one had identified this card as representing a non-function because it was unfimiliar to teachers. In spite of this, she pointed that it was a function because it was a one to one mapping and here the appeal was to a mapping.

A summary of the appeals is presented in **Table 6.4**. It was not surprising that all appeals were to mathematical reality since all were made by the Facilitator. This occurance supports the conclusion I made earlier pertaining to my analysis of scaffolding at this cluster when I found that the cluster experienced high scaffolding. The only appearace to mathematical appeareance came as an illustration from the Facilitator when she pointed that learners might say Card 24 was not a Function because it contained dots.

Table 6.4: A summary of legitimating criteria in Task 1 at the N-Cluster

Legitimating criteria	Appeal	Enacted object
for every one x there must be one y	Definition of function	$\boldsymbol{Mt_{re+}}$
if zero was an x value and had two y values then that would not be a function	Definition of function	$\boldsymbol{Mt_{re+}}$
because one x value was associated with more than one y	definition of function	$\boldsymbol{Mt_{re+}}$
Card 24 is not a function because it had dots	Graphical representation	$\boldsymbol{Mt_{ap-}}$
Because a vertical line test can be used to determine if a graph represented a function.	vertical line test	$\boldsymbol{Mt_{re+}}$
because one value of x was associated with more than one y values	Definition of function	$\boldsymbol{Mt_{re+}}$
because it was possible to apply the vertical line test on it	Mapping	$\boldsymbol{Mt_{re+}}$
that one to many meant that one x value mapped onto many y values	Mapping	$\boldsymbol{Mt_{re+}}$
even many to many mapping was not a function	Mapping	$\boldsymbol{Mt_{re+}}$
because it was a one to one mapping	Mapping	$\boldsymbol{Mt_{re+}}$

6.10. Discussion of what was made possible to learn at I- and N-Clusters

To understand what was made possible to learn I recruited patterns of variation which I used to visibilise the enacted object of learning. The first pattern of variation was embedded in the scaffolding where the discussion focused on the four representation of the same linear function. There, as I described in the description section, teachers had opportunities to discern similarities between the representations and in variation theory terms, this opened opportunities for generalisation-a pattern of variation where awareness is brought about by experiencing similarity between the critical features. As such, in this instance, there were opportunities to discern the same properties of functions across the four representations. By way of example, the multiple representations of a linear function were foregrounded in the scaffold and so opportunities to discern the same gradient, and intercepts in the four representations were opened up. The pattern of variation experienced here was generalisation because the focus was not on the discernment of difference but similarity between the critical features.

As the critical features (multiple representations of the linear function) were varied to show similarity, the dimension of variation (function) remained invariant. *Consequently, teachers were helped to notice that a representation is not a property of a linear function but the gradient and intercepts were*. This was further emphasised when the Facilitator explained at the end of Task 1, that a function can be anything, it does not need to be expressible in algebraic form, graph nor is it necessary for a function to have a known relationship. He concluded by saying that in a function relationship, one set might have body parts when the other set has names of flowers. He pointed to Card 7 to illuminate the essence of a function relationship.

A second pattern of variation (*contrast*) emerged from working on Task 1. There, teachers sorted representations of functions and non-functions into two piles. Sorting representations according to whether they were functions or not, made it possible for teachers to simultaneously work with and experience difference between functions and non-functions (a necessary condition for discernment of difference).

Experiencing difference is associated with *contrast*- awareness brought about by experiencing difference between two or more critical features. *As such sorting cards opened up opportunities for discernment of difference between function and non-function relationships.*

A third pattern of variation was present and explicit when the Facilitator worked with the teachers' incorrect productions where he explained with demonstrations, why most of the cards were incorrectly listed as non-functions. This was evident when Card 3 was in the fore. Card 3, 4, and 12 were multiple representations of the same quadratic function and were all listed as non-functions. So, discussions around Card 3 pointed to a misconception and thus a gap in teachers' knowledge of functions and quadratic functions in particular. It is clear that the listing of the cards suggested that teachers did not know that a quadratic function is a function. By way of example, Card 4 consisted of a quadratic algebraic representation but teachers could not discern the highest degree of the equation and even when the quadratic function was presented in graphical form as shown on Card 12 they also identified it as a non-function.

To help teachers discern that Card 3 was a function the Facilitator introduced contrast. To put things into perspective, the Facilitator was responding to an earlier explanation made by one teacher who concluded that for him, a many-to-one was not a function. This is evident in the excerpt where a teacher was explaining and demonstrating why Card 3 was a non-function (see Excerpt). Facilitator chose mapping as a dimension of variation and the types of mapping (one-to-one; many-to-one; and one-to-many) were three critical features he foregrounded to attend to the teachers' misconceptions. Here two patterns of variation in focus were ***generalisation*** and ***contrast***. Generalisation was in the fore when the focus was on mappings of a function (i.e. one to one and many to one) and contrast was in focus when many to one and one to many mappings were in the fore.

In terms of **generalisation**, the Facilitator explained verbally that a one to one and many to one mapping were functions. He used linear functions with gradients greater and less than zero as examples of one to one functions and a linear function with gradient equal to zero as an example of a many to one function. What makes the graphs of these mappings similar is that the vertical line would cut each graph at one place.

In terms of contrast, he explained the many-to-one and one-to-many diagrammatically to differentiate between a mapping of a function and of a non-function. The many to one function and one to many functions were simultaneously foregrounded on the same Cartesian plane (see **Figure 5.20**) to show the line f(x) = 5 (a many-to-one mapping) and x = 3 (a one to many mapping). What is different about the graphs of these mappings is that the vertical line would cut one at one place and the other at more than one place. Table shows what was made possible to learn from the pattern of variation.

It is interesting to note that understanding that many to one was also a mapping of a function was indeed a critical feature for teachers because as soon as this misconception was attended to, they were able to discern that the rest of the cards they had listed as non-functions were actually functions.

Table 6.5: Summary of what was made possible to learn in Task 1 at the I-Cluster

Invariant	**Varies**	**Discerned**	**Enacted object of learning**
Properties of a linear function	Multiple representations	Gradient and intercepts are properties of a linear function but multiple representations are not	$\boldsymbol{Mt_{re+}}$
Relationship	Type of relationship	A Function relationship has every input mapped onto a unique output. A non-function relationship has one input mapped to more than one output	$\boldsymbol{Mt_{re+}}$
Mapping	Types of mapping	Many to one are functions. One to many mapping are not functions	$\boldsymbol{Mt_{re+}}$

Table 6.6: Summary of what was possible to discern from Task 1 in the N-Cluster

Invariant	Varies	Discernment	Enacted object
Relationship	Type of relationship (function vs non-function)	In a functional relationship one x value maps onto one y value In a non functional relationship one x value maps onto more than one y value	$\mathbf{Mt_{re+}}$
Mapping	Type of mapping	One to one and many to one mappings represent functions One to many and many to many represent non-functions	$\mathbf{Mt_{re+}}$
Vertical line test	Number of times vertical line cuts a graph	Vertical line cuts a graph of a function once and of a non function more than once	$\mathbf{Mt_{re+}}$
Relationship	Type of relationship (function vs non-function)	In a functional relationship one x value maps onto one y value In a non functional relationship one x value maps onto more than one y value	$\mathbf{Mt_{re+}}$
Gradient and Intercepts	Multiple representations of the same linear function	Gradient and intercepts are properties of a linear function but the multiple representations are not its properties	$\mathbf{Mt_{re+}}$
True meaning of the definition	Words used in the definition	The essence of the definition of a function is not affected by variations in words used.	$\mathbf{Mt_{re+}}$

6.11. Summary of the analysis of Task 1 at the I-Cluster and N-Cluster

The analysis revealed that teachers at the I-Cluster experienced *moderate* scaffolding while the N-Cluster experienced high scaffolding following the adoption of the *deductive pedagogic strategy*.

Column 4 of **Tables 6.3 and 6.4** shows that the I-Cluster appeals were constituted by $\boldsymbol{Mt_{re+}}$ **,** $\boldsymbol{Mt_{re-}}$ and $\boldsymbol{Mt_{ap+}}$ while the N-Cluster appeals were largely $\boldsymbol{Mt_{re+}}$. The legitimating criteria were dominated by appeals to *mathematics reality* **(**$\boldsymbol{Mt_{re+}}$**)** across the two clusters.

Similarly, Column 4 of **Table 6.5** and **6.6** shows that patterns of variation openned up the space of learning which was constituted by *mathematical reality* $\boldsymbol{Mt_{re+}}$. Therefore the enacted object of learning in Task 1 at the two cluster was largely $\boldsymbol{Mt_{re+}}$. Hence teacher learning in these clusters was situated in the learning of *mathematical reality*. The $\boldsymbol{Mt_{re-}}$ was essentially, evidence of gaps in teachers' prior knowledge of Functions.

6.12. Conclusion

In the above analysis two pedagogic strategies emerged empirically from the data and these were evident where Facilitators were scaffolding tasks. Analysis of the pedagogic strategies revealed three levels of scaffolding where the E-Cluster, I-Cluster and the N-Cluster, respectively experienced low, moderate and high scafolding. Analysis with tools from Adler and Davis' methodology revealed that the legitimating criteria were in every respect, dominated by appeals to *mathematical reality* ($\boldsymbol{Mt_{re+}}$**).** In addition to this, the analysis done with patterns of variation revealed that matheamtical reality was dominant. While teachers' prior knowledge of Functions was constituted by $\boldsymbol{Mt_{ap+}}$, $\boldsymbol{Mt_{re-}}$ and $\boldsymbol{Mt_{ap-}}$**,** Facilitators were consistent in visibilizing th**e** $\boldsymbol{Mt_{re+}}$**.** Hence across the three clusters the enacted object of learning was dominated by *mathematics reality* ($\boldsymbol{Mt_{re+}}$)**.**

CHAPTER 7: ANALYSIS OF TASK 2 OF FUNCTIONS 1

7.1. Introduction

In this chapter I present the analysis of video data and the focus of my analysis was on the co-constitution of the enacted object of learning by Facilitators and teachers in the sessions. The chapter is focused on task 2 of Functions1. In Task 2 teachers worked with the same multiple representations as before but here the focus was on matching multiple representations of the same function. The pattern of analysis done here is the same as that of **Chapter 6**. Text containing evidence of scaffold is highlighted, legitimating criteria is underlined and patterns of variation is bold.

7.2. Description of Functions 1 focused on Task 2 at E-Cluster

- **Task 2-Event 0: Unpacking of Task 2 instructions**

At the beginning of Task 2, the facilitator asked teachers to remove Card 24 which was not a function, from the rest of the cards. He then put up a slide bearing task instructions detailing what teachers were expected to do (see **Figure 7.1**).

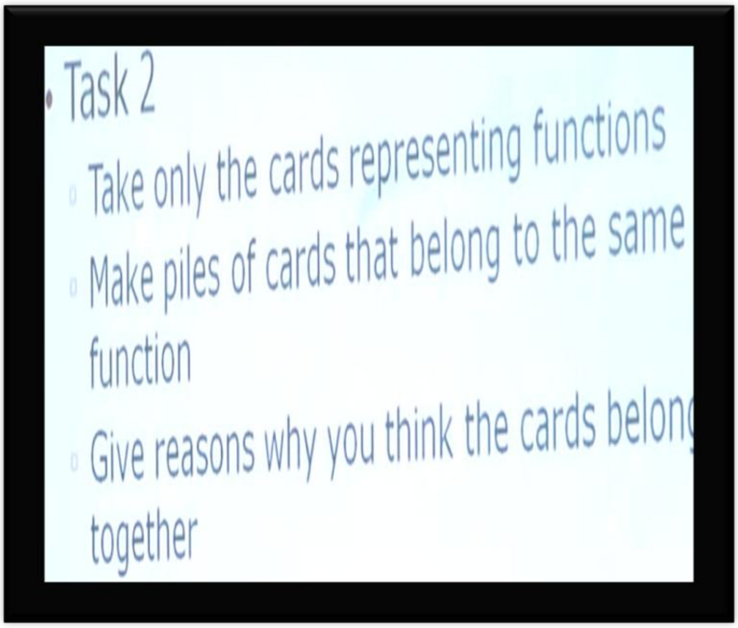


Figure 7.1: Task 2 Instructions

The task required teachers to sort cards into representations of the same function and teachers were also required to give reasons why they thought the cards belonged together. When Card 24 was removed, there were 23 cards bearing multiple representations of functions in the form of linear, quadratic, hyperbola and mapping. In **Excerpt 7.1** contain the facilitator's utterances in which he was unpacking the task instruction.

F:	Ok, so we are taking only the cards representing functions. You might not be sure about 19. You can decide whether you want it in or out. I want you to make piles of cards for cards that belong to the same function. And obviously then be able to give reasons. So when I'm saying the same function. I'm not saying these are all straight lines or these are all parabolas – it would be a specific parabola or specific straight line that would be in the pile ok. So if it were… If it's a different equation then it's not the same function. If it's a different graph then it can't be the same function. Ok. So can you put them in piles of or in rows or whatever the ones that you think belong together.

Excerpt 7.1: Facilitator unpacking Task 2 instructions

Teachers were also asked to insert the blank cards which they had put aside in Task 1. This task required teachers to write missing representations on blank cards. That is, in situations where there were only three representations teachers were to use the blank card to write down the missing representation.

- **Task 2-Event 1: First list of matching cards – 3, 4, 12 and 23**

Facilitator started the task by pointing out to teachers that in Task 1 they found that Card 3 and 4 represented the same quadratic function. In so doing the facilitator was helping teachers and this amounted to scaffold. He asked if there were any more cards that belonged with cards 3 and 4 and so teachers added cards 12 and 23 to complete the group (see **Excerpt 7.2**. and **Figure 7.2**).

F	Ag, on the flipchart. Based on our cards, we said that 3 and 4 belong together, not so?
T42	Yes
F	So do you have others that also belong together with 3 and 4? *[on the overhead:* *4.* *3.* *f(x) = 1-*x^2 *To get f(x) you square x and subtract it from 1]*
T43	12
F	You say 12? [F puts a graph on the overhead] I'm sorry, the text is really small. So those 3 belong together. Did you get anything else that belongs together? So it's 3, 4, 12. So I'm just going to call it group 1: 3, 4, 12. Did anybody get anything else?
T44	23
F	And 23. *[on the flipchart writes: (1) 3, 4, 12, 23]* So 23 is a table. Ok, let's just check quickly if it fits the rules that… *[F puts up an overhead with card 23]*

- **Excerpt 7.2: Discussion focussed on the first list of matching cards**

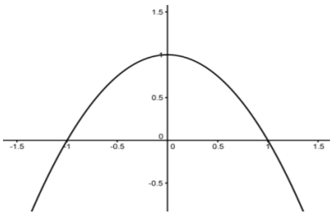
3

To get f (x) you square x and subtract it from 1

4

$f(x) = 1 - x^2$

12



23 x	**f(x)**
-2	-3
0	1
1	0
12	-143

Figure 7.2: Multiple representations of the same quadratic function

The number of each card was written down on the flipchart and the cards were displayed on the overheard projector (see **Figure 7.2**). The first complete list of matching cards involved cards 3, 4, 12 and 23 which were representations of the same quadratic function.

- **Task 2-Event 2: Second list of matching cards – 1, 5, 11 and 16**

The second complete list consisted of 1,5, 11 and 16 which were representations of the same linear function (see **Figure 7.3**).

<table>
<tr>
<td>1

y = – 2x + 1</td>
<td>5

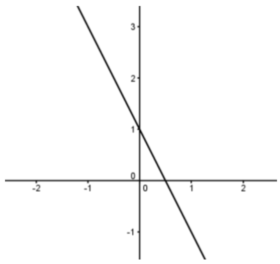
<table>
<tr><td>x</td><td>-3</td><td>0</td><td>1</td><td>5</td></tr>
<tr><td>y</td><td>7</td><td>1</td><td>-1</td><td>-9</td></tr>
</table>
</td>
</tr>
<tr>
<td>16
To get y you double x, multiply by -1 and then add 1</td>
<td>11
</td>
</tr>
</table>

Figure 7.3: Four cards that represent the same linear function

The second list was generated by teachers without scaffold from the facilitator. This is evident in **Excerpt 7.3** in which the second list was generated.

1045	F:	Give the ones that you guys are just doing? *[points to some teachers*]
1047	T45:	5, 11, 1, 16
1049	F:	Let's just write 1, 5, 11… 1, 5, 11. And 16?
1051	T45:	16. Yes.
1053	F:	Did you also get that? *[next to (2) on the flipchart writes: 1, 5, 11, 16]*
1055	T46:	Ja
1057	F:	Ok, 1, 5, 11, 16

Excerpt 7.3: Discussion focused on the second list of matching cards

A list involving cards 1, 5, 11 1nd 16 were accepted as correct and so were written down on the list of matching cards.

- **Task 2-Event 3: Third list of matching cards – 18, 19 and 20**

Facilitator asked for the third list of cards and cards 18 and 20 were provided (see **Excerpt 7.4**). Facilitator then called teachers' attention to the front as he was displaying cards 18 and 20 on the overheard projector. At the same time teachers said card 19 also belonged with 18 and 20. The cards were projected onto the whiteboard and the card numbers were written down on the flipchart. An instance of scaffold emerged when the facilitator converted the 18 and 19 from verbal to algebraic representations to make it possible for teachers to see how each was the same or different from card 20.

1069 1070 1072	F	Anyone got another one? It might be incomplete, that's fine. Let's just see what, what even partial groups we've got so far. What have you got?
1074	T47	T47: 18 and 20.
1076	F	F: 18 and 20?
1078	T47	T47: Ja
1080	F	Ok. Can we all take a look at those two together? 18 and 20.
1082		*[on the flipchart F writes:* (3) 18, 20 *]*
1084	Trs	And 19 *[some teachers responding in chorus]*
1086	F	F: And 19?
1088	Trs	Yes *[some teachers responding in a chorus]*
1090	T48	18, 19, 20

- **Excerpt 7.4: Discussion focused on the third list of matching cards**

The facilitator called teachers' attention to the three listed cards and began by reading the verbal representations of card 18 and card 19. Thereafter, the facilitator converted card 19 from verbal to algebraic representation (see **line 1105 – 1108 of Excerpt 7.5** below).

1105 1106 1107 1108 1111 1112 1114 1115	F:	Ok, let's take a look. *[reads the slides]* To get the input you take… Ag, to get the output you take the input and divide it into 2. Halve the input to get the output. And $y = \frac{x}{2}$ So halve the input to get the output would be taking the input, dividing it by 2. *[writes on the flipchart:* $\frac{x}{2}$ *]* Right? Which would be the same as saying a half x [*writes:* ½ x*] Ok*. So we would then say 19 and 20 belong together, ja. But what about 18?
1117	Trs:	*[some teachers discuss card 18 amongst themselves]*
1121 1122	F:	To get the output, you take the input… So that's taking x. How do you divide x into 2?
1124	T49:	x over 2
1126	F:	*[writes T49's response on the flipchart:* $\frac{x}{2}$ *]*
1129	T50:	2 over x
1131 1132	F:	I mean clearly the other option is 2 over x, right? [*Writes T50's response,* $\frac{2}{x}$ *on flipchart*]. So I mean we're 50:50 here. Is it x over 2 or is it 2 over x?
1178	F:	But I think what we are clear on is 19 and 20 definitely go together

Excerpt 7.5: Further discussion focused on the third list of matching cards

Following on this conversion, it was concluded that card 19 and 20 belong to the same list of matching cards (see **lines 1114-1115** of **Excerpt 7.5** above). The next discussion focused on converting card 18 from verbal to algebraic. Two different versions of the algebraic representation were generated by Teachers 49 and Teacher 50. Hence card 18 was removed from list because its wording was ambiguous. As such the third list comprised of card 19 and 20 as reflected in F**igure 7.4**.

While the conversion of card 18 into an algebraic representation for teachers to notice that it was different from card 20 opened an opportunity for discernment of contrast, the conversion of card 19 opened an opportunity for discernment of generalisation. It was through awareness of difference between cards 18 and 20 that teachers agreed that card 18 be removed from the list. Similarly, it was through discernment of similarity between card 19 and 20 that card 19 was retained.

19	20
Halve the input to get the output	$y = \frac{\mathbf{x}}{\mathbf{2}}$

Figure 7.4: Two cards bearing representations of the same linear function

The highlighted utterance in **Excerpt 7.4** and **7.5** show two instances of scaffold.

- **Task 2-Event 4: Fourth list of matching cards – 15 and 17**

Facilitator asked for the fourth list of cards and one teacher gave an incorrect match involving cards 10, 17 and 15. The three cards were representations of three different quadratic functions (see **Figures 7.5 and 7.6**)

17	**15**
$\mathbf{y = -2x^2 - 2}$	$\mathbf{y = -2(1 - x)^2}$

Figure 7.5: Cards 17 and 15 incorrectly matched

On reflection, a teacher noticed that 10 and 17 were not representing the same function. Facilitator put these up on the overhead project, together with card 15 to show why they did not belong together. In the discussion one teacher explained that cards 17 and 15 were different because in 17 the coefficient of x was zero while in 15 it was 4 (see **Excerpt 7.5**)

	F:	Convince me these aren't the same?
	T56:	Convince you?
	F:	Ja
	T14:	Because the co-efficient of x is 0 in card 15.
	F:	*[points to card 15 on the overhead]* The co-efficient here is zero so it would be: In card 15, we've got f of x equals negative 2x squared minus 2. Is that right? *[on the flipchart writes:* $f(x) = -2x^2 - 2$ *]*
		And in 17 we'd have y equals *[writes:* $y = -2x^2 + 4x - 2$ *]*
	T14:	minus 2x squared… negative 2x squared plus 4x subtract 2
	F:	We've got to multiply that bracket out first though and then multiply, distribute the negative 2's.

Excerpt 7.6: Discussion focused on cards 17 and 15

In the above excerpt, I have highlighted evidence of scaffold and underlined the legitimating criteria in which the Teacher 14 drew on to point out the difference between cards 15 and 17.

Episode 6: Fifth list of matching cards – 10 and 15

On cards 10 and 15, Facilitator wrote the verbal representation of Card 10 in algebraic form so that the difference between cards 15 and 10 was noticeable (see Figure 7.6).

10	15
To get the output, you take the input, square it, then double it, then decrease it by 2 $y = 2x^2 - 2$	$y = -2(1 - x)^2$

Figure 7.6: Cards 10 and 15 incorrectly matched

The following excerpt contains the discussion focused on the difference between cards 10 and 15. In the discussion the facilitator pointed out the difference between cards 10 and 15 (see **Excerpt 7.7**).

1261 1262	F:	Ok. 10, 15 now we're saying these don't work either?
1264	T45:	Ja
1268 1269	F:	To get the output you take the input, square it, so 10 would be, you take the input is x, square it, then double it, then decrease it by 2. So what's wrong?
1271		*[writes on the flipchart:*
1272		*10.* $2x^2 - 2$ *]*
1274	T58:	What's wrong with that?
1276 1277	F:	The co-efficient of x squared is 2, whereas in the others the co-efficient of x squared is negative 2. Ok. Alright

Excerpt 7.7: Discussion focused on cards 10 and 15

In **Excerpt 7.7** I have underlined the legitimating which the facilitator used to point out the difference between cards 10 and 10. He said the two were different because on card 10 the coefficient of x^2 was positive 2 and on Card 15 it was negative 2.

- **Task 2-Event 5: Fifth list of matching cards - 9 and 14**

Facilitator called attention to cards 9 and 14 by asking if all agreed that they belonged together. One teacher pointed that they did not belong and so the two cards were put up on the overhead for discussion (see **Figure 7.7**).

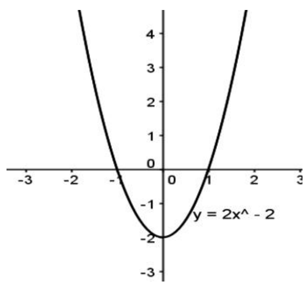
9	14
$y = x^2 – 1$	

Figure 7.7: Cards 9 and 14 incorrectly matched

Facilitator asked teachers to pay particular attention to the y-intercept of both cards. The Facilitator pointed to the y-intercept of card 9 which was negative 1 and to y-intercept of card 14 which was negative 2. Facilitator pointed that the two had the same roots but different y-intercepts.

1277	F:	Anyone want to… 9 and 14, do we agree?
1289	T59:	No, it's wrong
1981	F:	It's also wrong?
1283	T59:	Ja
1285	T60:	What?
1287	T59	14
1289	F:	*[F puts 2 slides on the overhead:*
1290		9. *graph*
1291		$y = x^2 – 1$]
1293		You have to look really carefully at the y intercept here
1295	T59	You have to
1297 1298	F:	So x^2 – 1, it's smiley, *[demonstrates the 'smiley' shape while he talks]* <u>it cuts at negative 1, it cuts the y axis at negative 1.</u>
1304		<u>This one *[points to the graph]* cuts at negative 2</u>, but its roots are ok, not so?
1306	T59:	T59?: Yes
1308 1309	F	Positive 1 and negative 1. This would have roots *[points to* $y = x^2 – 1$*]* (x–1), (x+1), so it's going to give us roots of plus or minus 1, ok?
1312	T59:	T59: Exactly, that's what I was thinking.

Excerpt 7.8: Discussion focused on cards 9 and 14

In **Figure 7.7** I have shown cards 9 and 14 as they appeared in Task 2 and I illustrate the Facilitator's explanation of the difference between the two cards in **Figure 7.8** below, where I transformed the algebraic representation of card 9 into graphical representation and placed it together with the graph of card 14 on the same axes to visibilize it.

The Facilitator's explanation is visible from **Figure 7.8** which shows that the roots of cards 9 and 14 are the same and the y-intercepts are different. By way of example, the roots for both cards are negative 1 and positive 1 and the graph of Card 9 meets the y-axis at negative 1 while that of Card 14 meets the same y-axis at negative 2. According to the Facilitator, the difference between the y-intercepts suggests that Card 9 and 14 are not representations of the same quadratic Function.

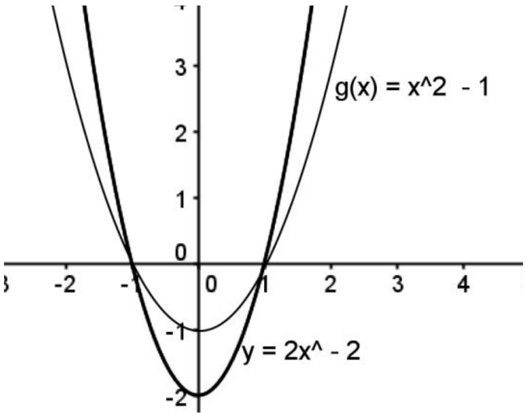


Figure 7.8: Graphs of card 9 and 14 drawn on the same axes

Figure 7.8 show evidence of both similarity and contrast between cards 9 and 14. The two are similar with respect to the x-intercepts but are in contrast with respect to the y-intercepts.

- **Task 2-Event 6: Sixth list of matching cards – 2 and 22**

The last group of matching cards came into being when one teacher suggested that cards 2 and 22 belonged to the same group. Facilitator wrote these down on the flipchart and also put them up on the overhead for the whole class to determine if the suggestion was valid (see **Figure 7.9** below). It was agreed that the two were representations belonged to the same hyperbolic function. Facilitator explained that another way would be to determine if the product of values in Card 22 was two. He said g (x) = $\frac{2}{x}$ may be written as y = $\frac{2}{x}$ such that it becomes xy = 2. As such one can find the product of x and y to determine if it is positive 2.

2

g (x) = $\frac{2}{x}$

22

x	**-1**	**½**	**2,5**	**10**
y	**-2**	**4**	**0,8**	**0,2**

Figure 7.9: Cards 2 and 22

- **Task 2-Event 7: List of all matching cards in Task 2**

The following **Figure 7.10** is where I provide a summary of Task 2 by showing all the group of cards that were suggested by teachers and written down on the flipchart by the Facilitator. The left hand side of the flipchart shows numbers put in circles. These are the numberings of each group, and the listed numbers beside each circled number are numbers on the cards and are card numbers that teachers had suggested were representations of the same function. Card number 17 was cancelled after it was agreed that it did not belong in the group containing 10 and 15. Later on in the discussion it was found that even cards 10 and 15 were not representations of the same quadratic function. Card 18 is in brackets because it was found to be ambiguous, and so it was put aside.

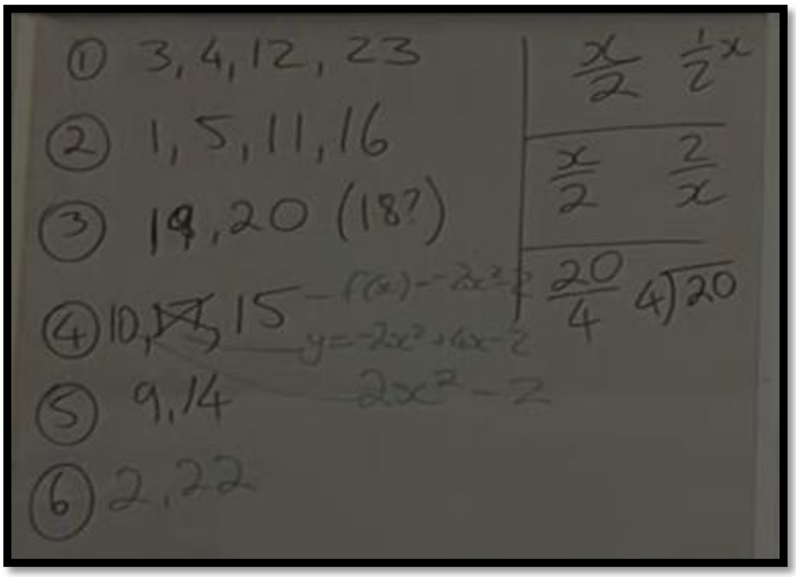


Figure 7.10: A list of groups of cards which were identified as correct matches

- **Task 2-Event 8: Reflections on Task 2**

At the end of Task 2 facilitator asked teachers to reflect on the task by responding to the questions on the slide (see **Figure 7.11**). Teachers were required to say which cards were the most difficult to match. They were also to say which representation was easy and which one was difficult to work with. In addition, teachers had to say if Grade 10 learners would cope with the task. Teachers were asked to discuss in their groups.

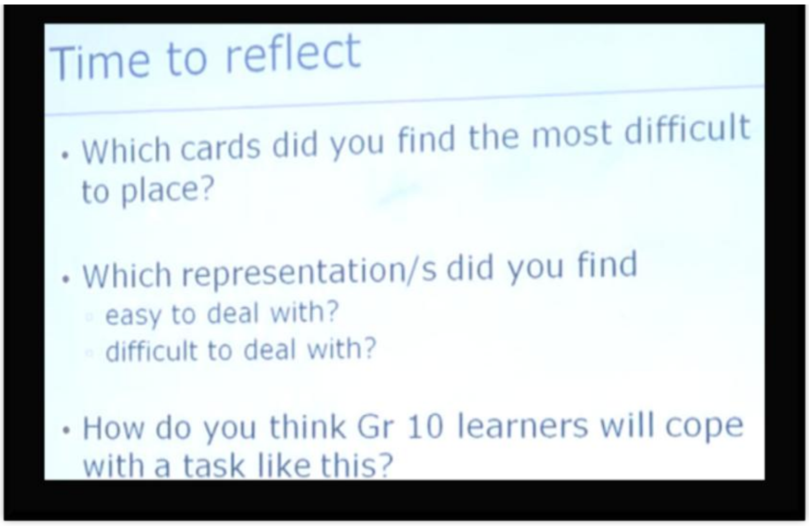


Figure 7.11: Questions used in the reflection at the end of Task 2

Facilitator pointed out that it was easy to recognise that an equation or a graph was a representation of a linear, a quadratic, a cubic or a hyperbolic function. He pointed out that it was not easy to tell which function was represented by table of values unless one first performs some mathematical working. By way of example, he said if one wanted to see if a table of values represented a linear function, then one must take two pairs and from it calculate the gradient. He emphasized that the gradients must be the same irrespective of the ordered pairs used in the calculations. He said to determine if it belonged to a quadratic one can look for the second difference, which must be the same.

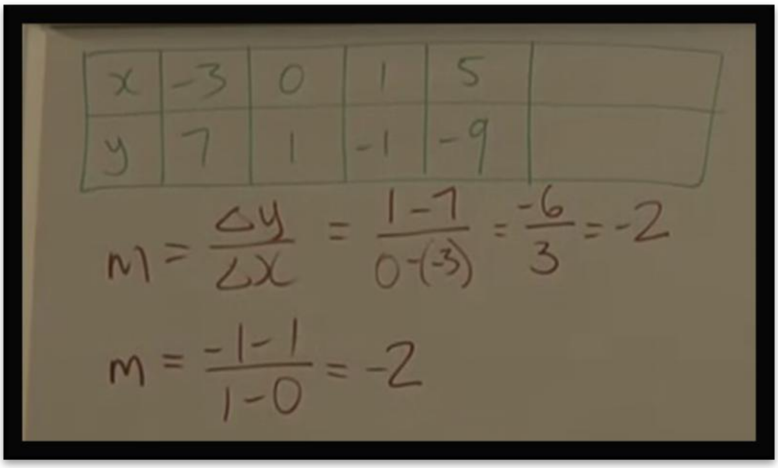


Figure 7.12: Example on how to obtain the gradient from a table of values

7.3. Discussion of the pedagogic strategy that was used at the E-Cluster

In Task 2 the Facilitator's natural attitude (taking it for granted that teachers would not encounter learning difficulties) was once more evident in his continued use of the inductive pedagogic strategy which constrains scaffolding. Here teachers were provided with *task instructions* only to work with (**see Figure 7.1**).

A form of scaffolding was noticeable at the beginning of Task 2 when the Facilitator reminded teachers that in Task 1 the class had found that Cards 3 and 4 were representations of the same quadratic function. He then asked them to find other cards that were representations of the same function as cards 3 and 4. Working from this scaffolding, cards 12 and 23 were identified, such that the first pack of 4 cards contained 3, 4, 12 and 23. Though the scaffolding was limited, teachers were able to make progress on the task because they were able to find packs of cards for other functions. By way of example, one group of teachers identified cards 1, 5, 11, and 16 as representations of the same linear function. I should point that this pack of cards was the only one which teachers were able to match easily. They found the four cards without first mismatching cards as they had done with the rest. I must also add that teachers were at the end able to identify all related representations, even though doing so was not easy. On the way to producing all the required packs teachers encountered situations where a group would first mismatch cards before producing the correct match. **Table 7.1** presents a summary of correct and incorrect matches which were produced as teachers were working on the task.

Table 7. 1: A summary of correct and incorrectly matched cards

Type of function	Correctly matched cards	Incorrectly matched cards
Quadratic	12 & 23 (3 and 4 were provided via the Facilitator's scaffold)	
Linear	1, 5, 11 & 16	
Linear	19 & 20	18 & 20
Hyperbolic	2 & 22	
		10, 15 & 17
		9 & 14

The purpose of this summary was to elaborate the fact that the inductive pedagogic and its low scaffold helped teachers learn from their mistakes and clarify misconceptions. It was also productive for the Facilitator, because it gave him insight into teachers’ knowledge of functions at the beginning of the theme work focused on functions. This insight was evident when Task 2 was conducted the following day in the I-Cluster and the N-Cluster, where a deductive pedagogic strategy was adopted. There, teachers experienced more scaffolding. I elaborate this further when I discuss the analysis of Task 2 at these two clusters. In conclusion, the analysis of the adopted pedagogy at the E-Cluster indicated that teachers there experienced low scaffolding, and this might have been a consequence of the inductive pedagogic strategy which the facilitator had chosen to work with.

The use of this strategy at the E-Cluster constrained scaffolding such that teachers’ experienced low scaffolding. Teachers approached the task differently from each other. The resultant responses were characterised by both correct and incorrect matching of the multiple representations (see Figure 7.10). When these results are viewed simultaneously with the adopted pedagogic strategy and its associated low scaffolding capacity, a temptation to conclude that these two are related is possible. However, looking across the three clusters, and in particular, to teachers’ responses and pedagogic strategies, this conclusion does not hold. This is so because teachers in the other two clusters produced similar responses in spite of experiencing the deductive pedagogic strategy which opened more scaffolding opportunities.

7.4. Discussion of the legitimating criteria at the E-Cluster

The pedagogic strategies were accompanied by legitimating criteria which were used to ground the meaning of the enacted object of learning for teachers. These criteria were characterised by appeals to mathematical reality?. I have elaborated this further with illustrations in the next paragraph.

The first legitimating criteria emerged when Facilitator translated the verbal representation of card 19 into an algebraic representation to help teachers discern sameness between Cards 19 and 20. In so doing, he was appealing to the constitution of the two representations with particular focus on the parameter. The second and third legitimating criteria came into being when attention was on cards 10, 17 and 15. In the second criteria, a teacher pointed out that cards 17 and 15 were not multiple representations of the same function because the coefficients of x in either card were different from each other. For instance, on Card 15 the coefficient of x was zero (0) and on Card 17 it was four (4). In so saying the teacher was *appealing to the constitution of the two representations with particular focus on parameter*. The third legitimating criteria was present when the Facilitator translated the verbal representation of on Card 10 into the algebraic representation to help teachers discern similarities between cards 10 and 15. Here the facilitator was *appealing to the constitution of the representations*. Following on this, he pointed out that the two cards were different because they each had a different coefficient of x^2. For instance, Card 10 had positive two (+2) and Card 15 had negative two (– 2). The appeal embedded in these criteria was an *appeal to the constitution of the representations with particular attention to parameter.*

The fifth and sixth legitimating criteria came into existence when attention was on cards 19 and 20. For instance the fifth criteria appeared when the verbal representation on Card 19 was translated into an algebraic representation to enable comparison between the two cards. In so doing an *appeal to the constitution of the representations* was unveiled. Translation of the verbal representation of Card 19 into algebraic made it possible for the Facilitator to visibilize its features and to help teachers to work with familiar representations. For instance, when the verbal representation were translated into algebraic formation, teachers were helped to notice and compare its coefficients and use these to decide if they the cards were representations of the same function or not.

The sixth legitimating criteria came into play when attention was called to cards 9 and 14 (see **Figure 7.8**). Here, while working from both the algebraic and graph, the facilitator pointed out that the two were different because they possessed different y-intercepts. For instance, on Card 9 the y-intercept was negative one (-1) and on Card 14 it was (-2). In this case the *appeal was to one property of a function* - which in this criteria was the y-intercept.

In summary (see **Table 7.2**), I can point that at the E-Cluster there were six legitimating criteria in the enactment and these characterised by appeals to mathematics objects.

Table 7.2: A summary of the legitimating criteria used in the E-Cluster

Legitimating criteria	Appeals	Enacted object of learning
Translation of the verbal representation of card 19 into an algebraic representation	Multiple representation	$\boldsymbol{Mt_{re+}}$
The two cards are different because the coefficients of x (gradient) in either card are different from the other.	parameter	$\boldsymbol{Mt_{re+}}$
Translation of the verbal representation of on Card 10 into its algebraic representation	multiple representation	$\boldsymbol{Mt_{re+}}$
The two cards are different because they each have a different coefficient of x^2.	Parameter	$\boldsymbol{Mt_{re+}}$
Translation of the verbal representation into an algebraic representation to enable comparison between the two cards.	Multiple representation	$\boldsymbol{Mt_{re+}}$
Facilitator pointed out that two cards were different because they possessed different y-intercepts.	Intercept	$\boldsymbol{Mt_{re+}}$

7.5. Discussion of what was made possible to learn at the E-Cluster

Analysis of the enacted object of learning in Task 2 opened up two patterns of variation that included *Contrast* and *Generalisation*.

The first pattern of variation was *generalisation* and this was evident in three instances. First, when teachers were matching cards 12, 3, 4 and 23. Here, teachers experienced similarities between the multiple representations of the same quadratic function. By way of example, teachers had to notice that the y-intercept in cards 3, 4, 12 and 23 were positive one (+1). The same property was earlier made possible to discern in Card 3 when its verbal representation was translated into the algebraic representation and this was evident in Task 1. Second, this pattern of variation was also present when attention was on cards 19 and 20. Teachers were helped to experience similarities between the coefficients of x (which was ½ in both cards 19 and 20**). As such, it was possible for teachers to discern that the coefficient of x (i.e. gradient) was a property of a linear function but its representations are not its properties.**

Third, *generalisation* was present when the focus was on cards 1, 5, 11 and 16. These cards were representations of the same linear function. For teachers to match these cards they had to discern the similar properties of the linear function across the four representations. These included the y-intercept and the gradient, which were positive 1 and negative 2 respectively. When cards 3, 4, 12 and 23 were in the fore, teachers experienced similarity in some properties of the same quadratic function. Similarly when cards 1, 5, 11 and 16 were in the fore. Now, in variation theory terms, for one to experience similarity the important feature must be invariant and less important features must vary. So here the properties of the quadratic function were invariant across its representations while the representations were varying and **this opened opportunities for teachers to discern that the y-intercept was a property of a quadratic function but its multiple representations were not its properties.** Similarly, focus on cards 1, 5, 11 and 16 **made it possible for teachers to discern that the y-intercept and gradient were properties of a linear function but its representations were not its properties.**

The last instance of generalisation was present when cards 2 and 22 were in the fore. Here there were opportunities to experience similarity in the product of x and y and values of x and y across the representations. Here **experiencing similarity in the values of x and y across the representation made it possible for teachers to discern that the product was 2.**

The second pattern of variation emerged where teachers experienced difference between the types of functions. In the first instance of *contrast* teachers worked with the linear, quadratic and hyperbolic functions. In this pattern of variation the dimension of variation was type of functions and the critical features were the linear, quadratic and hyperbolic functions. **Experiencing difference between the three functions made it possible for teachers to discern the difference between their constitutions.** The second instance of *contrast* was experienced where teachers' attention was called to the coefficients of x. and x^2. This emerged when attention was on cards 15 and 17. For instance, the coefficient of x was 4 in card 15: ($y = -2 + 4x - x^2$) and zero (0) in card 17: ($y = - -2x^2 - 2$). **Experiencing difference between the coefficients of x made it possible for teachers to discern that cards 15 and 17 were not representations of the same quadratic function.** The third instance of **contrast** was present when attention was on cards 10 and 17. Here **the experienced difference between the coefficients of x^2 made it possible to discern that cards 10 and 17 were not representations of the same quadratic function.**

The fourth instance involving contrast was when cards 9 and 14 were in the fore. Here teachers experienced difference between the y-intercepts on the algebraic representation of card 9 and the graphical representation of card 14. On card 9 the y-intercept was negative 1 and on card 14 it was negative 2. Facilitator had deliberately called teachers' attention to these to help them discern the difference between cards 9 and 14. **As such, experiencing difference between the y-intercepts made it possible for teachers to discern that cards 9 and 14 were not representations of the same linear quadratic function.**

Table 7.3: Summary of what was made possible for teachers to learn at E-Cluster

Pattern of variation	**Invariant**	**Varied**	**Discernment**	**Enacted object of learning**
Generalisation (3, 4, 12, 23)	y-intercept	Multiple representations of quadratic function	The y-intercept is a property of a quadratic function but its multiple representations are not its properties	$\mathbf{Mt_{re+}}$
Generalisation (1,5,11, 16)	y-intercept and gradient	Multiple representations of a linear function	The y-intercept and gradient are properties of a linear function but its multiple representations are not its properties	$\mathbf{Mt_{re+}}$
Generalisation (19, 20)	Gradient	Multiple representation	It was possible for teachers to discern that the coefficient of x (i.e. gradient) was a property of a linear function but its representations are not its properties.	$\mathbf{Mt_{re+}}$
Contrast (all cards)	Type of function	Linear, quadratic and, hyperbolic functions	Experiencing difference between the three functions made it possible for teachers to discern difference between their key features	$\mathbf{Mt_{re+}}$
Contrast (15, 17)	Quadratic function	Coefficients of x	Experiencing difference between the coefficients of x made it possible for teachers to discern that the gradient is a key feature in a linear function	$\mathbf{Mt_{re+}}$
Contrast (10, 17)	Quadratic function	Coefficients of x^2	Experiencing difference between the coefficients of x^2 made it possible to discern that difference between coefficients of x^2 imply difference between any two quadratic functions	$\mathbf{Mt_{re+}}$
Contrast	Quadratic	y-intercepts	Experiencing difference between the	$\mathbf{Mt_{re+}}$

(9, 14)	function		y-intercepts made it possible for teachers to discern that cards the y-intercept is a distinguishing feature between any two Functions	
Generalisation (2, 22)	Product of x and y (i.e. 2)	Multiple representati on	Experiencing similarity in the product of x and y across the two representations made it possible for teachers to discern that the product of x and y is a property of a hyperbolic function	$\mathbf{Mt_{re+}}$

7.6. Summary of the enactment of Task 2 of Functions 1 at the E-Cluster

Columns 3 and 5 of Tables 6.3 and 6.2 indicate that the enacted object of learning was constituted by appeals to $\boldsymbol{Mt_{re+}}$. Similarly, patterns of variation opened opportunities for discernment of Mt_{re+}. Two patterns of variation made several aspects of Functions possible to learn. By way of example, through *generalisation* four (4) aspects of functions (see Table 7.3) were made possible to learn. Similarly, the same number of aspects of functions was made possible to learn through *contrast*. In is interesting that the analysis of appeals does not show any of Mt_{re-}, Mt_{ap+} or Mt_{ap-} . It is probable that participation in Task 1 might have helped teachers to close the gaps in their prior knowledge of Functions.

7.7. Description of Task 2 of Functions 1 at the I-Cluster and N-Cluster

The enactment of Task 2, just like Task 1, was similar at the two clusters. The same presentation was used at both clusters. The difference was that this was enacted by two different Facilitators. However, the content and sequencing of the presentation were the same. Hence here I use the enactment at the I-Cluster to describe the enactment at both clusters. Combining the description of Task 2 at the two clusters is for the purpose of minimizing repetition. However, any difference between the two is highlighted in the analysis.

The Facilitator introduced Task 2 by asking teachers to remove Card 24 from the pack of cards so that they were only remaining with cards that represented functions. Now, the instructions for the task were modified. At the I-Cluster Facilitator decided how teachers worked on the task. How this was done was different from the E-cluster (see Figure 7.13). There, teachers were not provided a step-by-step procedure to work with.

Task 2

a). Take card 1, find 3 other cards that also represent this function. Give reasons why they represent this function

b). Take card 12, find 3 other cards that also represent this function

Figure 7.13: Modified Task 2 Instructions

- **Task 2-Event 1: First list of matching cards – 1, 5, 11 and 16**

The I-Cluster teachers were instructed to start with Card 1 and then proceed from there to find 3 other cards representing the same function (see **Excerpt 7.9**).

F:	And now what I'd like you to do is to take card number 1 *[teachers look for the card]* and find 3 other cards that represent this same function as card 1. So you've got card 1. Am I right it's -2x + 1? Ok that's card 1. *[circles* -2x + 1 *on paper on the easel]* Can you find three other cards in your collection that are linked to the same function as that one?

Excerpt 7.9: Evidence of scaffold on Task 2

Card 1 was an algebraic representation of a linear function. As such, teachers were helped to begin the task with easier matching cards. Facilitator wrote down Card 1 on the flipchart as teachers were using it to find three others. After some working, cards 16, 5 and 11 were identified as cards representing the same linear function as Card 1 (see **Figure 7.3**). Facilitator wrote the card numbers on the flipchart.

F	Can you find three other cards in your collection that are linked to the same function as Card 1 ?
Teachers	*[teachers work in groups]*
F	Card number 1. It seems like everybody agrees. The only cards that go with it are 16, 5 and 11. Yes?
T23	Yes
F	So 16 gives us the verbal, 11 gives us the graphical and 5 gives us the numeric. Ok? When we take number 12 that's the graphical. So here I asked you to start with number 1 which was the algebraic. And then you have to find the matching graph, the matching tables, the matching verbal descriptions.

Excerpt 7.9: Discussion focused on cards 1,5,11 and 16

In Excerpt 7.9 the highlighted text is evidence of scaffold on Task 2 at both the I- and N-clusters. Teachers at these clusters were told which cards to begin with. The excerpt also show the list of cards that matched with card 1.

- **Task 2-Event 2: Second list of mathich cards – 12, 3, 4 and 23**

Following this, teachers were asked to take Card 12 and then work with it to find 3 other cards representing the same function. Card 12 was a parabola with a maximum value at positive 1.

682	F:	Number 12
683	T22:	Number 4, number 16
684	T23:	Number 3 – do we have 3?
685	T24:	4, 3
686	F:	4 and 3. So 3 and 4.
687	T24:	23
688	F:	23. What about 16? Is it in this?
689	T25:	It's for the first one.
690	T24	Ja, no
691	F:	Can it be for both?
692	T24	No
694	F:	Can it be for both?
695	T24:	No.
696	T26:	I just think it's
697	T24:	He's just mixed his cards up.
699	T27:	16's a mistake
703 704 705 706	F:	Now this is doubling and it's not square root. So it's not 16. *[next to* 12 *writes:* 3, 4, 16, 23 *but then crosses out 16]* Ok. So in these two examples I deliberately chose card 1 and card 12 because here you can get a pile of 4 cards that all represent the same function.

Excerpt 7.10: Discussion focused on cards matching with card 12

Facilitator pointed out to teachers that he was asking them to start with card 12 which was a graph and that a graph was not easy to start with. Facilitator asked for the cards and 3, 4 and 23 were given as cards representing the same function as card 12 (see **Figure 7.2**). He explained that he deliberately wanted teachers to work with Cards 1 and 12 because each card would lead to a pile of four different representations. The highlighted text shows evidence of scaffold on the task.

- **Task 2-Event 3: Third list of matching cards – 2, 22, 6 and 18**

Facilitator then told teachers that of the remaining cards some were quadratic some hyperbolic and others were functions which were neither quadratic nor hyperbolic functions. He then instructed teachers to put aside those that were neither quadratic nor hyperbolic functions so that they only worked with them. Facilitator asked for a list of matching cards that were representations of a hyperbolic function. Teachers identified cards 2, 22, 6 and 18. However it was later discovered that only cards 2 and 22 were matching. Cards 6 and 18 were not the same hyperbolic function as 2 and 22. He explained that Card 6 was a different hyperbola because the product of coordinates of point P (2.5; 5) on the graph of Card 6 was not equal to 2 (see **Excerpt 7.10** and **Figure 7.14**).

805 806 807	F:	Does six represent the same hyperbola as 22 and 2? [*points to the numbers on the board*] So 6 is the graph. [*addressing the teacher*] Are you answering that question?
810	T39:	I want you to do ?
811	F:	Let's just finish my question.
812	T39:	No, 6…
813 814	F:	Ok. 6 is not right because the point that's plotted on that hyperbola is the point P($2\frac{1}{2}$, 5) and when you multiply those two together…
815	T39:	You don't get 2
816 817	F:	… you don't get 2. So although these are all hyperbolas (and let's not go to 18 yet) they are not the same hyperbola necessarily.

Excerpt 7.11: Discussion focused on cards 2, 22, 6 and 18

In the above excerpt, the legitimating criteria explaining why card 6 was different from cards 2 and 22 has been underlined.

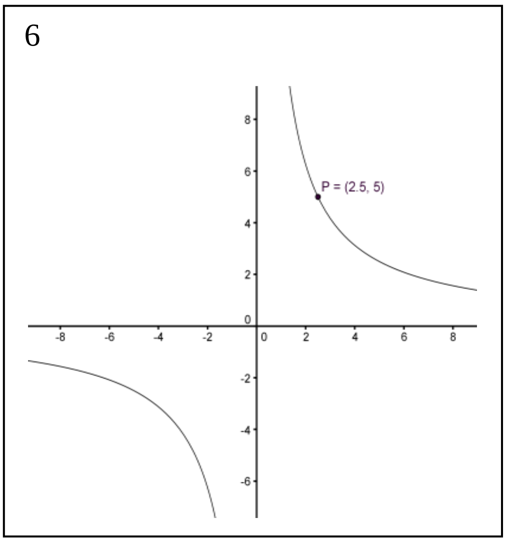


Figure 7.14: Card 6

Facilitator then asked teachers to produce the graphical representation that would match with cards 2 and 22. Cards 2 and 22 are shown in **Figure 7.9**).

- **Task 2-Event 4: Identifying a hyperbola**

One teacher asked how to determine if card 22 was a hyperbola. Another teacher suggested to her that she can substitute values of x into the algebraic representation shown on Card 2. Facilitator took this further and explained the procedure for determining the type of a function represented by a given table of values. He called attention to a set of parent functions which he had displayed on a slide (see **Figure 7.15**).

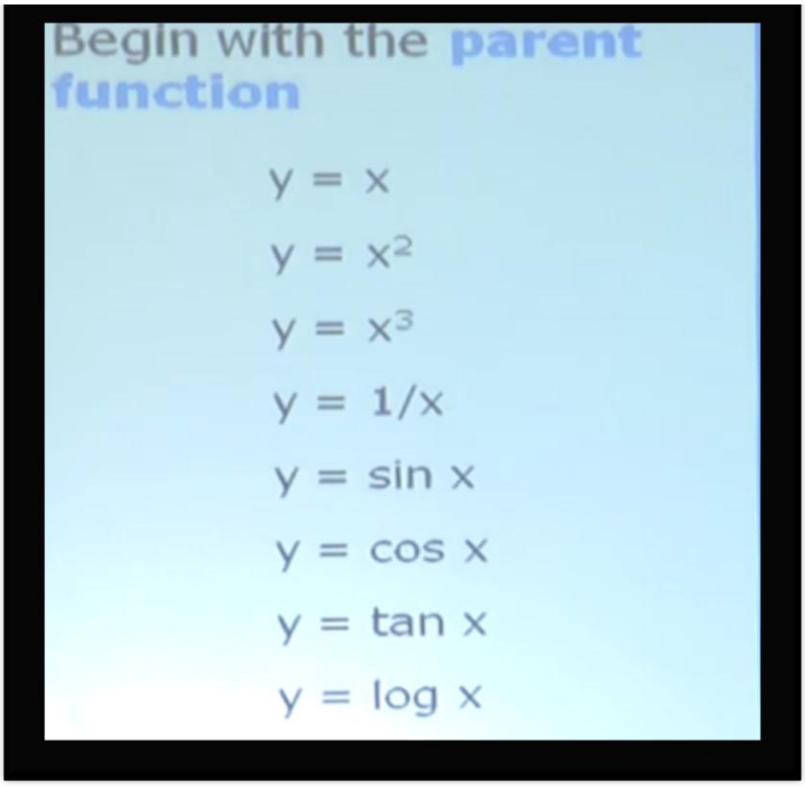


Figure 7.15: Parent functions

He said knowledge of parent functions was critical when determining the type of function corresponding to a given representation. He said one would need to appeal to the form of the function. By this he meant that if it is a table of values, one has to operate on the value to see if it yields a correct corresponding y value. If it is a graph one has to appeal to its shape and if it is algebraic one has to appeal to the degree of the function.

In addition, he said to determine if the table represents a linear function one has to look for the first common difference between the y values. For the quadratic one has to find the second common difference between the y values and for the hyperbolic function one has to find the product of each ordered pair. The following **Figure 7.15** shows the list of card numbers which the Facilitator wrote on the flipchart after teachers suggested were matching.

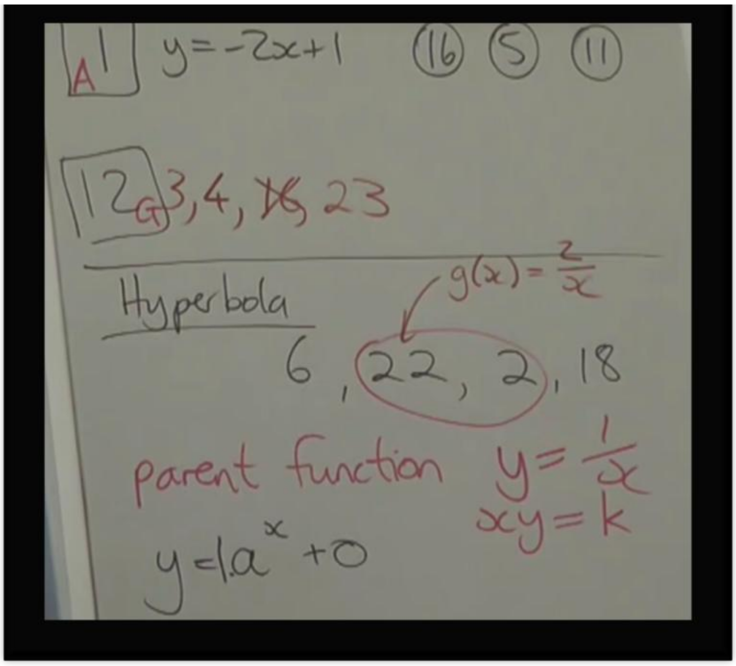


Figure 7.16: List of suggested matching cards

- **Task 2-Event 5: Summary of Functions 1**

At the end of Task 2 the Facilitator presented a summary of all that was done in session 2.1. First he mentioned that at the beginning he introduced the Grade 10 definition of function so that teachers would have something to refer to when working on Task 1. Secondly, he mentioned that he asked teachers to start Task 2 with Card 1 which was an algebraic representation of a linear function and proceed with it to find three other cards that matched with it. He said he did this as opposed to asking teachers to work anyhow. He said he wanted teachers to start with an easier type of function and also to begin with an algebraic representation because they were used to working from there. He pointed out that in the next instruction he deliberately chose Card 12 in order to raise the cognitive demand of the task. He said Card 12 moved teachers from working with a linear to quadratic function and this raised the demand of the task. In addition, he said the demand of the task was further raised by asking teachers to begin with a graph rather than the algebraic. He said he was aware that teachers were not familiar with working from the graph. Hence he wanted them to experience working in the reverse order. Facilitator pointed out that to further raise the cognitive demand of the task he asked teachers to put away cards that were not quadratic or hyperbolic and only work with the remaining cards. He said doing this was challenging because it was not easy to decide which cards were not quadratic or hyperbolic. He said matching the remain cards was also challenging because they were less structured

To conclude the session he put up a slide on which were three questions that required teachers to reflect on the task. The first and second questions were similar and they both asked teachers to name the representation which they had found difficult to work with (see **Excerpt 7.12**).

954 955 956	F:	One of the things you're meant to reflect on is which cards did you find the most difficult to place? Which kind of card? So which representation was the hardest for you to work personally?
957	Teachers:	Table
958	F:	Ok, that's 3 votes for table
959	T43:	The new way of working you had to test
960	F:	T44
961	T44:	Ja, table
962 963	F:	I think that's what many people found difficult to work with. It's just numbers, you know, it shouldn't be that hard but the issue is it's not just numbers.
964	T45:	Ja
965 966	F:	It's the relationships between the numbers. And we're not used to working with them like that.

Excerpt 7.12: Discussion focused on reflections on Functions 1

In **Excerpt 7.12** above most of the teachers found working with the table of values difficult. Facilitator agreed to this and pointed out that it was difficult because the table required teachers to find the relationship between the numbers. The algebraic representation was the easiest to work with. In the last question teachers were asked if learners would find the task difficult. There was no response from teachers. Facilitator pointed out that if Grade 10 learners can be able to do this task well, then they might perform better in the baseline test

7.8. Discussion of the pedagogic strategy that was used at the I-Cluster

Analysis of the pedagogic strategy adopted at the I-Cluster indicates that the deductive pedagogic strategy was preferred by the Facilitator. This strategy opened scaffolding and this was evident in the instructions which suggested how teachers may proceed on the task. Here teachers were instructed to begin the task in a particular way to help them progress. Hence teachers at the I-Cluster experienced *high scaffolding*. By way of example, Facilitator deliberately chose Card 1 and its partners as the starting point. Now Card 1 and its partners were multiple representations of a linear function and this was chosen because it was familiar to both General Education and Training (GET) and Further Education and Training (FET) teachers. In addition, the linear function is may be considered to be a basic type of function which all teachers should know.

In another form of scaffolding the Facilitator chose Card 1 which was an algebraic representation. An algebraic representation is useful for visibilizing properties of a function. Hence working from the algebraic to other representations was easier for teachers than if they had been asked to work from a table of values. It is easy to obtain a table of values from the equation by substitution and a graph is also easy when the resultant pairs of coordinates are plotted and joined to produce a line. The verbal is also obtainable when the mathematical symbols are verbalized (translated into words). The scaffolding done around cards 1, 5, 11 and 16 helped teachers identify the other three cards with less difficulty. When this was done, focus went to card 12 which was a graphical representation of a quadratic function. Teachers were instructed to find the table of values, verbal and algebraic representations that partner with the graphical representation of Card 12. Teachers are not familiar with working in the reverse (from graph back to algebraic or table of values). As such, asking teachers to work from the graphical raised the cognitive demand of the task and provided opportunities to learn to work in the reverse order. It can then be pointed out that instructing teachers to work in the reverse was another form of scaffolding the task because this helped teachers to be flexible on their working with multiple representations. This scaffolding was a way of forcing teachers to realise that all representations contained the properties of the same linear function and so working from anyone of representations had potential to produce its partners. I must mention that instructing teachers to work in the reverse order was deliberate, productive and intelligible because the rationale for engaging teachers in this particular task was to disrupt the tendency to always begin with the algebra and proceeding from there to other representations. The danger of teaching that follows this pattern is that learners who experience this teaching fail to interpret graphs and this has a negative impact on their ability to obtain solutions from the graph. I have provided evidence to support this in Chapter 2 where I discuss the context of the study. There I show evidence from the WMC-S learner baseline test where learners' performed below expectation on algebra and functions test items.

It is interesting that in spite of high scaffolding at the I-Cluster, teachers produced some incorrect matches just like teachers at the E-Cluster who experienced low scaffolding (see **Table 7.4**).

Table 7.4: Teachers' response on card matching at the I-Cluster

Type of function	Correctly matched cards	Incorrectly matched cards
Linear	1, 5, 11, 16	
Quadratic	12, 3, 4, 23	16
Hyperbolic	2 & 22	6, 18

The above table shows that teachers were able to match representations of the linear function but encountered some challenges when matching the quadratic and the hyperbolic functions. By way of example, Card 16 was incorrectly matched 12, 3, 4 and 23 and both 6 and 18 were incorrectly matched with 2 and 22.

7.9. Discussion of the legitimating criteria at the I-Cluster

The enactment was filled with several legitimating criteria which were used to fix meaning to the enacted object of learning. The criteria were characterised by appeals to mathematics objects. In **Table 7.5** I show both the criteria and their accompanying appeals to provide evidence that suggests that the enacted object of learning was a mathematics object.

