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Symmetries and Conservation Laws of Difference and Iterative Equations

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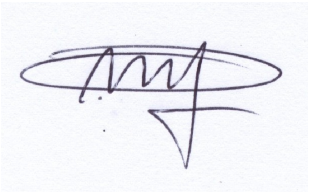
Johannesburg, in fulfilment of the requirements for the degree of Doctor of Philosophy.

Johannesburg, August 2015.



Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly. It is being submitted for the degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.



Mensah Kékéli Folly-Gbetoula, 12 August 2015.

Dedication

To my son John Dieudonné.

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Abstract

We construct, using first principles, a number of non-trivial conservation laws of some partial difference equations, viz, the discrete Liouville equation and the discrete Sine-Gordon equation. Symmetries and the more recent ideas and notions of characteristics (multipliers) for difference equations are also discussed.

We then determine the symmetry generators of some ordinary difference equations and proceed to find the first integral and reduce the order of the difference equations. We show that, in some cases, the symmetry generator and first integral are *associated* via the 'invariance condition'. That is, the first integral may be invariant under the symmetry of the original difference equation. We proceed to carry out double reduction of the difference equation in these cases.

We then consider discrete versions of the Painlevé equations. We assume that the characteristics depend on n and u_n only and we obtain a number of symmetries. These symmetries are used to construct exact solutions in some cases.

Finally, we discuss symmetries of linear iterative equations and their transformation properties. We characterize coefficients of linear iterative equations for order less than or equal to ten, although our approach of characterization is valid for any order. Furthermore, a list of coefficients of linear iterative equations of order up to 10, in normal reduced form is given.

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Chapter 1

Introduction

In the nineteenth century, symmetry attracted many scientists after the initial works by the prominent Norwegian mathematician Marius Sophie Lie (December 17 , 1842-February $18, 1899$). Thanks to him, the foundation of the theory of continuous groups of transformations that leaves differential equations invariant was laid.

His theory permits one to obtain exact solutions to differential equations by performing an algorithm now known as *symmetry analysis* of differential equations. After his death, G. Birkhoff, W. Killing, H. Weyl and V. Ovsiannikow, etc, worked on Lie groups and made significant contributions in the application of the theory in mathematical problems in which differential equations are used to model certain phenomena.

A German mathematician Emmy Noether (23 March $1882 - 14$ April 1935) later proved a connection between symmetries and the notion of conservation laws in physics. She came up with the theorem known as Noether's theorem that allows the construction of conservation laws for Euler-Lagrange equation by a formula, provided that the Noether's symmetries and the Lagrangian are known [1].

The theory, reasoning and algebraic structures dealing with the construction of symmetries for differential equations is now well established and documented. Moreover, the application of these in the analysis of differential equations, in particular, for finding exact solutions, is widely used in a variety of areas from relativity to fluid mechanics (see [2–5]). Secondly, the relationship between symmetries and conservation laws has been a subject of interest since Noether's celebrated work [1] for variational differential equations. The extension of this relationship to differential equations which may not be

variational has been done more recently [6, 7]. The first consequence of this interplay has led to the double reduction of differential equations [8–10].

A vast amount of work has been done to extend the ideas and applications of symmetries to difference equations in a number of ways - see [11–15] and references therein. Several techniques which are used to obtain a discretized differential equation have been a subject of study [16–20]. In some cases, the difference equations are constructed from the differential equations in a manner that the algebra of Lie symmetries remain the same [21]. Maeda [22, 23] was the first to develop a method for finding continuous point symmetries of ordinary difference equations. Recently, this method has been prolonged to difference equations by Levi and Winternitz [24], Quispel et al. [25, 26], Hydon [11, 12] and others. Unlike differential equations, powerful computer packages that are utilized to perform symmetry analysis of difference equations are presently not available.

In this thesis, we perform the symmetry analysis for difference equations both ordinary and partial. We study the association of conservation laws, integrability and symmetries for difference equations as well as the theory of double reduction. The latter is possible when the symmetries and conservation laws are associated via the invariance condition. As far as the double reduction of ordinary difference equations is concerned, the work is new and more work is needed to extend the theory to partial difference equations.

An Outline of the Chapters

In the second chapter, definitions and theorems are introduced to tackle our investigation. A method for calculating symmetries of differential equations and some well-known results and properties of differential equations with maximal dimension have been reviewed. We also review standard methods for constructing symmetries and conservation laws of difference equations already available in the literature.

In the third chapter, we study two non-autonomous partial difference equations, the discrete Liouville equation and the discrete Sine-Gordon equations. We see if we can obtain nontrivial symmetries and conservation laws of these equations and we discuss the more recent notion of multiplier of partial difference equations.

In the fourth chapter, we perform the symmetry analysis of some ordinary difference equations and proceed to find their conservation laws and then reduce their order. More importantly, we show that, in some cases, these symmetries and conservations are associated and we successfully perform the double reductions theory.

In the fifth chapter, we consider the discrete versions of Painlevé equations and we perform their symmetry analysis, and cases where exact solutions can be obtained are discussed.

Finally in the sixth chapter, we apply the same algorithm for finding symmetries to linear differential equations with maximal dimension. We review the work by Mohamed [27] and we aim to make contribution to his results. We proceed to look at how we can characterize these type of equations by their coefficients.

Publications associated with our work appeared in a number of articles, viz [41, 48, 52, 53]

Chapter 2

Preliminaries and Definitions

2.1 Introduction

We provide some background required to understand the concepts of transformations that defined Lie groups and the algorithm that generates Lie symmetries of differential equations and difference equations; we introduce some well-known properties of linear iterative equations needed to tackle their characterizations.

2.2 Analysis of Differential Equations via Symmetry

2.2.1 Basic Definitions

The definitions in this section are taken from a number of references, e.g., [28].

Definition 2.2.2 (Group). *[28] A group is a set G together with a group operation (usually called multiplication) such that for any two elements g and h of G, the product $g \cdot h$ is again an element of G. The group operator is required to satisfy the following axioms:*

- *Associativity. If g, h and k are elements of G, then*

$$g \cdot (h \cdot k) = (g \cdot h) \cdot k$$

.

- *Identity element. There is a distinguished element e of G, called the identity element, which has the property that*

$$e \cdot g = g \cdot e$$

for all g in G.

- *Inverses.For each g in G there is an inverse, denoted g^{-1}, with the property*

$$g \cdot g^{-1} = e = g^{-1} \cdot g.$$

Definition 2.2.3 (Manifold)**.** *[28] An m-dimensional manifold is a set M together with a countable collection of subsets $U_\alpha \subset M$, called coordinate charts, and one-to-one functions $\chi_\alpha : U_\alpha \mapsto V_\alpha$ onto connected open subsets $V_\alpha \subset \mathbb{R}^m$, called local coordinate maps, which satisfy the following properties:*

- *The coordinate charts cover M:*

$$\bigcup_\alpha U_\alpha = M.$$

- *On the overlap of any pair of coordinate charts $U_\alpha \cap U_\beta$ the composite map*

$$\chi_\beta \circ \chi_\alpha^{-1} : \chi_\alpha(U_\alpha \cap U_\beta) \mapsto \chi_\beta(U_\alpha \cap U_\beta)$$

is a smooth (infinitely differentiable) function.

- *If $x \in U_\alpha$, $\tilde{x} \in U_\beta$ are distinct points of M, then there exist open subsets $W \subset V_\alpha$, $\tilde{W} \subset V_\beta$, with $\chi_\alpha(x) \in W$, $\chi_\beta(\tilde{x}) \in \tilde{W}$, satisfying*

$$\chi_\alpha^{-1}(W) \cap \chi_\beta^{-1}(\tilde{W}) = \oslash.$$

Basically, a Lie group G is a group with an additional requirement that the map

$$G \times G \mapsto G,\ (g, h) \mapsto g \cdot h^{-1}$$

be smooth. A Lie group of dimension r is called an r-parameter Lie group. The reader can refer to the book by Olver for a better understanding of groups and manifolds.

Definition 2.2.4 (Lie Group). *[28] An r-parameter Lie group is a group G which also carries the structure of an r-dimensional smooth manifold in such a way that both the group operator*

$$m : G \times G \mapsto G, \qquad m(g, h) = g \cdot h, \qquad g, h \in G,$$

and the inversion

$$i : G \mapsto G, \qquad i(g) = g^{-1}, \qquad g \in G,$$

are smooth maps between manifolds.

Lie groups appear often as groups of transformations that act on manifolds and sometimes locally. That is, the group of transformations is only defined for some points on the manifold.

Definition 2.2.5 (Group of Transformations). *[28] Let M be a smooth manifold. A local group of transformations acting on M is given by a (local) Lie group G, an open subset $\mathcal{U}$, with*

$$\{e\} \times M \subset \mathcal{U} \subset G \times M,$$

which is the domain of definition of the group action, and a smooth map $\Psi : \mathcal{U} \mapsto M$ with the following properties:

- *If $(h, x) \in \mathcal{U}, (g, \Psi(h, x)) \in \mathcal{U}$, and also $(g \cdot h, x) \in \mathcal{U}$, then*

$$\Psi(g, \Psi(h, x)) = \Psi(g \cdot h, x).$$

- *For all $x \in M$,*

$$\Psi(e, x) = x.$$

- *If $(g, x) \in \mathcal{U}$, then $(g^{-1}, \Psi(g, x)) \in \mathcal{U}$ and*

$$\Psi(g^{-1}, \Psi(g, x)) = x.$$

Definition 2.2.6 (Symmetry Group). *[28] Let G be a local group of transformations acting on a manifold M. A subset $\mathcal{S} \subset M$ is called G-invariant, and G is called a symmetry group of $\mathcal{S}$, if whenever $x \in \mathcal{S}$, and $g \in G$ is such that $g \cdot x$ is defined, then $g \cdot x \in \mathcal{S}$.*

2.2.7 Lie Groups of Differential Equations and Calculation of Symmetries

The symmetry group of a differential equation is the largest local group of transformations that acts on the variables by mapping solutions onto other solutions. It is worth noting that for every Lie group that acts on a manifold, there exists a set of vector fields on the manifold called infinitesimal generators of the group operation.

Let us consider an nth-order differential equation

$$\mathbb{E}_n(x, y^{(n)}) = 0, \tag{2.1}$$

with the variables $x = (x^1, x^2, \ldots, x^p)$ that are independent and variables $y = (y^1, \ldots, y^q)$ that are dependent, where $y^{(n)}$ denote the collection of all first, second, third, $\ldots$, n-th order partial derivatives. Let us look for a one-parameter (ϵ) Lie group

$$\Psi(x, y) = (\Psi_1(x, y), \Psi_2(x, y)) = (\tilde{x}, \tilde{y}). \tag{2.2}$$

By expanding (2.2) to first-order in ϵ and by letting

$$\xi^i(x, y) = \frac{d}{d\epsilon}\Psi_1^i(x, y)|_{\epsilon=0}, \qquad i = 1, \ldots, p, \tag{2.3}$$

$$\phi^j(x, y) = \frac{d}{d\epsilon}\Psi_2^j(x, y))|_{\epsilon=0}, \qquad i = 1, \ldots, q, \tag{2.4}$$

where $\Psi_1 = (\Psi_1^1, \Psi_1^2, \ldots, \Psi_1^p)$ and $\Psi_2 = (\Psi_2^1, \Psi_2^2, \ldots, \Psi_2^p)$, the infinitesimal transformations take the form

$$\tilde{x} = x + \epsilon\xi(x, y) \tag{2.5}$$

$$\tilde{y} = y + \epsilon\phi(x, y). \tag{2.6}$$

The functions $\xi = (\xi^1, \ldots, \xi^p)$ and $\phi = (\phi^1, \phi^2, \ldots, \phi^q)$ are called the infinitesimals of (2.2) and the corresponding infinitesimal generator is given by

$$X = \sum_{i=1}^{p} \xi^i(x, y)\frac{\partial}{\partial x^i} + \sum_{j=1}^{q} \phi_j(x, y)\frac{\partial}{\partial y^j}. \tag{2.7}$$

Theorem 2.2.8 (General Prolongation Formula)**.** *[28] Let*

$$X = \sum_{i=1}^{p} \xi^i(x, y)\frac{\partial}{\partial x^i} + \sum_{j=1}^{q} \phi_j(x, y)\frac{\partial}{\partial y^j}$$

be a vector field defined on an open subset $M \subset X \times U$. The k-th prolongation of X is the vector field

$$X^{(k)} = X + \sum_{j=1}^{q} \sum_{J} \phi_j^J(x, y^{(k)}) \frac{\partial}{\partial y_J^j} \tag{2.8}$$

defined on the corresponding jet space $M^{(k)} \subset X \times U^{(k)}$, the second summation being over all (unordered) multi-indices $J = (j_1, j_2, \ldots, j_s)$, with $1 \leq j_s \leq p$, $1 \leq s \leq k$. The coefficient functions ϕ_j^J of $X^{(k)}$ are given by the following formula:

$$\phi_j^J(x, y^{(k)} = D_J \left(\phi_j - \sum_{i=1}^{p} \xi^i y_i^j \right) + \sum_{i=1}^{p} \xi^i y_{J,i}^j, \tag{2.9}$$

where

$$y_i^j = \partial y^j / \partial x^i, y_{J,i}^j = \partial y_J^j / \partial x^i$$

and

$$D_J = D_{j_1} D_{j_2} \ldots D_{j_s}.$$

Note that in the case $(t, x) \in X = \mathbb{R}^2$ and $y \in U = \mathbb{R}$, we have

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots,$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots.$$

The theorem that follows leads us to the algorithm that generates the infinitesimals of differential equations.

Theorem 2.2.9 (Infinitesimal Criterion)**.** *[28] Suppose*

$$\Delta(x, u^{(n)}) = 0,$$

is a differential equation of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M, and

$$X^{(n)}[\Delta(x, u^{(n)})] = 0, \tag{2.10a}$$

whenever

$$\Delta(x, u^{(n)}) = 0, \tag{2.10b}$$

for every infinitesimal generator X of G, then G is a symmetry group of the differential equation.

Note that (2.10) is a necessary and sufficient condition for G to be a symmetry group. In this thesis, we will only consider local point symmetries, i.e., we will assume that the infinitesimals are functions of x and u only. The invariance condition (2.10) leads to a single differential equation. Given that we are dealing with local point symmetries we can equate all coefficients of powers of derivatives of u that do not bear any constraints to zero. This naturally results into a system of determining equations whose solutions are the infinitesimals ξ and ϕ. Symmetries are important tools that can be used to reduce the order of a differential equation. In some cases, they can be used to reduce the number of variables as well as to obtain exact solutions via the invariance of the equation and to transform solutions into another solutions.

2.3 Symmetry Analysis of Difference Equations

A difference relation or recurrence relation is an equation that defines a sequence of terms, where initial terms are known and each successive term is obtained from the preceding terms. We shall assume in this work that all independent variables are in the set of integers $\mathbb{Z}$ and (i, j) shall be refereed to as a lattice point. A simple example of a difference equation is the Fibonacci sequence $u_n = u_{n-1} + u_{n-2}$ with $u_1 = u_2 = 1$. For ordinary difference equations, the order is the difference between the greatest and the lowest subscripts. For example,

$$u_{n+k} = \omega(u_n, u_{n+1}, \ldots, u_{n+k-1}), \tag{2.11}$$

is an ordinary difference equation of order k. Equation (2.11) is said to be linear if it is linear in u_n, u_{n+1}, $\ldots$, u_{n+k-1}. Since a special interest will be given to ordinary difference equations, we would like to briefly recall a basic method for solving linear ordinary difference equations with constant coefficients.

Consider a kth-order linear ordinary difference equation

$$u_{n+k} + a_1 u_{n+k-1} + a_2 u_{n+k-2} + \cdots + a_k u_n = 0. \tag{2.12}$$

The characteristic equation of (2.12) is given by

$$\lambda^k + a_1 \lambda^{k-1} + a_2 \lambda^{k-2} + \cdots + a_k \lambda = 0. \tag{2.13}$$

If we suppose that (2.13) has distinct roots λ_i with multiplicity r_i, $1 \leq i \leq m \leq k$, then the general solution of (2.12) is given by

$$u_n = \sum_{i=1}^{m} f_i(n) \lambda_i^{r_i}, \tag{2.14}$$

where f_i is a polynomial function of degree $r_i - 1$. Next, we aim to provide background on Lie symmetries analysis and the algorithm that generates symmetries of difference equations. The method is similar to the one for differential equations. In fact, in the case of differential equations the prolongation is done with respect to derivatives but with difference equations we prolong to points of the lattice that appear in the equation.

2.3.1 Symmetries

Consider a difference equation

$$\mathbf{E}(\mathbf{n}, \mathbf{u}_n, \mathbf{u}_{(n)}) = 0, \tag{2.15}$$

where $\mathbf{E}$ depend on the independent variable $\mathbf{n} = (n_1, n_2, n_3, \ldots, n_N) \in \mathbb{Z}^N$, the dependent variable $\mathbf{u}_n = \left(u_{n_1}, \ldots, u_{n_q}\right) \in \mathbb{R}^q$ and their shifts, $\mathbf{u}_{(n)}$. Consider also the transformations

$$\Gamma : (\mathbf{n}, \mathbf{u}_n) \mapsto (\hat{\mathbf{n}}, \hat{\mathbf{u}_n}). \tag{2.16}$$

Note that the operator $\mathcal{S}^{(i)}$ defined by

$$\mathcal{S}^{(i)}_j : n_j \mapsto n_j + \delta^i_j, \tag{2.17}$$

where δ is the Kronecker delta, will be referred to as the shift operator. We search for a one-parameter Lie group:

$$\hat{n_j} = n_j + \epsilon\, \xi_{n_j}(\mathbf{n}, \mathbf{u}_n), \tag{2.18a}$$

$$\hat{u}_{n_j} = u_{n_j} + \epsilon\, Q_{n_j}(\mathbf{n}, \mathbf{u}_n). \tag{2.18b}$$

$Q = \left(\xi_{n_1}, \xi_{n_2}, \ldots, \xi_{n_N}, Q_{n_1}, \ldots, Q_{n_q}\right)$ is the characteristic and ϵ the parameter of the group transformation Γ.

Here, the symmetry generator $\mathbf{V}$ is of the form

$$\mathbf{V} = \sum_{i=1}^{N} \xi_{n_i}(\mathbf{n}, \mathbf{u}_n) \frac{\partial}{\partial n_i} + \sum_{j=1}^{q} Q_{n_j}(\mathbf{n}, \mathbf{u}_n) \frac{\partial}{\partial u_{n_j}}, \tag{2.19}$$

and its infinite prolongation is given by

$$\mathbf{V}^{[\infty]} = \sum_{|J| \geq 0}^{\infty} \xi_{n_i + J}(\mathbf{n}, \mathbf{u}_n) \frac{\partial}{\partial (n_i + J)} + \sum_{|K| \geq 0}^{\infty} Q_{n_j + K}(\mathbf{n}, \mathbf{u}_n) \frac{\partial}{\partial u_{n_j + K}}. \tag{2.20}$$

Definition 2.3.2. *A symmetry generator,* $\mathbf{V}$*, of (2.15) is given by (2.19) and satisfies the symmetry condition*

$$\mathbf{V}^{[\infty]}\left(\mathbf{E}\right) = 0, \tag{2.21}$$

whenever (2.15) holds.

The imposition of the symmetry condition (2.21) leads to a single equation that involves functions with different arguments. Since ξ_{n_j} and Q_{n_j} are functions of $\mathbf{n}$ and $\mathbf{u}_n$, one can equate all coefficients of power of any new variable to zero. This leads to (usually after a number of derivations) an overdetermined system of differential equations from which ξ_{n_j} and Q_{n_j} are found.

Partial difference equations of order $1+1$

Where we have first-order difference equations in two variables ($N = 2$) we will let $n_1 = k$ and $n_2 = l$, and we will assume that (2.15) can be written in the form

$$u_{k+1}^{l+1} = \omega(k, l, u_k^l, u_k^{l+1}, u_{k+1}^l), \tag{2.22}$$

in which k, l are integers, u_k^l is a function of k and l. Note that k and l are independent variables. We redefine (2.17) as follows

$$S_m : u_m^n \mapsto u_{m+1}^n, \qquad S_n : u_m^n \mapsto u_m^{n+1}. \tag{2.23}$$

Since equation (2.22) depends explicitly on the lattice (k, l), we assume the same for the symmetries and we suppose that ω depends on all its dependent variables, that is,

$$\frac{\partial \omega}{\partial u_k^l} \neq 0, \quad \frac{\partial \omega}{\partial u_{k+1}^l} \neq 0 \quad \text{and} \quad \frac{\partial \omega}{\partial u_k^{l+1}} \neq 0. \tag{2.24}$$

Consider the transformation [11]

$$\Gamma : (k, l, u_k^l, u_{k+1}^l, u_k^{l+1}, u_{k+1}^{l+1}) \mapsto (k, l, \hat{u}_k^l, \hat{u}_{k+1}^l, \hat{u}_k^{l+1}, \hat{u}_{k+1}^{l+1}). \tag{2.25}$$

If Γ is a symmetry for (2.22) then we have

$$\hat{u}_{k+1}^{l+1} = \omega(k, l, \hat{u}_k^l, \hat{u}_{k+1}^l, \hat{u}_k^{l+1}) \tag{2.26}$$

whenever (2.22) holds. In order to get Lie symmetries we linearize (2.26) about the identity, as follows. We look for symmetries in the form

$$\hat{u}_k^l \;\; = \;\; u_k^l + \epsilon Q(k, l, u_k^l) + O(\epsilon^2). \tag{2.27a}$$

For partial difference equations we shall assume that the independent variables remain unchanged. We also have

$$\hat{u}^l_{k+1} = u^l_{k+1} + \epsilon Q(k+1, l, u^l_{k+1}) + O(\epsilon^2) \tag{2.27b}$$

$$\hat{u}^{l+1}_k = u^{l+1}_k + \epsilon Q(k, l+1, u^{l+1}_k) + O(\epsilon^2) \tag{2.27c}$$

$$\hat{u}^{l+1}_{k+1} = u^{l+1}_{k+1} + \epsilon Q(k+1, l+1, u^{l+1}_{k+1}) + O(\epsilon^2). \tag{2.27d}$$

Therefore, the linearized symmetry condition is given by

$$S_k S_l \, Q(k, l, u^l_k) - X\omega = 0, \tag{2.28}$$

where

$$X = Q\frac{\partial}{\partial u^l_k} + (S_k Q)\,\frac{\partial}{\partial u^l_{k+1}} + (S_l Q)\,\frac{\partial}{\partial u^{l+1}_k}. \tag{2.29}$$

We recall a useful theorem needed for obtaining symmetries and conservation laws.

Theorem 2.3.3. *[29] Suppose that ϕ is a real-valued functions defined on a domain D and continuously differentiable on an open set $D^1 \subset D \subset \mathbb{R}^n, (x^0_1, x^0_2, \ldots, x^0_n) \in D^1$, and*

$$\phi(x^0_1, x^0_2, \ldots, x^0_n) = 0. \tag{2.30}$$

Further suppose that

$$\frac{\partial \phi(x^0_1, x^0_2, \ldots, x^0_n)}{\partial x_1} \neq 0. \tag{2.31}$$

Then there exists a neighbourhood $V_\delta(x^0_1, x^0_2, \ldots, x^0_n) \subset D^1$, an open set $W \subset \mathbb{R}^1$ containing x^0_1 and a real valued function $\psi : V \mapsto W$, continuously differentiable on V, such that

$$x^0_1 = \psi_1(x^0_2, x^0_3 \ldots, x^0_n) \tag{2.32a}$$

$$\phi(\psi_1(x^0_2, x^0_3 \ldots, x^0_n), x^0_2, x^0_3 \ldots, x^0_n) \equiv 0. \tag{2.32b}$$

Furthermore for $k = 2, \ldots, n$,

$$\frac{\partial \psi_1(x^0_2, x^0_3, \ldots, x^0_n)}{\partial x_k} = -\frac{\frac{\partial \phi(x^0_1, x^0_2, x^0_3, \ldots, x^0_n)}{\partial x_k}}{\frac{\partial \phi(x^0_1, x^0_2, x^0_3, \ldots, x^0_n)}{\partial x_1}}. \tag{2.33}$$

Using Theorem 2.3.3, the reader can readily check that the differentiation with respect to u_k^l, keeping ω fixed and considering u_k^{l+1} to be a function of k, l, u_k^l, u_{k+1}^l and ω, is given by the differential operator L_0 as follows

$$L_0 = \frac{\partial}{\partial u_k^l} + \frac{\partial u_{k+1}^l}{\partial u_k^l}\frac{\partial}{\partial u_{k+1}^l} = \frac{\partial}{\partial u_k^l} - \frac{\omega_{u_k^l}}{\omega_{u_{k+1}^l}}\frac{\partial}{\partial u_{k+1}^l}, \tag{2.34a}$$

where

$$f_{u_j^i} = \frac{\partial f}{\partial u_j^i}.$$

Similarly, we have

$$L_1 = \frac{\partial}{\partial u_k^l} + \frac{\partial u_k^{l+1}}{\partial u_k^l}\frac{\partial}{\partial u_k^{l+1}} = \frac{\partial}{\partial u_k^l} - \frac{\omega_{u_k^l}}{\omega_{u_k^{l+1}}}\frac{\partial}{\partial u_k^{l+1}}, \tag{2.34b}$$

$$L_2 = \frac{\partial}{\partial u_{k+1}^l} - \frac{\omega_{u_{k+1}^l}}{\omega_{u_k^l}}\frac{\partial}{\partial u_k^l}, \tag{2.34c}$$

$$L_3 = \frac{\partial}{\partial u_k^{l+1}} - \frac{\omega_{u_k^{l+1}}}{\omega_{u_k^l}}\frac{\partial}{\partial u_k^l}. \tag{2.34d}$$

By choosing the appropriate operators, the symmetries can be obtained after a set of long calculations. For instance, by differentiating the symmetry condition (2.28) using the operator L_0 we obtain

$$\begin{aligned}
&\omega_{u_k^l} Q'(k+1, l, u_{k+1}^l) + \frac{\omega_{u_k^l}\omega_{u_k^l u_{k+1}^l}}{\omega_{u_{k+1}^l}} Q(k, l, u_k^l) + \frac{\omega_{u_k^l}\omega_{u_{k+1}^l u_{k+1}^l}}{\omega_{u_{k+1}^l}} S_k Q(k, l, u_k^l) \\
&+ \frac{\omega_{u_k^l}\omega_{u_k^{l+1} u_{k+1}^l}}{\omega_{u_{k+1}^l}} Q(k, l+1, u_k^{l+1}) - \omega_{u_k^l u_k^l} Q(k, l, u_k^l) - \omega_{u_k^l} Q'(k, l, u_k^l) - \\
&\omega_{u_k^l u_{k+1}^l} Q(k+1, l, u_{k+1}^l) - \omega_{u_k^l u_k^{l+1}} Q(k, l+1, u_k^{l+1}) = 0
\end{aligned} \tag{2.35}$$

which is the necessary condition for obtaining symmetries for equations that we will be considering in this work.