Table 7.5: Summary of legitimating criteria in Task 2 at I-Cluster

No	Legitimating criteria	Appeals	Enacted object
1	Because each card would lead to a pile of four different representations.	Representations of a linear and quadratic functions	Mt_{re+}
2	Because the product of coordinates of point P (2.5; 5) on the graph of Card 6 was not equal to 2.	Product of a pair of coordinates	Mt_{re+}
3	She suggested to her that she can substitute values of x into the algebraic representation shown on Card 2.	Mathematics procedure	Mt_{re+}
4	He called attention to a set of parent functions which he had displayed on a slide	Parent functions	Mt_{re+}
5	He said one would need to appeal to the form of the function	Parent functions	Mt_{re+}
6	if it is a table of values, one has to operate on the value to see if it yields a correct corresponding y value	Mathematics procedure	Mt_{re+}
7	If it is a graph, one has to appeal to its shape	Shape of graph	Mt_{re+}
8	if it is algebraic one has to appeal to the degree of the function	Index, Power, Exponent	Mt_{re+}
9	to determine if the table represents a linear function one has to look for the first common difference between the y values.	Common difference	Mt_{re+}
10	For the quadratic one has to find the second common difference	Common difference	Mt_{re+}
11	for the hyperbolic function one has to find the product of each ordered pair	Product of ordered pairs	Mt_{re+}
12	It was difficult because the table required teachers to find the relationship between the numbers	Relationship between table of values	Mt_{re+}

7.10. Discussion of what was made possible to learn at the I-Cluster

Analysis of what was made possible to learn revealed two patterns of variation which were at play at the I-Cluster. Just like at the E-Cluster, the first pattern of variation experienced was *generalisation* and this was evident when teachers had to match cards representing the same linear function. For teachers to discern that cards 1, 5, 11 and 16 were representations of the same linear function, they had to identify common features that made the four representations similar and in this case it were the gradient and the y-intercept. Since the gradient and y-intercept were common across the four representations it meant that they were invariant and moving from one representation to another caused the representations to vary. In addition, since the gradient and y-intercept were important features but at the same time invariant, then opportunity for generalisation was made possible. **What was possible to discern was that the gradient and the y-intercept were properties of linear functions and, that its representations were not.**

A second instance for generalisation was possible when attention was on finding cards that matched with Card 12. This card was a representation of a quadratic function and in a quadratic function features that might help teachers discern that representations were of the same function were the parameters and the y-intercept. The parameters included the coefficients of x^2 and x. For discernment to occur it meant that these parameters and the y-intercept had to be invariant while representations had to vary and in variation theory, when the important feature is invariant while less important features vary, then the pattern of variation which is availed is generalisation. **Hence when attention was on matching card 12 and its partners, it was possible for teachers to learn that parameters (coefficients of x^2 and x) and the y-intercept are properties of a quadratic function**. This pattern of variation also emerged when attention was on cards 2 and 22 which were representations of a hyperbolic function. Here the common feature was the product of x and y coordinates which was 2 and was discernible in both cards 2 and 22 which were an algebraic and a table of values respectively. **It was possible for teachers to learn that in $= \frac{a}{x}$, a is the product of the coordinates of x and y.**

The second pattern of variation was contrast and this was present when attention was on Card 6 when the Facilitator was explaining why this card was not a partner to cards 2 and 22. Contrast was on the product of the coordinates of x and y. Facilitator multiplied 2.5 and 5, coordinates of the point (2.5; 5) on the graph of Card 6 after showing that the product of pairs of coordinates on card 22 was 2, to show that the product was not equal to 2. In variation theory terms this constitutes contrast since the facilitator wanted teachers to experience difference between the products. **As such what was made possible for teachers to learn was that card 6 was not a partner of cards 2 and 22. Table 7.6** shows a summary what was made possible to learn at the I-Cluster. It is noticeable from this table that there were fewer opportunities for discernment of contrast at this cluster and this might be due to the high scaffolding which was opened up by the deductive pedagogic strategy which was adopted there.

Table 7.6: Summary of what was possible to learn at the I-Cluster

Pattern of variation	Invariant	Varied	Discernment	Enacted object of learning
Generalisation (3, 4, 12, 23)	Parameters (coefficients of x^2 and x) and y- intercept	Multiple representations of quadratic function	The y-intercept is a property of a quadratic function but its multiple representations are not its properties	Mt_{re+}
Generalisation 91,5,11, 16)	y-intercept and gradient	Multiple representations of a linear function	The y-intercept and gradient are properties of a linear function but its multiple representations are not its properties	Mt_{re+}
Generalisation (2, 22)	Product of x and y (i.e. 2)	Multiple representations	Experiencing similarity in the product of x and y across the two representations made it possible for teachers to discern that cards 2 and 22 were representations of the same hyperbolic function	Mt_{re+}
Contrast (working with all cards)	Type of function	Linear, quadratic and, hyperbolic functions	Experiencing difference between the three functions made it possible for teachers to discern the difference between their constitutions	Mt_{re+}
Contrast (Card 6, 2, 22)	Hyperbolic function	Product of x and y is 2 in cards 2 and 22 and is 12.5 in Card 6	Experiencing difference between the products of x and y made it possible for teachers to discern that card 6 was not a partner with cards 2 and 22	Mt_{re+}

For instance, teachers were instructed to work only with cards which were representations of linear, quadratic and hyperbolic functions. In addition, teachers were instructed on how they may work on the task and so this reduced cases of mismatching cards. Now in the E-Cluster there were more cases of mismatching cards and this provided opportunities for experiencing difference when the facilitator had to explain why the cards were a mismatch. Examples of mismatch were cards 15, 10 and 17 and cards 9 and 14. Explaining why 15 was not a partner led to contrast between the coefficient of x^2 in cards 15 and 17 and coefficient of x in 10 and 15. In cards 9 and 14 the contrast was between the y-intercepts which were different.

7.11. Discussion of the pedagogic strategy that was used at the N-Cluster

The Facilitator introduced the object of learning by posting an instruction that required teachers to "*Take Card 1, [and] find 3 other cards that represent the same function"* onto a power point slide. The Facilitator proceeds from here to unpack the instruction and concludes this by scaffolding the task using the deductive pedagogic strategy. By way of example, Facilitator first calls attention to the linear function which was familiar to all teachers. Doing this helped teachers to work from the easier and familiar linear function. Success on this was useful for motivating and building teachers' confidence for the next demanding stage of the task.

Second, by instructing teachers to begin with Card 1 reduced the level of difficulty of the task as a whole. For instance teachers had 20 different representations of different types of functions to choose from. However, by pointing that Card 1 was a linear function gave away other representations, particularly the graphs of other functions because of their shapes. Thirdly, by instructing teachers to begin with Card 1 the Facilitator employed a pedagogic strategy that made it possible for teachers to work from familiar contexts. Teachers are familiar with working from the equation to the table and then to the graph in their normal teaching. So the beginning with Card 1 helped teachers to draw from their prior knowledge of working with multiple representations. However, working with Card 12 increase the cognitive demand of the task because here teachers were made to work from the graph to other representations and this was unfamiliar to teachers. In spite of that, this constituted high scaffolding because teachers were deliberately inducted into a new way of seeing multiple representations. Since the task was made easier to comprehend it produced very few situations that needed the Facilitator to make some justifications. Hence the space for legitimating criteria was narrowed, and this had effects on appeals.

7.12. Discussion of what was made possible to learn at the N-Cluster

The first pattern of variation opened when teachers had to find 3 representations that matched Card 1 – i.e. the linear function $y = -2x + 1$. Teachers had the opportunity to experience contrast in the types of representations of a linear function. Variation theory contends that experiencing difference is a precondition for the discernment of difference between the critical features together with the critical aspect in which they are values. Hence the task made it possible for teachers to discern both the dimension of variation and its critical features simultaneously. In this task the critical features were the table, graph and the verbal description that teachers have to find. The critical features together constitute '*types of representations*' and so types of representations was the critical aspect in which the critical features were values. **As such it was possible for teachers to discern difference between the multiple representations of a linear function**

The second pattern related to finding cards that matched $y = -2x + 1$ was generalization. For teachers to identify cards that matched Card 1, they needed to discern the same properties of a linear function in the cards they were looking for. So here, the invariant feature was in the fore while the varying features were out of focus. **Hence it was possible for teachers to discern that the representations were not properties of a linear function but features the y-intercept and gradient was its properties.**

The third pattern of variation was contrast, and this was present when teachers were searching for the linear representations from a pile of cards containing representations of functions other than linear function. During the search the type of function varied while the fact that they were all functions was invariant. So in this task, moving from one card to another helped teachers experience difference between linear, quadratic and hyperbolic functions (critical features). The dimension of variation was function, and this was backgrounded since what was in focus was a type of function (linear function), i.e. critical feature. **AS such it was possible to discern difference between the types of functions**

In spite of the scaffolding that was done at the beginning of this task, finding the verbal and the table representations for Card 1 was challenging for most of the teachers. Finding these two representations posed challenges because generally, key features of functions are not as evident in the table and verbal descriptions as they are, in the graph and the equation. By way of example finding the graph was easier and this might have been due to the fact that the key features (gradient and intercept) of a linear function were self evident in it. The difficulty that the teachers were experiencing might be due to the fact that they were familiar with working from the equation to table and not the other way round. In phenomenology and variation theory the natural attitude – the tendency to impose one's preconceptions or suppositions onto the phenomena hinders the discernment of new meanings or knowledge.

So the teachers' preconceptions –(i.e. that a table is a product of an equation) were challenged and so working through the task helped to shape their perceptions and to realize that each representation can independently communicate all features the function. I must point that the need for teachers to realize this was deliberate and was influenced by evidence from professional developers' classroom observations of these teachers teaching functions. In those observations teaching always proceeded from the equation, to the table and ended with the graph- such that the table seemed like a step that you go through to produce a picture. There was never a situation where the teaching proceeded from the table or the graph and verbal descriptions were always absent in the teaching.

The fourth pattern of variation was generalisation, which was present when the facilitator instructed teachers to start the task with Card 12 and find 3 other matching cards. Finding the matching cards helped teachers to experience similarity between the critical features (multiple representations). While the representations of a quadratic function were varying, the function was invariant (i.e. they were all quadratic representations) **As such it was possible for teachers to discern similarity between the four representations of the same quadratic function.**

The fifth pattern of variation was contrast, and was opened up when teachers were searching through the pile of cards for 3 cards matching Card 12. Teachers had to identify representations of a quadratic function from a pile consisting of both quadratic and linear functions. The invariant feature was functions while the varying feature was type of function and what was varying was in the fore. **As such teachers had the opportunity to discern difference between the properties of linear and quadratic functions**

Table 7.7: Summary of what was made possible to learn at the N-Cluster

Invariant	Vary	Discernment	Enacted object of learning
Linear function	Table, linear graph, verbal, equation	Discernment of difference between the representations of a linear function.	Mt_{re+}
Llinear function	Table, linear graph, verbal, equation	Discernment of similarity between the representations	Mt_{re+}
Quadratic function	Graph, table, verbal, equation	Discernment of difference between the representations of a quadratic function.	Mt_{re+}
Quadratic function	Graph, table, verbal, equation	Discernment of similarity between the representations of a quadratic function	Mt_{re+}
Function	Linear, quadratic	Discernment of difference between linear and quadratic functions	Mt_{re+}

7.13. Summary of the analysis of Task 2 of Functions 1 at the I-Cluster and N-Cluster

The analysis of Task 2 at the two clusters revealed that teachers at these cluster experienced high scaffolding. While high scaffolding at the I-Cluster did not constrain production of legitimating criteria, this was the case at the N-Cluster. The legitimating criteria at the I-Cluster were constituted by appeals to Mt_{re+}. Across the two clusters contrast and generalisation were the dominant patterns of variation and the analysis of these showed that Mt_{re+} was the dominant Mt subcategory.

7.14. Conclusion

The analysis of Task 2 has revealed that the enacted object of learning was invariant across the three clusters and that it was an $\boldsymbol{Mt_{re+}}$. By way of example, across the E-Cluster and the I-Cluster the appeals were largely $\boldsymbol{Mt_{re+}}$. This might be due to the fact that evidence from Task 1, provided facilitators with insight into gaps in teachers' prior knowledge of functions which was dominated by Mt_{ap}. It is also probable that Task 1 made it possible for teachers to discern aspects of functions which were not clearer before the PD. Hence only Mt_{re+} was present in Task 2. The dominance of Mt_{re+} in both the appeals and patterns of variation suggest that in Task 2 of Functions 1 the enacted object of learning was *correct mathematical reality* ($\boldsymbol{Mt_{re+}}$). The dominance of Mt_{re+} across the three clusters was interesting because it reflected the goals of the PD. Teachers were *revisiting* the school mathematics and so the desired shift was a move from Mt_{ap} to Mt_{re+}. The shift was necessary because the analysis of legitimating criteria in **Chapter** 6 revealed that teachers came into the PD with both Mt_{ap+}, Mt_{ap-} and Mt_{re-}. So facilitators used Task 2 to close the gaps in the teachers'knowledge of functions and to deepen their SMK.

Though the enacted object of learning was the same, the PD was experienced differently by teachers. Firstly, there were differences in the adopted pedagogic strategies. The E-Cluster teachers experienced low scaffolding due to the inductive pedagogic strategy while both the I- and N-Cluster teachers experienced high scaffolding due to the deductive pedagogic strategy used in the two clusters. There were different facilitators. The E- and I-Clusters shared the same facilitators while the N-Cluster had a different facilitator. The PD was held in three different locations. However, and as noted above, the enacted object of learning was the same across the clusters in spite of the experienced differences.

CHAPTER 8: ANALYSIS OF TASK 1 OF FUNCTIONS 2

8.1. Introduction

This section of the analysis of the enacted object of learning is focused on the co-constitution of the enacted object of learning across the three Clusters where four tasks were in the fore. The goal of Functions 2 was to engage teachers in mathematical thinking and this was done using the concept of Transformation. Task 1 focused on the introduction of Domain and Range and this was done by developing links between these concepts with inputs and outputs- which were foregrounded in Functions 1. The connections were developed with the aid of the multiple representations of the same quadratic function and the multiple representations that were also foregrounded in Functions 1. Task 2 involved teachers finding functions with maximum value of 2 at the turning point. Task 3 was about finding a function with a minimum value at the point (5; 4) and Task 4 was about distinguishing between one-to-one and many-to-one functions. This analysis follows the same format across the three clusters. I begin in a grounded way to analyze the nature of the pedagogic strategy at play and the scaffolding it affords. This I followed with the analysis of the legitimating criteria and concluded with the analyses of patterns of variation to gain insight into what was made possible to learn. Thus, my analysis is divided into three subsections. I must point that at before I present my analysis I first provide a description of the lesson and I accompany this with examples.

8.2. Descriptive analysis of Task 1 of Functions 2 at the E-Cluster

- **Task 1-Event 1: Discussions focused on $f(x) = (x-1)^2 + 2$**

  - **Sub-Event 1.1: Connecting Functions 1 and 2**

Facilitator began Functions by building on what was taught in Functions 1. He reminds teachers that they learnt about inputs and outputs from the previous session and he does this with the aid of four multiple representations of the same quadratic function (see **Figure 8.1**). He pointed out at the same inputs and outputs across the three representations. By way of example, he substituted 4 into $f(x) = (x-1)^2 + 2$ to obtain 11 (see **Excerpt 8.1**). He then pointed out at the table of values which he then used to explain how 4 was the input and 11 the output.

F3:	Ok, so last lesson we were talking about functions and inputs and outputs into functions, yeah? So for example when we looked at the function, this function x minus 1 all squared plus 2, [points to this function written on the slide] an input might be 4 in a ? in and if I put 4 in then the output is 4 minus 1, so that's 3 squared, 9 plus 2, which is 11. *[points to the table]* And that's what we were looking at last lesson.

Excerpt 8.1: Creating connections between Functions 1 and 2

Building on teachers' familiarity with the terms input and output from Functions 1 the Facilitator introduced the terms domain and range by showing how they were related to input and output

- **Sub-Event 1.2: Introducing domain and range**

Working with the same table of values (see **Figure 8.1**) the facilitator pointed out that all possible inputs to $f(x) = (x-1)^2 + 2$ constituted the domain and all possible outcomes constituted the range (see **Excerpt 8.2**).

	F	Now I want to start by telling you some new language. *[as he speaks, he points to the relevant words on the screen]* We've got these things called domain and range. Now, 4 is an input, 3 is an input. If you think of all the possible inputs, every single one of them, then we call that the domain. It's the whole set of numbers that you can input. The range is all the numbers that can come out. All the different outputs together. Yeah? So in this example, um 3 is in the domain so right we can put 3 in, that's one of the many numbers in the domain. And in the range, the number 6 is in the range, and so are lots of others.

Excerpt 8.2: Introduction of domain and range terminology

In a way of illustration, facilitator called attention to 3 and 6, which he had drawn from the table of values and inserted into a mapping diagram (see **Figure 8.1**). He said 3 was one of the many numbers in the domain and 6 was one of the many numbers in the range (**see Excerpt 8.2**). In the above excerpt I have highlighted the facilitator's scaffold and also underlined his legitimating criteria.

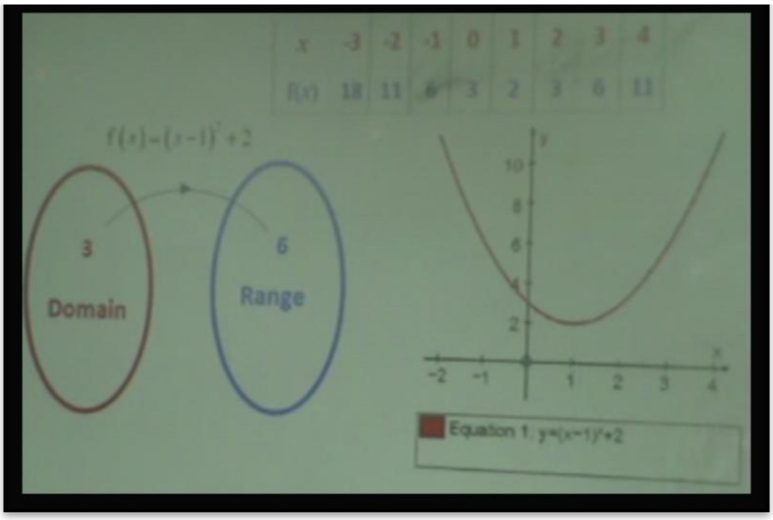


Figure 8. 1: Multiple representations of the same quadratic function

Working from the graphical representation of $f(x) = (x - 1)^2 + 2$ Facilitator asked teachers to identify the domain and one teacher said it was along the x-axis. Facilitator agreed with this and showed the rest of the teachers where the domain was located on the graph and on the table of values.

90 91	F:	Can you tell me on this graph *[points to the graph on the screen]* where do you see the domain? Can you see it anywhere on that graph?
93	T3:	I can see it on the x axis.
95 96 97 98	F:	Go on. So… So it's all of these numbers. Alright, so that's, that's, because that's the numbers you're putting in and it corresponds with those there. *[points to the table]* That's right. Ok? Somebody else, what about the range? Where can you see the range on this?
101	T3:	On the y axis.
102 103	F:	On the y axis. Can you tell me what the range is? Have you got a way of describing the range in words or numbers?
105	T3:	That is the answer that we get when we replace x at the domain
106	F:	Ok

Excerpt 8.3: Teacher 3's responses to the Facilitator's questions

Building on the discussion around the domain he asked teachers to identify the range from the graph on Figure 8.1. He followed this by asking the teacher to describe the range in words. Lines 93 and 106 of Excerpt 8.3 contain Teacher 3's responses to the facilitator's questions about the range. The teacher's response was that the range was a number that was obtained when x was replaced at the domain. The highlighted text in Excerpt 8.3 shows evidence of scaffold in the form of probes and the underlined text is the legitimating criteria which the facilitator drew on to ground the notion of domain.

- **Sub-Event 1.3: Discussion focused on zero from the table of values**

While still on the range, but now speaking to the table of values, Facilitator asked if the number zero was in the range. I must point out that the facilitator was asking because the table did not contain zero. He was deliberately not asking from the graph because there it would have been easy to discern that it was not in the range. Speaking from the graph, a teacher said zero was not in the range because the graph was turning above zero. The facilitator agreed with the response and demonstrated from the graph to the rest of the teachers why zero was not in the range.

113 114	F:	Ok, ok ja, good. So tell me… I can see that these numbers are in the range, *[points to the table]* is the number zero in the range?
116	T3:	No
118	F:	Why not?
120	T3:	Because the curve is above zero.
122 123 124	F:	Ok, so it goes back to what you say there that the range, if I'm looking on the y axis and because this number zero doesn't fall within the graph, then you say zero's not in the range.

Excerpt 8.4: Discussion focused on the range

Excerpt 8.4 contain evidence of scaffold and is in form of a probe and there are two instances of legitimation – with one coming from Teacher 3 and another from the facilitator

- **Sub-Event 1.4: Discussion focused on the smallest number in the range from the graph**

Following on the discussion focused on zero, facilitator asked for the smallest number in the range. After discussion one group's response (given by Teacher 5) was that 2 was the smallest number in the range and other groups were in agreement. Facilitator concluded that the range was all numbers greater or equal to two (**see Excerpt 8.5**).

126 127 128	F:	Ok, so what's the smallest number that's in the… Think about that for a minute, check with the people on the table that they think the same thing and I'll pick on one table to tell me what's the smallest number in the range
134	F:	Ok, let's, let's have a look. Let's see what you have to say about that?
136	T5:	2
138 139	F:	Please tell me what your answer is? Ok. Do the other tables agree? They say 2. Do you agree?
141	T6:	Yes
143 144	F:	Do you agree with 2? Because at the minute they're saying 2. So the range is all the numbers – all numbers greater or equal to 2. Yeah? Well I can see.*[points to the graph]*

Excerpt 8.5: Discussion focused on smallest number in the range

The above excerpt contains probes which constitutes scaffold and an instance of legitimation which was used to ground Teacher 5's response.

- **Sub-Event 1.5: Discussion focused on the number 4 from the table of values**

Moving on but building on Sub-Episode 1.3, the facilitator asked if 4 were in the range. The probe was prompted by the fact that the table of values in **Figure 8.1** did not explicitly include 4 in the range. Teacher 7 responding on behalf of one group said 4 was in the range and the Facilitator asked teachers to explain why 4 was in the range even though 4 was not among the outputs (see **Lines 149 – 151** of **Excerpt 8.6**).

145	F:	Is 4 in the range?
147	Trs	Ja
149 150 151	F:	It's not on this list *[points to the table]* But go on then, explain to me why 4 is in the range but it's not in here. *[points to the table]* What do you think? *[points to a teacher]* Can you see?
153 154	T7:	Yes, 4 it's also in the range by this reason that in the range we have got 2 and 3. And in between 2 and 3 is this… Ah, we've got 2… We have got 3 and 6
156	F:	Yeah
158	T7:	So between 3 and 6 obviously 4 is also there.
162	F:	Ok, so although it doesn't correspond to an integer
164	T7:	Yes, yes
166 167 168	F:	There will be a number somewhere between 2 and 3 that gives you 4. Perfect. Yeah. Very good. Excellent. Um, so all the numbers greater or equal to 2 are in the range.

Excerpt 8.6: Discussion focused on the number 4

Speaking from the table of values, Teacher 7 reasoned that since 3 and 6 were two numbers in the range, then the number 4 was between the two. Elaborating further on this, the facilitator pointed out that there was a number in the input set between 2 and 3 that corresponded to 4 in the output set. To this point the facilitator was satisfied by the teachers' progress and this was evident in his commendations (see **Lines 166 – 168**).

- **Sub-Event 1.6: Discussion focused on the number 2 in the equation**

Facilitator called attention to the number 2 in the equation $f(x) = (x - 1)^2 + 2$ and asked if it was coincidence that the smallest number in the range was 2. He went on to ask if the 2 in the equation and the smallest number in the range were perhaps the same value. A teacher responded that the 2 in the equation will always be the range. Facilitator asked him to explain why this will always be the case and he said because the 2 in the equation was the y-value in the turning point (**see Excerpt 8.7**).

170 171 172 173	F:	Look at that *[B points to the equation* $f(x) = (x - 1)^2 + 2$ *that is on the board]* I can see the number 2 there. Is that coincidence or can you explain if that 2 *[points to the* 2 *in the equation]* is the same as the 2 that you've just discussed? Is it… Is it surprising or is there a good reason why we'll get the same number? *[points to T3]* Ja?
175	T3	That, that will always be your range.
177	F:	That's a big statement. Go on. Why?
179	T3:	Because that is the y value of your turning point which determine where the graph turns
181 182	F:	How do you know? How do you know that the turning point is going to be this number? *[points to the* 2 *in the equation]*

Excerpt 8.7: Discussion focused on the number 2 in the equation

Taking the discussion further, the facilitator asked Teacher 3 why the turning point was always going to be 2, and his response was that the equation was in the form $\boldsymbol{p}(x - a)^2 + \boldsymbol{q}$. Hence $\boldsymbol{q}$ was the y-value at the turning point. To help teachers understand this, the facilitator asked teachers for the smallest value of $(x-1)^2$ which was found to be zero when x = 1. Facilitator asked whether the smallest value could be lower than zero and the teachers said it could not because the square of any number could never be less than zero. Facilitator pointed out that 2 was the minimum value of the graph and also the smallest in the range. Facilitator pointed out that he deliberately chose to work with $(x-1)^2 +2$ rather than (x-1)(x-1) + 2 or $x^2 -2x + 3$ because it would have been more challenging for teachers to recognise that 2 was the minimum value (see **Excerpt 8.8**)

F:	Now imagine if you'd come into the room *[on the flipchart, B crosses out* (x – 1) (x – 1) + 2 *]* and that was what I was asking you about *[points to:* $x^2 - 2x + 3$*]* What's the minimum value of this?
T12:	No
F:	That's not nice. That's hard.
T13:	Mmm

Excerpt 8.8: Discussion focused on choice of examples

Facilitator wrote (x)(x) and asked teachers which factors would go into the brackets. One teacher suggested 1 and 3 but another said there were no roots because though the product of 1 and 3 was 3, the sum of 1 and 3 was not negative 2. In response the facilitator pointed out that the reason why the graph turned above the x-axis was due to the nonexistence of real roots. Hence the function could not be written in the form (x)(x). To conclude the discussion on Episode 1 he explained that he had been working with a function with domain all real numbers and the range values of y greater or equal to 2.

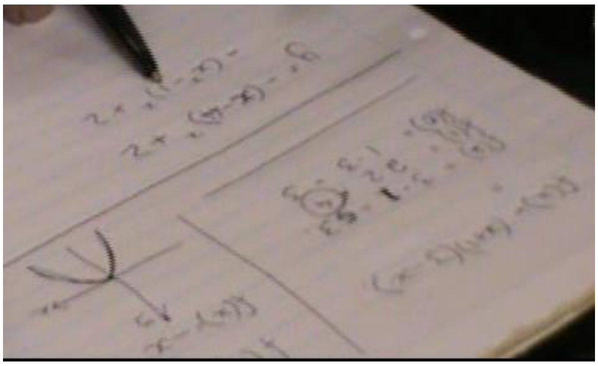
- **Task 1-Event 2: Functions with the range: $y \leq 2$**

Following on Episode 1 the facilitator challenged teachers to find a function with the range $y \leq 2$. Facilitator scaffolded the task by explaining that the function rises to 2 and then falls down. Teachers discuss in groups to find the required functions and this proves to be a challenging task for them. Different strategies are suggested in each group and when this does not lead anywhere, they are abandoned and replaced with others. At the end the responses were as follows:

- **Sub-Event 2.1: Group 1's response**

Group 1 produced $y = -(x-1)^2 +2$ as the function with the range $y \leq 2$ (see **Figure 8.2**)

Group 1



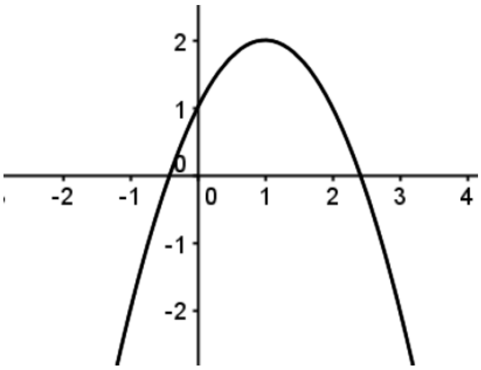


Figure 8.2: Group 1's working and the resultant correct graph

- **Sub-Episode 2.2: Group 2's response**

Group 2 produced $y = (x+1)^2 + 2$ but realised it was yielding y values greater than 2. As a result they inserted the minus sign before the brackets and the correct resultant equation was $f(x) = -(x+1)^2 + 2$. **Figure 8.3** shows Group 2's initial incorrect graph drawn on paper and the correct graph next to each other.

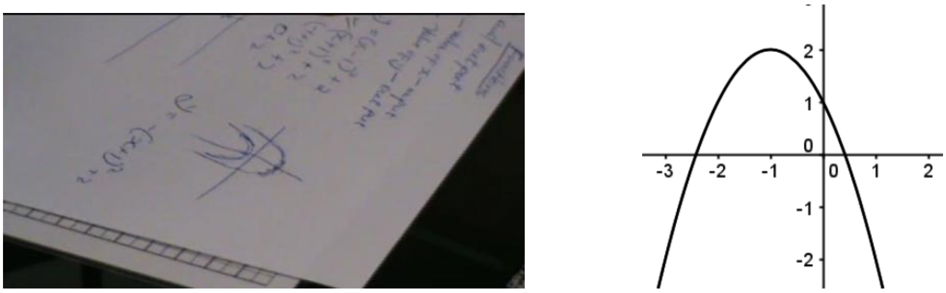


Figure 8.3: Group 2's working and the resultant correct graph

- **Sub-Event 2.3: Group 3's response**

Group 3 produced the equation $(x - 1)^2 - 2$, which was incorrect.

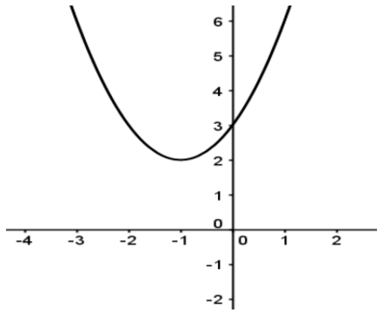


Figure 8. 4: Group 3's resultant incorrect graph

At the end a teacher from Group 2 was asked to explain on the flipchart how the group found the correct function. He explained that with the initial equation they realised that whenever they substituted a value of x into the brackets the result was greater than 2. As a result they inserted a negative before the brackets and this produced the correct equation.

551 552 553 554 555 556 557 558	T4	*[adds in a square to the equation on the flipchart]* Then we started to substitute. What happens is we were getting the values of, the values of y, being, going up, being above 2 *[thus on the flipchart T4 has written:* $y = (x + 1)^2 + 2$ *]* which will be 3, you know, further and whatever. Then as we were trying then I kind of… Oh now the graph is going down. <u>Oh, it means therefore that this graph must have a negative sign here so that it will turn down</u>. *[writes:* $y = -(x + 1)^2 + 2$ *]* Now then when we are working, substituting these values of y, then it allows us to have these values of y which are less than positive 2 going down.
562 563	F	Very good. Excellent. So I'm going to check on here now, make sure you're right. So you said y equals minus

Excerpt 8.9: Teacher 4's explanation of Group 2's response

Following on teacher 4's explanation the facilitator use the Geogebra to draw the graph of the function identified by Group 2 to confirm if it was correct. However, before doing this he asked teachers for the value of x where the graph was maximum and x = - 1 was suggested. One teacher asked why negative 1 was the correct value to substitute into the bracket and another said the reason was that when $x = -1$ the equation $y = -(x + 1)^2 + 2$ became $y = (-1 + 1) + 2$ and yielded $y = 0 + 2 = 2$. To confirm the teacher's response the Facilitator asked him if he was saying that to get the equation to produce a maximum value of 2 then $(x+1)^2$ should be zero and he agreed. The facilitator proceeded to produce the graph of $y = -(x+1)^2 +2$ (see **Figure 8.5**)

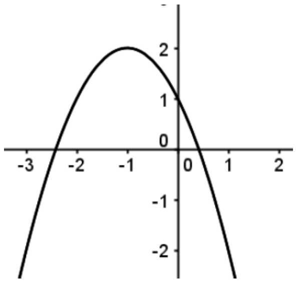


Figure 8. 5: Graph of y = - (x +1)2 +2 drawn by facilitator using Geogebra

From the graph, when x = -1 the graph has a maximum at positive 2. Facilitator then asked for the simplest example of a curve that has a maximum value at positive 2. One teacher suggested $-x^2+2$. Building on this, the Facilitator pointed out that adding 1 to x inside the brackets, i.e. -(x +1) changed the shape of $y = -x^2 + 2$. He then rewrote this equation as $y = 2 - x^2$.

- **Event 3: A function with the minimum-value at the point (5; 4)**

Facilitator gave teachers the second task in which he wanted teachers to produce a function with a minimum value at the point (5; 4). Teachers worked in their groups to find the required function. **Excerpt 8.10** contains a discussion between Group 1 members. It is evident in the discussion that the group was able to produce the correct function.

668	T3	What's the lowest point?
670	F	(5; 4) its lowest point.
676	T3	So that should be 0 because then it's
678 679	T2	'T3 what are you saying? At this point here, that this there, at the break there is zero.
681	T3	Ja
683	T2	Right? Then we shove it up 4.
685	T3	Ja. So that's what that's doing. But now for it to be, the y to be 5
687	T1	The y is 4
689	T2	The y is 4
691	T3	And it's 5. No man. No, it's right. That should be 4.
693 694	T2	Because we're starting now. If we started putting 5 in there and 4 in there. *[points to the equation T3 has written: $y = (x-5)^2 + 4$]* Right?
696	T3	Ja
698 699	T2	So this had better be 4. <u>You're right with the 4 because this whole thing is zero.</u> So if, if a kid says to you…
703	F:	Ok. It is interesting. All 3 tables have come up with exactly the same answer.

Excerpt 8.10: Group 1 discussion

Following teachers' discussions on the task, the facilitator pointed out that all the 3 groups found $f(x) = (x-5)^2+4$ which was the correct function. He drew the graph using the Geogebra (see **Figure 8.6**).

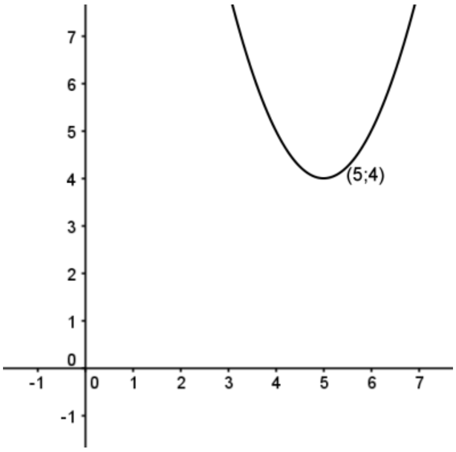


Figure 8. 6: Quadratic Graph with a turning point at (5; 4)

Facilitator asks if there are more such functions, and one teacher suggested that inserting a positive number in front of $f(x) = (x – 5)^2+4$ would generate more graphs (see **Excerpt 8.11** and **Figure 8.8**.). Facilitator worked from the Geogebra to generate more such functions with the aid of a slider.

717 718	F	Are there other ones, other curves with the Minimum value at (5;4)?
720 721	T3	If you put any number, positive number in front of the bracket, the turning point will stay the same.
723	F	Ah ha
725	T3	It will just change the shape.
727 728 729 730 731 732	F	Ok, *[inputs this into the computer]* Well, I'll use geogebra to do this and geogebra's giving me a value base. I'm going to put b. Oh, it's gone because of course it's taken away from the completed square form, so I'll do it again. y equals this number b which at the moment is 1. *[points to the overhead]* So geogebra is going to interpret b as being 1 until I play with that slide bar. Times x minus 5 squared plus 4. <u>And yeah, when b is 1, you can't see because it's hiding behind the original</u>

Excerpt 8.11: Teacher 3's response to the probes

In further discussion, the facilitator explained what happened when the value of *b* changed from 2 to 1 and to 0. The Facilitator also generated graphs for negative values of *b* (see **Figure 8.7**). One teacher observed that when b became zero then the graph was a straight line. Adding to this observation the Facilitator explained that when $b = 0$ in $y = 4 + b(x – 5)^2$ then $b(x – 5)^2$ got wiped out and the equation became $y = 4$. Hence a straight line was observed. He also explained that when b was negative the graph dropped down below positive 4 because then all numbers coming out of the function were less than 4 (see **Figure 8.7**).

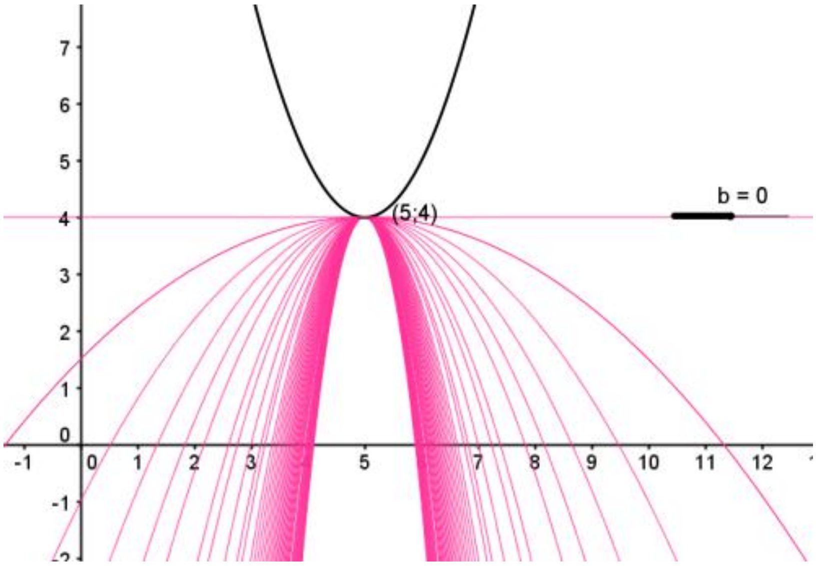


Figure 8. 7: Multiple graphs with the turning point at (5; 4)

This explanation marked the end of work and discussions around the range of a function with the next foci on domain.

- **Event 4: Functions with domain all real numbers**

Facilitator asked teachers to produce functions with the domain not all real numbers. He advised then not to be contented with finding one function but each group should find more than one. Teachers worked on the task as the facilitator went around noting teachers' productions and writing these up on the flipchart for later discussion (see **Figure 8.8**).

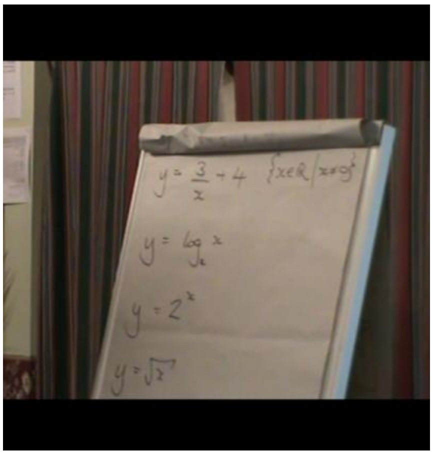


Figure 8.8: Functions with domain not all real numbers

When the Facilitator had obtained a sufficient number of teacher productions and written these on the flipchart he called all the groups to attention. Each group was asked to choose one function from the flipchart and identify its Domain.

- **Sub-Event 4.1: Discussions on domains of three different functions**

The first group chose the hyperbolic function $y = \frac{3}{x} + 4$. A teacher from the group said the Domain for this function was all real numbers except zero. The Facilitator wrote this formally in mathematical notation as $\{x \in \mathbb{R} \mid x \neq 0\}$. The second group chose $y = 2^x$. A teacher from the group pointed out that the function would take in any number. Hence it was concluded that its domain was all real numbers and was written formally as $\mathbb{R}$ (see **Excerpt 8.13**).

894 895	F:	What about… I'll give you some choice *[points to all the options on the flipchart],* Group 1, your table, take your pick.
897	T3:	*[chuckles]* Ag, it doesn't matter: y is equal to 2 to the power of x.
899 900	F:	*[points to:* $y = 2^x$ *]* This one, ok? I saw this answer somewhere, so what's your thoughts on it? Group 1 table, tell me what the domain is for this one.
904	T3:	If I put negative 1, it's a half. Negative 10, you can put any, any number.
906 907 911 912	F:	So for this one the domain is our… And T35 gave me this one and said, 'Cross it out.' I refused. *[laughter]* Because it's always worth seeing wrong answers. And he realised it was wrong. So in this case the domain is the real numbers.

Excerpt 8.13: Teacher 3's response on the domain of $y = 2^x$

Facilitator mentioned that he obtained the function from one group and when the group realised it had no restriction they asked him to delete if from the list but he refused because it was always worth working with wrong responses. Building on this the facilitator asked if the function could ever be zero or a negative and the response was that the range will not include zero or negative values because it will always have numbers greater than zero (i.e. the range was $y > 0$). The third Group chose $y = \sqrt{x}$ and led the discussion about its domain. A teacher from the group mentioned that the domain was x greater or equal to zero. This response was written down formally on the flipchart as $\{x \in \mathbb{R} \mid x \geq 0\}$. Facilitator explained that the domain was non-negative real numbers and not positive real numbers because zero was a member of the domain but was neither positive nor negative.

- **Sub-Event 4.2: Discussion focused on the domain of $y = \sqrt{(x-5)}$**

Facilitator called attention to the function $y = \sqrt{(x-5)}$ for the whole class to talk about its domain. A teacher suggested that the domain was all numbers greater or equal to 5 and the Facilitator wrote this down on the flipchart as $\{x \in \mathbb{R} \mid x \geq 5\}$. Following on this response Facilitator asked teachers if they had an idea of how the graphs of $y = \sqrt{x}$ and $y = \sqrt{(x-5)}$ would look like (see **Excerpt 8.14**).

965 966	F:	What about this? No, let's do this together *[points to:* y = √ x-5 *]* What do you think?
968	T33:	Um, it's any number greater or equal to 5
970 971 972	F:	Ah, that's interesting. *[next to:* y = √ x-5 *writes:* {x ∈ ℝ | x ≥ 5} *]* They look quite similar these two. And any idea what those two graphs look like? *[on the flipchart points to:* y = √ x *and* y = √ x–5 *]*
974	T34:	Um…
976	T35:	They're sitting… That one is sitting at zero and going up. That on is sitting at 5
978	T4:	Say a bit louder please?
980 981 982 983 984	F:	Ok, so this one sits at zero. *[points to:* y = √ x *]* It starts from zero and goes that way *[demonstrates the direction using his hands]*. It doesn't actually have the bottom bit because the definition of this *[again points to* y = √ x *]* is the positive. To get, to get a bit below you would have to put in a minus and then it would not be a function. Ok, and that is to the right of 5.*[points to:*y=√(x–5)*]*

Excerpt 8.14: Discussion focused on graphs of $y = \sqrt{x}$ and $y = \sqrt{(x-5)}$

One teacher said the graph of $y = \sqrt{x}$ would start from zero and move up to the right and Facilitator added that for $y = \sqrt{(x-5)}$ the graph would start from positive 5 and move up to the right (see **Figure 8.9**).

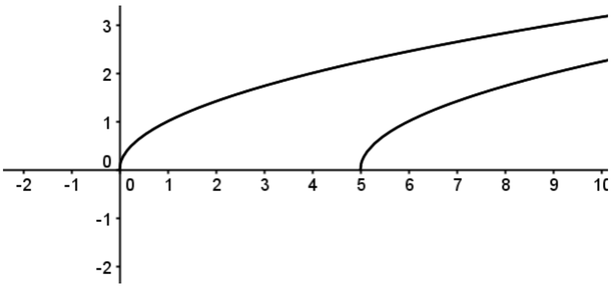


Figure 8. 9: Graphs of y=√(x) and y=√(x-5)

He explained that the graphs do not have negative values of y because by definition the square root symbol (i.e. √) implies that all y values are non-negative. He pointed out that when the negative values are included such that the equation becomes $y = \pm\sqrt{x}$ then the graph will not represent a function (see **Figure 8.10**).

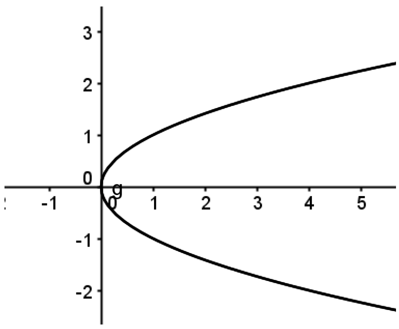


Figure 8. 10: Graph of y=±√x

- **Sub-Event 4.3: Discussion focused on $y = \log_2 x$**

The last discussion focused on the equation $y = \log_2 x$. A teacher pointed out that the log function was related to the function $y = 2^x$. Building on this observation the Facilitator drew the graph of $y = 2^x$ so that it can provide insight into the nature of the graph of $= \log_2 x$. Once the graph was up Facilitator drew teachers' attention to $y = \log_2 x$ which he then rewrote as $x = 2^y$ (see Figure) which was an inverse function of $y = 2^x$.

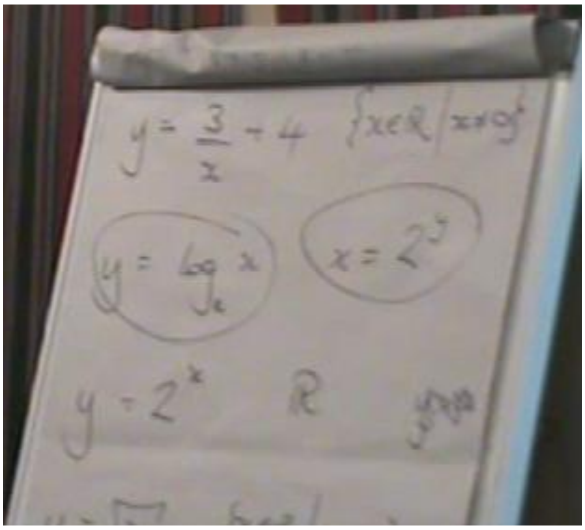


Figure 8. 11: The circled equations are equivalent

The facilitator pointed out that $y = \log_2 x$ would produce the inverse graph of $y = 2^x$. He called a teacher to the front to sketch the inverse graph. Following on this the facilitator drew the actual inverse graph using the Geogebra and was successful. **Figure 8.12** shows a red graph of $y = \log_2 x$ which the facilitator had drawn. The graph is similar to the sketch which was earlier drawn on the board by one teacher. This graph shares the same axes with the exponential graph of $y = 2^x$.

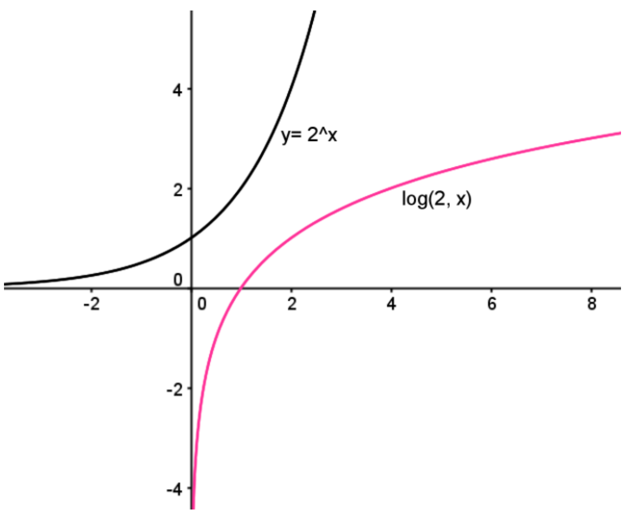


Figure 8. 12: An exponential and a logarithmic graph

- **Event 5: Discussion focused on one-to-one and many-to-one functions**

Facilitator introduces the notion of one-to-one function which he said would be important in the second task. He pointed out that knowing one-to-one functions was an important concept in mathematics because it allowed one to work backwards. That is, when given the output, one can work backwards to obtain the input. He followed this explanation by putting up a slide showing two graphs. One representing one-to-one and another many-to-one function (**see Figure 8.13**). He illustrated this point by saying that when he inputs a number into $y = \frac{12}{x}$ the output was 2. He asked teachers to find the input and they found 6 (see **Figure 8.13**).

1090 1091 1092 1093	F:	A one-to-one function is important in mathematics because it allows you to work backwards. Now look at this one. *[points to a graph on the overhead]* I've put in a number into this function 12 over x, and out has come the number 2. What number did I put in?
1095	Trs:	6
1097	F:	So it's 6. Ja, Out came the number 2 and the number *[looks at the graph]* is 6.

Excerpt 8.15: Discussion focused on on-to-one function

He then called attention to a many-to-one function and asked teachers to find the input if the output were 2. Teachers found three possibilities and the Facilitator observed that it was difficult to say which of the three was the input because it could have been any of the three (see **Excerpt 8.16**).

1098 1099	F:	What about this one? *[points to another graph on the overhead]* This one I've got a number and out came 2. What number did I put in?
1101	T41	Negative a half
1103	F:	No
1107	T3:	or positive x.
1109	F:	There's all sorts of things. If 2 comes out…
1111	T3:	Sorry
1113 1114	F:	… it could have been that one or that one. You'd better go for all three then *[chuckles]* Well, the point is that you can't tell.
1116	T3:	Ja
1118 1119 1120 1121 1122	F:	You can't be sure. If you'd been lucky and I'd said 'Well out comes the number 4' then sure enough, there's only 1 possible input. So if we've got a function like this and it's got the horizontal ruler test, as the horizontal ruler goes up the page it never crosses the curve more than once. Sometimes horizontal will miss it altogether. It doesn't matter. But here, wherever it goes, the horizontal ruler cuts it only once.

Excerpt 8.16: Discussion focused on many-to-one function

Going back to the hyperbola he demonstrated how the horizontal line test could be used to determine if a function was one-to-one or many-to-one. He showed on the hyperbola that the horizontal line test cut the graph once but cut the cubic graph more than once.

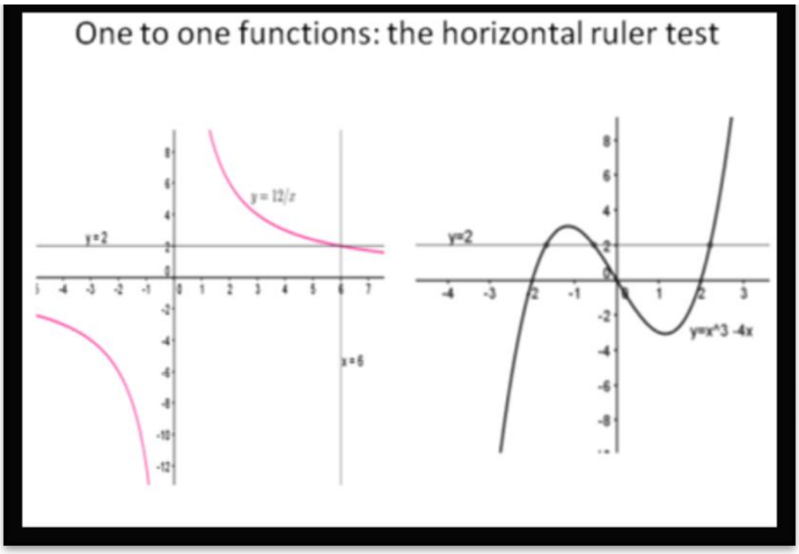


Figure 8. 13: Graphs of one-to-one and many-to-one functions

He, however, pointed out that there are instances where the horizontal ruler test would cut a many-to-one function once and he gave 4 as an example of the output which has one input.

8.3. Discussion of pedagogic strategies used in Task 1 of Functions 2 at E-Cluster

The facilitator began the session by creating links between what was made possible to learn in Functions 1 and he did this through a resource which had aspects of Functions 1 and aspects of the new session. The purpose of the resource was to help teachers to work with and see connections between the past and current sessions. The task required teachers to bring into focus their prior knowledge of input and output which were the mathematical language which was foregrounded in Functions 1. These terminologies were central in the definition of function during Functions 1. The link that the Facilitator wanted to create was between these terminologies (input and output) and domain and range- to which input and output were elements respectively. Creating the link between the two sessions was the first form of scaffolding and the second was evident through the resources which were foregrounded at the beginning of the session (see **Excerpts 8.1, 8.2** and **Figure 8.1**). These were the multiple representations of the same quadratic function and they included a table of values, an algebraic equation and a graph. The Facilitator used these representations interchangeably to show the links between input and domain and between output and range. For instance he obtained 4 from a table of values and substituted it into the algebraic equation to produce 11 which he said was the output. I have provided a detailed description of how this was done in the above section where I present a description of Functions 2 at the E-Cluster. It is noticeable in **Excerpts 8.1** and **8.2** that the facilitator's *natural attitude* was bracketed. That is, he did not take it for granted that the object of learning was easy for teachers to comprehend with little or no scaffold. Hence the facilitator worked with the *deductive pedagogic strategy* and this strategy was evident in the amount of scaffold that was provided to teachers across the tasks.

The facilitator ensured that all the necessary information and assistance were ever present for teachers to progress on tasks. By way of example, work on most tasks was preceded by the facilitator's explanations, demonstrations, and illustrations which teachers later drew from to work on their own. In addition to this, the Facilitator moved around providing help in the form of cues and prompts when teachers are working on assigned tasks. Facilitator provided resources which he was using to explain and also for teachers to use while working tasks. He also moved about rendering individual assistance to teachers working in groups.

It is interesting that unlike in Functions 1 where teachers did not appear to draw from resources which Facilitators made available - here teachers were working with the scaffold that was provided. Consequently, teachers experienced high scaffold and their productions on assigned tasks were mostly correct. By way of example, all the three groups were able to produce a function where the domain was not all real numbers; and when given a function with the domain not including all real numbers, each group was able to give the correct domain for that particular function. In another task all the groups were able to find a function where the graph had a minimum value at the point (5, 4) and this was made possible by the fact that previously the Facilitator had worked with teachers using the Function $f(x) = (x-1)^2 + 2$. The scaffolding that the Facilitator performed was when he helped teachers to discern that in this function the graph had a minimum value at the point (1, 2). So when working on the task involving (5;4) teachers were able to produced the required function by just replacing 1 with 5 and 2 with 4 to get $f(x) = (x-5)^2 + 4$. Hence I have pointed out above that teachers appeared to be working with availed scaffolding and this may have contributed to the correct responses which teachers produced across the tasks.

8.4. Discussion of the legitimating criteria at the E-Cluster

The enactment was also characterised by the legitimating criteria which were co-constituted by the Facilitator and the teachers. The first legitimating criteria came into existence when a teacher said that zero was not in the range of $f(x) = (x-1)^2 + 2$ because its graph did not touch the x-axis where $y = 0$. The appeal associated with these criteria was an appeal to a graphical representation of a function. This criteria arose from the Facilitator's question where had asked if zero was in the range of $f(x) = (x-1)^2 + 2$. The second legitimating criteria emerged when a teacher was responding to a similar question where the facilitator had asked if 4 were in the range of the same Function. The Facilitator was basing his question on the fact that 4 was not one of the outputs in the table of values just as was the case with zero. The response was that since 3 and 6 were in the range and 4 were a number between these two, then 4 was also in the range even though it did not appear in the table of values. The teacher was appealing to the numerical representation with attention to the relationship between the three numbers (i.e.3, 4, and 6) (see **Excerpt 8.6**). The third legitimating criteria came from a teacher who was explaining that in $f(x) = (x-1)^2 + 2$ the number 2 will always be the smallest value in the range because when the function is in the form $f(x) = a(x-p)^2 + q$ and a is positive, then q is the y value at the turning point (see **Excerpt 8.7**).

The fourth legitimating criteria was present when the focus was on $(x-1)^2$. Here the Facilitator asked teachers find the smallest possible value of $(x-1)^2$. One teacher found that zero was the smallest possible value and so he was asked if it was possible that the lowest value can be smaller than zero. His response was that zero was the smallest because it is not possible for the square of any number to be less than zero. The teacher's appeal here was to his knowledge of square numbers. The fifth legitimating criteria appeared when one teacher was responding to another teacher's earlier response. Here, the facilitator had asked for factors that would go into $(x\ \ )(x\ \ )$ such that when the brackets are expand they would produce $f(x) = x^2 - 2x + 3$. The first teacher had suggested that 1 and 3 were the required factors and the second said that was not possible because 1 and 3 would multiply to produce 3 but their sum would not be negative 2. The appeal here was to knowledge of factorization of quadratic expressions.

The sixth legitimating criteria came from the facilitator in his support of the teacher's explanation about 1 and 3 not being factors of $f(x) = x^2 - 2x + 3$. He added that the function did not have real roots and that was the reason why the graph turned above the x-axis. The Facilitator was appealing to the nature of the roots of a function. The seventh legitimating criteria came into being when attention was on teachers producing a function where the maximum value was 2. One function which was produced by one group was $y = -(x+1)^2 + 2$. Building on this facilitator asked for the value of x at which the function had a maximum. One teacher's response was that it had a maximum value of 2 when x = -1 because $-(x+1)^2$ becomes 0 and 0 + 2 = 2 and this left function with a maximum value when y = 2.

The teacher here was appealing to mathematics procedure. The eighth legitimating criteria was present when the discussion was around graphs of Functions with turning point at (5;4). Facilitator had asked teachers to generate functions with turning point at this point and all the three groups produced the same function (i.e. $y = (x - 5)^2 + 4$). The Facilitator then worked from the Geogebra software to produce multiple functions that met the condition. To make this possible he set up a slider such that many functions were produced when he moved the slider. He then had to explained that the graph of $y = (x - 5)^2 + 4$ turned into a straight line because the value of b was zero such that the part $(x - 5)^2$, became zero and the remaining part was only y = 4, which was a straight line. The appeal associated with this was an appeal to mathematics procedure. The ninth legitimating criteria were present in the same discussion but this was related to why the graph dropped below 4 such that 4 was the maximum value. Facilitator explained that the graph dropped below 4 because when b was negative in $y = b(x - 5)^2 + 4$ all values coming out of the function were less than 4. As such the largest value was 4 and this was at the turning point (see **Figure 8.7**). The appeal here was still to mathematics procedure. The tenth legitimating criteria came into being when attention was on the domain. Here the Facilitator explained that the domain $\{x \in \mathbb{R} \mid x \geq 0\}$ should not be read as positive real numbers but non-negative real numbers because it included zero and zero was not a positive or negative number. The Facilitator was appealing to the constitution of integers.

The eleventh legitimating criteria was present when the Facilitator explained that the graphs of $y = \sqrt{x}$ and $y = \sqrt{(x-5)}$ did not include negative numbers because by definition the square root symbol (√) implied that all y values were non-negative (see **Excerpt 8.14**). The appeal here was to the definition of the square root symbol. The twelventh legitimating criteria came into being when the discussion focused on one-to-one and many-to-one functions. Here the Facilitator mentioned that one-to-one Functions were important because they make it possible for one to work backwards. That is, when one knows the output, it is possible to go back and find the output. This is not possible in a many-to-one Function. The appeal associated with these criteria was to behaviour of one-to-one Functions.

Table 8.1: Summary of the distribution of appeals at E-Cluster

Legitimating criteria	**Appeal**	**Enacted object**
zero is not in the range of $f(x) = (x-1)^2 + 2$ because its graph did not touch the x-axis where y = 0.	appeal to graphical representation	Mt_{ap+}
since 3 and 6 were in the range and 4 were a number between these two, then 4 was also in the range even though it did not appear in the table of values.	appealing to the numerical representation	Mt_{ap+}
zero is the smallest because it is not possible for the square of any number to be less than zero.	appeal to knowledge of square numbers	Mt_{re+}
not possible because 1 and 3 would multiply to produce 3 but their sum would not be negative 2.	appeal to parameters of a quadratic function	Mt_{re+}
the function did not have real roots and that was the reason why the graph turned above the x-axis.	appeal to the nature of the roots of a function	Mt_{re+}
was that it had a maximum value of 2 when x = -1 because $-(x+1)^2$ becomes 0 and 0 + 2 = 2 and this left function with a maximum value when y = 2.	appeal to mathematics procedure	Mt_{re+}
, became zero and the remaining part was only y = 4, which was a straight line.	appeal to mathematics procedure	Mt_{ap+}
the graph dropped below 4 because when b was negative in $y = b(x-5)^2 + 4$ all values coming out of the function were less than 4.	appeal to mathematics procedure	Mt_{ap+}
because it included zero and zero was not a positive or negative number.	appeal to the constitution of integers	Mt_{re+}
because by definition the square root symbol (√) implied that all y values were non-negative.	appeal to the definition of the square root symbol	Mt_{re+}
One-to-one Functions are important because they make it possible for one to work backwards.	appeal to behaviour of one-to-one Functions	Mt_{re+}

From Table 8.1, it can be discerned that at the E-Cluster the co-constitution of the enacted object of learning was characterised by appeals to both correct mathematical reality ($\boldsymbol{Mt_{re+}}$) and correct mathematical appearance ($\boldsymbol{Mt_{ap+}}$), and a larger proportion of the appeals were to $\boldsymbol{Mt_{re+}}$.

8.5. Discussion of what was made possible to learn at the E-Cluster

To gain insight into what was made possible for teachers to learn in Functions 2 I recruited tools from Variation Theory and in particular I worked with patterns of variation. These patterns are types of awareness which the learner might experience in the enactment of the object of learning. Experiencing these patterns increases opportunities to discern the critical features of the object of learning. Elsewhere, I elaborated on these patterns and here I am only working with them to gain insight into what was made possible to learn. In this analysis I worked with the same generalisation and contrast which were the dominant patterns of variation in the enactment of Session 2.2 at the E-Cluster.

The enactment began with the Facilitator working with the multiple representations of **f(x) = (x – 1)^2 + 2.** There the focus was on helping teachers to see the link between input and domain and between output and range. In so doing a pattern of variation was opened up and this was **generalisation**. This pattern emerged as the Facilitator was working with inputs and outputs from the table and go on to derive the same inputs from the graph, equation, and from the mapping diagram. This was evident when the Facilitator obtained 4 from the table and substituted this into the equation to produce 11. The two values (i.e. 4 and 11) were already present in the table of values and even on the graph (see **Figure 8.1**). As such, working in this way-with the multiple representations helped teachers to discern similarities between them. In another similar instance, the Facilitator brought 3 and 6 into the fore and to do this he used the table of values and the mapping of sets. He talked about 3 and 6 from each of the two representations and as such teachers were helped to discern similarity between the two representations. Consequently, the resultant pattern of variation was generalisation. Unpacking this pattern reveals that what was invariant here were the notion of input and output and what was varying were the multiple representations. Since the focus was on discernment of inputs and outputs across the four representations (see **Figure 8.1**) it suggest that what was important was therefore that which was invariant and not what was varying. **What was made possible to learn through this pattern of variation was that inputs and outputs are properties of a function but its representations are not its properties.**

In the second opportunity for **generalisation** attention was on domain and range. Here Facilitator worked with teachers to help discern the notions of domain and range across the multiple representations of the same quadratic function. Firstly, when working from the table of values he pointed out that all possible values of x that could go into f(x) = (x-1)^2 + 2 constituted the domain of the function and those that can come out constituted the range. Secondly, when working from the mapping diagram he illustrated the domain using 3 which was also present in the table of values and pointed out that it was one of the many numbers in the domain. Similarly with 6, he used this to illustrate the constitution of the range and mentioned that it was one of the many numbers in the range. It is evident here that the Facilitator is talking from two representations (mapping diagram and table of values) simultaneously to enact and visibilize the notions of domain and range.

Thirdly, when working from the graph, he asked teachers to locate the domain and range from the graph and this they did very well. Fourthly, when speaking from the table of values, he asked if zero was in the range and correct response was given. He was speaking from the table rather than from the graph because it was not easy to notice that it was not. However, in response, a teacher spoke from the graph to point that the fact that the graph did not touch the x-axis suggested that zero was not in the range. Fifthly, he spoke from the table to ask if 4 was in the range and a correct response was given from the same table. The way the Facilitator worked with the multiple representations about domain and range helped teachers to discern the same domain and range across the multiple representations of the same quadratic function and this was evident in their responses when they could answer questions related to domain or range from whichever of the present representations. In variation terms, since the important feature was invariant and what was less important was made to vary, this constituted the pattern of variation known as generalisation. **Consequently, what was made possible to learn was that domain and range are properties of a quadratic function but its representations were not its properties.**

A third occurrence of **generalisation** was present when attention was on the equation ($f(x) = (x - 1)^2 + 2$) but with particular attention on the number 2. In this instance Facilitator asked if it was coincidence that two in the equation and the turning point on the graph were the same number. By drawing attention to this value, Facilitator helped teachers to experience the same property in the equation and on the graph and this experiencing facilitated opportunity for generalization. The invariant feature, which was also the focus of the discussion, was the number 2 which was the minimum value in both the equation and the graph. In the graph this value is the y-coordinate at the turning point. The less important feature which was also varying was the equation and the graph. **As such, what was made possible to learn was that the value of q in a quadratic function of the form $f(x) = a(x - p)^2 + q$ is a property of a quadratic function but the representations were not its properties.**

The second pattern of variation which contributed to the co-constitution of the enactment was **contrast**. This pattern appeared first when attention was on the different ways to write the equation $f(x) = (x-1)^2 +2$. Facilitator presented the other two ways in which he could have chosen to work with this equation but chose not to because it would have been more challenging for teachers to notice the value of q from the these forms. All the four forms of the equation are shown below with the second, third and fourth on the list being the forms which the Facilitator said would have masked the minimum value if he had chosen to work with anyone of them instead of the first one. The Facilitator pointed out that the difficulty of identifying the minimum value increases as one moves from the first form to the fourth. Note that the fourth form has unknown values because the Function under discussion does not have real roots and while the facilitator chose not to insert any value, I have inserted unknown values (i.e. ***a*** and ***b***)

1. $f(x) = (x - 1)^2 + 2$
2. $f(x) = (x - 1)(x - 1) + 2$
3. $f(x) = x^2 - 2x + 3$
4. $f(x) = (x - \boldsymbol{a})(x - \boldsymbol{b})$

By bring all four forms of the same equation opened an opportunity for experiencing difference between the different written forms of the same Function with particular focus on what important features each visibilize or mask. By way of example, the last three forms are not in the form $\mathbf{a(x - p)^2 + q}$ and so the visibility of each parameter diminishes as one moves down from the first to the fourth form. The values of a, p and q determine the essential meaning structure of the Function (.i.e. its constitution). As an illustration - ***a*** determines the size of the graph; p determines horizontal shift; q is the minimum or maximum value of the Function at the turning point. Since attention was onto the different equations with respect to how each visibilize or mask parameters, then these constituted the critical features with the Quadratic Function as the dimension of variation in which the four equations are values. **Teachers were helped to experience difference between the four equations of the same Quadratic Function and this made it possible to discern that some equations visibilize and others mask the Quadratic Function's essential meaning structure.**

An instance where we notice the fourth occurrence of **generalisation** when attention was on finding functions with the range $y \leq 2$. The following functions were produced by teachers working in groups

1. $y = -(x-1)^2 + 2$
2. $y = -(x + 1)^2 + 2$
3. $y = 2 - x^2$

Engaging teachers in this task helped them to experience similarity between the Functions that shared the same range. What was invariant was the range and the critical features were the functions. Attention was on the range (invariant feature) and not on what was varying (critical features). **As such what was made possible to learn that different functions can share the same range.**

A fifth happening involving generalisation was present when attention was on finding functions with a minimum value point at (5;4). The first was produced by all the three groups and the second was added by the Facilitator who worked with it on the Geogebra to generate multitudes of graphs with a minimum value at (5; 4).