Ordinary difference equations

As mentioned earlier, we will be conducting the work with a special emphasis on second-order difference equations, therefore for the sake of clarification we would like to provide

a brief summary of concepts and notation that will be used for second-order ordinary difference equations. Let us consider the pth-order ordinary difference equation

$$u_{n+p} = \omega(n, u_n, \dots, u_{n+p-1}) \tag{2.36}$$

for some smooth function ω such that $\frac{\partial \omega}{\partial u_n} \neq 0$, that is, $N = q = 1$. The solution of (2.36) can be phrased in the form

$$u_n = f(n, c_1, \dots, c_p) \tag{2.37}$$

where $c_1, \dots, c_p$ are arbitrary constants. For the sake of simplification, we redefine (2.17), for $N = q = 1$, as follows:

$$\mathcal{S}^k : n \mapsto n + k. \tag{2.38}$$

That is to say, if $u_n = f(n, c_1, \dots, c_p)$ then,

$$\mathcal{S}(u_n) \quad = \quad u_{n+1}.$$

Considering the transformation

$$\Gamma : (n, u_n, u_{n+1}, \dots, u_{n+p}) \mapsto (\hat{n}, \hat{u_n}, \hat{u}_{n+1}, \dots, \hat{u}_{n+p}), \tag{2.39}$$

we can write the transformed equation obtained from (2.36) using (2.39) as follows

$$\hat{u}_{n+p} = \omega(\hat{n}, \hat{u_n}, \hat{u}_{n+1}, \dots, \hat{u}_{n+p}). \tag{2.40}$$

We are searching for a one-parameter Lie group of transformations:

$$\hat{n} \quad = \quad n + \epsilon\, \xi(n, u_n, u_{n+1}, \dots, u_{n+p}), \tag{2.41a}$$

$$\hat{u}_{n+i} \quad = \quad u_{n+i} + \epsilon\, \mathcal{S}^i Q(n, u_n, u_{n+1}, \dots, u_{n+p}) \tag{2.41b}$$

for $0 \leq i \leq p$. Then a symmetry generator of (2.36) is

$$\begin{aligned} X =& \xi(n, u_n, \dots, u_{n+p-1}) \frac{\partial}{\partial n} + Q(n, u_n, \dots, u_{n+p-1}) \frac{\partial}{\partial u_n} + (\mathcal{S}Q) \frac{\partial}{\partial u_{n+1}} \\ &+ \cdots + \left(\mathcal{S}^{p-1} Q(n, u_n, \dots, u_{n+p-1})\right) \frac{\partial}{\partial u_{n+p-1}}, \end{aligned} \tag{2.42}$$

and satisfies

$$S^p Q - X\omega = 0. \tag{2.43}$$

We recall that the linearized symmetry condition

$$\mathcal{S}^{(p)} Q = X\omega \tag{2.44}$$

is obtained by expanding (2.40) to first-order in ϵ.

Using symmetries to obtain the general solution.

As in the case of differential equations, symmetries are utilised to obtain solutions of ordinary difference equations. We refer to ideas given by Hydon in [11].

Definition 2.3.4. *Given a symmetry generator for a second-order OΔE,*

$$X = Q(n, u_n, u_{n+1})\frac{\partial}{\partial u_n} + Q(n+1, u_{n+1}, \omega(n, u_n, u_{n+1}))\frac{\partial}{\partial u_{n+1}},$$

there exists an invariant,

$$v_n = v(n, u_n, u_{n+1}), \tag{2.45}$$

satisfying

$$Xv_n = 0, \qquad \frac{\partial v_n}{\partial u_{n+1}} \neq 0. \tag{2.46}$$

Recall that the invariant satisfies

$$\left[Q\frac{\partial}{\partial u_n} + \mathcal{S}Q\frac{\partial}{\partial u_{n+1}}\right] v_n \;=\; 0.$$

Suppose that in (2.45) u_{n+1} can be expressed as a function of n, u_n and v_n, i.e.,

$$u_{n+1} = f_0(n, u_n, v_n) \tag{2.47}$$

To solve (2.47) involves obtaining

$$s_n = s(n, u_n) \tag{2.48}$$

which satisfies $Xs_n = 1$. The most widely used s is

$$s(n, u_n) = \int \frac{\mathrm{d}u_n}{Q(n, u_n, f_0(n, u_n, f(n; c_1)))} \tag{2.49}$$

whose solution takes the shape

$$s_n = c_2 + \sum_{k=n_0}^{n-1} g(k, f(k; c_1)),$$

in which n_0 is an integer and arbitrary [11].

2.3.5 Conservation Laws

A conservation law of (2.15) is an expression that can be written in the form

$$Div(\mathcal{F}) \equiv \sum_{i=1}^{N} (S_i - \mathrm{id})\, F^i(\mathbf{n}, \mathbf{u}_n, \mathbf{u}_{(n)}) = 0, \tag{2.50}$$

whenever (2.15) holds, for some smooth function $\mathcal{F} = (F_1, \ldots, F_N)$.

Partial difference equations

We assume that the partial difference equation is in the form (2.22). A conservation law for the partial difference equation (2.22) is then an expression of the form

$$(S_k - id)F + (S_l - \mathrm{id})G = 0 \tag{2.51}$$

that is satisfied by all solutions of the equation [30]. Note that F and G are functions of the dependent and independent variables, id is the identity mapping. We are looking for conservation laws that lie on the quad-graph and we are interested in finding nontrivial conservation laws. We then assume that the functions F and G are of the form:

$$F = F(k, l, u_k^l, u_k^{l+1}), \quad G = G(k, l, u_k^l, u_{k+1}^l). \tag{2.52}$$

The conservation laws (2.51) amount to

$$(S_k - id)F(k, l, u_k^l, u_k^{l+1}) + (S_l - id)G(k, l, u_k^l, u_{k+1}^l) = 0. \tag{2.53}$$

Ordinary difference equations

In [11], Hydon presents a methodology to construct the first integrals of ordinary difference equations. In this construction knowledge of symmetries of the ordinary difference equation is not necessary. If ϕ is a first integral (conservation law) for the ordinary difference equation then, it is constant on its solutions and hence satisfies

$$\mathcal{S}(\phi(n, u_n, \ldots, u_{n+p-1})) \;=\; \phi(n, u_n, \ldots, u_{n+p-1}) \tag{2.54}$$

in which $\mathcal{S}$ is the operator defined in (2.38). We construct first integrals using (2.54) and an additional condition, viz.,

$$\phi(n+1, u_{n+1}, \omega(n, u_n, u_{n+1})) \;=\; \phi(n, u_n, u_{n+1}), \qquad \frac{\partial \phi}{\partial u_{n+1}} \neq 0. \tag{2.55}$$

Using the notation

$$P_1(n, u_n, u_{n+1}) = \frac{\partial \phi}{\partial u_n}, \tag{2.56a}$$

$$P_2(n, u_n, u_{n+1}) = \frac{\partial \phi}{\partial u_{n+1}}, \tag{2.56b}$$

we obtain

$$P_1 = \mathcal{S} P_2 \frac{\partial \omega}{\partial u_n} \tag{2.57}$$

by differentiating (2.55) with respect to u_n. On the other hand, we differente (2.55) with respect to u_{n+1} to get

$$P_2 = \mathcal{S} P_1 + \frac{\partial \omega}{\partial u_{n+1}} \mathcal{S} P_2. \tag{2.58}$$

In virtue of (2.57) and (2.58), we can eliminate P_1 to get a second-order linear functional equation, or *first integral condition*,

$$\mathcal{S}\left(\frac{\partial \omega}{\partial u_n}\right) \mathcal{S}^2 P_2 + \frac{\partial \omega}{\partial u_{n+1}} \mathcal{S} P_2 - P_2 = 0 \tag{2.59}$$

satisfied by P_2. After solving for P_2 and constructing P_1, we need to check that the *integrability condition*

$$\frac{\partial P_1}{\partial u_{n+1}} = \frac{\partial P_2}{\partial u_n} \tag{2.60}$$

is satisfied. Hence if (2.60) holds, the first integral takes the form

$$\phi = \int (P_1 \mathrm{d}u_n + P_2 \mathrm{d}u_{n+1}) + G(n). \tag{2.61}$$

To get the solution of $G(n)$, substitute (2.61) into (2.55). Then the resulting equation is a first order ordinary difference equation which can easily be solved.

2.4 Iterative Equations

It is well-known that the maximal number of point symmetries of second-order equations is equal to eight and that all second-order differential equations are reducible to the canonical form $y'' = 0$ by point transformations [31]. These properties are not retained

for higher order equations and characterizations become more complicated when $n > 2$. Laguerre [32, 33] and Lie [34] proved that a third-order differential equation

$$y''' + ay'' + by' + c = 0$$

can be reduced to the canonical form if and only if

$$54a - 18bc + 4b^3 - 27a' + 18bb' + 9b'' = 0.$$

Lie also proved that the dimension of the Lie algebra of the symmetry group of an ordinary differential equation of order n is less than or equal to $n + 4$ and that only equations that have maximal dimension can be reduced to the canonical form. Krause and Michel [35] also studied the case of maximal dimension and proved the following theorem

Theorem 2.4.1. *[35] For linear differential equations of order $n > 2$ the following three properties are equivalent:*

(a) the equation is reducible to the form $y^{(n)} = 0$ by a local diffeomorphism of a plane $P(x, y)$,

(b) the Lie algebra of its symmetry group has maximal dimension $n + 4$,

(c) the equation is iterative.

Linear iterative equations are characterized by the iterations $L^n[y] = 0$ of the linear first order equation of the form

$$L[y] \equiv r(x)y' + q(x)y = 0, \tag{2.62a}$$

$$L^n[y] \equiv L^{(n-1)}\left[L\left[y\right]\right], \tag{2.62b}$$

where n is a natural number. Without loss of generality assume that a linear iterative equation of a general order is of the form

$$L^n\left[y\right] \equiv K_n^0 y^{(n)} + K_n^1 y^{(n-1)} + K_n^2 y^{(n-2)} + \cdots + K_n^{n-1} y' + K_n^n y = 0. \tag{2.63}$$

Equation (2.63) can be transformed into

$$y^{(n)} + A_n^2 y^{(n-2)} + \cdots + A_n^j y^{(n-j)} + \cdots + A_n^n y = 0 \tag{2.64}$$

by using the so-called Laguerre transformation

$$y \mapsto y \exp\left(n^{-1} \int_{x_0}^{x} K_n^1(t)dt\right). \tag{2.65}$$

We will refer to (2.64) as the normal reduced form of equation (2.63). The general coefficient K_n^j is given by the following theorem [36] .

Theorem 2.4.2. *In terms of the parameters r and q of the source equation, the general coefficient K_n^j of the iterative equation Eq. (2.63) has the form*

$$K_n^j = \sum_{k_j=j}^{M_j} \sum_{k_{j-1}=j-1}^{M_{j-1}} \cdots \sum_{k_2=2}^{M_2} \sum_{k_1=1}^{M_1} r^{k_j-j} L\left[r^{k_{j-1}-j-1} L\left[\ldots L\left[r^{k_1-1} L r^{\alpha_j}\right]\ldots\right]\right], \quad (2.66)$$

for $n \geq 1$ and $1 \leq j \leq n$, and where the expressions for M_l, β_{ij} and α_j are given by

$$\begin{aligned}
\beta_{ij} &= \sum_{u=i+1}^{l} K_u, k_u \in \mathrm{Z}, \\
M_l &= n + \binom{j}{2} - \binom{i}{2} - \beta_{ij}, \\
\alpha_j &= n + \binom{j}{2} - \beta_{0j} = M_0.
\end{aligned}$$

It is also known that the n solutions of (2.64) are

$$y_k = u^{n-k-1} v^k, \quad 0 \leq k \leq n-1,$$

where u, v satisfy

$$y'' + A_2^2\, y = 0.$$

Mohamed and Leach in their paper [27] performed the Lie analysis on equation (2.64) and found that its infinitesimals are given by

$$\xi = a(x), \quad (2.67a)$$

$$\eta = \left(\frac{n-1}{2} a' + \alpha\right) y + b(x), \quad (2.67b)$$

in which α is a constant, a and b are functions of x satisfying

$$\frac{(n+1)!}{(n-2)!4!} a^{(3)} + A_n^2 a' + \frac{1}{2} A_n^{2\prime} a = 0, \quad (2.68)$$

$$b^{(n)} + \sum_{k=2}^{n} A_n^k b^{(n-k)} = 0, \quad (2.69)$$

respectively. They then applied the Leibniz rule for differentiation to η together with the invariance criterion to show that the coefficients A_n^s in (2.64) verify the relation

$$\begin{aligned}&\frac{(n+1)!(i-1)}{(n-i)!(i+1)!2}y^{(i+1)}+iy^{(1)}A_n^i+yA_n^{i\,(1)}\\&+\sum_{j=2}^{i-1}A_n^j\frac{(n-j)![n(i-j-1)+i+j-1]}{(n-i)!(i-j+1)!2}y^{(i-j+1)}=0,\quad i=1,\dots,n.\end{aligned}\tag{2.70}$$

They noted that relation (2.70) can be used to express A_n^j recursively in terms of A_n^2 and they performed this for $3\leq j\leq 8$.

Chapter 3

Symmetries, Conservation Laws and 'integrability' of Partial Difference Equations

3.1 Introduction

The role of symmetries of difference equations is now well established and the applications of the symmetries in the analysis (especially reduction) of the equations are also well documented (see [11, 13–15]). However, the role and construction of conservation laws for partial difference equations, to the best of our knowledge, is somewhat new but the preliminary concepts and definitions are available even in the context of variational equations (see [12, 30, 37]). These conservation laws, as in the case of differential equations, have a variety of applications especially as another tool in the reduction of the equation under scrutiny.

The aim of this work is to obtain the conservation laws of partial difference equations which are of interest, viz., the discrete Liouville equation and the discrete Sine-Gordon equation. These equations were studied in [38], [39] and [40], inter alia. The method for the construction of the conservation laws employed here follows that introduced in [30]. The variational approach, not done here, uses the equivalent of Noether symmetries and can be found in [37].

We regard the domain of a given partial differential equation as a fibre bundle $M = X \times U$, where X is the base space of independent variables and U is the vertical space, i.e., the fibre of dependent variables **u** over each **x** $\in X$. The direct method for constructing conservation laws of partial differential equations requires the domain M to be topologically trivial, which occurs if each fibre U and the base space X are star-shaped (see Poincaré's Lemma).

As far as partial difference equations are concerned, we can write the domain as $M = X \times U$, but now X is the set of integer-valued multi-indices **n** that label each lattice point (we assume that the lattice points are labeled sequentially, without jumps; this does not require the lattice to be uniform).

The content of this chapter appeared in the article [41].

3.2 Conservation Laws of Liouville and Sine-Gordon Equations

In order to find F and G, we consider the theory provided by Hydon in [30]. In his paper, Hydon applied the method to scalar partial difference equations that are second order in one variable but in this paper we are dealing with first order difference equations in two variables.

The discrete Liouville equation

Consider the discrete Liouville equation

$$u_{k+1}^{l+1} = \frac{u_k^{l+1} u_{k+1}^l + z_k^l}{u_k^l}. \tag{3.1}$$

It is an equation which is not integrable through the inverse scattering transform techniques (see [38]). The application of this method to equation (3.1) requires z_k^l to satisfy

$$z_{k+1}^{l+1} z_k^l - z_k^{l+1} z_{k+1}^l = 0. \tag{3.2}$$

We therefore assume that condition (3.2) holds for any k and l. Note that equation (2.51) can be written as follows

$$F(k+1,l,u_{k+1}^l,\omega)-F(k,l,u_k^l,u_k^{l+1})+G(k,l+1,u_k^{l+1},\omega)-G(k,l,u_k^l,u_{k+1}^l)=0 \tag{3.3}$$

where ω is the right-hand side of the discrete Liouville equation (3.1).

As we said earlier, we are not interested in nontrivial conservation laws. It is worthwhile to mention that if F and G are solutions to (3.3), then $F+F_1(k,l)$ and $G+G_1(k,l)$ are also solutions for any functions F_1 and G_1. One can see from equation (3.3) that F and G take different arguments. To overcome this, we eliminate terms that depend on ω, by differentiating with respect to u_k^{l+1} and u_{k+1}^l, respectively, keeping ω fixed. The derivative of (3.3) with respect to u_{k+1}^l is given by

$$\begin{aligned}&\frac{u_k^l u_k^{l+1}}{u_k^{l+1}u_{k+1}^l+z_k^l}\Big[-F_{,3}(k,l,u_k^l,u_k^{l+1})-G_{,3}(k,l,u_k^l,u_{k+1}^l)\Big]+\\&F_{,3}(k+1,l,u_{k+1}^l,\omega)-G_{,4}(k,l,u_k^l,u_{k+1}^l)=0.\end{aligned} \tag{3.4}$$

The differentiation of (3.3) with respect to u_k^{l+1} leads to

$$\begin{aligned}&\frac{u_k^l u_{k+1}^l}{u_k^{l+1}u_{k+1}^l+z_k^l}\left[-F_{,3}(k,l,u_k^l,u_k^{l+1})-G_{,3}(k,l,u_k^l,u_{k+1}^l)\right]-F_{,4}(k,l,u_k^l,u_k^{l+1})\\&+G_{,3}(k,l+1,u_k^{l+1},\omega)=0.\end{aligned} \tag{3.5}$$

Differentiating (3.5) with respect to u_{k+1}^l we get

$$\begin{aligned}&\frac{u_k^l u_k^{l+1}}{u_k^{l+1}u_{k+1}^l+z_k^l}\left[\frac{-u_{k+1}^l\left(F_{,3}(k,l,u_k^l,u_k^{l+1})+G_{,3}(k,l,u_k^l,u_{k+1}^l)\right)}{u_k^{l+1}u_{k+1}^l+z_k^l}\right.\\&\left.-\frac{u_k^l u_{k+1}^l\left(F_{,33}(k,l,u_k^l,u_k^{l+1})+G_{,33}(k,l,u_k^l,u_{k+1}^l)\right)}{u_k^{l+1}u_{k+1}^l+z_k^l}-F_{,43}(k,l,u_k^l,u_k^{l+1})\right]\\&+\frac{z_k^l u_k^l\left(-F_{,3}(k,l,u_k^l,u_k^{l+1})-G_{,3}(k,l,u_k^l,u_{k+1}^l)\right)}{(u_k^{l+1}u_{k+1}^l+z_k^l)^2}\\&-\frac{u_k^l u_{k+1}^l G_{,34}(k,l,u_k^l,u_{k+1}^l)}{u_k^{l+1}u_{k+1}^l+z_k^l}=0.\end{aligned} \tag{3.6}$$

We multiply by $(u_k^{l+1}u_{k+1}^l+z_k^l)^2$ to clear fractions. This gives

$$\begin{aligned}&(u_k^{l+1}u_{k+1}^l+z_k^l)\left[F_{,3}(k,l,u_k^l,u_k^{l+1})+G_{,3}(k,l,u_k^l,u_{k+1}^l)\right]\\&+u_k^l u_{k+1}^l u_k^{l+1}\left[F_{,33}(k,l,u_k^l,u_k^{l+1})+G_{,33}(k,l,u_k^l,u_{k+1}^l)\right]+u_k^{l+1}(u_k^{l+1}u_{k+1}^l\\&+z_k^l)F_{,43}(k,l,u_k^l,u_k^{l+1})+u_{k+1}^l(u_k^{l+1}u_{k+1}^l+z_k^l)G_{,43}(k,l,u_k^l,u_{k+1}^l)=0.\end{aligned} \tag{3.7}$$

In order to get rid of G we differentiate twice with respect to u_k^{l+1}. This yields

$$\begin{aligned} &u_{k+1}^l \left[F_{,3}(k,l,u_k^l,u_k^{l+1}) + G_{,3}(k,l,u_k^l,u_{k+1}^l)\right] + u_k^l u_{k+1}^l \left[F_{,33}(k,l,u_k^l,u_k^{l+1})\right. \\ &\left. + G_{,33}(k,l,u_k^l,u_{k+1}^l)\right] + u_k^l u_k^{l+1} u_{k+1}^l F_{,334}(k,l,u_k^l,u_k^{l+1}) + (3u_k^{l+1}u_{k+1}^l \\ &+ 2z_k^l)F_{,43}(k,l,u_k^l,u_k^{l+1}) + u_k^{l+1}(u_k^{l+1}u_{k+1}^l + z_k^l)F_{,344}(k,l,u_k^l,u_k^{l+1}) \\ &+ {u_{k+1}^l}^2 G_{,34}(k,l,u_k^l,u_{k+1}^l) = 0 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &4u_{k+1}^l F_{,34}(k,l,u_k^l,u_k^{l+1}) + 2u_k^l u_{k+1}^l F_{,334}(k,l,u_k^l,u_k^{l+1}) + (5u_k^{l+1}u_{k+1}^l + \\ &3z_k^l)F_{,334}(k,l,u_k^l,u_k^{l+1}) + u_k^l u_k^{l+1} u_{k+1}^l F_{,3344}(k,l,u_k^l,u_k^{l+1}) \\ &+ ({u_k^{l+1}}^2 u_{k+1}^l + z_k^l u_k^{l+1})F_{,3444}(k,l,u_k^l,u_k^{l+1}) = 0, \end{aligned} \tag{3.9}$$

respectively. The function F does not depend on u_k^{l+1}, therefore we can separate by powers of u_k^{l+1} to get

$$1 : 3F_{,344}(k,l,u_k^l,u_k^{l+1}) + u_k^{l+1}F_{,3444}(k,l,u_k^l,u_k^{l+1}) = 0, \tag{3.10a}$$

$$\begin{aligned} u_{k+1}^l : &4F_{,34}(k,l,u_k^l,u_k^{l+1}) + 2u_k^l F_{,334}(k,l,u_k^l,u_k^{l+1}) + 2u_k^{l+1}F_{,344}(k,l,u_k^l,u_k^{l+1}) \\ &+ u_k^l u_k^{l+1} F_{,3344}(k,l,u_k^l,u_k^{l+1}) = 0. \end{aligned} \tag{3.10b}$$

From equation (3.10a) we get

$$F_{,344}(k,l,u_{k+1}^l,u_k^{l+1}) = f_1'(k,l,u_k^l)\frac{1}{{u_k^{l+1}}^3}, \tag{3.11}$$

for some function f_1. Hence,

$$F(k,l,u_k^l,u_k^{l+1}) = \frac{f_1(k,l,u_k^l)}{2u_k^{l+1}} + f_2(k,l,u_k^l)u_k^{l+1} + f_3(k,l,u_k^l) + f_4(k,l,u_k^{l+1}), \tag{3.12}$$

where f_2, f_3 and f_4 are arbitrary functions. On the other hand, we differentiate (3.6) with respect to u_{k+1}^l to get an equation involving G only. The first derivative with respect to u_{k+1}^l gives

$$\begin{aligned} &- u_k^{l+1}\left[(F_{,3}(k,l,u_k^l,u_k^{l+1}) + G_{,3}(k,l,u_k^l,u_{k+1}^l)\right] - (u_k^{l+1}u_{k+1}^l + z_k^l)G_{,34} \\ &- u_k^l u_k^{l+1}\left[F_{,33}(k,l,u_k^l,u_k^{l+1}) + G_{,33}(k,l,u_k^l,u_{k+1}^l)\right] - \\ &u_k^l u_k^{l+1} u_{k+1}^l G_{,334}(k,l,u_k^l,u_{k+1}^l) - (2u_k^{l+1}u_{k+1}^l + z_k^l)G_{,43}(k,l,u_k^l,u_{k+1}^l) \\ &- u_{k+1}^l(u_k^{l+1}u_{k+1}^l + z_k^l)G_{,344}(k,l,u_k^l,u_{k+1}^l) - {u_k^{l+1}}^2 F_{,34}(k,l,u_k^l,u_k^{l+1}) = 0 \end{aligned} \tag{3.13}$$

and the second derivative gives

$$\begin{aligned}&-4u_k^{l+1}G_{,34}(k,l,u_k^l,u_{k+1}^l)-(5u_k^{l+1}u_{k+1}^l+3z_k^l)G_{,344}(k,l,u_k^l,u_{k+1}^l)\\&-2u_k^{l+1}u_k^lG_{,334}(k,l,u_k^l,u_{k+1}^l)-u_k^lu_k^{l+1}u_{k+1}^lG_{,3344}(k,l,u_k^l,u_{k+1}^l)\\&-u_{k+1}^l(u_k^{l+1}u_{k+1}^l+z_k^l)G_{,3444}(k,l,u_k^l,u_{k+1}^l)=0.\end{aligned}\tag{3.14}$$

Similarly, the function G does not depend on u_{k+1}^l. This allows us to equate the coefficients of powers of u_l^{k+1} to zero. We get

$$1:3G_{,344}(k,l,u_k^l,u_{k+1}^l)+u_{k+1}^lG_{,3444}(k,l,u_k^l,u_{k+1}^l)=0,\tag{3.15a}$$

$$\begin{aligned}u_k^{l+1}:&4G_{,34}(k,l,u_k^l,u_{k+1}^l)+5u_{k+1}^lG_{,344}(k,l,u_k^l,u_{k+1}^l)+\\&2u_k^lG_{,334}(k,l,u_k^l,u_{k+1}^l)+u_k^lu_{k+1}^lG_{,3344}(k,l,u_k^l,u_{k+1}^l)+\\&{u_{k+1}^l}^2G_{,3444}(k,l,u_k^l,u_{k+1}^l)=0.\end{aligned}\tag{3.15b}$$

The general solution of (3.15a) and (3.15b) is given by

$$G(k,l,u_k^l,u_{k+1}^l)=\frac{g_1(k,l,u_k^l)}{2u_{k+1}^l}+g_2(k,l,u_k^l)u_{k+1}^l+g_3(k,l,u_k^l)+g_4(k,l,u_{k+1}^l),\tag{3.16}$$

where g_1, g_2, g_3 and g_4 are arbitrary functions. Substituting the expressions of F and G, given by (3.12) and (3.16), into equations (3.9) and (3.14) gives

$$f_2(k,l,u_k^l)=-\frac{c_1(k,l)}{u_k^l}+c_2(k,l)$$

and

$$g_2(k,l,u_k^l)=-\frac{c_3(k,l)}{u_k^l}+c_4(k,l),$$

respectively. Therefore, we have

$$F(k,l,u_k^l,u_k^{l+1})=\frac{f_1}{2u_k^{l+1}}+\left(-\frac{c_1(k,l)}{u_k^l}+c_2(k,l)\right)u_k^{l+1}+f_3+f_4\tag{3.17}$$

$$G(k,l,u_k^l,u_{k+1}^l)=\frac{g_1}{2u_{k+1}^l}+\left(-\frac{c_3(k,l)}{u_k^l}+c_4(k,l)\right)u_{k+1}^l+g_3+g_4.\tag{3.18}$$

Substituting the above results into (3.13), and separating by powers of u_{k+1}^l and u_k^{l+1},

we obtain

$$1 : \frac{f_1''}{2} u_k^l + 2c_3 \frac{z_k^l}{{u_k^l}^2} = 0,$$

$$u_{k+1}^l u_k^{l+1} : \frac{c_3}{{u_k^l}^2} - \frac{c_3}{{u_k^l}^2} = 0,$$

$$u_k^{l+1} : f_3' + g_3' + u_k^k (f_3'' + g_3'') = 0,$$

$$u_k^{l+1\,2} : \frac{c_1}{{u_k^l}^2} - \frac{c_1}{{u_k^l}^2} = 0,$$

and therefore

$$f_1(k,l,u_k^l) = -\frac{2c_3(k,l) z_k^l}{u_k^l} + c_5(k,l) u_k^l + c_6(k,l), \tag{3.19a}$$

$$(f_3 + g_3)(k,l,u_k^l) = c_7(k,l) \ln(u_k^l) + c_8(k,l). \tag{3.19b}$$

On the other hand, substituting (3.17) and(3.18) into (3.8), and separating by powers of u_k^{l+1}, u_{k+1}^l, we get (after solving the resulting system)

$$g_1(k,l,u_k^l) = -\frac{2c_1(k,l) z_k^l}{u_k^l} + c_9(k,l) u_k^l + c_{10}(k,l). \tag{3.19c}$$

Results (3.17),(3.18) and (3.19) lead to

$$\begin{aligned} F(k,l,u_k^l,u_k^{l+1}) = &\left(-\frac{2c_3(k,l) z_k^l}{u_k^l} + c_5(k,l) u_k^l + c_6(k,l) \right) \frac{1}{2u_k^{l+1}} \\ &- \left(\frac{c_1}{u_k^l} - c_2 \right) u_k^{l+1} + f_3(k,l,u_k^l) + f_4(k,l,u_k^{l+1}) \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} G(k,l,u_k^l,u_{k+1}^l) = &\left(-\frac{2c_1(k,l) z_k^l}{u_k^l} + c_9(k,l) u_k^l + c_{10}(k,l) \right) \frac{1}{2u_{k+1}^l} + \Bigg(-\frac{c_3}{u_k^l} \\ &+ c_4(k,l) \Bigg) u_{k+1}^l - f_3(k,l,u_k^l) + c_7(k,l) \ln(u_k^l) + c_8(k,l) \\ &+ g_4(k,l,u_{k+1}^l), \end{aligned} \tag{3.21}$$

where c_i, $1 \leq i \leq 10$, depend on k and l. The substitution of (3.20) and (3.21) in (3.6) did not provide any new information on the unknown functions. We then continue our computations by substituting (3.20) and (3.21) into (3.4) to get

$$\begin{aligned}
&-\frac{c_9u_k^lu_k^{l+1}}{2u_{k+1}^l}-c_7u_k^{l+1}-\frac{c_5u_k^l}{2}-\frac{c_1zu_k^{l+1^2}+c_3z_k^l+c_3u_{k+1}^lu_k^{l+1}}{u_k^l}\\
&+\frac{c_5(k+1,l)u_k^l}{2}+({u_{k+1}^l}^2u_k^{l+1^2}+2z_k^lu_k^{l+1}u_{k+1}^l+{z_k^l}^2)\frac{c_1(k+1,l)}{u_k^l{u_{k+1}^l}^2}\\
&+\frac{c_3(k+1,l)z_{k+1}^lu_k^l}{{u_{k+1}^l}^2}+u_k^{l+1}u_{k+1}^lf_3'(k+1,l,u_{k+1}^l)+z_k^lf_3'(k+1,l,u_{k+1}^l)\\
&-\frac{c_1z_k^lu_k^{l+1}}{u_k^lu_{k+1}^l}+\frac{c_9u_k^lu_k^{l+1}}{2u_{k+1}^l}+\frac{c_3u_k^{l+1}u_{k+1}^l}{u_k^l}-c_4u_k^{l+1}u_{k+1}^l-u_k^{l+1}u_{k+1}^lg_4'\\
&+\frac{c_{10}u_k^l}{2u_{k+1}^l}-\frac{c_1{z_k^l}^2}{{u_{k+1}^l}^2u_k^l}+\frac{c_9z_k^lu_k^l}{2{u_{k+1}^l}^2}+\frac{c_3z_k^l}{u_k^l}-c_4z_k^l-z_k^lg_4'+\frac{c_{10}z_k^l}{2{u_{k+1}^l}^2}=0.
\end{aligned}\tag{3.22}$$

Separating the resulting equation (3.22) by powers of u_k^{l+1} we have

$$\begin{aligned}
1:&z_k^lf_3'(k+1,l,u_{k+1}^l)-c_4(k,l)z_k^l-z_k^lg_4'(k,l,u_{k+1}^l)+\frac{c_{10}(k,l)z_k^l}{2{u_{k+1}^l}^2}\\
&+\frac{{z_k^l}^2c_1(k+1,l)}{u_k^l{u_{k+1}^l}^2}-\frac{{z_k^l}^2c_1(k,l)}{u_k^l{u_{k+1}^l}^2}+\frac{[-c_5(k,l)+c_5(k+1,l)]u_k^l}{2}\\
&+\frac{(2c_3(k+1,l)z_{k+1}^l+c_9z_k^l)u_k^l}{2{u_{k+1}^l}^2}=0,
\end{aligned}\tag{3.23a}$$

$$\begin{aligned}
u_k^{l+1}:&-c_7(k,l)+u_{k+1}^lf_3'(k+1,l,u_{k+1}^l)-c_4(k,l)u_{k+1}^l-u_{k+1}^lg_4'(k,l,u_{k+1}^l)\\
&+\frac{c_{10}(k,l)}{2u_{k+1}^l}-\frac{2c_9(k,l)u_k^l}{2u_{k+1}^l}+\frac{c_9(k,l)u_k^l}{2u_{k+1}^l}-\frac{c_1(k,l)z_k^l}{u_k^lu_{k+1}^l}+\frac{2c_1(k+1,l)z_k^l}{u_k^lu_{k+1}^l}\\
&-\frac{c_1z_k^l}{u_k^lu_{k+1}^l}=0,
\end{aligned}\tag{3.23b}$$

where $'$ denotes the derivative with respect to the dependent variable. Again we can split (3.23a) and (3.23b) by powers of u_k^l since f_3, g_4 and c_i, $1 \leq i \leq 10$, do not depend

on u_k^l. We redo the same thing with (3.5) to get

$$\begin{aligned} 1 : & - z_k^l f_4'(k,l,u_k^{l+1}) - c_2(k,l) z_k^l + z_k^l g_3'(k,l+1,u_k^{l+1}) + \frac{c_6(k,l) z_k^l}{2 {u_k^{l+1}}^2} \\ & + \frac{{z_k^l}^2 c_3(k,l+1)}{u_k^l {u_k^{l+1}}^2} - \frac{{z_k^l}^2 c_3(k,l)}{u_k^l {u_k^{l+1}}^2} + \frac{[-c_9(k,l) + c_9(k,l+1)] u_k^l}{2} \\ & + \frac{[2c_1(k,l+1) z_k^{l+1} + c_5 z_k^l] u_k^l}{2 {u_k^{l+1}}^2} = 0, \end{aligned} \tag{3.24a}$$

$$\begin{aligned} u_{k+1}^l : & - c_7(k,l) + u_{k+1}^l f_3'(k+1,l,u_{k+1}^l) - c_4(k,l) u_{k+1}^l - u_{k+1}^l g_4'(k,l,u_{k+1}^l) \\ & + \frac{c_{10}(k,l)}{2u_{k+1}^l} - \frac{2c_9(k,l) u_k^l}{2u_{k+1}^l} + \frac{c_9(k,l) u_k^l}{2u_{k+1}^l} - \frac{c_1(k,l) z_k^l}{u_k^l u_{k+1}^l} + \frac{2c_1(k+1,l) z_k^l}{u_k^l u_{k+1}^l} \\ & - \frac{c_1(k,l) z_k^l}{u_k^l u_{k+1}^l} = 0. \end{aligned} \tag{3.24b}$$

After rearranging, simplifying and solving the resulting equations (3.23a), (3.23b), (3.24a) and (3.24b), we get some new information on the unknown functions:

$$c_1(k,l) = K_1,$$

$$c_3(k,l) = K_3,$$

$$c_5(k,l) = -\frac{2c_1(k,l+1) z_k^{l+1}}{z_k^l},$$

$$c_7(k,l) = 0,$$

$$c_9(k,l) = -2\frac{c_3(k+1,l) z_{k+1}^l}{z_k^l},$$

$$f_3(k+1,l,u_{k+1}^l) = g_4(k,l,u_{k+1}^l) + \frac{c_{10}(k,l)}{2u_{k+1}^l} + c_4(k,l) u_{k+1}^l + c_{11}(k,l),$$

$$g_3(k,l+1,u_k^{l+1}) = f_4(k,l,u_k^{l+1}) + \frac{c_6(k,l)}{2u_k^{l+1}} + c_2(k,l) u_k^{l+1} + c_{12}(k,l).$$

The last step consists of substituting all the information we got so far into (3.3). This yields

$$c_1(k,l) = K_1,$$

$$c_3(k,l) = K_3,$$

$$c_5(k,l) = K_5,$$

$$c_9(k,l) = K_9,$$

$$c_{13}(k,l) = 0,$$

$$c_4(k,l) + c_{14}(k,l) = -K_2,$$

$$\begin{aligned} &c_{11}(k,l) + c_{15}(k+1,l) - c_{16}(k-1,l) - c_{11}(k-1,l) + c_{12}(k,l) \\ &+ c_{16}(k,l-1) - c_{12}(k,l-1) = 0. \end{aligned}$$

where K_1, K_3, K_5 and K_9 are constants. Summarizing these results, we have obtained the conservation laws,

$$\begin{aligned} F(k,l,u_k^l,u_k^{l+1}) = &- K_1\left(\frac{z_k^{l+1}u_k^l}{z_k^l u_k^{l+1}} + \frac{u_k^{l+1}}{u_k^l}\right) - K_2(k-1,l)u_k^l - K_3\frac{z_k^l}{u_k^l u_k^{l+1}} \\ &+ c_{16}(k-1,l) + c_{11}(k-1,l) + K_2 u_k^{l+1} + c_{15}(k,l), \end{aligned} \tag{3.25}$$

$$\begin{aligned} G(k,l,u_k^l,u_{k+1}^l) = &- K_1\frac{z_k^l}{u_k^l u_{k+1}^l} - K_3\left(\frac{z_{k+1}^l u_k^l}{z_k^l u_{k+1}^l} + \frac{u_{k+1}^l}{u_k^l}\right) - K_2 u_{k+1}^l - \\ &K_2(k,l-1)u_k^l - c_{15}(k,l-1) + c_{12}(k,l-1) + c_{16}(k,l). \end{aligned} \tag{3.26}$$

Therefore all independent conservations laws for the discrete Liouville equation are given

by

$$\begin{aligned} F_1(k,l,u_k^l,u_k^{l+1}) &= -\frac{z_k^{l+1}u_k^l}{z_k^l u_k^{l+1}} - \frac{u_k^{l+1}}{u_k^l}, \\ G_1(k,l,u_k^l,u_{k+1}^l) &= -\frac{z_k^l}{u_k^l u_{k+1}^l}; \end{aligned} \tag{3.27a}$$

$$\begin{aligned} F_2(k,l,u_k^l,u_k^{l+1}) &= K_2(k,l)u_k^{l+1} - K_2(k-1,l)u_k^l, \\ G_2(k,l,u_k^l,u_{k+1}^l) &= -K_2(k,l)u_{k+1}^l - K_2(k,l-1)u_k^l; \end{aligned} \tag{3.27b}$$

$$\begin{aligned} F_3(k,l,u_k^l,u_k^{l+1}) &= -\frac{z_k^l}{u_k^l u_k^{l+1}}, \\ G_3(k,l,u_k^l,u_{k+1}^l) &= -\frac{z_{k+1}^l u_k^l}{z_k^l u_{k+1}^l} - \frac{u_{k+1}^l}{u_k^l}, \end{aligned} \tag{3.27c}$$

where K_2 is a function of k and l. Note that (F_2, G_2) is a trivial one.

The discrete Sine-Gordon equation

The discrete Sine-Gordon equation is given by [39, 40]

$$u_{k+1}^{l+1}u_k^l = \frac{1 + z_k^l u_k^{l+1}u_{k+1}^l}{u_k^{l+1}u_{k+1}^l + z_k^l}. \tag{3.28}$$

Again, z_i^j must satisfy

$$z_k^l z_{k+1}^{l+1} = z_k^{l+1}z_{k+1}^l. \tag{3.29}$$

Equation (3.28) represents a non-autonomous generalization of the lattice Sine-Gordon equation. The limit of this non-autonomous form is $w_{x,t} = f(x)g(t)\sin w$ [38].

In this section, we jump the first few steps since we have explained them in the first example. Let us take it from the condition of the conservation laws (3.3),

$$\begin{aligned} &F(k+1,l,u_{k+1}^l,\omega) - F(k,l,u_k^l,u_k^{l+1}) + G(k,l+1,u_k^{l+1},\omega) \\ &- G(k,l,u_k^l,u_{k+1}^l) = 0, \end{aligned} \tag{3.30}$$

where ω is given by

$$\omega = u_{k+1}^{l+1} = \frac{1 + z_k^l u_k^{l+1}u_{k+1}^l}{u_k^l(u_k^{l+1}u_{k+1}^l + z_k^l)}. \tag{3.31}$$

We differentiate (3.30) with respect to u_k^{l+1} by applying the operator $(\partial/\partial u_k^{l+1}) - (\omega_{u_k^{l+1}}/\omega_{u_k^l})(\partial/\partial u_k^l)$ to get

$$\begin{aligned}&(z_k^{l\,2}-1)u_k^l u_{k+1}^l\left[F_{,3}(k,l,u_k^l,u_k^{l+1})+G_{,3}(k,l,u_k^l,u_{k+1}^l)\right]+\\&(z_k^l u_k^{l+1\,2}u_{k+1}^{l\;\;2}+u_k^{l+1}u_{k+1}^l+z_k^{l\,2}u_k^{l+1}u_{k+1}^l+z_k^l)\big[F_{,4}(k,l,u_k^l,u_k^{l+1})\\&-G_{,3}(k,l+1,u_k^{l+1},\omega)\big]=0.\end{aligned}\tag{3.32}$$

We repeat the same procedure but now we are planning to eliminate F to obtain an equation that involves G only. For the same reason as before, we equate coefficients of powers of u_k^{l+1}. The resulting system can be summarized as follows

$$4G_{,3444}(k,l,u_k^l,u_{k+1}^l)+u_{k+1}^l G_{,34444}(k,l,u_k^l,u_{k+1}^l)=0,\tag{3.33a}$$

$$\begin{aligned}&(9z_k^{l\,2}+3)G_{,344}(k,l,u_k^l,u_{k+1}^l)+(7z_k^{l\,2}+3)u_{k+1}^l G_{,3444}(k,l,u_k^l,u_{k+1}^l)+\\&(3z_k^{l\,2}-3)u_k^l G_{,3344}(k,l,u_k^l,u_{k+1}^l)+(z_k^{l\,2}-1)u_k^l u_{k+1}^l G_{,33444}(k,l,u_k^l,u_{k+1}^l)\\&+(z_k^{l\,2}+1)u_{k+1}^{l\;\;2}G_{,34444}(k,l,u_k^l,u_{k+1}^l)=0,\end{aligned}\tag{3.33b}$$

$$\begin{aligned}&12G_{,344}(k,l,u_k^l,u_{k+1}^l)+8u_{k+1}^l G_{,3444}(k,l,u_k^l,u_{k+1}^l)\\&+u_{k+1}^{l\;\;2}G_{,34444}(k,l,u_k^l,u_{k+1}^l)=0\end{aligned}\tag{3.33c}$$

whose solution is given by

$$\begin{aligned}G(k,l,u_k^l,u_{k+1}^l)=&\frac{g_1(k,l,u_k^l)}{u_{k+1}^l}+g_2(k,l,u_k^l)u_{k+1}^l+g_3(k,l,u_k^l)\\&+g_4(k,l,u_{k+1}^l),\end{aligned}\tag{3.34}$$

for some functions g_1, g_2, g_3 and g_4. To know the dependency among the g_i's, $1 \leq i \leq 4$, we have to substitute F and G into the previous equations. After a set of long calculations, we find that the conservation laws for the discrete Sine-Gordon equation

are given by

$$\begin{aligned} F_1(k,l,u_k^l,u_k^{l+1}) &= \frac{-c_1(k+1,l)}{({z_{k+1}^l}^2-1)u_k^l u_k^{l+1}} - \frac{c_1(k,l)u_k^l u_k^{l+1}}{{z_k^l}^2-1}, \\ G_1(k,l,u_k^l,u_{k+1}^l) &= \frac{-z_k^l c_1(k,l)u_k^l}{({z_k^l}^2-1)u_{k+1}^l} - \frac{c_1(k+1,l)z_k^l u_{k+1}^l}{({z_{k+1}^l}^2-1)u_k^l}; \end{aligned} \tag{3.35a}$$

$$\begin{aligned} F_2(k,l,u_k^l,u_k^{l+1}) &= u_k^{l+1}-u_k^l, \\ G_2(k,l,u_k^l,u_{k+1}^l) &= -u_{k+1}^l+u_k^l; \end{aligned} \tag{3.35b}$$

$$\begin{aligned} F_3(k,l,u_k^l,u_k^{l+1}) &= \frac{-z_k^l c_3(k,l)u_k^l}{({z_k^l}^2-1)u_k^{l+1}} + \frac{z_k^l c_3(k,l-1)u_k^{l+1}}{(1-z_k^{l-1^2})u_k^l}, \\ G_3(k,l,u_k^l,u_{k+1}^l) &= \frac{c_3(k,l-1)}{(1-z_k^{l-1^2})u_k^l u_{k+1}^l} - \frac{c_3(k,l)u_k^l u_{k+1}^l}{({z_k^l}^2-1)}; \end{aligned} \tag{3.35c}$$

where c_1 and c_3 are such that

$$c_1(k+1,l+1) = \frac{({z_{k+1}^{l+1}}^2-1)z_k^l}{({z_{k+1}^l}^2-1)z_k^{l+1}} c_1(k+1,l) \tag{3.36a}$$

$$c_1(k,l+1) = \frac{(z_k^{l+1^2}-1)z_k^l}{({z_k^l}^2-1)z_k^{l+1}} c_1(k,l) \tag{3.36b}$$

$$c_1(k+2,l) = \frac{({z_{k+2}^l}^2-1)}{({z_k^l}^2-1)} c_1(k,l) \tag{3.36c}$$

and

$$c_3(k+1,l-1) = \frac{(1-z_{k+1}^{l-1^2})z_k^l}{(1-z_k^{l-1^2})z_{k+1}^l} c_3(k,l-1) \tag{3.36d}$$

$$c_3(k,l+1) = \frac{(1-z_k^{l+1^2})}{(1-z_k^{l-1^2})} c_3(k,l-1) \tag{3.36e}$$

$$c_3(k+1,l) = \frac{(1-{z_{k+1}^l}^2)z_k^l}{(1-{z_k^l}^2)z_{k+1}^l} c_3(k,l). \tag{3.36f}$$

Here, (F_2, G_2) is trivial.

3.3 On multipliers/characteristics

The relationship between the characteristics and conserved vectors of partial differential equations was well known for variational equations and was generalized relatively recently. In fact, the characteristics are the conserved vectors. The use of these in the construction of conservation laws has been discussed in detail in [42, 43] and [44] for the symmetry underlying relation that exists. Very recently, this idea has been initiated and discussed for partial difference equations in [45]. Below, we present the multipliers for the nontrivial cases of conservation laws that arise above.