1. $f(x) = (x-5)^2 + 4$
2. $f(x) = b(x - 5)^2 + 4$

Working with these functions made it possible for teachers to experience similarity between functions with a minimum value at (5; 4). In this situation, the minimum value was invariant while the different functions varied and these were the critical features. **Since what was foregrounded was that which was invariant and what was varying was not in focus,**

then what was made possible for teachers to learn was that different functions can share the same minimum value.

The second emergence of contrast occurred in the same situation where attention was on functions that had a turning point at (5; 4). The instance of contrast was revealed when attention was on the different values of b and in particular when $b = 0$; $b < 0$; $b > 0$;. Here teachers were helped to experience difference in the values of b. Here attention was to the critical features which were $b = 0$; $b < 0$; $b > 0$ and the dimension of variation was b itself. **What was made possible to discern was that when $\mathbf{b > 0}$ the function had a minimum value at the turning point; when $\mathbf{b = 0}$ the function had neither a maximum nor minimum value at the turning point and when $\mathbf{b < 0}$ the function had a maximum value at the turning point**

Still with the same focus on the point (5; 4) the third instance of contrast was present when attention was on the shape and size of the graphs. Teachers were helped to experience difference in the shapes and sizes of graphs as the value of b varied from $b < 0$ to $b > 0$. Attention was on the shape and size and these were the dimension of variation and critical features in this dimension parabola facing up; straight line and parabola facing down. Another dimension was size and the critical features for this dimension were contraction and expansion. **What was possible to discern was that graphs of functions with the same turning point might be a parabola facing down or up or a straight line. It was also possible to learn that graphs of functions with the same turning point might be contracted or expanded**

When the focus shifted Functions where the domain was not all real numbers another instance of generalisation opened up. The source of this similarity was the domain since all equations needed to each have the domain not all real numbers. Focusing on this dimension of variation presented teachers with the opportunity to experience similarity between different functions. In variation theory terms, the equations were the critical features were the domain was invariant because it had to be the same for all equations. By "the same" I am referring to the domain not being all real numbers. **As such what was possible to discern due to experienced similarity was that domain is a property of functions but an equation is not a property. This seems to add to the message of Functions 1 where the facilitator explained that a function does not have to be expressed algebraically nor have a known algebraic representation to be a Function.**

When the focus was still on finding equations with domain not all real numbers a pattern of variation was infused by the facilitator and this was contrast. This came about when the facilitator chose to work with an incorrect equation from one of the groups. I must mention that the Group had initially thought its domain was not all real numbers but later realised that it was not. In spite of that, the facilitator deliberately included it among other equations and allowed a discussion around it. Bringing into focus an equation which had the domain with all real numbers against those with domain not all real numbers opened opportunities for discernment of difference. The dimension of variation was domain and the critical features were types of domain. Facilitator led teachers into the discussion intended to help them notice that the equation $y = 2^x$ had the domain containing all real numbers and the rest of the equations having domain not all real numbers (see Figure). **Consequently, what was made possible to learn was that the domain of some functions does not include all real numbers but for some it does include them**

The sixth instance where contrast was the opened pattern of variation was present when attention was on the graphs of $y = \sqrt{x}$ and $y = \sqrt{(x-5)}$. The pattern opened up when the Facilitator was explaining why the graphs of these equations did not take negative values. The reason, which I have elaborated on in more details where I present the description of the session, was that the graph would not represent a Function. Pointing to what constitutes a function what does not, helped teachers to experience difference between what is and what is not a Function. This aspect was covered in Session 2.1 and bring it into focus here was useful and productive because it shows the link between the sessions and also helps teachers to integrate what they might have learnt there and here. From the three graphs we notice that the dimension of variation is types of relationships and the critical features are function and non-function. **As such, what was made possible to discern that $y = \sqrt{(x)}$ was a representation of Function because every x value in the domain has a corresponding unique y value in the range and $y = \pm\sqrt{(x)}$ was not a representation of a Function because most x values in the domain corresponds to two y values in the range**

The eight instance of generalisation emerged when attention was on the graph of $y = \log_2 x$. To draw the graph the Facilitator first converted this Function into its equivalent form and this was $x = 2^y$. Rewriting it in this form enable teachers to construct the image of the graph and this was evident when one teacher came to the board and correctly drew the graph of $x = 2^y$ which was a reflection of the graph of $x = 2^y$. Since the focus was on the nature of the graph of $y = \log_2 x$ and to visibilize its equivalent equation was used, this helped teachers to experience similarity between the two equations and what was the same was the graph. Similarity was experienced by keeping the graph invariant while varying the two equations (i.e. converting $y = \log_2 x$ to $x = 2^y$. **As such, what was made possible to learn was that the graphs of the two equations are congruent**

The seventh instance of contrast was present after the graph of $x = 2^y$ was drawn opposite the graph of $y = 2^x$ (See Figure). Putting these together on the same axes helped teachers to experience difference between the two graphs. Here the dimension of variation was inverse functions and the critical features were $x = 2^y$ and $y = 2^x$. **What was made possible to learn by bringing the graphs of these two equations was that the domain of $y = 2^x$ is all real numbers (i.e. $\{x \in \mathbb{R}\}$ and the domain of $x = 2^y$ does not include zero (i.e. $\{x \in \mathbb{R} \mid x \neq 0\}$)**

The ninth occurrence of contrast was in existence when the focus was on the types of functions. The Facilitator simultaneously brought the one-to-one and many to one graphs to the fore and explained the difference between the two types of functions using the horizontal ruler test. In so doing he helped teachers to experience difference between a one-to-one and a many –to-one function. The ruler test cut the one-to-one function in one place each time but cut the many-to-one function in more than one places. In this pattern the dimension of variation was types of functions and the critical features were one-to-one and many-to-one functions. **What was made possible to learn was that the horizontal ruler test cuts the graph of a one-to-one function in one place and cuts that of a many-to-one function in more than one place.**

Table 8.2: Summary of what was made possible to learn at E-Cluster

Invariant	**Varies**	**Discernment**	**Enacted object of learning**
Input and output	Multiple representations	What was made possible to learn through this pattern of variation was that inputs and outputs are properties of a function but its representations are not its properties.	$\mathbf{Mt_{re+}}$
Domain and range	Multiple representations	Consequently, what was made possible to learn was that domain and range are properties of a quadratic function but its representations were not its properties.	$\mathbf{Mt_{re+}}$
Turning point	Multiple representations	what was made possible to learn was that the value of q in a quadratic function of the form $f(x) = a(x – p)^2 + q$ is a property of a quadratic function but the representations were not its properties	$\mathbf{Mt_{re+}}$
Quadratic function	equations	this made it possible to discern that some equations visibilize and others mask the Quadratic Function’s essential meaning structure	$\mathbf{Mt_{ap+}}$
Range	Equations	As such what was made possible to learn was that different functions can share the same range	$\mathbf{Mt_{ap+}}$
Minimum value	Equations	what was made possible for teachers to learn was that different functions can share the same minimum value	$\mathbf{Mt_{ap+}}$
Coefficient of $(x – 5)^2$	$b > 0;\ b < 0; b = 0$	What was made possible to discern was that when $\mathbf{b > 0}$ the function had a minimum value at the turning point; when $\mathbf{b = 0}$	$\mathbf{Mt_{re+}}$

		the function had neither a maximum nor minimum value at the turning point and when $\mathbf{b} < \mathbf{0}$ the function had a maximum value at the turning point	
Size and shape	Contraction; expansion Facing down; facing up; straight line	What was possible to discern was that graphs of functions with the same turning point might be a parabola facing down or up or a straight line. It was also possible to learn that graphs of functions with the same turning point might be contracted or expanded	$\mathbf{Mt_{ap+}}$
Domain	Equations	what was possible to discern due to experienced similarity was that domain is a property of functions but an equation is not a property	$\mathbf{Mt_{re+}}$
Inverse function	$\mathrm{x} = 2^{\mathrm{y}}$; $\mathrm{y} = 2^{\mathrm{x}}$	What was made possible to learn by bringing the graphs of these two equations was that the domain of $\mathbf{y} = \mathbf{2^x}$ is all real numbers (i.e. $\{\mathbf{x} \in \mathbb{R}\}$ and the domain of $\mathbf{x} = \mathbf{2^y}$ does not include zero (i.e. $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \neq \mathbf{0}\}$)	$\mathbf{Mt_{re+}}$
Types of functions	One-to-one Many-to-one	What was made possible to learn was that the horizontal ruler test cuts the graph of a one-to-one function in one place and cuts that of a many-to-one function in more than one place.	$\mathbf{Mt_{re+}}$
Horizontal ruler test	Number of places it cuts a graph of a function	What was possible to discern was that the ruler test cuts graphs of one-to-one graphs at one place and many-to-one graph more than once	$\mathbf{Mt_{re+}}$

From **Table 8.2**, it is discernible that at the E-Cluster, just as was the case in **Table 1** above, what was made possible to learn was constituted by both *correct mathematical reality* ($\boldsymbol{Mt_{re+}}$) and *correct mathematical appearance* ($\boldsymbol{Mt_{ap+}}$). This suggests that the enacted object of learning was constituted by $\boldsymbol{Mt_{ap+}}$ and $\boldsymbol{Mt_{re+}}$ with $\boldsymbol{Mt_{re+}}$ dominant.

8.6. Descriptive analysis of Task 1 of Functions 2 at the I-Cluster

Functions 2 was conducted simultaneously at the I-Cluster and N-Cluster following its enactment at the E-Cluster the previous day. The enactment of this session at the two clusters was informed by the outcomes at the E-Cluster. Functions 2 had Domain and Range as its first focus.

- **Event 1: Creating connections between Functions 1 and Functions 2**

At the I-Cluster the Facilitator began the session by writing the equation f(x) = x^2 + 2 on the flipchart and following on that with its graph which he drew from the Autograph and was projected on the screen (see **Excerpt 8.17**).

B:	Ok, so in last week's lesson we were looking at inputs and outputs in functions, yeah? we looked at things like, um, f of x is equal to x squared plus 2. *[on the flipchart B writes:* f (x) = x^2 + 2*]* and we drew the graph. And I'm going to draw the graph on here now. *[B points to the screen which already has a set of axes]* Not with this *[B puts down the pen]*. But now I want you to imagine before I draw it (well this draws it) *[pointing to the computer]* what this is going to look like.

Excerpt 8.17: Facilitator's introduction of Task 1

Building on this he asked what would happen to the graph if he were to replace the 2 in the $f(x) = x^2 + 2$ by 5 to obtain x^2 + 5. The response was that it would shift up 3 units. Following on this he shifted attention back to $f(x) = x^2 + 2$ which he then transformed into $f(x) = (x-1)^2 + 2$ by replacing x^2 with $(x – 1)^2$. He then asked teachers to discuss in their groups how the graph of the new equation would look like. Following the teachers' discussions he called a teacher from one group to come and explain verbally and graphically how the graph looked like.

66 67 68	F:	Ok, so I've seen lots of things going on at tables, I was interested in that table *[points to the table that T2 sat at]* because they were drawing graphs. Can you talk to us about what you decided? Can everybody hear this? You're gonna hear some, some thinking now. Go on.
71 72 73 74	T2:	Ok. Now this *[points to the screen]* negative 1 is going to affect our graph especially for the values of x, so that our outcome for the value of x is going to change. So it's going to change to the left, to the right *[facing the screen, T2 points to the right]* 1 unit.
75	F:	Will you draw that graph there please?
76	T2:	*[draws the graph on the flipchart]*
77	F:	Ok

Excerpt 8.18: Teacher 2's explanation

The teacher explained (**see Excerpt 8.4**) that the negative 1 would cause the graph to shift to the right one unit. He was asked to sketch the graph and his response was correct (see **Figure 8.14**).

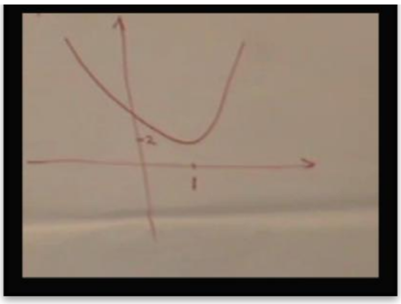


Figure 8. 14: Teacher 2's sketch of f(x) = (x-1)² + 2

Following on this he worked from the Autograph to confirm the teacher's sketched graph (see **Figure 8.15**). Facilitator pointed out that it was surprising that after inserting negative 1 one would have expected the graph to shift to the left instead of right.

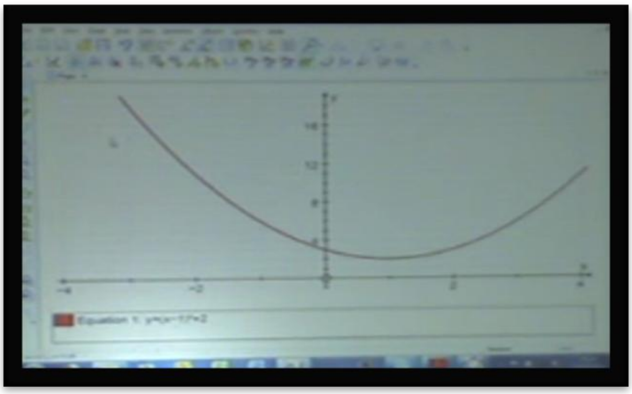


Figure 8. 15: Facilitator's actual graph of f(x) = (x-1)²+2

Speaking from the graph, Facilitator pointed out that its minimum value was 2, and this occurred when x was positive 1.

- **Sub-Event 1.1: Using multiple representations to connect Functions 1 and 2**

Facilitator put a slide showing four multiple representations of $f(x) = (x-1)^2 + 2$ to take the earlier discussion on input and output further and to introduce the notions of domain and range (see **Excerpt 8.19** and **Figure 8.16**).

F:	**And I have a table of values (like you did in your last lesson last week) and I found that for example *[B points to the table]* when x is 3, y is 6. *[B points to the graph]* x is 3, y is 6. *[B points to the equation]* x is 3, 3 take away 1, 2 squared plus 2 is 6.** **Yeah?Alright. Now you've known that these numbers *[B points to the top row of the table: x -3 -2 -1 0 1 2 3 4]* are inputs and these numbers *[B points to the bottom row of the table: 18 11 6 3 2 3 6 11]* are outputs. 4's an input, 11's an output. Two new words I wanted to use today are the Domain and the Range. *[B points to those words]***

Excerpt 8.19: Facilitator's showing connections between multiple representations

Working with these representations he demonstrated from the table of values that 3 was the input and 6 was the output. He moved to the graph and performed the same demonstration with 3 and 6. He then proceeded with the same into the equation to show that when you input 3 the outcome is 6. He concluded this demonstration with the mapping diagram where 3 was in the domain set and 6 was in the range set. He moved back to the table of values and mentioned that 4 was the input and 11 the output.

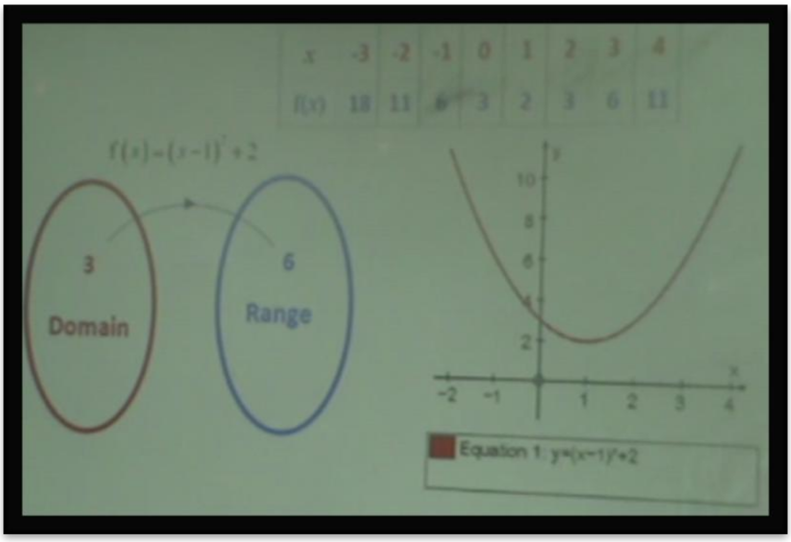


Figure 8. 16: Four multiple representations of f(x)=(x-1)^2+2

Moving on to domain and range he explained that 4 or 3 were each an input, and not the domain. He defined the domain as all numbers that could go into the function. Similarly, he said 11 or 6 were not the range. Instead, the range was all the numbers that could come out. Now, speaking from the table he showed teachers what constituted the domain and the range.

B:	Now the domain is not just 4 or it's not just 3, it's every number you can put in. Altogether. The set of numbers. So that's the domain [points to the red domain 'egg'] there. *[B points to the table]* The range is all the numbers that come out, so not just 11, not just 6, it's every number that can come out. Ok? So I can see the domain up here. *[B points to the top row of the table]* and I can see the range in there. *[B points to the bottom row of the table]* Can you tell me, where do you see the domain and range in this graph? *[B points to the graph on the screen]* Think about it.

Excerpt 8.20: Facilitator's explanations on domain and range

He then asked teachers to identify the domain and the range from the graph. One teacher said the domain was all values of x and the range all values of y. Another teacher said the range were numbers bigger than 2 because zero can never come out. Facilitator then asked if 1 and 1.9 could come out, and teachers said they could not. When he asked if 2 could come out they agreed. From here he wrote $\{\mathbb{R}\}$ and $\{y \geq 2\}$ as the domain and range respectively.

- **Event 2: Finding functions with the range $y \leq 3$**

After the discussion involving the introduction of domain and range, Facilitator asked teachers to work in groups and find a function where the range was $\{y \leq 3\}$ (see **Excerpt 8.21**).

B:	Ok, so on your table what I want you to do is find a different function where the range is not this anymore *[B points to:* $y \geq 2$ *]* I want a new function where the range is y is less than or equal to 3 *[B writes on the flipchart:* $y \leq 3$ *]* Can you find it? You'll have the domain as real numbers... What function is that? *[points to* $y \leq 3$ *]* Decide in your tables.

Excerpt 8.21: Instruction for teachers to find functions with the range y≤3

Teachers worked in their groups to find the function with the specified range. **Figure 8.17** shows the equation that was produced by one of the groups. I must point that all the groups had produced the same equation (i.e. $f(x) = -(x-1)^2 + 3$) (see **Figure 8.17**).

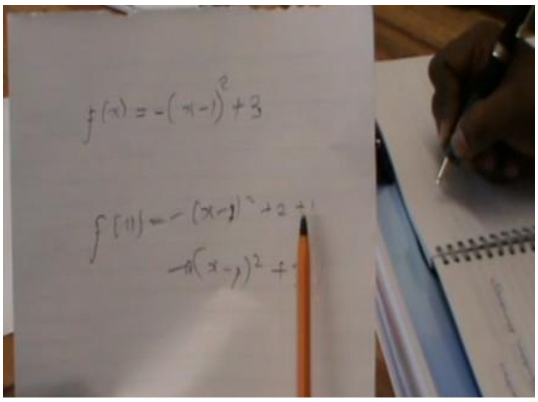


Figure 8. 17: A production from T8's group

After seeing each group's production the Facilitator called one teacher to come to the front to sketch and explain their production to the rest of the teachers (see **Figure 8.18**). The teacher explained that the graph would face downwards because 3 was the maximum value. He then sketched the graph correctly (see **Excerpt 8.22**).

179 180 181 182	F	Ok there's been a lot of things going on round the tables. I'm going to ask (T8) to come out and don't just tell us the answer, talk us through how you came up with your answer. Your answer is correct, tell everyone how you got it. Come on. And you can write on that if you need to on the screen over there. *[F points to the flipchart]*
184 185 186 187 188 189 190 191 192 193 194 195 196 197 198 199	T8:	Alright, they ask us about the numbers, the domain that will be less than 3, né? Ja, let's go. In other words we're looking for the maximum of the turning point we call it in that particular way because on the parabola it says… **When you do the parabola you say y is equal to x squared** ***[T10 writes on the flipchart:*** **$y = x^2$*]*** **It's facing upwards. And then you say y is equals to minus x squared, it's facing downwards.** ***[writes:*** **$y = -x^2$*]*** We are looking for the number that will be less than 3. So this one, because we are adding zero *[writes:* + 0*]* because we turn at zero, né? We're adding nothing so we don't have to write nothing. So if we add 3 in other words it's going to turn at 3 and face up because it doesn't have a negative number, it's gonna face downwards. All the numbers are. *[draws a set of axes on the flipchart and marks* 3*]* That's 3 and then this must be the intercept. It must be less than this particular number. The graph must not go above that particular one. Even if it does have a positive number, wherever it's going to turn it's gonna be this particular number. So in other words this particular graph must face downwards so that all the numbers above this one must not be involved. So we choose any number. Let's say y is equal to minus x plus… We choose 2, hey?
200	T9	Minus 2, ja
201 202 203 204	T8:	Minus 2 squared plus 3. *[writes:* $y = -(x-2)^2 + 3$ *]* See, we're looking at the negative to see this is our x^2 né? It's looking at the negative to say it's gonna face downwards. So this is gonna be our turning point, it's (2;3) in this particular one and it's facing downwards. *[sketches the graph]* ??
206 207 208	F	Actually, before you go... Thanks, well done. But before you go can you tell me that point there *[B points to the graph T8 has drawn]* where it's crossed the y axis. Or anybody wants to help ?
209	T8	y intercept?
220	F	Ok, very good. That's wonderful. So this is y intercept and is at -1 *[B writes* -1 *on the graph]*

Excerpt 8.22: Teacher 8's explanation

In **Excerpt 8.22** Teacher 8 opened a pattern of variation known as **contrast** when he drew attention to the graphs of $y = x^2$ and $y = -x^2$. The underlined text was part of the teacher's legitimation. The highlighted text shows the facilitator's probe, which was part of the scaffold on the task. Following the teacher's explanation the facilitator asked him to state the y-intercept. After discussion it was agreed that the y intercept was negative 1.

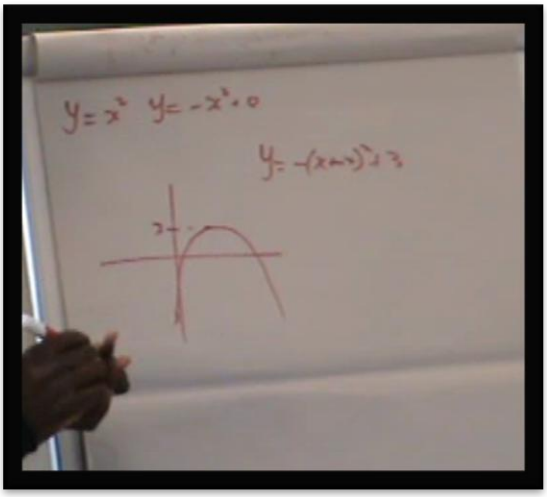


Figure 8. 18: Teacher 8's sketch of the graph of f(x)=-(x-2)²+3

Facilitator then drew the actual graph using Autograph (see **Figure 8.19**).

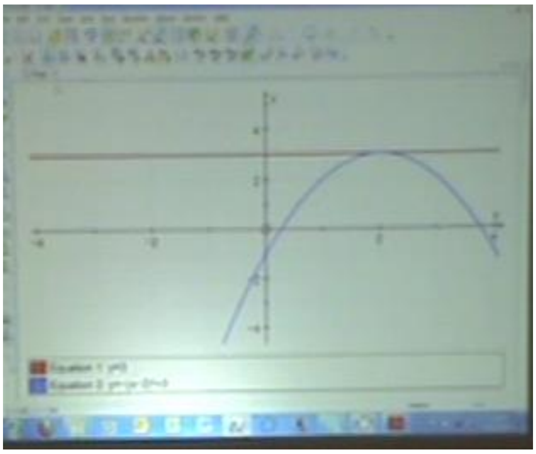


Figure 8. 19: Facilitator's actual graph of f(x)=-(x-2)²+3

Speaking to the graph, facilitator asked teachers for the domain, and they responded correctly (see **Excerpt 23**). Bringing the range and domain simultaneously from the same graph opened a pattern of variation known as fusion.

B:	Ok. This, this function, can you tell me what the domain is?
T11:	The domain our-?
B:	The domain is?
T11:	All real numbers
B:	All real numbers?
T11:	Yes
B:	**That's true because we can put any value… We have 2 here I think.** ***[on the flipchart writes: ??]*** **Alright. So you can put in a number and the numbers that come out are less than or equal to 3.** ***[points to the flipchart]*** **This is the domain, that's the range.** Ok. I want now each table to come up with a function where the domain is not all real numbers, that there are some numbers that is impossible to put in.

Excerpt 8.23: Discussion focused on domain of $f(x) = -(x-2)^2 + 3$

The bold text in the above excerpt contains an instance where the facilitator talked about the range and domain simultaneously to create **fusion**

- **Event 3: Finding functions with the domain not all real numbers**

The next task required teachers to find a function where the domain was not all real numbers.

F:	Ok. I want now each table to come up with a function where the domain is not all real numbers, that there are some numbers that is impossible to put in.

Excerpt 8.24: Finding functions with domain all real numbers

After working in their groups a list of functions was produced and then written down on the flipchart by the Facilitator (see **Figure 8.20**). In the discussion around the list Facilitator asked for values of x that could not go into $y = \frac{2}{x}$, and zero was given as the number that x cannot be. Facilitator drew the graph of this Function with Autograph (see **Figure 8.21**).

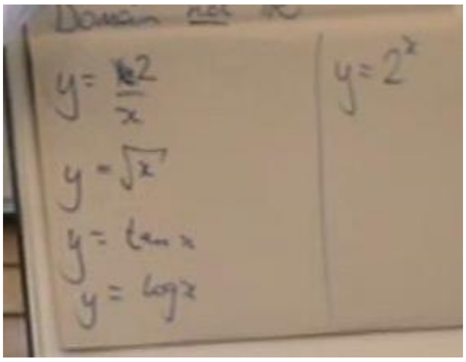


Figure 8. 20: A list of functions with the domain not real numbers

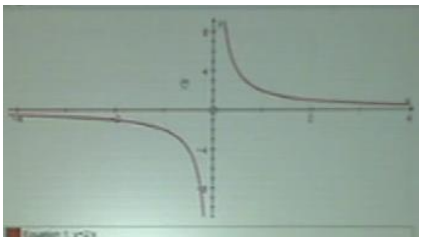


Figure 8. 21: A graph of y=2/x

On the $y = \sqrt{x}$ teachers said x could not be negative. Facilitator drew the graph and it appeared as shown in **Figure 8.22**. Based on the shape of the graph the facilitator asked teachers if the graph rises up forever to include all non-negatives, or if it flattens up along the way. The response was that it includes all non-negatives. A reason from one teacher was that there is no line of asymptote to prevent the graph from going upwards.

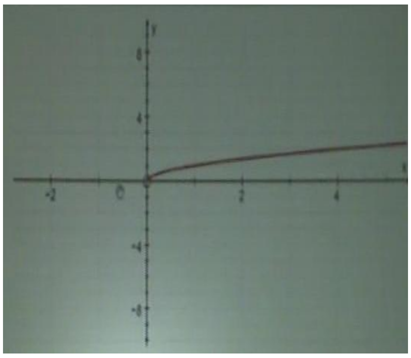


Figure 8. 22: A graph of y=√x

Facilitator explained that the symbol $\sqrt{\ }$ meant that only positive square roots were considered. Facilitator explained that if the negative square root is considered then the graph would not represent a function.

316 317 318 319	B:	Well then you've got to be careful about what this symbol means, because the symbol, that symbol means positive square root. *[B points to the question on the flipchart]* If you want to include both, you'd have to put plus or minus and then we don't get function because we'd get it going 2 ways.

Excerpt 8.25: Facilitator's explanation about the square root symbol

The next discussion centred on $y = tanx$ (see **Figure 8.23**). The log function was not discussed. The discussion ended with $y = 2^x$and its graph was drawn.

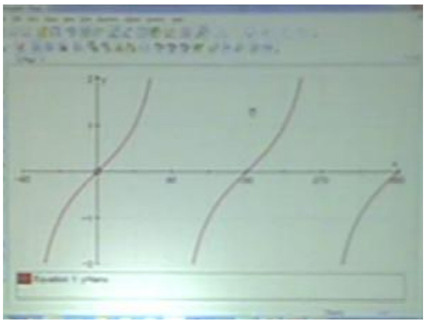


Figure 8. 23: A graph of y=tanx

- **Event 4: Discussion involving one-to-one and many-to-one functions**

The next task involved one-to-one and many-to-one functions. Here the facilitator put up the graphs of these types of functions simultaneously on the same slide and went on to explain the differences between the two. He showed how the horizontal ruler test may be used to distinguish the two functions. He pointed out that the ruler test cuts the graph of a one-to-one function once at most (see **Figure 8.24**) but the many-to-one more than once.

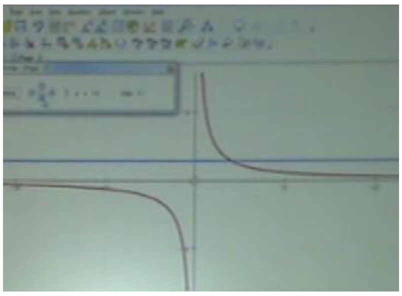


Figure 8. 24: A graph of a one-to-one function

He said one-to-one functions are special because you can work backwards. He said in a many-to-one function it is not possible to work backwards since several inputs map on to the same output, and so when one has an output it is not easy to trace it back to a particular input that produced it (see **Excerpt 8.26** and **Figure 8.25**).

F:	**If I now have a line that goes up and down here, everywhere it will only cut the curve - either it misses the curve altogether, the zero, or it cuts it only once. Yeah? So if you tell me that your line cuts the curve at 3, I don't have to look, I know 12 over 3 it cuts at 4. Ok? That's a special function because I can come backwards.**
F:	**But if your function was something like x times x minus 2, times x plus 2. 2 was there, then sometimes, we're ok… But if the horizontal line is down here and you tell me your output is let's say 1, I don't know which input you have. It could be this, *[B points to points on the graph as he speaks]* it could be this or it could be this – I don't know. And so I can't work backwards from your answer.**

Excerpt 8.26: Facilitator's explanation of one-to-one and many-to-one functions

The bold text in **Excerpt 8.26** shows an instance of contrast in the facilitator's explanation of one-to-one functions and many-to-one functions. **Figures 8.24** and **8.25** were displayed simultaneously on the same slide.

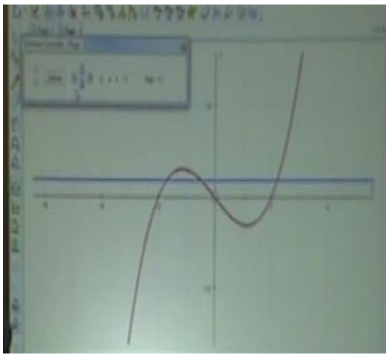


Figure 8. 25: A graph of a many-to-one function

Figure 8.26 shows the graph of a many-to-one function which the facilitator drew before demonstrating a horizontal ruler test to show that it was not easy to move backwards on it.

8.7. Discussion of pedagogic strategies used at I-Cluster

In this session where the focus was on Domain and Range, Facilitator adopted a deducative pedagogic strategy. This is strategy is adopted when the Faciliator has bracketed his natural attittude such that it does not interfere with his decisions about the amount of scaffolding to provide in th task. There was high scaffolding, and this was evident in the availed resources which were used to help teachers make progress on the task . The resources were present before, during and after teachers worked on the task, and they included the use of a familiar equation (i.e. $f(x) = x^2 + 2$) to introduce a complex equation ($f(x) = (x – 1)^2 + 2$), and to help teachers notice similarities between the two equations. In particular he wanted to help teachers to discern that the complex one was in effect a transformation of the familar equation. It included working with multiple representations of the same quadratic function to invoke teachers' prior knowledge of inputs and outputs from Functions 1 (see **Figure 8.17**).

It included working from Autograph and flipchart to visibilize important mathematical ideas, concepts and relationships between mathemtical objects and it also included moving between groups providing help in the form of reminders and probes. Here teachers' responses were probed and this was evident when teachers were often called to go to the front and illustrate their responses on the flipchart (see Figure 8.15). There the teacher was called to explain by means of drawing, what the graph of $f(x) = (x – 2)^2 + 2$ would look like. This followed from the teacher's initial verbal explanation which she was asked to illustrate on the flipchart. Interestingly, every teacher's response was followed by the Facilitator's illutsrating of the same with a graph from the Autograph (see Figure 8.16). On the basis of the availed resources and help that was present at the I-Cluster, teachers there experienced high scaffolding, and this was made possible by the deductive pedagogic strategy adopted there.

8.8. Discussion of legitimating criteria at the I-Cluster

The first legitimating criteria emerged after ateachers were asked to explain where the domain and range were located on the axes. A teacher said that the range of $f(x) = (x – 1)^2 + 2$ was all numbers bigger than 2 because zero was not one of the outputs. The teacher was appealing to the outputs of the function. In the second legitimating criteria a teacher was asked if the graph of $y = \sqrt{x}$ would rise and reach a point where it no longer rises and stay constant. He reasoned that the graph would continue to rise up infinitely because it did not have an asymptote to prevent it from rising. The teacher was appealing to his knowledge of asymptotes and was probably drawing from his knowledge of hyperbolic functions where asymptotes are like barriers which the graph does not go across. The third legitimating criteria was present when the Facilitator was using the horizontal ruler test to explain the difference between one-to-one and many-to-one functions. As it can be discerned, he was appealing to the horizontal ruler test The fourth legitimating criteria was present when the Facilitator was explaining illunminating the difference between one-to-one and many-to-one functions. He pointed out that one-to-one were important because they allowed one to work backwards. The Facilitator was appealling to the nature of one-to-one functions

Table 8.3: Summary of legitimating criteria at the I-Cluster

Legitiamting criteria	Appeal	Enacted object of learning
Oh, right. Now that's interesting. So you're saying that the range is every number bigger than 2 because we can never get out zero. Can we get out 1?	Range as a property of functions	$\boldsymbol{Mt_{re+}}$
Well then you've got to be careful about what this symbol means, because the symbol, that symbol means positive square root	Mathematical symbol	$\boldsymbol{Mt_{ap+}}$
So if I don't look and you put a horizontal line down there and you tell me that it cuts the, it cuts the curve at a value of 6, I know the x value will be 2, because there's only one place	Horizontal line test	$\boldsymbol{Mt_{re+}}$
If we've got a one-to-one function then I can work backwards. So if you tell me that your line cuts the curve at 3, I don't have to look, I know 12 over 3 it cuts at 4. Ok? That's a special function because I can come backwards.	Knowledge of one –to-one function	$\boldsymbol{Mt_{re+}}$

From **Table 8.3,** it can be noted that the legitimating criteria were constituted by both $\boldsymbol{Mt_{re+}}$ and $\boldsymbol{Mt_{ap+}}$ with the former dominant.

8.9. Discussion of what was made possible to learn at the I-Cluster

The first pattern of variation was contrast, and it resulted from working with these two equations:

(i) $f(x) = x^2 + 2$
(ii) $f(x) = (x-1)^2 + 2$

The facilitator first drew the graph of $f(x) = x^2 + 2$ and then asked teachers to discuss how the graph of $f(x) = (x-1)^2 + 2$ would look like. Following this she explained that it would shift one unit to the right and provided its corrrect sketch. Engaging teachers with the two equations hepled them to experience differences between them and and also between their graphs (see **Figure 8.26**). **As such, what was made possible to discern was that transformation of x^2 to $(x-1)^2$ shifted the graph of $f(x) = x^2 + 2$ one unit to the right and not to the right**

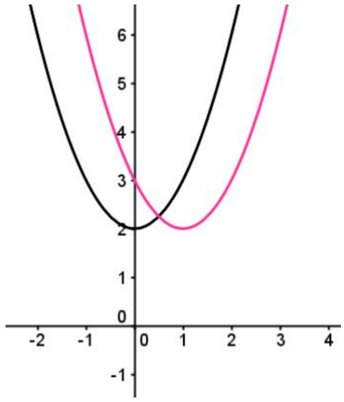


Figure 8.26: Graphs of $f(x) = x^2 + 2$ and $f(x) = (x - 1)^2 + 2$

The second pattern of variation was generalisation, and it occured when the Facilitator brought four multiple representations of the same quadratic function to the fore (see **Figure 8.17**). Here teachers experienced similarity between the four representations when the Facilitator kept the domain and range invariant while varying the representations. The important features were domain and range, and these were invariant. What was varying were the critical features, but were backgrounded. In variation theory terms, generalistion emerges when the invariant feature is foregrounded and the varying features are backgrounded. In particular, teachers were helped to discern the same domain and range from different representations. By way of example, Facilitator worked from the table to illuminate 3 as an input and 6 as the ouput. He went to the equation and subtituted 3 to obtain 6. He then went and did the same from the graph. Similarly, he did the same demonstration for 4 and 11 as the input and output respectively. This movement between the representations helped teachers to notice the similarity between them, and what was similar was that they shared the same domain and range. **What was made possible to learn was that domain and range were properties of a function, but its representations were not its properties**

The third pattern of variation was contrast, and was present where the Facilitator was explaining the difference between input and domain and between output and range. There the facilitator demonstrated across the representations what constituted input in a function. Having done that he pointed out that an individual input such as 4 or 3 in the table of values was not the domain, but instead the domain were all inputs that could go into the function. Similarly with output, he pointed out that 6 and 11 were not the range but instead, the range was all numbers that could come out of the function. The distinction between these concepts helped teachers to experience difference between input and domain and between output and range. **Consequently, it was made possible for teachers to discern that inputs were elements in the domain and outputs were elements in the range**

The fourth pattern of varition was generalisation, and was present when attention was on finding functions with the domain not all real numbers (see Figure 8.21). Teachers were helped to experience similarity between the equations, and this was that their domains were not all real numbers. In this pattern of variation the invariant feature was domain, and the critical features were the four equations. **What was made possible to learn was that domain was a property of Functions, but equations were not its properties**

The fifth pattern of variation was present when attention was still on the four equations, particularly when the domain of each equation was foregrounded. Teachers were helped to experience difference between the types of domains that do not include all real numbers. In this instance what was in the fore were the different domains that do not include all real numbers, and these were the critical features. The invariant feature was the dimension of varition, and was not in the fore. The dimension of variation was functions with different domains. **So what was made possible to discern was the difference between domains that do not include all real numbers.**

The sixth pattern of variation was present when attention was on one-to-one and many-to-one functions. Facilitator worked with the horizontal ruler test to show its behaviour on the graph of each type of function. Here teachers were helped to experience difference in how the ruler test behaved towards each type of function. By way of example it was shown that the horizontal ruler test cut one-to-one functions once and many-to-one functions more than once, and that it is possible to work backwards in one-to-one, but not in many-to-one functions (see **Figure 8.25** and **8.26**). **What was made possible to learn was that the horizontal line test cut one-to-one functions once and many-to-one more than once.** In **Table 8.4** I provide a a summary of what was made possible to learn in the I-Cluster.

Table 8.4: Summary of what was made possible to learn in the I-Cluster

Invariant	Varies	Discernment	Enacted object
Quadratic function	Equations	What was made possible to discern was that transformation of x^2 to $(x-1)^2$ was a horizontal shift to the right, and not the left	Mt_{re+}
Domain & Range	Multiple representations	What was made possible to learn was that domain and range were properties of a function, but its representations were not its properties	Mt_{re+}
Quadratic function	Input versus domain and output versus range	Consequently, it was made possible for teachers to discern that inputs are elements in the domain and outputs were elements in the range	Mt_{re+}
Domain	Quadratic equations with domain not all	What was made possible to learn was that domain	Mt_{re+}

	real numbers	was a property of Functions, but equations were not properties	
Function	Domains that do not include all real numbers	So what was made possible to discern was the difference between domains that do not include all real numbers.	$\boldsymbol{Mt_{re+}}$
Function	One-to-one versus many-to-one	What was made possible to learn was that the horizontal line test cut one-to-one functions once, but many-to-one more than once, and that it is possible to work backwards in one-to-one, but not in many-to-one functions	$\boldsymbol{Mt_{re+}}$

From **Table 8.4**, it is discernible that what was made possible to learn was constituted by *mathematical reality* $\boldsymbol{Mt_{re+}}$.

8.10. Descriptive analysis of Functions 2 at the N-Cluster

I should point out to the reader that presentation of Functions 2 in the N-Cluster was the same as the one conducted in the I-Cluster. Hence the following descriptive analysis will not differ much from that of the I-Cluster.

- **Event 1: Introducing the notions of domain and range**

Facilitator began his presentation by showing the links between input and domain and between output and range. While input and out were central in Functions 1, domain and range were central in Functions 2.

- **Sub-Event 1.1: Introducing domain and range with a mapping diagram**

Excerpt 8.27 contains an explanation of domain and range which were made by the facilitator as a way to introduce Functions 2 ad the N-Cluster.

87 89 90	F:	Right, good afternoon everybody. This was the beginning of your homework that I gave you last lesson, ok? Thinking about inputs and thinking about outputs. *[S points to the flipchart]*
103 104 106 107 108		Here *[points to the red oval]* what we have is the collection of all the numbers that can go into the function that we have, the equation down here *[points to the equation at the bottom of the screen]* This is the whole collection of different numbers that can be fed into the function. We call it the domain. So we're dealing with some language here, some mathematical language which you haven't encountered before – this is domain. Ok?
110 111 112 113 114		Here *[points to the blue oval]* we have all the numbers that are output. *[S points to the red oval]* This is input *[S points to the blue oval]* and these are the outputs from each of the numbers that we put in as in input. This we call the range, ok? So that's the domain *[points to the red oval]* and this is the range *[points to the blue oval]*. So can I ask you where in the table of values at the top you might be able to see the domain and the range?

Excerpt 8.27: Facilitator's introduction of domain and range

In **Excerpt 8.27** the facilitator speaks to **Figure 8.26** to show the links between the notions of input and domain, and between out and range from a mapping diagram.

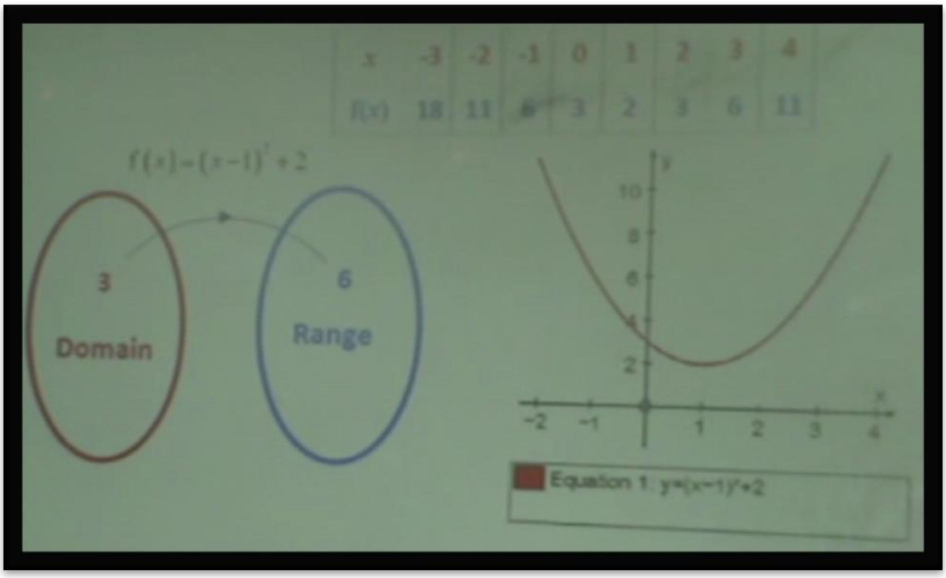


Figure 8. 26: Multiple representations of the same quadratic function

Beginning from the mapping diagram the facilitator pointed to the equation and mentioned that the whole collection of numbers that could go into $f(x) = (x-1)^2 + 2$ constituted the domain. Still working from the mapping diagram he pointed out that the whole collection of numbers that could come out of the equation constituted the range.

- **Sub-Event 1.2: Identifying domain and range from a table of values**

From the mapping diagram the facilitator shifted attention to a table of values. Working from the table of values he asked teachers to identify the domain and the range. The table of values is evident in **Figure 8.26** above.

113 114	S:	So can I ask you where in the table of values at the top you might be able to see the domain and the range
116		*[silence for about 6 seconds]*
118	S:	T2
120	T2:	The domain is of x, x the red ones, x represents the domain
122 123		Right. Right, he's colour coding it so its the domain is the x. Good. And the domain is the set of numbers at the top, the x values. And the range?
125	Trs:	Blue ones
127	F:	Ja, the blue ones, here. That's ok.

Excerpt 8.28: Identifying domain and range from table of values

Now, the x values on the table were coloured red and the y-values coloured blue. Similarly with the mapping diagram - the domain was red and range blue. Hence in response to the Facilitator's question a teacher said that the domain were the red values on the table and the range was blue values.

- **Sub-Event 1.3: Identifying domain and range from the graph**

Building on these responses he moved to the graph and asked where the domain and range were located on the axes. A teacher's response was a gesture with his right hand moving across from the left to right, suggesting that the domain was along the x-axis.

127 128 129	F:	So now let me ask you whether you can see the domain and the range on the graph or not? Let's start with the domain. Where's the domain on the graph? Anybody? Ja.
131	T3:	Across *[T3 gestures with his right hand to demonstrate going across]*
133 134 135	F:	Across all these numbers here, that's great. Yes, all of those. *[points to the slide]* That's the domain. Those are all the numbers that can be put into the function, that's right. How about the range? Where's the range? *[Asking Teacher 4]*
137	T4:	Along the Y-axis.
139	F:	Along the Y-axis, that's right, ja.

Excerpt 8.29: Discussion focused on domain and range from the graph

Facilitator confirmed this by saying that those were all numbers that could be put into the function. Another teacher's response related to the range was that the range was found along the y-axis, and the facilitator was in agreement with this. Following on this the Facilitator reminded teachers that on the table the range for the function $f(x) = (x-1)^2 + 2$ was numbers in the bottom line. One teacher said it was on the curve, and facilitator agreed with this response. He said the curve contained the all the outputs, and these were y values. Following on this he asked teachers to identify the smallest value in the range. One teacher said 2 was the smallest, and facilitator asked how this was recognisable on the graph, and the teacher said it was at the turning point of the curve (see **Excerpt 8.30**).

168	F:	Now what is the smallest number in the range?What's the smallest number in the range?
170	T5:	2
172 173	F:	2, ok. Can you describe just how you see it? How you can spot it? It's fairly obvious. Somebody tell us?
175	T6	It's the turning point.
177 178 179 180	F:	Yes, it's right at the bottom *[points to the graph on the screen]* right there. You can see it right there. That's the turning point, ok? And… Ok, um, could it be 0? If we input 0 then we get a range value of 3, so it's certainly not that. It comes down here at 1. Put in an x value of 1, the input is 1 and the range, the output value is, is 2, as you said. Great. Ok.

Excerpt 8.30: Discussion focused on identifying the minimum value

Facilitator then asked for the value of x where the graph had a minimum value of 2. The response was that the x-value at the turning point was 1. Following on this he asked if it was coincidence that the minimum value on the curve was 2 and that there was 2 in $f(x) = (x-1)^2 + 2$. A teacher said it was not coincidence, because as long as the equation was in the form $(x – 1)^2 + 2$ the smallest value of the function will always be 2 (see **Excerpt 8.31**).

184 185 186 187 188	F:	Is it a coincidence that the minimum value here at the bottom of the curve is a 2 and we've got a 2 there? Have a chat around your tables and see what you think about that. Is it just a coincidence or is it something we could have seen? Talk to the people around your tables. Make a suggestion and see if you can explain why.
252 253 254	T2:	According to that function, it's not a, <u>it's not a coincidence because even as long as the function is always like x minus 1 squared, even if you change the 2 to 3, the turning point will change also to 3</u>
264 265	F:	Why is that? You said it's something to do with the form that it's in. Why is that? Can you explain?
269 270	T2:	We said <u>if we don't change the function, the equation x minus 1 to say the f of x is equal to x minus 1all squared , the lowest value is 1 minus 1, you'll get zero</u>

Excerpt 8.31: Teacher 2 explaining why the minimum value was 2

Building on Teacher 2's response the facilitator explained that the minimum value was 2, because the smallest value of $(x – 1)^2$ was zero and that happened when x was 1.

- **Sub-Event 1.4: Comparing the form $\boldsymbol{f(x) = p(x-a)^2 + q}$ with other forms**

Taking the explanation further, the facilitator rewrote $f(x) = (x-1)^2 + 2$ as $y = x^2 – 2x + 3$ and pointed out that it would have not been easy to discern the range and the lowest value of y from this equation. He pointed out that factorising the equation would be problematic, because the function does not have real roots and that was discernible from the graph because it was not crossing the x-axis.

324 325 326	F:	Right. But if we just… if we didn't do that and if instead we chose to find out what it looks like, like we know it's a parabola, where it might cross the axes. So without going into the calculus and just staying with the algebra, what would you try to do?
328	T15:	Factorise it.
330 331	F:	Factorise it, ok. So would you like to factorise that? What's going to happen? Can anybody tell me?
333	T16	There's no roots
335 336 337 338 339	F:	There are no roots. You see, you can see there are no roots by the graph, *[points to the graph on the screen]* it doesn't cross the axis. *[Now S points to the equation on the flipchart:* $y = x^2 - 2x + 3$*]* So we have real trouble doing this, but rewriting it in this form *(S points to* $y = (x - 1)^2 + 2$) we can see straight away that the minimum of the graph is going to be when x is 1 and the minimum value's going to be 2.

Excerpt 8.32: Discussion focused of different ways to write $f(x) = (x - 1)^2 + 2$

He pointed out that writing the equation in the form $y = (x -1)^2 + 2$ made it easy to discern the range and the value of y at the turning point compared to writing it in other forms. Facilitator asked what were the domain and the range of $y = (x -1)^2 + 2$, and the response was that the domain was all real numbers and the range was $y \geq 2$.

- **Event 2: Finding functions with the range $y \leq 2$.**

The next task was for teachers to find the function with the maximum value of 2 at the turning point. Teachers worked on this in groups. Teachers worked on this task for 3 minutes and produced responses (**see Figure 8.27**) which were listed on the flipchart.

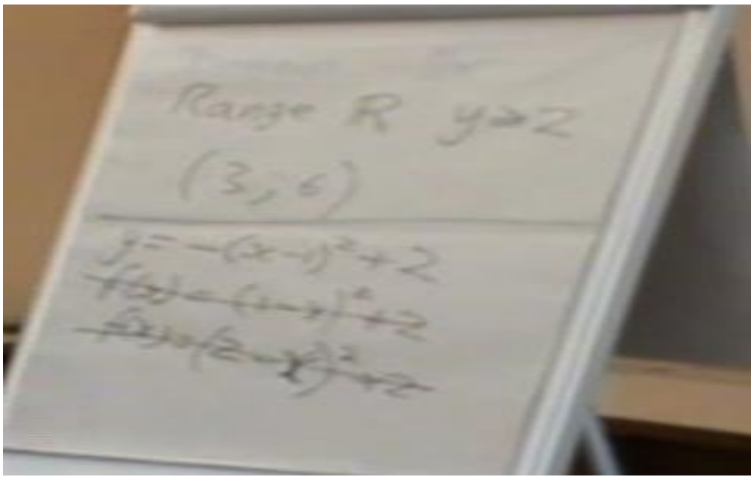


Figure 8. 27: List of functions with maximum value as 2

The first group produced $y = -(x -1)^2 + 2$ (see **Figure 8.28**), and Facilitator wrote it on the flipchart. Facilitator asked the group to explain how they knew that it would have a maximum value of 2, and the response was that the negative in front of the bracket showed that it would have a maximum value.

472 473	F:	Ok, let's see what we get then. *[points to a group in the front of the class]* Can I ask this group please what function you got?
475	T20:	The same function *[points to the flipchart]* but you put a minus $y = -(x - 1)^2 + 2$
477 478 479 480 481	F:	*[F writes this down on the flipchart]* Ok, there's an answer from this group, ok. This is the function from this group. *[points to:* $y = -(x-1)^2 + 2$*]* Is that going to work? Is that going to give a maximum of 2? Why will it work? Just explain to us why that's? Why is that going to give us...How do you know that is going to give us a graph with a maximum of 2? You've still got a plus 2 on the end as we had before?
	T20:	*[nods]* Because of the negative at the beginning. That one tells you if your graph is going to have a maximum or a minimum. A negative sits with a maximum *[moves her hand to opposite way of a smiley face]*, the positive is like that *[moves her hands to form a 'smiley face']*. Smiley face *[chuckles]* *[some of the other members of her group also smile]*

Excerpt 8.33: Group 1's explanation

Following on Teacher 20's explanation the facilitator added that whenever a number other than 1 was put in, the value of the function was less than 2. By way of example, he said if $x = 2$, the value of $y = -(2-1)^2 + 2$ becomes 1and 1 is less than 2. **Figure 8.28** shows the graph of Group 1's suggested function. The graph has a maximum value at its turning point and this indicated that the response was correct.

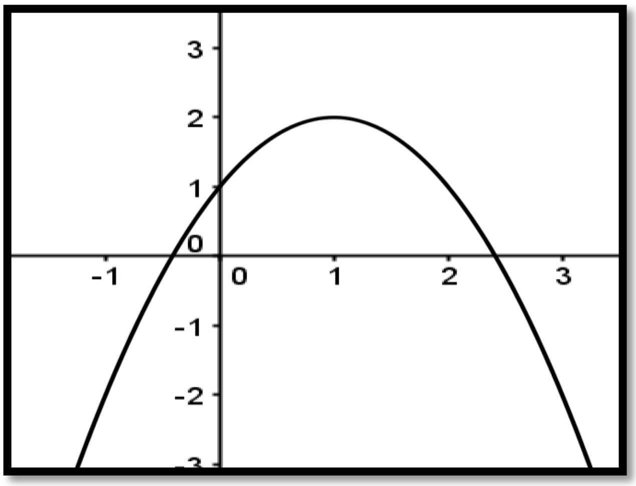


Figure 8. 28: Group 1's correct response

The second group's response was $f(x) = (1 - x)^2 + 2$ (see **Figure 8.27**)and realised it was an incorrect function when they were explaining how they recognised that it would yield 2 as its maximum (see **Excerpt 8.34**).

522	S:	Great. Ok, now tell us why it works?
524 525 526	T21	Ah, since we, we looked at that graph we saw that the, the maximum points will be the, you can find when you make a reflection on the *[holds up his two hands with the fingers pointing towards each other]*… Oh, it's wrong.
528	F:	It's wrong? *[class laughs and someone makes a comment]*
532	F:	Why is it wrong?
536	T21	Because I .. on there.
538 539	F:	Sure. But why is this not going to work? What's going to happen if we look at that graph?
541	T22	It is positive
543 544 545 546 548	F:	It's still going to be positive. If x is 5, 1 minus 5 is minus 4 squared is 16, so it's going to be a minimum of 2 rather than a maximum of 2. Ok. We can confirm that y equals 1 minus x squared *[types the equation into the computer and it appears on the screen]* plus 2 and there it's the same one as before which is in fact 1 minus x squared or x minus 1 squared is going to be the same. Ok, so that one we want to say no thanks. Alright, I'll delete that in a moment.

Excerpt 8.34: Discussion focused on Group 2's response

Working with the same incorrect function facilitator asked the rest of the groups why it was incorrect, and the response was that because all the values will be positive, and so will have values greater than 2. Adding to this the facilitator pointed out that instead of a maximum of 2 the Function will have a minimum of 2. As a way to confirm his explanation he drew the graph of $f(x) = (1 - x)^2 + 2$ to show that it was the same as $f(x) = (x - 1)^2 + 2$.

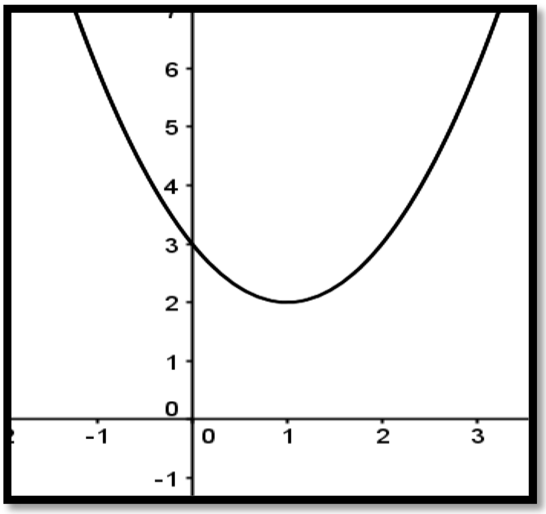


Figure 8. 29: Group 2's incorrect response

The third group produced $f(x) = (2 – x)^2 + 2$, but also realised that it was incorrect. Facilitator then cancelled productions from Groups 2 and 3 because they were incorrect (see **Figure 8.27). Figure 8.30** shows Group 3's incorrect response.

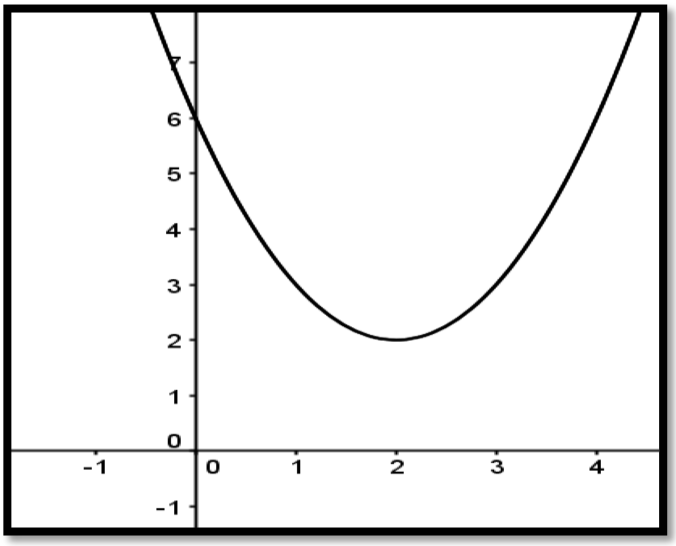


Figure 8. 30: Group 3's incorrect response

The response from Group 4 was that there were several functions that could be produced. A teacher from this group pointed out that if the 1 inside the brackets was replaced by 2 or 3 or any other number, the graph would shift horizontally, but the maximum value would still be 2 (see **Excerpt 8.35**). **Figure 8.31** is where I have illustrated the teacher's explanation.

586 587 588 589 590 591	T8:	Ja, we still ... but now explanation. On that function you had y equal to minus bracket x minus 1 squared plus 2. So like it's just a shift along the axis. *[demonstrates by moving his hands along]* And whatever you are going to put where there's 1. Whether you put 2 or 3 or 4, in other words you are shifting the graph *[moves his arms to the right to indicate how the graph is shifting]*. But the maximum is going to be 2. You can shift again that way round *[moves his arms to the left]* but the maximum is going to be 2.

Excerpt 8.35: Group 4's explanation on their response

To exemplify Teacher 8's explanation, the facilitator replaced 1 with 3 in $f(x) = -(x - 1)^2 + 2$ to generate $f(x) = -(x - 3)^2 + 2$, which was one of several functions suggested by Group 4.

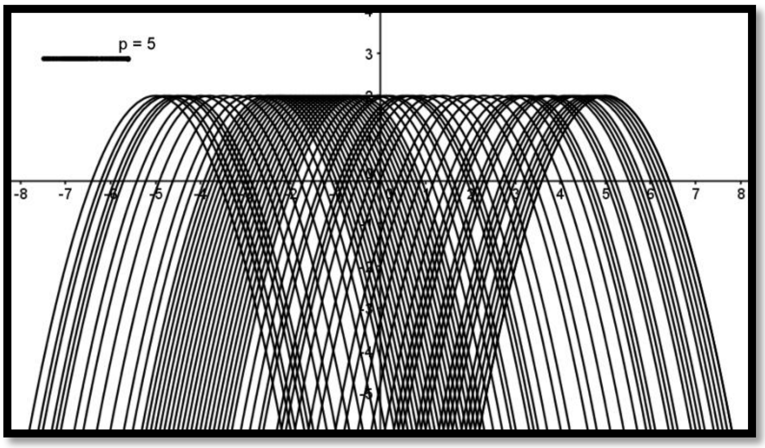


Figure 8. 31: Group 4's correct response

Facilitator emphasised that the function would have a maximum value of 2 because the largest value of $-(x-3)^2$ would still be zero, and when it is zero then the maximum value for $f(x) = -(x-3)^2 + 2$ would be 2. The fifth group produced $y = 2 - x^2$ and was accepted (**see Figure 8.32**).

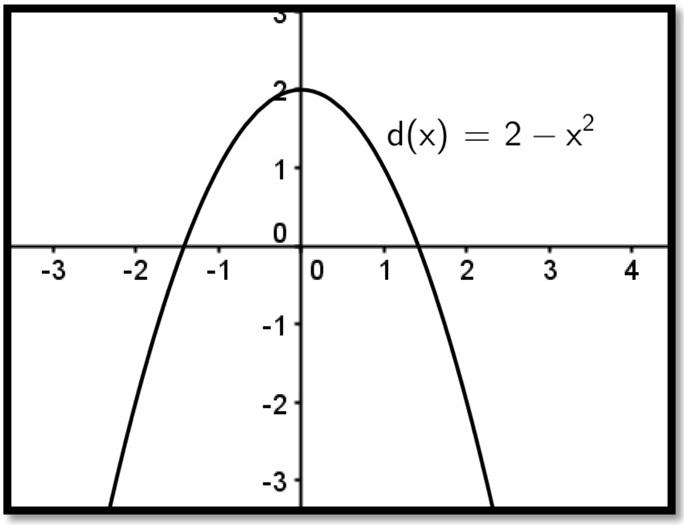


Figure 8. 32: Group 5's correct response

- **Event 3: Finding functions with minimum value at (5; 4)**

The third task involved teachers finding a function which had a minimum value at the point (5; 4). Teachers worked on this for 3 minutes in their groups. **Excerpt 8.36** contains the response from Group 1.

637 638 639 640 641	F:	So now let's try something else. Supposing… Supposing I took the number, the point here of 5, 4. *[gets this onto the screen]* Ok, there's the point 5, 4. Can you find a curve whose lowest point is at 5, 4? Find a curve with the lowest point at 5, 4. Ok, round your tables and do it. Minimum at point 5, 4. Can you find a function whose lowest point is here, 5, 4?
645	Trs:	*[teachers discuss this in groups. F walks round the groups]*
649 650 651	F:	Right. Are we ready for an answer? Right, let's see what answers we get then. Let's start at the back this time. *[F points to the back of the class]* What's your function here? What have you got?
659	T24:	x minus 5 all squared plus 4
661	F:	minus 5
663	T24:	all squared
665		all squared plus 4.
667	T24:	plus 4
669 670	F:	*[puts this onto the computer]* Ok, x minus 5 all squared plus 4. There we go. That would certainly work, wouldn't it? Good.

Excerpt 8.36: Group 1's response

The response from the group was $f(x) = (x-5)^2 + 4$, and this was transformed into a graph by the facilitator to check if it will have a minimum value at (5; 4) (see **Figure 8.33**). This response was confirmed to be correct by the facilitator.

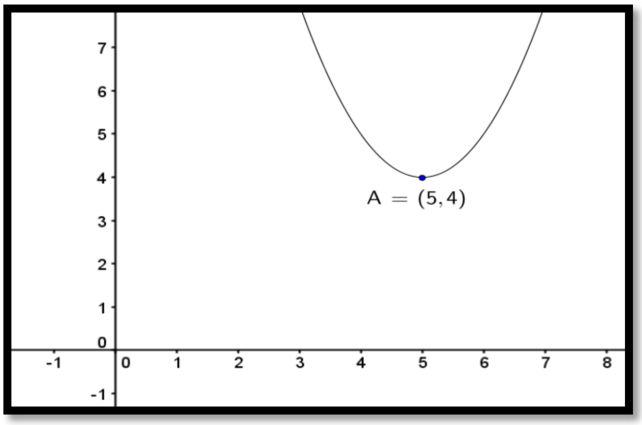


Figure 8. 33: Group 1's correct response

Group 2's response was $f(x) = a(x - 5)^2 + 4$ where **a** was any natural number (see **Excerpt 8.37**). Speaking on behalf of Group 2, Teacher 25 read out the function for the facilitator to write down on the flipchart. What was interesting about the group's response was the selection of the variable as the coefficient, because this produced all possible functions having a minimum value at (5; 4).

708	T25:	a bracket x minus 5 squared.
710	F:	*[after the* a *S writes:* $a(x - 5)^2$ *]*
712	T25:	plus 4
714	F:	*[S writes:* + 4 *] [Thus the equation is:* $f(x) = \boldsymbol{a}(x - 5)^2 + 4$ *]*
716	T25:	Where ***a*** is the set of real numbers
718	F:	Where ***a*** could be any number?
720	T28:	No
722	T25:	Natural numbers
724	F:	Any natural number?
726	T25:	Yes
728	F:	Ok

Excerpt 8.37: Group 2's response

Working from the Geogebra environment the facilitator generated graphs of functions with a minimum value at the point (5;4), and he did this with the aid of a slider which produced different graphs as the value of **a** varied between zero and positive 5 **(see Figure 8.34).**

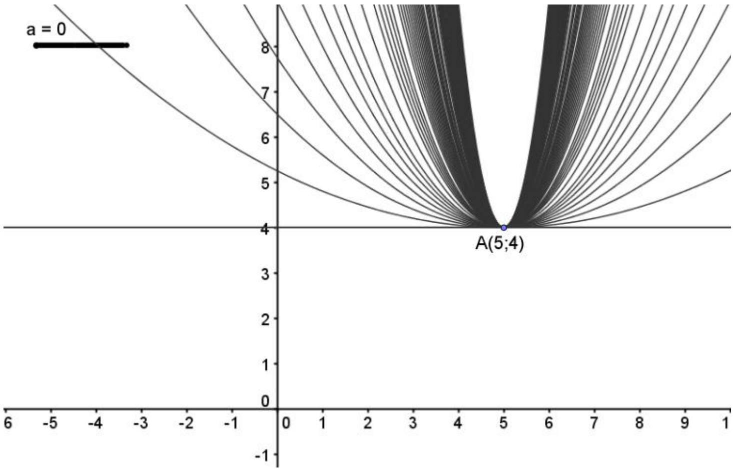


Figure 8. 34: Group 2’s correct response

At the end of the task facilitator pointed out that in the previous tasks the domain was all real numbers while the range was varying. Hence in the next task the domain was not going to be all real numbers.

- **Event 4: Finding functions with domain not all real numbers**

To begin the next focus he asked teachers to find functions with the domain not all real numbers. Teachers worked on this task for 3 minutes in groups, and at the end the Facilitator wrote down responses on the flipchart (see **Figure 8.35**). The following list of functions was produced by different groups:

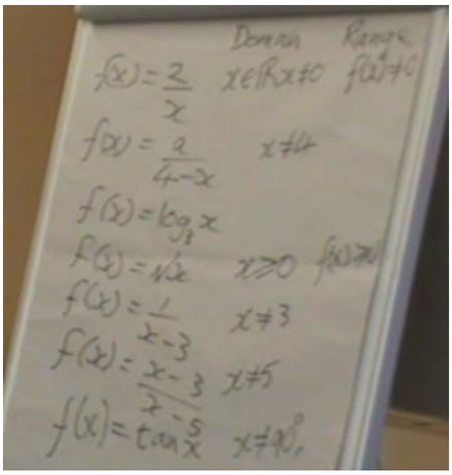


Figure 8. 35: List of equations with domain not all real numbers

- **Sub-Event 4.1: Discussion focused on domain and range of $\boldsymbol{f(x) = \frac{2}{x}}$**

When all responses had been listed the facilitator asked for the domain and range of each of the listed functions. The first focus was on $f(x) = \frac{2}{x}$ (see **Excerpt 8.38**).

F:	Ok, let's just check the domain and range and see what they are. *[F refers to the equations he has just written on the flipchart.* $f(x) = \frac{2}{x}$. What's the domain then in this one?
T37:	It does not cover all numbers
F:	Why does it not cover all of the numbers? Who gave us the 2 over x? *[points to a teacher]* You did, yes? Domain?
T32:	If x is zero it's not defined.
F:	Right, so it's not defined when x is zero. So we have to call the domain
T32:	x
F:	Right. So we say x is a real number but x is not equal to zero. *[under Domain S writes:* $x \in \mathbb{R}, x \neq 0$ *]* Ok. And the range, by the way, what's the range?
T39:	Same applies
F:	Same applies. The same applies, you just inverse these. So the range is also f(x) never equals zero. *[under Range writes* $f(x) \neq 0$ *]* Ok, think of the graph of this.

Excerpt 8.38: Discussion focused on domain and range of $f(x) = \frac{2}{x}$

The above excerpt shows that indeed the domain of $f(x) = \frac{2}{x}$ was not all real numbers, since it did not include zero. It was also agreed that the range did not include zero. Hence the group that suggested this function had produced a correct response. The excerpt also contains evidence of scaffold from the facilitator and legitimation from Teacher 32.

- **Sub-Event 4.2: Discussion focused on domain and range of $f(x) = \sqrt{x}$**

The next discussion centred on $f(x) = \sqrt{x}$ after the facilitator decided he was not going to discuss the suggested logarithmic function because it was unfamiliar to most teachers and not within the focus of the presentation. **Excerpt 8.39** contains the discussion focused on the domain and range of $f(x) = \sqrt{x}$.

942 943	F:	The root of x. *[points to:* f(x) = √ x *]* Domain and range. Who was it who offered the root of x?
945	T41:	x cannot be negative
947 948	F:	x cannot be negative?. Ok, when we write it like this we're taking, right, where x cannot be negative we can't underscore it so x must be, how shall we describe it?
950	T42:	Greater than
952 953	F:	Greater than or equal to zero. *[writes:* $x \geq 0$ *]* What's the range going to have to be? Sorry?
955	Trs:	The same *[teachers commenting]*
957	F:	S: The same?
959	Trs:	The same *[teachers continue making comments]*
960 961 962 963	F:	In fact the convention is unless you have the plus or minus here we actually need just the plus, ok? If we had plus or minus, as you said, we would expect every possibility. But because we don't have a plus or minus then we treat it just the positive values and so we're going to have to have f of x greater or equal this one. Yes?

Excerpt 8.9: Discussion on domain and range

The discussion also focused on the meaning of the square root sign. There the facilitator explained that when the square root symbol did not carry any sign, it meant only the positive square root was considered. The excerpt contains an instance of scaffold, which was a probe and instances of legitimation made by the facilitator

Later in the discussion one teacher asked why the negative square root was not considered, and the facilitator explained that by mathematical convention the square root sign that did not carry a specific sign suggested a positive square root only. Another teacher added that if the negative square root was considered then $f(x) = \sqrt{x}$ will cease to be a function (see **Excerpt 8.40** and **Figure 8.11**).

1021 1022	T22:	And obviously it can't be positive and negative and still call it a function because then it wouldn't be a function.
1024	F:	That's right. That's another problem, that's right.

Excerpt 8.40: Teacher 22's explanation

It is interesting here that the teacher was able to realise that taking both the positive and negative square roots would produce a non-function. This is so since in Functions 1 teachers identified non-functions as functions. The explanation contained a legitimation, which I have underlined. Below is **Figure 8.36** which I have provided to illustrate Teacher 22's explanation.

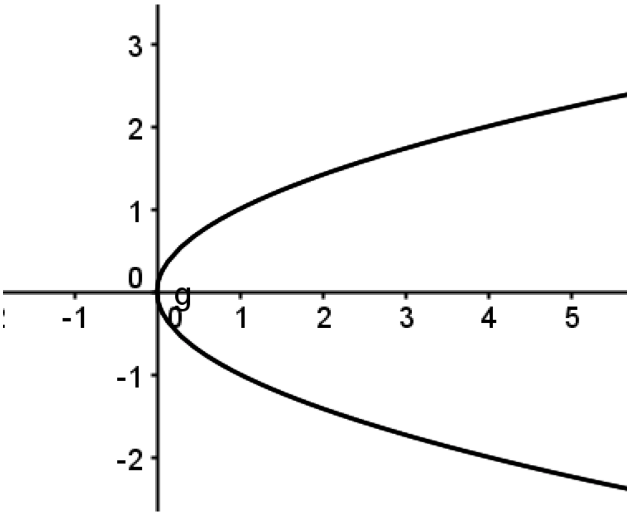


Figure 8. 36: Graph of f(x)=±√x is a non-function

According to Teacher 22, **Figure 8.36** represented a non-function, and he was correct because the vertical line test would cut the graph in more than one place, and this was contrary to the definition of function. The teacher appears to have drawn from his experiencing of Functions 1 where **Figure 8.36** was used to exemplify a non-function.

- **Event 5: Discussion focused on one-to-one and many-to-one functions**

At the end of the discussion focused on domain and range attention went to one-to-one functions. Facilitator simultaneously brought into the fore a hyperbolic and a cubic function which were a one-to-one and a many-to-one functions respectively. The purpose of the task was to introduce the former functions to teachers. However, Facilitator chose to bring them both together for teachers to be able to distinguish one from the other (see **Figure 8.14**).

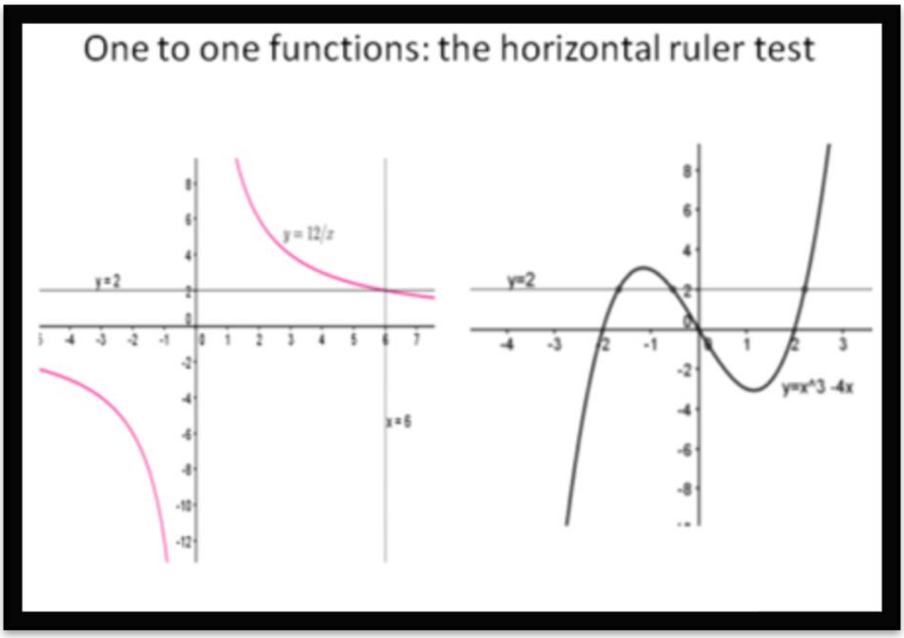


Figure 8.14: Graphs of one-to-one and many-to-one functions

Speaking from the hyperbola on **Figure 8.14**, the facilitator asked for the value of the input if the output was 2, and the response was that it was 6. Speaking from the cubic function he said it was not easy to say what the input was because there were three possibilities (see **Excerpt 8.41**).

1086 1087 1088 1089	F:	*[another slide appears on the screen:* One-to-one functions: the horizontal ruler test. *Two graphs are on the slide]* Ok, if I were to ask you what is the value of this first graph when x is 2, what is the value of the function? What is the output if the input is 2?On this first one, what would you tell me?*[points to the graph on the screen]*
1091	T45:	When x is 2?
1093	F:	When x is 2
1095	Trs:	6
1097 1098 1099 1100	F:	*[points to the graph]* It's going to be 6. And if I asked you on this graph *[points to the second graph on the screen]* if… Ok, no, I want to ask you something on the domain. If I were to say that the, um, that the output is 2 what is the input, what would you say? The output is 2, what is the input?
1102	T45:	6
1104 1105	F:	The output is 2 what's the input it is also 6. Quite right. And here *[pointing to the second graph on the screen]* the output is 2, what is the input?
1107	Trs:	*[teachers discuss this]*
1119 1120 1121 1122 1123	F:	We've in fact got three possibilities, ok? Now both of these are functions but they're different kinds of functions. *[points to the first graph]* This one we call a one-to-one function because we're only going to get one output… one input for any output, whereas on this graph *[points to the second graph]* we've got, this is called a many-to-one because we'll have three possible inputs for a particular output.

Excerpt 8.41: Explanation of one-to-one and many-to-one functions

In the excerpt, facilitator pointed out that the hyperbola was a one-to-one function and the cubic a many-to-one function. To explain the difference between the two types of functions further he said a horizontal ruler test would cut the one-to-one once, whereas it would cut the many-to-one function more than once, and he demonstrated on each of the graphs. In the excerpt instances of scaffold are highlighted and legitimation underlined.

8.11. Discussion of pedagogic strategies used at N-cluster

Facilitator's natural attitude was bracketed in this section of Session 2.2. He did not take it for granted that the task would be easy for teachers, and this was evident in the availed resources and group assistance by the Facilitator. By way of example, he put up four multiple representations which he used to check teachers' prior knowledge of inputs and outputs and to introduce the new content focused on Domain and Range (see Figure 8.2). Group assistance or help was present was provided when teachers were working on assigned tasks when the Facilitator moved between groups checking their progress. Since the were resources and help rendered to teachers then it can be said that teachers at this cluster experienced high scaffolding, and this was due to the deductive pedagogic strategy which was adopted there. As I have described before, a deductive pedagogic strategy is one where there is a step-by-step provision of scaffold to enable progress on the task.

Another example of scaffolding was in the choice of $f(x) = (x-1)^2 + 2$ as an example to work with across the task involving domain and range. Facilitator pointed out that he chose this form of the equation because it was easier to discern the minimum and maximum values from this equation than from $f(x) = (x-1)^2 + 2$. In addition, teachers' verbal responses were illustrated in the Geogebra environment to confirm or illuminate them. It is evident from teachers' correct and incorrect responses to tasks that they were working with the availed scaffolding, unlike in Session 2.1 where they appeared to be working from their prior knowledge of functions. In this session teachers' responses were modeled around the Facilitator's examples. By way of example, the Facilitator worked with $f(x) = (x-1)^2 + 2$ to link Functions 1 and Functions 2, and went on to introduce the notions of Domain and Range still using the same equation. Consequently, when teachers were asked to find functions with the maximum value of 2, Group 1 produced a response which was an adopted version of the facilitator's example. The group introduced a negative sign in front of $f(x) = (x-1)^2 + 2$ to obtain $f(x) = -(x-1)^2 + 2$, and the response was correct (see **Figure 8.28**).

Groups 2 and 3 produced incorrect responses, and were able to realize and correct themselves. The groups' incorrect responses were also modeled along the facilitator's examples (see **Figures 8.29 and 8.30**). I must point that available scaffolding helped teachers to notice the mistakes in their responses, and this happened when they were asked to explain why their productions would have a maximum value at 2.

Following on the teacher's recognition of his group's mistake the facilitator pointed out that the graph would have a minimum value of 2 instead of a maximum and he said this was because $(1 - x)^2$ would yield a positive value. He pointed out that in actual fact there was no difference between the group's production, ($f(x) = (1 - x)^2 + 2$ and the original function, $f(x) = (x - 1)^2 + 2$. He went on to illustrate his explanation from the Geogebra environment by showing that graphs from the two equations were coincidental (see **Figure 8.37**).

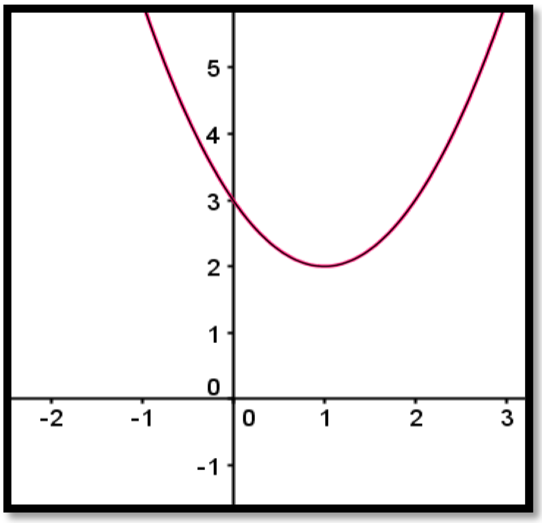


Figure 8. 37: Facilitator's illustration of two coinciding graphs

Group 3 also noticed that their production was incorrect after a discussion of Group 2's production, because there was little difference between the two productions. Group 3 had produced $f(x) = (2 - x)^2 = 2$ and Group 2 had $f(x) = (1 - x)^2 = 2$. As such the Facilitator's explanation on Group 2's response helped Group 3 members to notice that their production was incorrect.

It is noticeable from **Figure 8.27** that after the discussions focused on Groups 2 and 3's productions, that the two were cancelled from the list of Functions with a maximum value of 2. As I mentioned earlier, it was interesting to notice teachers drawing from the provided scaffolding in the session, and this seems to have helped them to progress in the task. By way of example, the teachers' examples were modeled around the facilitator's example, and those who produced incorrect responses were able to notice and self-correct. A major contribution of the scaffold was evident in Group 4's response. This group was able to notice that changing the value of **p** in $f(x) = a(x - \boldsymbol{p})^2 + q$ generates all possible functions with the maximum value of 2. The discussion led by Group 4 led to the co-production of another function, $f(x) = -(x - \mathbf{3})^2 + 2$ and an agreement that a vertical shift of the original function would yield all possibilities. The last to offer a response was Group 5, which produced $f(x) = 2 - x^2$. The following Excerpt 8.39 covers the conversation between a Group 5 member and the facilitator, who was writing down the response on the flipchart as the teacher was talking.

In conclusion, it can be pointed out that the deductive pedagogic strategy and its resultant high scaffolding helped teachers to work on task successfully. It is 'successfully' in the sense that three groups were able to produce correct responses on the first task, and the other two noticed their mistakes and corrected them. Furthermore, teachers were able to notice that changing the value of **p** would generate the required functions. It suffices to mention at this point that it was evident that working with the provided scaffoldings was productive for, and useful to teachers. I keep repeating this viewpoint because in Session 2.1 teachers produced and defended their incorrect responses, and did not appear to be working with the provided scaffoldings. As such their progress on tasks was not productive. However, here we notice teachers working with resources and help from the Facilitator to produce positive responses. It was also evident from incorrect responses that the teachers were working with the provided high scaffolding. For instance, their incorrect functions still maintained the features of the Facilitator's original function (see **Figure 8.37**).

8.11. Discussion of legitimating criteria at the N-Cluster

The first legitimating criteria was present when a teacher was responding to the Facilitator's question, in which he asked teachers if it was coincidence that the minimum value on the graph (see Figure 8.27) was 2, and the value of q in the function $f(x) = (x-1)^2 + 2$ was also 2. A teacher's response was that it was not coincidence, because as long as the equation was in the form $f(x) = (x-1)^2 + 2$ the minimum value on the graph would always be 2. In this criteria the teacher was appealing to the $f(x) = a(x-p)^2 + q$ formation of the function, where **q** is the value of the function at the turning point and is minimum when **a** is positive and maximum when **a** is negative. The second legitimating criteria was present when the facilitator was explaining why the minimum value on the graph was 2. He said it was 2 because the smallest value $(x - 1)^2$ could be was zero and that happened when x was 1. The Facilitator was appealing to the values of $(x -1)^2$. The third legitimating criteria surfaced when $f(x) = (x-1)^2 + 2$ was rewritten in the form $f(x) = x^2 - 2x + 3$. Facilitator explained that this form of the equation would have made it difficult to factorise because the equation did not have real roots. The appeal here was to the nature of roots. The fourth legitimating criteria emerged in the same explanation when the Facilitator mentioned that the function did not have real roots because the graph did not cross the x-axis. The appeal here was to the behaviour of the graphical representation.

The fifth legitimating criteria came from a member of Group 1 when she explained that the function $f(x) = -(x-1)^2 + 2$ had a maximum value of 2 at the turning point because of the negative sign in front of the equation. The teacher was appealing to the effect of the negative sign on the shape of the parabola. The sixth legitimating criteria was present when the discussion was on the incorrect response from Group 2 which had offered $f(x) = (1-x)^2 + 2$ as one example of a function with a maximum value of 2. Facilitator asked why the response was incorrect, and one teacher said it was incorrect because all values coming out of $(1-x)^2$ would be positive, and so 2 would be the minimum value instead of maximum value. The teacher was appealing to effect of exponent on (1-x). The seventh legitimating criteria came into being when the discussion was on the range of $f(x) = \sqrt{x}$. A teacher wanted to know why the negative square root was excluded from the range, and the facilitator explained that this was <u>because by mathematical conversion the square root sign (√) denote a positive square root</u>. The appeal here was to mathematical conversion.

Table 8. 5: A summary of legitimating criteria at the N-Cluster

Legitimating criteria	**Appeal**	**Enacted object**
because as long as the equation was in the form $f(x) = (x-1)^2 + 2$ the minimum value on the graph would always be 2.	Appeal to the general formation of $f(x) = a(x-p)^2 + q$	$\boldsymbol{Mt_{re+}}$
because the smallest value $(x – 1)^2$ could be, was zero and that happened when x was 1	appealing to the values of $(x -1)^2$	$\boldsymbol{Mt_{re+}}$
because the equation did not have real roots	appeal to nature of the roots of a quadratic function	$\boldsymbol{Mt_{re+}}$
because the graph did not cross the x-axis	behaviour of the graph	$\boldsymbol{Mt_{re+}}$
The graph has a maximum because of the negative sign in front of the equation	Appeal to effect of the negative sign on the shape of the parabola.	$\boldsymbol{Mt_{re+}}$
because all values coming out of $(1-x)^2$ would be positive	appeal to the effect of two, the exponent, on (1-x).	$\boldsymbol{Mt_{re+}}$
because by mathematical conversion the square roots sign (√) denote a positive square root.	Appeal to mathematical conversion	$\boldsymbol{Mt_{re+}}$

Table 8.5 shows that the seven appeals that characterised the legitimating criteria drew from different aspects of functions that included (i) form of equation; (ii) nature of values from a particular part of an equation, (iii) nature of roots, (iv) behaviour of parabola, (v) shape of parabola, (vi) effect of exponent, and (vii) mathematical conversion. Since functions are a mathematics object, it is safe to conclude that the enacted object of learning was a mathematics object. In Chapter 11, I reveal that a mathematical object is twofold - it can be personal mathematics (Mt-PL) or professional mathematics (Mt-pl). Consequently, the mathematics object that was enacted was personal mathematics because this mathematics was meant to strengthen the teachers' own knowledge of functions, and is evident in the Facilitator's instructions, tasks and teachers' productions (particularly their incorrect responses), that they were learning mathematics for personal growth.

8.13. Analysis of what was made possible to learn at the N-Cluster

The enactment of Functions 2 in the N-Cluster was characterised by two patterns of variation, namely, generalisation and contrast. The first pattern of variation to emerge was generalisation, and this was present where attention was on domain and range. There the Facilitator worked with multiple representations of the same quadratic function to help teachers experience the same domain and the same range across the four representations (see **Figure 8.25**). By way of example he called attention to the mapping representation where two sets were labelled Domain and Range respectively. Moving from there, attention went to the equation where he pointed out that all inputs that could go into the equation constituted the domain and those that could come out constituted the range. Speaking from the table of values he asked teachers to identify the domain and range. The response was that the domain was represented by the x values and the range by the y-values in the table. Attention was then shifted to the graph, and speaking from there, Facilitator asked teachers where the domain and range were located on the graph, and the response was that the domain was along the x-axis and the range along the y-axis. Hence, working from the four representations with attention on domain and range helped teachers to experience similarity between the representations. In variation terms, this constituted a pattern of variation known as generalisation because the important feature (i.e. domain and range) was invariant when the less important features (multiple representations) were varying. **As such what was possible to learn was that domain and range were properties of a quadratic function but its representations were not its properties.**

The second pattern of variation was present where the focus was on whether it was coincidence that there was 2 in the equation and the minimum value at the turning point of graph was also 2. Drawing attention to this opened opportunities for teachers to experience similarity between the equation and the graph with particular focus on 2. Here the value of two was similar across the two representations, and was an important feature to which attention was drawn. As such 2 was invariant and the equation and the graph were varying. Since that which was important was invariant, and what was less important was varying, the pattern of variation was generalisation. **What was made possible to learn through this pattern was that the minimum value of the function at the turning point was a property of a quadratic function, and its representations were not its properties.**

The third pattern of variation was contrast, and this was present when attention was on the different ways of writing equations of the same quadratic function. There the Facilitator pointed out that he chose to work with $f(x) = (x - \mathbf{1})^2 + 2$ instead of $f(x) = x^2 - 2x + 3$, because it was easier to discern the range and the value of the function at the turning point from the former, and difficult to discern the same from the later equation. Working with these two equations and in particular, highlighting the differences between them, helped teachers to experience difference between the two equations. From variation theory perspective, experiencing difference is a necessary condition for discernment of difference between two or more critical features. In this example the critical features were the two equations, and these were equations of the same quadratic function. As such this makes quadratic function the dimension of variation in which the two equations are values. **Consequently, what was made possible to discern was that it is easier to identify the range and the value of the quadratic function at the turning point when the equation is in the form $\mathbf{f(x) = a(x-p)^2 + q}$ than when it is in the form $\mathbf{f(x) = ax^2 + bx + c}$.**

The fourth pattern of variation was generalization, and this was present when teachers were finding functions with the maximum value of 2. What was invariant was the maximum value of 2 at the turning point, and what was varying were the equations which the teachers produced. What was interesting about the teachers' responses was that in both the correct and incorrect responses the value of the function at the turning point was invariant. This value was a minimum in incorrect responses and a maximum in correct responses, but what was similar across the responses was that the value of the function at the turning point was 2 (see **Figure 8.38**).

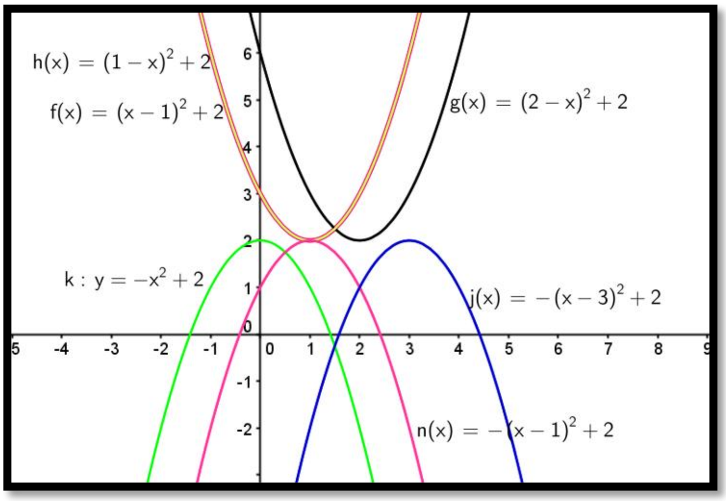


Figure 8. 38: Teacher productions that show opportunity to experience similarity

As such, working on the task where they were finding the function with a maximum value of 2 helped teachers to experience similarity in the responses. **What was made possible to learn was that the value of the function at the turning point was a property of a quadratic function, but equations were not properties of a function.**

The fifth pattern of variation was present when attention was on teachers finding functions with the minimum value at the point (5; 4). The pattern was generalization, and here teachers experienced similarity between functions with the same minimum value. The invariant feature was important, and this was the minimum value at the point (5;4). The varying features were the equations, and these were not in focus. **As such what was possible to learn was that the value at the turning point was a property of a quadratic function.**

The sixth pattern of variation was present when the focus was on teachers finding functions with the domain not all real numbers. There the pattern of variation was still generalization, and so what was similar and invariant was the domain not all real numbers, and what was varying were the different functions (see **Figure 8.35**). **As such what was possible to learn was that the domain with not all real numbers was a property of a function.**

The seventh pattern of variation was contrast and was also present when attention was on domain not all real functions. Teachers had opportunity to experience difference between the functions when they were listed together on the flipchart (see **Figure 8.27**). The critical features were the functions with domain not all real numbers and the dimension of variation was types of functions. **What was possible to discern was that there different types of functions with the domain not all real numbers.**

The eighth pattern of variation was also present where attention was on functions with domain not all real numbers, and this was still contrast. Teachers were helped to experience difference between the domains that were not all real numbers (see Figure 8.38). There it was possible for teachers to notice that the domains which do not include all real numbers are different from each other. By way of example, the domain of $f(x) = \frac{2}{x}$ is $\{x \in \mathbb{R} | x \neq 0\}$ and this is different from the domain of $f(x) = \sqrt{x}$ which is $\{x \in \mathbb{R} \mid x \geq 0\}$. The dimension of variation was domain not all real numbers, and the critical features were the domains of individual functions. **What was possible to discern was that domains which do not include all real numbers differ according to the function they associate with.**

The ninth pattern of variation was contrast, and was present when attention was on one-to-one and many-to-one functions. Facilitator worked simultaneously with the two types of functions to show the difference between the two. He demonstrated the horizontal ruler test to show that it cuts the graph of a one-to-one function once and of many-to-one more than once. As such working with the ruler test helped teachers experience difference between the two types of functions. Here the dimension of variation was the horizontal ruler test, and the critical features were the number of times the ruler cuts the graph of a function. **What was made possible to discern was that a horizontal ruler test cuts the graph of a one-to-one function once and of many-to-one more than once.**

The tenth pattern was still contrast, and was also present when attention was on the one-to-one and many-to-one functions. In particular, the facilitator pointed out that it is possible to work backwards in a one–to-one function, but not possible to do so in a many-to-one function (see **Figure 8.14**). The dimension of variation was functions and the critical features were one-to-one and many-to-one functions. **What was possible to discern was that working from a one-to-one function backwards produces an inverse function, and working backwards from a many-to-one does not produce an inverse function**

Table 8.6: Summary of what was made possible to learn at the N-cluster

Invariant	Varies	What is discerned	Enacted object of learning
Domain and range	Representations	Domain and range are properties of a quadratic function, but its representations are not its properties.	Mt_{re+}
Minimum/maximum value	Representations [e.g. equations with turning point at f(x) = 2 and equations with turning point at (5; 4)],	that the minimum value of the function at the turning point was a property of a quadratic function and its representations were not its properties	Mt_{re+}
Quadratic function	Forms of writing the equation	A quadratic equation in the form $f(x) = a(x - p)^2 + q$ visibilizes the range and the value of f(x) at the turning point than when it is in the form $f(x) = ax^2 + bx + c$.	Mt_{ap+}
Domain not all real numbers	Equations of different Functions	That the domain with not all real numbers was a property of a function	Mt_{re+}
Domain not all real numbers	Functions	That there were different types of functions with the domain not all real numbers.	Mt_{ap+}
Domain not all real numbers	Domains of individual functions	that domains which do not include all real numbers differ according to the type function they associate with.	Mt_{ap+}
Horizontal ruler test	Number of times ruler test cuts a graph of a function	that a horizontal ruler test cuts the graph of a one-to-one function once and of many-to-one more than once.	Mt_{re+}
Functions	Types of functions (one-to-one and many-to-one)	that working from a one-to-one function backwards produces an inverse function but a many-to-one does not	Mt_{re+}

From **Table 8.6**, it is discernible that what is possible to learn is constituted by both *mathematical reality* (**Mt_{re+}**) and *mathematical appearance* (**Mt_{ap+}**), with **Mt_{re+}** slightly dominant. Therefore, the result of **Table 8.6** suggests that the enacted object of learning was constituted by *mathematical reality* and *appearance*.

8.14: Conclusion

The analysis of Functions 2 revealed similarities in the teachers' experiencing of Functions 2. Firstly, across the three clusters facilitators adopted the deductic pedagogic strategy, which openned up scaffold for teachers to experience high scaffold. The consequence of high scaffold was more correct teacher productions. Secondly, the object of learning was invariant across the clusters. Thirdly, the legitimating criteria and what was possible to discern were characterised by both $\boldsymbol{Mt_{re+}}$ and $\boldsymbol{Mt_{ap+}}$. As such the enacted object of learning, (which was constituted by *correct mathematical reality* and *correct mathematical appearance)*, was the same across the three clusters.

In spite of the articulated similarity in experiencing Functions 2, the contexts under which the experiencing occurred were different. As noted in **Chapter 7**, the clusters were facilitated by different facilitators at different locations and on different days.

In this chapter I have analyzed Task 1 of Functions 2 in detail. In **Chapter 9** I present a summary of the analysis of the remainder of Functions 2 – i.e. Tasks 2 – 4. Detailed analysis of these tasks is available in the attached CD (see **Appendix E**).

CHAPTER 9: ANALYSIS OF TASK 2 - 4 OF FUNCTIONS 2

9.1. Introduction

Tasks 2, 3 and 4 were group activities. Each task began with an introduction of the task followed by teachers working in groups on the task. The Facilitator then moved between the groups monitoring progress and providing scaffold where it was needed. Discussions between the facilitator and each group were largely out of the public channel, and where the discussions were in the public space, they were not clear because all groups were talking simultaneously. Consequently, the legitimating criteria that might have been present in those discussions were not captured in the video recordings. Furthermore, for the larger part of the time teachers were working on tasks the video was switched off, since the discussions were out of the public space. Hence, there were limited legitimating criteria in the analyzed tasks.

In Task 1 (see **Chapter 8**), though teachers worked in groups, each group was called to present to the whole class. Hence there the legitimating criteria were in the public channel and were captured by the video. Therefore the analysis of these three tasks includes the analysis of pedagogic strategies and patterns of variation. Suffice to note here that while Task 2 was the same across the three clusters, Task 3 and Task 4 were not the same (see **Appendix E** in the attached CD). As noted in the conclusion section of **Chapter 8**, this chapter presents a summary of the analysis of Tasks 2 – 4. Here I only present a discussion on pedagogic strategies and patterns of variation. As mentioned above, legitimating criteria were not captured in the video recordings. Hence their analysis were not possible, and so are absent in the following summary. The detailed analysis that includes the descriptions of Tasks 2 – 4 can be found in **Appendix E**. The purpose of presenting a summary rather than the detailed analysis was to ensure that the thesis was within the official recommended word count. I must mention here that the **Figures** and **Excerpts** that I refer to in this chapter are in **Appendix E**.

9.2. Discussion of pedagogic strategies used in Tasks 2, 3 and 4 at E-Cluster

Facilitator's natural attitude was bracketed throughout the task. This is evident from the resources and help which were availed to teachers before and during work on the task. At the beginning of the task he explained Task 2 to Group 3 by finding three cards containing range, equation and graph and asking the group to write down the domain on the blank card. The cards were all related to $\mathrm{y} = \tan x$ (see **Figure 9.7 in Appendix E in the attached CD**). The Facilitator continued to move around assisting groups through probes and explanations. By way of example when Group 2 was looking for the range of $\mathrm{y} = \sin x - 4$, he reminded the group about the graph of $\mathrm{y} = \sin x$, which he drew for them to work with to find the range of $\mathrm{y} = \sin x - 4$ (see **Figure 9.1**). There were several instances where probes and explanations were used to scaffold the task. Most of the excerpts contain highlighted text which constitutes probes and explanations.

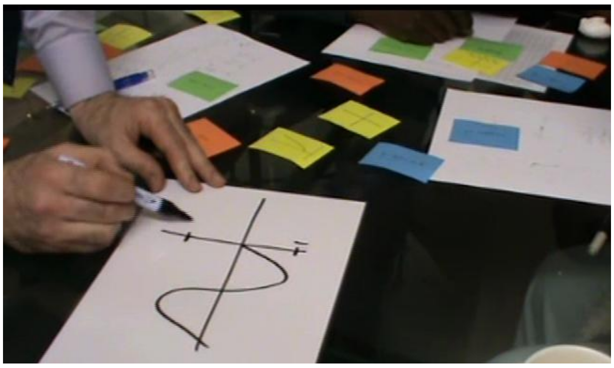


Figure 9. 1: Facilitator using the graph of y = sinx to scaffold a task

Since the Facilitator chose to bracket his natural attitude it was evident that he was working with the deductive pedagogic strategy and this was evident in the high scaffolding that was present throughout the task. Prior to working on this task teachers worked on Task 1 which served as an introduction to Task 2. There, teachers were finding domains or range when given an equation, or working from the domain and range to find the equation or its graph. Teachers were also introduced to one-to-one and many-to-one functions. Knowledge of these types of functions was useful in Task 2, because when the Facilitator was moving between the groups he kept asking if the function was one-to-one or many-to-one. In this task it was evident that teachers were drawing from the availed scaffold, and this was evident in the rate of success on tasks. Teachers did not seem to be drawing from their prior knowledge as they did in Functions 1. Here they were working from the availed scaffold.

9.3: Discussion of legitimating criteria at the E-Cluster

The first legitimating criteria emerged when Group 1 was finding the range of $y = \frac{1}{x} + 1$. Firstly, facilitator asked which x-values were not in the domain, and one member said x could not be zero. Following on the response, facilitator asked if $\{y \in \mathbb{R} \mid y \geq 1\}$ was the range, and the same teacher said yes, because x cannot be zero. The teacher thought that the range could not include zero because x did not include zero. However, the range included zero, and this happened when x was -1. The appeal characterizing the legitimating criteria was to mathematical logic. Since the appeal was incorrect this constituted an appeal to *incorrect mathematical reality*. The second legitimating criteria emerged when the discussion was focused on the graph of $y = Sin\, x^2$. Here the facilitator was explaining that the graph was crowded compared to the graph of $y = Sin\left(\sqrt{x}\right)$, because squaring was the opposite effect of taking square root. The facilitator was appealing to the inverse property of operations. Hence this contituted an appeal to *correct mathematical reality*. **Table 9.2** contains a summary of the legitimating criteria.

Table 9.1: Summary of legitimating criteria at the E-Cluster

Legitimating criteria	Appeal	Enacted object
Because x does not include zero	Mathematical logic	Mt_{re-}
Because the square is the opposite effect of square root	converse property	Mt_{re+}

9.4. Discussion of what was made possible to learn at E-Cluster

Task 2 contained patterns of variation, and these included fusion and contrast. The first pattern of variation was fusion, and it was present when teachers were helped to experience simultaneous variation between three dimensions of variation that included representations (i.e. graph and equation), domain and range. This was possible when teachers were matching domain, range, graph and equation of the same function (see **Figure 9.7 in Appendix E in the CD**). The experienced pattern of variation was fusion because three dimensions of variation were made to vary, and when this happens fusion is experienced. Representation was a dimension of variation because it contained two critical features (i.e. graph and equation). Domain was a dimension of variation because it contained many critical features. By way of example $\{x \in \mathbb{R}\}$ and $\{x \in \mathbb{R} \mid x \neq 0\}$ are types of domains. Similarly with the range. It is a dimension of variation because there are many types of ranges (e.g. $\{2 \leq y \leq 5\}$ and $\{y \in \mathbb{R} \mid y \neq 0\}$). Experiencing this pattern of variation made it possible for teachers to discern the relationship between the three dimensions of variation and between them and the function to which they belong. **As such it was possible to learn that each representation contains the domain and range of a function**

The second pattern of variation was Contrast. This was present where teachers were helped to experience difference between the kinds of functions which they worked with in the card activity. By way of example teachers worked with quadratic, tangent, sine, square root and hyperbolic functions, and this provided opportunities for discernment of difference between them. In this pattern of variation, the critical features were the kinds of functions, and these were in the fore and varying. The dimension of variation was functions, and this was invariant. Contrast is experienced when that which is varying is in focus and that which is invariant is out of focus, but is experienced simultaneously with that which is in focus. Similarly, teachers experienced difference between the different kinds of graphs that were availed to them to work with. **As such what was possible to learn was that there are different kinds of functions.**

The third pattern of variation was fusion, and this was present in Task 3 where Group 1 teachers had to generate equations that matched given domain and range (see **Table 9.1**). The teachers were helped to experience simultaneous variation between domain and range, and this provided opportunity for discernment of the relationship between domain and range of the same function. By way of example, the Facilitator's demonstration with $y = xSinx$ and

$y = sin\sqrt{x}$ helped teachers to notice variation in the domain and range of $y = Sin\,x$. The domain of $y = Sinx$ was $x \in \mathbb{R}$, and it became $x \in \mathbb{R} \mid x \geq 0$ when $y = Sinx$ was transformed into $y = sin\sqrt{x}$ (see **Figure 9.13 in the CD**). Here the range was invariant. Similarly, in $y = Sinx$ the range was $\{y \in \mathbb{R} \mid -1 \leq y \leq 1\}$ and became $\{y \in \mathbb{R}\}$ in $y = xSinx$. In this case the domain was invariant. **As such what was possible to learn was that domain and range of a function may change separately when a function is transformed.**

Still in Task 3, a fourth pattern variation was present when teachers were helped to experience difference in domains with and domains without restrictions. This pattern was contrast. Teachers were able to notice that working with domains with restrictions was more challenging than where there are no restrictions. In this pattern the critical features were domain with restrictions and domain without restrictions, and these were in the fore. The dimension of variation was domain. **As such what was possible to learn was that a domain may have restrictions or may not, and that domains with restrictions are difficult to work with. Similarly with the range, the same was discernible.**

The fifth pattern of variation was present when attention was on transformation of the graph of y = Sinx. The pattern of variation here was Contrast. Teachers were helped to experience difference between graphs involving Sin(x) (see **Figures 9.10 – 9.15 in Appendix E in the CD**). There were several dimensions of variation in the graphs, and these included variations with respect to domain, range and shape of graph. Similarly, there were dimensions of variation in the facilitator's examples shown below. In one dimension of variation the focus was on the product of two functions - i.e., $y = xsinx$ and $y = (\sqrt{x})sinx$). In the other dimension the focus was on the sine itself, but with variations in the nature of the angle – i.e. $y = \sin(\sqrt{x})$ and $y = \sin\, x^2$.

$$y = sinx$$

$$y = xsinx$$

$$y = (\sqrt{x})sinx$$
$$y = \sin(\sqrt{x})$$

$$y = \sin\, x^2$$

The equations were also characterized by variations, and these were mainly in structure. What was varying was the formation of graphs involving sin x, and the invariant dimension of variation was sin x. **What was possible to learn was that graphs of functions involving sin x behave differently with respect to shape, domain and range.**

The sixth pattern of variation was present in Task 4 when the Facilitator was scaffolding the task to teachers. There the Facilitator worked with two different equations of a quadratic function to demonstrate the properties of quadratic equations. The resultant pattern of variation was contrast. Teachers were helped to experience difference between properties of different equations of the quadratic function. By way of example, teachers noticed that $y = x^2 - 5$ contained y= -1 in its range, but $y = x^2$ did not. In addition, the graph of $y = x^2 - 5$ passed through the fourth quadrant and graph of $y = x^2$ did not. **As such what was possible to learn was that equations of the same type of function possess different properties**

Table 9. 2: Summary of what was made possible to learn at the E-Cluster

Invariant	Varies	What is possible to discernment?	Enacted object
Function	Domain/Range	A function is constituted by domain and range	Mt_{re+}
Functions	Representations (Equations and graphs)	Functions are constituted by representations	Mt_{re+}
Function	Domain/range	Domain and range of a function may change differently when a function is transformed	Mt_{re+}
Domain Range	Domain with restriction and domain with no restrictions Range with restriction and range with no restriction	Domain may have restrictions or not and domains with restrictions are difficult to work with. Similarly with the range.	Mt_{re+}
Sin x	Graphs of functions involving sin x	Graphs of functions involving sin x behave differently with respect to shape, domain and range.	Mt_{re+}
Function	Equations	Equations of the same type of function posses different properties	Mt_{re+}

Table 9.3 shows that the object of learning was largely constituted by Mt_{re+}. Hence there were opportunities for the E-Cluster teachers to discern *mathematical reality*.

9.5. Discussion of pedagogic strategies used at the I-Cluster

The Facilitator availed resources and moved between groups providing help to enable progress on the Task. In addition to this, each of the four groups had a WMC-S project member sitting in and participating in group discussions. Provision of resources such as cards for teachers to work with, assistance from the Facilitator and additional guidance from project members are an indication that the Facilitator's natural attitude was bracketed - he did not take it for granted that the task would be easy for teachers. For instance, guidance from a project member in Group 3 was invaluable in enabling progress. By way of example, in the discussion focused on identifying the range of $y = \cos x + 1$ the facilitator probed the group to help them clarify their verbal description of the range (see **Excerpt 9.12)**

The WMC-S team members were familiar with the task because they were present when it was enacted at the E-Cluster the previous day. Their participation involved probing teachers' responses, and when progress was slow, they provided cues from what was experienced before to enable progress. I must also add that before presentations in clusters team members and Facilitators came together to design the presentation. The design stage involved planning, organisation of content and trial presentation to team members for feedback. As such they were as knowledgeable as the Facilitator in terms of the indirect and direct objects of learning.

The availed resources and assistance from team members and Facilitator suggest that the teachers were experiencing a deductive pedagogic strategy. This was evident in the step-by-step provision of resources and assistance on the task, and the success rate on the task across the four groups (see **Figures 9.18 – 9.25 in Appendix E in the CD**). As such teachers in this cluster experienced high scaffolding first because of the bracketing of the natural attitude and the subsequent adoption of the deductive pedagogic strategy. In the I-Cluster, just as in E-Cluster, teachers across the four groups worked with the provided high scaffolding, and this was evident in the teachers' correct productions (see **Figures 9.17 – 9.25 in the CD)**

9.6. Discussion of what was made possible to learn at the I-Cluster in Task 2

The enactment of Task 2 was constituted by patterns of variation. In my analysis the patterns of variation reveal what was made possible to discern, and all that which was made possible to learn constitutes the enacted object of learning. In other words, patterns of variation are used in this analysis to visibilise the enacted object of learning.

The first pattern of variation was fusion, and was present when attention was on matching representations (an equation and a graph) to their domains and ranges (e.g. see **Figure 9.2** below).

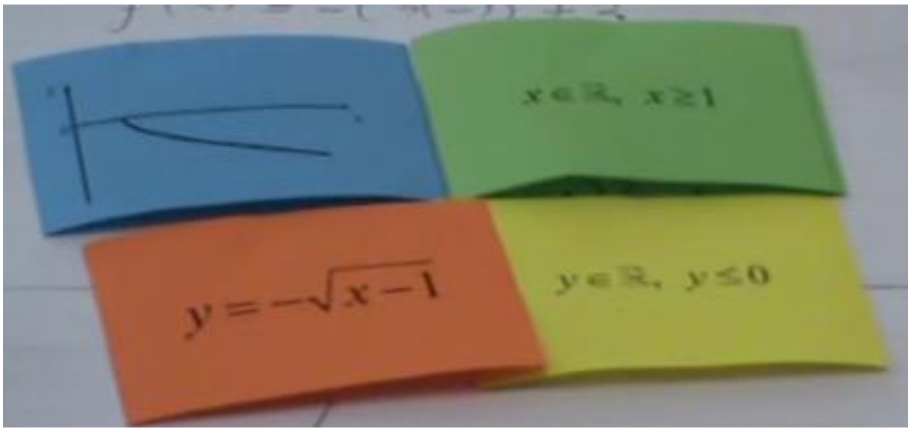


Figure 9. 2: Group 2's correct response

Matching these four cards required teachers to pay attention to domain and range simultaneously, but these two aspects of a function were each a dimension of variation, because each has values (critical features) in it. For instance, the domain has two critical features that include (i) domain with all real numbers and (ii) domain with not all real numbers. Similarly, the range is a dimension of variation because it has values and these include (i) range with all real numbers and (ii) range with not all real numbers. In variation theory terms, when two or more dimensions of variation vary simultaneously, we experience **fusion**. I must remind the reader here, once again, that **Fusion** is a pattern of variation which makes it possible for discernment of the relationship between the parts themselves and between the parts and the whole (Lo, 2012). As such, the matching of representations with their domains and ranges helped teachers to experience the relationship between the parts (dimensions of variation - domain and range) and between the parts (dimensions of variation) and the whole (function).

Still using **Figure 9.2**, it is discernible from there that both the domain and range do not include all real numbers, and so it was possible for teachers to discern four possible relations between the domain and range of the same function. That is to say, (i) both may not include all real numbers, (as is the case in **Figure 9.2**); (ii) both may include all real numbers (as is the case in **Figure 9.19 in Appendix E**); (iii) the range may not include all real numbers when the domain includes all real numbers and (iv) domain may not be all real numbers when the range is all real numbers. By way of example, Group 3 found that the range of $y = \cos x + 1$ was $0 \leq y \leq 2$, and the domain was all real numbers (see **Excerpt 9.6 and Figure 9.25 in Appendix E)** which shows the graph of $y = \cos x + 1$ as a vertical shift of $y = \cos x$).

Similarly, Group 1 found that the range for $y = \tan x$ was all real numbers when the domain was not all real numbers (see **Figure 9.1**). The above examples illustrate the relationship between the range and domain and to the function to which they are parts. **As such what was made possible to learn was that a function might have a range which is made up of all real numbers or not all real numbers. Similarly with the domain - a function might have the domain constituted by all real numbers or not all real numbers.**

The second pattern of variation was contrast, and this was present when teachers were working with different types of functions and graphs (see **Figure 9.21 in Appendix E**). Working with cards containing different functions helped teachers to experience difference between equations and between graphs of different functions. For example, **Figure 9.26** shows three types of equations and graphs which teachers had worked with in the card-matching task. In the task teachers worked with 8 different equations, and they also worked with 8 different graphs. **As such what was possible to learn was that there are different ways to represent functions algebraically and graphically.**

The third pattern of variation was contrast, and was also present when teachers were matching cards. In the task teachers experienced different types of domains and different types of ranges. There were domains that involved all real numbers and others which did not, and there were ranges that involved all real numbers and others which did not. **As such what was possible to learn was that the domain of a function might be all real numbers or not all real numbers. Similarly with the range.**

The fourth pattern of variation was present when Task 3 was in focus. Task 3 focused on transformation - with particular focus on the horizontal shift. Facilitator demonstrated the horizontal shift to the left and helped teachers realise conditions under which the horizontal shift to the right might occur. This helped teachers to experience difference in conditions that facilitate shifts to the left or to the right.

	F:	So your tables. What do you think would have happened if instead of g(x-1) you had g(x+3)? [on the flipchart F writes: g(x +3)
	T31:	We know[laughter]
	F:	It will shift left

Excerpt 9.8: Facilitator probing a group

For example, g(x-1) would produce a horizontal shift to the right, and g(x+3) would produce a shift to the left. This is also evident in **Excerpt 9.8** where the Facilitator is asking teachers about the graphs of g(x-1) and g(x+3).

As such what was possible to learn was that there is a horizontal shift to the right when there is subtraction and to the left when there is addition.

The fifth pattern of variation was generalisation, and was present when the focus was on the transformation of g(x). Teachers were helped to experience similarity in the images produced by two different types of transformation on the graph of g(x) = x(x-4). In particular, teachers experienced the same image, which was a reflection and also a horizontal shift to the right on the graph of g(x) (see **Figure 9.3** where graphs of g(-x) and h(x) are coincidental).

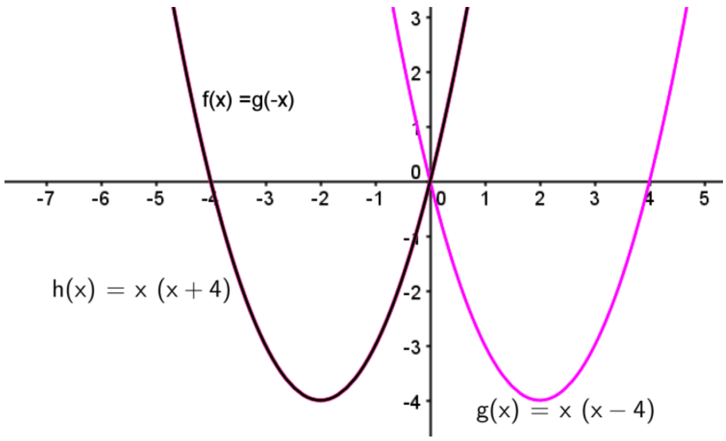


Figure 9. 3: Graphs of g(x) with g(-x) and h(x) coincidental

In this pattern of variation reflection and horizontal right shift were two critical features, but were not in the fore because what was in focus were the images and these were invariant. The pattern observed here was generalisation because what was in the fore was that which was invariant, and not what was varying. **As such what was possible to learn was that the same image may be produced from two different transformations.**

Table 9.3: Summary of what was made possible to learn at the I-Cluster

Invariant	Varies	What is possible to discernment?	Enacted object
Function	Domain /range	A function might have a range which is made up of all real numbers or not all real numbers. Similarly with the domain - a function might have the domain constituted by all real numbers or not all real numbers.	$\boldsymbol{Mt_{re+}}$
Function	Representations (graphs and equations)	There are different ways of representing functions algebraically and graphically	$\boldsymbol{Mt_{ap+}}$
Horizontal shift	Shift to the right and shift to the left	There is a horizontal shift to the right when there is subtraction and to the left when there is addition.	$\boldsymbol{Mt_{re+}}$
Image of g(x)	Types of transformation (reflection and horizontal right shift)	The same image may be produced from two different transformations	$\boldsymbol{Mt_{ap+}}$

From **Table 9.3** it is discernible that what was possible to learn were mathematical reality Mt_{re+} and appearance Mt_{ap+}. Hence the enacted object of learning was constituted by both mathematical reality and appearance.

9.7. Discussion of the pedagogic strategies at the N-Cluster

At the N-Cluster resources and help were availed for teachers to work with across the Task 2, 3 and 4. Here teachers had opportunities to experience high scaffolding. Facilitator provided demonstrations and explanations before teachers worked on a task, and even during work on the task he moved between the groups probing and prompting teachers to think about the task. Resources used in earlier tasks were useful in later tasks. By way of example, when working on Task 2, Group 5 adopted the $y = a(x-p)^2 + q$ formation which was dominant in Task 1 to find the missing equation, even though it was not productive (see **Figure 9.44 in Appendix E)**. In another example, when working on Task 3, Group 1 used Task 2 cards to think about the functions which they were required to find in Task 3. For instance this Group was observed studying the Function $y = \sqrt{(x-1)}$ (see **Figure 9.4**) before writing the function $y = \sqrt{x} + 1$

Figure 9. 4: Cards used by Group 1 to scaffold Task 3

In conclusion it can be noted that the high scaffolding was an indication that the Facilitator chose to bracket his natural attitude, and as a result worked with the deductive pedagogic strategy which availed resources and assistance. Since teachers were making satisfactory progress across tasks, this limited opportunities to fix meaning by the facilitator and teachers. As such the analysis of legitimating criteria and its associated appeals was not possible in task 2, 3 and 4. This occurrence was present across the three clusters.

9.8. Discussion of what was possible to learn at the N-Cluster

The first pattern of variation was fusion, and this was present in Task 2 when teachers were helped to experience the relationship between a representation, domain and range of the same function. The three are dimensions of variation, and in the card-matching task these three were varying simultaneously; and when two or more dimensions of variation vary simultaneously fusion occurs. Fusion reveals the relationship between the parts themselves and between the parts and the whole. For instance, in Task 2 some groups worked from the graph to find domain and range, and others worked from the equation to find domain and range. **As such what was made possible to learn was that any representation of a function contains the domain and range.**

The second pattern of variation was generalisation, and this was present in Task 2 when teachers were working with equations of linear, quadratic, hyperbolic and tangent, sine, and cosine functions. This helped them to experience similarity between the equations. Similarity was experienced when the different equations were varying, and the fact that each was a function was invariant. **As such what was possible to learn was that functions exist in different algebraic formations.**

The third pattern of variation was contrast, and was present when teachers were working on Task 2. Teachers experienced difference between the equations and graphs of different functions. By way of example, it was possible for teachers to discern difference between equations of the quadratic and square roots function. Similarly with graphs, there were opportunities to experience difference between a hyperbolic and a tangent graph. **As such it was possible to discern the difference between the representations of functions**

The fourth pattern of variation was contrast, and was present in Task 2, when teachers were matching the cards. There were opportunities to experience difference between domains of different functions. Similarly with the range. **As such it was possible to discern difference between the domain and range of functions.**

The fifth pattern of variation was fusion, and this was present in Task 3, when teachers were finding missing equations. Working on this task availed opportunities for teachers to experience the relationship between equation, domain and range. Teachers normally work from the equation to find the domain and range. However, this task helped teachers realise that the relationship is reciprocal. **As such what was possible to learn was that one can work from the domain and range to find a missing equation.**

The sixth pattern of variation was contrast, and was present when the Facilitator went about asking teachers to identify cards that were one-to-one and those that were many-to-one. Here teachers had opportunities to experience difference between these two types of mapping. **As such what was made possible to learn was that the ruler test cut one-to-one functions once and many-to- one functions more than once.**

The seventh pattern of variation was generalisation, and was present when attention was on Task 4. In this task teachers were asked to find functions that satisfied three conditions. This pattern of variation was generalisation because attention was on the three conditions, and these were invariant. What were varying were functions, but these were not in the fore. What went into the eight spaces was determined by the conditions, and this made them an important feature. Since what was in focus was that which was invariant, the resultant pattern of variation was generalisation. **As such what was made possible to learn was that different functions can satisfy the same conditions.**

Table 9. 4: Summary of what was made possible to learn at the I-Cluster

Invariant	Varies	What is possible to discernment?	Enacted object
Representation	Domain/range	a representation of a function contains the domain and range.	$\boldsymbol{Mt_{re+}}$
Algebraic representations	Types of equations	that functions exist in different algebraic formations.	$\boldsymbol{Mt_{ap+}}$
Functions	Representations	the difference between the representations of functions	$\boldsymbol{Mt_{ap+}}$
Functions	Domain/range	The difference between the domains of functions. Similarly with range	$\boldsymbol{Mt_{re+}}$
Domain/range	Equation	One can work from the domain and range to find a missing equation.	$\boldsymbol{Mt_{re+}}$
Ruler test	Number of times the ruler test cuts a graph	The ruler test cut one-to-one functions once and many-to- one functions more than once	$\boldsymbol{Mt_{re+}}$
Criteria	Kinds of functions	Different functions can satisfy the same criteria.	$\boldsymbol{Mt_{ap+}}$

From **Table 9.8** it is discernible that what was possible to learn were *mathematical reality* ($\boldsymbol{Mt_{re+}}$) and *mathematical appearance* ($\boldsymbol{Mt_{ap+}}$). Hence the enacted object of learning was constituted by both *mathematical reality* and *appearance*.

9.9. Conclusion

The analysis of pedagogic strategies across the three clusters has shown that all clusters experienced high scaffold across the tasks. This was possible because facilitators adopted the deductive pedagogic strategy, which opened up scaffold on tasks. Teachers were provided resources to work with, and help by facilitators, who moved between groups monitoring progress. It is interesting, but not surprising, that legitimating criteria were almost entirely absent across the three clusters. The fact that most of the discussions took place in the groups meant that most could not be captured on video, and even those that were captured were inaudible because all the groups were talking simultaneously. However, this does not suggest that there were no legitimations, but only that they were not in the public channel. Analysis of patterns of variation revealed that the enacted object of learning was constituted by both *correct mathematical appearance* ($\boldsymbol{Mt_{ap+}}$) and *correct mathematical reality* ($\boldsymbol{Mt_{re+}}$) with the latter dominant.

The analysis of the enacted object of learning across **Chapters 6 – 9** has shown that the enacted object of learning was constituted by mathematical appearance and mathematical reality, with the latter dominant in Chapters **7 – 9**. Following on what we already know from the analysis done in these chapters, it suffices to ask this question: 'If the enacted object of learning was $\boldsymbol{Mt_{ap+}}$ and $\boldsymbol{Mt_{re+}}$, then what became the lived object of learning for the teachers? I respond to this question in the next chapter, where I present my analysis of the teachers' lived object of learning. It must be noted that all the nine teachers, irrespective of the cluster they belonged to, were provided with similar opportunities to learn.

CHAPTER 10: ANALYSIS OF INTERVIEWS

10.1. Introduction

This chapter presents the analysis of teachers' lived object of learning after experiencing the Functions theme work. As noted earlier in Chapter 5, the source of data related to this analysis was in-depth interviews conducted with the nine teachers. The analysis of the interview had two foci. In the first focus the analysis was done with respect to resources that were part of the context in which teachers were learning in the PD. In particular, I focused the analysis on resources that teachers used before and after experiencing PD. Still in this focus, I analysed resources that facilitated teachers' learning in the PD. The analysis with respect to resources appears in Sections 10.3 – 10.5. In the second focus of my analysis of teachers' lived object of learning I analysed the teachers' experiencing of tasks in the PD. The analysis related to the second focus appears in Section 10.6. I have summarised the analysis of the teachers' experiencing of PD tasks in Section 10.7. The distribution of themes that emerged from the analysis of teachers' experiencing of PD tasks is presented in Section 10.8. The conclusion of the analysis appears in Section 10.9.

10.2. Operationalization of terminology used in the analysis of interview data

As noted earlier in Chapter 5, the tools used in the analysis of interview data were drawn from phenomenology and the methodology for studying teacher education. From phenomenology I worked with these particular tools: ***horizontalization***; ***meaning units***; ***meaning structure*** and ***essential invariant composite structure,*** and for further analysis I recruited the notion of ***duality of the object of learning*** from the methodology for studying mathematics teacher education.

10.3. Operationalisation of four themes that emerged from analysing interview data.

The four themes emerged from my analysis with tools which I have presented in the preceding section. I have put the themes early in my analysis because in later sections I used them to describe each teacher's lived object of learning. Now, the analysis that produced the following four themes was done with tools from phenomenological research (Creswell, 1998) and others from the methodology for studying teacher education (Adler & Davis, 2006). The four themes included the following:

(a). Personal learning of functions (**Mt-pl**):- In the analysis of interviews where teachers talked about learning functions there were instances where a teacher would talk about his/her own learning of functions. This was usually evident where a teacher talked about how he/she did not understand a particular content area of functions. The essence of these talks was a characteristic of teachers learning of functions for their personal growth. An example of such teacher talk is from Teacher 4, who talked more about her learning of functions in the PD. In **Excerpt 10.1** below, the teacher talked about what she did not know before, which has now become simple for her to understand. In particular, the teacher talks about his opportunity to learn about the gradient of a straight line through the notion of “rise over run”, and matching multiple representations of functions. In Functions 2 there were opportunities for teachers to learn about the horizontal shift on graphs, and so a table was used to help teachers discern why the graph of g(x-1) tends to shift to the right rather than to the left. Teacher 4 pointed out that in her earlier learning prior PD, she did not understand why the graph of g(x-1) would shift to the right and not to the left. Task 3 of Chapter 9 contains activities which provided opportunities for the discernment of the horizontal shift of graphs and Teacher 4 participated in these activities. In particular, **Table 9.4** in Chapter 9 is the one which Teacher 4 refers to in **Excerpt 10.1**.

335 336 337 338 339 340 341	T4:	Ok. The gradient one, the one with the multiple representations and the table one especially that one I was like, it’s like this ja, because when I was also at school I think I was just cramming for the sake of passing but not… I didn’t understand why do we do this when we graph a function? But after that I was like, all this time I was struggling and it’s just staying, it’s simple. Ja, especially that one. Ja, but I think after attending these functions I think when I compare myself after attending the workshops and when we started, I can say I was at minus 1 [laughs].