The discrete Liouville equation

For the discrete Liouville equation,

$$u_k^l u_{k+1}^{l+1} = u_k^{l+1} u_{k+1}^l + z_k^l, \tag{3.37}$$

we have shown that

$$\begin{aligned}(S_k - id)F_1 + (S_l - id)G_1 = &\left(-\frac{1}{u_k^l u_{k+1}^l} + \frac{z_k^{l+1}}{z_k^l u_k^{l+1} u_{k+1}^{l+1}}\right)(-u_k^{l+1}u_{k+1}^l - z_k^l \\ &+ u_k^l u_{k+1}^{l+1}).\end{aligned} \tag{3.38}$$

Therefore, the multiplier is given by

$$\lambda_{l1} = -\frac{1}{u_k^l u_{k+1}^l} + \frac{z_k^{l+1}}{z_k^l u_k^{l+1} u_{k+1}^{l+1}}. \tag{3.39}$$

Similarly,

$$\begin{aligned}(S_k - id)F_3 + (S_l - id)G_3 = &\left(-\frac{1}{u_k^l u_k^{l+1}} + \frac{z_{k+1}^l}{z_k^l u_{k+1}^l u_{k+1}^{l+1}}\right)(-u_k^{l+1}u_{k+1}^l - z_k^l \\ &+ u_k^l u_{k+1}^{l+1})\end{aligned} \tag{3.40}$$

and the multiplier is given by

$$\lambda_{l3} = -\frac{1}{u_k^l u_k^{l+1}} + \frac{z_{k+1}^l}{z_k^l u_{k+1}^l u_{k+1}^{l+1}}. \tag{3.41}$$

The discrete Sine-Gordon equation

Consider the Sine-Gordon equation

$$u_{k+1}^{l+1} = \omega = \frac{1 + z_k^l u_k^{l+1} u_{k+1}^l}{u_k^l (u_k^{l+1} u_{k+1}^l + z_k^l)}. \tag{3.42}$$

We have

$$\begin{aligned}(S_k - id)F_1 + (S_l - id)G_1 = \left(u_k^{l+1} u_{k+1}^l + z_k^l\right) \Bigg[& \frac{c_1(k+1,l)}{(1 - {z_{k+1}^l}^2) u_k^l u_k^{l+1}} \\ & - \frac{c_1}{(1 - {z_k^l}^2) u_{k+1}^l u_{k+1}^{l+1}} \Bigg] \left(-u_{k+1}^{l+1} u_k^l + \frac{1 + z_k^l u_k^{l+1} u_{k+1}^l}{u_k^{l+1} u_{k+1}^l + z_k^l} \right)\end{aligned} \tag{3.43}$$

and

$$\begin{aligned}(S_k - id)F_3 + (S_l - id)G_3 = -\left(u_k^{l+1} u_{k+1}^l + z_k^l\right) \Bigg[& \frac{c_3(k,l-1)}{(1 - {z_k^{l-1}}^2) u_k^l u_{k+1}^l} - \\ & \frac{c_3}{(1 - {z_k^l}^2) u_k^{l+1} u_{k+1}^{l+1}} \Bigg] \left(-u_{k+1}^{l+1} u_k^l + \frac{1 + z_k^l u_k^{l+1} u_{k+1}^l}{u_k^{l+1} u_{k+1}^l + z_k^l} \right).\end{aligned} \tag{3.44}$$

The multipliers are then given by

$$\lambda_{Sg1} = \left(u_k^{l+1} u_{k+1}^l + z_k^l\right) \left(\frac{c_1(k+1,l)}{(1 - {z_{k+1}^l}^2) u_k^l u_k^{l+1}} - \frac{c_1}{(1 - {z_k^l}^2) u_{k+1}^l u_{k+1}^{l+1}} \right) \tag{3.45}$$

and

$$\lambda_{Sg2} = -\left(u_k^{l+1} u_{k+1}^l + z_k^l\right) \left(\frac{c_3(k,l-1)}{(1 - {z_k^{l-1}}^2) u_k^l u_{k+1}^l} - \frac{c_3}{(1 - {z_k^l}^2) u_k^{l+1} u_{k+1}^{l+1}} \right), \tag{3.46}$$

respectively.

This method for constructing characteristics does not work for all difference equations. For instance, logarithmic and exponential conservation laws can not be put in the shape

$$(S_k - id)F + (S_l - id)G = \lambda \left(u_{k+1}^{l+1} - \omega\right)$$

unless we introduce a singularity into λ. However, It has be shown in [45] that this difficulty can be resolved by considering a general divergence expression

$$C\left(\mathbf{z}, [\mathcal{E}]\right) = Div(F(\mathbf{z})), \tag{3.47}$$

where $\mathcal{E}$ is the partial difference equation in its *Kovalevskaya form* (also known as solved form), $\mathbf{z}$ is the set of all uneliminated variables, and $[\mathcal{E}]$ stands for $\mathcal{E}$ and their shifts. Equation (3.47) gives conservation laws if and only if

$$C\left(\mathbf{z}, [0]\right) = 0.$$

Of course $[0]$ is where the difference equation is equal to zero. So by the Fundamental Theorem of calculus

$$C\left(\mathbf{z}, [\mathcal{E}]\right) = \int_{\lambda=0}^{1} \frac{d}{d\lambda} C\left(\mathbf{z}, [\lambda\mathcal{E}]\right) d\lambda \tag{3.48}$$

$$= \mathcal{E} \int_{\lambda=0}^{1} \left(E_{\mathcal{E}}\left(C\left(\mathbf{z}, [\mathcal{E}]\right)\right)\right)|_{\mathcal{E}\mapsto\lambda\mathcal{E}} d\lambda + Div(P(\mathbf{z}, [\mathcal{E}])), \tag{3.49}$$

where $E_{\mathcal{E}}$ is the difference Euler-Lagrange operator for $\mathcal{E}$,

$$E_{\mathcal{E}}\left(C\left(\mathbf{z}, [\mathcal{E}]\right)\right) = \sum_{i,j} S_m^{-i} S_n^{-j} \frac{\partial C\left(\mathbf{z}, [\mathcal{E}]\right)}{\partial S_m^i S_n^j \mathcal{E}}$$

for two independent variables, and $Div(P(\mathbf{z}, [\mathcal{E}])) = 0$. Therefore, given any conservation law the characteristic can be defined as

$$Q\left(\mathbf{z}, [\mathcal{E}]\right) = \int_{\lambda=0}^{1} \left(E_{\mathcal{E}}\left(C\left(\mathbf{z}, [\mathcal{E}]\right)\right)\right)|_{\mathcal{E}\mapsto\lambda\mathcal{E}} d\lambda.$$

Some of main results from [45] are:

- if $\mathcal{E}$ is in Kovalevskaya form then a conservation law is trivial if and only if

$$Q\left(\mathbf{z}, [0]\right) \equiv \left(E_{\mathcal{E}}\left(C\left(\mathbf{z}, [\mathcal{E}]\right)\right)\right)|_{\mathcal{E}\mapsto 0} = 0$$

- if $\mathcal{E}$ is in Kovalevskaya form then 'equivalent characteristics' is equivalent to 'equivalent conservation laws'.

3.4 Symmetries of Liouville and Sine-Gordon Equation

Using the theory provided in Chapter 2 we aim to find symmetry of both non-autonomous discrete equations of order $(1+1)$ Liouville and Sine-Gordon.

The discrete Liouville equation

Consider the discrete Liouville equation [38]

$$u_k^l u_{k+1}^{l+1} = u_k^{l+1} u_{k+1}^l + z_k^l. \tag{3.50}$$

This equation can be written in the form

$$u_{k+1}^{l+1} = \omega(k, l, u_k^l, u_k^{l+1}, u_{k+1}^l), \tag{3.51}$$

where $\omega = \left(u_k^{l+1} u_{k+1}^l + z_k^l\right)/u_k^l$. We aim to find one-parameter Lie groups of symmetries by assuming that the characteristic is of the form

$$Q = Q(k, l, u_k^l). \tag{3.52}$$

A linearised symmetry condition for (3.51) is given by

$$(S_k S_l)\ Q(k, l, u_k^l) - X\omega = 0, \tag{3.53}$$

i.e.,

$$Q(k+1, l+1, \omega) - X\omega = 0, \tag{3.54}$$

where X is the symmetry generator given by

$$X = Q(k, l, u_k^l)\frac{\partial}{\partial u_k^l} + Q(k+1, l, u_{k+1}^l)\frac{\partial}{\partial u_{k+1}^l} + Q(k, l+1, u_k^{l+1})\frac{\partial}{\partial u_k^{l+1}}.$$

The symmetry condition (3.54) becomes

$$\begin{aligned} &Q(k+1, l+1, \omega) + \left(\frac{u_k^{l+1} u_{k+1}^l + z_k^l}{{u_k^l}^2}\right) Q(k, l, u_k^l) - \\ &\left(\frac{u_k^{l+1}}{u_k^l}\right) Q(k+1, l, u_{k+1}^l) - \left(\frac{u_{k+1}^l}{u_k^l}\right) Q(k, l+1, u_k^{l+1}) = 0. \end{aligned} \tag{3.55}$$

We differentiate (3.55) with respect to u_k^l (keeping ω fixed), i.e., we apply the differential operator $L_0 = (\partial/\partial u_k^l) + (\partial u_{k+1}^l/\partial u_k^l)(\partial/\partial u_{k+1}^l)$. This gives

$$\left(\frac{\partial}{\partial u_k^l} + \frac{u_k^{l+1}u_{k+1}^l + z_k^l}{u_k^l u_k^{l+1}}\frac{\partial}{\partial u_{k+1}^l}\right)\left[Q(k+1,l+1,w) + \left(\frac{u_k^{l+1}u_{k+1}^l + z_k^l}{{u_k^l}^2}\right)Q\right.$$
$$\left. - \left(\frac{u_k^{l+1}}{u_k^l}\right)Q(k+1,l,u_{k+1}^l) - \left(\frac{u_{k+1}^l}{u_k^l}\right)Q(k,l+1,u_k^{l+1})\right] = 0, \tag{3.56}$$

i.e.,

$$\left(\frac{Q'(k,l,u_k^l){u_k^l}^2 - 2u_k^l Q(k,l,u_k^l)}{{u_k^l}^2}\right)(u_k^{l+1}u_{k+1}^l + z_k^l) + Q(k+1,l,u_{k+1}^l)\,u_k^{l+1}$$
$$+ Q(k,l+1,u_k^{l+1})u_{k+1}^l + \left(\frac{u_k^{l+1}u_{k+1}^l + z_k^l}{u_k^{l+1}}\right)\left[Q(k,l,u_k^l)\frac{u_k^{l+1}}{u_k^l}\right. \tag{3.57}$$
$$\left. - Q'(k+1,l,u_{k+1}^l)u_k^{l+1} - Q(k,l+1,u_k^{l+1})\right] = 0.$$

We now differentiate (3.57) with respect to u_k^l to get

$$(u_k^{l+1}u_{k+1}^l + z_k^l)\frac{d}{du_k^l}\left[\frac{Q'(k,l,u_k^l){u_k^l}^2 - 2u_k^l Q(k,l,u_k^l)}{{u_k^l}^2}\right] +$$
$$(u_k^{l+1}u_{k+1}^l + z_k^l)\frac{d}{du_k^l}\left[\frac{Q(k,l,u_k^l)}{u_k^l}\right] = 0. \tag{3.58}$$

This simplifies to

$$\left[\frac{Q(k,l,u_k^l)}{u_k^l}\right]' = -\left[\frac{Q'(k,l,u_k^l)u_k^l - 2Q(k,l,u_k^l)}{u_k^l}\right]'. \tag{3.59}$$

Integrating (3.59) with respect to u_k^l, and rearranging the resulting equation leads to

$$Q'(k,l,u_k^l) - \frac{1}{u_k^l}Q(k,l,u_k^l) - c_1 = 0. \tag{3.60}$$

The general solution to equation (3.60) is given by

$$Q(k,l,u_k^l) = c_1(k,l)u_k^l \ln(u_k^l) + c_2(k,l)u_k^l, \tag{3.61}$$

where c_1 and c_2 are functions of k and l. We then substitute (3.61) in equation (3.55) to get the following constraint

$$\begin{aligned}&[c_2(k+1,l+1)+c_2(k,l)-c_2(k+1,l)-c_2(k,l+1)]\,u_k^{l+1}u_{k+1}^l\\&+[c_2(k+1,l+1)+c_2(k,l)]z_k^l=0.\end{aligned}\tag{3.62}$$

The function c_2 does not depend on u_k^{l+1} and u_{k+1}^l therefore we can say that

$$c_1=0,\tag{3.63}$$

$$c_2(k+1,l+1)=-c_2(k,l),\tag{3.64}$$

$$c_2(k+1,l)=-c_2(k,l+1).\tag{3.65}$$

It can easily be verified that the functions that satisfy (3.64) and (3.65) are given by

$$c_2(k,l)=(-1)^k,\tag{3.66}$$

$$c_2(k,l)=(-1)^l.\tag{3.67}$$

Therefore, the symmetries of the Liouville equation (3.1) are as follows:

$$X_1 = (-1)^k u_k^l\frac{\partial}{\partial u_k^l}-(-1)^k u_{k+1}^l\frac{\partial}{\partial u_{k+1}^l}+(-1)^k u_k^{l+1}\frac{\partial}{\partial u_k^{l+1}}\tag{3.68}$$

$$X_2 = (-1)^l\left(u_k^l\frac{\partial}{\partial u_k^l}+u_{k+1}^l\frac{\partial}{\partial u_{k+1}^l}-u_k^{l+1}\frac{\partial}{\partial u_k^{l+1}}\right).\tag{3.69}$$

The discrete Sine-Gordon equation

The discrete Sine-Gordon equation is given by [38]

$$u_{k+1}^{l+1}u_k^l=\frac{1+z_k^l u_k^{l+1}u_{k+1}^l}{u_k^{l+1}u_{k+1}^l+z_k^l}.\tag{3.70}$$

Imposing the symmetry condition (2.28) to the discrete Sine-gordon equation, we obtain

$$\begin{aligned}&Q(k+1,l+1,u_{k+1}^{l+1})+\left[\frac{z_k^l u_k^{l+1}u_{k+1}^l+1}{{u_k^l}^2(z_k^l+u_k^{l+1}u_{k+1}^l)}\right]Q-\frac{Q(k+1,l,u_{k+1}^l)}{u_k^l}\\&\left[\frac{{z_k^l}^2u_k^{l+1}-u_k^{l+1}}{(u_{k+1}^l u_k^{l+1}+z_k^l)^2}\right]-\left(\frac{Q(k,l+1,u_k^{l+1})}{u_k^l}\right)\left[\frac{{z_k^l}^2u_{k+1}^l-u_{k+1}^l}{(u_{k+1}^l u_k^{l+1}+z_k^l)^2}\right]=0.\end{aligned}\tag{3.71}$$

The function in above equation is associated with different arguments. To overcome this, we differentiate the above equation with respect to u_k^l. This yields

$$\begin{aligned}
&\frac{({z_k^l}^2-1)\left[u_k^{l+1}Q(k+1,l,u_{k+1}^l)+u_{k+1}^l Q(k,l+1,u_k^{l+1})\right]}{(u_{k+1}^l u_k^{l+1}+z_k^l)^2 {u_k^l}^2}+\\
&\frac{(z_k^l u_{k+1}^l u_k^{l+1}+1)}{(u_{k+1}^l u_k^{l+1}+z_k^l)}\frac{d}{du_k^l}\left(\frac{Q(k,l,u_k^l)}{{u_k^l}^2}\right)+\frac{(z_k^l u_{k+1}^l u_k^{l+1}+1)Q(k,l,u_k^l)}{{u_k^l}^3(u_k^{l+1}u_{k+1}^l+z_k^l)}\\
&-\frac{(z_k^l u_{k+1}^l u_k^{l+1}+1)(u_{k+1}^l u_k^{l+1}+z_k^l)}{{u_k^l}^2}\left[\frac{d}{du_{k+1}^l}\left(\frac{Q(k+1,l,u_{k+1}^l)}{(u_{k+1}^l u_k^{l+1}+z_k^l)^2}\right)\right.\\
&\left.-\frac{1}{u_k^{l+1}}\frac{d}{du_{k+1}^l}\left(\frac{u_{k+1}^l}{(u_{k+1}^l u_k^{l+1}+z_k^l)^2}\right)\right]=0.
\end{aligned}\tag{3.72}$$

We can solve for Q after a series of steps. Firstly, we multiply (3.72) by $(u_k^l)^2$. Secondly, we differentiate the resulting equation with respect to u_k^l. Finally, we simplify to get the equation

$$\frac{d}{du_k^l}\left(\frac{Q'(k,l,u_k^l)u_k^l-Q(k,l,u_k^l)}{u_k^l}\right)=0. \tag{3.73}$$

The general solution of (3.73) is given by

$$Q(k,l,u_k^l)=B(k,l)u_k^l\ln u_k^l+C(k,l)u_k^l, \tag{3.74}$$

where B and C are functions of k and l. The substitution of (3.74) in equations (3.73) and (3.71) puts more constraints on the unknown functions B and C. After a set of long calculations we get

$$Q(k,l,u_k^l)=(-1)^k u_k^l,\quad Q(k,l,u_k^l)=(-1)^l u_k^l. \tag{3.75}$$

Therefore, the symmetries are given by

$$X_1 = (-1)^k\left(u_k^l\frac{\partial}{\partial u_k^l}-u_{k+1}^l\frac{\partial}{\partial u_{k+1}^l}+u_k^{l+1}\frac{\partial}{\partial u_k^{l+1}}\right) \tag{3.76a}$$

$$X_2 = (-1)^l\left(u_k^l\frac{\partial}{\partial u_k^l}+u_{k+1}^l\frac{\partial}{\partial u_{k+1}^l}-u_k^{l+1}\frac{\partial}{\partial u_k^{l+1}}\right). \tag{3.76b}$$

Remark 3.4.1. *(Association)* Recall that one of the conservation laws of the discrete Liouville equation is given by

$$F_1(k,l,u_k^l,u_k^{l+1})=-\frac{z_k^{l+1}u_k^l}{z_k^l u_k^{l+1}}-\frac{u_k^{l+1}}{u_k^l}, \tag{3.77a}$$

$$G_1(k,l,u_k^l,u_{k+1}^l)=-\frac{z_k^l}{u_k^l u_{k+1}^l}; \tag{3.77b}$$

and that it has a symmetry given by

$$X_1 = (-1)^k u_k^l \frac{\partial}{\partial u_k^l} - (-1)^k u_{k+1}^l \frac{\partial}{\partial u_{k+1}^l} + (-1)^k u_k^{l+1} \frac{\partial}{\partial u_k^{l+1}}. \tag{3.78}$$

The symmetry (3.78)acts on (3.77a) as follows

$$\begin{aligned} X_1 F_1 =& \left[(-1)^k u_k^l \frac{\partial}{\partial u_k^l} - (-1)^k u_{k+1}^l \frac{\partial}{\partial u_{k+1}^l} + (-1)^k u_k^{l+1} \frac{\partial}{\partial u_k^{l+1}}\right] \left(- \frac{z_k^{l+1} u_k^l}{z_k^l u_k^{l+1}} - \frac{u_k^{l+1}}{u_k^l} \right) \\ =&(-1)^k \left[u_k^l \left(-\frac{z_k^{l+1}}{z_k^l u_k^{l+1}} + \frac{u_k^{l+1}}{{u_k^l}^2} \right) - u_{k+1}^l(0) + u_k^{l+1} \left(\frac{z_k^{l+1} u_k^l}{z_k^l {u_k^{l+1}}^2} - \frac{1}{u_k^l} \right) \right] \\ =&0. \end{aligned} \tag{3.79}$$

Similarly, we have

$$\begin{aligned} X_1 G_1 =& \left[(-1)^k u_k^l \frac{\partial}{\partial u_k^l} - (-1)^k u_{k+1}^l \frac{\partial}{\partial u_{k+1}^l} + (-1)^k u_k^{l+1} \frac{\partial}{\partial u_k^{l+1}}\right] \left(- \frac{z_k^l}{u_k^l u_{k+1}^l} \right) \\ =&(-1)^k \left[u_k^l \left(\frac{z_k^l}{{u_k^l}^2 u_{k+1}^l} \right) - u_{k+1}^l \left(\frac{z_k^l}{u_k^l {u_{k+1}^l}^2} - \frac{1}{u_k^l} \right) + u_k^{l+1}(0) \right] \\ =&0. \end{aligned} \tag{3.80}$$

Therefore, we conclude that the symmetry given in (3.78) and the conserved vector given in (3.77) are associated. The implication is that this symmetry and conserved vector could be used to perform double reductions of the discrete Liouville equation. We have shown that the symmetries and conservaton laws of the discrete Liouville equation satisfy

$$(X_1 F_1,\, X_1 G_1) = (0, 0),$$

$$(X_2 F_1,\, X_2 G_1) \neq (0, 0),$$

$$(X_1 F_2,\, X_1 G_2) \neq (0, 0),$$

$$(X_2 F_2,\, X_2 G_2) \neq (0, 0),$$

$$(X_1 F_3,\, X_1 G_3) \neq (0, 0),$$

$$(X_2 F_3,\, X_2 G_3) = (0, 0).$$

We have carried out the same calculations for the discrete Sine-Gordon and we have found that

$$(X_1F_1,\ X_1G_1) \neq (0,0),$$

$$(X_2F_1,\ X_2G_1) = (0,0),$$

$$(X_1F_2,\ X_1G_2) \neq (0,0),$$

$$(X_2F_2,\ X_2G_2) = (-1)^l(F_2, G_2) \neq (0,0),$$

$$(X_1F_3,\ X_1G_3) = (0,0),$$

$$(X_2F_3,\ X_2G_3) \neq (0,0).$$

3.5 Concluding Remarks

In this chapter, we have obtained conservation laws, multipliers and symmetries of the discrete Liouville equation and the discrete Sine-Gordon equation. It should be noted that when we were finding the conservation laws, the differentiations have created a hierarchy of functional difference equations that F and G satisfied. The unknown functions were naturally found by going up the hierarchy but surprisingly we were also able to find another constraint on z. In fact the shift operator, S_l, acts on equation (3.36d) to produce the following equation

$$S_l\left(c_3(k+1,l-1)\right) = c_3(k+1,l) = \frac{(1-{z_{k+1}^l}^2)z_k^{l+1}}{(1-{z_k^l}^2)z_{k+1}^{l+1}}c_3(k,l). \tag{3.81}$$

By replacing in (3.81) the function $c_3(k+1,l)$ with its expression given by (3.36f), we obtained the constraint (3.2), i.e,

$$z_{k+1}^{l+1}z_k^l - z_k^{l+1}z_{k+1}^l = 0. \tag{3.82}$$

The constraint (3.82) turns out to be a sufficient condition for the discrete Liouville equation and the discrete Sine-Gordon equation to have non trivial conservation laws (Note that the substitution of (3.27a), (3.27b) and (3.27c) into (3.3) leads to the same

constraint (3.2)). This condition was obtained in [46] using the singularity confinement condition. It is also precisely the one obtained in [38] using the techniques of study of the degree of the iterate.

We note that the association of symmetries, conservation laws and integrability for difference equations is as important and conclusive as was established for differential equations even in the non-variational case.

Chapter 4

Symmetries, Associated First Integrals, and Double Reduction of Difference Equations

4.1 Introduction

Symmetries and conservation laws are useful tools for finding exact solutions to differential equations. The association of symmetries, conservation laws and integrability was established for differential equations [42–44]. It has been shown that when the symmetry generator and the first integral (conservation laws) are associated via the invariance condition, one may proceed to double reduction of the equation. These properties should be retained when discrete analogs of such equations are constructed. As far as conservation laws of difference equations goes, the work is more recent - see [12, 37, 41, 47, 48]. In this chapter, we construct symmetries and conservation laws for some ordinary difference equations, utilize the symmetries to obtain reductions of the equations and show, in fact, that the notion of 'association' between these concepts can be analogously extended to ordinary difference equations. That is, an association between a symmetry and first integral exists if and only if the first integral is invariant under the symmetry. Thus, a 'double reduction' of the difference equation is possible.

The content of this chapter appeared in the article [48].

4.2 Symmetry Reductions

In this section we consider two instances, find their symmetries, conservation laws and their solutions. A brief discussion of the meaning of double reduction and association is given.

4.2.1 First Example

Consider the equation [11],

$$\omega = u_{n+2} = \frac{n}{n+1}u_n + \frac{1}{u_{n+1}}. \tag{4.1}$$

Symmetry generator

Here, the symmetry condition, given by (2.43) becomes

$$Q(n+2,\omega) + Q(n+1,u_{n+1})\frac{1}{u_{n+1}^2} - Q(n,u_n)\left(\frac{n}{n+1}\right) \;=\; 0. \tag{4.2}$$

Firstly, we differentiate (4.2) with respect to u_n (keeping ω fixed) and we assume that u_{n+1} ia a function of n, u_n and ω. By the implicit function theorem differentiating u_{n+1} with respect to u_n yields

$$\frac{\partial u_{n+1}}{\partial u_n} = -\frac{\frac{\partial \omega}{\partial u_n}}{\frac{\partial \omega}{\partial u_{n+1}}} = \frac{nu_{n+1}^2}{n+1}. \tag{4.3}$$

Secondly, we apply the *differential operator*, given by

$$L = \frac{\partial}{\partial u_n} + \frac{\partial u_{n+1}}{\partial u_n}\frac{\partial}{\partial u_{n+1}}, \tag{4.4}$$

to equation (4.2) to get

$$-\frac{2n}{(n+1)u_{n+1}}Q(n+1,u_{n+1}) + \frac{n}{n+1}Q'(n+1,u_{n+1}) - \frac{n}{n+1}Q'(n,u_n) = 0. \tag{4.5}$$

To solve (4.5), we differentiate it with respect to u_n and keep u_{n+1} fixed. Hence we get

$$\frac{\mathrm{d}}{\mathrm{d}u_n}\left(\frac{n}{n+1}Q'(n,u_n)\right) \;=\; 0 \tag{4.6}$$

whose solution is given by

$$Q(n,u_n)=\left(\frac{n+1}{n}\right)A(n)u_n+B(n). \tag{4.7}$$

For the sake of simplification assume that $B(n)=0$. Next we substitute (4.7) into (4.5) and we simplify the resulting equation to obtain

$$\left[\frac{-n(n+2)}{(n+1)^2}\right]A(n+1) \;=\; A(n). \tag{4.8}$$

Thus,

$$A(n)=\left(\frac{n}{n+1}\right)2c(-1)^{n-1}, \tag{4.9}$$

where c is a constant. Substituting (4.9) into (4.7) leads to

$$Q(n,u_n) \;=\; \left(\frac{n+1}{n}\right)\left(\frac{n}{n+1}\right)2c(-1)^{n-1}u_n=2c(-1)^{n-1}u_n. \tag{4.10}$$

Therefore, the symmetry generator is given by

$$X=2c(-1)^{n-1}u_n\frac{\partial}{\partial u_n}. \tag{4.11}$$

First integral

Suppose that $P_2=P_2(n,u_n)$, then equation (2.59) becomes

$$\left(\frac{n+1}{n+2}\right)P_2(n+2,u_{n+2})-\frac{1}{u_{n+1}^2}P_2(n+1,u_{n+1})-P_2(n,u_n) \;=\; 0. \tag{4.12}$$

We apply the differential operator L, given by (5.27), to (4.12) to get

$$\frac{n}{n+1}\frac{2}{u_{n+1}}P_2(n+1,u_{n+1})-\frac{n}{n+1}P_2'(n+1,u_{n+1})-P_2'(n,u_n)=0. \tag{4.13}$$

Next we differentiate (4.13) with respect to u_n, keeping u_{n+1} fixed to get $(\mathrm{d}/\mathrm{d}u_n)\left(P_2'(n,u_n)\right)=0$ whose solution is given by

$$P_2(n,u_n) \;=\; B(n)u_n+c=B(n)u_n \tag{4.14}$$

if we set $c=0$. We carry out substitution of equation (4.14) into equation (4.13) to obtain

$$B(n+1) \;=\; \frac{n+1}{n}B(n). \tag{4.15}$$

We choose $B(1) = 1$ to get

$$B(n) = n. \tag{4.16}$$

The next step consists of substituting equation (4.16) into (4.14) to get

$$P_2(n, u_n) = nu_n. \tag{4.17}$$

From equation (2.57) we get

$$P_1(n, u_n, u_{n+1}) = \mathcal{S}P_2\frac{\partial\omega}{\partial u_n} = nu_{n+1} = P_1(n, u_{n+1}). \tag{4.18}$$

The condition for integrability is satisfied and so we compute the first integral ϕ. Using equations (4.17) and (4.18) we obtain

$$\phi = \int (P_1 \mathrm{d}u_n + P_2 \mathrm{d}u_{n+1}) + G(n) = nu_n u_{n+1} + G(n). \tag{4.19}$$

In order to find $G(n)$ we use (4.19) and (2.55). We get

$$G(n+1) - G(n) + n + 1 = 0 \tag{4.20}$$

whose solution is

$$G(n) = -\frac{n(n+1)}{2}. \tag{4.21}$$

Finally the substitution of (4.21) into (4.19) gives

$$\phi = nu_n u_{n+1} - \frac{n(n+1)}{2}. \tag{4.22}$$

Note: The symmetry generator given by equation (4.11) acts on the first integral, ϕ to produce the following equation

$$\begin{aligned} X\phi &= Q(n, u_n)\frac{\partial\phi}{\partial u_n} + Q(n+1, u_{n+1})\frac{\partial\phi}{\partial u_{n+1}} \\ &= 2c(-1)^{n-1}(nu_n nu_{n+1} - nu_n nu_{n+1}) \\ &= 0. \end{aligned}$$

We say X and ϕ are *associated* and this property has far reaching consequences on 'further' reduction of the equation.

Symmetry reduction

Recall that

$$X = 2c(-1)^{n-1}u_n\frac{\partial}{\partial u_n}$$

is a symmetry generator in (4.11). If $v_n = v(n, u_n, u_{n+1})$ is invariant under X, then

$$Xv_n = \left(Q(n,u_n)\frac{\partial}{\partial u_n} + \mathcal{S}Q(n,u_n)\frac{\partial}{\partial u_{n+1}}\right)v_n = 0. \tag{4.23}$$

We can use the characteristics

$$\frac{\mathrm{d}u_n}{2c(-1)^{n-1}u_n} = \frac{\mathrm{d}u_{n+1}}{2c(-1)^n u_{n+1}} = \frac{\mathrm{d}v_n}{0} \tag{4.24}$$

to find v_n. The independent and dependent variables are given by

$$\alpha = u_n u_{n+1} \qquad \text{and} \quad \gamma = v_n \tag{4.25}$$

respectively. Therefore by equation (4.25), the dependent variable, v_n, is given by

$$v_n = u_n u_{n+1}. \tag{4.26}$$

Applying the shift operator on v_n and solving the resulting equation we get

$$v_n = \frac{n+1}{2} + \frac{c}{n}, \tag{4.27}$$

where c is a constant. Then equating (4.26) and (4.27), and solving for u_{n+1} we obtain

$$u_{n+1} = \frac{n+1}{2u_n} + \frac{c}{nu_n}. \tag{4.28}$$

Note: Equation (4.1) has been reduced by one order into equation (4.28). Solving (4.28) for c, gives

$$c = nu_n u_{n+1} - \frac{n(n+1)}{2} = \phi. \tag{4.29}$$

The first integral ϕ, given by equation (4.22), and the reduction are the same. This is another indication of a relationship between ϕ and X. In fact, this is the association, i.e., ϕ is *invariant* under X.

4.2.2 Second Example

Consider the linear difference equation [11]

$$\omega = u_{n+2} = 2u_{n+1} - u_n. \tag{4.30}$$

Symmetries

Let $Q = Q(n, u_n)$, then (2.26) gives

$$Q(n+2, \omega) - 2Q(n+1, u_{n+1}) + Q(n, u_n) \quad = \quad 0. \tag{4.31}$$

Similarly, we apply the operator L to (4.31) and we differentiate the resulting equation,

$$Q'(n, u_n) - Q'(n+1, u_{n+1}) \quad = \quad 0, \tag{4.32}$$

with respect to u_n to get $Q''(n, u_n) = 0$. Therefore,

$$Q(n, u_n) \quad = \quad A(n)u_n + B(n). \tag{4.33}$$

We then find $A(n)$ via substitution of equation (4.33) into (4.32). This gives

$$A(n+1) \quad = \quad A(n) = a, \tag{4.34}$$

in which a is constant. Setting $A(n) = a$ in (4.33) leads to

$$Q(n, u_n) \quad = \quad au_n + B(n). \tag{4.35}$$

The substitution of (4.35) into (4.31) yields

$$B(n+2) - 2B(n+1) + B(n) \quad = \quad 0. \tag{4.36}$$

Thus,

$$B(n) \quad = \quad bn + c, \tag{4.37}$$

for arbitrary constants c and b. Using (4.37) and (4.35) we get

$$Q(n, u_n) = au_n + bn + c. \tag{4.38}$$

Thus

$$X_1 \quad = \quad u_n \frac{\partial}{\partial u_n}, \qquad X_2 = n\frac{\partial}{\partial u_n}, \qquad X_3 = \frac{\partial}{\partial u_n}. \tag{4.39}$$

First integrals

Assume P_2 is a function of n and u_n, i.e. $P_2 = P_2(n, u_n)$. Applying the condition for *first integral* yields

$$P_2(n+2,\omega) - 2P_2(n+1,u_{n+1}) + P_2(n,u_n) \;=\; 0. \tag{4.40}$$

The solution to (4.40) is given by

$$P_2(n,u_n) = ku_n + pn + q \tag{4.41}$$

in which k, p and q are constants. Using (2.57), we get

$$P_1(n,u_{n+1}) \;=\; \mathcal{S}P_2(n,u_n)\frac{\partial\omega}{\partial u_n} = -ku_{n+1} - pn - p - q. \tag{4.42}$$

Substituting (4.41) and (4.42) into (2.61) we obtain the first integral

$$\phi \;=\; pn(u_{n+1}-u_n) + q(u_{n+1}-u_n) - pu_n + G(n). \tag{4.43}$$

Then we have

$$\mathcal{S}\phi \;=\; pn(u_{n+1}-u_n) + q(u_{n+1}-u_n) - pu_n + G(n+1). \tag{4.44}$$

Equating (4.43) and (4.44). This gives

$$G(n+1) \;=\; G(n) \;=\; r, \tag{4.45}$$

in which r is constant. We thus write ϕ as

$$\phi \;=\; (pn+q)u_{n+1} - (pn+p+q)u_n + r. \tag{4.46}$$

We find out whether there is an association between ϕ and symmetry generators in equation (4.39).

- $X_1 = u_n\frac{\partial}{\partial u_n}$. One can readily verify that

  $$X_1\phi \;=\; -u_n\left(pn+q+p\right) + u_{n+1}\left(pn+q\right).$$

  So $X\phi = 0$, if and only if the following are satisfied,

  $$pn + q + p = 0$$

  and

  $$pn + q = 0.$$

  As a result, for ϕ to be associated with X_1,

  $$\phi = r. \tag{4.47}$$

- $X_2 = n\frac{\partial}{\partial u_n}$. We have

$$X_2\phi = n\frac{\partial \phi}{\partial u_n} + (n+1)\frac{\partial \phi}{\partial u_{n+1}} = q.$$

As a result, ϕ and X_2 are associated if $q = 0$, i.e.,

$$\phi = pnu_{n+1} - (pn+p)u_n + r. \tag{4.48}$$

- $X_3 = c\frac{\partial}{\partial u_n}$. Then,

$$X_3\phi = c\frac{\partial \phi}{\partial u_n} + c\frac{\partial \phi}{\partial u_{n+1}} = -cp.$$

Thus ϕ and X_3 are associated as long as

$$\phi = q(u_{n+1} - u_n) + r.$$

General Solution

We now find the general solution of (4.30). We determine the commutators of the symmetries to indicate the order of the symmetries in the reduction procedure.

- Since

$$[X_1, X_2] = -X_2,$$

we reduce (4.30) using X_2 first. Assume that $v_n = v(n, u_n, u_{n+1})$ is invariant under X_2. We have

$$X_2 v_n = \left[n\frac{\partial v_n}{\partial u_n} + (n+1)\frac{\partial v_n}{\partial u_{n+1}}\right] = 0.$$

Using the method of characteristic we obtain

$$v_n = nu_{n+1} - (n+1)u_n. \tag{4.49}$$

The shift operator acts on v_n to give

$$\mathcal{S}(v_n) = v_{n+1} = v_n,$$

that is,

$$v_{n+1} = v_n = c_1, \tag{4.50}$$

in which c_1 is constant. Equating equations (4.49) and (4.50), and solving for u_{n+1}, we have

$$u_{n+1} = \frac{c_1}{n} + \left(1 + \frac{1}{n}\right) u_n \tag{4.51}$$

whose solution is given by

$$u_n = nc_2 + c_1(n-1), \tag{4.52}$$

in which c_2 is constant. Note that (4.52) is the solution to (4.30) where c_1 is given by

$$c_1 = nu_{n+1} - (n+1)u_n. \tag{4.53}$$

Thus, ϕ in (4.48) and the reduced equation (4.53) are identical if $p = 1$ and $r = 0$.

- The solution of (4.30) can also be found by utilising a different symmetry generator. Here,

$$[X_1, X_3] = -X_1,$$

so that (4.30) is reduced using X_1 first. In a similar manner as done before, assume $v_n = v(n, u_n, u_{n+1})$ is invariant under X_1. We have

$$X_1 v_n = \left[u_n \frac{\partial v_n}{\partial u_n} + u_{n+1} \frac{\partial v_n}{\partial u_{n+1}}\right] = 0$$

which gives

$$v_n = \frac{u_{n+1}}{u_n} \tag{4.54}$$

via the method of characteristics. Acting the shift operator on v_n leads to

$$v_{n+1} = 2 - \frac{1}{v_n} \tag{4.55}$$

whose solution is given by

$$v_n = \frac{1 + 2c_1 + nc_1}{1 + c_1 + nc_1}, \tag{4.56}$$

in which c_1 is constant. Considering (4.54) and (4.56) we get

$$u_{n+1} = \left(\frac{1 + 2c_1 + nc_1}{1 + c_1 + nc_1}\right) u_n. \tag{4.57}$$

Hence the solution of (4.30) is given by

$$u_n = \frac{(1 + c_1 + nc_1)\, c_2}{1 + c_1} \tag{4.58}$$

in which c_2 is constant.

- We then look at the commutator of X_2 and X_3. We have

$$[X_2, X_3] \;=\; 0.$$

Here, we have a choice of either reducing first with X_2 or X_3. Using X_3 we finally get

$$X_3 v_n \;=\; \left[c\frac{\partial v_n}{\partial u_n} + c\frac{\partial v_n}{\partial u_{n+1}} \right] = 0,$$

where we assumed that $v_n = v(n, u_n, u_{n+1})$. Applying the method of characteristics, we have

$$v_n = u_{n+1} - u_n. \tag{4.59}$$

When $\mathcal{S}$ is applied on (4.59) and the resulting equation solved, we get

$$\mathcal{S}(v_n) \;=\; v_{n+1} \;=\; v_n = c_1. \tag{4.60}$$

Equating (4.59) and (4.60) gives

$$u_{n+1} \;=\; u_n + c_1. \tag{4.61}$$

We solve (4.61) and find

$$u_n = nc_1 + c_2 \tag{4.62}$$

which is a general solution of (4.30). It has to be noted that (4.61) is identical to ϕ in (4.47) as long as $r = 0$ and $q = 1$.

4.3 Conclusion

We have recalled the procedure to calculate the symmetry generators of some ordinary difference equations and proceeded to find the first integral and reduce the order of the difference equations. We have shown that, in some cases, the symmetry generator, X, and first integral, ϕ, are associated via the invariance condition $X\phi = 0$. When this condition is satisfied, we may proceed to double reduction of the equation.

Chapter 5

Symmetry Reductions and Exact Solutions of Painléve Equations

5.1 Introduction

Ramani in his paper [49] presented his method for obtaining the discrete versions of the Painlevé equations and showed that the mappings for the discrete versions satisfy the same reduction relations as in the case of the continuous ones. In this chapter, we construct symmetries of these discrete versions of Painlevé equations and in some cases the symmetries are used to find exact solutions to the equations. To ease our computation we shall assume that the characteristic depends on n and u_n only. It is worthwhile to mention that not all equations admit such characteristics. Here, the corresponding generator will be

$$X = \xi(n, u_n)\partial_n + Q(n, u_u)\partial_{u_n} \tag{5.1}$$

and in this chapter the painlévé equations presented in paper [49] will always be put in the form

$$u_{n+p} = \Omega(n, u_n, u_{n+1}, \ldots, u_{n+p-1}), \tag{5.2}$$

where Ω is a smooth function such that $\partial\Omega/\partial u_n = 0$. In order to solve equation (5.2), we require the pth extension of X and we further impose the symmetry condition (2.44) whenever (5.2) holds. As said earlier, this leads to an equation that involves functions with different arguments which makes the obtention of solutions difficult. However, the solutions can be obtained after a series of lengthy calculations. Despite the fact that

Painlevé equations are second-order difference equations, the procedure for obtaining symmetries is still lengthy and some details of computations will not be presented in this paper. The main steps involved are three; firstly differentiate (5.2) implicitly with respect to u_n keeping ω fixed; secondly differentiate with respect to u_n (keeping u_{n+1} fixed) as many as possible in order to get rid of any undesirable arguments; thirdly we equate all coefficients of powers of u_{n+1} to zero since the characteristic Q does not depend on u_{n+1}. These steps normally lead to a system of determining equations whose solutions give the expressions of ξ and Q. The number of independent unknown constants determines the dimension of the group. Once the symmetries are found we use the method of characteristics given by

$$\frac{du_n}{Q} = \frac{du_{n+1}}{\mathcal{S}Q} = \frac{du_{n+2}}{\mathcal{S}^{(2)}Q} = \cdots = \frac{du_{n+p-1}}{\mathcal{S}^{(p-1)}Q} \tag{5.3}$$

to obtain the invariant functions which are actually the first integrals of (5.3).

5.2 Symmetries of the Discrete Painlevé Equations

In this section we consider the discrete forms of the Painlevé equations and we aim to find their symmetries using the method explained above. As mentioned earlier, we shall assume that the characteristics ξ and Q depend on n and u_n. In some cases the exact solutions are found.

The discrete Painlevé I

The discrete form is known to be

$$u_{n+1} + u_{n-1} + u_n = \frac{An + B}{u_n} + C, \tag{5.4}$$

where A, B and C are arbitrary constants (we shall assume that C is zero). This is a second-order difference equation which can be written in the form

$$u_{n+2} = \Omega = -u_n - u_{n+1} + \frac{an + b}{u_{n+1}} \tag{5.5}$$

for some constants a and b.
Assume that the symmetry generator is of the form $X = \xi(n, u_n)\partial_n + Q(n, u_n)\partial_{u_n}$.

Imposing the symmetry condition (2.44) we have

$$Q(n+2,u_{n+2})+Q(n+1,u_{n+1})\left(1+\frac{an+b}{u_{n+1}^2}\right)+Q(n,u_n) \\ +\frac{a}{u_{n+1}}\xi(n,u_n)=0. \tag{5.6}$$

To solve for Q and ξ we first differentiate equation (5.6) implicitly with respect to u_n to get

$$Q'(n,u_n)-\xi'(u,u_n)\frac{a}{u_{n+1}}-\frac{{u_{n+1}}^2}{{u_{n+1}}^2+an+b}\Bigg[-\frac{2(an+b)}{{u_{n+1}}^3}Q(n+1, \\ u_{n+1})+\left(1+\frac{an+b}{u_{n+1}^2}\right)Q'(n+1,u_{n+1})+\frac{a}{{u_{n+1}}^2}\xi(n,u_n)\Bigg]=0. \tag{5.7}$$

Secondly, we differentiate (5.7) with respect to u_n keeping u_{n+1} fixed. This leads to a second-order differential equation that involves two different functions with same arguments n and u_n:

$${u_{n+1}}^2Q''+(an+b)Q''-\left(au_{n+1}+\frac{an^2+ab}{u_{n+1}}\right)\xi''-a\xi'=0. \tag{5.8}$$

In the equation above, Q and ξ do not depend on u_{n+1}, therefore we can equate all the coefficients of all the powers of u_{n+1} to zero. We get

$${u_{k+1}}^{-1} \;:\; -a(an+b)\xi''=0 \tag{5.9a}$$

$$1 \;:\; (an+b)Q''-a\xi'=0 \tag{5.9b}$$

$$u_{k+1} \;:\; -a\xi''=0 \tag{5.9c}$$

$${u_{k+1}}^2 \;:\; Q''=0. \tag{5.9d}$$

Case $a\neq 0$

We have

$$Q=\alpha u_n+\beta, \qquad \xi=\gamma, \tag{5.10}$$

where α, β and γ are functions of n. We substitute (5.14) back in equations (5.8), (5.7) and (5.6). We obtain

$$\alpha = A, \qquad \beta = 0, \qquad \gamma = \frac{2\alpha}{a}(an+b). \tag{5.11}$$

Hence, the symmetry generator of (5.5) is given by

$$X = 2(an+b)\partial n + au_n\partial u_n + au_{n+1}\partial u_{n+1}. \tag{5.12}$$

Case $a = b = 0$

Equation (5.5) becomes

$$u_{n+2} = -u_n - u_{n+1} \tag{5.13}$$

and solution to system (5.9) in this case is given by

$$Q = c_1 u_n + \theta, \qquad \xi = c_2, \tag{5.14}$$

where c_1, c_2 are constant and θ is a function of n that satisfies the original equation. Here, we obtain four symmetries given as follows

$$X_1 = u_n\frac{\partial}{\partial u_n} + u_{n+1}\frac{\partial}{\partial u_{n+1}}, \tag{5.15a}$$

$$X_2 = (-1)^n\left[\cos\left(\frac{n\pi}{3}\right)\frac{\partial}{\partial u_n} - \cos\left(\frac{(n+1)\pi}{3}\right)\frac{\partial}{\partial u_{n+1}}\right], \tag{5.15b}$$

$$X_3 = (-1)^n\left[\sin\left(\frac{n\pi}{3}\right)\frac{\partial}{\partial u_n} - \sin\left(\frac{(n+1)\pi}{3}\right)\frac{\partial}{\partial u_{n+1}}\right], \tag{5.15c}$$

$$X_4 = \frac{\partial}{\partial n}. \tag{5.15d}$$

General solutions: If we assume that $v_n = v(n, u_n, u_{n+1})$ is invariant under $X = Q\partial u_n + Q(n+1, u_{n+1})\partial u_{n+1}$, we may obtain the solution of (5.13) by using the method of characteristics.

- Reduction using $X_1 = u_n\frac{\partial}{\partial u_n} + u_{n+1}\frac{\partial}{\partial u_{n+1}}$. Using the characteristic equation given by

$$\frac{du_n}{u_n} = \frac{du_{n+1}}{u_{n+1}} = \frac{dv_n}{0}, \tag{5.16}$$

one can readily check that $v_n = f(u_{n+1}/u_n)$ for some function f. We choose f to be the identity function, i.e.,

$$v_n = \frac{u_{n+1}}{u_n} \tag{5.17}$$

and we act the shift operator on v_n to get the first-order difference equation

$$v_{n+1} = -\frac{1}{v_n} - 1 \tag{5.18}$$

whose solution is given by

$$v_n = c_1 e^{\frac{2\pi i}{3}} + c_2 e^{\frac{-2\pi i}{3}}. \tag{5.19}$$

Equating (5.17) and (5.19), we obtain

$$u_{n+1} = e^{\frac{2\pi i}{3}} u_n \quad \text{and} \quad u_{n+1} = e^{\frac{-2\pi i}{3}} u_n. \tag{5.20}$$

We note that equation (5.13) has been reduced by one order to (5.20) which follows a geometric progression. Therefore,

$$u_n = e^{\frac{2n\pi i}{3}} u_0 \quad \text{and} \quad u_n = e^{\frac{-2n\pi i}{3}} u_0 \tag{5.21}$$

are solutions to (5.13).

- Reduction using $X_2 + iX_3$. Applying the same method, we have obtained

$$v_{n+1} = -e^{\frac{\pi i}{3}} v_n, \tag{5.22}$$

where

$$v_n = u_{n+1} - e^{\frac{2i\pi}{3}} u_n. \tag{5.23}$$

Equation (5.23) is further reduced to

$$u_n = (-1)^{\frac{2(n-1)}{3}} \left(u_1 - e^{\frac{2i\pi}{3}} u_0\right) \left[1 + \frac{e^{\frac{-4in\pi}{3}}(1 - e^{\frac{4in\pi}{3}})}{-1 + (-1)^{\frac{2}{3}}}\right] \tag{5.24}$$

which is also a solution to equation (5.13).

- Naturally, the reduction using $X_2 - iX_3$ leads to the conjugate of (5.24).

The discrete Painlevé II

Consider the Painlevé equation II

$$u_{n+2} = \omega = -u_n + \frac{u_{n+1}(an+b)+c}{1-{u_{n+1}}^2}. \tag{5.25}$$

In order to determine the symmetry generator of (5.25) we need to find the *characteristic* Q. Again, we assume that Q depends on n and u_n only. Here, the symmetry condition (2.44) becomes

$$\begin{aligned} &Q(n+2,\omega) - \left[\frac{(an+b)(1+{u_{n+1}}^2)+2cu_{n+1}}{(1-{u_{n+1}}^2)^2}\right] Q(n+1,u_{n+1}) \\ &+ Q(n,u_n) - \frac{au_{n+1}}{1-{u_{n+1}}^2}\xi(n,u_n) = 0. \end{aligned} \tag{5.26}$$

We now apply the *differential operator*, L, given by

$$L = \partial u_n + \frac{\partial u_{n+1}}{\partial u_n}\partial u_{n+1}, \tag{5.27}$$

to equation (5.26) and differentiate the resulting equation with respect to u_n, keeping u_{n+1} fixed. We obtain

$$\begin{aligned} &anQ''(n,u_n) - a\xi'(n,u_n) + bQ''(n,u_n) + \left[(a^2n^2+2abn+b^2+2c)Q''(n,u_n)\right. \\ &\left. - (a^2n+ab)\xi''(n,u_n)\right]u_{n+1} - \left[(an+b)Q''(n,u_n) + 2ac\xi''\right]{u_{k+1}}^2 - \left[(a^2n^2\right. \\ &\left. + 2abn + b^2 + 2c)Q''(n,u_n) + (a^2n+ab)\xi(n,u_n)\right]{u_{k+1}}^3 + \\ &a\xi'(n,u_n){u_{k+1}}^4 = 0. \end{aligned} \tag{5.28}$$

Similarly, after equating the coefficients of powers of u_{n+1} to zero, we have solved the resulting determining system and have noticed that the condition for having non-zero characteristics that depend on n and u_n only is when $a = b = c = 0$, that is,

$$u_{n+2} = -u_n. \tag{5.29}$$

Under this condition, the symmetries are given by

$$X_1 = \partial_n, \qquad X_2 = \alpha(n,u_n)\partial u_n + \alpha(n+1,u_{n+1})\partial u_{n+1}, \tag{5.30}$$

where α satisfies the original equation (5.29), i.e.,

$$\alpha(n+2,u_{n+2}) = -\alpha(n,u_n). \tag{5.31}$$

Note: If we assume that the function α does not depend on u_n, we obtain three symmetries:

$$X_1 = \partial n, \tag{5.32a}$$

$$X_2 = \cos\left(\frac{n\pi}{2}\right)\frac{\partial}{\partial u_n} - \sin\left(\frac{n\pi}{2}\right)\frac{\partial}{\partial u_{n+1}}, \tag{5.32b}$$

$$X_3 = \sin\left(\frac{n\pi}{2}\right)\frac{\partial}{\partial u_n} + \cos\left(\frac{n\pi}{2}\right)\frac{\partial}{\partial u_{n+1}}. \tag{5.32c}$$

General solutions: We assume $v_n = v(n, u_n, u_{n+1})$ is invariant under (5.32).