Excerpt 10.1: Exemplification of personal learning of functions

(b). Professional learning of functions (**Mt-PL**):- is where teachers talked about learning functions for utilitarian purposes. In the following excerpt Teacher 2 talks about his learning of functions with respect to what is important for learners to know about functions. Most of what the teacher pointed out as important to learn were what the WMCS PD had identified as key features of functions. Teacher 2 pointed out the need to foreground the definition of function, connections between the multiple representations of a function and connections between domain and range as important for learners to know about functions (see Task 2 of **Chapter 9**). What makes the extract exemplify **Mt-PL** is the fact that the teacher’s talk foregrounded the object of learning with respect to the learner – i.e. what learners needed to know about functions. So in the PD the teacher had the opportunity to learn what learners need to know when they learn functions. Hence this constitutes **Mt-PL.**

271 272 273 274 275 276 277 278 279 280 281	T2:	Ja, what I can vividly recall from those sessions is that when we teach functions we need to be very clear, ja. We need to… Let's connect them from the definition, those representations, ja. Let's… Let's make it such that the learners should be able to move from one end to the other. And also connect it with the, with the domains and the range, especially where it is defined and where it is not, where it is undefined, ja. Why are we saying strictly x must be greater than this? And between this and that? Ja, at times it is because we want to make, to continue defining it as a function. If we just allow it to take any value we may not end up having what? A function. This is something which I said myself, ja, really it is very important as we teach, even learners themselves as they define it, they should be able to define it, ok it is only valid for this and this and that. And it also helped me because I also teach grade 12.

Excerpt 10.2: Exemplification of professional learning of functions

(c). Personal learning of teaching functions **(Tm-pl)**:- this subcategory is concerned with professional learning of teaching mathematics for personal growth, rather than for use or application in the class. In the following excerpt Teacher 1 is contrasting his normal teaching of functions with the way functions were presented in PD. He saw PD presentation as a shift from his monotonous teaching where the representations were never connected and were presented in isolation. So here the teacher is reflecting on his practice, and by doing so he is learning from his own practice as well as from PD. The PD offered him the opportunity to critique his own practice, and this contributed to his personal growth.

265 266 267 268 269 270	T1:	Ok, I think for me in terms of maybe imparting functions as a topic it was the cards. Matching those cards, because I, I found it one like shifting, you know, from your monotonous way of, you know, teaching like ok you start with these, you go to that, you go… I mean ok, whether you start with a table, then you go to the equation, from the equation then you go to the graph or whichever other way. You see I think we were not… Or personally I hadn't come to a point whereby I connect the 4 representations or functions.
274 275 276 277 278		You do these different representations in isolation and then I don't think there would be any time when I would try to maybe put these together or try to emphasise to the learners that, 'Ok, this is another way of representing this same thing' or 'This is representing what we did there.' And that link was lacking. So from the cards, you know, I was actually able to learn that.

Excerpt: 10.3: Exemplification of personal learning of teaching functions

(d). Professional learning of teaching mathematics (Tm-PL):- The Tm-PL is learning about teaching mathematics for utilitarian concerns. By way of example in the extract shown below, Teacher 7 talks about the usefulness of the cards and grid in helping learners learn functions in his class.

207 208 209 210	T7:	As, as I'm saying, cards played a very important one. The use of the grid on the board that was provided it actually made the lesson to be fascinating – not for me as a teacher, but for learners – because I… it made me… It made possible for me to, you know, to get information out of the learners instead of giving learners the information.
212 213		Yes. Sometimes they had to argue. It was fine and… But at the end of the day there was 100% commitment in the class

Excerpt 10.4: Exemplification of professional learning of teaching functions

10.4. Analysis of resources used by teachers before professional development

During PD teachers were given resources such as readings (e.g.book chapters and other literature on functions). They were also given a CD containing Geogebra software, Cards, and Grids. In the interview I asked teachers to talk about resources that they used when they taught functions before and after PD. I also asked them to talk about resources that facilitated their own learning in the PD. Excerpt 10.5 contains a conversation between the researcher and Teacher 1 about the resources that he used before PD. I have used my conversation with Teacher 1 as an example of similar conversations I held with the other 8 teachers. My conversations with the rest of the teachers have been appended (see **Appendix G**). The shaded text contains evidence of what teachers said they used before PD. It must be noted that similar shading has been used in other sections where teachers talked about resources used after and resources that facilitated their learning in the PD.

40	R:	What are the resources that you use when you teach functions?
41 42 43	T1:	Ok. Well obviously the first one will be the textbook and specifically Classroom Maths because that's the one that each learner has a copy of. Ja, so that's the first resource that we're using in the school. And the second one, every classroom has a graph board.
183 184 185	R:	Now where else, apart from prescribed learner books or teacher textbooks, do you obtain information on functions – for your own knowledge and for planning your lessons?
186 187 188 190 191 192 193 194 195 196	T1:	Ok, I mean I do go to the media centre because there are other textbooks that, you know, we don't normally have in the classroom. Like you mentioned Focus and then there's this other one which I like a lot, I mean it's just that gosh, I'm forgetting… Study and Master, you know. Study and Master, I use that from time to time. Ja, I mean to, whatever it is that maybe I don't like with Classroom Mathematics, maybe in Study and Master it's put, you know, rather differently. Then I would actually keep that one up and because like with the learners I know that ok in Classroom Maths this is what they have. So what they lack for instance or Classroom Mathematics doesn't have, then I'll photocopy from Study and Master, for instance, and give ourselves additional something in terms of information to the learners.

Excerpt 10.5: Teacher 1's utterance about resources that he used before PD

In the excerpt Teacher 1 said he used Classroom Mathematics, which is the prescribed textbook, a graph board, a text book called Focus and another text book from the media centre (i.e. school library) called Study Master. Though the rest of the conversations have been appended, **Table 10.1** contains a summary of the resources used before PD by the 9 teachers.

Table 10.1: Resources used by teachers before experiencing PD

Teacher	Resources used before PD
T1	Classroom Maths, Focus, Study & Master), Graph board, Chalkboard
T2	Chalkboard, Geo-board , Classroom Maths
T3	Classroom maths , Worksheets, Chalkboard, Classroom Mathematics, Study guide in Classroom Mathematics, Other books, Worksheets
T4	Classroom maths, Graph paper, Past Exam Papers, Question papers, Chalkboard
T5	Classroom maths, Other textbooks, Non Prescribed-Mind Action, Graph paper, Internet, Chalkboard
T6	Classroom maths, other textbooks, worksheet, Chalkboard
T7	Classroom maths, Worksheets , Chalkboard
T8	Classroom Mathematics , Sowetan-(vertical functions, Chalkboard
T9	Overhead projector, Data projector, Classroom maths, Cambridge University-books, Website for mathematics, Department of Education-General Mathematics question papers, Districts-tests, Chalkboard

The data in **Table 10.1** has been compressed into four categories. The first category included the prescribed text book – i.e. Classroom Mathematics. The second category included textbooks which were used, but not prescribed. The third category were what I referred to as other materials and they included: resources from the internet; worksheets; graph paper; past exam papers and geo-board. The fourth category of resources was chalkboard. **Chart 10.1** shows the distribution of resources used before PD.

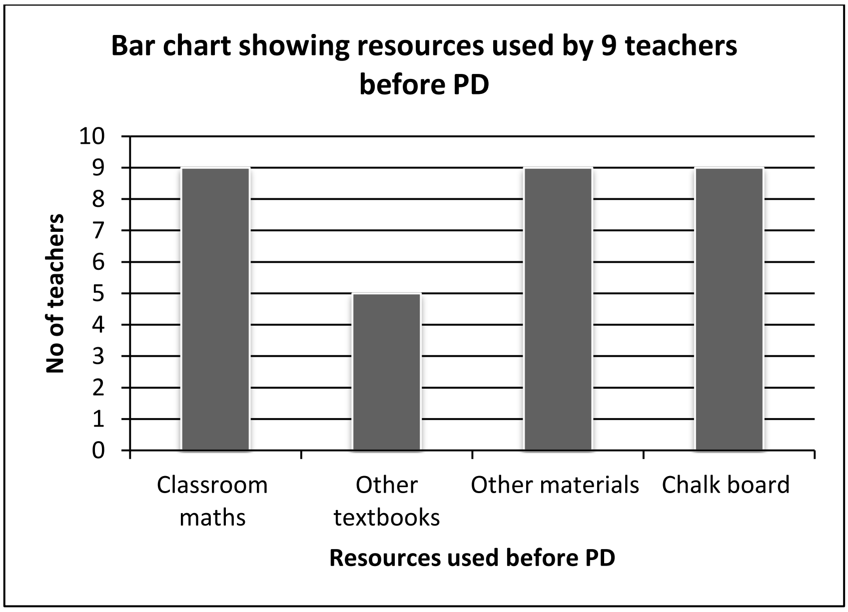


Chart 10. 1: Distribution of resources used by teachers before PD

The chart shows that all nine (9) teachers used the prescribed textbook, other materials and the chalkboard. There were five (5) teachers who used other textbooks as additional resources. This therefore suggests that four teachers did not use other textbooks.

10.5. Analysis of resources used by teachers after professional development

I asked teachers to talk about resources they used to teach functions after PD. In particular, I was interested in PD resources that teachers used in their teaching. I have used Teacher 5' talk about resources she used after PD to illustrate similar conversations with other teachers. **Excerpt 10.2** contains my conversation with Teacher 5 about the resources that she used after PD. She identified the large grid and domain and range cards as resources that she used after PD.

R:	Ok. Now which of the resources have you used in your class after PD?
T5:	The big grid now stays behind my board. Yes I make use of it ? for teaching functions.
R:	Mmm
T5:	See. It's now like a permanent thing
R:	You've put it. Ja
T5:	Because I use it more, more than expected now.
R:	Mmm
T5:	And I've also made use of the… Like I said, I'm still trying to get my hands on the geogebra *[laughs]* and get to understand it. But I've also made use of the domain, range cards as well. I gave them, not necessarily to the grade 10s, but in my revision classes with the grade 12s
R:	Ok
T5:	When we're now revising every day we'd have questions and answers, questions and answers. And one day I thought, 'You know what? Too many questions and answering. Like every test, every day it's a test.' One day it was a break, I gave them the cards and they enjoyed it. And they did get a better idea of the domain and the range.

Excerpt 10.2: Exemplification of teacher talk about resources used before PD

The analysis across the nine teachers revealed that in addition to resources used before PD, teachers recruited resources from PD to teach functions. **Table 10.2** shows PD resources used by each teacher. From the table it is evident that Teacher 6 did not use any of the resources from the PD. In her talk, she explained that by the time she went to participate in the PD she had already covered a section of the curriculum focused on functions (see **Excerpt 10.3**). However, she intended to use them the following year when she would be teaching functions again.

273	R:	Which resources did you use in your class?
274	T6:	With the card games? No.
275	R:	Card game or any other…
276	T6:	No, not yet.
277	R:	Ok
278 279	T6:	Not yet, because at that time when we did the functions, the schedule we have already passed that part of the schedule so maybe I will try them next year in the classes.

Excerpt 10.3: Teacher 6's explanation

Table 10.2 also shows that the Grid was the most used PD resource followed by Other. The table also shows that the resources were of two types. The first type were tangible resources such as Grids, Cards, Geogebara, Readings and Calculators. The second type were intangible resources such as the vetical line test.

Table 10.2:

Table 10.2: Distribution of resources used after PD

Teacher	**WMC-S Resources used after PD**
T1	WMC-S Grids
T2	WMC-S Grids
T3	WMC-S Grid Matching cards
T4	Cards; Bernard's Table; Readings ; Grid
T5	WMC-S readings; WMC –S learner grid Non-permanent marker; WMC PD Laminated Grids Cards on domain and range
T6	None used
T7	Vertical line test Grid - Large and small
T8	Cards Mapping diagrams WMC-S Readings Grids
T9	Calculators from PD Geogebra

In the excerpt Teacher 7 explains his use of the grid and its impact on both teaching and learning. He drew graphs of different functions using different colors to help learners discern that several graphs can be drawn simultaneously on the same Cartesian plane. This was also meant to help them discern the difference between graphs of different functions (e.g. hyperbola and exponential graphs) (see **Excerpt 10.4**).

347 348 349 350 351 352	T7:	They were actually effective, especially when it comes to identifying the domain, the range, the asymptotes – they should know exactly what those things mean and how can they read that from the graph. So the grid actually helped a lot because it makes life easier for one also to explain and identify. For example if you say this is a graph of a hyperbola, they should know what the domain is. They should also be able to visualise and conceptualise that from the graph.

Excerpt 10.4: Teacher 7's use of WMCS Grid

Chart 10.2 provides a summary of the PD resources used across the teachers. The chart shows that the resources fell into four categories that included Grid, Cards, Readings and Other. Most teachers used the Grid after PD. This was followed by Other and Cards. The least-used resource was Readings which was used by 3 teachers. In the following Excerpt I have used Teacher 7's talk about his use of the Grid to illustrate its use across the teachers. Similar talk from other teachers is appended.

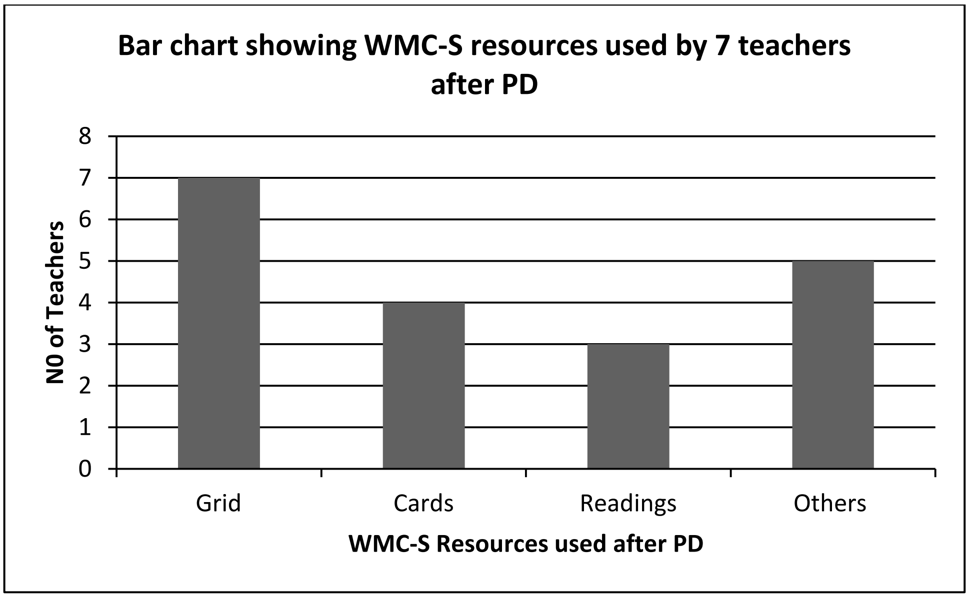


Chart 10. 2: WMCS resources used after PD

10.6. Analysis of *resources* which facilitated teacher learning in PD

This section presents detailed analysis of resources which facilitated the teachers' learning in the PD. Unlike the previous analysis where I showed evidence from one teacher and appended the rest, here I have incorporated evidence from all the 9 teachers. Now, in the interview I asked teachers which resources made it possible for them to learn in the PD. I must point out here that both Functions 1 and Functions 2 involved teachers in card activities. In particular, Functions 1 focused on card matching and sorting involving multiple representations only and Functions 2 focused on card matching involving two multiple representations (i.e. graph and equation), domain and range.

10.6.1. Teacher 1

10.6.1.1. Descriptive analysis of Teacher 1's learning with respect to resources

Excerpt 10.5 contains evidence that supports Teacher 1's learning from resources used in the PD. The rest of the utterances related to this have been appended. The first was a conversation between the researcher and Teacher 1 (below).

271	R:	Which of these resources would you say helped you understand functions more?
272	T1	Ok, I think for me in terms of maybe imparting functions as a topic it was the cards.
273	R:	The cards
274	T1:	Mmm
275	R:	Mmm
276 277 278 279 280 281	T1	Matching those cards, because I, I found it one like shifting, you know, from your monotonous way of, you know, teaching like ok you start with these, you go to that, you go… I mean ok, whether you start with a table, then you go to the equation, from the equation then you go to the graph or whichever other way. You see I think we were not… Or personally I hadn't come to a point whereby I connect the 4 representations or functions.
282	R:	Mmm
283	T1:	Do you understand?
284		Mmm
285	R:	Mmm
286 287 288 289	T1:	You do these different representations in isolation and then I don't think there would be any time when I would try to maybe put these together or try to emphasise to the learners that, 'Ok, this is another way of representing this same thing' or 'This is representing what we did there.' And that link was lacking. So from the cards, you know, I was actually able to learn that.
290	R:	Mmm
291	T1:	Ja

Excerpt 10.5: Teacher 1's utterance about resources that facilitated his learning in the PD

Teacher 1 talks about what he learnt from working with cards in the PD sessions, and he says cards helped him with respect to imparting or teaching functions. In other words, his learning from cards was in relation to teaching functions. He explains that the way cards were used in the PD was a shift from his normal way of teaching functions, which he says is monotonous. He observed multiple representations presented in a connected that allows one to begin with either representation. In his teaching he normally teaches the representations in isolation and never creates connections between them. It is interesting to note that Teacher 1's learning from card activities facilitated his learning of teaching functions instead of functions themselves. It is interesting because his learning as articulated here is contrary to the PD's intended object of learning. The PD sought to strengthen teachers' subject matter knowledge of functions.

The analysis of Teacher 1's utterances using phenomenological research tools and the notion of the duality of objects of learning (Adler & Davis, 2006) revealed the following three aspects which constitute the meaning structure of Teacher 1's experiencing of PD resources. These meanings were constructed from the utterance above, which I have dissected and distributed below among the four aspects to illustrate the source of the meaning.

(i) Card activity develops understanding of teaching functions

First, the teacher's involvement in card activities helped him develop a deeper understanding of teaching functions. This is evident from his criticism of his practice, which he sees as a monotonous way of teaching. The teacher's critique of his previous practice constitute his reflection on his practice and this is essential for shifts to occur in future practice. This reflection is attributable to experiencing an object-focused PD.

(ii) Card activity facilitates reflection on teaching practices

Second, for this teacher the PD provided him with tools which he is using to reflect on his practice. Teacher 1's involvement introduced him to a novel way of teaching functions. He describes this as a shift from his normal way of teaching functions. His participation has provided him with an alternative approach to teaching functions. However, it is not clear if this teacher might work with ideas from PD in his class in spite of his criticisms of his ***Normal*** practice. Notwithstanding, there is evidence of shifts in Teacher 1's Post-PD lesson plan. The shift was evidenced by the presence of the WMCS key features of functions. In the context of PD, it is believed that teachers' beliefs and attitudes impede shifts from old to new practices. However, teachers who are able to reflect on their practice stand a better chance of trying newly-learnt ideas and practices in their classrooms with learners.

(iii) Card activity is a connected way of teaching functions

Third, Teacher 1's experiencing of card activity endured him with the ability to see connections between the representations. His involvement in this PD helped him realize that multiple representations can be taught such that connections are visible to learners. In the context of object-focused PD, showing connections between multiple representations brings the object of learning into focus. In the case of functions, to do this requires the teacher to privilege *key features* of functions. The importance of focusing on key features is that the object of learning does not get out of focus or get lost during enactment. Hence for Teacher 1 to have discerned the need to present the multiple representations in a connected way is a positive outcome of his experiencing the PD.

10.6.1.2. What constitute Teacher 1's lived object of learning with respect to resources

The meaning structure of Teacher 1's experiencing of resources suggests that his learning was in relation to the teaching object (**Tm**). Now it is evident in his talk that he is contrasting is normal teaching approach with the teaching done in the PD. So this suggests that his learning from PD resources was *Personal learning of teaching functions* (**Tm-pl**).

10.6.2. Teacher 2

10.6.2. Descriptive analysis of Teacher 2's learning with respect to resources

In the conversation shown below, Teacher 2 talks about resources that facilitated his learning in the PD. Teacher 2 begins by asserting that he knew functions prior to participating in the PD and he says what was new to him was how functions were taught in the PD. However, evidence from Chapter 6 contradicts Teacher 2's assertion that he knew functions prior to PD. Evidence from Task 1 of Chapter 6 pointed to gaps in his knowledge of functions per se. This is evident in Functions 1, where he listed several functions as non-functions, and it was surprising that he listed and even went to the front to demonstrate that a parabola was not a function. I have presented the teacher's explanation in later sections where I present the analysis of what teachers said they learnt in the PD sessions.

R:	Would you say amongst the resources that were used, there were some that helped you understand functions in the PD?
T2:	The understanding, should I say, was there, of functions. But the way, how it was… like the use of cards, the way it brought in more activities which will make more understanding especially when it comes to learners. Like you start to match – this goes with this, this goes with this, this then does not go. And in itself, being able to do that it involves a lot of reasoning. And if a learner can be able to match from this to this, to that to that then one can safely say that this one has well understood what do we mean by functions..
R:	Ok
	Ja, obviously because you will see that all these were linked. You could get, should I say a graphical representation. You'll get a description which is in words. You can also have a table. And they are mixed. You start to connect from the… wherever you start, it can be from the graph to the table, from the table to the algebra and all that. It was quite involving and very interesting. Ja, really.

Excerpt 10. 6: Teacher 2's talk about resources that facilitated his learning

In the conversation above Teacher 2 talks about how the way functions were presented in the PD would help learners understand functions. He realised that matching cards introduced more learning opportunities, especially where the learner would have to work with multiple representations simultaneously, and he believes card matching can develop reasoning in learners. What is evident in Teacher 2's utterance is that his learning from card matching activities is with respect to teaching functions, and this is different from Teacher 1's learning which was also with respect to teaching functions. The difference between the two is that Teacher 1's learning was with respect to his personal way of teaching functions, and this falls in the category of personal teaching of functions. Teacher 2's learning is not related to how he teaches, but how the teaching of functions as was done in the PD might benefit learners. As such his learning is with respect to teaching functions for utilitarian purposes, and I have defined this as professional teaching of functions. So if Teacher 1 adopts the way functions were taught in the PD it would be because his way is inadequate, and if Teacher 2 adopts this way of teaching it would not be because his way of teaching is inadequate, but because he wants to develop learners' reasoning. So for him it is another way of teaching functions.

Analysis of Teacher 2's utterances about his learning form PD resources produced six aspects which constitute the meaning structure of his experiencing of resources used in the PD. The six aspects summarize his learning from PD resources. Card activity is a connected way to teach functions:

(i) Card activities create more opportunities to learn

The first aspect is in relation to connections between the representations. Prior to engaging teachers in the PD professional developers observed these teachers teaching with no connections between the representations. Hence multiple representations were used to help teachers discern connections between them. So it is interesting that Teacher 2 appears to have noticed these connections and also considering them as important for learners to discern.

(ii) Card activity involves reasoning

The teacher also noticed that card matching might improve learners' ability to reason. In part of his data which is not shown here, he mentioned that working on PD tasks was challenging for Educators. So it is probable that he is talking is informed by his own experiencing of this activity in the PD.

(iii) Card activity leads to understanding of functions

He also realized that working with this activity helps one to understand functions. It is probable that he is talking from experience, because later in Section 10.6 I use gaps in his knowledge to suggest that teachers came into the PD with knowledge gaps, and that these narrowed with more participation in the PD.

(iv) Card activity is involving and interesting

Teacher 2 noticed that card activity was involving and also interesting to use as a teaching resource.

10.6.2.2. What constitute Teacher 2's lived object of learning with respect to resources?

Teacher 2's utterance is characterised by talk about teaching of functions, and the teaching is talked about with respect to how it might benefit learners. So if Teacher 2 were to adopt this way of teaching he would be doing so because of potential benefits it might provide the learner. As such the teacher's lived object of learning was *Professional learning of teaching functions* (**Tm-PL**).

10.6.3. Teacher 3

10.6.3.1. Descriptive analysis of Teacher 3's learning with respect to resources

Teacher 3 talked about her learning from resources, which were availed in the PD. Below is a conversation between the researcher and Teacher 3 in relation to her learning from PD resources:

R:	Ok. And then now which of these resources which were available in the sessions you attended helped you to learn functions?
T3:	This card ma… card matching – what are you calling it?
R:	Card matching activity
T3:	Ja, it was, they were interesting and then they were giving us enough information to define the functions. But then in, like I don't know, for me to use it in my classroom I think it was going to be a waste of, you don't have enough time. You have only 60 minutes
R:	30
	Or 30, so if I have to use it, say now ok let's do that for 60 minutes I won't finish the syllabus if I keep on using card matching. Maybe once we are done everything then we are using them as revision, yes. They were good.
R:	Ok

T3:	But that time it was fun. We had fun, we've learned a lot, but then I think in a classroom even if my class is not that big, but still I don't think… I don't… After we're done, maybe a lesson/less than during the functions, I'm done with the functions, then at the end when I'm about to close the chapter on functions, then I give it to them.

Excerpt 10.7: Teacher 3's talk about resources that facilitated her learning

There are four themes that describe Teacher 3's learning from the resources that were available in the PD sessions that she attended. I describe these below:

(i). ***Card matching is interesting***

Teacher 3 points out that learning about functions through card matching was an interesting experience. In addition, she had enough information to use to define functions. Although it was an interesting way to learn functions, the teacher thinks that using cards would delay curriculum coverage.

(ii). ***Card matching can be used to revise functions***

Teacher 3 perceives card matching as a way to revise functions. The teacher says she would use this way of teaching functions once she has taught everything that learners need to know about functions, because using it in between would make it difficult to complete the curriculum.

(iii). ***Card matching is good***

Teacher 3 says card matching is a good way to learn functions. It is interesting that while the teacher sees card matching as a good way to learn functions, she is not in a position to use it in her class with learners.

(iv) ***Card matching is fun***

Teacher three thinks that card matching is a fun way to learn functions. She adds that she was able to learn a lot about functions through card matching.

10.6.3.2. What constitute Teacher 3's lived object of learning with respect to resources?

The specific content that the teacher says she learnt was to define functions. There are four main meanings emerging from Teacher 3's talk. The first meaning was that card matching was interesting. Secondly, it is useful as a revision tool. Thirdly, it is a good way to learn functions. Lastly, it is a fun way to learn functions. There are three instances in the teacher' utterance that points to the teacher's own learning of functions. These are where the teacher says card matching is interesting, good and fun. These three meanings, together with the specific content that she says she learnt pointed to *personal learning of functions* (**Mt-pl**). Seeing card matching as a way to revise functions pointed to *professional learning of teaching functions* (**Tm-PL**).

10.6.4. Teacher 4

10.6.4.1. Descriptive analysis of Teacher 4's lived object of learning with respect to resources

Analysis of teacher 4's utterances resulted in 5 categories which constituted her lived object of learning in PD. The categories were content focused and they are presented below. It must be pointed out that the categories are the researcher's summaries of the teacher's utterances about her learning in the WMC-S project's professional development. Below is Teacher 4's utterance about her learning from resources

247 248	R:	Ok. Then where else, apart from prescribed learner and teacher textbook do you obtain information and functions for your own knowledge and for planning your lessons?
319 320 321 322 323	T4:	The run and rise we were never really touching on that. I think that was also very helpful because it gave learners a problem. Also when they were given a question where you are supposed to give the equation of a straight line graph, and now to get the gradient, whereas it was just a simple thing, they would struggle and struggle. I think that was also a very, very important resource, yes.
338	R:	Then can you specifically say which of these helped you understand functions more,
342 343 344 345 346 347 348	T4:	Functions more. Ok. The gradient one, the one with the multiple representations and the table one – especially that one I was like, it's like this ja, because when I was also at school I think I was just cramming for the sake of passing but not… I didn't understand why do we do this when we graph a function? But after that I was like, all this time I was struggling and it's just staying, it's simple. Ja, especially that one. Ja, but I think after attending these functions I think when I compare myself after attending the workshops and when we started, I can say I was at minus 1 *[laughs]*
352	R:	Ja
352 354 355 356	T4:	But now I think I'm very comfortable with functions. I like them because now I see they are so simple. The resources that you gave us it makes them simple and you also are happy because you are explaining to learners what is happening. You now know, you have the knowledge after you have given us these resources. So…

Excerpt 10.8: Teacher 4's talk about resources that facilitated her learning

In **theme (i)** below the teacher was talking about her experiencing of PD activity where the method of "Rise over Run" was used to find the gradient from a linear graph. Having observed and used this method the teacher talked about it in the interview. The teacher also talked about her learning in the card matching activity.

(i) Rise over Run approach to finding slope strengthens understanding on gradient

In PD the notion of "Rise over Run" was present, where it was presented as one way of making gradient visible without need for calculation. This way of seeing gradient seems to have deepened Teacher 4's understanding of Gradient.

This category of lived object is evident where the teacher explains the kind of resources which helped her understand functions more than she did before participating in PD. In her talk Teacher 4 noted that the notion of Rise over Run deepened her understanding of Gradient. In PD, this notion was brought to the fore so as to ensure that Gradient, as the object of learning which is usually presented through calculations using two coordinate pairs seem to obscure the essence of gradient as a concept. Hence Gradient was introduced as a Rise over Run in PD sessions. In PD, certain features of functions were privileged because of their importance over others. In particular, Gradient was a privileged feature of functions because it is integral in the relationship between the input and output in a functional relationship.

(ii). Multiple representations facilitate more understanding of functions

Teacher 4 makes mention of how multiple representations helped strengthen her understanding of functions. In PD activities these were visible in most activities, especially in activities that involved the use of card-activities, which were anchored on multiple representations. Teacher 4's learning included multiple representations. This is evident in the utterance the teacher makes about her learning in PD. Multiple representations was a dominant privileged feature in PD. The teacher engaged in card activities where the central message was to present functions as constituting a relationship rather than operations aimed at producing a graph. In her utterance, Teacher 4 appears to have discerned functions as a relationship and this was the intended object of learning in PD.

(iii). Bernard's Table facilitates more understanding on concepts that were not well understood during high school

Bernard's Table: This appears to have been the main highlight of Teacher 4's learning in PD. The teacher talks more about how the use of this table helped her overcome difficulties she had while she was a student. The table is called after one of the facilitators who invented it to explain why a graph of $y = (g - 1)$ shifts horizontally to the right and not to the left (see **Figure 2.3, Chapter 2, p18 - 19**).

In the excerpt Teacher 4 describes her knowledge of functions before and after participation in PD. By way of example, she gives examples of some aspects of functions which she did not understand well, even though she had to teach to her learners. The teacher also pointed to the source of her difficulties as not having grasped the concept well during her schooling. However, she is not explicit as to whether this was at secondary or tertiary level. After participating in PD activities where Bernard's table was used in the learning of horizontal shift, the teacher considers it simple to understand. There is a point which Teacher 4 is raising where she talks about the tendency to rush over a topic when one does not have a firm grasp of content. Now it can be speculated that Teacher 4 might previously not have been teaching this aspect of functions properly, but that she now feels confident with her knowledge. This is evident in **Lines 352 – 356** of **Excerpt 10.8** where the teacher articulates her knowledge in a positive light.

It suffices to note aspects of Teacher 4's learning as they emerge from analysis of her utterances in terms of Bernard's table's contribution to her understanding of functions. From the utterances it is clear that the teacher sees functions, and in particular why a quadratic function of the form $f(x-1) = x^2$ undergoes a horizontal shift to the right and not to the left. This realisation resulted in improved confidence in the teacher who believes this understanding is sufficient to facilitate explanations in her class. I must point that Teacher 4 went and taught this aspect of functions using the same Bernard's Table she had experienced in PD. What is also evident in Teacher 4's learning is her ability to reflect on her past knowledge and practice. In her reflection she talks about her past unpleasant learning experiences and her limited knowledge of functions, and how it impeded her teaching of functions. So it can be said that Teacher 4, just like Teacher 1, obtained tools with which she could talk fluently about her subject matter knowledge of functions. Another noteworthy point is that there seems to have been a shift in Teacher 4's perception of her knowledge of functions. In the excerpt above the teacher mentions that she is now comfortable with functions following her participation in PD. In **Lines 338 – 348** of **Excerpt 10.8** the teacher talks about how her knowledge was very limited before PD, but in **Lines 352 – 356** she talks positively about her knowledge of functions. The last point related to Teacher 4's encounter with Bernard's table is that her confidence in her own knowledge of functions confirms her learning in PD.

(iv). **Resources make functions simple to understand**

In PD almost every activity was accompanied by resources that were either for use by teachers or by PD presenters. In her talk Teacher 4 does acknowledge the contribution from other resources, and attributes her learning and understanding of functions to all resources which were available to her in PD. By way of example, Rise over Run involved a visual resource, in which the teacher was able to discern the essence of gradient. In the case of multiple representations, card activities that included matching equivalent representations facilitated hands-on experiencing of functional relationship. Bernard's Table is itself a resource which the teacher has talked about more than others in her utterances. So it can be concluded that the multiplicity of resources in PD was essential for Teacher 4's learning of functions in PD. In particular, resources made functions simple to teach and improved the teacher's subject matter knowledge.

(V). Growth in SMK elicits reflections on past teaching practices

It is evident from the teacher that there seems to have been knowledge growth, especially SMK, and I attribute her capacity to reflect on her knowledge as an indicator of knowledge growth (see **line 352 – 356** in **Excerpt 10.8**). I think this is so because the teacher highlights aspects of her previous knowledge and how these affected her knowledge of teaching. Post-PD the teacher believes she is empowered to understand for herself, and even to explain a concept which was previously problematic in her knowledge, and even her teaching. Hence I attribute her ability to reflect on her practice as a consequence of growth in knowledge or understanding of functions. What is different for Teacher 4 is that her reflection is in relation to her knowledge of functions, while Teacher 1 reflected on his knowledge of teaching functions. It can also be speculated that growth in SMK contributed to teacher's self-confidence, which is evident where the teacher says that she is now happy because she is able to explain to learners what is happening in a vertical shift when a graph shifts unexpectedly.

10.6.4.1. What constituted Teacher 4's lived object of learning with respect to resources

Further analysis of Teacher 4's lived object of learning revealed that her lived object of learning was constituted by two theoretical themes that include *Professional learning of functions* ***(Mt-PL)*** and *personal learning of functions* (***Mt-pl***). The former was evident when she talked about the importance of helping learners learn to find the gradient of a straight line through the notion of rise over run (see **Lines 319 – 323** in **Excerpt 10.8**) The latter is evident when the teacher talked about her own knowledge of functions, which she says was at "minus 1" prior to PD (**see Lines 342 – 356**).

10.6.5. Teacher 5

10.6.5.1. Descriptive analysis of Teacher 5's lived object of learning with respect to resources

Teacher 5 was asked to talk about her learning from PD resources, and what she said is captured below in a conversation with the researcher.

270 271	R:	Ok. Then which of these resources that were used in the workshops helped you to understand functions?
272	T5:	The card.
273	R:	The card activities?
274	T5:	The card activities.
275	R	Mmm
276 277 278 279 280 281	T5:	I'll honestly tell you that prior to the work, this theme, I used to teach functions as graph, equation. Inasmuch as I know there is the, the verbal interpretation, there's also the tabular interpretation, but I kind of pushed it away to say, 'Ok, that's grade 9, so what.' But I find that if you start from there and bring in that relationship, it helps a lot. And also it's, it's fun. For once it's a fun way of… When I gave my learners to do that trust me, the next day they said, 'Ma'am we want to do it again.'
282	R:	Ja
283 284	T5:	So it's a fun way of learning maths, other than problem solving, problem solving every day, you know. So I found the cards much more useful.

Excerpt 10.9: Teacher 5's talk about resources that facilitated her learning

In this conversation Teacher 5 talks about how she used to teach functions, and she says she used to work with graphs and equations only. She mentions that she knew that the verbal description and table of values could be used, but she never considered them in her teaching. However, after participating in the PD she noticed the value of working with all representations. She also noticed that working with multiple representations was fun, and she allowed her learners to experience the same fun in her lesson. There are three main meanings that emerged from Teacher 5's talk about the resource that helped her learn functions.

(i) Card matching brings out the relationship between representations

The teacher talks about the importance of showing the relationship between the representations. She mentions that in her teaching she only moved between equation and graph and never incorporated other representations.

(ii) Card matching is a fun way to learn functions

The teacher points out that learning through card matching is fun. She goes on to cite an instance from her own class involving learning through card matching.

(iii) **Card matching is useful**

The teacher found it useful to work with cards in her class. The activity compels learners to want to learn engage more with funcions. The teacher saw card matching as an alternative to problem solving or testing learners. There is more evidence pertaining to the usefulness of card matching further in the larger transcript, on **Lines 384 – 392** of the full transcript which I have appended.

10.6.5.2. What constituted Teacher 5's lived object of learning with respect to resources

In the utterance Teacher 5 contrasts her teaching with the way she saw it done in the PD. She points out that in her teaching she focused on two representations only, while the PD worked with several. She also talked about card matching and its impact on learners. Hence the first focus of her talk was on *personal learning of teaching functions* (**Tm-pl**) and the second was on *professional learning of teaching functions* (**Tm-PL**).

10.6.6. Teacher 6

10.6.6.1. Descriptive analysis of Teacher 6's lived object of learning with respect to resources

Teacher 6 was asked to talk about her learning with respect to resources which were provided in the PD. Below is a conversation between her and the researcher.

R:	Ok. Then which of the resources that were used helped you learn functions?
T6:	Ja, the card activity really opened up in terms of what is it, what is not, you know. Because now if you say half the input to get the output is it a function? Why is it not? If it's not it is a function. So the card game really helped because what I've discovered, that teaching from a textbook it becomes so straight line, ja. Then it becomes monotonous. But once we branch into cards and all that then it helps in terms of deeper understanding of the whole thing. And it challenges you to say, to stand up and say, 'This is it,' you know. 'I think this is a function' and say the reasons why it is or it's not a function. Ja, I think that was the most important. The activities, the game, the card game was the more effective one. Because reading, if you give me a reading, I will read 1, 2 pages, but the card game will force me to read it. And when you do these card games I observed the time, you don't even look at the time, then you say, 'Ah, it's now time to pack up' and everybody's still interested inside, you know, in the game. Ja.

Excerpt 10.10: Teacher 6's talk about resources that facilitated her learning

There are six main meanings that emerge from Teacher 6's talk about the resource that helped her learn functions.

(i). ***Card matching makes it possible to learn what is a function***

The teacher points out that card activity helped her to learn what is and is not a function. In the PD teachers worked on a card sorting activity which required teachers to sort cards into functions and non-functions. Hence it is possible that this is the activity that helped her to learn what is and is not a function.

(ii) ***Card matching is not a straight way of teaching functions***

The teacher talks about her teaching which, she says, seems to have been tied to the textbook approach. She describes the textbook approach, which might be the way she taught, as monotonous.

(iii) ***Card matching deepens understaning of functions***

The teacher she says card activity helps to deepen understanding of functions. Here the teacher was engaged in activities that required her to notice what is and is not a function, and doing this helped to deepen her understanding.

(iv) ***Card matching is challenging, effective and interesting***

Teacher 6 sees card matching as challenging because it requires one to distinguish between a function and non-function, and one is required to justify what one says is or is not a function. In addition, Teacher 6 sees this activity as effective and interesting resouce to learn by.

10.6.5.2. What constitutes Teacher 6's lived object of learning with respect to resources?

Now, from her talk we can discern both the mathematics object, which is dominant, and the teaching object. For instance, in the above four meaning categories, the first, third and fourth speak to the teachers' own learning of functions (***Mt-pl***), and the second speaks to her own learning of teaching functions (**Tm-pl**). Hence we can conclude that her lived object of learning was constituted by **Mt-pl** and **Tm-pl**.

10.6.7. Teacher 7

10.6.7.1. Descriptive analysis of Teacher 7's lived object of learning with respect to resources

Teacher 7 talks about his learning from resources used in the PD, and below is a conversation where he talks about it (see **Excerpt 10.10**)

T7:	I'd constantly be using grids and especially with markers of different colours so that learners could… If, if… Like they should be aware that on the same Cartesian plane you can have more than one graph. And in that they should be able to see and classify the graph that is given to them. For, for example some people could be mistaken for like a hyperbola – you can mistake a hyperbola for an exponential graph.
R:	Mmm.
T7:	So ja, it helps, especially with colours, different colours. It's very easy for them to see. It makes things clearer for them to see this is the same graph, even if it's two pieces. Some people could think it's two graphs…

Excerpt 10.10: Teacher 7's talk about resources that facilitated his learning

There seems to be one meaning category in the teacher's talk.

(i). Grids make it possible to discern that different graphs can be drawn on the Cartesian plane

Teacher 7's talk is characterised by the the teaching object. This is evident when he mentions that he would use the grid together with coloured markers to draw graphs on the same cartesian plane, to help learners discern that more than one graph can be drawn on the same set of axes.

(ii). Grids make it possible for learners to see and classify graphs

Teacher 7 pointed out that working with the grid makes it possible for learners to discern difference between properties of graphs of functions and categorize them accordingly.

(***iii). Grids are helpful***

Working with graphs of functions on the grid is helpful, particularly when different colours are used for learners to distinguish one graph from the other.

(iv) Grids make things clearer

For Teacher 7 grids make it easy for learners to discern similar graphs drawn on the same Cartesian Plane.

10.6.7.2 What constituted Teacher 7's lived object of learning with respect to resources?

The teacher talks about his learning from resources with respect to teaching. Since his talk about learning from the Grid was for the benefit of learners, his lived object of learning constituted **Tm-PL**.

10.6.8. Teacher 8

10.6.8.1 Descriptive analysis of Teacher 8's lived object of learning

The following excerpt contains Teacher 8's talk about resources that facilitated his learning in the PD.

R:	Then which of the resources that were used altogether, all in all in the sessions helped you understand functions better?
T8:	The use of the cards made me understand the concept we had. Because…
R:	Ok. And would you say… Ok, continue
T8:	Because you give them the cards, you ask them to match. So when they are matching they are doing, there is reasoning taking place there. It's to say I need to match this graph. Given a graph, I'm just given a graph, you need to match the equation, you need to look at the graph and you need to identify is it a hyperbola, is it an exponential graph? So from there there's reasoning taking place. So it was very helpful. Even to the learners it was very interesting for the learners to identify 'This is a parabola.' So… 'So this is a parabola. So it's a function of a parabola, it's in this form.' Ja.

Excerpt 10.11: Teacher 8's talk about resources that facilitated his learning

In the excerpt Teacher 8 says the use of cards helped him learn functions. He points out that the process of matching cards involves reasoning. He also says it is a helpful way to learn functions. Furthermore, he noticed that card matching was interesting to learners. As such there are three meanings embedded in the teacher's talk.

(i). Card matching facilitates understanding of functions

This is evident when the teacher mentions that cards made him understand functions. The teacher participated in both Functions 1 and 2, where there were card matching activities.

(ii). Card matching involves reasoning

According to Teacher 8, graphs to equations and deciding whether the graph is hyperbola or not requires reasoning. In the PD card matching involved working with representations of different functions and moving between representations without always beginning from the algebraic, as is normally the case in the classroom teaching of functions.

(iii). Card matching is interesting to learners

Teacher 8 noted how card matching was interesting to learners. In his talk about his Pre-and Post PD lesson planning the teacher said he used cards in his class with learners. Hence it is probable that he noticed the learners' interest in his class.

10.6.8.2. What constitutes Teacher 8's lived object of learning with respect to resources?

Teacher 8 talks about his learning of functions for his own understanding. This is evident when he says that working with cards helped him understand functions and that cards required him to reason as he went about searching for matching cards. He also found it helpful to learners. Since he talks about card matching with respect to his own learning and learners' learning, then his lived object of learning was constituted by both **Mt-pl** and **Mt-PL**.

10.6.9. Teacher 9

10.6.9.1. Descriptive analysis of Teacher 9's lived object of learning

Teacher 9 was also asked to talk about his learning from resources that were available in the PD. Below is his talk in a conversation with the researcher.

R:	Then which of these resources which were present in the sessions you attended helped you learn functions?
T9:	I liked the issue of cards – the matching of cards.
R:	Mmm
T9:	Ja. Because normally as teachers we didn't do, especial with the issue of matching. We normally do the equation and the graph only without using the verbal, yes.
R:	Mmm. Then do you mind explaining maybe what exactly did these resources help you understand? Maybe you can pinpoint to say, 'This particular idea about functions I was able to understand it even more through the use of this particular resource.'
T9:	I liked the issue of this one, the issue of Geogebra because if you, the way how you write on it, the way how you put the equation into the programme, if you put it another way it will not show you the graph. So it means you must know how to put it. You must first of all convert it to y is equals to. If it starts with x you must convert it so that y become the subject of the what? Of the formula. That's what I liked about this thing because if you put like that it will not show you the what? The graph. And what I liked about the Geogebra, even if you draw 5 graphs, all those 5 graphs the appear on the same what? On the same graph. You want to delete the one that you don't want, you can delete it. But all of them they will be there, yes.

Excerpt 10.12: Teacher 9's talk about resources that facilitated his learning

There are two main meanings that emerged from Teacher 9's talk and these included the following:

(i) Card matching involves other representations

The teacher pointed out out that he liked the idea of matching cards, and he contrasted this with his normal way of teaching, which focuses only on equations and graphs only. He found card matching to be productive because it allowed for the use of all representations.

(ii). Geogebra is fascinating

Teacher 9 liked the way Geogebra works to produce the different graphs. The teacher is particularly fascinated by the technical aspect of the software with respect to what it does or does not allow.

10.6.9.2. What constitutes Teacher 9's lived object of learning with respect to resources?

The teacher makes reference to teaching and this teaching is a contrast between how he teaches functions and how he saw this done in PD. He also talks about his learning to use the Geogebra to produce graphs of functions. So his learning was constituted by **Tm–pl** and **Mt-pl.** The former is evident when the teacher appears to have realised the need to incorporate other representations in his teaching of functions. The latter is evident in teacher's narration of ways that produce graphs and ways do not, when working from the Geogebra software.

10.6.10. Summary of teachers' learning with respect to resources

The following table contains a summary of the resources teachers said they used in their teaching**.** The table shows that a plethora of resources were used before PD, and the prescibed textbook was used by all teachers. In addition to this resource each teacher drewn on additional resources. The table also shows that among the resources used after PD were some resources from the WMCS PD. Furthermore, the third column of the table shows that most teachers' learning in the PD was facilitated by the card matching activity.

Table 10.3: Summary of resources teachers used in their classrooms

Teacher	***Resources used before PD***	***WMC-S Resources used after PD***	***Resources that facilitated teacher learning in PD***
T1	Classroom Maths, Focus Study & Master) Graph board Chalkboard	WMC-S Grids	Cards facilitated more understanding
T2	Chalkboard Geo-board Classroom Maths	WMC-S Grids	Cards
T3	Classroom maths Worksheets Chalkboard Classroom Mathematics Study guide in Classroom Mathematics Other books Worksheets	WMC-S Grid Matching cards	Card matching activity Enough information provided
T4	Classroom maths Graph paper Past Exam Papers Question papers Chalkboard	Cards Bernard's Table Readings Grid	Rise over Run Card Matching activities Bernard's Table
T5	Classroom maths Other textbooks Non Prescribed-Mind Action Graph paper	WMC-S readings WMC –S learner grid Non-permanent marker; WMC PD	Card activity helped the teacher understand functions more

	Internet Chalkboard	Laminated Grids Cards on domain and range	
T6	Classroom maths another one worksheet Chalkboard		Card activity enabled the teacher realise what a function is and what it is not
T7	Classroom maths Worksheets Chalkboard	Use of grids The use of different colours on the grid Using the Vertical line test Grid charts ; Large grids and smaller grids	Cards played a very important part in the teacher's understanding of functions Grid made the teacher's lesson fascinate
T8	Classroom Mathematics Sowetan-(vertical functions Chalkboard	Cards Mapping diagrams WMC-S Readings Grids	Cards made the teacher understand functions
T9	overhead projector data projector Classroom maths Cambridge University-books website for mathematics, Department of Education-General Mathematics 4question papers Districts-tests Chalkboard	Calculators from PD Geogebra	Card matching was the teacher's preferred resource

Table 10.4 and **Chart 10.3** show the distribution of themes related to learning from PD resources. Both sources indicate that teachers' learning from resources was dominated by teaching objects. In particular, **Table 10.4** shows that a larger proportion of the themes were Tm-pl – this is evident where teachers talked more about how they taught functions and contrasting this with how functions were handled in the PD. **Chart 10.3** shows the dominance of themes related to teaching objects.

Table 10. 4: Summary of themes in teachers' utterances about learning from resources

Teacher	Mt-pl	Mt-PL	Tm-pl	Tm-PL
T1			8	
T2			5	3
T3			6	4
T4	11	4	1	
T5			4	6
T6			10	3
T7			1	5
T8			7	1
T9	4		3	1

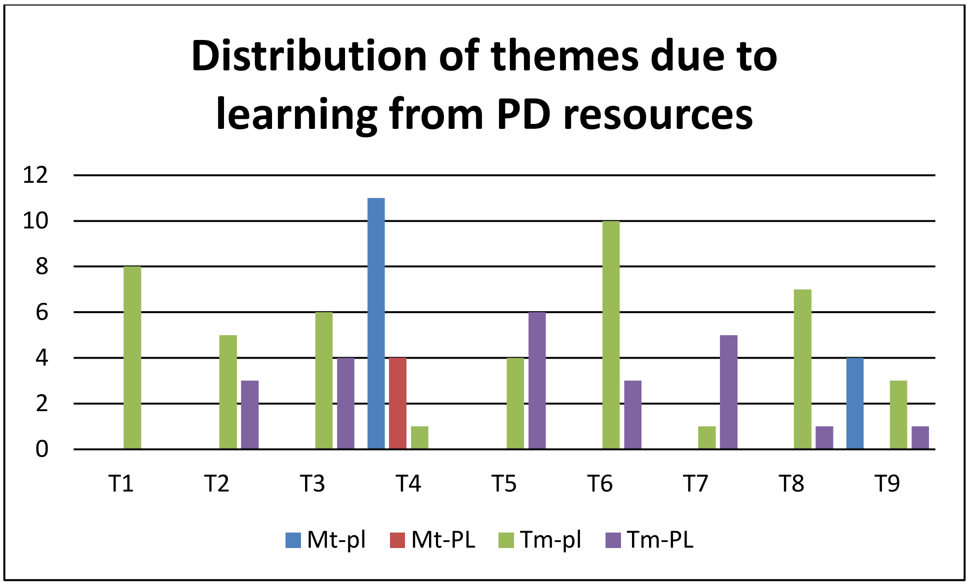


Chart 10. 3: Teacher learning from PD resources

10.7. What was the status of teachers' knowledge of functions before PD?

I have preceded my analysis by foregrounding gaps in Teacher 2's knowledge of functions at the beginning of the Functions theme work in order to provide an insight into teachers' knowledge before PD. I have selected Teacher 2 because his knowledge of mathematics was believed to be stronger than most of the teachers in PD. Illustrating gaps in the teachers' knowledge is fundamental to the claim that I make about the teachers' mathematics knowledge before and after PD later in **Chapter 12.**

10.7.1. Using Teacher 2 to illustrate the teachers' knowledge of functions

Teacher 2 was a participant at the I-Cluster. His misconception got illuminated in Task 1 of Functions 1 after the facilitator had given teachers 24 cards to sort into two piles of functions and non-functions. Working together with members of his group they produced several cards which were incorrectly classified as non-functions when they were functions, and among these were Card 3 (see **Chapter 6**). In a way to help teachers discern that Card 3 was a function the facilitator translated the verbal representation on Card 3 into an algebraic representation and then to a parabola. He then asked Teacher 2 to explain why his group thought Card 3 was not a function. **Excerpt 10.13** contains a whole class discussion led by Teacher 2 in which he is incorrectly explaining why Card 3 was a non-function. Below are Cards 3 and 4 respectively. Card 4 is the algebraic representation of Card 3, and I must mention that they were among cards listed as non-functions at the E- and I-Clusters (see **Figures 6.2** and **6.20** in **Chapter 6**).

3 **To get f (x) you square x and subtract it from 1**	**4** **f(x) = 1 – x²**

Figure 10.1: Card 3 and 4

The following conversation centered on Card 3, which Teacher 2 said was not a function. In the conversation Teacher 3 was called to the front to explain with illustrations why he thought a parabola was not a function. Card 3 was a verbal representation of a quadratic function. In his explanation he argued that the horizontal ruler test cut the graph (i.e. parabola) in two places instead of one, and that made it a non-function.

328 329	F:	So you're saying this is the graph. But now tell us why it's not a function. [*After drawing the parabola*]
330	T2:	Right. I used the horizontal test. The horizontal test, ok
331	F:	Which says what?
332	T2:	A function should have one input, one output.
333	F:	Ok
334	T2:	Ja. So if we take our horizontal test we discover that one input
335	F:	I'm going to just do it there. [*Facilitator drew the horizontal line on the parabola to illustrate what Teacher 2 was explaining*]
336	T2:	Exactly, ja. It will have two inputs versus one output.
337	T8:	2 inputs, 1 output?
338	T2:	One output

Excerpt 10.13: Teacher 2's explanation with respect to horizontal line test

Teacher 2 used the parabola which was drawn by the facilitator to show that the horizontal line cut the graph of $f(x) = 1 - x^2$ in two places. The following conversation centered on **Figure 10.2** which was the graphical representation of Cards 3 and 4.

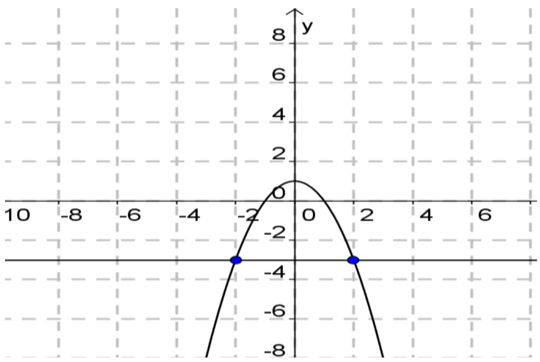


Figure 10.2: Graphical representaion of Card 3

Teacher 2 was called to the front to explain his viewpoint pertaining to Card 3. **Excerpt 10.14** contains Teacher 2's explanation and illustration using a mapping diagram (see **Figure 10.3**). Working from **Figure 10.2**, Teacher 2 showed that both negative 2 and positve two mapped onto negative 3. Figure 10.2 shows the horozontal line cutting the parabola twice.

339	F:	Do you want to come and show us what you mean. [*Teacher 2 went to the flipchart and drew two points on the parabola to show that -2 and 2 were mapped onto -3 and for him this made Card 3 a non-function*]
341	T2:	*[goes to the front]* Let's assume x is negative 2 and I come here, I have what?
342	T4:	2
343 344	T2:	2, isn't it. So I have negative 2 and 2. *[draws on the board]* But here I have, let's take this to be y. What do you think?
345	T9	minus 4, negative 3
346	F:	Negative 3
347	T9:	Minus 4
348	T2:	Ja, negative 3. So I have…
349	T4:	So where's your axis?
350	T2:	The brown line is the axis. So that will be negative 2 squared is 4
351	T9:	Minus
352 353 354 355 356 357 358	T2:	Negative 4 plus 1. Ja, it's negative 3. *[continues writing on the board]* Right, I have negative 3, isn't it? Right. So it means it will be like this. That's my functions, do you see it? *[T2 has drawn a large circle containing -2 and under -2 is written 2. An arrow points from -2 to* -3 *which is in another circle. Another arrow points from 2 to the same* -3. *Along the arrow of -2 is written* x_1*) and along the arrow from 2 is written x]*

Excerpt 10.14: Teacher 2's explanation with respect to inputs and outputs

To further illustrate hs argument, Teacher 2 used a mapping diagram to show that negative 2 and positive 2 mapped onto negative 3 (see **Figure 10.3**). In the above excerpt, lines **352 – 358** contain the teacher's explanation pertaining to the mapping diagram.

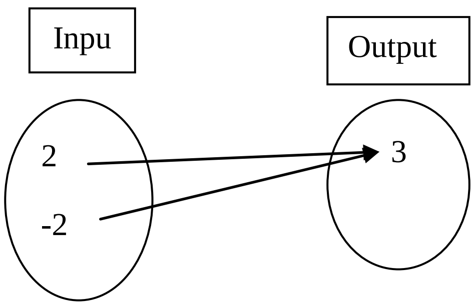


Figure 10.3: Mapping diagram used to show that two inputs mapped onto one output

Further in the conversation he said a function should be one-to-one for it to be called a function, and also said that a many-to-one was not a function. In a many-to-one function two inputs mapped onto the same output –an occurrence which showed that the output was not unique. For Teacher 2 there was no univalance when two inputs mapped onto the same output. This argument is located in lines 334 and 336. In these lines the teacher is saying that the horizontal line test will show that two inputs mapped onto the same output and this made Card 3 a non-function.

The essence of Teacher 2's explanation was that since -2 and 2 are mapped onto -3, Card 3 was not a function. First he used a horizontal line test to show that the parabola was not a function. He also used a mapping diagram to show that two inputs were mapped onto one output, and this, he believed, was contrary to the definition of a function. In the conversation he pointed out that he would prefer each input to have its different output and not share the same output. In so saying he was misinterpreting the definition when it says each input should have a unique output. This aspect of the definition says that an input should not have more than one output, and not that two inputs cannot not map onto the same output. Teacher 2 is misreading the meaning of unique as used in the definition.

As I pointed out out at the beginning, teachers came into the PD with gaps in their knowledge of functions, and some got closed as teachers' participation in the WMC-S project' PD continued. At the end of the theme work teachers participated in in-depth interviews to talk about their learning in the PD. In the following conversation Teacher 2 was responding to interview tasks which were designed to tap into his knowledge of functions following his participation in the PD. I have drawn from responses which serve to demonstrate that earlier misconceptions appear to have been redressed. In the above conversation he pointed out that he used the horizontal line which showed that a parabola was a non-function. However in the interview he mentioned that he would use a vertical line test to show that a graph was a function or not a function. The teacher was responding to item 7 of the interview protocol (see Item 7 below).

7. In Ms Moalosi's lesson focused on functions a learner reasoned that "a function is only that which can be represented by a straight line or a smooth curve. Any function whose graph is neither a straight line nor a smooth curve is not a function".
   a) Do you think the learner's reasoning is true/false?
   b) If the learner's reasoning is:
      i. True, please give an example to support the learner's reasoning
      ii. False, how would you correct the learner
   (c) What could be the source of the learner's reasoning?

Excerpt 10.15: Scenario 7 from teacher interview

The following conversation captures Teacher 2's correct response to item 7

861 862 863 864	R:	A learner reasoned, you know, that a function is only that which can be represented by a straight line or a smooth curve. Any function whose graph is neither a straight line nor a smooth curve is not a function. So do you think the learner's reasoning is true?
874 875	T2:	Right. I was thinking of a counter example, ja, where we can say is this a function after we have drawn it?
877 879		Like the first one where we came up with this, of a reflection. A reflection like that. It is a smooth curve, but it's not a function.
881 882		Ja, it's not a function. Suppose we take this as our XY [plane], ja, where we have one input and what? 1 input versus 2 outputs.
884 885 886 887 888 889 890		Ja. What I would rather say to this learner is for him or her to be satisfied that this is a function the learner must be emphasising on the vertical testing. For vertical testing the graph, you know, that vertical line test must touch the graph once. If it touches twice (like in this case) it's no longer a what? A function. Like in this case it's touching here and also touching here. For horizontal testing we may find that it is, yes, it is only touching once, but it's not satisfying the vertical test. So if both are not satisfied, hence it's not a function. So there we are

Excerpt 10.16: Teacher 2's response to Scenario 7

In his response Teacher 2 is no longer appealing to the horizontal line test to show that a graph is not a function, but to vertical line test. He appears to be aware that the horizontal line test is not useful for testing if a graph represents a function. As a matter of fact, in **Lines 888 – 889** of **Excerpt 10.16** the teacher points out that even when the horizontal line cut the graph once, as long as the vertical line cut it twice, it is not a function.

I should point out that following Teacher 2's explanation in which he demonstrated that Card 3 was not a function the facilitator used the vertical line test to explain and demonstrate that the parabola of Card 3 was a function, together with other cards which were incorrectly classified as non-functions. So it was there that Teacher 2 was helped to experience that the vertical line test, and not horizontal line test, is used to test if a graph is a function or not a function. Now in the response to Item 7 Teacher 2 is drawing from that experience which he did not have before participating in the PD. He goes further to say that he would use a vertical line test on a counter-example to show that it cuts the graph of a non-function more than once, and he used the following example which was used by the facilitator when he was redressing Teacher 2' misconception (see **Figure 10.4**).

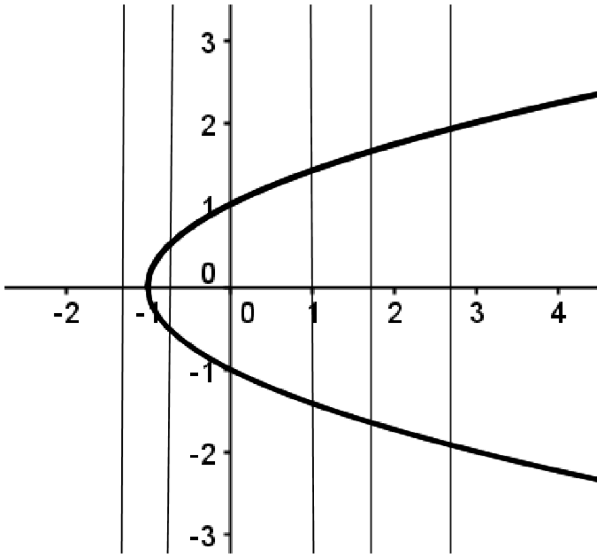


Figure 10.4: Graph of a non-function

When responding to Item 7 most of the teachers said they would use this as a counter-example of a smooth curve, which was not a function. In addition, just like Teacher 2, they said they would use the vertical line test to show that this smooth curve was not a function. Just to remind the reader - the purpose of foregrounding Teacher 2's responses before and after PD was to show that teachers came with gaps, and these had narrowed down after the Functions theme work.

10.8. Analysis of teacher learning in PD activities

This analysis is focused on Teachers' utterances about their learning in PD activities. These were activities that were inherent in the tasks that were used in the enactment. There were two tasks in Functions 1 and five tasks in Functions 2. The analysis done here focused on what teachers said they learnt from those tasks. Following participation in these tasks I asked each teacher to talk about his/her experiencing of activities in these tasks "***What did you learn in the two sessions (i.e. Functions 1 and Functions 2) where you were a participant?***.

10. 8.1. Teacher 1

10.8.1. 1. Descriptive analysis of Teacher 1's lived object of learning with respect to PD activities

Teacher 1 was asked to talk about what he learnt in PD sessions. The teacher says he learnt the different representations which he found refreshing, because he never experienced then before participation in the PD. The teacher is talking about his own learning of functions which was challenging.

R:	What did you learn in the two sessions where functions were in focus?
T1:	Ok. I think I mentioned earlier like when we have to match the cards, alright, the different representations of functions. I think I found that one quite refreshing. Ja, because that was something that I had not, you know, thought of or seen before. Ja, so I think that was a… Ja, I mean it was like something really wonderful that I, you know, I learned. . And because even myself, you know, there were times when I probably struggled to match, you know, one representation with the other, you know those… It sort of exposed my weakness, I mean, if you like. So I would say that was the… Ja, something out of the box. I mean, ja, I think that was the highlight for me.

Excerpt 10.17: Teacher 1's talk about learning in PD activities

Analysis of Teacher 1's interview utterances resulted in 5 content-focused categories. These I have presented and discussed below by linking them to the PD where the teacher was helped to experience them. The teacher's talk about his learning was predominately related to his experiencing of card activities involving multiple representations.

i. Card Matching makes it possible to learn the different representations of functions

The first category of the teacher's lived object of learning was where the teacher talked about learning about the relationship between the different representations of the same function. Multiple representations of functions were central in both Functions 1 and Functions 2. In the former, teachers worked on two tasks – one involving sorting cards into functions non-functions, and the other task involved matching cards which represented the same function. In Session 2.2 teachers matched two representations to the domain and range.

ii. Card matching refreshes knowledge of functions

Participating in the PD helped to stimulate Teacher 1's own knowledge of functions. From the teacher's talk it is clear that he had not experienced functions before in the same fashion as he did in the PD. The teacher is not pointing to particular aspects for functions which he has learnt. In spite of that it can be inferred that Teacher 1 was able to experience that which was in the foreground, and this was to help teachers to realize that a function is a relationship between inputs and outputs, and that each multiple representation of a function constitutes this relationship

iii. Card matching is wonderful

Teacher 1 also talked about his learning of multiple representations which he said was a wonderful experience. He had never encountered functions in this fashion before PD.

iv. Card matching demand great effort

The teacher found it challenging to match multiple representations of functions. In the excerpt he pointed out that he struggled to identify related representations. It is evident in this utterance that card activities which were present in the two sessions helped him to learn functions even though his learning was fraught with difficulties. The teacher confessed to gaps in his knowledge of functions by pointing out that working with multiple representations exposed some weaknesses in his knowledge of functions.

v. Card matching is a highlight

Teacher 1 mentions that working with multiple representations brought to light a novel way of seeing functions, and he referred to this as a highlight in his learning of functions. He asserts that this was something which he had not seen before coming to the PD.

(vi). Card Matching is a ingenious way to learn functions

Teacher 1 sees learning about, with and from multiple representations of functions as a revolution from the normal way. Hence he refers to learning in this manner an out-of-the-box way to experience functions

10.8.1.2. Analysis of Teacher 1's lived object of learning with respect to PD activities

It is evident in Teacher 1's talk that his lived experience is wholly characterized by talk focused on learning mathematics (i.e. functions). So bringing my analytical tool to bear on Teacher 1's utterances I notice that all his talk falls in the **Mt** category, suggesting that the above five statements are instances of learning functions and not teaching functions. In the above extract the teacher mentions that he encountered challenges when matching cards and this was where he struggled to match the representations. The teacher is also stating that his weaknesses were exposed when he could not match cards, and instances that required teachers to match cards were related to strengthening their content knowledge of functions. So the teacher's lived object of learning was constituted by personal learning of functions (**Mt-pl**).

10.8.2. Teacher 2

10.8.2.1. Descriptive analysis of Teacher 2's lived object of learning with respect to PD activities

Teacher 2 talked about his learning in PD sessions and this is what he said:

R:	So in which session were you engaged? Would you say you were engaged in meaningful learning of functions, learning for yourself?
T2:	Ja, really in all the sessions it's like I was, we were a bit fortunate because most of the functions we had done here it was very exact, so I had the pleasure of attending all of them and participating in almost everything. I really enjoyed it. Ja, really. It was quite interesting, especially with this other guy from where?
R:	From the UK, Bernard.
T2:	From the UK, Bernard. You know we really, we really enjoyed it.
R:	Is there anything specific that you could say even if you cannot pinpoint, you are not able to tell which session it was, to say 'But this is the thing which I can say I got from the sessions"?
	Ja, what I can vividly recall and, from those sessions is that when we teach functions we need to be very clear, ja. We need to… Let's connect them from the definition, those representations, ja. Let's… Let's make it a, that the learners should be able to move from one end to the other. And also connect it with the, with the domains and the range, especially where it is defined and where it is not, where it is undefined, ja. Why are we saying strictly x must be greater than this? And between this and that? Ja, at times it is because we want to make, to continue defining it as a function. If we just allow it to take any value we may not end up having what? A function. This is something which I said myself, ja, really it is very important as we teach, even learners themselves as they define it, they should be able to define it, ok it is only valid for this and this and that. And it also helped me because I also teach grade 12.

Excerpt 10.18: Teacher 2's talk about learning with respect to PD activities

Teacher 2 says that he learnt functions in all PD sessions, particularly Session 2.2. The teacher's experiencing of PD activities brings to light several things that constitute his lived object of learning.

(i). Teacher 2 enjoyed PD activities

Professional development activities were a source of pleasure for the teacher. Thus is evident when he pointed out that he enjoyed all PD activities in which he was a participant.

(ii). PD activities were interesting

The teacher found PD activities interesting, and this was particularly so in Functions 2. Functions 2 privileged four key features. First was the relationship between input and domain, and output and range. Second was the relationship between domain, range, graph and equation of a function. Third was transformation of functions with focus on the horizonatal shift of a parabola, and the fourth was mathmatical thinking involving intersecting circles.

(iii). Teaching of functions must show connections between mutiple representations

Teacher 2 noticed the value of linking the representations of functions in the PD and so he believes it is important to do the same in the class with learners. The representations must be connected to the definition of a function, and also to domain and range. Showing the connections is importnt because it will allow learners to move between the representations.

10.8.2.2. Analysis of Teacher 2's lived object of learning with respect to PD activities

It is evident that the teacher talks about both his learning of functions with respect to own growth and use in the class with learners. Personally, he mentions that he enjoyed participating in the PD activities, which he also says were interesting. Professionally, he talks about what should be privileged in the class for learners to discern. In particular, he emphasises connections between multiple representations and definition, and between multiple representations and domain and range. He also says learners should know the definition of a function. His talk about learners is rooted in mathematics. This is evident when he talks about the mathematics that learners need to know. So his lived object is constituted by **Mt-pl** and **Mt-PL**.

10.8.3. Teacher 3

10.8.3.1. Descriptive analysis of Teacher 3's lived object of learning with respect to PD activities

Teacher 3 was asked to talk about her learning in the Sessions and she responded as follows:

454 455	R:	Now in which of the 2 sessions would you say were effectively engaged in the learning of functions, you know, for yourself?
456 457 458 459 460 461 462	T3:	The first one. The ones that we were doing card matching, I've learnt a lot. And even the second one. The second one is actually they were teaching us, we were not participating much. They were, they came here and then the teacher was, there was this doctor who - I don't know from where – but then he was teaching us. But I don't know, we were participating, he was giving us a chance to ask, but then most of the time he was the one who was talking. But then, then in a way participating. But then I think I learnt a lot. I put myself in the learners' shoes so that if I'm a learner and then, ja.
464 465	R:	Ok. Now can you give some examples of what you would say you learnt in the theme work where you were actively involved?
466 467 468 469 470	T3:	What I've learnt is that I've… most of the, like I've already said, the domains. We are not spending enough time on the domain. What I've learnt is that once I've introduced the… before I even go to plotting of the graph I must spend time on domains and range. So I think that was the most important thing. Before I even go to how to plot the graph.
471	R:	Ok
472	T3:	Ja, so...

Excerpt 10.20: Teacher 3's talk about learning with respect to PD activities

The teacher is talking about her learning in the two sessions, and mentions that she learnt through card matching in Functions 1. She says in Functions 2 the Facilitator was teaching and she was learning through participation in PD activities. She says she learnt a lot of things in those sessions. She says she learnt about domains and she realized that she does not spend enough time on domain and range. Functions 2 engaged teachers in a matching activity that required teachers to match an equation, to graph, domain and range.

10.8.3.2. Analysis of Teacher 3's lived object of learning with respect to PD activities

The teacher talks about her learning of domain and range, and also talks about them with respect to how much time she should spend on them. It is probable that the teacher sees the need to spend more time on domain and range because she now knows more about these than before. So here the focus is on mathematics with respect to own learning of functions and learning functions for utilitarian purposes. So her lived object of learning is constituted by **Mt-pl** and **Mt-PL**.

10.8.4. Teacher 4

10.8.4.1. Descriptive analysis of Teacher 4's lived object of learning with respect to PD activities

Teacher 4 engaged in a conversation with the researcher to talk about her learning in the two sessions. The conversation is provided below:

R:	In which session were you engaged in the learning of functions for yourself?
T4:	I would think all of them.
R:	All of them?
T4:	Yes
R:	Ok
T4:	I was getting different things from the, from the workshops. Yes, I would think all of them.
R:	Is there any example that you could say 'No, in this…' even if you cannot pinpoint the, attach the example to the function, but to say there is somewhere, one session where I got this particular thing?
T4:	Ok *[laughs]* I'll be repeating myself.
R:	Ja. No, it's fine. I know you've already highlighted some things.
T4:	The gradient one and where Craig asked learners the gradient of a given graph and they were giving him different answers and, yes, that one. And also ok, when we were matching cards those… When we were matching those cards there was an instance where there was a graph with 2 functions that we could not match, but that was also very interesting for me because we had put a totally different function and after that we were like 'It is this function', you know. I think those instances.

Excerpt 10.21: Teacher 4's talk about learning with respect to PD activities

The teacher says she learnt across the sessions, and she singles out the gradient as the thing that she learnt in one of the sessions where learners were present. In that session gradient was derived from the graph using the notion of rise over run and this was useful knowledge to the teacher. The teacher also learnt functions through card matching in Functions 1.

10.8.4.2. Analysis of Teacher 4's lived object of learning with respect to PD activities

In the teacher's talk she mentions learning about gradient when it was taught to learners. This is an instance of learning functions, but with respect to use or for utilitarian purposes. Hence this constitutes **Mt-PL**. She also talks about her learning of functions for personal growth and this constitutes **Mt-pl**. So her lived object of learning was constituted by **Mt-pl** and **Mt-PL**.

10.8.6. Teacher 6

10.8.6.1. Descriptive analysis of Teacher 6's lived object of learning with respect to PD activities

Teacher 6 talked about her learning in PD sessions. Below is her conversation with the researcher:

R:	In which session were you engaged in meaningful learning of functions where you can say, 'Personally in this session I was meaningfully engaged in learning functions for myself'?
T6:	I think that one of Bernard and, because he didn't talk for the whole session. He talked, he explained, then we engaged in activities the whole period. And we were like giving feedback. I think the most important was when teachers were giving feedback on the board, reporting, then that's where you look at what this group thought is the answer, and what that group thought it is. And so it's like sharing different ideas from different groups. So I think the report back session was more important. Ja.
R:	If you were to give an example of what exactly would you say you learnt in that theme and an example to say, 'No this is what I can point at and say this is what I learnt.
T6:	I think the most one is the domain and the range. Ja. The domain and the range where Bernard was explaining about the domain and the range. Ja, the relationship between the two. And the parameters, yes. I think that was the most, ja.

Excerpt 10.22: Teacher 6's talk about her learning in with respect to PD activities

Teacher 6 says she was engaged in the learning of functions in Functions 2, and she mentions instances when groups had to make presentations to others from the board as the most important. She identifies her learning of domain and range as the most important aspect of functions that she learnt.

10.8.6.2. Analysis of Teacher 6's lived object of learning with respect to PD activities

Teacher 6's talk is centered on her learning of functions in activities that were provided in the sessions. In Functions 2 teachers worked in groups to find a function when given its minimum or maximum value. So the teacher was engaged in the learning of functions in group work. She also learnt about parameters and the relationship between domain and range. So her lived object of learning was characterized by **Mt-pl**.

10.8.7. Teacher 8

10.8.7.1. Descriptive analysis of Teacher 8's lived object of learning with respect to PD activities

Teacher 8 talked at length about his learning in PD sessions. I have captured him in the following conversation:

148	R:	What did you learn for yourself, in the PD ?
149 150 151 223 224 225 226 227 230 231 233 234	T8:	What I learned from that is that you want to be able to identify given a problem you need to identify whether it is a function or it's not a function. So you'll be able to know this one is a function, but this one is not a function. It made me learn to know that if you are saying a function, maybe vertical shifts and horizontal shifts, it will be easy to see what is taking place. The graph will be shifted and you will see its new position from the previous position say it has shifted 2 places upwards or 2 places downwards or 2 places to the left or to the right. You would easily say from the software that is exactly what is happening. Yes, maybe I give an example of a function f of x is equals to x squared plus 1. So it means it is going up a place. One place upwards, yes. I am able to see that the graph has shifted from the point of origin and it shifted one place upwards or it, the other way, it can go downwards as well.
	R:	Ok. So in 2.1, the first one, do you remember anything that you could say, 'This is what I got from the, the workshop'?
	T8:	Ok, the first one was about identifying whether a given function - non-functions and functions. Ja. Identifying. It was identifying…
	R:	Ja
	T8:	given questions. Is it a function, a non-function or given diagrams. You're given a diagram you need to see is it a function or is it a non-function? Yes.
	R:	Ok. And then in session… the second session, 2.2, the one which was resourced by B
	T8:	From this one here I learnt that when given a function it's very important for one to analyse, to analyse it fully. You need to know everything about the graph.
	R:	Mmm
509 510 511 512 513 514 515 516 517	T8:	Given the function, you need to know what the range of it is, what is the domain of it – everything about the graph. So it was very, very useful when it comes to the classroom situation because, you know, the questions when they are coming in the exams, they will be testing those things. So if the learner knows more about the graph it will be very easy for them to know if they are asking about the domain, the range. Given a graph the first thing that they need to know everything about the graph. That is what I learned from that workshop. I learned that given a function you need to know all about it, especially the terminology, the range, the domain - everything. Anyway, you know everything about it, then when it comes to questions it varies

Excerpt 10.23: Teacher 8's talk about his learning with respect to PD activities

Teacher 8 says he learnt what constitutes a function and what does not. This aspect was forgrounded in Functions 1, when teachers had to sort cards in to piles of functions and non-functions. He was able to notice through the visual scaffolding of Geogebra when does a graph shift vertically and when does it shift horizontally. Geogebra was one of the resources used to help teachers discern key features of functions. He has also realized the need to know how to analyze a graph of a function to learn its properties. He explains this by giving range and domain as examples of things that need to be known in a function. Knowing all properties of a function is important in examinations. He adds to this, the need to know function terminology.

10.8.7.2. Analysis of Teacher 8's lived object of learning?

Teacher 8 talked about his learning of functions in the sessions, and he does this by providing examples of aspects of functions which he learnt. In addition, he talks about the importance of learners knowing all aspects of functions for examination purposes. So the teacher seems to be talking about learning functions and learning of functions for utilitarian purposes. So his lived object of learning is constituted by **Mt-pl** and **Mt-PL** with the former dominant in the talk.

10.8.8. Teacher 9

10.8.8.1. Descriptive analysis of Teacher 9's lived object of learning with respect to PD activities

Teacher 9 also had the opportunity to talk about his learning in PD sessions. His conversation with the researcher is shown below.

530	R:	Ok. Now in which session were you engaged in meaningful learning of functions?
531 532	T9:	The functions… the session that we did in the staffroom when we had teachers from other schools, not the one that we had learners in.
533	R:	Ja, ja
534 535 5536 627 628 629 630 631 632 633 634 635	T9:	Because that one which was sort of, you learn new methods of doing things than the one that your normally what? do. Like for example the issue of the ruler test. Yes. I learned it from there The use of the card activities. Yes, the use of the card activities because from what I saw here if you look at the cards, these cards are different colours. So if you ended up having the same colour it means we had one more, we had one representation instead of having different. And I like the way how they did, they used different colours. If you had used the same colours you end up with learners having the same what? Maybe the same equations or verbal expressions without them looking and saying, 'Ah, we've got the same things, guys, and this is wrong.' But the use of different colours it will let them say, 'Hey, this is wrong. You must have 4 different what? colours.' Ja. The issue of colour it was very important.

Excerpt 10.24: Teacher 9's talk about his learning with respect to PD activities

Teacher 9 says he learnt functions in Functions 1 and 2. These sessions were offered to teachers only, and his school was the host. He says he was able to learn new ways of teaching functions. He identifies the ruler test which is used to determine if a graph represents a function as one of the methods he learnt. Teacher 9 says knowledge of the ruler test is important for learners to have. His talk is dominated by learning of functions both for utilitarian and personal growth. This is evident where he keeps making mention of methods of learning functions. Of course, he does contrast his normal way of teaching functions and how it was presented in the PD.

10.8.8.2. Analysis of Teacher 9's lived object of learning with respect to PD activities

It is evident from the teacher's talk that his learning was constituted by learning of functions for use and for personal growth. Learning for use is evident where he talks about learning the ruler test to use by the teacher and learners. Learning for personal growth is evident where he talks about teachers defending their views on assigned tasks. So his lived object of learning was constituted by ***Mt-PL*** and ***Mt-pl***.

10.9. Summary of themes constituting the lived object of learning in PD sessions

The following table and chart show the distribution of themes in the lived object of learning –with respect to learning in Functions 1 and 2. **Table 10.5** shows that generally, mathematics objects (**Mt**) were dominant over teaching objects (**Tm**). In particular, **Mt-pl** were present in larger proportions followed by **Mt-PL**

Table 10.5: Distribution of themes with respect to PD sessions

Teacher	Mt-pl	Mt-PL	Tm-pl	Tm-PL
T1	7			
T2	4	11		
T3	6	3		1
T4	3	1		
T6	7			
T8	15	6		
T9	11	3	1	

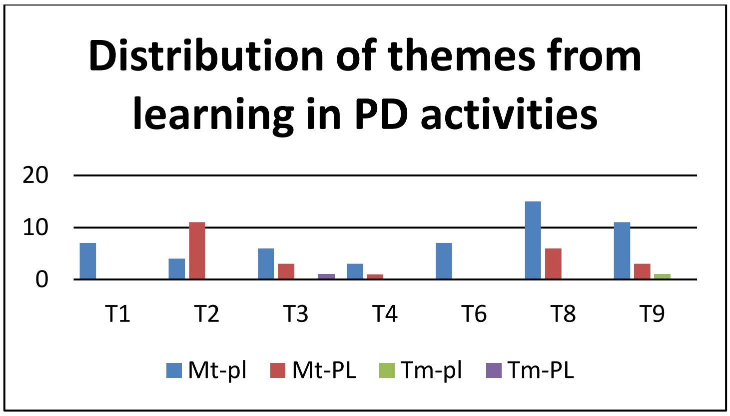


Chart 10. 4: Teacher learning from PD sessions

10.10. Distribution of themes across learning from resources and PD sessions

The following chart shows a combined distribution of themes across the two learning sites (i.e., resources and PD activities). The chart shows that teacher learning in the PD as a whole was dominated by mathematics objects and Mt-pl was present in larger proportions followed by Mt-PL.

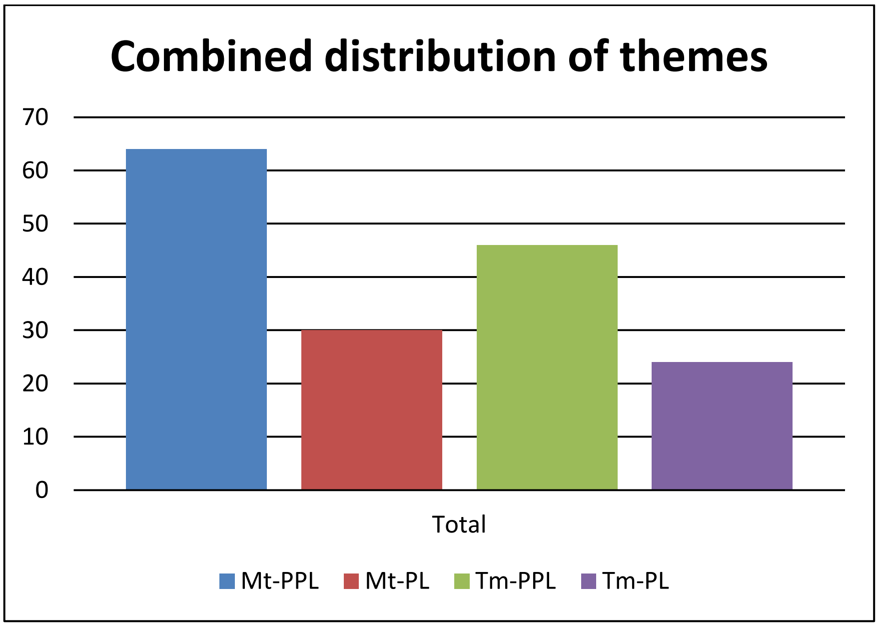


Chart 10. 5: Combined distribution themes

The dominance of mathematics objects resonate well with the expectations of PD, since the goal of the PD was to strengthen teachers' subject matter knowledge of functions. Later I return to this in Chapter 12 to talk about this with respect to Variation Theory's first principle.

10.11. Conclusion

The analysis of teacher learning from resources suggests that when resources were in focus, **Tm** was dominant over **Mt**, and **Mt** was dominant when PD sessions were in the fore. The presence of **Tm** was not surprising since this confirmed Adler and Davis' (2006) earlier findings. However, it was interesting that **Tm** was dominant over **Mt** when resources were in foregrounded. In spite of this occurrence, the combined distribution of themes show the dominance of **Mt** over **Tm**. The analysis shows that the lived object of learning for teachers was constituted by both mathematics (**Mt-pl, Mt-PL**) and teaching objects (**Tm-pl, Tm-PL**). I return to talk more about the presence of teaching objects in the lived object of learning in **Chapter 12**.

Now, looking back from my analysis of interviews with respect to teacher learning with respect to resources and PD activities I have insight into the teachers' lived object of learning in the PD. Now it is interesting to go into their lesson planning and see what they did in their planning with respect to what they say they learnt in the PD. Remember, I asked each teacher to plan a Pre-PD and a Post –PD lesson plan – i.e. before and after participating in PD.

CHAPTER 11: ANALYSIS OF LESSON PLANS

11.1. Introduction

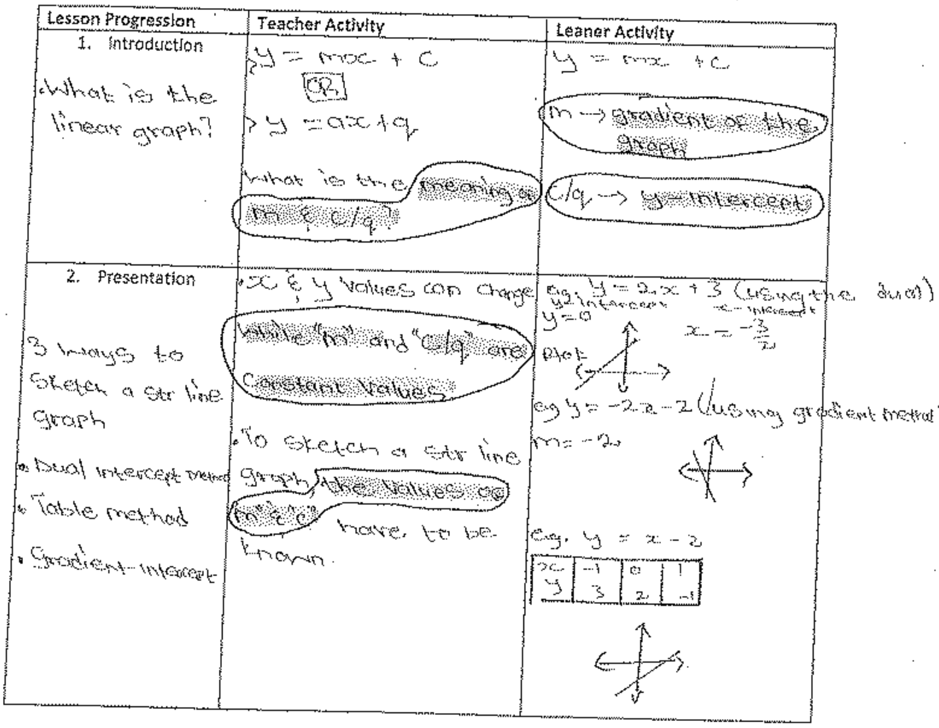
This chapter presents my analysis of the teachers' Pre- and Post –PD lesson planning. For the purpose of being within the official prescribed word count I used Teacher 3 and Teacher 5's lesson plans to illuminate my detailed analysis on the nine Pre- and nine Post-PD lesson plans. The detailed analysis is followed by a summary of the analysis of the rest of the lesson plans. Suffice it to point out that each lesson plan was my unit of analysis. Analysis of the lesson plans was my second analysis focused on the teachers' lived object of learning. In my analysis I worked empirically by shading and enclosing the different foci of each lesson plan and worked theoretically with variation theory tools and notions of privileged features and key features to describe and theorise those foci. Variation theory and the two notions were operationalised in **Chapter 5**. Copies of the rest of the lesson plans and their analysis are in **Appendix F** in the attached CD.

11.2. Illustration of my analysis of Pre- and Post-PD Lesson Plan

11.2.1. Teacher 3

11.2.1.1. Descriptive analysis of Teacher 3's Pre-PD lesson plan

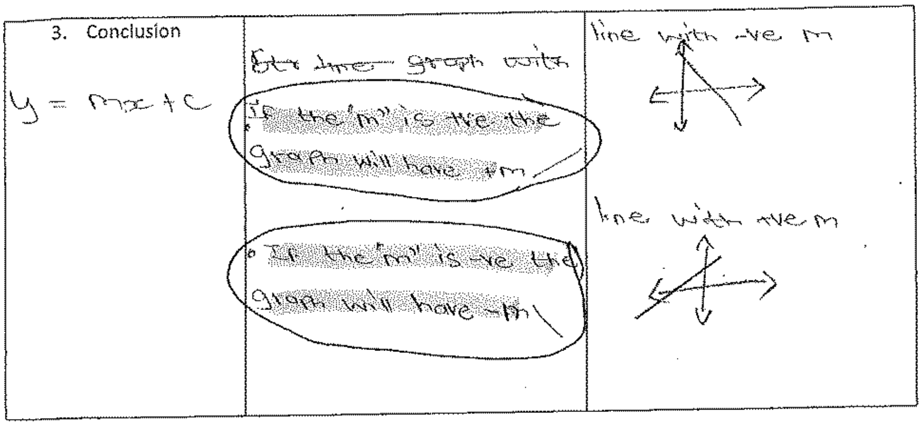
Excerpt 11.1 show Teacher 3's Pre-PD lesson plan. Part of the data in the lesson plan is shaded and also encircled. This particular data constitutes regions in which the teachers' privileged features and key features coincided. It must be noted that the number of regions does not necessarily suggest the number of key features in the plan because the same key feature might appear in several regions. Similarly with privileged features. There are seven regions in the lesson plan and these together were directed around two main foci that included gradient and y-intercept.



Lesson Progression	Teacher Activity	Leaner Activity
1. Introduction •What is the linear graph?	$y = mx + c$ OR $y = ax + q$ What is the meaning of m & c/q?	$y = mx + c$ m → gradient of the graph c/q → y-intercept
2. Presentation 3 ways to sketch a str line graph • Dual intercept method • Table method • Gradient-intercept	•x & y values can change while "m" and "c/q" are constant values. •To sketch a str line graph, the values of "m" & "c" have to be known.	eg. $y = 2x + 3$ (using the dual) y-intercept x-intercept $y = 0$ $x = -\frac{3}{2}$ Plot eg $y = -2x - 2$ (using gradient method) $m = -2$ eg. $y = x - 2$ x: -1, 0, 1 y: 3, 2, -1

Excerpt 11.1: Teacher 3's Pre-PD lesson plan

The seven regions included the two that appear in **Excerpt 11.2** below. **Excerpt 11.2** focused on the effect of the gradient on the shape of a linear graph.



3. Conclusion $y = mx + c$	~~Str line~~ graph with •If the 'm' is +ve the graph will have +m / •If the 'm' is -ve the graph will have -m \	line with -ve m line with +ve m

Excerpt 11.2: Continuation of Teacher 3's Pre-PD lesson plan

In the interview I asked teachers to reflect on their lesson planning. In the following excerpt Teacher 3 talked about the differences between her Pre- and Post-PD lesson plans. In particular, she said while the focus of the first lesson plan was on the shape of the parabola her second planning focused on domain and range. She points out that the first planning did not foreground the definition, but following her learning in the PD she was now able to work from the definition to domain and range. She noted that she has not been paying much attention to domain and range, and as a result her learners found domain and range challenging at Grade 11 – 12. Now, both definition and domain and range were part of the WMCS key features.

512 513 514	T3:	Oh, this one is the first one. Ok. Purpose of the lesson: To teach learners about the shape of the graph when the graph is positive and negative. Ok, can you see here I'm talking about the graph. Can you see here I'm talking about the domain and the range.
517 518 519 520 521 522 523 524 525		Once I introduce the functions so that the learners they know what is a function, then I can go to domain and the range. I don't rush to the graph. Here – can you see here – I was also, after, maybe I didn't talk about the definition of the function, I just talk about the shape of the graph. Can you see that? After that session I learned a lot. Can you see I first define it, then after the definition I go to the domain and then the range because once I get there I'm not going to talk more about the domain and the range, that is why in grade 11 and 12 they are also struggling. They are still struggling about the domain and the range because we didn't talk much about it.

Excerpt 11.3: Teacher 3's talk about her Pre- and Post- lesson plans

The teacher's talk about the two lesson plans seem to agree with what is present in the two lesson plans. Evidence abounds in either plan to support her utterance about the difference between the lesson plans. In addition, what the teacher said was the focus of the Post-PD resonates well with what the teacher said she learnt from her participation in PD activities (see **Chapter 10, Excerpt 10.20, lines 466 – 470**).

11.2.1.2. Interpretive analysis of Teacher 3's Pre-PD lesson plan

Teacher's privileged features: Teacher 3 has chosen to privilege (i) sketching a linear graph – using three different approaches (ii) investigating shapes of linear and (iii) meaning of parameter 'm' and intercept 'c'. In particular, the first feature involves the use of dual intercept; Table and Gradient-intercept methods in the sketching of linear graphs, and to achieve this the teacher provides three linear equations, with each equation having its own method. The second feature is where the teacher intends for learners to see how the value of 'm' impact on the shape of the graph. The teacher intend for learners to discern that when the gradient is positive the graph slants towards the right and to the left when the gradient is negative. The third privileged feature is where the lesson focus is on the meaning of 'm' and 'c' in the equation $y = mx + c$.

Space of Variation: Teacher 3's lesson plan opens three dimensions of variation where the first dimension is on how to use the notion of $y = 0$ to find the x coordinate; and $x = 0$ to find the y coordinate to find two points where the linear graph intercept the y and the x-axes. These two points are useful for construction of a linear graph. The use of a table of values from which ordered pairs are obtained and plotted on the Cartesian plane to construct a graph and the Gradient-intercept are components of this dimension, and when all the three are viewed together, constitute variation on methods. The second dimension of variation is on the effect of 'm' on the shape of the graph, and this dimension constitutes variation on the sign of 'm'. The third dimension is concerned with the essence of parameters 'm' and 'c' in a linear equation.

Table 11.1: Analysis of Teacher 3's Pre-PD lesson plan

Teacher 3's Pre Plan	Intended Object of Learning The Linear Function: $\boldsymbol{y = mx + c}$	
Teacher 3' s privileged features	**Space of variation**	**WMC's key Features/ Predetermined typologies** **(Domain/Range; Multiple Representations; Parameters; Intercepts)**
Three approaches to Sketching a linear graph -Dual intercept -Table method -Gradient -intercept Sketch of each method is provided in the lesson plan	1. Dimension of Variation Example -Dual intercept method: $y = 2x + 3; y = 0; x = \frac{-3}{2}$ -Gradient-intercept $y = -2x - 2$ -Table method $y = x - 2$	Intercepts - To be used in the sketching of linear graphs
Shape of linear graph when gradient is positive & negative	2. Dimension of Variation Example: - If m is (+ve) the graph will slant to the right - If the m is (–ve) the graph will slant to the left	Parameters: - Negative & positive gradients used to determine the shape of the graph
Learners will be asked the meaning of m, and c in y = mx + c	3. Dimension of Variation - Meaning of m, and c -M is gradient -C is y-intercept	Parameters Meaning of 'm' and 'c'

WMC-Key features: There are two key features discernible in the lesson plan, and these are (i) intercepts and (ii) parameters. Intercepts are present where they are used in the sketching of the graph and this is visible in column 2 in the first dimension of variation. The second key feature is visible where the effect of 'm' and the meanings of 'm' and 'c' are made mention of in the second and third dimensions of variation.

11.2. 1.3. Descriptive analysis of Teacher 3's Post-PD lesson plan

The Post-PD lesson plan shown below contains eight shaded and enclosed regions where privileged and key features coincide. The shaded and enclosed regions have definition of function, multiple representations, definition of domain and range, and ordered pairs – i.e. mapping (see **Excerpts 11.4 and 11.5**).

Lesson Progression	Teacher Activity	Leaner Activity
1. Introduction Definition of a function	Function can be represented in several ways: › Mapping diagram › table › graphically › Modelling/word problems	Giving some of the example about the function.
2. Presentation eg x y eg $y = x+1$	Define Domain & Range • give the ordered pairs eg what is the domain from the graph • what is the range from the graph	eg $(-2;4)(-1;1)(0;0)$ eg $D = x \in \mathbb{R} / (-\infty; \infty)$ $R = y \in \mathbb{R} / (-\infty; \infty)$

Excerpt 11.4: Teacher 3's Post-PD lessonplan

The concluding section of the lesson was also charaterized by domain and range, with particular focus on the defintion of a linear function

3. Conclusion	When the linear graph Domain is always $-\infty$ to ∞	Domain is the x-values Range is the y-value

Excerpt 11.5: Continuation of Teacher 3's Post-PD lesson plan

11.2.1.4. Interpretive analysis of Teacher 3's Post-PD lesson plan

Teacher 3's privileged features: The features that have been privileged by the teacher in her post-lesson plan include; (i) Definition of function; (ii) multiple representations; and (iii) domain and range. The lesson is to begin with a definition of function, and this is to be followed by multiple representations of function to show the different ways in which a function can be presented.

Space of variation: Teacher 3's space of learning is constituted by four dimensions of variation, and where the dimension focuses on definition of function. In this dimension the teacher defines a function, and this is followed by provision of examples of functions. This is done using the multiple representations (mappings; table; graphic; and verbal) of functions, and learners are to give examples of functions. In the second dimension the teacher defines domain and range, and gives an example of domain and range from a linear function. **Excerpt 11.4** contains information from the actual Post-PD Lesson plan that supports assertions on the first and second dimensions of variation described above. The third dimension focuses on ability to discern domain and range from the multiple representations (mapping diagram; algebraic; ordered pairs; and graph) of a linear function. In the fourth dimension of variation the focus is on using a given representation to obtain the domain and range and the teacher began with graphical representation**. Excerpt 11.5** is another excerpt from the actual lesson plan, where the lesson plan makes reference to aspects mentioned in the third and fourth dimensions.

Table 11.2: Analysis of Teacher 3's Post-PD lesson plan

Teacher 3 Post Plan	Intended Object of Learning **Topic: Linear Function $\boldsymbol{f(x) = ax + q}$**	
Teacher 3' s privileged features	**Space of variation**	**WMC's key Features/ Predetermined typologies** **(Domain/Range; Multiple Representations; Parameters; Intercepts; Definition)**
Definition of a function	1. Dimension of Variation - Teacher defines a function	**Definition**
Multiple Representations	- Representing functions in several ways: (i) Mapping diagram (ii) Table; (iii) Graphically; (iv) verbal 2. Learners give examples of functions	**Multiple representations**
Defining Domain and Range	3. Dimension of Variation - Teacher define the Domain Range as follows: - $Domain: x \in R/(-\infty;\infty)$ $Range: y \in R/(-\infty;\infty)$	**Domain and Range**
Domain and Range from different representations	4. Dimension of Variation - Teacher produces different representations of functions showing domain and range	**Multiple representations /Domain and Range**
Finding domain and Range from given representation	5. Dimension of Variation - From a given graphical representation learners respond to questions: (i) what is the domain from the graph? What is range from the graph?	**Domain and Range**

WMCS key features which are present in the post lesson plan are definition; multiple representations; multiple representations involving domain and range. Multiple representations are dominant in the lesson plan. An excerpt from Teacher 3's Post-PD lesson plan has traces of these key features.

11.2.2. Teacher 5

11.2.2.1. Descriptive analysis of Teacher 5's Pre-Lesson Plan

There are four shaded or encircled regions in the lesson plan (see **Excerpt 11.6**). The first region focused on parameters (i.e. a, b, and c). The second also focused on parameters at the turning point (i.e. p and q). The third was on defining line of line of symmetry, and the last focused on intercepts.

LESSON PLAN

GRADE: 10																	
LOs	1			2				3				4					
ASs	1.1	1.2	1.3	2.1	2.2	2.3	3.1	3.2	3.3	3.4	4.1	4.2	4.3	4.4	4.5	4
TOPIC: FUNCTIONS – PARABOLA							DURATION:						DATE: 04/05/2011				

	TEACHER ACTIVITIES/CONTENT	LEARNER ACTIVITIES	RESOURCES
	• The shapes of a parabola ie. ☺ or ☹ • U +a and ∩ −a	• Practise free hand drawing of the shape of a parabola both +ve or −ve.	CHARTS NEWSPAPERS TEXTBOOK
	• The "a" is from the standard equation: $y = ax^2+bx+c$ / $y = a(x-p)^2+q$	• Standard quadratic eqn is taken down and from eqs they practise to identify a, b and c.	MAGAZINE CASS DOCUMENT/ EDU. SOFTWARE
	• (p; q) are co-ordinates of the turning points, found by $x = -\frac{b}{2a} = p$	• Practise determining the turning point of a parabola given the equation.	INDUSTRY CASE STUDY LOCAL PEOPLE
	• Parabola's have a line of symmetry given by $x = p$ and it cuts the graph at the turning point.	• Define line of symmetry and one learner demonstrates/draws it on the board.	COUNCIL VIDEO SPECIALIST
	• Parabola is a graph you get from a quadratic equation ie. an equation with the highest exponent of 2 on the variable.	• Draw/sketch a parabola showing all intercepts and give equation of line of symmetry.	G.O.L OHP RADIO

Excerpt 11.6: Teacher 5's Pre-PD lesson plan

In the interview I asked the teacher to talk about the two lesson plans. In her talk about the difference between the two lesson plans, Teacher 5 pointed out that the former was teacher-centered lesson in which learners were listening to a lecture. She said the latter lesson plan was learner-centered, particularly with the introduction of the card matching activity. The teacher also pointed out that while the Pre-PD plan foregrounded parabola with respect to turning point and shape, the Post-PD was focused on what was important to know about parabola. The latter also emphasised working back from graph to equation.

568	T5:	The original lesson plan, this is a university lecturer. Stands on the podium, give
569		content and give.
571		This is more of teacher centred.
573		It's like me giving information and learners writing, like secretaries.
575		Ok. Yes, there are activities here and there which learners will make use of. In the
576		second one you'll find that now the card exercise is coming in. That's bringing in
577		the learner centred approach. And this, the original, talks of the parabola in general
578		– what's a parabola, how to find the turning point, what's the shape? Already in the
579		first part I'm talking of smiley face and sad face, you know, the equation - all that
580		kind of stuff. Yet in here you identify specifically what's the most important things
581		in a parabola? The turning points? The intercepts? All that. And also, if I'm not
582		mistaken, this one also goes to the extent of determine the equation given the graph.

Excerpt 11.7: Teacher 5's talk about her Pre-and Post PD lesson planning

Suffice it to point out that in **Chapter 10** there was an instance in which this teacher talks about the impact of card matching activity on her learners (see **Excerpt 10. 9 lines 280 – 281**).

11.2.2.2. Interpretive analysis of Teacher 5's Pre-PD lesson plan

Teacher 5's Privileged content: There are four content areas in the teacher's Pre-lesson plan, and these include attention on (i) Effect of 'a' on the shape of a parabola; (ii) Coordinates of a turning point; (iii) Line symmetry; and (v) the nature of a parabola.

Space of variation: Each content area has contributed one dimension of variation on the object of learning. Consequently, there are four dimensions of variation emerging from the Pre-lesson analysis. The first dimension of variation is where attention is on the shape of the parabola as a result of the 'a'. The lesson is focused on the direction the parabola will face when 'a' is positive and when it is negative. In addition, learners are given opportunities to draw both the positive and negative parabola. In the second dimension the lesson focuses on identifying 'p' and 'q' – two parameters that determine the turning point of a parabola. The focus of the third dimension of variation is on defining the line of symmetry of a parabola and finding its equation which is in the form; x = p. Sketching and identifying intercepts constitute the fourth and last dimension from Teacher 9' Pre-lesson plan.

WMC – Key features: Column 3 of the Pre-lesson plan contains only three key features of functions and these are parameters, definition and intercepts.

Table 11.3: Analysis of Teacher 5's Pre – Lesson Plan

Intended Object of Learning Topic: Function Notation $\boldsymbol{f(x) = ax + q}$ Teacher 5 Pre Plan		
Teacher 5' s privileged content	**Space of variation**	**WMC's key Features/ Predetermined typologies** **(Domain/Range; Multiple Representations; Parameters; Intercepts)**
- Shape of a parabola	1. Dimension of Variation - Parabola facing up (+a) - Parabola facing down (-a) Free hand drawing of both positive and negative parabola by learners	
- Relationship between $ax^2 + bx + c$ and $a(x - p) + q$	2. Dimension of variation Identifying a, b and c	**Parameters 'a', 'b' and 'c'**
- Turning point of parabola	3. Dimension of variation Determining the turning point	**Parameters 'p' and 'q'**
- Coordinates of turning point	4. Dimension of Variation - Identifying parameters that give turning point; (p ; q) $x = -\frac{b}{2a} = p$	**Parameters 'p' and 'q'**
- Line of symmetry	5. Dimension of Variation - Definition of line of symmetry - Line of symmetry given by x = p	**Parameters 'p'** **Definition**
- Nature of parabola	6. Dimension of Variation - Sketching parabola - Identifying intercepts	**Intercepts**

11.2.2.3. Descriptive Analysis of Teacher 5's Post PD Lesson Plan

There are ten regions on the Post-PD lesson plan where privileged features and key features coincide. There are four main foci of the lesson plan, and these are definition of function, intercepts, multiple representations and parameters.

Lesson Progression	Teacher Activity	Leaner Activity
1. Introduction	• A parabola is a name given to graphs of quadratic functions. • Use the table method to draw a graph of $f(x) = x^2$. • Highlight the key points of the graph ie. (t.p ; x & y-intercepts)	• Listern and take down notes. • Deduce table of values for $f(x) = x^2$. • Plot the graph using the table method. • Identify the key points of the graph.
2. Presentation	• Turning point => $x = \frac{-b}{2a}$, x-intercept => y = 0, y-intercept => x = 0. • Give 9 cards with graphs and some with the corresponding equations. NOTE: the graphs only have the t.p and intercepts. • $f(x) = (x - x_1)(x - x_2)$ where x_1 and x_2 are x-intercepts • $f(x) = ax^2 + q$ where t.p = (0; q) and use another point on the graph to determine a.	• Answer questions on intercepts and take down the formula for the t.p. • Match each graph to the correct equation. • Draw graphs of equations with missing graphs. Determine the equations of graphs with missing equations, given only the intercepts. Determin the equations of graphs with missing equations, given the turning point and another point on the graph.

Excerpt 11.8: Teacher 5's Post-PD lesson plan

11.2.2.4. Interpretive analysis of Teacher 5's Post-PD lesson plan

Teacher 5's privileged content: There are six content areas which Teacher 9 has selected to focus her Post-lesson on. These content areas are; (i) Explaining a parabola; (ii) Use of table method to draw graphs; (iii) Key points to learn about a parabola; Card matching- with both representations known; (iv) Card matching – with one representation known; (v) relationship between quadratic equation and parabola.

Space of variation: There are six dimensions in Teacher 9's Post-lesson plan. In the first dimension the teacher gives an explanation of a parabola, and learners take notes on the explanation. The teacher uses table method in the second dimension to draw a quadratic graph of $f(x) = x^2$, and learners use the same to plot a graph of a quadratic function. Learners are also to deduce a table of values from the $f(x) = x^2$. The third dimension is where the teacher call learners' attention to the turning point; x-intercepts; and y-intercept which she considers as key points in a parabola. The teacher has selected card matching with both representations known as the focus of the fourth dimension of variation. Learners are provided with some cards containing algebraic and others graphical representations to match. Graphical representations will be containing information about the key points of a parabola. The fifth dimension of the lesson also focuses on card matching, but with one representation unknown. Here learners obtain x-intercepts from given graphs and substitute them into $f(x) = (x- x_1) (x – x_2)$. In addition, the fifth dimension pays attention to using the turning point of a graph, and a second point from the graph to determine missing equations. In the sixth dimension, attention is on the value of ‘a’ and how this value affects the shape of the graph. According to the teacher’s lesson plan a negative value of ‘a’ produces a parabola with a sad face and a positive value of ‘a’ produces a parabola with smiley face.

WMC- Key features: There are four key features of the WMC present in Teacher 5’s Post-lesson plan. First is the definition, and this is implied where the teacher has chosen to give an explanation of a parabola. The second key feature is the intercepts, which are present where the lesson focuses on the teacher’s key points (x-intercepts; y-intercept; and turning point). The third key feature is multiple representations, which is discernible in dimensions 4 and 5 which focus on card matching. The fourth key feature from the WMC present in the lesson plan is parameter ‘a’. Focus on parameter ‘a’ is evident in the sixth dimension where the focus is on the shape of the parabola when parameter ‘a’ is positive and when it is negative.

Table 11.4: Analysis of Teacher 5’ Post PD– lesson plan

Teacher 5 Post Plan	Intended Object of Learning **Topic: Functions: Parabola**	
Teacher 5’ s privileged features	**Space of variation**	**WMC’s key Features/ Predetermined typologies** ***(Domain/Range; Multiple Representations; Parameters; Intercepts)***
Explaining a Parabola	1. Dimension of Variation - Teacher explain a parabola - Learners listen and take down notes	**Definition**
Use of table method to draw graphs	2. Dimension of Variation - Teacher uses table method to draw a graph of $\boldsymbol{f(x) = x^2}$ - Learners use table method to plot the graph - Learners deduce table of values for $\boldsymbol{f(x) = x^2}$	
Key points to learn about a Parabola	3. Dimension of Variation - Teacher highlights key points about the graph – turning point is where $x = \frac{-b}{2a}$; - x-intercepts is where $y = 0$; and y-intercepts is where $x = 0$ - Learners identify key points from the graph	**Intercepts** - X and Y intercepts are considered key points in learning about Parabola
Card matching - where both representations are known	4. Dimension of Variation - Teachers issues cards containing graphical and algebraic representations. Graphs contain information about their turning points and intercepts. - Learners match corresponding representations.	**Multiple Representation** - Matching representations which are both present in the given cards.

Card matching - where one representation is known	5. Dimension of Variation - Learners determine missing equations by substituting the x- intercepts of given graphs into $\boldsymbol{f(x) = (x - x_1)(x - x_2)}$. - Learners determine equations of missing graphs by using turning point **(0, q)** and another point from the graph to find '**a**' using $\boldsymbol{f(x) = ax^2 + q}$.	**Multiple Representation** - Determine the missing corresponding representation
Every quadratic equation gives a parabola	6. Dimension of Variation - A parabola can be a sad or smiley face depending on the value 'a'; $\boldsymbol{f(x) = -ax^2}$ **is a sad face** $\boldsymbol{f(x) = +ax^2}$ **is a smiley face**	**Parameter** - Parameter 'a' is used to describe a parabola

11.3. Summary of the analysis of teachers' lesson planning

11.3.1. Teacher 1

11.3.1.1. Interpretive analysis of Teacher 1's Pre-PD lesson plan

Teacher 1's privileged features: In his Pre-PD lesson plan Teacher 1 selected four aspects of functions to focus the lesson on (see **Appendix F**). These are (i) calculation of gradients; (ii) finding the y-intercept; (iii) finding the x-intercept; and (iv) drawing of line graphs.

Space of variation: In the space of learning there were four dimensions of variation that were opened up. In the first dimension the lesson plan focused on calculating gradients to check previous knowledge. The second dimension of variation was concerned with finding the y-intercept. To do this, the value of x was set at 0 – (i.e. x = 0) and this value was then substituted into a linear equation and by so doing it was possible to find the y-value. In the third dimension of variation the lesson focused on finding the x-intercept – a value of x where the linear graph crossed the x-axis. The fourth and last dimension of variation focused on the use of two points to construct line graphs.

WMC Key features: Two key features were present in Teacher 1's Pre-Lesson Plan, and these included (i) Parameter- Gradient and (ii) Intercepts - both x and y-intercepts were included.

In the following excerpt Teacher 1 talks about the foci of the two lesson plans. In the talk the teacher points out that the first lesson plan focused on the gradient and y-intercept. He contrasted this focus with that of the Post-PD when he said following his participation in the PD activities he realized the importance of showing connections between the representations. Hence his Post-PD lesson plan focused on the relationship between representations.

T1	Ok, the first one I think the emphasis is more like on the, on the determining, or determination of the gradient. And ja, from there, from there calculating the y intercept so that the learners are able to maybe relate the function with the, with the graph. But then I think what I would say I found it different maybe after the theme on functions is the fact that, I mean, when I developed this lesson plan here I think the connection, I wasn't really… The focus, you know, between those different representations, wasn't really there. Ja. Because I was looking at maybe like let's say the calculation of a gradient, you know, or maybe the finding of the intercepts and everything. You look at this, you know, sort of like in isolation, whereas with the second lesson plan, I mean if the learners have to plot the graph from the table I mean you actually, you are showing them the link or rather the relationship that ok, there is this table. I mean it's representing specific, you know, ok the table and the graph is one and the same thing, so there has to be a relationship between the table and the graph, alright. I mean that link or the different forms, you know, of functions I think for me that is what came out most in the second, you know, session, or rather the second lesson plan…

Excerpt 11.9: Teacher 1's talk about his Pre & Post PD lesson plans

It is interesting that the teacher's utterance about the Post-PD is not different from what he said he learnt in the PD (see **Chapter 10, Excerpt 10.5**).

11.3.1.2. Interpretive analysis of Teacher 1's Post-PD lesson plan

Teacher 1's privileged content: Teacher 1 has privileged three content areas for discussion in the Post-PD lesson plan: (i) Plotting graphs of linear functions, (ii) multiple representations and (iii) Exchange rate (see **Appendix**).

Space of Variation: The space of variation is constituted by three dimensions of variation. In the first dimension learners are required to work in pairs to plot points on graph papers to produce graphs of linear functions, and this is accompanied by emphasis on the use of appropriate scales. The second dimension is where multiple representations will be brought to learners' awareness and in the third dimension of variation learners find the Rand/ Dollar exchange rate.

WMC-Key features: The only key feature of functions which is present in the teacher's Post-PD lesson planning is multiple representations. This is not surprising because in his talk about his learning in PD activities where these were in focus he pointed out that working with multiple representations was refreshing, wonderful, and an out-of-the-box activity. **Excerpt 11.9** contains the teacher's talk about the difference between the two lesson plans.

11.3.2. Teacher 2

11.3.2.1. Interpretive analysis of Teacher 2's Pre-PD lesson plan

Teacher's privileged features: Teacher 2 privileged (i) identifying exponential from non-exponential functions; (ii) line of asymptote (iii) y-intercept; (iv) sketching exponential graphs and (v) transformation of exponential functions.

Space of variation constitutes what is possible to learn on implementation of the Pre-lesson plan, and is characterized by five dimensions of variation. The first dimension of variation points to developing learners' capability to distinguish exponential from non-exponential functions. The second dimension focuses on the ability to find the line of asymptote, and one example the teacher brings to focus is y = 0. The line y = 0 is the equation of an asymptote to the graph of $y = 2^x$, and this is evident in the Pre-PD- Lesson Plan (see **Appendix F**). The teacher facilitates this by providing an exponential function graph of $y = 2^x$ from which it is possible to discern the line of asymptote.

The third dimension of variation is the y- intercept at (0, 1.) (i.e., where x = 0). This is a point where the exponential graph of $y = 2^x$ cuts the y-axis. This point is useful in the sketching of the exponential graph. Here the teacher brings into prominence the value of b when it is greater than 1 ($b > 1$) and when it is between 0 and 1 ($0 < b < 1$). This is accompanied by provisions of corresponding graphs that illustrate the nature of the graph when $b > 1$ and when $0 < b < 1$.

The fourth dimension of variation is the transformation of a specific exponential function (i.e. $f(x) = 3^x$). The graph of the function is to be subjected to several transformations that include vertical shift down ($y = 3^x – 3$); reflection along the y-axis ($y=3^{-x}$); reflection along the x-axis ($y = - 3^x$); and combined transformation constituted by reflection along the y-axis, followed by a vertical shift up ($y = 3^{-x} + 2$).

WMCS Key features: There are two key features of functions in the lesson plan: (i) ***Intercept*** and (ii) ***Parameter*** (different values of b, i.e. $b > 1$ and $0 < b < 1$ are brought into focus.

In the following utterances Teacher 2 talks about differences between his lesson plans

T2:	Alright. The first one… The first one I did not emphasise much on the, on the presentations, on those different presentations. Ja. The first one was more like a normal, you know, a flow of where I have an algebraic function. From there I have a table, I draw the graph, I give learners an activity which is almost similar to what I have taught, ok. But after we had had those sessions and those workshops I realised that no, its not always possible. Ja, I should always start it from even the verbal representation itself, then from the verbal learners would be able to come up with the algebraic form of the function and end up with a graph. Or else I would start with the graph, from the graph they should be able to come up with the algebraic form of what – of the function - and also even come up with the table, as is the last one. But really, that’s how this is if you read this you can see that

Excerpt 11.10: Teacher 2’s utterance about the Pre & Post lesson plans

In the excerpt the teacher points out that he did not foreground multiple representations of functions in the Pre-PD lesson plan as he did in the Post-PD lesson plan. He described the first plan as a normal lesson plan. He described ‘normal’ as moving from the equation to table to graph. However, after participating in PD activites he reaslised the need to start from any representation, such as verbal descriptions or graphs. As a matter of fact, in his talk about his learning in the PD activitues he said he that it was important to always show connections between multiple representations (see **Chapter 10, Excerpt 10.6** and **10.18 lines 271 - 281**).

11.3.2.2. Interpretive analysis of Teacher 2’s Post-PD lesson plan

Teacher’s privileged features. The teacher has selected to focus the lesson on (i) evaluating exponents; (ii) definition of function; (iii) multiple representations – (from equation to table to graph; (iv); (v) multiple representation – (from graph to table to equation); and (vi) line of asymptote

Space of variation: The teacher broke down the privileged content into small chunks and each chunk focused on a unique aspect of the content. The first dimension of variation foregrounds capability to evaluate exponents. In the second dimension of variation we see the teacher privileging the definition of an exponential function. Moving on to the third dimension the teacher begins with $y = 5^x$and wants the equation to be represented into table of values and graph. The fourth dimension of variation is the reciprocal of the third. Here the plan begins with a graph and moves back to table of values and to equation. Specifically, points are to be obtained from the graph to form a table of values. Thereafter the points are to be used to find the value of $\boldsymbol{a}$ in $y = \boldsymbol{a}^x$. The fifth dimension of variation is an exploration on the relationship between f(x) and x. In particular, the teacher intends for learners to determine whether f(x) decreases as x increases. The sixth dimension is about stating the equation of asymptote.

WMCS Key features: The key features of functions which are evident in the teacher's Post-Lesson Plan include (i) definition of function; (ii) multiple representation; and (iii) parameter 'a' **(see Appendix F)**. For instance, in the lesson plan there is a statement where the teacher says he will define the exponential function. The parameter '***a***' is also mentioned explicitly in this lesson plan, and it is to be calculated using points (-2, a); (-1,3); (1,$\frac{1}{3}$); (0,1), which are also present in the lesson plan. The multiple representations are present in the Post-PD lesson plan and in the interview conversations when the teacher talks about the differences between the Post and Pre –PD lesson plans. In the Post-Lesson plan the teacher provides a graph with three points clearly written, and the fourth one (0, 1), the intercept point, can be read off the graph. All the four points are listed in this paragraph above. The teacher makes reference to the point that learners will read off points from the graph. Second, the points will be used to produce a table of values and thirdly, the table of values will be used to calculate parameter '***a***' in $\boldsymbol{y} = \boldsymbol{a}^x$ which will make known the actual algebraic equation of the exponential function. The teacher wants learners to work backwards from graph to table and to algebraic equation. Hence multiple representations are present in the lesson plan in the sense that there is a graph to begin with, a table is generated from the graph, and an equation is formed from the table. The inclusion and emphasis on multiple representations and nature of definition of functions seem to be the two main features setting apart the two lesson plans. My in-depth interview with the teacher where I asked the teacher to explain the differences between the two and how the differences might improve the teaching of the lesson, reveals more on the centrality of multiple representations on the post lesson plan (see **Excerpt 11.10**)

11.3.3. Teacher 4

11.3.3.1. Interpretive analysis of Teacher 4's Pre-PD lesson plan

Teacher's privileged features: In her planning Teacher 4 selected two areas to focus her teaching. The areas contained (i) Domain and Range and (ii) Function notation.

Space of variation: In the plan there are three dimensions of variation that constitute the space of learning. The constituents of the first dimension are on the relationship between domain and range followed by substitution of inputs to obtain outputs. The second dimension of variation is focused on function notation and its application (see **Appendix F**).

WMC Key features: This teacher's Pre-lesson plan contains two features related to what the WMC-S project considers important in the teaching of functions. Domain and Range is a key feature to teach, together with the use of function terminology such as input and output. The WMCS sees the use of correct mathematical language in the teaching of mathematics as critical.

The following excerpt contains the teacher's talk about the differences between the Pre- and Post-PD lesson plans.

695 696 697	T 4:	Ok. Ok the first lesson plan it's demonstrate the relationship between range and domain and how to find the range given domain and substitute the different input values to get the output values.
701 737 738 739 740 741		And… The second lesson plan is about different representation methods. Yes. In the second lesson plan the focus is now on different representation methods. The flow diagram, graphical table values, algebraic expression and verbal and also input and… input, output, domain, range, dependent and independent values together with the function notation. And in the first one it was just function notation, domain and range, input and output.
746 747		I think after, after attending the workshops now I know what learners are supposed to know on functions, what is important on functions, yes.

Excerpt 10.11: Teacher 4's utterance about lesson planning

In the excerpt the teacher says she is now aware of what is important to know about functions. She mentions multiple representations as the focus of the Post-PD lesson plan and also as what is important to know about functions.

11.3.3.2. Interpretive analysis of Teacher 4's Post-PD lesson plan

Teacher 4's privileged content: The Post–PD lesson plan of Teacher 4 contains four content areas, and they include (i) Multiple representations (ii) Relationships between output, dependent and range; (iii) Relationship between input, independent and domain; (iv) function notation; and (v) definition of function.

Space of variation: There are 7 dimensions of variation in the teacher's lesson plan. In the first dimension the teacher chose to focus the lesson on the different ways of representing functions, and these include using (i) flow diagrams; (ii) graphs; (iii) table of values; and (iv) verbal representations.

The second dimension of variation focuses on learners working in pairs to make flow diagrams and table of values from given algebraic equation. In the third dimension the terms: input, independent, domain, x-values, y-values, independent and range are explained, and learners sort them into two groups.

The fourth dimension is focused on the use of Function Notation. There are not enough examples in the lesson plan that explain or show how this will be done by the teacher or learners. However, the teacher intends to show learners how to substitute into the Function Notation. The fifth dimension of variation is where the teacher defines the term Function.

WMC- Key features: Three key features are embedded in the post lesson plan of Teacher 5. Here we see evidence of (i) multiple representations; (ii) function terminology is discernible in the third and fourth dimensions; and (iii) definition of function. Teacher 4 explains the differences between her lesson plans in the next utterance.

11.3.4. Teacher 6

11.3.4.1. Interpretive analysis of Teacher 6's Pre-PD lesson plan

Teacher's privileged features: The teacher privileged four content areas in her Pre-PD lesson plan, and these are (i) coordinates; (ii) Cartesian plane; (iii) plotting points; and (iv) joining of points to produce a graph

Space of variation: The space of learning in Teacher 6's lesson is constituted by four dimensions of variation. First, the lesson plan begins with identifying the x- and the y-coordinates from an ordered pair. The second dimension of variation focuses on construction of the Cartesian plane. The third dimension is on plotting points. The fourth dimension of variation points is on joining points to produce a linear graph.

WMC-key features: Teacher 6's Pre-PD lesson plan does not contain any key feature of functions and in WMCS PD terms, the lesson plan does not contain what is important to learn about functions. The absence of key features is also evident in the teacher's talk about the difference between the Pre- and Post-PD lesson plans (see **Excerpt 11.12**)

T6:	So on the first one I didn't emphasise on the table, ok. Then on the second one I even drew the table to say learners and the relationship. On the first one I didn't even have a relationship. There was no relationship of like giving the learners a relationship there. And then on the second one there was a relationship given and a table given and even more explanation of how learners were going to find the value of x, of y. You know, they were going to do the substitution, ja.

Excerpt 11.12: Teacher 6's utterance about lesson planning

According to the teacher, the difference between the two plans was the absence of table of values in the former plan while the latter had a table of values.

11.3.4.2. Interpretive analysis of Teacher 6's Post-PD lesson plan

Teacher 6's privileged content: In the post-lesson plan the teacher selected (i) coordinates, (ii) completing a table of values, (iii) substitution, (iv) plotting points, and (v) joining points. Altogether there are 5 selected content areas for the Post-lesson plan.

Space of variation: The first dimension of variation foregrounds knowledge of coordinates. In the second dimension of variation a table of values containing x-values is put up and the teacher demonstrates how y-values are to be obtained by substituting x-values into an algebraic equation. Learners are then to be given three equations to practice substitution. In the third dimension of variation the lesson focuses on plotting points. The fourth and last dimension of variation is on joining points to obtain graphs of linear functions.

The two lesson plans are basically the same, because they foreground plotting and joining points to produce a graph. This was what was noticed by professional developers during classroom observations prior to organizing PD for teachers. While it appears most teachers have shifted from this practice, Teacher 6 is the only one whose planning has not shifted.

WMC-Key features: In Teacher 6's Post-lesson plan we fail to see any of the Key features of functions. This is evident in column 3 of Table 13 shown above. In the following utterance Teacher 6 talks about the difference between her two lesson plans.

11.3.5. Teacher 7

11.3.5.1. Interpretive analysis of Teacher 7's Pre-PD lesson plan

Teacher's privileged features: The first focus of Teacher 7's lesson is on sketching the graph of a quadratic function. A particular focus in this lesson is for learners to be able to transform a general quadratic function in equation form ($y = ax^2 + b\ x + c$) into ($y = a(x - p)^2 + q$), a form known as completing the square, to facilitate the sketching the graph. I analyze the lesson plan by describing what is written in the columns, and I do this by discussing each column under its heading. The lesson begins with a focus on completing the square, and three different forms of the quadratic function in equation form are provided under this sub-topic of the lesson (see Appendix F).

The second is on sketching quadratic graphs using turning points; axis of symmetry and intercepts. The third focus is on the effect of parameters (***a, p*** and ***q***) on the graph.

Space of variation/learning: In this space we observe three dimensions of variation on the object of learning. The first foregrounds expressing $2x^2 +10x +12$ in the form $a(x –p)^2 +q$. The second is about sketching graphs of $f(x) = -x^2 - 5 - 6$; $g(x) = 2(x - 4) + 5$; and $h(x) = (x - 3)(x + 2$. The third dimension of variation is transformations involving the effects of parameters (a, p, and q) on the shape of the parabola.

WMC Key Features: In Teacher 2's Pre Lesson Plan there are two key features, and these include; (i) Intercepts – the intercept is implicitly present in the lesson plan; and (ii) Parameters. Below is where Teacher 7 talks about the difference between his lesson plans.

554 555 557 558 559 560 561 562	T7:	In the first lesson you are teaching them completing a square which is an algebraic manipulation. Yes. And to get it's, it's standard form. I mean a certain form of an equation which learners should be able to see that that is a parabola. And immediately using that after just completing the square, interpreting out of the equation to get the graph. And looking at the graph itself, look at the shape, look at the influence of these parameters. For example, if a is positive one should already interpret that what I'm drawing here is, it's a smile, not a sad. Ja.
574 575 576 577 578 579		And when it comes to the second one a learner should know now that to sketch a graph I can be given an equation, I can be given a table to complete. And how do I complete that table? Complete the table, can plot a graph. After plotting a graph you write the equation. Now how do you derive, get the equation? Are you able to get it from the table? Are you able to get it from the graph itself? Right?
581 582 583 584 586 587 588 589 590 591		And when you look at the second one now there is a graph. It's given to you, you have to find the equation. Is it possible to find the equation? Is it possible? Maybe, would that be possible to find the table - which most of the learners were asking me. This, this creates some sense of, you know, involvement, you see. So it's raising, you can see whoever was following what is happening, you see. But we talked about this. Am I able to… From the graph am I able to get the equation? If yes, am I able to find… the table? Am I able to find the turning, the co-ordinates of the turning point? Am I able to find the x intercepts? y intercept? So that sort of solidifies or consolidates the terminology there. Am I able to find the, the domain? Am I able to find the range?

Excerpt 11. 13: Teacher 7's utterance about lesson planning

In the talk the teacher explains that the first lesson plan focused on completing the square and using this knowledge to sketch graphs. He says the focus of the second was on working from graph to obtain equation, table of values, turning points, x- and y-intercepts, domain and range.