- Reduction using $X_2 + i\,X_3$.
  The characteristic equation

$$\frac{du_n}{i^n} = \frac{du_{n+1}}{i^{n+1}} = \frac{dv_n}{0} \tag{5.33}$$

  can be solved to get the invariants $\gamma = v_n$ and $\alpha = u_{n+1} - e^{\left(\frac{i\pi}{2}\right)}u_n$. We have,

$$v_n = u_{n+1} - e^{\left(\frac{i\pi}{2}\right)}u_n \tag{5.34}$$

  so

$$v_{n+1} = S(v_n) = e^{\left(\frac{i\pi}{2}\right)}v_n = e^{\frac{i(n+1)\pi}{2}}v_0. \tag{5.35}$$

  Equations (5.34) and (5.35) lead to a first-order difference equation

$$u_{n+1} = e^{\left(\frac{i\pi}{2}\right)}u_n + e^{\frac{in\pi}{2}}\left(u_1 - e^{\left(\frac{i\pi}{2}\right)}u_0\right). \tag{5.36}$$

  It has to be noted that equation (5.29) has been reduced by one order to (5.36). We deduce that the general solution in this case is given by

$$u_n = (u_1 - iu_0)\left[\frac{3}{2} - \frac{1}{2}(-1)^n\right]i^{n-1}. \tag{5.37}$$

- The reduction using $X_2 - i\,X_3$ leads to the conjugate of (5.37).

The discrete Painlevé III

Consider the discrete painlevé equation III

$$u_{n+2} = \frac{1}{u_n}\left(\frac{au_{n+1}^2 + bu_{n+1} + c}{u_{n+1}^2 + du_{n+1} + e}\right), \tag{5.38}$$

where a, b, c, d and e are constant. We seek characteristics of the form $Q = Q(n, u_n)$. The symmetry condition (2.44) gives

$$\begin{aligned} &Q(n+2,\omega) + \frac{au_{n+1}^2 + bu_{n+1} + c}{u_n^2({u_{n+1}}^2 + du_{n+1} + e)}Q(n, u_n) \\ &- \frac{(ad-b)u_{n+1}^2 + (2ae-2c)u_{n+1} + eb - cd}{u_n({u_{n+1}}^2 + du_{n+1} + e)^2}Q(n+1, u_{n+1}) = 0. \end{aligned} \tag{5.39}$$

Here, the procedure for obtaining the system of determining equations can be summarized as follows

- Apply the differential operator L on (5.39).
- Multiply through to clear fractions.
- Differentiate with respect to u_n twice, keeping u_{n+1} fixed.
- Divide by ${u_{n+1}}^3$
- Separate by powers of u_{n+1} and equate to zero.

These steps lead to the system

$$u_{n+1}{}^5 : (-2a^2d + 2ab)Q''(n, u_n) = 0, \tag{5.40a}$$

$$u_{n+1}{}^4 : (-4a^2d^2 + 2abd - 4a^2e + 2b^2 + 4ac)Q''(n, u_n) = 0, \tag{5.40b}$$

$$\begin{aligned} u_{n+1}{}^3 : &(a^2d - ab)u_nQ^{(3)}(n, u_n) + (-2a^2d^3 - 2abd^2 - 12a^2de + 3a^2d \\ &+ 4b^2d + 8acd - 2abe - 3ab + 6bc)Q''(n, u_n) = 0, \end{aligned} \tag{5.40c}$$

$$\begin{aligned} u_{n+1}{}^2 : &(a^2d^2 + 2a^2e - b^2 - 2ac)u_nQ^{(3)}(n, u_n) + (-2abd^3 - 8a^2d^2e \\ &+ 3a^2d^2 + 2b^2d^2 + 4acd^2d - 12abde - 8a^2e^2 + 14bcd + 6a^2e + 2b^2e \\ &+ 4ace - 3b^2 - 6ac + 4c^2)Q''(n, u_n) = 0, \end{aligned} \tag{5.40d}$$

$$\begin{aligned} u_{n+1} : &(abd^2 + 3a^2de - b^2d - 2acd + 2abe - 3bc)u_nQ^{(3)}(n, u_n) + \\ &(-10abd^2e - 10a^2de^2 + 3abd^2 + 10bcd^2 + 9a^2de - 10abe^2 - 3b^2d \\ &- 6acd + 10c^2d + 6abe + 10bce - 9bc)Q''(n, u_n) = 0, \end{aligned} \tag{5.40e}$$

$$\begin{aligned} 1 : &(4abde + 2a^2e^2 - 4bcd - 2c^2)u_nQ^{(3)}(n, u_n) + (2bcd^3 - 2b^2d^2e \\ &- 4acd^2e - 14abde^2 - 4a^2e^3 + 8c^2d^2 + 12abde + 12bcde + 6a^2e^2 \\ &- 2b^2e^2 - 4ace^2 - 12bcd + 8c^2e - 6c^2)Q(n, u_n) = 0 \end{aligned} \tag{5.40f}$$

which reduce to two possibilities:

$$ad = b, \quad ae = c \tag{5.41a}$$

and

$$Q''(n, u_n) = 0. \tag{5.41b}$$

Case $ad = b$**,** $ae = c$

Equation (5.38) simplifies to

$$u_{n+2} = \frac{a}{u_n} \tag{5.42}$$

and its symmetries are given by

$$X_1 = \left(1 - \frac{u_n^2}{a}\right)\frac{\partial}{\partial u_n} - \left(1 - \frac{u_{n+1}^2}{a}\right)\frac{\partial}{\partial u_{n+1}}, \tag{5.43a}$$

$$X_2 = \sin\left(\frac{n\pi}{2}\right)\left(1 + \frac{u_n^2}{a}\right)\frac{\partial}{\partial u_n} + \cos\left(\frac{n\pi}{2}\right)\left(1 + \frac{u_{n+1}^2}{a}\right)\frac{\partial}{\partial u_{n+1}}, \tag{5.43b}$$

$$X_3 = \cos\left(\frac{n\pi}{2}\right)\left(1 + \frac{u_n^2}{a}\right)\frac{\partial}{\partial u_n} - \sin\left(\frac{n\pi}{2}\right)\left(1 + \frac{u_{n+1}^2}{a}\right)\frac{\partial}{\partial u_{n+1}}, \tag{5.43c}$$

$$X_4 = (-1)^n\left(1 - \frac{u_n^2}{a}\right)\frac{\partial}{\partial u_n} - (-1)^n\left(1 - \frac{u_{n+1}^2}{a}\right)\frac{\partial}{\partial u_{n+1}}, \tag{5.43d}$$

$$X_5 = \cos\left(\frac{n\pi}{2}\right)u_n\frac{\partial}{\partial u_n} - \sin\left(\frac{n\pi}{2}\right)u_{n+1}\frac{\partial}{\partial u_{n+1}}, \tag{5.43e}$$

$$X_6 = \sin\left(\frac{n\pi}{2}\right)u_n\frac{\partial}{\partial u_n} + \cos\left(\frac{n\pi}{2}\right)u_{n+1}\frac{\partial}{\partial u_{n+1}}, \tag{5.43f}$$

$$X_7 = \frac{\partial}{\partial n}. \tag{5.43g}$$

Note: By making the change of variable $w_n = \ln u_n$, equation (5.42) reduces to (5.29).

Case $Q''(n, u_n) = 0$

The general solution in this case is given by

$$Q(n, u_n) = \alpha(n)u_n + \beta(n), \tag{5.44}$$

where α and β are functions of n. The substitution of Q, given by (5.44), in equation (5.39) leads to two cases:
Case $\beta = 0$**:** The characteristic becomes $Q = \alpha(n)u_n$ and for the equation (5.39) to be satisfied we must have

$$\alpha(n+2) = -\alpha(n), \quad ae = c, \quad ad = b. \tag{5.45}$$

Thus, equation (5.38) simplifies to $u_{n+2} = a/u_n$ which has two symmetries given by (5.43e) and (5.43f).
Case $d^2 = -2e$**:** For the equation (5.39) to be satisfied we must have

$$\alpha(n+2) + \alpha(n+1) + \alpha(n) = 0, \quad a = c = d = e = \beta = 0. \tag{5.46}$$

Thus, the equation (5.38) simplifies to

$$u_{n+2} = \frac{b}{u_n u_{n+1}} \tag{5.47}$$

and the symmetries in this case are given by

$$X_1 = (-1)^n \left[\cos\left(\frac{n\pi}{3}\right) u_n \frac{\partial}{\partial u_n} - \cos\left(\frac{(n+1)\pi}{3}\right) u_{n+1} \frac{\partial}{\partial u_{n+1}} \right], \tag{5.48a}$$

$$X_2 = (-1)^n \left[\sin\left(\frac{n\pi}{3}\right) u_n \frac{\partial}{\partial u_n} - \sin\left(\frac{(n+1)\pi}{3}\right) u_{n+1} \frac{\partial}{\partial u_{n+1}} \right] \tag{5.48b}$$

$$X_3 = \frac{\partial}{\partial n}. \tag{5.48c}$$

Note: By making the change of variable $U_n = \ln u_n$, equation (5.47) reduces to (5.4) when a and b are equal to zero.

The discrete Painlevé IV

For Painlevé IV we shall consider the result obtained in paper [49] and we shall assume that $\gamma_0 = a = b = 0$. Let the equation take the shape

$$u_{n+2} = \omega = \frac{1}{u_n + u_{n+1}} \left(-u_n u_{n+1} + \frac{\mu}{{u_{n+1}}^2} + \epsilon_0 \right). \tag{5.49}$$

Here, we present the results without details since the method is similar to the one presented above. After a set of long calculations we obtain the following symmetries

$$X_1 = \cos\left(\tfrac{2n\pi}{3}\right) ({u_n}^2 + \epsilon_0) \tfrac{\partial}{\partial u_n} + \cos\left(\tfrac{2(n+1)\pi}{3}\right) ({u_{n+1}}^2 + \epsilon_0) \tfrac{\partial}{\partial u_{n+1}}, \tag{5.50a}$$

$$X_2 = \sin\left(\tfrac{2n\pi}{3}\right) ({u_n}^2 + \epsilon_0) \tfrac{\partial}{\partial u_n} + \sin\left(\tfrac{2(n+1)\pi}{3}\right) ({u_{n+1}}^2 + \epsilon_0) \tfrac{\partial}{\partial u_{n+1}}, \tag{5.50b}$$

$$X_3 = \tfrac{\partial}{\partial n}. \tag{5.50c}$$

Note: If we assume that μ and ϵ_0 are zero, equation (5.49) becomes

$$u_{n+2} = -\frac{u_n u_{n+1}}{u_n + u_{n+1}} \tag{5.51}$$

and has four symmetries given as follows

$$X_1 = u_n\frac{\partial}{\partial u_n} + u_{n+1}\frac{\partial}{\partial u_{n+1}}, \tag{5.52a}$$

$$X_2 = (-1)^n\left[\cos\left(\frac{n\pi}{3}\right){u_n}^2\frac{\partial}{\partial u_n} - \cos\left(\frac{(n+1)\pi}{3}\right){u_{n+1}}^2\frac{\partial}{\partial u_{n+1}}\right], \tag{5.52b}$$

$$X_3 = (-1)^n\left[\sin\left(\frac{n\pi}{3}\right){u_n}^2\frac{\partial}{\partial u_n} - \sin\left(\frac{(n+1)\pi}{3}\right){u_{n+1}}^2\frac{\partial}{\partial u_{n+1}}\right], \tag{5.52c}$$

$$X_4 = \frac{\partial}{\partial n}. \tag{5.52d}$$

The general solution of equation (5.51) can be constructed by applying the method of characteristics using the symmetry X_1 given by (5.52a). In fact, the invariant $v_n = u_{n+1}/u_n$ obtained by using $X_1 = u_n\partial/\partial u_n + u_{n+1}\partial/\partial u_{n+1}$ satisfies the equation

$$v_n = \frac{u_{n+1}}{u_n} = -\frac{2}{2(-1)^n v_0 - (-1)^n + 1}. \tag{5.53}$$

Note: Equation (5.51) has been reduced by one order into equation (5.53), a first-order ordinary difference equation, which can easily be solve using *Mathematica*.

The solution to the first-order difference equation (5.53) is also the general solution to equation (5.51). It is given by

$$u_n = (-1)^{n-1}2^{n+\lceil\frac{1-n}{2}\rceil-1}\left(2 - 2\frac{u_0}{u_1}\right)^{\lceil\frac{2-n}{2}\rceil-1}\left(\frac{u_0}{u_1}\right)^{\lceil\frac{1-n}{2}\rceil+1}, \tag{5.54}$$

where $\lceil x\rceil$ is the ceiling function.

The discrete Painlevé V

The discretization of the continuous case of the Painlevé V was studied in [49] and the discrete form was given. Our aim is to find the symmetries of the discrete form. To ease our computations, we shall assume that the parameters α_0, σ, ρ_0, θ and μ are equal to zero, that is, we consider the equation

$$u_{n+2} = \omega = \frac{u_n u_{n+1}}{2u_{n+1}u_n - u_n - u_{n+1}}. \tag{5.55}$$

If we impose the invariance condition, assuming that the characteristic is of the form $Q = Q(n, u_n)$, we obtain

$$Q(n+2,\omega) + \frac{1}{\left(2u_{n+1}u_n - u_n - u_{n+1}\right)^2}\left[{u_{n+1}}^2 Q + {u_n}^2 Q(n+1, u_{n+1})\right]. \quad (5.56)$$

Again we proceed by applying the operator L to equation (5.56) and then we differentiate with respect to u_n four times to get a system of differential equations that involves u_n and a function of u_n only. The lengthy calculations for solving the resulting system lead to the following symmetries:

$$X_1 = \left(u_n - \frac{2}{3}{u_n}^2\right)\frac{\partial}{\partial u_n} + \left(u_{n+1} - \frac{2}{3}{u_{n+1}}^2\right)\frac{\partial}{\partial u_{n+1}}, \quad (5.57a)$$

$$X_2 = (-1)^n\left[\cos\left(\frac{n\pi}{3}\right){u_n}^2\frac{\partial}{\partial u_n} - \cos\left(\frac{(n+1)\pi}{3}\right){u_{n+1}}^2\frac{\partial}{\partial u_{n+1}}\right], \quad (5.57b)$$

$$X_3 = (-1)^n\left[\sin\left(\frac{n\pi}{3}\right){u_n}^2\frac{\partial}{\partial u_n} - \sin\left(\frac{(n+1)\pi}{3}\right){u_{n+1}}^2\frac{\partial}{\partial u_{n+1}}\right], \quad (5.57c)$$

$$X_4 = \frac{\partial}{\partial n}. \quad (5.57d)$$

5.3 Conclusion

We have presented a procedure for obtaining symmetries of difference equations. We have assumed that ξ and Q are functions of n and u_n only (not all equations admit such characteristics) and we have derived a number of symmetries that were used, in some cases, to solve the difference equations. In the case of Painlevé I, we have shown that ξ is not zero whenever a is non-zero.

Chapter 6

Symmetry and Properties of Linear Ordinary Differential Equations with Maximal Dimension

6.1 Introduction

Consider nth order linear ordinary differential equations $\Delta_n(y) = 0$, where $y = y(x)$. It has been proved that for $n = 1$ and 2, all equations can be written in the canonical form $y' = 0$ and $y'' = 0$, respectively. Lie showed that a linear ordinary differential equation of a general order $n > 2$ is equivalent to $y^{(n)} = 0$ if and only if its symmetry algebra has the maximal dimension $n + 4$ [50]. It has also been shown in [35] that a LODE of order $n > 2$ has a symmetry algebra of maximal dimension if and only if it is iterative. In [51] a list of three linear equations (in their normal form) of order $n = 3, 4$ and 5 that are iterative is provided. In this chapter, we review the expression of the symmetry generator obtained in [27]. We wish to obtain solutions in terms of the parameters of the first-order source equation. Expression of coefficients of LODE with maximal dimension in terms of the parameter of the second-order source equation and transformation properties will be considered. This chapter is a contribution to the work done in paper [52], although new results have been obtained.

The content of this chapter appeared in the article [53].

6.2 Symmetry Analysis

In this section, the application of the algorithm for finding symmetries of differential equations is considered. It has to be noted that the nth prolongation of the infinitesimal generator for iterative equation (2.64). It is given by [27]

$$\begin{aligned}\mathbf{X}^{[n]} =& \mathbf{v} + \sum_{k=1}^{n}\left[\sum_{i=0}^{k-1}\binom{k}{i}\left(\frac{n-1}{2}y^{(i)}f^{(k+1-i)} - y^{(i+1)}f^{(k-i)}\right)\right.\\ &\left.+\left(\frac{n-1}{2}f^{(1)}+\alpha\right)y^{(k)} + h^{(k)}\right]\partial_{y^{(k)}}.\end{aligned} \tag{6.1}$$

Imposing the invariance criterion $\mathbf{X}^{[n]}(E) = 0$ (whenever $(E) = 0$ holds) and applying the algorithm of finding symmetry it was shown in [27] that

$$\mathbf{X} = \quad f\,\partial_x + \left[\left(\frac{n-1}{2}\,f' + c\right)\,y + h\right]\,\partial_y, \tag{6.2}$$

where

$$\frac{(n+1)!}{(n-2)!4!}f^{(3)} + A_n^2 f^{(1)} + \frac{1}{2}A_n^{2\,(1)} f = 0, \tag{6.3a}$$

and

$$h^{(n)} + \sum_{i=0}^{n-2} A_n^{n-i}\,h^{(i)} = 0. \tag{6.3b}$$

In [52] it was proven that (6.3a) can be simplified further independent of order and it was shown that condition (6.3) can be reduced to

$$f^{(3)} + 4A_2^2 f^{(1)} + 2A_2^{2\,(1)} f = 0, \tag{6.4a}$$

$$h^{(n)} + \sum_{i=0}^{n-2} A_n^{n-i} h^{(i)} = 0. \tag{6.4b}$$

It is well-known that n linearly independent solutions of (2.64) are

$$y_k \quad = \quad u^{n-(k+1)}v^k, \qquad 0 \le k \le n-1, \tag{6.5}$$

where u and v are the solutions of

$$y^{(2)} + A_2^2\,y = 0. \tag{6.6}$$

6.2.1 Expressions of the A_n^i in terms of A_2^2

It follows from the work in paper [27] that all the coefficients of iterative equations of order n can be expressed in terms of the coefficient A_n^2 and its derivatives only. In fact, using the symmetry condition and splitting the resulting equation with respect to $y^{(n-i)}$,$1 \leq i \leq n$,it has been shown in [27] that

$$\begin{aligned}&\frac{(n+1)!(i-1)}{(n-i)!(i+1)!2}y^{(i+1)} + iy^{(1)}A_n^i + yA_n^{i\,(1)} \\ &+ \sum_{j=2}^{i-1} A_n^j \frac{(n-j)![n(i-j-1)+i+j-1]}{(n-i)!(i-j+1)!2}y^{(i-j+1)} = 0, \quad i = 1,\ldots,n.\end{aligned} \tag{6.7}$$

We now want to investigate how we can make use of (6.7) to write down all coefficients of equation (2.64) up to $n = 10$ in terms of A_2^2.
For $i = 2$, (6.7) becomes

$$\frac{(n+1)!}{(n-2)!4!}y^{(3)} + A_n^2 y^{(1)} + \frac{1}{2}A_n^{2\,(1)}y = 0. \tag{6.8}$$

Invoking [27]

$$A_n^2 \;=\; \left(\begin{array}{c} n+1 \\ 3 \end{array}\right) A_2^2, \tag{6.9}$$

Equation (6.8) reduces to

$$y^{(3)} + 4A_2^2 y^{(1)} + 2A_2^{2\,(1)}y \;=\; 0. \tag{6.10}$$

Note that (6.10) is independent of the order n.
For $i = 3$, (6.7) becomes

$$\frac{(n+1)!2}{(n-3)!4!2}y^{(4)} + 3y^{(1)}A_n^3 + yA_n^{3\,(1)} + A_n^2\frac{(n-2)!4}{(n-3)!2!2}y^{(2)} = 0. \tag{6.11}$$

We can obtain the fourth derivative of y, $y^{(4)}$, by differentiating (6.10) with respect to x. Equating all coefficients of derivatives of y to zero we get

$$A_n^3 = \frac{n-2}{2}A_n^{2\,(1)} = \frac{n-2}{2}\left(\begin{array}{c} n+1 \\ 3 \end{array}\right) A_2^{2(1)} = 2\left(\begin{array}{c} n+1 \\ 4 \end{array}\right) A_2^{2(1)}. \tag{6.12}$$

This gives an expression of A_n^3 in terms of the parameters A_2^2, i.e.,

$$A_n^3 = 2\left(\begin{array}{c} n+1 \\ 4 \end{array}\right) A_2^{2\prime}.$$

Similarly for $i = 4$, equation (6.7) reduces to

$$\frac{(n+1)(n)(n-1)(n-2)(n-3)}{40}y^{(5)} + \frac{(n-2)(n-3)(n+5)}{3!}A_n^2 y^{(3)} + 3(n-3)_n^3 y^{(2)} + 8A_n^4 y^{(1)} + 2A_n^4 y = 0. \tag{6.13}$$

We replace the fifth derivative of y (obtained by differentiating (6.8) with respect to x twice) in equation (6.13) and then we equate the coefficients of derivatives of y to zero again. These steps, taking into consideration equation (6.9), lead to

$$A_n^4 = 3\begin{pmatrix} n+1 \\ 5 \end{pmatrix} A_2^{2(2)} + \frac{1}{3}(5n+7)\begin{pmatrix} n+1 \\ 5 \end{pmatrix}\left(A_2^2\right)^2. \tag{6.14}$$

Repeating the same idea for $i = 5, \ldots, 10$, we obtain all coefficients A_n^l, $2 \leq l \leq 10$, in terms of A_2^2 :

$$A_n^2 = \begin{pmatrix} n+1 \\ 3 \end{pmatrix} A_2^2, \tag{6.15a}$$

$$A_n^3 = \begin{pmatrix} n+1 \\ 4 \end{pmatrix}\left[2A_2^{2(1)}\right], \tag{6.15b}$$

$$A_n^4 = \begin{pmatrix} n+1 \\ 5 \end{pmatrix}\left[3A_2^{2(2)} + \frac{1}{3}(5n+7)A_2^{2^2}\right], \tag{6.15c}$$

$$A_n^5 = \begin{pmatrix} n+1 \\ 6 \end{pmatrix}\left[4A_2^{2(3)} + 2(5n+7)A_2^2 A_2^{2(1)}\right], \tag{6.15d}$$

$$A_n^6 = \begin{pmatrix} n+1 \\ 7 \end{pmatrix}\left[5A_2^{2(4)} + (21n+29)A_2^2 A_2^{2(2)} + \frac{5}{2}(7n+10)A_2^{2(1)^2} + \frac{1}{9}(35n^2+112n+93)A_2^{2^3}\right], \tag{6.15e}$$

$$A_n^7 = \begin{pmatrix} n+1 \\ 8 \end{pmatrix}\left[6A_2^{2(5)} + \frac{4}{3}(35n^2+112n+93)A_2^{2^2}A_2^{2(1)} + 12(7n+10)A_2^{2(1)}A_2^{2(2)} + \frac{8}{3}(14n+19)A_2^2 A_2^{2(3)}\right], \tag{6.15f}$$

$$A_n^8 = \binom{n+1}{9}\Big[7A_2^{2(6)} + 20(3n+4)A_2^2A_2^{2(4)} + \frac{63}{5}(9n+13)A_2^{2(2)^2} + 14(12n+17)A_2^{2(1)}A_2^{2(3)} + \frac{1}{15}(175n^3+945n^2+1769n+1143)A_2^{2^4} + \frac{2}{5}(315n^2+996n+817)A_2^{2^2}A_2^{2(2)} + 2(105n^2+339n+284)A_2^2A_2^{2(1)^2}\Big], \tag{6.15g}$$

$$A_n^9 = \binom{n+1}{10}\Big[8\,A_2^{2(7)} - \frac{112}{3}\Big((5n+7)A_2^2A_2^{2(1)} + 2A_2^{2(3)}\Big)\big(2A_2^{2^2} - A_2^{2(2)}\big)(3n+13) - \frac{280}{9}\Big((5n+7)A_2^{2^2} + 9A_2^{2(2)}\Big)\Big(8A_2^2A_2^{2(1)} - A_2^{2(3)}\Big)(n+3) + \frac{70}{3}\Big(48A_2^{2^2}A_2^{2(1)} - 34A_2^{2(1)}A_2^{2(2)} - 20A_2^2A_2^{2(3)} + A_2^{2(5)}\Big)(3n+5)A_2^2 + \frac{20}{9}\Big(2\left(35n^2+112n+93\right)A_2^{2^2}A_2^{2(1)} + 18\left(7\,n+10\right)A_2^{2(1)}A_2^{2(2)} + 4\left(14n+19\right)A_2^2A_2^{2(3)} + 9\,A_2^{2(5)}\Big)(n+15)A_2^2 + \frac{40}{3}\Big(16\,A_2^{2^3} - 27\,A_2^{2(1)^2} - 40\,A_2^2A_2^{2(2)} + 3\,A_2^{2(4)}\Big)(5\,n+11)A_2^{2(1)} + \frac{10}{9}\Big(2\left(35\,n^2+112\,n+93\right)A_2^{2^3} + 45\left(7\,n+10\right)A_2^{2(1)^2} + 18\left(21\,n+29\right)A_2^2A_2^{2(2)} + 90\,A_2^{2(4)}\Big)(n+7)A_2^{2(1)} + 1440\,A_2^{2(1)^3} + 6464\,A_2^2A_2^{2(1)}A_2^{2(2)} + 1792\,A_2^{2^2}A_2^{2(3)} - \frac{3248}{3}\,A_2^{2(2)}A_2^{2(3)} - 720\,A_2^{2(1)}A_2^{2(4)} - \frac{896}{3}\,A_2^2A_2^{2(5)} - 2048A_2^{2^3}A_2^{2(1)}\Big], \tag{6.15h}$$

$$A_n^{10} = \binom{n+1}{11}\Big[9\,A_2^{2(8)} + \frac{22}{5}\Big(\left(5\,n+7\right)A_2^{2^2} + 9\,A_2^{2(2)}\Big)\Big(8\left(2\,A_2^{2^2} - A_2^{2(2)}\right)A_2^2 - 27\,A_2^{2(1)^2} - 32\,A_2^2A_2^{2(2)} + 3\,A_2^{2(4)}\Big)(5\,n+13) - 154\Big((5\,n+7)A_2^2A_2^{2(1)} +$$

$$2\,A_2^{2^{(3)}}\Big)\big(8\,A_2^2A_2^{2^{(1)}}-A_2^{2^{(3)}}\big)(2\,n+7)-\frac{22}{5}\Bigg(2\,\big(35\,n^2+112\,n+93\big)A_2^{2^3}$$
$$+45\,(7\,n+10)A_2^{2^{(1)^2}}+18\,(21\,n+29)A_2^2A_2^{2^{(2)}}+90\,A_2^{2^{(4)}}\Bigg)\Big(2\,A_2^{2^2}-A_2^{2^{(2)}}\Big)(n$$
$$+5)-\frac{11}{3}\Bigg(4\Big(16A_2^{2^3}-27A_2^{2^{(1)^2}}-40A_2^2A_2^{2^{(2)}}+3A_2^{2^{(4)}}\Big)A_2^2-65\big(8A_2^2A_2^{2^{(1)}}$$
$$-A_2^{2^{(3)}}\big)A_2^{2^{(1)}}-144\,\Big(2\,A_2^{2^2}-A_2^{2^{(2)}}\Big)A_2^{2^{(2)}}+165\,A_2^{2^{(1)}}A_2^{2^{(3)}}+100\,A_2^2A_2^{2^{(4)}}$$
$$-4\,A_2^{2^{(6)}}\Bigg)(7\,n+11)A_2^2+\frac{11}{45}\Bigg(\big(175\,n^3+945\,n^2+1769\,n+1143\big)A_2^{2^4}$$
$$+30\,\big(105\,n^2+339\,n+284\big)A_2^2A_2^{2^{(1)^2}}+6\,\big(315\,n^2+996\,n+817\big)A_2^{2^2}A_2^{2^{(2)}}$$
$$+189\,(9\,n+13)A_2^{2^{(2)^2}}+210\,(12\,n+17)A_2^{2^{(1)}}A_2^{2^{(3)}}+300\,(3\,n+4)A_2^2A_2^{2^{(4)}}$$
$$+105\,A_2^{2^{(6)}}\Bigg)(n+17)A_2^2+\frac{33}{2}\Bigg(2\,\big(35\,n^2+112\,n+93\big)A_2^{2^2}A_2^{2^{(1)}}+18\,(7\,n$$
$$+10)A_2^{2^{(1)}}A_2^{2^{(2)}}+4\,(14\,n+19)A_2^2A_2^{2^{(3)}}+9\,A_2^{2^{(5)}}\Bigg)(n+8)A_2^{2^{(1)}}$$
$$+\frac{99}{2}\Bigg(20\,\Big(8\,A_2^2A_2^{2^{(1)}}-A_2^{2^{(3)}}\Big)A_2^2+88\,\Big(2\,A_2^{2^2}-A_2^{2^{(2)}}\Big)A_2^{2^{(1)}}-150\,A_2^{2^{(1)}}A_2^{2^{(2)}}$$
$$-120\,A_2^2A_2^{2^{(3)}}+7\,A_2^{2^{(5)}}\Bigg)(n+2)A_2^{2^{(1)}}-\frac{1071}{5}\Bigg(48\,A_2^{2^2}A_2^{2^{(1)}}-34\,A_2^{2^{(1)}}A_2^{2^{(2)}}$$
$$-20A_2^2A_2^{2^{(3)}}+A_2^{2^{(5)}}\Bigg)A_2^{2^{(1)}}-\frac{1152}{5}\Bigg(16A_2^{2^3}-27A_2^{2^{(1)^2}}-40A_2^2A_2^{2^{(2)}}$$
$$+3A_2^{2^{(4)}}\Bigg)A_2^{2^{(2)}}+1260\Big(8A_2^2A_2^{2^{(1)}}-A_2^{2^{(3)}}\Big)A_2^{2^{(3)}}+\frac{7056}{5}\Big(2A_2^{2^2}-A_2^{2^{(2)}}\Big)A_2^{2^{(4)}}$$
$$-\frac{4914}{5}A_2^{2^{(1)}}A_2^{2^{(5)}}-\frac{2016}{5}\,A_2^2A_2^{2^{(6)}}-\frac{9}{40}\,A_2^2\Bigg(9216\,A_2^{2^2}A_2^{2^{(2)}}-128\,\Bigg(16\,A_2^{2^3}$$
$$-27\,A_2^{2^{(1)^2}}-40\,A_2^2A_2^{2^{(2)}}+3\,A_2^{2^{(4)}}\Bigg)A_2^2+2080\,\Big(8\,A_2^2A_2^{2^{(1)}}-A_2^{2^{(3)}}\Big)A_2^{2^{(1)}}$$
$$-4608\,A_2^{2^{(2)^2}}-5280\,A_2^{2^{(1)}}A_2^{2^{(3)}}-3200\,A_2^2A_2^{2^{(4)}}+128\,A_2^{2^{(6)}}\Bigg)\Bigg]. \tag{6.15i}$$

In paper [52], they let Ω_n, $n \geq 3$, be the operator that corresponds to the linear iterative equation of order n and they proved for n up to 6 that its expression is given by

$$\Omega_n(y_k) = \left(\prod_{j=0}^{n-1}(k-j)\right) \cdot u^{2\,n-(1+k)} v^{k-n} \left[\left(\frac{v}{u}\right)'\right]^n = 0, \tag{6.16}$$

for $0 \leq k \leq n-1$.Now that we have obtained the expressions of the A_n^j's, $1 \leq j \leq 10$, the proof of (6.16) for $n \leq 10$ becomes straightforward but lengthy. Below is the proof for $n = 7$ and 10.

Proof 6.2.2. *Let us consider the 7th order iterative equation*

$$y^{(7)} + A_7^2 y^{(5)} + A_7^3 y^{(4)} + A_7^4 y^{(3)} + A_7^5 y^{(2)} + A_7^6 y^{(1)} + A_7^7 y = 0 \tag{6.17}$$

and let

$$\begin{aligned}(\Omega_7)\,(y_k) =& \big(D^{(7)} + A_7^2 D^{(5)} + A_7^3 D^{(4)} + A_7^4 D^{(3)} + A_7^5 D^{(2)} + A_7^6 Y^{(1)} \\ &+ A_7^7\big)(y_k),\end{aligned} \tag{6.18}$$

where D denotes the differentiation operator d/dx and $y_k = u^{6-k}v^k$. The expansion of (6.18) leads to an expression that involves higher derivatives of u and v. So, we must eliminate the dependency among higher derivatives by expressing them in terms of u, v, u' and v'. Using equation (6.6) we get

$$\gamma^{(2)} = -A_2^2\gamma, \tag{6.19a}$$

$$\gamma^{(3)} = -A_2^{2^{(1)}}\gamma - A_2^2\gamma^{(1)}, \tag{6.19b}$$

$$\gamma^{(4)} = -2A_2^{2^{(1)}}\gamma^{(1)} + (A_2^{2^2} - A_2^{2^{(2)}}\gamma, \tag{6.19c}$$

$$\gamma^{(5)} = \left(A_2^{2^2} - 3A_2^{2^{(2)}}\right)\gamma^{(1)} + \left(4A_2^2 A_2^{2^{(1)}} - A_2^{2^{(3)}}\right)\gamma, \tag{6.19d}$$

$$\gamma^{(6)} = \left(6A_2^2 A_2^{2^{(1)}} - 4A_2^{2^{(3)}}\right)\gamma^{(1)} + \left(7A_2^2 A_2^{2^{(2)}} + 4A_2^{2^{(1)^2}} - A_2^{2^{(4)}} + A_2^{2^3}\right)\gamma, \tag{6.19e}$$

$$\begin{aligned}\gamma^{(7)} =& \left(10A_2^{2^{(2)}} + 13A_2^2 A_2^{2^{(2)}} - A_2^{2^{(4)}} + A_2^{2^3}\right)\gamma^{(1)} + \Bigg(3A_2^{2^2} A_2^{2^{(1)}} + 11A_2^2 A_2^{2^{(3)}} \\ &+ 15A_2^{2^{(1)}} A_2^{2^{(2)}} - A_2^{2^5}\Bigg)\gamma,\end{aligned} \tag{6.19f}$$

for $\gamma = u, v$. Using in (6.18) the results (6.15), for $n = 7$, and (6.19) we get after simplification

$$\begin{aligned}(\Omega_7)\,(y_k) =&(k^7 - 21k^6 + 175k^5 - 735k^4 + 1624k^3 - 1764k^2 + 720k)\times \\ &\Big(u^{-k+6}v^{k-7}{v^{(1)}}^7 - u^{-k-1}v^k{u^{(1)}}^7 - 7u^{-k+5}v^{k-6}u^{(1)}{v^{(1)}}^6 + \\ &21u^{-k+4}v^{k-5}{u^{(1)}}^2{v^{(1)}}^5 - 35u^{-k+3}v^{k-4}{u^{(1)}}^3{v^{(1)}}^4 + 35u^{-k+2}v^{k-3} \\ &{u^{(1)}}^4{v^{(1)}}^3 - 21u^{-k+1}v^{k-2}{u^{(1)}}^5{v^{(1)}}^2 + 7u^{-k}v^{k-1}{u^{(1)}}^6v^{(1)}\Big)\end{aligned} \tag{6.20}$$

which can be written in the form

$$\Omega_7(y_k) = \left(\prod_{j=0}^{6}(k-j)\right)\cdot u^{13-k}v^{k-7}\left[\left(\frac{v}{u}\right)'\right]^7, \text{ for } \quad 0 \le k \le 6. \tag{6.21}$$

We have verified that (6.16) is also true for $n = 8, 9$ and 10. Below is the verification for degree $n = 10$.

Proof 6.2.3. *Consider the tenth-order linear iterative equation*

$$\begin{aligned}&y^{(10)} + A_{10}^2\,y^{(8)} + A_{10}^3\,y^{(7)} + A_{10}^4\,y^{(6)} + A_{10}^5\,y^{(5)} + A_{10}^6\,y^{(4)} + A_{10}^7\,y^{(3)} \\ &+ A_{10}^8\,y^{(2)} + A_{10}^9\,y^{(1)} + A_{10}^{10}\,y = 0.\end{aligned} \tag{6.22}$$

Let

$$\begin{aligned}\Omega_{10} =&D^{(10)} + A_{10}^2\,D^{(8)} + A_{10}^3\,D^{(7)} + A_{10}^4\,D^{(6)} + A_{10}^5\,D^{(5)} + A_{10}^6\,D^{(4)} \\ &+ A_{10}^7\,D^{(3)} + A_{10}^8\,D^{(2)} + A_{10}^9\,D^{(1)} + A_{10}^{10}\,D\end{aligned} \tag{6.23}$$

and let

$$y_k = u^{9-k}v^k.$$

We have

$$\begin{aligned}\Omega_{10}(y_k) =&(D^{(10)} + A_{10}^2\,D^{(8)} + A_{10}^3\,D^{(7)} + A_{10}^4\,D^{(6)} + A_{10}^5\,D^{(5)} + A_{10}^6\,D^{(4)} + \\ &A_{10}^7\,D^{(3)} + A_{10}^8\,D^{(2)} + A_{10}^9\,D^{(1)} + A_{10}^{10}\,D)(u^{9-k}v^k) \\ =&\sum_{l=0}^{10}\binom{10}{l}(u^{9-k})^{(l)}(v^k)^{(10-l)} + A_{10}^2\sum_{l=0}^{8}\binom{8}{l}(u^{9-k})^{(l)}(v^k)^{(8-l)} \\ &+ \cdots + A_{10}^{10}u^{9-k}v^k.\end{aligned} \tag{6.24}$$

Firstly, we expand (6.24) and we substitute higher derivatives of u, v by their expressions in terms of u', v' using (6.19) and

$$\begin{aligned}\gamma^{(8)} =& -\Big(12A_2^{2^2}A_2^{2'} - 24A_2^2A_2^{2^{(3)}} - 48A_2^{2'}A_2^{2^{(2)}} + 6A_2^{2^{(5)}}\Big)\gamma^{(1)}\\ &-\Big(22A_2^{2^2}A_2^{2^{(2)}} + 28A_2^2A_2^{2'^2} - 16A_2^2A_2^{2^{(4)}} + A_2^{2^4} - 26A_2^{2'}A_2^{2^{(3)}}\\ &- 15A_2^{2''^2} + A_2^{2^{(6)}}\Big)\gamma,\end{aligned} \tag{6.25a}$$

$$\begin{aligned}\gamma^{(9)} =& -\Big(10A_2^2A_2^{2^{(2)}} + 34A_2^{2^2}A_2^{2^{(2)}} - 36A_2^2A_2^{2^{(4)}} + A_2^{2^4} + 42A_2^2A_2^{2'^2}\\ &- 98A_2^{2'}A_2^{2^{(3)}} - 63A_2^{2^{(2)^2}} + 7A_2^{2^{(6)}}\Big)\gamma^{(1)} - \Big(10A_2^{2^3}A_2^{2'} + 46A_2^{2^2}A_2^{2^{(3)}}\\ &+ 146A_2^2A_2^{2'}A_2^{2^{(2)}} - A_2^{2^6} + 28A_2^{2'^3} - 42A_2^{2'}A_2^{2^{(4)}} - 56A_2^{2^{(2)}}A_2^{2^{(3)}}\\ &- 21A_2^2A_2^{2^{(5)}} + A_2^{2^{(7)}}\Big)\gamma,\end{aligned} \tag{6.25b}$$

$$\begin{aligned}\gamma^{(10)} =& -\Big(80A_2^{2^2}A_2^{2^{(3)}} - 4A_2^{2^3}A_2^{2'} + 320A_2^2A_2^{2'}A_2^{2^{(2)}} - 62A_2^2A_2^{2^{(5)}}\\ &+ 80A_2^{2'}A_2^{2^{(2)}} - 148A_2^{2'}A_2^{2^{(4)}} - 280A_2^{2^{(2)}}A_2^{2^{(3)}} + 8A_2^{2^{(7)}}\Big)\gamma'\\ &-\Big(6A_2^{2^3}A_2^{2^{(2)}} - 44A_2^{2^2}A_2^{2'^2} + 86A_2^{2^2}A_2^{2^{(4)}} - A_2^{2^5} + 338A_2^2A_2^{2'}A_2^{2^{(3)}}\\ &+ 211A_2^2A_2^{2''^2} - 29A_2^2A_2^{2^{(6)}} + 232A_2^{2'^2}A_2^{2^{(2)}} - 8A_2^{2^{(1)}}A_2^{2^5}\\ &- 98A_2^{2^{(2)}}A_2^{2^{(4)}} - 56A_2^{2^{(3)^2}} - 56A_2^{2'}A_2^{2^{(5)}} + A_2^{2^{(8)}}\Big)\gamma\end{aligned} \tag{6.25c}$$

for $\gamma = u, v$. Secondly, we replace the A_{10}^i's, for $1 < i < 11$, in the resulting equation by their expressions given by (6.15) for $n = 10$. Here, the computation becomes lengthy

but thanks to Mathematica, the expansion and the simplification lead to

$$
\begin{aligned}
(\Omega_{10})\,(y_k) =&(k^{10}-45k^9+870k^8-9450k^7+63273k^6-269325k^5\\
&+723680k^4-1172700k^3+1026576k^2-362880k)u^{19-k}v^{k-10}\Bigg[\\
&\frac{(v')^{10}}{u^{10}}-\frac{10\,v\,(u')\,(v')^9}{u^{11}}+\frac{45\,v^2\,(u')^2\,(v')^8}{u^{12}}-\frac{120\,v^3\,(u')^3\,(v')^7}{u^{13}}+\\
&\frac{210\,v^4\,(u')^4\,(v')^6}{u^{14}}-\frac{252\,v^5\,(u')^5\,(v')^5}{u^{15}}+\frac{210\,v^6\,(u')^6\,(v')^4}{u^{16}}-\\
&\frac{120\,v^7\,(u')^7\,(v')^3}{u^{17}}+\frac{45\,v^8\,(u')^8\,(v')^2}{u^{18}}-\frac{10\,v^9\,(u')^9\,(v')}{u^{19}}+\\
&\frac{v^{10}\,(u')^{10}}{u^{20}}\Bigg]
\end{aligned}
\tag{6.26a}
$$

$$
\begin{aligned}
=&(k^{10}-45k^9+870k^8-9450k^7+63273k^6-269325k^5+\\
&723680k^4-1172700k^3+1026576k^2-362880k)\\
&\Big[u^{9-k}v^{k-10}v'^{10}-10\,u^{8-k}v^{k-9}u'v'^9+45\,u^{7-k}v^{k-8}u'^2v'^8-\\
&120u^{6-k}v^{k-7}u'^3v'^7+210u^{5-k}v^{k-6}u'^4v'^6-252u^{4-k}v^{k-5}u'^5v'^5\\
&+\,u^{3-k}v^{k-4}u'^6v'^4-120\,u^{2-k}v^{k-3}u'^7v'^3+45\,u^{1-k}v^{k-2}u'^8v'^2\\
&-10u^{-k}\,v^{k-1}u'^9v'+u^{-1-k}v^ku'10\Big]
\end{aligned}
\tag{6.26b}
$$

$$
=\left(\prod_{j=0}^{9}(k-j)\right)\cdot u^{19-k}v^{k-10}\left[\left(\frac{v}{u}\right)'\right]^{10},\ \textit{for}\quad 0\le k\le 9. \tag{6.26c}
$$

It worthwhile to mention that the validity of (6.16) for an arbitrary n is unknown.

6.2.4 Symmetry Generators

In this section, we are interested in expressing symmetry generators in terms of the parameter r, and for r as large as possible. In [52] we showed that this reduces to finding

solutions of the second-order source equation in terms of the parameter r. Recall that

$$A_2^2 = \frac{r'^2 - 2\,r\,r''}{4\,r^2} \tag{6.27a}$$

provided that

$$q = \frac{(1-n)}{2} r'. \tag{6.27b}$$

Recall that n linearly independent solutions of (2.64) are given by (6.5). Therefore, the symmetry algebra of Eq. (2.64) is spanned by

$$X_0 = y\partial_y, \tag{6.28a}$$

$$X_1 = u^2\partial_x + (n-1)uu'y\partial_y, \tag{6.28b}$$

$$X_2 = uv\partial_x + \frac{n-1}{2}(uv)'y\partial_y, \tag{6.28c}$$

$$X_3 = v^2\partial_x + (n-1)vv'y\partial_y, \tag{6.28d}$$

$$X_k = u^{n-1-k}v^k\partial_y, \quad 4 \leq k \leq n+3. \tag{6.28e}$$

Firstly, we express u and v in terms of r. Considering (6.28), this results to solving (6.27) for r. Note also that in virtue of (6.28), we only need to express solutions of (6.6) as functions of r. This approach leads to the results summarized in Table 6.1.

Table 6.1 For given values of r

r	$u,\ v$	$X_1, X_2, X_3, X_k, 4 \leq k \leq n+3$
c	$u = 1,\ v = x$	$y\,\partial_y,\ \partial_x,\ x\,\partial_x + \frac{n-1}{2}y\partial_y,$ $x^2\partial_x + (n-1)\,x\,y\,\partial_y,$ $x^k\,\partial_y$
$c_1x + c_2$	$u = r^{\frac{1}{2}},\ v = r^{\frac{1}{2}}\ln r$	$y\partial_y,\ r\partial_x - qy\partial_y,\ r\ln r\partial_x - (1+\ln r)qy\partial_y,$ $r\ln^2 r\partial_x - \left(\ln^2 r + 2\ln r\right)qy\partial_y,$ $r^{\frac{n-1}{2}}\ln^k r\partial_y$
$(c_1x + c_2)^m$	$u = r^{\frac{1}{2}},\ v = r^{\frac{2-m}{2m}}$	$y\partial_y,\ r\partial_x - qy\partial_y,\ r^{\frac{1}{m}}\partial_x - \frac{2}{m(n-1)}r^{\frac{1-m}{m}}qy\partial_y$ $r^{\frac{2-m}{m}}\partial_x - \left(\frac{2-m}{m}\right)r^{\frac{2-2m}{m}}qy\partial_y,$ $r^{\frac{k}{m}+\frac{n-1-2k}{2}}\partial_y$
e^{mx}	$u = r^{\frac{1}{2}},\quad v = r^{-\frac{1}{2}}$	$y\partial_y,\ r\partial_x - qy\partial_y,\ \partial_x,$ $\frac{1}{r}\partial_x + \left(\frac{q}{r^2}\right)y\partial_y,$ $r^{\frac{n-1-2k}{2}}\partial_y$

Secondly, we record values for A_2^2. For various values of A_2^2 we solve the source equation. We restrict ourselves to values of A_2^2 that have not been previously considered by a choice of r. Our findings using the second approach are summarized in Table 6.2.

Table 6.2 For given values of A_2^2

A_2^2	$u,\ v$	$X_1, X_2, X_3, X_k, 4 \le k \le n+3$
c	$u = \cos(\sqrt{c}x),$ $v = \sin(\sqrt{c}x)$	$y\partial_y,\ \partial_x,\ x\partial_x + \frac{n-1}{2}y\partial_y,$ $x^2\partial_x + (n-1)xy\partial_y,$ $x^k\partial_y$
αx^m	$u = \sqrt{x}J_{\frac{1}{m+2}}\left(\frac{2x^{\frac{m}{2}+1}}{m+2}\sqrt{\alpha}\right),$ $v = \sqrt{x}Y_{\frac{1}{m+2}}\left(\frac{2x^{\frac{m}{2}+1}}{m+2}\sqrt{\alpha}\right)$	$r\partial_x - qy\partial_y, r\ln r\partial_x - (1+\ln r)qy\partial_y,$ $y\partial_y, r\ln^2 r\partial_x - \left(\ln^2 r + 2\ln r\right)qy\partial_y,$ $r^{\frac{n-1}{2}}\ln^k r\partial_y$
$(c_1x + c_2)^m$	$u = r^{\frac{1}{2}},\ v = r^{\frac{2-m}{2m}}$	$r\partial_x - qy\partial_y,\ r^{\frac{1}{m}}\partial_x - \frac{2}{m(n-1)}r^{\frac{1-m}{m}}qy\partial_y$ $y\partial_y,\ r^{\frac{2-m}{m}}\partial_x - \left(\frac{2-m}{m}\right)r^{\frac{2-2m}{m}}qy\partial_y,$ $r^{\frac{k}{m}+\frac{n-1-2k}{2}}\partial_y$
e^{bx}	$J\left(\frac{2}{b}\sqrt{e^{bx}}\right),\ Y\left(\frac{2}{b}\sqrt{e^{bx}}\right)$	$y\partial_y,\ r\partial_x - qy\partial_y,\ \partial_x,$ $\frac{1}{r}\partial_x + \left(\frac{q}{r^2}\right)y\partial_y,$ $r^{\frac{n-1-2k}{2}}\partial_y$

6.3 Transformation Properties

Consider the equation

$$L^n y \equiv K_n^0\, y^{(n)} + K_n^1\, y^{(n-1)} + K_n^2\, y^{(n-2)} + \cdots + K_n^{n-1}\, y^{(1)} + K_n^n\, y = 0, \tag{6.29}$$

where $r\, y^{(1)} + q\, y \equiv L(y)$ is the original source equation.