11.3.5.2. Interpretive analysis of Teacher 7's Post-PD lesson plan

Teacher's privileged content: In the plan the teacher chose to focus the lesson (i) using table of values to plot a graph and its equation; and (ii) obtaining equation from graph.

Space of Variation: The first dimension of variation is where the lesson plan focuses on proceeding from a table of values to graph. In the second dimension the lesson focuses on moving from table of values to algebraic equation. The third dimension of variation focuses on moving from graph to algebraic equation.

WMC Key Features: Multiple representations is the dominant and only key feature in the lesson plan. This key feature of functions appears across the three dimensions of variation. Multiple representations are discernible in a movement from table of values to graph to equation. Rather than beginning with a table, moving to plotting a graph, and using the graph to write the equation, the lesson moves from a table to graph, and also moves from the same table to equation.

11.3.6. Teacher 8

11.3.6.1. Interpretive analysis of Teacher 8's Pre-PD lesson plan

Teacher 8's privileged content: The lesson gives prominence to four content areas that include (i) completing the table; (ii) plotting points; (iii) joining points; and (iv) labeling the graph.

Space of variation: There are four dimensions of variation which constitute the space of learning. In the first dimension of variation the teacher's lesson focuses on finding corresponding values of y to complete a table of values, where values of x are already known. The process involves substituting values of x in a given linear equation. The second dimension of variation is where the lesson focuses on drawing the Cartesian plane and labeling the axes. The third dimension is focused on joining points, and this is to be done with the aid of a ruler. Labeling the graph is the focus in the fourth dimension of variation.

WMC-Key features: There are no key features of functions in Teacher 8's Pre-lesson plan.

11.3.6.2. Interpretive analysis of Teacher 8's Post-PD lesson plan

Teacher 8's privileged content: There are four focus areas that Teacher 7 has privileged in his Post-lesson plan. These include; (i) relationship between input and output; (iii) plotting points on the Cartesian plane; and (iv) identifying gradient and y-intercepts.

Space of variation: this space is characterized by four dimensions of variation. The first dimension is where the teacher brings a table of values to the fore for learners to identify inputs and outputs. The second dimension of variation is where the attention is on the teacher explaining how to find the output and range when given the input and domain. With this explanation learners find output and range to complete a table. In the third dimension of variation ordered pairs of coordinates are presented and learners plot these on the Cartesian plane. The fourth dimension of variation is concerned with teacher explanations on identifying gradient and y-intercept from a linear Function, and learners do work where they identify these to parameters.

WMC Key features: There are three key features in Teacher 8's Post PD lesson plan. These include (i) use of function terminology (i.e. input, output, range, and domain); (ii) parameters (gradient) and the (iii) intercepts (i.e. y-intercept). Teacher 8 was asked to explain the difference between his two lesson plans and this is what he said:

575 576 577 578	T8:	The first one there wasn't any use of the terminology of functions, the mathematical language of approaching and dealing with functions. But the second one, this terminology, these terms like input, range, output values of a given table of values, given the input, find the output. So that's one of the differences here.
599 560 530 531 633 637 638		What I considered to be important in this first lesson plan was the concept of substitution. In the second lesson plan the focus was the relationship between the input and the output. What was very important is for learners to see what is happening to the input, not to the output. Something is happening to the input to get the output. That's the relationship between the two. In substitution they're just taking the value and saying, 'This is this. This is this,' so they cannot easily find out what the relationship, what is happening between the two

Excerpt 11.14: Teacher 8's utterance about lesson planning

According Teacher 8, the first lesson plan did not focus on functions terminology like the second lesson plan. Still in the first lesson plan, he considered substitution as important to know about functions. In the second he saw the relationship between input and put as important to know about functions.

11.3.7. Teacher 9

11.3.7.1. Interpretive analysis of Teacher 9's Pre-PD lesson plan

Teacher's Privileged features: There were two privileged features in the lesson plan. These included (i) using table of values to draw parabola and (ii) investigating the effect of parameter 'a' on the parabola.

Space of Variation: The four dimensions of variation were (i) completing the table (ii) plotting points (iii) labeling the graph and (iv) transformation.

WMC-Key features: Teacher 9's Pre-lesson plan contains one key feature. In the lesson plan the parameter 'a' was privileged. The effect of 'a' (in $f(x) = ax^2 + q$) on the parabola is investigated by making 'a' assume different values. Varying values of 'a' might have effect on the shape, size, and direction the graph might face, and this amount to transformation on the graph. The teacher was asked to talk about his two lesson plans (see **Excerpt 10.5).**

655 656 657 658	T9:	Then also on this one, the second one, which is the post one I had to also use the issue of card matching so that they know which one is a parabola and which one is not. And they also, on the second one, I also used the issue of the verbal, the different method of representing what? A parabola. Yes.
667 668 669		On the second one now I'm telling that if a is greater than zero how does the graph look like? Is it a positive or a negative? I'm now using the language that you use positive or negative – the mathematical language.

Excerpt 10.15: Teacher 9's utterance about lesson planning

The teacher explains that he used card matching activity in the second lesson plan for learners to know what is and is not a parabola. He also used multiple representations of a quadratic function. The teacher also points out that in the second lesson plan he uses proper mathematical language like positive and negative graphs instead of sad and smiley faces.

11.3.7.2. Interpretive analysis of Teacher 9's Post –PD lesson plan

Teacher's privileged features: From **Table 10.18** it can be seen that the teacher has privileged seven content areas of the parabola to focus learner attention on. The seven include (i) drawing of parabola without table of values; (ii) gradient; (iii) computing values of functions using a calculator; (iv) drawing of graphs; (v) investigating effect of 'a' on the graph; (vi) effect of 'q' on the graph ; (vii) turning points and ; (viii) Functions and their graphs

Space of variation: The Post-lesson plan contains eight dimensions of variation on the object of learning which I am describing in this paragraph. The first dimension of variation is where company logos are used for learners to identify those that resemble a parabola and Mc Donald's company logo is among the teacher's examples. Still within the first dimension the teacher provides a list of company logos for learners to identify those that match a parabola. The second dimension is focused on finding gradients and a card matching activity focused on transformation. In the third dimension of variation the lesson focuses on the use of calculator to obtain ordered pairs for a given function. In the fourth dimension learners are to draw graphs of functions using calculator generated ordered pairs. The effect of the sign of 'a' – (i.e. negative sign) is investigated in the fifth dimension of variation and four quadratic functions have been selected for this. The selected functions are characterized by negative coefficients of 'x^2'. The sixth dimension focuses on another investigation, and this time it is on the effect of the y-intercept 'q' on the parabola of the form; $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{a}\boldsymbol{x}^2 + \boldsymbol{q}$. In the seventh dimension of variation the focus is now on the minimum and maximum and the eighth dimension of variation is where we see attention shifting onto matching functions with their graphs.

11.4. Summary of the analysis

The analysis of the nine teachers' Pre PD lesson plans indicates that 8 of the 9 teachers had the WMC-S key features present in their lesson plans. *Parameters* and *intercepts* were dominant WMC-S key features in the Pre PD lesson plans. In the Post PD lesson plans the most dominant WMCS key feature was *multiple representations* of functions, followed by *function terminology*. Teacher 6 is the only one who does not have any of the key features in her Pre and Post PD lesson plans (see **Table 11.5**).

Table 11.5: Distribution of WMC-S key features across the nine teachers

Teacher	**WMC-S key features present in Pre-PD lesson planning**	**WMC-S key features present in Post –PD lesson planning**
Teacher 1	Parameter Intercepts	Multiple Representations
Teacher 2	Intercepts Parameters	Definition Multiple Representation Parameter
Teacher 3	Parameters Intercepts	Definition Multiple Representations Domain and Range
Teacher 4	Domain & Range	Multiple Representations Mathematical Language Definition
Teacher 5	Parameters ‘p’ and ‘q’	Definition Intercepts Multiple Representation Parameter
Teacher 6	None	None
Teacher 7	Intercepts Parameters,	Multiple Representations
Teacher 8	None	Function terminology Parameters Intercept
Teacher 9	Parameter	Parameter Intercept Multiple Representations

Table 11.5 provides evidence that shows that participating in the WMC-S project's PD activities facilitated shifts in the teachers' lived object of learning. By way of example, Pre-PD lesson planning did not have multiple representations, but this key feature was dominant in the Post-PD planning. In their talks where they talk about the differences between the two planning, teachers talk more about the need to help learners discern relationships between the multiple representations. In his talk, Teacher 1 pointed out that his Pre-PD presented the representations in an isolated way, but the Post-PD brought them together in a way that illuminated the relationship between them.

Teacher 2 said after participating in the PD sessions he realised the need to change from always working from the algebraic representation "But after we had had those sessions and those workshops I realised that no, its not always possible. Ja, I should always start it from even the verbal representation itself, then from the verbal learners would be able to come up with the algebraic form of the function and end up with a graph"

Teacher 4 said "..I think after, after attending the workshops now I know what learners are supposed to know on functions, what is important on functions, yes"

Teacher 9 pointed out that he used card matching to help learners discern difference between parabola and a non-parabola. He also used it to show the different representations of the parabola. Similar viewpoints were raised by other teachers. The above sections contain each teacher's detailed utterance about his/her planning. In their utterances teachers are talking about how teaching in ways suggested by Post-PD plans would help learners learn functions better. For instance, Teacher 4 says her Post-PD lesson plan activities are learner centred and Teacher 2 talks about how his learners would work with the multiple representations. Teacher 9 also talks about showing learners difference between parabola and non-parabola. As such the lived object of learning which is exhibited by the lesson plans and supported by utterances is was a teaching object (Tm). In particular, the lived object of learning associated with teachers' capacity to incorporate WMCS PD key features in their planning was *professional learning of functions (***Mt-PL***).* It is Mt-PL because the lesson plans show evidence of shift in the object of learning, which is functions. The shift is a consequence of the realisation of what is important for learners to know about functions. In my operationalisation of the four categories I explained that the Mt object is professional if its learning is for utilitarian purposes. So lesson plans provide evidence of the teachers' professional knowledge of functions.

11.5. Conclusion

In this chapter I analyzed nine teachers' Pre- and Post-PD lesson planning to study their lived object of learning. In particular, I was interested in whether there were any shifts in what they saw as important to learn about functions. To do this, I worked both empirically and theoretically on lesson plan data to visibilze what teachers considered as important to learn about functions, and this I theorized as teachers' privileged features. Further analysis of privileged features with the notion of key features revealed that there was correspondence between the teachers' privileged features and the WMCS key features. The analysis done in this chapter, together with my interview analysis in the Chapter 10, seem to support the claim that I later make in Chapter 12 about teacher learning in an object focused PD. Suffice it to point out here that, there is a large amount of data evidencing the claim in **Appendix F**.

CHAPTER 12: CONCLUSION

12.1. Introduction

This chapter is the culmination of a comprehensive research which sought to gain insight into the nature of the enacted object of learning in an object–focused model of professional development, and what became the lived object of learning for participating teachers. The immediate goal of the PD was to help teachers to strengthen their own subject matter knowledge of mathematics in general and functions in particular. As such, to bring this research to a close I begin the chapter with Section 12.2 below, in which I describe the foci of the study by talking more about the analysis process. In 12.3 I answer the research questions, and I bring the contributions of the study to the fore in 12.4 focusing in on both methodology and literature on mathematics education, professional developemnt and teacher knowledge. In 12.5 I present my thesis – i.e. the argument of this research, after which, I discuss the implications for policy for professional development in 12. 6. Of course, all research has its limitations, hence I discuss these in 12.7 as well as what could have be done to improve the research, and new ideas for further research are discussed in 12.8.

12.2. What were the foci of the research?

This study had two main foci, and these were the enacted and the lived object of learning with more detailed analysis of the enacted object of learning. The limitations to what the study might reveal about the lived object of learning has been discussed both in this chapter and **Chapter 5**. These objects were studied in the context of the Wits Maths Connect Secondary project that worked with teachers from 11 secondary schools in one district in Johannesburg. While I have described the relation of these objects to the intended object of learning, in this research my interest was in the constitution of the enacted and the lived object of learning. Notwithstanding, the intended object of learning has been discussed as part of the context of the study in **Chapter 2**.

As noted earlier, the enacted object of learning was studied from videos of professional development where functions were in the fore, and was studied from three sites. Studying the enacted object of learning from three different instances of enactments has helped to illuminate what was key in the enactments. Therefore rich insight from all the three into the relationship between the enacted and the intended objects of learning. My study of the enacted object of learning had three foci. First focus was on the pedagogic strategies which facilitators worked with to open and/or close scaffolding on tasks. The second focus was on the nature of the legitimating criteria which facilitators and teachers used to fix the meaning (illuminate) of the enacted object of learning. The third focus was on patterns of variation which facilitators worked with to visibilize the object of learning for teachers. As noted earlier, these 3 foci were both process and product of the thesis. For instance, while pedagogic strategies and scaffolding were introduced after studying video data, the combination of legitimating criteria and patterns of variation together came to be what enabled me to describe what was made possible to learn and what was enacted. I did not set out with these three foci as clear as they seem to be here. Hence they are both process and product of the thesis, and have become the major part of my study

In order to study the lived object of learning I analyzed two sets of data. The first came from in-depth teacher interviews and the focus of my analysis of this data was on the meanings which teachers attached to their experiencing of object focused professional development. To study these meanings I recruited analytical tools from phenomenological research and I worked with these to construct each teacher's meaning structure. At the end of this analysis I pulled together the nine individual meaning structures to construct the essential invariant composite meaning structure. Knowing the essential invariant composite meaning structure was critical for answering the research question that inquired about the teachers' lived object of learning. That said, the level of the lived object in terms of what teachers say is an important portion of the analysis, but it did not tell much about their practice. What did tell something about the teachers' practice was the Pre- and Post-PD lesson plans, but these were plans of teaching and not teaching itself. Nevertheless, the Pre- and Post-PD lesson plans made it possible to observe the extent to which the key features of functions (i.e. the focus of the PD) had become part of the teachers' consciousness in terms of how they planned. Suffice it to point out here, that the data on the lived object of learning had some limitations, and I return to talk about this later in this chapter when I discuss the limitation of the study.

The second set of data came from teachers' Pre- and Post-PD lesson plans. I recruited the notion of 'space of variation' from Variation theory to map what teachers privileged in each lesson plan onto the WMCS project's key features of Functions. Essentially, I was interested in whether or not, what teachers privileged in their lesson planning, aligned with the WMCS's key features of functions. This was necessary to understand whether there were shifts in what teachers considered to be important to teach and learn about functions after participation in the PD. The increased presence of key features of functions in the Post-PD lesson plans served as evidence of teacher learning in the PD.

12.3. What are the empirical contributions of this research?

1. **What is constituted as the object of learning in object focused professional development and how is it constituted. i.e., what is the space of variation and what is the nature of the object of learning?**

   (a). **What is the primary enacted object? i.e., is it mathematics object or a teaching object, or a co-constituted object?**

Why was it important to explore the nature of the enacted object of learning? The need to explore the nature of the enacted object of learning was informed by two perspectives. Firstly, it was informed by the duality of the object of learning in mathematics teacher education, and so in PD. Although the PD was subject matter (**Mt**)-focused, it was important to explore the nature of the enacted object of learning to see the extent to which pedagogy (**Tm**) might have infiltrated the enactment and so disrupted or enhanced or expanded its message about the object. The analysis revealed that the enacted object of learning was constituted by both *mathematical reality* ($\boldsymbol{Mt_{re}}$) and *mathematical appearance* ($\boldsymbol{Mt_{ap}}$). Research (e.g. Vinner, 1983) has shown that both ($\boldsymbol{Mt_{re}}$) and ($\boldsymbol{Mt_{ap}}$) are important components of **Mt** and that one without the other is not sufficient for one to successfully learn a mathematics concept. This is evident in the lesson plans where Pre-PD lesson plans were dominated by *mathematical appearance.*

Secondly, according Variation theorists, the intended and enacted objects seldom cohere - i.e. what is planned is not necessarily what gets taught in the class. Hence it was important to explore the extent to which the enacted and the intended object might cohere. As noted above, the results of the analysis pertaining to enacted object of learning in an object focused professional development revealed that *the primary enacted object of learning* was a *mathematics object (***Mt**)*,* and this cohered with the intended object of learning, which was also **Mt**. There is evidence to support this finding across the chapters that contain the analysis of the enacted object of learning (see **Chapters 6 – 9**).

By way of example, in Chapters 6 – 7 I analysed the enacted object of learning in Tasks 1 and 2 of Functions 1. In my analyis of Task 1 I found that the legitimating criteria were characterized by *mathematics reality* ($\boldsymbol{Mt_{re}}$) and *mathematics appearance* ($\boldsymbol{Mt_{ap}}$) - which this research has shown to be the *what* of **Mt** in an object focused PD.

In addition to that, subsequent analysis on Task 1 with patterns of variation produced similar results. These results revealed that what was made possible to learn in Functions 1 and 2 was **Mt** which was predominatly $\boldsymbol{Mt_{re+}}$ and of interest in strengthening the teachers' SMK.

Similar analysis of Task 2 of Functions 1 revealed that the **Mt** was primarily $\boldsymbol{Mt_{re}}$ and with the $\boldsymbol{Mt_{ap}}$ seemingly absent. Now, does this mean that the $\boldsymbol{Mt_{ap}}$ were not present in Task 2? The response is that the $\boldsymbol{Mt_{ap}}$ were present except that they were not in the public space. The absence of $\boldsymbol{Mt_{ap}}$ in Task 2 might be explained in three possible ways: Firstly, in Task 2 teachers worked in groups and the Facilitator moved from one group to another providing scaffold and the consequence was inaudible discussions between teachers and between teachers and the Facilitator. This meant that legitimating criteria in which the $\boldsymbol{Mt_{ap}}$ were probably located were not clearly captured in the video recordings. Secondly, there were times when the video camera was switched off when teachers were working in groups on the task and later switched on when the discussions went back to the public space at the end of the Task for the Facilitator to summarise the work done. The Facilitator's legitimaing criteria were always constituted by appeals to $\boldsymbol{Mt_{re+}}$**.** Thirdly, there was high scaffold on the task, and when this happened the public space mainly consisted of $\boldsymbol{Mt_{re+}}$, because high scaffold tended to obscure the teachers' learning difficulties which often manifested themselves as any of these: $\boldsymbol{Mt_{ap+}}$ or $\boldsymbol{Mt_{ap-}}$ or $\boldsymbol{Mt_{re-}}$. The high scaffold in Task 2 was prompted by teachers' incorrect responses in Task 1.

In Chapters (8 - 9) I analyzed Tasks 1, 2, 3 and 4 of Functions 2. Similar analysis involving the legitimating criteria and patterns of variation were conducted in these chapters and the results still showed that the *primary enacted object of learning* was **Mt** dominated by $\boldsymbol{Mt_{re+}}$.

According to Pournara (2013) there are two components of SMK, and these include *new mathematics* and *revisited mathematics*. While new mathematics refers to mathematics never learnt before, revisited mathematics refers to the mathematics that was learnt before. In the PD the focus was to provide the opportunity for teachers to (re)learn or revisit the secondary school mathematics, and so this was the foregrounded component of SMK in the PD. Therefore unpacking the primary enacted object of learning with respect to whether it was new mathematics or revisited mathematics suggests that it was the latter. The analysis showed that the revisited or (re)learnt mathematics was constituted by *personal mathematics (*$\mathbf{Mt - pl}$) and *professional mathematics* ($\mathbf{Mt - PL}$). What this tells us about SMK is that *revisiting/(re)learning mathematics* makes it possible for teachers to learn both **Mt-pl** and **Mt-PL**. In **Mt–pl** teachers learn mathematics to deepen their SMK, while in **Mt-PL** they learn the relevance and pertinence of mathematics to the learner and to learning. *Professional learning of mathematics* (**Mt-PL**) does not suggest that one knows how to teach mathematics, but an understanding of what is salient and critical in variation theory terms, for learners to learn a particular mathematics object.

In the Functions theme work it was made possible for teachers to learn Functions per se (**Mt-pl**), as well as what is important for learners to know in and about functions (**Mt-PL**). Since **Mt-pl** and **Mt-PL** are components of **Mt** – the PD's intended object of learning, it suggests that there was corresponcdence between the intended and the enacted object of learning. As noted ealier, after noticing gaps in learners' knowledge and teachers' knowledge of functions the WMCS professional developers designed PD activites to help teachers to strenghten their own knowledge of Functions. The intended object of learning was constituted by key features of functions which were, among others, properties of Functions (e.g. definition, parameters and intercepts) and correct use of function terminology. Properties of Functions were consituted by $\boldsymbol{Mt_{re}}$ while other features such as function terminology constituted $\boldsymbol{Mt_{ap}}$.

Variation Theorists (e.g. Runesson, 2005) are in agreement that the desired goal of any enactment is to attain resemblance between the intended, enacted and lived objects of learning. Notwithstanding, they caution that attaining this resemblance is fraught with challenges which are almost always present in every enactement as a result of several intervening factors. It is interesting that the WMCS project was able to harmonize the enacted and intended objects of learning in spite of the challenges that often disrupt this occurrence.

As a form of reflection, it is interesting that the analysis of the enacted object of learning in Chapters 6 – 9 could not on its own, illuminate the *personal* and *professional mathematics* components of **Mt**. Instead, the componets were illuminated by foregrounding the relationship between the enacted and the lived object of learning. Hence the variation theorists' arguement that it is important to look at one object of learning with respect to others is important. Suffice it, at this point, to pose the question; '*What does this tell us about studying the enactment of SMK in general, or the enacted object of learning in particular?*

(b). How was the primary enacted object of learning constituted?

The primary enacted object of learning was facilitated by the deliberate use of two pedagogic strategies. In the first contact session the Facilitators worked an inductive pedagogic strategy which constrained initial scaffolding and this was productive in helping them to gain insight into teachers' prior knowledge of functions. After noticing gaps in teachers' knowledge Facilitators worked with the deductive pedagogic strategy which opened up scaffolding for teachers to progress on tasks. Also present in the enactment were legitimating criteria which were used to fix meaning on the enacted object. In their legitimating criteria Facilitators consistently appealed to mathematics and this consistency might have contributed to the alignment between the intended and the enacted objects of learning. The enactment was also characterized by the deliberate use of patterns of variation which are known for making the object of learning visible. Working with these patterns helped Facilitators to continually see if the enactment was in alignment with the intended object because whatever was being enacted was always visible to Facilitators and teachers.

Another productive way of working with patterns of variation is the fact that Variation Theory keeps the object of learning in focus. In the PD variation theory was inherent in the planning and enactment. By way of example, patterns of variation were used to visibilize the object of learning. Working with contrast, generalization and fusion meant that Facilitators were concerned with helping teachers to discern difference and similarity between critical features and relationships between dimensions of variation.

One other potential contributor to the alignment might be the fact that the participants were revisiting Functions. I derive this line of reasoning from the fact that variation theory has been largely used to study teaching and learning where the object of learning was new mathematics. As such the caution that Variation Theorists pose above might be more relevant to enactments in classrooms where there is an overabundance of intervening factors such as unfamiliarity with the object of learning. This research has shown that when the focus of the PD was on revisiting mathematics the enacted object of learning aligned with the intended object of learning. Hence is possible that the alignment between these two might be due to the focus on *revisiting mathematics*.

As noted, Pournara (2013) has found that *revisiting mathematics* is an important part of SMK. Hence any PD in mathematics teacher education has to be clear on whether its focus is on revisiting mathematics or construction of new mathematics. For instance, my research has shown that when revisiting mathematics was the focused SMK the enacted object of learning was constituted by *mathematical reality* and *mathematical appearance*, with the former dominant. Though new mathematics was not the focus of this research, it is probable that when the focus is on the construction of new mathematics the enacted object of learning might be dominated by *mathematical appearance* because this is the first time teachers encounter the new mathematics and so are likely to appeal to $\boldsymbol{Mt_{ap}}$ than $\boldsymbol{Mt_{re}}$.

Figure 12.1 shows an alignment in the intended and enacted object of learning in the WMCS project PD. This is evident at the intersection of the Venn diagram.

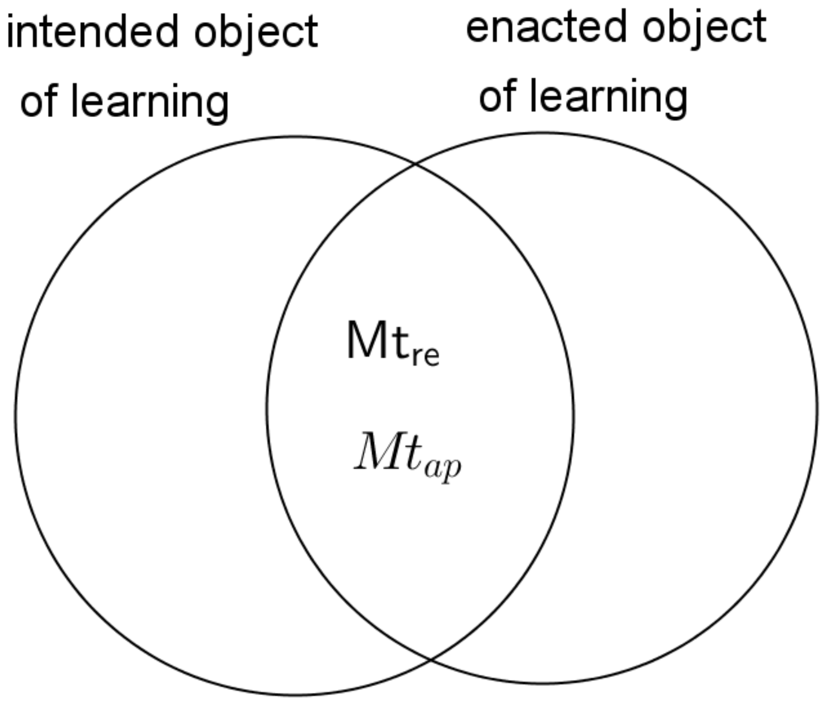


Figure 12.1: A diagrammatic representation of the alignment

What becomes the lived object for teachers? - i.e., what capabilities with respect to the mathematical object and its teaching are reflected in teachers' lesson planning and interviews?

To answer this question I analyzed two sets of data (i.e. lesson plans and interviews).

Results of the analysis of Pre- and Post-PD lesson plans showed that there were shifts in what teachers' considered to be important to know about Functions. Teachers' planning shifted from being predominately $\boldsymbol{Mt_{ap}}$ to reflect both $\boldsymbol{Mt_{ap}}$ and $\boldsymbol{Mt_{re}}$. Therefore the shift suggested that their lived object of learning at the level of lesson planning was both $\boldsymbol{Mt_{ap}}$ and $\boldsymbol{Mt_{re}}$. In addition, results of the analysis of teacher interviews has revealed that the lived object of learning associated with teachers' participation in the WMC-S project's PD was characterized by (i) Personal learning of functions (Mt-pl) (ii) Professional learning of functions (Mt-PL), (iii) Personal learning of teaching functions (Tm-pl) and (iv) Professional learning of teaching functions (Tm-PL). It must be noted that the results are limited to the nine teachers who participated in the study.

(a). How do these capabilities mediate teachers' professional knowledge (SMK, PCK)?

The results of the analysis of the Pre-PD and Post-PD lesson plans revealed that there was a shift in the teachers' notion of Functions. The Post-PD lesson plans incorporated multiple representations. In particular, the teachers came out of professional development with the awareness of key features of Functions. For instance, the teachers' lesson planning foregrounded Functions as a special relationship between input and output, and that the relationship can take many forms. These formations were multiple representations, each of which was a Function and not a step in the procedure, as it was the case prior to participation in the PD. Suffice it to mention that the shift in the notion of Function did not mean that this was how they taught Functions, because this research did not study the teachers' classroom practice. On the basis of evidence from lesson planning the shift in the teachers' notion of Function pointed to improvement in their SMK, particularly Mt-pl.

The results of my analysis of Part 2 of the interview, where teachers were asked to compare and contrast the two lesson plans, also pointed to shifts in the notion of Function. By way of example, both Teacher 1 and Teacher 2 pointed to the need and importance of helping learners to discern the connections between the representations. Teacher 1 said he taught the representations of Functions in isolation, and there was never a point where he ever showed connections between them - and he saw this as a weakness which he said made his teaching monotonous. Similarly, Teacher 5 pointed to some deficiencies in her Pre-PD lesson plan which she said was a lesson plan designed for a lecture. On the other hand, she described the Post-PD lesson plan as learner–centered because it provided opportunities for hands-on activities. In the interview some (e.g. Teachers 5, 4, and 7) talked about how they recontextualised the WMCS PD way of enacting Functions in their own classrooms. For instance, in a lesson involving matching the multiple representations of Functions, Teacher 5 noticed that even learners who seldom participated were engaged and willing to learn. Evidence from interviews suggests that there was improvement in both teachers' SMK and PCK. In SMK both the Mt-pl and Mt-PL improved, while the PCK components which included Tm-pl and Tm-PL also improved. Hence both teachers' SMK and PCK were mediated, and this was done through scaffolding, legitimating criteria and patterns of variation.

The significance of the shift in the teachers' notion of Functions implied that if the Post-PD plans were enacted they might provide opportunities for learners to: (i) discern each representation as a mathematical object which contained all features of a Function; (ii) discern Functions as a special relationship between input and output and not as a modus operandi for producing graphs from algebraic equations.

The presence of Mt-pl and Mt-PL in both interview and lesson plan data suggests that there was alignment between the intended and the lived object of learning. I have illustrated the alignment of the three objects in **Figure 12.2** below.

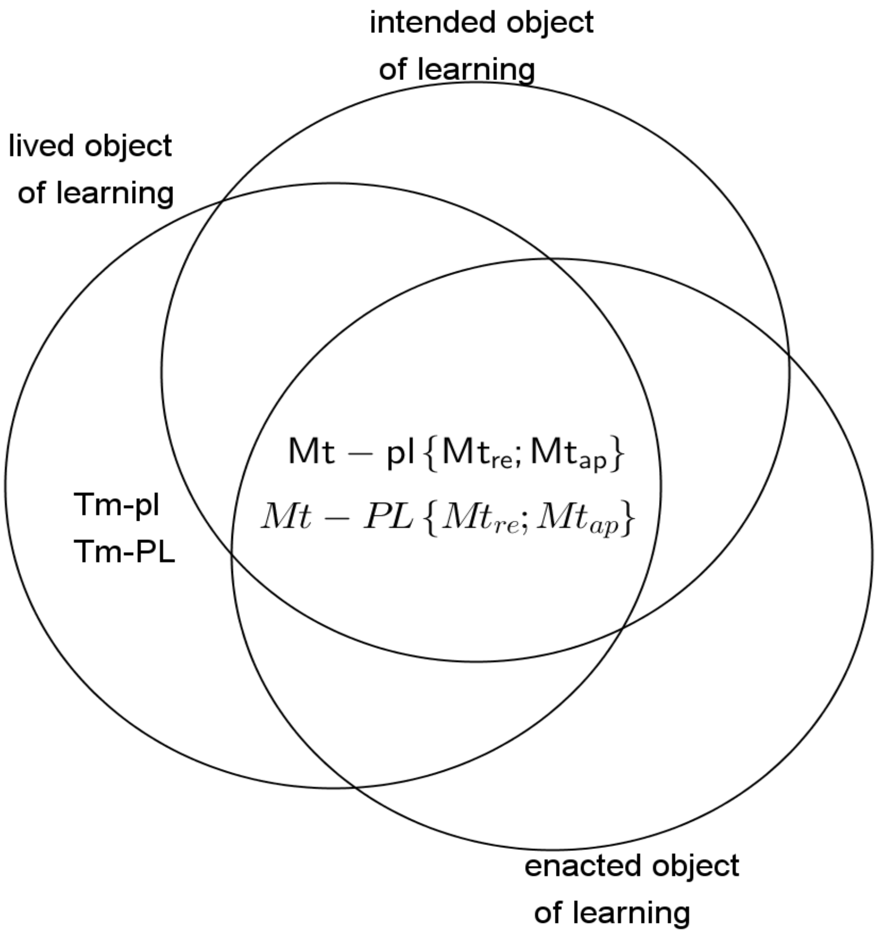


Figure 12.2: A diagrammatic representation of the alignment

As noted earlier, in the WMCS PD the intended object of learning was Mt, and this was informed by the observed gaps in teachers' knowledge of Functions. The enacted object of learning was both Mt-pl and Mt-PL which was constituted by both $\boldsymbol{Mt_{re}}$ and $\boldsymbol{Mt_{ap}}$. The presence of $\boldsymbol{Mt_{ap-}}$ and $\boldsymbol{Mt_{re-}}$ in the teachers' appeals substantiated gaps observed earlier, while the presence of $\boldsymbol{Mt_{re+}}$ and $\boldsymbol{Mt_{ap+}}$ with $\boldsymbol{Mt_{re+}}$ dominant, was a way to close the gaps. The lived object of learning revealed evidence of $\boldsymbol{Mt_{re+}}$ and $\boldsymbol{Mt_{ap+}}$ in lesson planning. Furthermore, the presence of Mt-pl in the lived object of learning suggested that it was a constituent of all three objects. Since the Mt-pl was a component of all three objects, the findings of the study suggests that the WMCS PD was successful at achieving coherence/harmony between the three objects of learning – a feat which is fraught with challenges.

(b). How might the mediation of teachers' professional knowledge be explained?

Explaining the mediation of teachers' professional knowledge in the PD can be understood through discussions of the way dilemmas/tensions were managed. To do this well, I draw two tensions (i.e. *ethos and "what"*) that provided opportunities for teachers to experience functions in new ways, and this I do below:

The Ethos tension: How were teachers positioned in the PD?

In the planning stage of the PD questions about the position of teachers and their role in the PD were discussed and resolved by professional developers. It was resolved that though teachers were coming as learners there was a need to treat them as equal partners, and this was informed by past PD practices where teachers were positioned as deficient, and so coming in to get knowledge. Research of literature (CTL, 2009; Mundry & Loucks-Horsley, 1999; Scheerens & Sleegers, 2009) on conventional professional development has found fault in this view, and even gone further to associate it with the failures of one-day in-service workshops. As such teachers in the PD were constituted as bring valuable experiences into the PD for everyone (including professional developers to learn from. Hence the WMC-S PD positioned teachers as equal partners who had something to share and for professional developers to learn from.

Teachers also acknowledged that they were positioned as equals when they talked about how the PD helped them to learn functions. According to Teacher 1 the WMCS PD provided an environment where professional developers and teachers were learning together and from each other through discussions. Teachers were not positioned as deficient in knowledge but as learning partners. Professional developers were open-minded and so encouraged divergent views. He concluded by saying that the PD was conducted professionally.

The 'what' (SMK) tension: Opportunity to experience functions in new ways:

First, teachers had opportunities to experience Functions in new ways. In particular the teachers' notion of Function shifted from a sequence of procedures to Functions as particulars that generated the universal. This was made possible first through the sensible use of two pedagogic strategies that closed or opened scaffolding. I mentioned earlier that teachers were perceived as coming with useful experiences to the PD for professional developers to learn from. Hence in the first contact session a deliberate move was made to close the scaffolding of tasks in order to gain insight into the teachers' knowledge of functions. They were provided with low scaffolding through an inductive pedagogic strategy and the consequence was numerous incorrect responses. The value of these incorrect responses was that they informed subsequent planning and modifications to the initial design. Building on this information Facilitators worked with the deductive pedagogic strategy in the remaining sessions to provide either high or moderate scaffolding. The success is discerned from teachers' utterances about how the PD helped them to learn and I showed this information in Chapter 11.

Second, the high or moderate scaffolds which were provided to teachers were characterized by an assortment of resources that included (i) experiencing visual images of the big ideas that were discussed and this was made possible through an extensive use of Geogebra software, (ii) Use of cards bearing representations of functions (ii) readings on functions, and resources for use in the classroom such as large and small WMCS grids. Teachers were also provided group or individual assistance whenever progress on the task was slow.

In summary it can be said that teachers were provided with scaffolding in the form of resources, and there were opportunities for knowledge conception through legitimating criteria and patterns of variations. As such scaffold, legitimating criteria and patterns of variation were central to how the PD made it possible for teachers to learn.

12.4. What are other contributions of the study?

In addition to empirical, other significant contributions were to theory, methodology and literature.

12.4.1. Theoretical contributions of the study

This study has made two main theoretical contributions and these are largely located in the analytical tools used in the analysis. These contributions were (i) *extension on the the notion of duality of object of learning in mathematics teacher education* and on *(ii) the notion of legitimating criteria.*

In their earlier research, Adler and Davis (2006) explored the object of learning in mathematics teacher education, and found that the enacted object of learning was either **Mt**, **Tm** or **both.** Building on this research and working with their first two categories to study the enacted object in professional development I found evidence which did not only confirm this duality, but also extended their work. In particular, my research has empirically shown that the **Mt** is constituted as $\boldsymbol{Mt_{re}}$ and $\boldsymbol{Mt_{ap}}$ at the level of enactment and as **Mt-pl** and **Mt-PL** at the level of the lived. In this research the **Tm** was not studied at the level of the enacted due to focus of the WMCS PD and the scope of the study. However, this was studied at the level of the lived through interviews. Evidence from the interviews showed that **Tm** was constituted as **Tm-pl** and **Tm-PL.** It is probable that analysis of Functions 3 and 4, which was not within the scope of this research, might have shed light into **Tm** at the level of enactment. Hence there is need for future research to study **Tm** at the level of the enacted.

In this research I also worked with the notion of legitimating criteria and patterns of variation as separate entities, to study the constitution of the enacted object. For instance, I worked with legitimating criteria as constituted by verbal statements only (see **Figure 12.3**).

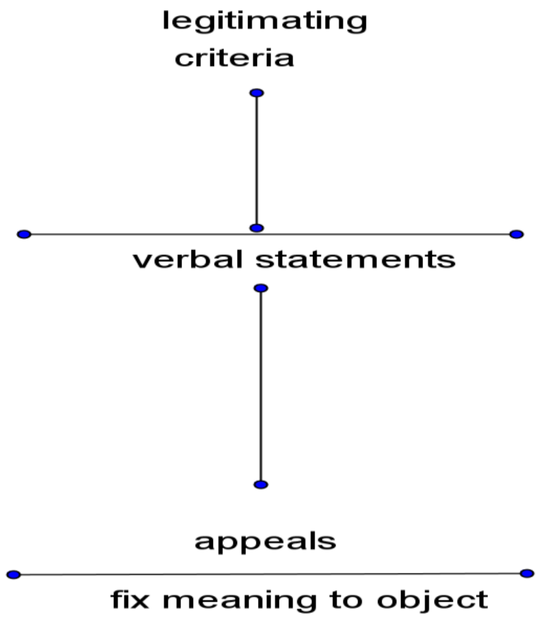


Figure 12.3: Initial view of legitimating criteria

When working with this notion I observed similarity between legitimating criteria and patterns of variation with respect to how they were used to scaffold tasks. Further observations revealed that in effect, patterns of variation were legitimating acts. This realisation resulted in the extension of legitimating criteria to include patterns of variation (see **Figure 12. 4**).

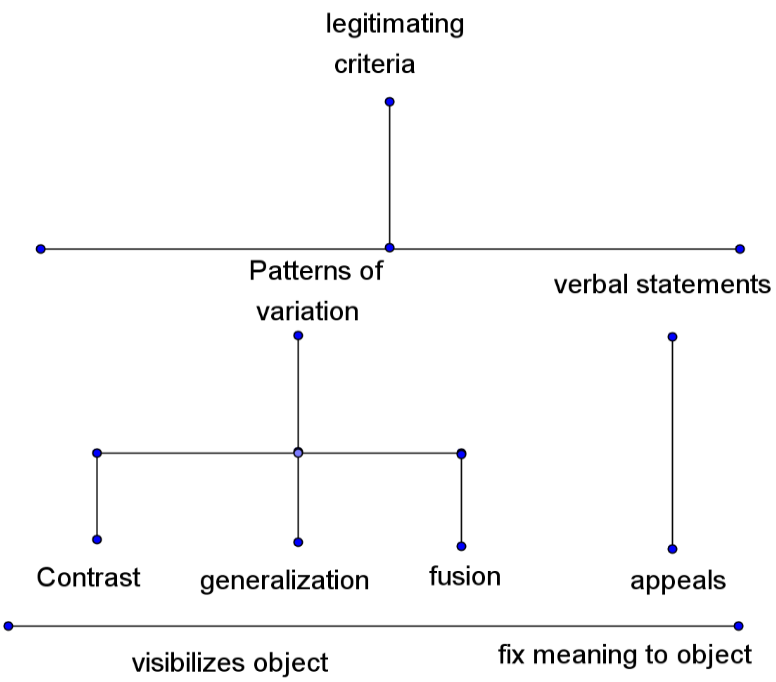


Figure 12.4: Alternative view of legitimating criteria

12.4.2. Methodological contributions of the research

To work with three different perspectives in a way that has not been done before and still work productively is not trivial. Two analytical tools were introduced into the data, and one emerged inductively from the data. Interestingly, I was able to work productively with the three. On the one hand, my inductive analysis helped to explain why there were more legitimating criteria in some enactments and fewer in others. On the other hand, separate analysis with patterns of variation and with legitimating criteria pointed to the same enacted object of learning and this was a mathematics object (**Mt-pl**). As such the relationship between these three components constitutes a methodology for studying the enacted object of learning where the object of learning is a mathematics object (**Mt**). The methodology can be used to study enactments in mathematics classrooms**.** The underlying assumption is that working with these three tools enables movement across the intended, enacted and lived objects of learning. Further research is proposed to explore this assumption.

Figure 12.5 shows the methodology, where I show that the enactment begins with decisions about whether to open or close scaffold. I also show that patterns of variation and legitimating criteria (inputs) are aspects of scaffold where one *visibilizes* and the other *illuminates* the boundaries of the enacted object of learning. Visibilization takes place through Patterns of variation particularly when they operate on a visual mathematics object. This makes it possible for the learner to literally see difference, similarity or relationships between the aspects of the object of learning. For instance it is possible to see difference between graphs of a one-to-one and many-to-one Function when these are simultaneously present. Visibilizing the object does not imply automatic discernment. Patterns of variation should be accompanied by verbal statements which clarify the observed difference, similarity or relationship. It is this clarification with verbal statements that I refer to as illuminating the object of learning. In effect, both patterns of variation and appeals are two legitimating criteria, where one shows and the other clarifies what the object is and is not.

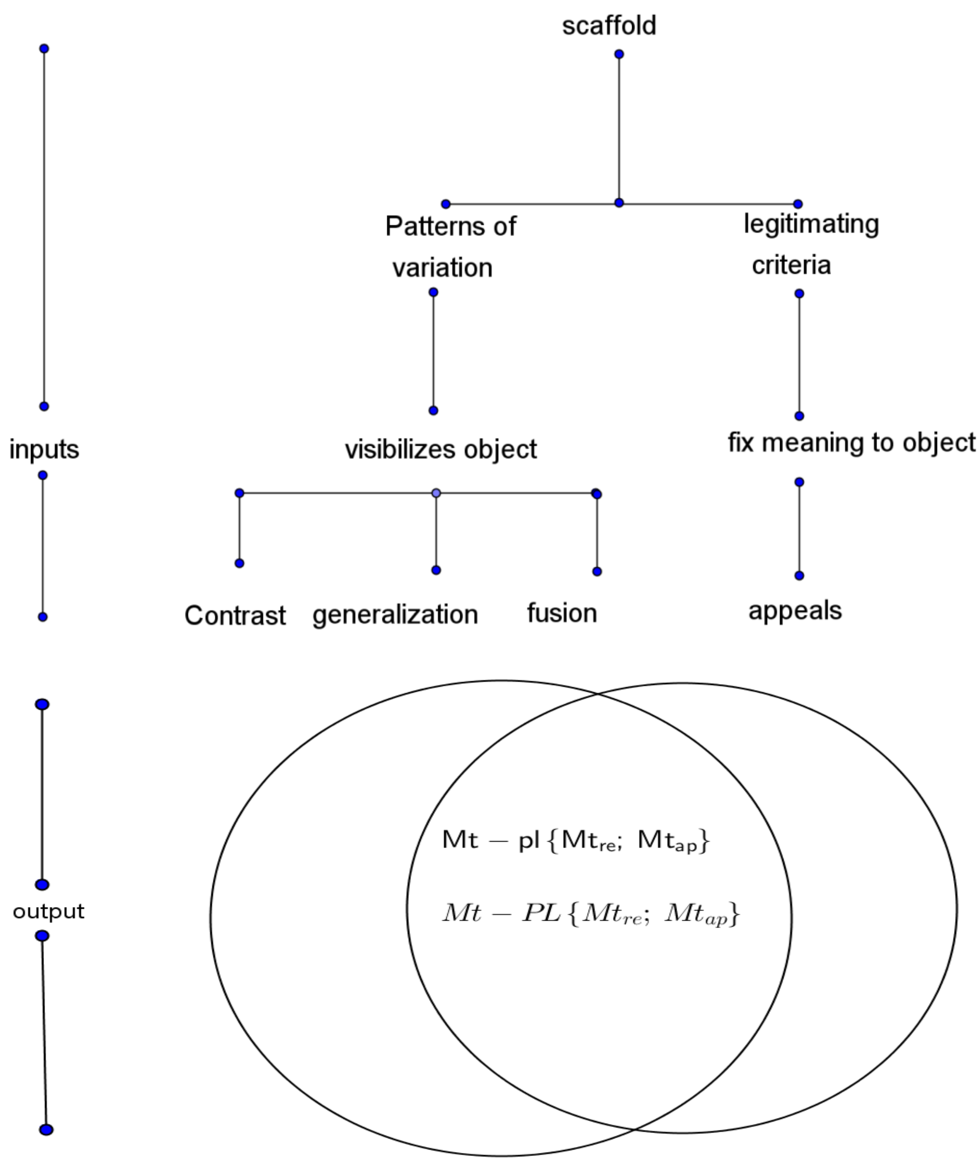


Figure 12.5: Methodology for studying enactment of Mt in an object focused PD

The model shown above depicts legitimating criteria and patterns of variation as categories of scaffold. There is apparent disctinction between the two when percieved from their origins. The former has roots in a social theory (Bernstein, 2000), which has focus on power relations, and the latter is rooted in phenonomenology (Creswell, 1998; Stumpf, 1994). In spite of this, there seem to be cohesion between the two tools.

My argument is that patterns of variation are essentially a category of legitimating criteria. This argument arises from the fact that patterns of variation visibilise the object of learning through discernment of difference, similarity and relationships. Discernment of difference makes it possible to notice what something is and what it is not, as such I consider showing contrast as a way to fix meaning on the enacted object of learning. Similarly, through separation it is possible to notice similarity that leads to generalisation. Hence separation is a legitimating act because it fixes meaning to the foregrounded invariant feature during the enactment. On the same note, fusion helps one to notice relationships between critical aspects of the enacted object and between these aspects and the object itself. So noticing this pattern helps to fixing meaning on the enacted object of learning. As such these patterns are essentially, legitimating criteria. Previous work (see Adler & Davis, 2006) has characterised legitimating criteria as verbal statements and in this work I worked with this notion as such (**see Figure 12.3**).

However, in my work with both verbal statements and patterns of varition, I noticed that the two are complementary aspects of legitimating criteria – verbal statements illuminate and patterns of variation visibilize the enacted object of learning. In my analysis of video data I noticed that what was illunimated was the same thing that was made possible to discern (i.e. what was visibilized). This was evident in the the fact that all verbal statememts contained appeals to mathematics object, and what was made possible to learn was also a mathematics object. In other words, that which patterns of variation made possible to learn was the same thing that was legitimated via verbal stametments. By way of example, in Session 2.2 contrast was used to visibilize difference between a one-to-one and many-to-one functions. In addition to this, a verbal statement was used to illuminate the difference when it was explained that the two were different because a horizontal line test cut one graph once and the other more than once. Hence I argue that they are two complementary legitimating criteria. Consequently, this study extend the notion of legitimating criteria to include patterns of variation (see **Figure 12. 4**).

12.4.3. What does the research contribute to literature?

This research contributes to different areas of research that include mathematics education, professional developemnt and teacher knowledge. Literature of research in mathematics education suggests that this area is under-researched, and so there is need for research to inform the practice of teacher educators.

This work, as I have earlier mentioned, has confirmed the duality of the object of learning in mathematics education and also in professional development. I have reviewed literature of research done in the 1990s on teachers' subject matter knowledge and pedagogical content knowledge. Most of that research found gaps in teachers' SMK and PCK (e.g.Ball, 1990; Eisenhart, et al., 1993). The teachers who participated in the WMC-S PD came into the PD with gaps in their knowledge of functions, but these gaps had narrowed by the time the functions theme work ended. This information suggests that engaging teachers in an object focused PD is a possible solution to closing gaps in teachers' SMK. While earlier research showed that SMK levered PCK and recent research showing that PCK levered SMK, this research has shown that *an object focused PD levered SMK directly.*

This research also provides possible explanations to the failures of once-off in-service workshops. I have drawn my explanation from the tension of duration (Adler, June 2013). In my analysis I noticed that at the beginning of the PD teachers across the three clusters, irrespective of the level of scaffolding provided, were not working from the provided definition of Function but instead drew from their $\boldsymbol{Mt_{ap-}}$ to produce incorrect responses. This happened on the first contact session, but with continuous participation they were able to work with the provided definiton, and there were improvements in their responses. This occurrence confirmed the articulated complexity in teacher learning which once-off workshops failed to manage. This finding suggests that continuous learning in professional developmet is critical for improvement to occur.

Failure to work with the definition confirmed Vinner's viewpoint which states that the correct mental image of the concept is fundamental to the development of the concept image. Vinner explained that it is not possible to work with the definition when the mental image is weak. Similarly, the failure of teachers in my research to work with the definition of Function might have been due to the presence of $\boldsymbol{Mt_{ap-}}$ in their prior knowledge of functions. While Vinner's work has elaborated on the the notions of *mental image* and *concept properties*, this research has developed and elaborated on $\boldsymbol{Mt_{ap}}$ and $\boldsymbol{Mt_{re}}$**.** Vinner arrived at his conclusions while working in the cognitivist while my conclusions were rooted in the social constructionist perspective.

Literature also benefits when attention is shifted to pedagogy. Teachers in this study talked about how the way the PD was enacted helped them to learn. For instance Teacher 1 talked about being treated as an equal partner in the PD, and that things were not imposed on them but instead Facilitators had an open mind – knowing that teachers also had something to contibute. Teacher 1's description resonates well with Mason's (2013) notion of learner-friendly classrooms. As such it can be said that the PD was in line with this view.

This work also contributes to professional development literature in two main ways. In the initial stage of my research I searched for gaps in the literature of research on professional development literature for gaps to pursue in this research. During the search I came across the work of Buddin and Zamarro (2009) where they pointed out that there was little evidence in professional development literature to explain what teachers learn in professional development. As such this research has shed light into this domain by revealing that in an object focused professional development it is possible for teachers to learn SMK explicitly and PCK implicitly. In general the SMK was **Mt** and the particular SMK was a combination of $\boldsymbol{Mt_{ap}}$ and $\boldsymbol{Mt_{re}}$. What became the lived object of learning was **Mt-pl**, **Mt-PL**, **Tm-pl** and **Tm-PL**. While this research was able to fill out **Mt** in both the enacted and lived object of learning it was able to fill out **Tm** at the level of the lived object of learning.

Teacher knowledge literature has also benefited from this research, and the benefit is in two ways. First, the research has revealed what teachers learn in object focused PD, and when these are seen as aspects of knowledge then Mt-pl; Mt-PL; Tm-pl; and Tm-PL may be construed as *personal knowledge of functions, professional knowledge of fucntions, personal knowledge of teaching functions, and professional knowledge of teaching functions* respectively. These categories adds to the emergent literature of research (Bair & Rich, 2011; Ball & Bass, 2003; Ball, et al., 2004; Ball, et al., 2009) in this domain. Another dimension in which this research contributes to teacher knowledge is located in earlier research which pointed to gaps in teachers'SMK and PCK (e.g. Moalosi & Kaino, 2010a). In this research I studied teachers who came into the PD with evidence of gaps in their knowledge of functions. However, through participation in an object focused PD there were opportunities for teachers to close the gaps in their knolwedge of functions, and there is evidence that suggests that by the end of the PD some of the gaps were closed. Hence this research provides a possible solution to the problem of limited teacher knowledge.

Another contribution to literature might be the paralles between my research and a recent study (Adler, et al., 2013). In their study of prospective teachers of mathematics, Adler, et. al (2013) looked into the constitution of understanding mathematics in depth and found that this notion was constituted by connectedness, reasoning and mathematical disposition. In my review of their work, particularly the students' utterances, I noticed that there were similarities between their participants' utterances and my participants' utterances. So what I might have described as either **Mt-pl** or **Mt-PL** was described in their work as *understanding with connectedness, or reasoning or mathematical disposition*. Of course the distinctions were a consequence of the lenses used. It is interesting that both themes emerged empirically through inductive analysis from in-depth interview data. It is also interesting that both sets of participants had learnt mathematics where they were *revisiting* school mathematics

12.4.4. What does this research contribute to education reforms?

Most countries do not seem to be achieving their national educational targets, and this is associated with many factors, such as limited teacher knowledge, one-day in-service workshops and socio-economic factors. In South Africa the problem of limited teacher knowledge is known (Adler & Davis, 2006; Simkins, 2013; Spaull, 2013). In their work Adler and Davis pointed to the need to redress the problem of under-qualified mathematics teachers. Recently, South African teachers experienced curriculum changes which introduced new content which some of the teachers have never experienced. This reseach has produced evidence which shows that teachers can learn school mathematics content, but under certain conditions. These conditions include participation in ongoing longitudinal professional development which works with extensive resources and scaffold tasks using legitimating criteria (i.e. patterns of variation and appeals to mathematics objects).

Another important contibution pertaining to educational reforms is a response to literature (Ball & Cohen, 1999). These authors have complained about the way teachers are usually blamed for low learner achievement. Ball and Cohen (1999) pointed out that teachers are usually blamed for failing to meet the expectations of the new reforms, yet teachers were the products of the very education system which everyone wanted to change. They go futher and ask how could teachers be expected to teach in new reform-oriented ways which they have never experienced. The authors' question seem to suggest that teachers need to experience particular content for them to be able to teach it well. This research contains evidence to support this view. Experiencing an object focused PD helped teachers to notice that their ways of teaching functions lacked connections and were monotonous. It also helped them realise that working with multiple representations could develop reasoning among learners and encouraged learner participation. Some of these teachers were able to help learners experience functions in new ways because they had experienced it themselves. Even their Post-PD lesson planning showed shifts what they percieved as important to teach about functions.

12.5: What is the main argument of this research?

As noted earlier, the purpose of the study reported in this thesis was to explore what was enacted in pedagogic practice, and became the lived object for the participating teachers. Object-focused professional development was a model where selected mathematical content in the secondary school curriculum was in focus for teachers to revisit. Consequently, the thesis of my research is stated as follows:

In an object focused PD which privileges key features of Functions and uses extensive resources, that works with inductive and deductive pedagogic strategies to close and open scaffold and legitimating criteria (patterns of variation and appeals to mathematics objects) to visibilize and illuminate the object of learning, the following can be realised: (i) the enacted object of learning is constituted by both correct mathematics appearance ($\boldsymbol{Mt_{ap+}}$) *and mathematics reality* $\boldsymbol{Mt_{re+}}$ *and (ii) the lived object of learning is consituted by* ***Mt-pl; Mt-PL; Tm-pl*** *and* ***Tm-PL****, which together constitute the essential invariant meaning structure of experiencing object focused PD (i.e. what it means to experience an object focused PD that privilege mathematics content in the secondary school curricum.*

The thesis was rooted in the claim that says that:

Teachers came into PD with gaps in their knowledge of functions, but by the end of the functions theme work the gaps had narrowed. This claim has both an empirical and theoretical base. For empirical support I have used Teacher 2 in **Chapter 10** to illustrate the rest of the teachers' knowledge of functions at the beginning of PD. For theoretical support I reffered to the teachers' legitimating criteria in which gaps in the teachers' knowledge were manifested as *incorrect mathematics reality* $\boldsymbol{Mt_{re-}}$ and *incorrect mathematics appearance* $\boldsymbol{Mt_{ap-}}$**.** Theoretical evidence is evident in **Chapter 6 (**see **Table 6.1, 6.3 and 6.4).**

12. 6. Implications of this research to policy on professional development and educational reforms

Firstly, the findings of this research speak to South Africa's policy on professional development in two main ways. First is the nature of teacher professional development initiatives. Evidence of teacher learning in the WMCS PD provides grounds from which a call can be made for shifts from once-off in-service professional development to continuous professional development. In my description of the WMCS PD in Chapter 1, I showed that it met the criteria of a continous professional development. Now, the findings of this research seem to confirm that the WMCS PD is a CPD initiative because there is evidence to show that it helped teachers to learn.

The persistence of once-off in-service workshops in South Africa might rest on the lack of local scientific evidence which point to their failure. In spite of that, professional development in the mould of the one being researched here might be a possible solution to redressing gaps in teachers' knowledge. Hence CPD in the mould of the WMCS might be an alternative route to strengthening teachers' subject mattter knowledge for the purpose of improving teaching quality and low learner achievement in mathematics in South Africa. Hence there is need for a policy which advocates CPD.

Secondly, other CPDs might borrow a leaf from the design of the WMCS PD in several ways. By way of example, in this PD resources were used extensively to help teachers discern the object of learning and my research provides empirical evidence which indicate that resources helped teachers to learn Functions. Teaching objects were backgrounded in the PD, but were implicitly foregrounded by the use of resources. Deliberate use of variation theory in the design of PD activities facilitated the use of patterns of variation.

Thirdly, the findings of my research provide information that might be used as part of the evalution of WMC-S project, particularly the work done in the functions theme work.

Fourthly, there are implications for further research to look into other theme works which were done in this PD following the functions theme work. Information coming out of that research will be useful for further illuminating the (i) importance of extensive use of resources in CPDs, (ii) value of positioning teachers as equal partners (iii) learning in CPD as a two-way enterprise and the (iv) value of foregrounding the object of learning.

12.7. Limitations of the study

In this research I was interested in sessions where the intended object of learning was a mathematics object. Hence I analysed videos of Functions 1 and Functions 2 where **Mt** was foregrounded. The purpose of my research was to study the enacted and lived object of learning when the object of learning was **Mt**. Due to the focus and scope of my research I did not conduct a detailed analysis of Functions 3 and Functions 4, which foregrounded **Tm.**

In spite of this, my analysis of interviews showed that **Tm-pl** and **Tm-PL** were components of the teachers' lived object of learning. The presence of teaching objects in the lived object and their absence in the enacted object of learning meant that I could not fillout the **Tm** because of lack of evidence at the level of the enacted. The reader must note that I studied the lived object of learning from interviews while the enacted object of lerning was studied from videos. Now, the Functions 1 and Functions 2 focused on **Mt**.

Perhaps if I had analysed Functions 3 and 4 I might have been able to fill out the **Tm**. Hence the absence of evidence at the level of enacted has imposed limitations on conclusions that I might want to make about the constitution of **Tm** in an object focused PD**.** In addition, this research is not able to say whether there was an enacted object of learning which was **Tm**. What might be said can only amount to speculation. In this research, deliberate decisions were made to exclude Functions 3 and 4 to narrow the scope of the research. Hence there is need for further research to study videos of Functions 3 and 4 to search for evidence which might provide insight into the constitution of **Tm** in an object focused PD.

12.8. Retrospection

As noted in Chapter 5, my interview protocol had three sections. The first focused on resources. The second focused on teachers' Pre- and Post-PD lesson plans, and the third on seven SMK-focused scenarios. However, of the seven scenarios, only Scenario 7 provided useful data which I have since analyzed and used in **Chapter 10**. Data from other scenarios were not used.

In retrospection, I strongly believe the scenarios could have been more productive had they been more directed around the specific instances of teachers' misconceptions that were evident in their legitimating criteria. In my construction of the scenarios I was guided by literature (e.g. Ball, et al., 2009; Blum & Krauss, 2008). This led my scenarios to tap into SMK in general and not the specific instances of the PD. Hence linking the teachers' responses on the scenarios to their learning in the PD was fraught with challenges. This points to a weakness in my methodology. A better way to deal with this challenge would have been to analyse the enacted object of learning before conducting interviews. This would have provided insight into the teachers' knowledge of functions at the beginning of PD, and the insight would have been used to guide the focus and construction of scenarios.

Furthermore, the presence of teaching objects in the teachers' lived object suggest perhaps that I should have also analyszed video data from Functions 3 and 4. Perhaps the analysis of this of enacted object in these two PD sessions might have shared light into the origin of the teaching objects.

Looking back to the design of the research, I strongly believe that beginning my analysis with interview data and ending with video data enabled me to notice and distiguish my notions of **Mt-pl, Mt-PL, Tm-pl and Tm-PL**. As it might be expected, I went into this research with certain expectations, particularly that I was a member of the WMCS PD team that conducted the PD. However, working from the interview served to bracket (to a reasonable degree) my preconceptions. Hence these notions were able to emerge empirically from the data. As such I also strongly believe that had I began my analysis with video data, it would have not been possible for these notions or themes to emerge, because the outcomes would have strengthened what I was expecting teachers to say, and so would have greatly influenced my probes. Hence the sequencing of my data analysis was a strength of the research.

12. 9. Possible improvements and new ideas for further research

This research did not follow teachers to the class to observe how shifts in the lived object of learning influenced their teaching practices. As such an improvement to this research would be to explore evidence of teacher learning in the classroom, and particular attention would be on the quality of teaching and learner achievement.

The results of this research revealed that there was coherence between the three objects of learning (intended, enacted and lived). According to Variation Theorists this is not a common feature in most enactments. Another idea emerged from noticing that though the PD foregrounded a mathematics object, the teaching object was experienced, and so was present in the lived object of learning. In what could be seen as the third idea, after working with patterns of variation I noticed that there is a possibility that patterns of variation, have potential to simultaneously foreground both the mathematics and teaching objects (**MT**). The conditions under which this happens are limited in the literature. As such there is need for research to explore these issues througth the following questions:

1. How did the enactment at the WMC-S project's professional development make it possible for the three objects of learning to align together?
2. Does backgrounding one object when foregrounding the other matter in an object focused PD?
3. Are patterns of variation capable of visibilising mathematics and teaching objects (MT) simultaneously?
4. What is the primary enacted object of learning in Functions 3 and Functions 4?

References

Adler, J. (2008). Adapted NRF proposal, *Mathematics for teaching: description, explanation and implications for mathematics teacher education*. Johannesburg: Marang Center, Wits School of Education.

Adler, J. (2009). A methodology for studying mathematics for teaching. *recherches en Didactique des Mathematiques, 29*.

Adler, J. (July 2011). *A continuing research and development learning curve: The work of the Wits FRF Mathematics Chair and the Wits Maths Connect-Secondary (WMC-S) project*. Johannesburg: University of the Witwatersrand.

Adler, J. (July 2012). *Research and development learning curve continued:Report of the Wits FRF Mathematics Chair and Wits Maths Connect (WMC-S) project*. Johannesburg: University of the Witwatersrand.

Adler, J. (June 2013). *The research and development learning curve continued: Report of the Wits FRF Mathematics Chair and the Wits Maths Connect - Secondary (WMC-S) project*. Johannesburg: University of the Witwatersrand.

Adler, J. (November 2010). *A research and development learning curve: The work of the Wits FRF Mathematics Chair and the work of the Wits Mathematics Connect-Secondary (WMC-S) project*. Johannesburg: University of the Witwatersrand.

Adler, J., & Davis, Z. (2006). Opening Another Black Box: Researching Mathematics for Teaching in Mathematics Teacher Education *Journal for Research in Mathematics Education, 37*(4), 270-296.

Adler, J., & Davis, Z. (2011). Modelling Teaching in Mathematics Teacher Education and the Constitution of Mathematics for Teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical Knowledge in Teaching* (pp. 139 - 160). Australia: Springer.

Adler, J., Davis, Z., Kazima, M., Parker, D., & Webb, L. (op cit). Working with learners' mathematics: Exploring a key element of mathematical knowledge for teaching

Adler, J., Hossain, S., Stevenson, M., Clarke, J., Archer, R., & Grantham, B. (2013). Mathematics for teaching and deep subject matter knowledge: voices of Mathematics Enhancement Course students in England *Springer, 16*(6).

Adler, J., & Hulliet, D. (2008). The social production of mathematics for teaching In T. Wood & P. Sullivan (Eds.), *International handbook of mathematics teacher education* (pp. 195 - 222). Netherlands: Sense Publishers.

Artigue, M. (2013). *Mathematics Teacher Development.* Paper presented at the 4th Africa Regional Congress Lesotho College of Education, Maseru

Bair, S. L., & Rich, B. S. (2011). Characterizing the Development of Specialized Mathematical Content Knowledge for Teaching in Algebraic Reasoning and Number Theory. *Mathematical Thinking and Learning, 13*(4), 292 - 321

Ball, D. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*, 132 -144.

Ball, D., & Bass, H. (2003). *Toward a practice-based theory of mathematical knowledge for teaching. Proceedings of the 2002 annual meeting* Paper presented at the Canadian Mathematics Education Study Group, Edmonton, AB:.

Ball, D., Bass, H., & Hill, H. (2004). *Knowing and using mathematical knowledge in teaching: learning what matters.* Paper presented at the Southern African Association for Research in Mathematics, Science and Technology Education, Cape Town.

Ball, D., & Cohen, D. (1999). Developing practice, developing practitioners: Toward practice-based theory of professional education. In L. D.-H. a. G. Sykes (Ed.), *Teaching as the learning profession* (pp. 3 - 33). San Francisco: Jossey-Bass.

Ball, D., Hill, H., & Bass, H. (2005). Who knows mathematics well enough to teach third grade, and how can we decide? . *American Educator, 5*(3), 14-17 20- 22 43-46.
Ball, D., Thames, M. H., & Phelps, G. (2009). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.
Bernstein, B. (2000). *Pedagogi, Symbolic Control and Identity: Theory, Research and Critique (Revised Edition)*. Lanham: Rowman & littlefield Publishers, Inc.
Blum, W., & Krauss, S. (2008). The Professional Knowledge of German Secondary Mathematics Teachers: Investigations in the Context of the COACTIV project Retrieved 13 July, 2010, from http://www.unige.ch/math/EnsMath/Rome2008/WG2/Papers/BLUMKR.pdf
Bogdan, R. C., & Biklen, S. R. (1982). *Qualitative Research for Education*. Boston: Allynand Baco, Inc.
Borko, H. (2004). Professional Development and teacher learning: mapping the terrain. *Educational Researcher* Retrieved 10, 2012, from http://www.aera.net/uploadedFiles/Journals_and_Publications/Journals/Educational_Researcher/, Volume_33_No_8/02_ERv33n8_Borko.pdf
Brodie, K. (2012). *Professional Learning Communities and Teacher Change*. Paper presented at the 12th International Congress on Mathematics Education.
Brodie, K., & Shalem, Y. (2011). Accountability conversations: mathematics teachers' learning through challenge and solidariy. *Journal of Mathematics Teacher Education, 14*, 419 - 439.
Bromme, R. (1994). Beyond subject matter: A psychological topology of teachers' professional knowledge. In R. Biehler, R. W. Scholz, R. Straber & B. Winkelmann (Eds.), *Mathematics didactics as a scientific discipline: state of the art* (pp. 73 - 88). Dordrecht: Kluwer.
Buddin, R., & Zamarro, G. (2009). *Teacher Qualifications and Student Achievement in Urban Elementary Schools* (No. CA 90407-2138). Santa Monica, CA: Institute of Education Science.
Chauraya, M., & Brodie, K. (2011). Mathematics teachers' learning and teaching practices in a professional learning community. 1-10
Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. [Electronic]. *Teaching and Teacher Education, 18*, 947-967.
Cohen, L., & Manion, L. (1980). *Research Methods in Education*. London: CroomHelm Ltd.
Cohen, L., Manion, L., & Morrison, K. (2001). *Research Methods in Education* (5 ed.). London: Routledge.
Creswell, J. W. (1998). *Qualitative Inquiry and Research Design: Chosing Among Five Traditions*. London: SAGE Publications.
Crotty, M. (2009). *The Foundations of Social Research:meaning and perspective in the research process* London: SAGE Publications.
CTL. (2009). Systemic vs. one-time teacher professional development: what does research say? *Research Note, 15*,
David, G. (2007). *The challenge of phenomenological research: Recovering the teacher-student relationship*. Paper presented at the Australian Association of Research in Education
Davis, Z. (2001). Measure for measure: evaluative judgement in school mathematics pedagogic texts. *Pythagoras, 56*, 2-11.
Davis, Z. (2005). *Pleasure and pedagogic discourse in school mathematics: A case study of the problem-centered pedagogic modality. Unpublished PhD Thesis.* University of Cape Town, Cape Town.

Davis, Z., Adler, J., & Parker, D. (2007). Identification with images of the teacher and teaching in formalised in-service mathematics teacher education and the constitution of mathematics for teaching. *Journal for Research in Mathematics Education, 42*.
DeMarois, P., & Tall, D. (1996). *Facets and layers of the Function concept.* Paper presented at the PME 20, Valencia.
Dobey, D., & Schafer, L. (1984). The effects of knowledge on elementary science inquiry teaching. . *Science Education, 68*, 39 - 51.
Eisenhart, M., Borko, H., Underhill, R. G., Brown, C. A., Jones, D., & Agard, P. C. (1993). Conceptual Knowledge Falls Through The Cracks: Complexities of Learning To Teach Mathematics for Understanding. *Journal for Research in Mathematics Education, 24*(1), 8-40.
Enerst, P. (1991). *The Philosophy of Mathematics Education*. London: Famer Press.
Enerst, P. (Ed.). (1999). *Forms of Knowledge in Mathematics and Mathematics Education:Philosophical and Rhetorical perspectives*. Netherlands: Kluwer Academic Publishers.
Even, R. (1999). Integrating academic and practical knowledge in a teacher leaders' development program. . *Educational Studies in Mathematics, 38*, 235-252.
Even, R., & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject matter. *Educational Studies in Mathematics, 29*(1), 1-20.
Fennema, E., & Romberg, T. A. (1999). *Mathematics classrooms that promote understanding*. Mahwah, N.J: Erlbaum.
Gay, L. R., & Airasian, P. (2003). *Educational Research: Competencies for Analysis and Applications* (7th ed.). New Jersey: Merril Prentice Hall.
Glazer, E. M., & Hannafin, J. (2006). The collaborative apprenticeship model:Situated professional development within school settings. *Teacher Education, 22*, 179 - 193.
Goos, M. (2013). Knowledge for teaching secondary school mathematics: what counts? *International Journal of Mathematical Education in Science and Technology, 44*(7), 972 - 983.
Graven, M. (2005). Dilemmas in the design of in-service education and training for mathematics teachers In R. Vithal, J. Adler & C. Keitel (Eds.), *Researching mathematics education in South Africa*. Cape Town: HSR press.
Hart, W. D. (1996). *The Philosophy od Mathematics* Toronto: Oxford University Press.
Hatch, A. J. (2010). *Doing Qualitative Research in Education Settings*.
Hill, H., Blunk, M., Charalambous, C., Lewis, J., Phelps, G., Sleep, L., et al. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430-511.
Hill, H., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement *American Educational Research Journal, 42*(2), 371-406.
Huillet, D., Adler, J., & Berger, M. (2011). Teachers as researchers: Placing mathematics at the core. *Education as Change, 15*(1), 17-32. doi: http://dx.doi.org/10.1080/16823206.2011.580769
Kaino, L. M., & Moalosi, S. S. (2013). Teachers' mathematical pedagogical content knowledge:some trends in search of adequate knowledge for effective teaching *Journal of social sciences, 37*(1), 189 -196.
Kazemi, E., & Franke, M. L. (2003). Using Student Work to Support Professional Development in Elementary Mathematics. *Center for the Study of Teaching and Policy, Document W-03-1*, 44

Leinhardt, G., Zaslavsky, O., & Stein, M. (1990). Functions, graphs and graphing: Tasks, learning, and teaching. *Review of Educational Research, 1*, 1 - 64.

Lo, M. L. (2012). Variation Theory and the Improvement of Teaching and Learnig Available from https://gupea.ub.gu.se/

Ma, L. (1999). *Knowing and teaching mathematics: Teachers' understanding of fundamental matheamtics in China and the United States*. Mahwah, NJ: Erlbaum.

Marton, F., & Booth, S. (1997). *Learning and Awareness*. New Jersey: Lawrence Erlbaum Associates.

Marton, F., & Pang, M. F. (2006). On Some Necessary Conditions of Learning *The Journal of the Learning Sciences 15*(2), 193-220.

Marton, F., Runesson, U., & Tsui. (2002). Space of Leaning. In F. Marton & A. M. R. Tsiu (Eds.), *Classroom Discourse and the Space of Learning*: Lawrence Erlbaum Associates.

Marton, F., Runesson, U., & Tsui, B. M. (2004). Space of Leaning. In F. Marton, A. M. R. Tsiu, (with, P. P. M. Chik, P. Y. Ko, M. L. Lo, I. A. C. Mok, F. P. Ng, M. F. Pang, W. Y. Pong & U. Runesson (Eds.), *Classroom Discourse and the Space of Learning*. Mahwah: Lawrence Erlbaum Associates, Inc.

Mason, J. (2013). *Resposible Teaching of Mathematics: Learner-friendly and Mathematically-respectful.* Paper presented at the Fourth Africa Regional Congress of the International Commision on Mathematical Instruction, Maseru, Lesotho.

Maton, K. (2006). On knowledge structures and knower structures. In R. Moore, Arnot, M, Beck, J and Daniel, H (Ed.), *Knowledge, Power and Educational Reform: Applying the sociology of Basil Bernstein* (pp. 44 - 59). London: Routledge.

Maxwell, J. A. (2005). *Qualitative research design: An interactive Approach* (2 ed. Vol. 42). London: SAGE Publications.

McCarthy, J., & Oliphant, R. (2013). *MATHEMATICS OUTCOMES IN SOUTH AFRICAN SCHOOLS What are the facts? What should be done?* Johannesburg: Centre for Development Enterprise.

McCarthy, J., & Rebecca, O. (2013). *Mathematics outcomes in South African schools. what are the facts? What should be done?* Johannesburg: CDE.

Moalosi, S. S. (2008). An investigation of mathematics teachers' subject matter and pedagogical content knowledge in linear systems. Unpublished M.Ed Disertation. University of Botswana.

Moalosi, S. S., & Kaino, L. M. (2010a). *School teachers' lack of subject matter and pedagogical content knowledge of Algebra: Botswana Junior Secondary Schools. Proceedings of the third AFRICME conference.* Paper presented at the AFRICME 3, University of Botswana, Gaborone.

Moalosi, S. S., & Kaino, L. M. (2010b). *School teachers' subject matter and pedagogical content knowledge of Algebra: Botswana Junior Secondary Schools.* . Paper presented at the AFRICME 3, University of Botswana.

Moses, R. P. a. C., Jr. C. E. (2001). *Radical equations:Civil Rights from Mississipi to the Algebra Project*. Boston, MA: Beacon Press.

Mullis, I. V. S., Martin, M. O., Foy, P., & Arora, A. (2012). *TIMSS 2011 International Results in Mathematics:TIMSS & PIRLS International Study Center and IEA.* Chesnut Hill, USA.

Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., & Chrostowski, S. J. (2004). *TIMSS 2003 International Mathematics Report: Findings from the IEA's Trends in International Mathematics and Science Study at the Eighth and Fourth Grades*. Chesnut Hill, MA: Boston College.

Mundry, S., & Loucks-Horsley, S. (1999). Designing professional Development for Science and matheamtics Teachers: Decision Points and dilemmas. *3*(1). Retrieved from www.wcerwisc.edu/nise

Parker, D. (2009). *The specialisation of pedagogic identities in initial mathematics teacher education in post-apartheid South Africa. Unpublished PhD thesis.* University of the Witwatersrand, Johannesburg, South Africa.

Parker, D., & Adler, J. (2012). Sociological tools in the study of knowledge and practice in mathematics teacher education. *Educational Studies in Mathematics*. doi:10.1007/s10649-012-9421-y

Pillay, V. (2013). *Enhancing mathematics teachers' mediation of a selected object of learning through participation in a learning study:the case of functions in Grade 10.* Unpublished Research, University of the Witwatersrand, Johannesburg.

Pournara, C. (2013). *Mathematics-for-teaching in pre-service teacher education:the case of financial mathematics.* Unpublished Research, University of the Witwatersrand, Johannesburg.

Pournara, C., & Adler, J. (2014). *Revisiting school mathematics: A key opportunity for learning mathematics-for-teaching.* Paper presented at the British Congress of Mathematics Education.

Roehler, L. R., Duffy, G. G., Conley, M., Hermann, B. A., Johnson, J., & Michelson, S. (1987). Paper presented at the American Educational Research Association, Washington DC.

Rubin, A., & Babbie, E. (1997). *Research Methods for Social Work* (3 ed.). London: Brooks/Cole Publishing Company.

Runesson, U. (1999). *Teaching as constituting a space of variation*. Paper presented at the 8th EARLI -Conference.

Runesson, U. (2005). Beyond discourse and interaction. Variation: a critical aspect for teaching and learning mathematics. *Cambridge Journal of Education, 35*(1), 69-87.

Scheerens, J., & Sleegers, P. (2009). CHAPTER 2: Enhancing educational effectiveness through teachers' professional development. Retrieved 13 July 2010, from http://ec.europa.eu/education/school-education/doc/talis/chapter2_en.pdf

Schifter, D. (1998). Learning mathematics for teaching: From a teachers'seminar to the classroom. *Journal of Mathematics Teacher Education 1*(1), 55 - 87.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*, 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*, 1-22.

Shulman, L. S. (2004). *Teaching as Community property*. San Fracisco: Jossey-Bass.

Simkins, C. (2013). *Performance in the South African Educational System: What do we know?* Johannesburg: Center for Development & Enterprise.

Singh, P. (2002). Pedagogising Knowledge: Bernstein's theory of the pedagogic device *British Journal of Sociology of Education, 23*(4), 571 - 582.

Smith, H. W. (1975). *Strategies of Social Research*. New Jersey: Prentice Hall, Inc.

Spaull, N. (2013). *South Africa's Education Crisis:The quality of education in South Africa 1994-2011*. Johannesburg: Center for Development & Enterprise.

Stumpf, S. E. (1994). *Philosophy: Histrory and Problems*. New York: MacGraw Hill, Inc.

Tall, D. (Ed.). (1997). *The transition to advance mathematical thinking*. New York: Macmillan.

Thomas, H., L. (1969). *An Analysis of stages in the attainment of a concept of Functions.* Columbia University, Michigan.

Tirosh, D., Even, R., & Robinson, E. (1998). Simplifying Algebraic Expressions: Teacher Awareness and Teaching Approaches. *Educational Studies in Mathematics, 35*, 65-83.

Veal, W., & Maknister, J. (1999). Veal, W. R., & Maknister, J. G. (1999). Pedagogical Content Knowledge Taxonomies. Retrieved from http://unr.edu/homepage/crowther/ejese/vealmak.html

Vescio, V., Ross, D., & Adams, A. (2008). A review of research on the impact of professional learning communities on teaching practice and student learning. *Teaching and Teacher Education, 24*, 80 - 91. Retrieved from www.sciencedirect.com

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematics Education, Science and Technology, 14*(3), 293 - 305.

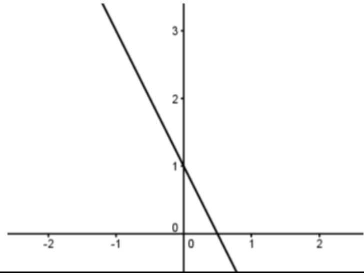
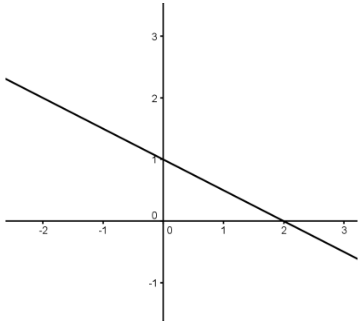
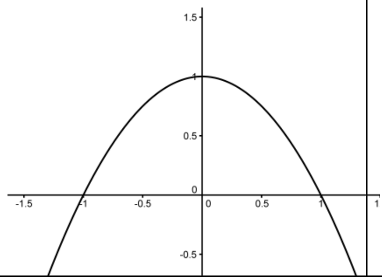
Watson, A., & Harel, G. (2013). The Role of Teachers' Knowledge of Functions in Their Teaching: A Conceptual Approach With Illustrations From Two Cases. *Canadian Journal of Science, Mathematics and Technology Education, 13*(2), 154 - 168.

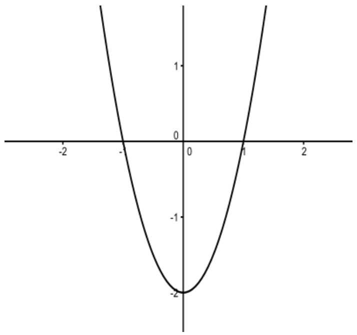
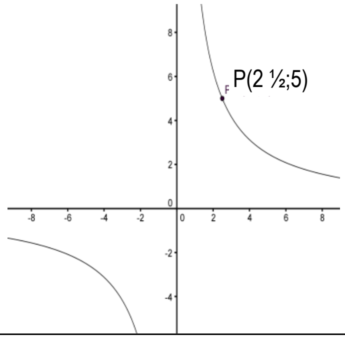
Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge, England: Cambridge University Press.

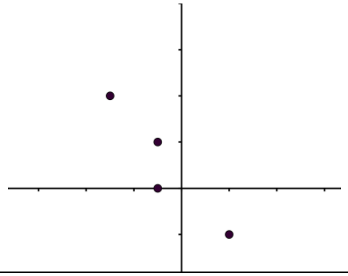
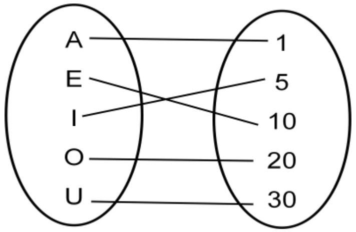
Wood, D., & Wood, H. (1996a). Vygotsky and Education. *Oxford Review, 22*(1), 5 - 16.

Wood, D., & Wood, H. (1996b). Vygotsky, Tutoring and Learning. *Oxford Review of Education, 22*(1), 5-16. Retrieved from http://www.jstor.org/stable/1050800

APPENDIX A: RESOURCES USED IN FUNCTIONS 1

<table>
<tr><th></th><th>Algebraic</th><th>Numeric</th><th>Graphical</th><th>Verbal</th><th>Extra (may be correct/ incorrect)</th></tr>
<tr><td>Linear</td><td>1
$y = -2x + 1$</td><td>5
<table><tr><td>x</td><td>-3</td><td>0</td><td>1</td><td>5</td></tr><tr><td>y</td><td>7</td><td>1</td><td>-1</td><td>-9</td></tr></table></td><td>11
</td><td>16
To get y you double x, multiply by -1 and then add 1</td><td>8
</td></tr>
<tr><td>Quadratic 1</td><td>4
$f(x) = 1 - x^2$</td><td>23
<table><tr><th>x</th><th>f (x)</th></tr><tr><td>-2</td><td>-3</td></tr><tr><td>0</td><td>1</td></tr><tr><td>1</td><td>0</td></tr><tr><td>12</td><td>-143</td></tr></table></td><td>12
</td><td>3
To get f (x) you square x and subtract it from 1</td><td>9
$y = x^2 - 1$</td></tr>
</table>

<table>
<tr><th></th><th>Algebraic</th><th>Numeric</th><th>Graphical</th><th>Verbal</th><th>Extra (may be correct/ incorrect)</th></tr>
<tr><td>Quadratic 2</td><td>15
$f(x) = -2x^2 - 2$</td><td>21
<table><tr><th>input</th><th>output</th></tr><tr><td>-2</td><td>6</td></tr><tr><td>1</td><td>0</td></tr><tr><td>4</td><td>30</td></tr></table></td><td>14
</td><td>10
To get the output you take the input, square it then double it then decrease it by 2</td><td>17
$y = -2(1-x)^2$</td></tr>
<tr><td>Hyperbola</td><td>2
$g(x) = \frac{2}{x}$</td><td>22
<table><tr><th>x</th><th>-1</th><th>½</th><th>2,5</th><th>10</th></tr><tr><td>y</td><td>-2</td><td>4</td><td>0,8</td><td>0,2</td></tr></table></td><td>6

</td><td>18
To get the output you take the input and divide it into 2</td><td>20
$y = \frac{x}{2}$</td></tr>
</table>

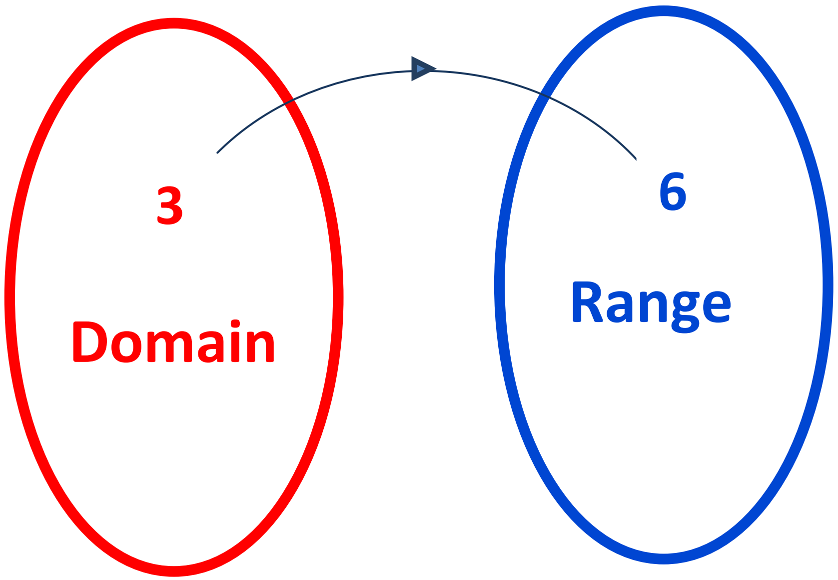
<table>
<tr><th></th><th>Algebraic</th><th>Numeric</th><th>Graphical</th><th>Verbal</th><th>Extra (may be correct/ incorrect)</th></tr>
<tr><td>Other cards</td><td>24
</td><td>7
</td><td>13
<table><tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>y</td><td>2</td><td>4</td><td>8</td><td>-1</td></tr></table></td><td>19
Halve the input to get the output</td><td>25 make your own card</td></tr>
<tr><td>Blank cards</td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

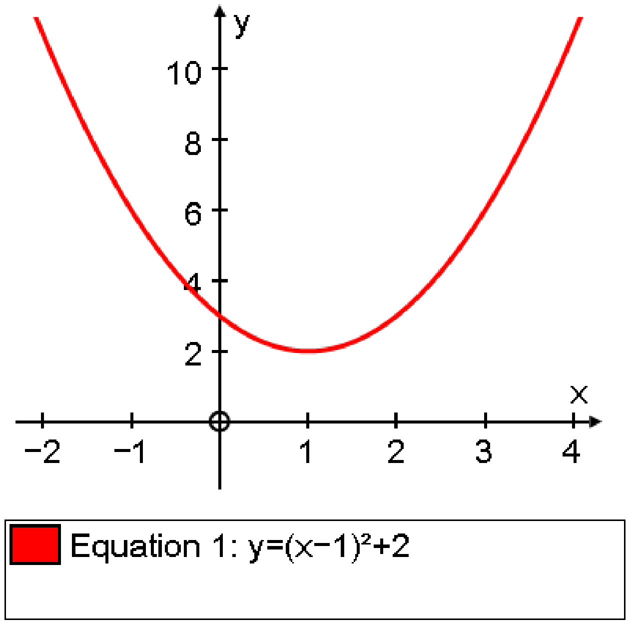
APPENDIX B: RESOURCES USED IN FUNCTIONS 2

Thinking about inputs and outputs

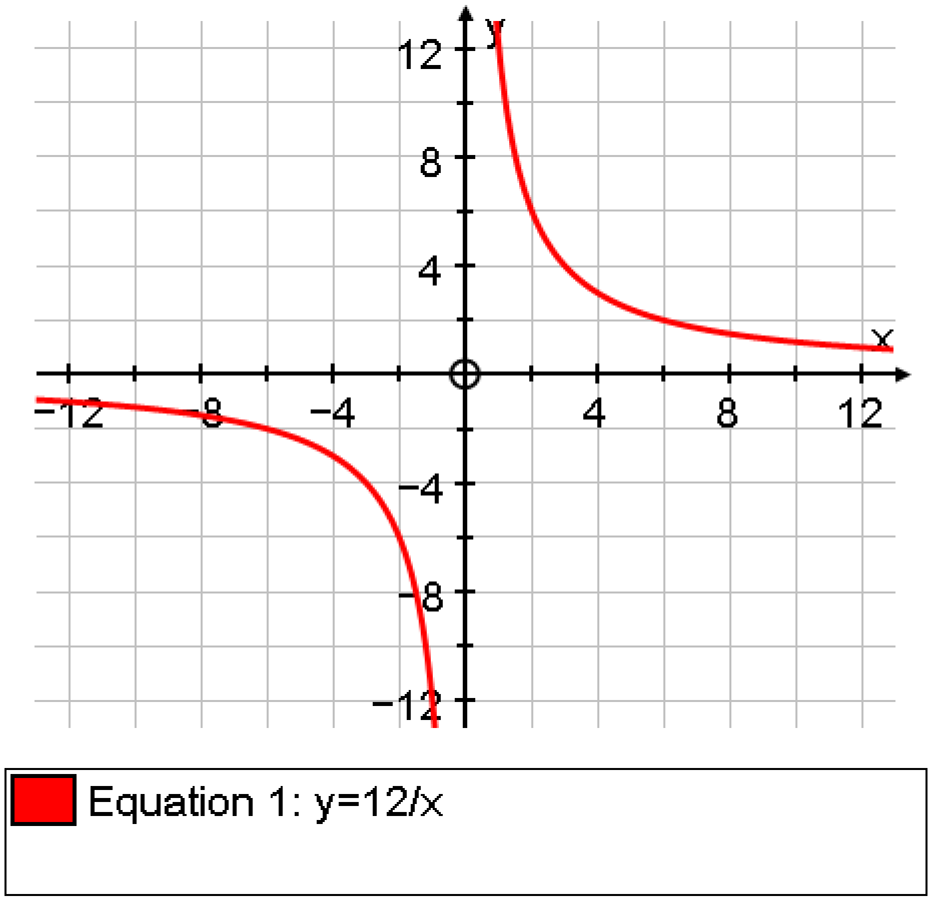
$$f(x) = (x-1)^2 + 2$$

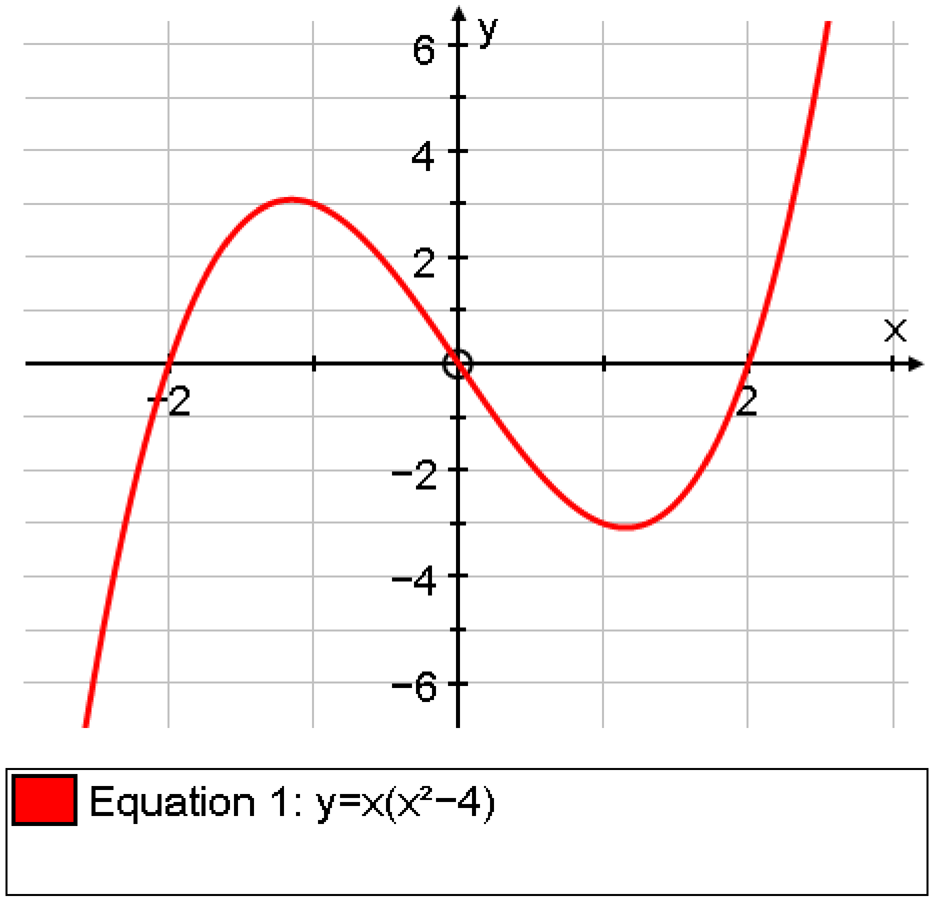
x	-3	-2	-1	0	1	2	3	4
$f(x)$	18	11	6	3	2	3	6	11





One-to-one functions: the horizontal ruler test.

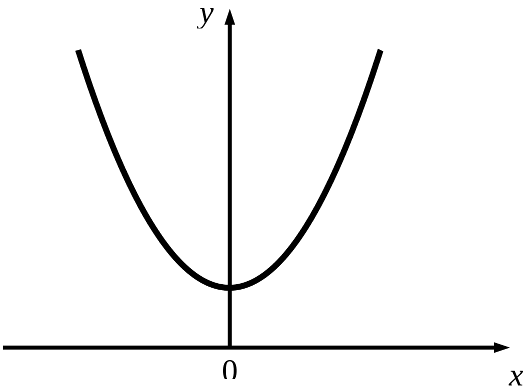


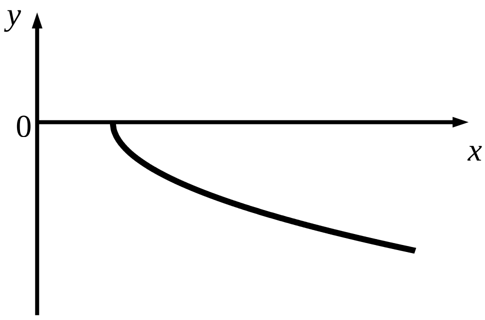


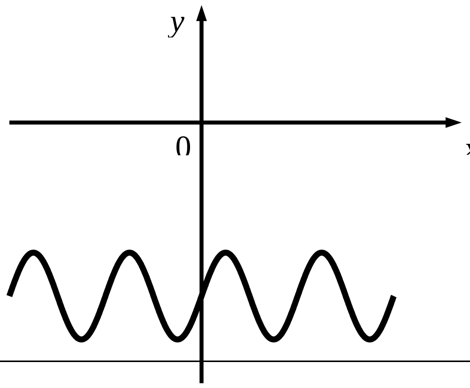
RESOURCES USED IN TASK 2

$y = \ldots\ldots\ldots\ldots$	$y = -\sqrt{x-1}$
$y = 2x - 1$	$y = \tan x^{\circ}$
$y = \dfrac{1}{x} + 1$	$y = \cos x^{\circ} + 1$
$y = 2\cos x^{\circ}$	$y = \sin x^{\circ} - 4$
$x \in \mathbb{R},\ \ x \geq 1$	$x \in \mathbb{R}$
$x \in \mathbb{R}$	Domain is:

$x \in \mathbb{R}$	$x \in \mathbb{R}$
$x \in \mathbb{R},\ x \neq 0$	$x \in \mathbb{R}$
$y \in \mathbb{R}$	$y \in \mathbb{R},\ 0 \leq y \leq 2$
$y \in \mathbb{R}$	$y \in \mathbb{R},\ y \geq 1$
$y \in \mathbb{R},\ y \neq 1$	$y \in \mathbb{R},\ y \leq 0$
Range is:	$y \in \mathbb{R},\ -2 \leq y \leq 2$


y
0
x


y
0
x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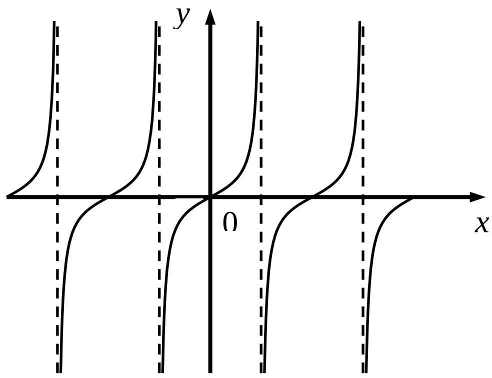
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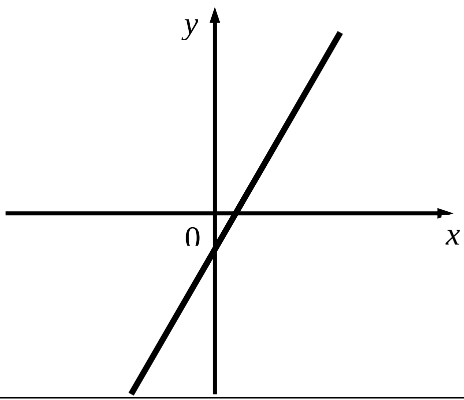
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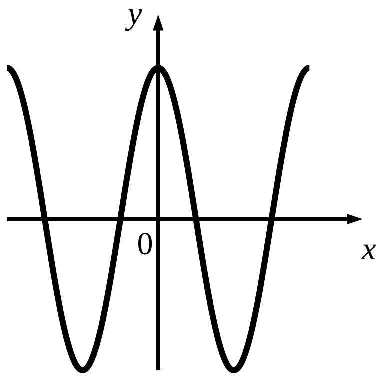
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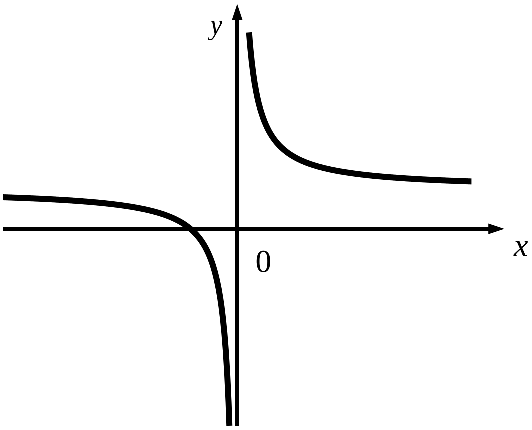
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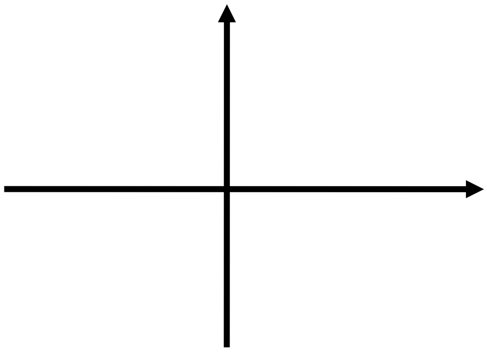
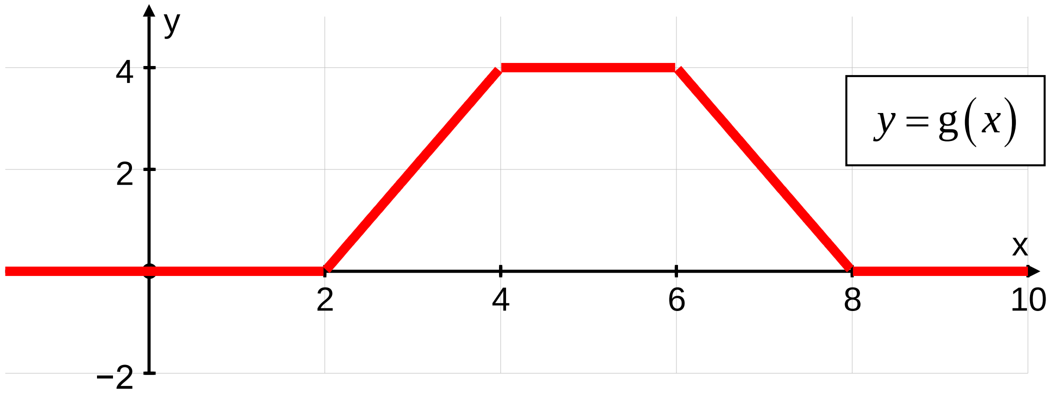


RESOURCES USED IN TASK 3

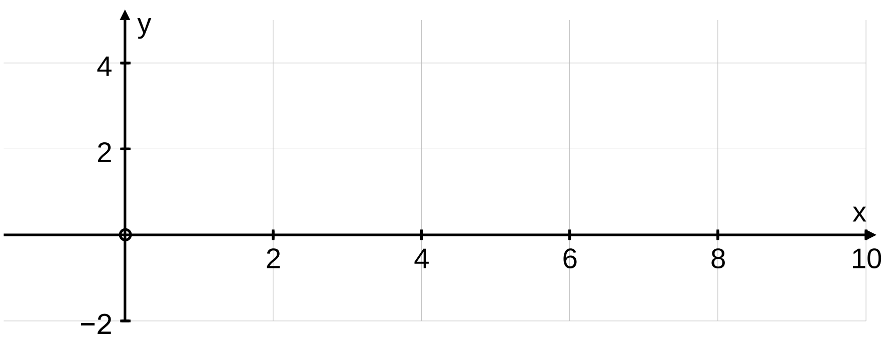
Can you find a function for each domain-range combination?

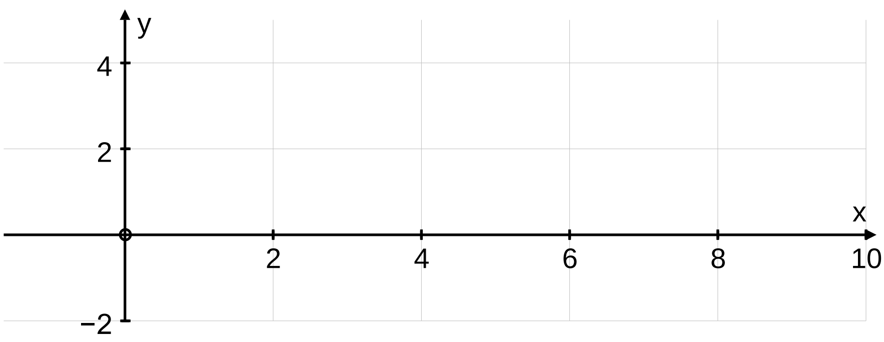
Domain / Range	$x \in \mathbb{R}$	$x \in \mathbb{R}, x \geq 0$
$y \in \mathbb{R}$		
$y \in \mathbb{R}$, $y \geq 1$		
$y \in \mathbb{R}$, $-1 \leq y \leq 1$		

RESOURCES USED IN TASK 4



x	0	1	2	3	4	5	6	7	8	9	10
$g(x)$											
$g(x-1)$											
$g(2x)$											





RESOURCES USED IN TASK 4

Categorising transformed functions

Think about all transformations of $\mathrm{f}(x) = x^2$ and the three properties:

A: The range includes the value $y = -1$

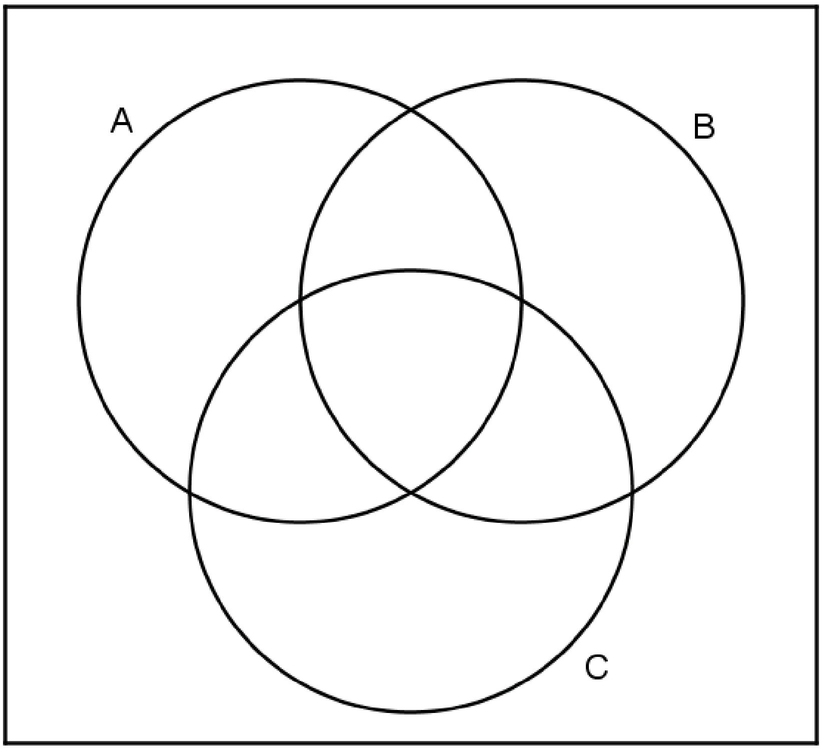
B: The graph is of the form $y = \mathrm{f}(x-3) + a$

C: The graph passes through the fourth quadrant

Can you find a transformation of $\mathrm{f}(x) = x^2$ which satisfies all three properties, A, B and C? If so, write this function in the central region where the three circles overlap.

How about a transformation of $\mathrm{f}(x) = x^2$ which doesn't satisfy any of the requirements? Fill this in the region outside the three circles.

The task: To find one example for each region. Is it possible to find an example for every region?



APPENDIX C: INTERVIEW RATIONALE AND PROTOCOL

INTERVIEW RATIONALE

The aim of the interview is to gather information about teachers' experiences of the Professional development model focused on mathematical objects to understand their lived object of learning. Hence all the information gathered through this protocol will provide data for answering the following critical question:

How does teachers' participation in professional development mediate their professional knowledge? What comes to be their lived object?

Note: The interview protocol has three parts and each part is constitutes a set of questions.

Part 1

Question 1: In PD teachers were exposed to a variety of discursive resources that were used to mediate teachers' mathematical knowledge for teaching with more focus on SMK. Through this question I intend to gather information that will make it possible for me to explore how these resources might have contributed to the way teachers experienced the PD. Hence I ask teachers to say which of these resources were useful to them, and how they made it possible for them to extend their knowledge of functions.

Question 2: Through this question I seek to explore teachers' experiences of the PD with respect to the message of Theme 2. The theme identified the need to know key features and associated terminology in functions as important aspects in the teaching and learning of functions. Hence I want to use this question to explore what and how these features enhanced teachers' experiencing and whether, at the end of participation in PD, teachers shared the view that function terminology and key features are important aspects that should be fore-grounded in the teaching and learning of functions. This is important since before teachers can begin to adopt ideas from PD into their classrooms, there must be evidence of shifts in their earlier perceptions about functions.

Part 2

Question 1: The question is a follow-up to the two lesson plans that teachers have already developed. In this question I purely seek to probe into why teachers planed differently or otherwise and how the differences if any, might lead to improved teaching and learning of functions. Information gathered will contribute to the overall understanding of teachers' experiencing of the object focused PD.

Part 3

Question 1: Teachers' understanding and comprehension of the definition of a function is critical in subsequent questions. The purpose of the question therefore is to provide the teacher with a tool that will facilitate responses to subsequent questions.

Question 2: Through this question I seek to probe further into the teacher's comprehension of the definition. The teacher's response to this particular question will be critical when I begin to interpret responses to questions 3, 4, and 5. How the teacher responds to questions 3, 4, 5, and 7 hinges on his/her comprehension of the definition. The questions require the teacher to apply the definition of a function to respond positively.

Question 3: In this question I probe into the teacher's knowledge of the definition with particular focus on his/her understanding that a functional relationship does not necessarily have to have a know graph or equation. In one of the sessions professional developers engaged teachers in activities where this fact was explicit. As a matter of fact, this particular task was present in one of the activities. Including this task in the interview protocol is relevant because it make it possible to directly tap into the teachers' lived object of learning. In other words, it is directly related to what teachers were exposed to in PD, and asking teachers about things that are familiar to them might enhance the quality of responses that they might give.

Question 4: In this question I want to tap into the teacher's knowledge of student and content. I am trying to get insights into how teachers might use their knowledge of functions in a classroom context. This question is on PCK (knowledge of student and content) and is different from question 4b which taps into SMK. Though the content of the two questions are similar, each has a different focus.

Question 5: I have chosen to include this task in the protocol for the following reason. In session 2.1, teachers were given a pile of 25 cards and were then asked to sort them into two piles of those representing functions and those that did not. All the cards given to teachers represented functions. During the sorting teachers identified some cards as representing non-functions. In subsequent activities of the PD teachers engaged in activities where they were able to discern that all the cards (including the ones they had earlier said were not functions) were all functions. The cards that I have chosen to use in the card sorting activity are those that teachers had said were not functions. I believe this question is important for understanding what teachers think about these cards long after PD.

Question 6: The question probes into teachers' knowledge of one of the key features of functions (multiple representations). Both session 2.1 and 2.2 engaged teachers in activities where they had to match representations. Through this question I want to tap into teachers' lived object of learning through a familiar activity. The card matching activity was also present in 2.3 where learners were asked to match the representations. Teachers were present in these activities. The intention of this question is to find out whether teachers might adopt this activity into their teaching of functions. I do this by asking them where they would slot in the activity in their lesson (beginning, middle, or end). I also want to find out whether teachers think the activity can improve teaching and learning in their classes.

Question 7: This question is closely related to question 3. Hence it may be rightly construed as a follow up question to question 3. Failure to find more lines or curves that pass through points A and B in question 3 might be a result of learners thinking that graphs are either straight lines or curves. Therefore the teachers' responses to these two questions are expected to corroborate each other.

INTERVIEW PROTOCOL

Thank you for honouring this interview. Your contribution through this interview is highly appreciated and will be acknowledged.

Please note that you are not the focus of this interview. Instead I want you to assist me to obtain information about the Wits Maths Connect model of professional development. In particular, the interview will focus on Functions Theme work in which you were a participant. I have stated in the consent letter that the purpose of this interview is to gather information which I will use in my research through which I want to understand what and how you might have experienced the WMC professional development focused on Functions. The interview is expected to last for an hour. Thank you once more for your willingness to participate in my research.

Information

The interview is divided into three Parts. Part 1 focuses on resources that were used in Theme 2 (Functions). Part 2 focuses on the two lesson plans that you have developed. Part 3 focuses on functions themselves and the teaching and learning of functions. Each part is expected to last for 20 minutes.

Attached here are the questions we will discuss in Part 3, so that you have time to think about them before the interview. I will bring copies of your lessons plans for discussion. It will be helpful if you bring along with you the resources you used when teaching functions this year in your various classes (e.g. other lesson notes or plans, textbook(s), other worksheets, any tests you set, etc). These will be useful in our discussion.

Part 1 - Discursive Resources (materials and terms)

1. There were four kinds of resources that were used in the functions sessions. I have here with me some of them.

   The first kind is resources that were given to teachers to deepen their own understanding of functions:

   a. ***Readings on functions***;
      i. What important information on functions did you obtain from the readings?
      ii. Where else, apart from prescribed learner and teacher text books do you obtain information on functions for your own knowledge and for planning your lessons?

   The second kind is resources that were present in sessions 2.1 and 2.2 that were attended by teachers only.

   b. ***Card Activities***-i.e. card sorting, and card matching;
   c. ***Use of teaching resources***-(e.g. use of Geo-gebra facility to construct and investigate graphs of functions).
   d. ***Emphasis on Key features and function terms-***
      i. multiple representations;
      ii. Definition - univalence; arbitrariness
      iii. parameters; and intercepts
      iv. Terminology and language: domain; range; input; output; dependent; independent.

      v. Can we begin by you telling me which of the resources was available in the sessions you attended?
      vi. Which of these helped you understand functions more?
      vii. If you have not tried them yet, can you tell me which of these resources may be useful in your functions lesson?

   The third kind is resources that were used by learners in the schools that hosted theme 2.3 and theme 2.4.

   e. ***Learner tasks (card matching and writing class tasks***)
   f. ***Homework writing tasks***
   g. ***Learner's Personal Grid***
   h. ***Kokies***
      i. Which of the resources would you say were effective in helping learners understand the function concept?
      ii. Why do you think so?
      iii. What did you learn from your observations of sub-themes 2.3 and 2.4?

   The fourth kind is resources that were given to teachers to use in their own function lessons

   i. ***Cards*** (matching multiple representations of functions; matching a representation to its domain and range)
   j. ***Large Grids*** (for use in classes without gridded-green-boards)
   k. ***CD*** (Geogebra Software)

i. Which of the resources have you used in your lessons?
ii. What did you exactly do with them?
iii. Were they useful or effective?
iv. Why did you not use others?

2. In the theme work, functions were presented in a way that emphasized the use of the ***mathematical language:***
   i. (input for x-value,
   ii. output for y-value,
   iii. domain, for all x-values,
   iv. and range-for all y-values)

   also in focus were ***key features*** of functions:
   v. multiple representations;
   vi. Definition - univalence; (one-to-one correspondence)
   vii. Definition - arbitrariness; (functional relationship can be between any two objects)
   viii. parameters;
   ix. and intercepts

   a) From your experience of teaching functions, what appeared to be similar to or different from:
      i. what you consider to be important to learn about functions
      ii. the mathematical language that you use when teach functions

   b) Suppose you were to teach functions tomorrow, which mathematical language and key features would you introduce into your lesson?
   c) The functions theme had 4 sessions, 2.1; 2.2; 2.3; and 2.4. In which session were you engaged in meaningful learning of functions?
   d) Can you give me examples of what you learnt in the theme(s) you just mentioned?
   e) Do you mind explaining how Professional Development made this possible for you to learn in that particular session?
   f) Among all things that may have interested you in the theme work, which of these caught your attention the most? Please explain.

Part 2 - Information about lesson plans

The section focuses on the two lesson plans that you have developed. In case you have forgotten what is entailed in each lesson plan, I brought copies of each lesson plan to make our discussion productive.

1. Maybe you have forgotten what you did in your Pre-Lesson plans, but here are both of them. I have had an initial look at them so that I could see what your focus was in each case. Please share with me what you see as different in the two plans, and why you think you made these changes? Your own views on what you were doing are useful to me to think about as well.
   a. How might the changes or differences if any, improve your teaching of this lesson
   b. How might the changes lead to a better understanding of the concept by your learners?

Part 3 – Functions and the teaching of functions

The section focuses on Functions themselves. I asked you to familiarize yourself with these tasks to facilitate the discussion in our forthcoming meeting, and hope you have had a chance to do so?

1. Here are two definitions of a function taken from our South African text books:

    - *A function is a special relationship between input values and output values where every input value has only one output value*

    - *A function relates each element of a set with exactly one element of another set*

    a) Do you think they are each a valid definition of function?

2. a) Which of the two definitions would you use in your lesson?

    i. How is it better than the other?

    b) As a teacher in your class, how do you normally define a function to your learners?

    c) A learner says that he/she does not understand this definition. How would you use a diagram to help the learner understand?

3. A learner is asked to give an example of a graph of a function that passes through the points A and B (see Fig. 1)

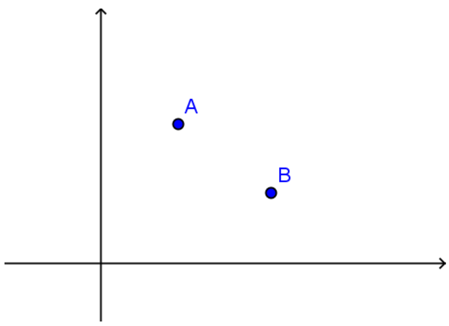


Figure 1

The learner gives the following answer (see Fig. 2)

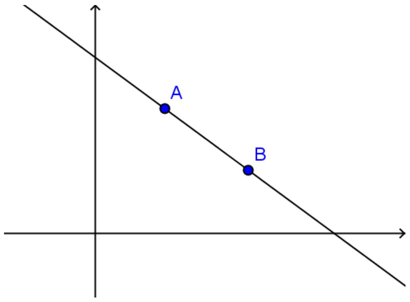


Figure 2

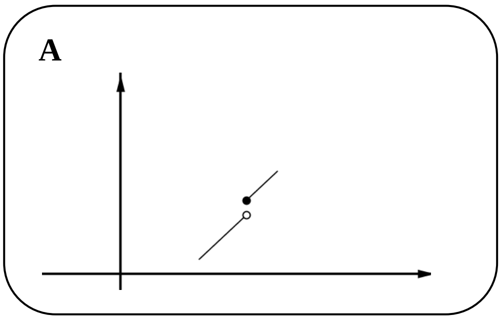
When asked if there is another way the learner says: “No”. How would you respond to this learner?

a) If you think the learner is right - explain why

b) If you think the learner is wrong – how many functions which satisfy the condition can you find? Explain

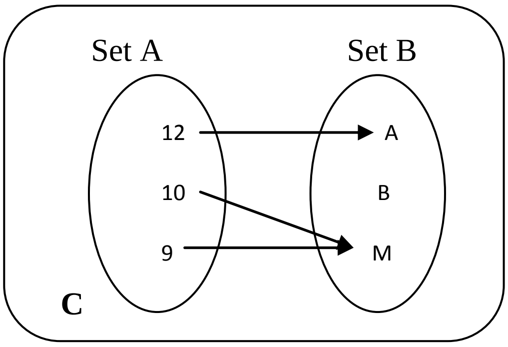
c) How do you think your learners in Gr __ will respond to the above task?

4. a) Imagine that in your class your learners had a task to work on in groups where they had to look at a set of cards each representing a possible function. They had to select those that were functions in one pile, and those that were not functions in another pile. Some learners had the following in their pile that they thought were functions but could not explain why. What would you say to the group about the cards in their pile?



B

$\{(1, 4), (2, 5), (3, 9)\}$



D

$f(x) = 4$

i. For each case decide whether the learner was right or wrong. Give reasons for each one of your decisions
ii. In cases where you think the learner was wrong, try to explain what the learner was thinking that could cause the mistake

5. b) Card sorting activity: Here is an activity you may be familiar with from Theme 2.

i. Here are 9 cards, please sort the cards into two piles, one representing functions and the other non-functions.
ii. In the case of non-functions, can you explain to me why you think they are non-functions?

6. Now, here are ten cards. Please take card 3, find three other cards that also represent this function
   a) What can this activity help learners understand about functions?

b) If you were planning to teach functions and you wanted to adopt the activity, would you insert it at the beginning or middle or end of the topic?
c) When would you insert it into a lesson (beginning or middle, or end of lesson)?

7. In Ms Moalosi's lesson focused on functions a learner reasoned that "a function is only that which can be represented by a straight line or a smooth curve. Any function whose graph is neither a straight line nor a smooth curve is not a function".
   a) Do you think the learner's reasoning is true/false?
   b) If the learner's reasoning is:
      i. True, please give an example to support the learner's reasoning
      ii. False, how would you correct the learner
   c) What could be the source of the learner's reasoning?

APPENDIX D: LETTERS AND CONSENT FORMS

Wits School of Education

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Tel: +27 11 717-3007 • Fax: +27 11 717-3009 • E-mail: enquiries@educ.wits.ac.za • Website: www.wits.ac.za

Student number: 509428
Protocol number: 2011ECE034**C**

07 June 2011

Shadrack Moalosi
shadrack.moalosi@wits.ac.za

Dear Shadrack Moalosi

Application for Ethics Clearance: Doctor of Philosophy

I have a pleasure in advising you that the Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate has agreed to approve your application for ethics clearance submitted for your proposal entitled:

Enhancing teacher knowledge through an object focused professional development model

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

Yours sincerely

M Mabeta

Matsie Mabeta
Wits School of Education

Cc Supervisor: Prof. J Adler (via email)

25 March, 2011

Dear Teacher

INVITATION TO PARTICIPATE IN MY PHD STUDY

As part of the research area of the Wits Maths Connect Secondary Project, I am conducting an in-depth PhD study in which I evaluate the Wits Maths Connect project's professional development. My study will look at how the WMC project assist in helping you grow professionally and I intend to evaluate its professional development activities in which you participate and to understand your experiences about participating in this professional development.

I would like to invite you to participate in this study. In order to contain the study so that it is not too big, I have decided to work with the teaching of functions, at Grade 10. Functions as we know are important and permeate almost all areas of mathematics.

Your participation in the study, if you agree to be part of it will involve the following:

a) Developing a Lesson Plan on a selected Sub-topic on Functions before participating in the Wits Maths Connect Cluster workshop on Functions and another Lesson Plan on the same Sub-topic following your participation in the workshop.

b) Participate in an interview based on the Pre- and Post-PD lesson plans described above

The interviews will take place in your school in the months of May and June 2011

In 2009, a consent form for, **video-taping** and **audio-taping** of interviews was sent to each teacher in the Wits Maths Connect Secondary Project, and we are grateful that you have given consent for us to work with you.

I acknowledge that this study adds more load in terms of the time you are already devoting to the Wits Maths Connect Project.

I look forward to working with you.

Yours sincerely

Shadrack Moalosi

Tel: (011)717-73236 (0ffice)

04 May, 2011

Dear Teacher

INVITATION TO DEVELOP PRE & POST LESSON PLANS ON FUNCTIONS

As you know, we start our theme work on Functions in May, and are looking forward to your participation. We also wish to evaluate the benefits of the theme work for teachers. For this purpose we humbly request you to develop ***one Lesson Plan on Grade 10 Linear Functions before the Wits Maths Cluster workshop on Functions.***

We request that you bring your plan to the first cluster workshop.

You might not be teaching Grade 10. If so, you can write a plan for a lesson on functions at a Grade 9 (linear function) or Grade 11 level.

Could you please include in your plan

1. The topic and purpose of the lesson
2. The grade it is aimed at
3. The steps you plan to follow in the lesson and what you expect learners to be doing. You might find the template attached helpful.

What the lesson plans are for

The information in the lesson plans will be used in two Doctoral studies which have been designed to evaluate the Wits Maths Connect project's professional development. One study is conducted by Shadrack and one by Vasen. We aim to evaluate the WMC professional development activities in which you participate and the way in which these impact on your planning. That is why we will also ask you to write a second lesson plan after the theme work is completed.

Evaluating the project is essential since information gathered will be useful in planning future professional development activities.

We look forward to your usual cooperation.

Yours sincerely

Shadrack Moalosi

For further information you may contact Shadrack Moalosi

Tel: (011)717-73236 (0ffice)

November, 2011

Dear Teacher

INVITATION TO AN INTERVIEW

By copy of this letter I wish to invite you to an interview in which I seek to understand your experiences of the Wits Maths Connect professional development activities focused on functions.

The interview is divided into three Parts. Part 1 focuses on resources that were used in Theme 2 (Functions). Part 2 focuses on the two lesson plans that you have developed. Part 3 focuses on functions themselves and the teaching and learning of functions. Each part is expected to last for 20 minutes but may take longer.

Please note that you are not the focus of this interview. Instead I want you to assist me to obtain information about the Wits Maths Connect model of professional development. In particular, the interview will focus on Functions Theme work in which you were a participant. I have stated in the consent letter that the purpose of this interview is to gather information which I will use in my research through which I want to understand what and how you might have experienced the WMC professional development focused on Functions. The interview is expected to last for an hour. Thank you once more for your willingness to participate in the interview.

I acknowledge that this interview adds more load in terms of the time you have already devoted to the Wits Maths Connect Project.

I am looking forward to our meeting.

Thank you

Yours honestly

Shadrack Moalosi

Tel: (011)717-3236 (0ffice)

INFORMATION SHEET AND CONSENT LETTERS: TEACHERS

Invitation to participate in the SHADRACK MOALOSI's Research Project

Titled

'Enhancing teachers' knowledge through an object- focused professional development: What and how does it mediate teachers' professional knowledge?

Undertaken by myself **SHADRACK MOALOSI**, a Doctoral student at the University of the Witwatersrand

Dear Mr/Mrs/Ms [Teacher ……….. individual names will be inserted into all the teacher letters]

As per your verbal consent to participate in my research project I hereby formalize my request for your consent. This letter provides you with information about the various aspects of the research that will be done, and asks for your consent to participate. All the information is on the next page, which is for you to keep. Below I formally ask you to indicate whether you are willing to take part in the research

If you are happy to take part in the research, please sign below.

I am happy to participate in SHADRACK MOALOSI's Doctoral research subsequent to my participation in the Wits Maths Connect project professional development

Signed………………………………………………………………………………………

Date ………………………………………………………………………………………

Name ………………………………………………………………………………………

The project has the support of the Gauteng Department of Education, and the Johannesburg East District Office.

WHAT WILL THE RESEARCHER DO?

I will be observing the Wits Maths Connect project professional development activities where teachers sit in a class where professional developers demonstrate ways of presenting certain mathematical concepts. Prior to the presentation I will ask you to develop a pre-lesson plan to demonstrate how you usually present the concept of Functions. After you have participated in Wits Maths Connect professional development activities I will request you to develop a post-lesson plan on the same concept.

I will need to collect data from you, specifically:

- To evaluate the Wits Maths Connect project's impact on your professional knowledge and so what you say and do pertaining to your participation in this project is important for my research. In particular, discussions emanating from your observations on the presentations and from your lesson planning will provide invaluable data for my research. Data will be collected in term 2 of the GDE school calendar between May and June 2011.

HOW WILL THE INFORMATION BE USED

I will analyze data for my research from the pre and post-lesson plans as well as from follow up interviews that I will audiotape.

HOW WILL INFORMATION BE USED

From the data I will write a Thesis which I will submit to the University of the Witwatersrand for the award of a Doctoral Degree qualification. The benefits of participating in this research is that the information you provide will enable the Wits Maths Connect professional developers improve ways in which they conduct their activities and also know the extent to which the objectives of the project are realized.

All videotapes will be used for the duration of my research project and later returned to the Wits Maths Connect project who will store them for a limited period. Thereafter they will be destroyed.

YOUR RIGHTS

I will not use your name in my Thesis or in any reports or articles that will emerge from this research. That is, anonymity and confidentiality is guaranteed.

The research is independent of your professional responsibilities in school. At no point will any of the information obtained for research purposes affect your appraisal.

There is no problem if you do not wish to take part in the research.

If you agree now to participate, and later decide that you no longer want to continue participating in the study, you are free to withdraw at any time.

For more information please do not hesitate to contact myself at the following addresses:

SHADRACK MOALOSI: University of the Witwatersrand, Wits School of Education. Email: shadrack.moalosi@wits.ac.za . Mobile: 0739379804

TEACHER CONSENT FORM FOR DEVELOPING PRE AND POST LESSON PLANS

If you are happy to develop the pre and post lesson plans subsequent to your participation in the Wits Maths Connect professional development activities please sign below

I am happy for the interviews with me to be audio-taped

Signed

………………………………………………………………………………………… ……………………………………………………

Date

………………………………………………………………………………………… ……………………………………………………

Name

………………………………………………………………………………………… ……………………………………………………

WHAT WILL THE RESEARCHER DO?

The researcher would like to collect information about the Wits Maths Connect professional development from you through your pre and post lesson planning. The purpose of the lesson plans will be to make visible your professional learning from the project. I will analyze the data emerging from the lesson plans and this data will contribute towards my thesis that I will submit to the University of the Witwatersrand for the award of a Doctoral degree. The data will be stored among the larger data of the Wits Maths Connect project and will later be disposed off in accordance with the laid down ethical procedures that ensure and maintain both your anonymity and confidentiality.

YOUR RIGHTS

You can withdraw your consent at any point in the research and this research will not influence in any negative way, your professional development and progress as a career teacher. Your withdrawal will not have any negative impact in your aspirations or ambitions as a teacher. Hence it will be unconditional.

APPENDICES IN THE CD

APPENDIX E: CHAPTER 9 WITH DETAILED ANALYSIS OF TASKS 2,3 & 4

APPENDIX F: CHAPTER 11 WITH DETAILED ANALYSIS OF LESSON PLANS

APPENDIX G: EVIDENCE OF RESOURCES USED BEFORE PD

APPENDIX H: EVIDENCE OF TEACHERS' EXPERIENCING OF PD ACTIVITIS

APPENDIX I: CODING TEACHER UTTERANCES IN NVIVO