We assume that

$$Z_n^0\, t^{(n)} + Z_n^1\, t^{(n-1)} + Z_n^2\, t^{(n-2)} + \cdots + Z_n^{n-1}\, t^{(1)} + Z_n^n\, t = 0 \tag{6.30}$$

is the transformed equation obtained from Eq.(6.29) by the equivalence transformations

$$x = f(z), \tag{6.31a}$$
$$y = g(z)t, \tag{6.31b}$$

for some functions f and g. It is known that under an equivalence transformation, the symmetry algebras of equivalent equations are isomorphic which means that (6.30) is an iterative equation with source equation

$$M(t) = R(z)t^{(1)} + Q(z)t. \tag{6.32}$$

That is

$$M^n(t) \equiv Z_n^0\, t^{(n)} + Z_n^1\, t^{(n-1)} + Z_n^2\, t^{(n-2)} + \cdots + Z_n^{n-1}\, t^{(1)} + Z_n^n\, t = 0. \tag{6.33}$$

It has been proved that if the linear iterative equation is in its normal reduced form the parameter, r, of the source equation and the parameter, R, of the transformed equation are such that

$$\frac{R^{(1)^2} - 2RR^{(2)}}{R^2} = \frac{1}{f^{(1)^2}}\left[\frac{\left(r^{(1)}(f)\right)^2 - 2r(f)r^{(2)}(f)}{r(f)^2} f^{(1)^4} - 3f^{(2)^2} + 2f^{(1)}f^{(3)}\right], \tag{6.34}$$

where $f = f(z)$[52]. Note that if the equation is in its normal form the computations become easier since we are looking only at how the parameter r is changing. We aim to establish a similar relationship for linear iterative equation in its standard form and to retrieve (6.34) using the well-known group of transformations,

$$x = f(z), \qquad y = \lambda \left[f^{(1)}(z)\right]^{\frac{n-1}{2}} t, \tag{6.35}$$

of the general linear equation in normal reduced form (see [54, 55]). Firstly, we use M to iterate n times. Secondly, we use L to iterate and we then transform the resulting equation using (6.31). Finally, we use the fact that K_m^j, Z_n^p, f and g are independent of t and t' to get the relationship that we are looking for.

Case $n = 2$

By definition

$$L(2)y \equiv K_2^0\, y^{(2)} + K_2^1\, y^{(1)} + K_2^2\, y. \tag{6.36}$$

From (6.31) we have

$$y^{(1)} = \frac{g^{(1)}}{f^{(1)}} t + \frac{g}{f^{(1)}} t^{(1)} \tag{6.37}$$

and

$$y^{(2)} = \frac{1}{f^{(1)^3}}\left[f^{(1)} g\, t^{(2)} + \left(2f^{(1)}g^{(1)} - f^{(2)}g\right) t^{(1)} + \left(f^{(1)}g^{(2)} - f^{(2)}g^{(1)}\right) t\right]. \tag{6.38}$$

According to (6.37) and (6.38), one can rewrite (6.36) as follows

$$\begin{aligned} L^{(2)}y =& K_2^0 \frac{g}{{f^{(1)}}^2} t^{(2)} + \left(\frac{2f^{(1)}g^{(1)} - f^{(2)}g}{{f^{(1)}}^3} K_2^0 + K_2^1 \frac{g}{f^{(1)}} \right) t^{(1)} \\ &+ \left(\frac{f^{(1)}g^{(2)} - f^{(2)}g^{(1)}}{{f^{(1)}}^3} K_2^0 + K_2^1 \frac{g^{(1)}}{f^{(1)}} + K_2^2 g \right) t. \end{aligned} \tag{6.39}$$

Setting $n = 2$ in (6.30) we get

$$Z_2^0\, t^{(2)} + Z_2^1\, t^{(1)} + Z_2^2\, t \;\; = \;\; 0. \tag{6.40}$$

The substitution of $Z_2^0 t^{(2)} = -Z_2^1 t^{(1)} - Z_2^2 t$ into (6.39) leads to

$$\begin{aligned} &\left(-K_2^0 Z_2^1 \frac{g}{{f^{(1)}}^2} + K_2^0 Z_2^0 \frac{(2f^{(1)}g^{(1)} - f^{(2)}g)}{{f^{(1)}}^3} + K_2^1 Z_2^0 \frac{g}{f^{(1)}} \right) t^{(1)} + \\ &\left(-K_2^0 Z_2^2 \frac{g}{{f^{(1)}}^2} + K_2^0 Z_2^0 \frac{(f^{(1)}g^{(2)} - f^{(2)}g^{(1)})}{{f^{(1)}}^3} + K_2^1 Z_2^0 \frac{g^{(1)}}{f^{(1)}} + K_2^2 Z_2^0 g \right) t = 0. \end{aligned} \tag{6.41}$$

We can equate the coefficients of t and $t^{(1)}$ to zero to yield

$$\mathcal{Z}_2^1 = f^{(1)}\mathcal{K}_2^1 + \frac{2f^{(1)}g^{(1)} - f^{(2)}g}{f^{(1)}g}, \tag{6.42a}$$

$$\mathcal{Z}_2^2 = {f^{(1)}}^2\mathcal{K}_2^2 + \frac{f^{(1)}g^{(1)}}{g}\mathcal{K}_2^1 + \frac{f^{(1)}g^{(2)} - f^{(2)}g^{(1)}}{f^{(1)}g}, \tag{6.42b}$$

where

$$\mathcal{K}_n^j = \frac{K_n^j}{K_n^0}, \tag{6.43a}$$

$$\mathcal{Z}_n^j = \frac{Z_n^j}{Z_n^0}. \tag{6.43b}$$

Case $n = 3$

By definition,

$$L^{(3)}y \;\; = \;\; K_3^0 y^{(3)} + K_3^1 y^{(2)} + K_3^2 y^{(1)} + K_3^3 y = 0. \tag{6.44}$$

From (6.38) we have

$$y^{(3)} = \left(\frac{g}{f^{(1)^3}}\right) t^{(3)} - \left(\frac{3f^{(2)}g}{f'^4} - \frac{3g^{(1)}}{f^{(1)^3}}\right) t^{(2)} - \Big(\frac{6f^{(2)}g^{(1)}}{f^{(1)^4}} - \frac{3f^{(2)^2}g}{f^{(1)^5}} - \frac{3g^{(2)}}{f^{(1)^3}} + \frac{f^{(3)}g}{f^{(1)^4}}\Big)t^{(1)} - \left(\frac{3f^{(2)}g^{(2)}}{f^{(1)^4}} - \frac{3f^{(2)^2}g^{(1)}}{f^{(1)^5}} - \frac{g^{(3)}}{f^{(1)^3}} + \frac{f^{(3)}g^{(1)}}{f^{(1)^4}}\right)t. \tag{6.45}$$

We substitute (6.38) and (6.45) in (6.44) and we replace $Z_3^0 t^{(3)}$ by $-(Z_3^1 t^{(2)} + Z_3^2 t^{(1)} + Z_3^3 t)$. Then, letting all the coefficients t, $t^{(1)}$ and $t^{(2)}$ equal to zero in the resulting equation leads to the following:

$$\mathcal{Z}_3^1 = f^{(1)}\mathcal{K}_3^1 - \frac{3f^{(2)}}{f^{(1)}} + \frac{3g^{(1)}}{g}, \tag{6.46a}$$

$$\mathcal{Z}_3^2 = f^{(1)^2}\mathcal{K}_3^2 + \left(\frac{2f^{(1)}g^{(1)} - f^{(2)}g}{g}\right)\mathcal{K}_3^1 - \frac{6f^{(2)}g^{(1)}}{f^{(1)}g} + \frac{3f^{(2)^2}}{f^{(1)^2}} + \frac{3g^{(2)}}{g} - \frac{f^{(3)}}{f^{(1)}}, \tag{6.46b}$$

$$\mathcal{Z}_3^3 = f^{(1)^3}\mathcal{K}_3^3 + \frac{f^{(1)^2}g^{(1)}}{g}\mathcal{K}_3^2 + \left(\frac{f^{(1)}g^{(2)} - f^{(2)}g^{(1)}}{g}\right)\mathcal{K}_3^1 - \frac{3f^{(2)}g^{(2)}}{f^{(1)}g} + \frac{3f^{(2)^2}g^{(1)}}{f^{(1)^2}g} + \frac{g^{(3)}}{g} - \frac{f^{(3)}g^{(1)}}{f^{(1)}g}. \tag{6.46c}$$

Case $n = 4$

By definition the expressions of $L^4 y$ and $M^4 t$ are given by

$$K_4^0\, y^{(4)} + K_4^1\, y^{(3)} + K_4^2\, y^{(2)} + K_4^3\, y^{(1)} + K_4^4\, y = 0, \tag{6.47}$$

$$Z_4^0\, t^{(4)} + Z_4^1\, t^{(3)} + Z_4^2\, t^{(2)} + Z_4^3\, t^{(1)} + Z_4^4\, t = 0, \tag{6.48}$$

respectively. Using the same idea we get

$$\mathcal{Z}_4^1 = f^{(1)}\mathcal{K}_4^1 + \frac{4g^{(1)}}{g} - \frac{6f^{(2)}}{f^{(1)}}, \tag{6.49a}$$

$$\mathcal{Z}_4^2 = f^{(1)^2}\mathcal{K}_4^2 + \left(\frac{3f^{(1)}g^{(1)}}{g} - 3f^{(2)}\right)\mathcal{K}_4^1 + \frac{6g^{(2)}}{g} - \frac{18f^{(2)}g^{(1)}}{f^{(1)}g} - \frac{4f^{(3)}}{f^{(1)}} + \frac{15f^{(2)^2}}{f^{(1)^2}}, \tag{6.49b}$$

$$\begin{aligned}\mathcal{Z}_4^3 =& {f^{(1)}}^3\mathcal{K}_4^3 + \left(\frac{2{f^{(1)}}^2 g^{(1)}}{g} - f^{(2)}f^{(1)}\right)\mathcal{K}_4^2 + \Bigg(- \frac{6f^{(2)}g^{(1)}}{g} + \frac{3{f^{(2)}}^2}{f^{(1)}} \\ &+ \frac{3f^{(1)}g^{(2)}}{g} - f^{(3)}\Bigg)\mathcal{K}_4^1 - \frac{8f^{(3)}g^{(1)}}{f^{(1)}g} - \frac{18f^{(2)}g^{(2)}}{f^{(1)}g} + \frac{30{f^{(2)}}^2 g^{(1)}}{{f^{(1)}}^2 g} + \\ &\frac{10f^{(3)}f^{(2)}}{{f^{(1)}}^2} - \frac{15{f^{(2)}}^3}{{f^{(1)}}^3} + \frac{4g^{(3)}}{g} - \frac{f^{(4)}}{f^{(1)}},\end{aligned} \tag{6.49c}$$

$$\begin{aligned}\mathcal{Z}_4^4 =& {f^{(1)}}^4\mathcal{K}_4^4 + \frac{{f^{(1)}}^3 g^{(1)}}{g}\mathcal{K}_4^3 + \left(\frac{{f^{(1)}}^2 g^{(2)}}{g} - \frac{f^{(2)}f^{(1)}g^{(1)}}{g}\right)\mathcal{K}_4^2 + \\ &\left(\frac{f^{(1)}g^{(3)}}{g} - \frac{3f^{(2)}g^{(2)}}{g} + \frac{3{f^{(2)}}^2 g^{(1)}}{f^{(1)}g} - \frac{f^{(3)}g^{(1)}}{g}\right)\mathcal{K}_4^1 + \frac{g^{(4)}}{g} - \frac{6f^{(2)}g^{(3)}}{f^{(1)}g} \\ &- \frac{f^{(4)}g^{(1)}}{f^{(1)}g} - \frac{4f^{(3)}g^{(2)}}{f^{(1)}g} + \frac{10f^{(3)}f^{(2)}g^{(1)}}{{f^{(1)}}^2 g} + \frac{15{f^{(2)}}^2 g^{(2)}}{{f^{(1)}}^2 g} - \frac{15{f^{(2)}}^3 g^{(1)}}{{f^{(1)}}^3 g}.\end{aligned} \tag{6.49d}$$

Case $n = 5$

A straightforward but tedious computations for $n = 5$ lead to

$$\mathcal{Z}_5^1 = f^{(1)}\mathcal{K}_5^1 + \frac{5g^{(1)}}{g} - \frac{10f^{(2)}}{f^{(1)}}, \tag{6.50a}$$

$$\begin{aligned}\mathcal{Z}_5^2 =& {f^{(1)}}^2\mathcal{K}_5^2 + \left(\frac{4f^{(1)}g^{(1)}}{g} - 6f^{(2)}\right)\mathcal{K}_5^1 + \frac{10g^{(2)}}{g} - \frac{40f^{(2)}g^{(1)}}{f^{(1)}g} - \frac{10f^{(3)}}{f^{(1)}} \\ &+ \frac{45{f^{(2)}}^2}{{f^{(1)}}^2},\end{aligned} \tag{6.50b}$$

$$\begin{aligned}\mathcal{Z}_5^3 =& {f^{(1)}}^3\mathcal{K}_5^3 + \left(\frac{3{f^{(1)}}^2 g^{(1)}}{g} - 3f^{(2)}f^{(1)}\right)\mathcal{K}_5^2 + \Bigg(- \frac{18f^{(2)}g^{(1)}}{g} + \frac{15{f^{(2)}}^2}{f^{(1)}} \\ &+ \frac{6f^{(1)}g^{(2)}}{g} - 4f^{(3)}\Bigg)\mathcal{K}_5^1 - \frac{30f^{(3)}g^{(1)}}{f^{(1)}g} - \frac{60f^{(2)}g^{(2)}}{f^{(1)}g} + \frac{135{f^{(2)}}^2 g^{(1)}}{{f^{(1)}}^2 g} + \\ &\frac{60f^{(3)}f^{(2)}}{{f^{(1)}}^2} - \frac{105{f^{(2)}}^3}{{f^{(1)}}^3} + \frac{10g^{(3)}}{g} - \frac{5f^{(4)}}{f^{(1)}},\end{aligned} \tag{6.50c}$$

$$\begin{aligned}\mathcal{Z}_5^4 =& {f^{(1)}}^4\mathcal{K}_5^4 + \left(\frac{2{f^{(1)}}^3 g^{(1)}}{g} - {f^{(1)}}^2 f^{(2)}\right)\mathcal{K}_5^3 + \Bigg(3{f^{(2)}}^2 + \frac{3{f^{(1)}}^2 g^{(2)}}{g} \\ & - \frac{6f^{(2)}f^{(1)}g^{(1)}}{g} - f^{(1)}f^{(3)}\Bigg)\mathcal{K}_5^2 + \Bigg(- f^{(4)} + \frac{10f^{(2)}f^{(3)}}{f^{(1)}} - \frac{15{f^{(2)}}^3}{{f^{(1)}}^2} \\ & + \frac{4f^{(1)}g^{(3)}}{g} - \frac{8f^{(3)}g^{(1)}}{g} - \frac{18f^{(2)}g^{(2)}}{g} + \frac{30{f^{(2)}}^2 g^{(1)}}{f^{(1)}g}\Bigg)\mathcal{K}_5^1 + \frac{5g^{(4)}}{g} \\ & - \frac{f^{(5)}}{f^{(1)}} + \frac{15f^{(2)}f^{(4)}}{{f^{(1)}}^2} - \frac{10f^{(4)}g^{(1)}}{f^{(1)}g} + \frac{10{f^{(3)}}^2}{{f^{(1)}}^2} - \frac{105f^{(3)}{f^{(2)}}^2}{{f^{(1)}}^3} + \\ & \frac{105{f^{(2)}}^4}{{f^{(1)}}^4} - \frac{210{f^{(2)}}^3 g^{(1)}}{{f^{(1)}}^3 g} - \frac{40f^{(2)}g^{(3)}}{f^{(1)}g} - \frac{30f^{(3)}g^{(2)}}{f^{(1)}g} + \frac{120f^{(3)}f^{(2)}g^{(1)}}{{f^{(1)}}^2 g} \\ & + \frac{135{f^{(2)}}^2 g^{(2)}}{{f^{(1)}}^2 g},\end{aligned} \tag{6.50d}$$

$$\begin{aligned}\mathcal{Z}_5^5 =& {f^{(1)}}^5\mathcal{K}_5^5 + \frac{{f^{(1)}}^4 g^{(1)}}{g}\mathcal{K}_5^4 + \left(\frac{{f^{(1)}}^3 g^{(2)}}{g} - \frac{{f^{(1)}}^2 f^{(2)}g^{(1)}}{g}\right)\mathcal{K}_5^3 + \\ & \left(\frac{{f^{(1)}}^2 g^{(3)}}{g} - \frac{f^{(1)}f^{(3)}g^{(1)}}{g} - \frac{3f^{(1)}f^{(2)}g^{(2)}}{g} + \frac{3{f^{(2)}}^2 g^{(1)}}{g}\right)\mathcal{K}_5^2 + \\ & \Bigg(\frac{f^{(1)}g^{(4)}}{g} - \frac{f^{(4)}g^{(1)}}{g} - \frac{6f^{(1)}f^{(2)}g^{(3)}}{g} - \frac{4f^{(3)}g^{(2)}}{g} + \frac{10f^{(2)}f^{(3)}g^{(1)}}{f^{(1)}g} \\ & + \frac{15f^{(2)}g^{(2)}}{f^{(1)}g} - \frac{15{f^{(2)}}^3 g^{(1)}}{{f^{(1)}}^2 g}\Bigg)\mathcal{K}_5^1 + \frac{g^{(5)}}{g} - \frac{f^{(5)}g^{(1)}}{f^{(1)}g} - \frac{10f^{(2)}g^{(4)}}{f^{(1)}g} \\ & - \frac{5f^{(4)}g^{(2)}}{f^{(1)}g} - \frac{105{f^{(2)}}^3 g^{(2)}}{{f^{(1)}}^3 g} + \frac{105{f^{(2)}}^4 g^{(1)}}{{f^{(1)}}^4 g} - \frac{10{f^{(3)}}g^{(3)}}{f^{(1)}g} + \\ & \frac{15f^{(2)}f^{(3)}g^{(1)}}{{f^{(1)}}^2 g}\frac{10{f^{(3)}}^2 g^{(1)}}{{f^{(1)}}^2 g} + \frac{45{f^{(2)}}^2 g^{(3)}}{{f^{(1)}}^2 g} + \frac{60f^{(2)}f^{(3)}g^{(2)}}{{f^{(1)}}^2} \\ & - \frac{105{f^{(2)}}^2 f^{(3)}}{{f^{(1)}}^3 g}.\end{aligned} \tag{6.50e}$$

As said earlier, it has to be noted that for equations in normal form Eq. (2.64), expressing A_n^i for $2 \leq i \leq n$ in terms of A_2^2 and its derivatives is always possible for $2 \leq i \leq n$. Therefore, it suffices to make use of the coefficients A_2^2 and B_2^2. Such relationship between parameters of two given equivalent equations in their standard form is not known. However, we have found the relationship for equations of order $n = 2, 3, 4$ and $n = 5$.

For the sake of clarification, let us consider the equation of second-order in its standard form and use the above results.

Example 1. For $n = 2$, the transformation Eq. (6.35) implies that $g = \lambda\sqrt{f'}$. Therefore, expression Eq. (6.42) becomes

$$\mathcal{Z}_2^1 = f^{(1)}\mathcal{K}_2^1 \tag{6.51}$$

$$\mathcal{Z}_2^2 = {f^{(1)}}^2\mathcal{K}_2^2 + \frac{1}{2}f^{(2)}\mathcal{K}_2^1 - \frac{1}{{f^{(1)}}^2}\left(\frac{3}{4}{f^{(2)}}^2 - \frac{1}{2}f^{(1)}f^{(3)}\right). \tag{6.52}$$

By letting $\mathcal{K}_2^1 = 0$ (taking into consideration the notations $A_n^j = \mathcal{K}_n^j|_{\mathcal{K}_n^1=0}$ and $B_n^j = \mathcal{Z}_n^j|_{\mathcal{Z}_n^1=0}$) we get $\mathcal{Z}_2^1 = 0$, as expected, and

$$B_2^2 = \frac{1}{{f^{(1)}}^2}\left(A_2^2{f^{(1)}}^4 - \frac{3}{4}{f^{(2)}}^2 + \frac{1}{2}f^{(1)}f^{(3)}\right). \tag{6.53}$$

Recall that

$$B_2^2 = \frac{{R^{(1)}}^2 - 2RR^{(2)}}{4R^2}, \tag{6.54}$$

$$A_2^2 = \frac{{r^{(1)}}^2 - 2rr^{(1)}}{4r^2}. \tag{6.55}$$

Then, from Eq. (6.53) we get the relation

$$\frac{R'^2 - 2RR''}{R^2} = \frac{1}{f'^2}\left[\frac{r'(f)^2 - 2r(f)r''(f)}{r(f)^2}f'^4 - 3f''^2 + 2f'f'''\right] \tag{6.56}$$

obtained in [52] using the reduced form.

6.4 Concluding Remarks

Fazal and Leach partially characterized iterative equations by expressing the A_n^k, $3 \leq k \leq 8$, in terms of A_n^2. We have proved that these equations can be further expressed in terms of A_2^2. Therefore, we made a contribution which consists of prolonging the characterization by expressing the A_n^k's as functions of A_2^2 for $3 \leq k \leq 10$. For instance, using our result (6.15), we obtain a list of normal forms of iterative equations of order

starting from 3 to 10 in which coefficients are constant:

$$y^{(2)} + A_2^2 y = 0,$$

$$y^{(3)} + 4A_2^2 y^{(1)} = 0,$$

$$y^{(4)} + 10A_2^2 y^{(2)} + 9\left(A_2^2\right)^2 y = 0,$$

$$y^{(5)} + 20A_2^2 y^{(3)} + 64\left(A_2^2\right)^2 y^{(1)} = 0,$$

$$y^{(6)} + 35A_2^2 y^{(4)} + 259\left(A_2^2\right)^2 y^{(2)} + 225\left(A_2^2\right)^3 y = 0,$$

$$y^{(7)} + 56A_2^2 y^{(5)} + 784\left(A_2^2\right)^2 y^{(3)} + 2304\left(A_2^2\right)^3 y^{(1)} = 0,$$

$$y^{(8)} + +84A_2^2 y^{(6)} + 1974\left(A_2^2\right)^2 y^{(4)} + 12916\left(A_2^2\right)^3 y^{(2)} + 11025\left(A_2^2\right)^4 y = 0,$$

$$y^{(9)} + 120A_2^2 y^{(7)} + 4368\left(A_2^2\right)^2 y^{(5)} + 52480\left(A_2^2\right)^3 y^{(3)} + 147456\left(A_2^2\right)^4 y^{(1)} = 0,$$

$$y^{(10)} + 165A_2^2 y^{(8)} + 8778\left(A_2^2\right)^2 y^{(6)} + 172810\left(A_2^2\right)^3 y^{(4)} + 1057221\left(A_2^2\right)^4 y^{(2)} + 893025\left(A_2^2\right)^5 y = 0.$$

This extension further allowed us to prove the validity of formula (6.16), i.e.,

$$\Omega_n(y_k) = \left(\prod_{j=0}^{n-1}(k-j)\right) \cdot u^{2n-(1+k)} v^{k-n} \left[\left(\frac{v}{u}\right)'\right]^n = 0,$$

$0 \leq k \leq n-1$, for n up to 10. Besides, it is interesting to note that this relation is another way of characterizing iterative equations by their coefficients without using the symmetry approach.

Chapter 7

Conclusion

In this thesis, a special emphasis was placed on difference equations and a symmetry approach for solving these difference equations was considered. We have utilized the symmetry analysis to investigate difference equations and their infinitesimal properties in much the same way as differential equations.

In chapter three, we studied two non-autonomous partial difference equations, viz, the discrete Liouville and the discrete Sine-Gordon equations. Using first principles, we got in each case two non-trivial and one trivial conservation laws. These conservation laws were used to construct characteristics of these equations. One of the roles of characteristics is to identify equivalent conservation laws. They can also be constructed via homotopy [45]. It is important to mention that a compatible discretization of continuous characteristics can preserve a number of conservation laws. Lastly, in this chapter we found some symmetries and we showed that some of these conservation laws and symmetries are associated.

In chapters four and five, we turned our attention to ordinary difference equations. In chapter four, we calculated the symmetry generators of some second-order ordinary difference equations. We then constructed their first integrals using the method developed by Hydon. We showed in some cases that the reduction (using symmetry) of the equation and the first integral are the same, and the implication of this is that the first integral is invariant under the corresponding symmetry. This association between first integrals and symmetries has initiated a second reduction of the reduced equations, we call it double reduction. A similar work has been done in chapter five with the discrete versions of the painleve equations but here knowledge of the symmetries was sufficient

for the obtention of exact solutions. We observed that the theory of double reduction works perfectly for ordinary difference equations but more work has to be done to extend the theory to partial difference equations.

In the sixth chapter, we expressed the Lie generator in terms of the parameters r and q. We showed that for $n = 1, \ldots, 10$, the linear operator that corresponds to the iterative equation satisfies Eq. (6.16). All A_n^j, $2 \leq j \leq 10$, expressed in terms of A_2^2 and a list of linear iterative equations in normal form of order up to 10 were given. Parameters of the transformed linear iterative equations (in standard form) of order $n = 2, 3, 4$ and 5 were also obtained.

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