3.4. **Regular Economies.** *Regular economies* were introduced into economic theory by Debreu [3]. They capture the desired properties of existence and stability of equilibrium prices. (There is a an intimate relationship between the theory of regular economies and methods of computing economic equilibria. We are however unable to examine this relationship more here. See for example Smale [27], Smale [31] and Eaves and Scarf [9].)

The following analysis depends on the *marginal rates of substitution* for each agent. Such marginal rates of substitution are defined as

$$g_i(x_i) = \lambda_i D u_i(x_i)$$

where λ_i is chosen so as to normalise $Du_i(x_i)$.

Let

$$\psi_u : \mathcal{S} \to (S)^{m+1} : (x, p) \mapsto (g_1(x_1), \ldots, g_m(x_m), p). \tag{3.2}$$

(S the unit spere in $\mathbb{R}^l$). By the hypothesis (C'), this expression is not zero. Note that ψ_u is C^1 and, in particular, differentiable at points on the boundary of $\mathcal{S}$.

To obtain a large set of economies which have a finite number of extended price equilibria, all of them "stable" with respect to changes of endowments and utilities, a set of utility functions which must first be excluded.

Define the diagonal of $(S)^{m+1}$ to be

$$\Delta = \left\{ (y_1, \ldots, y_{m+1}) \in (S)^{m+1} \middle| y_1 = \ldots = y_{m+1} \right\}.$$

Then (x, p) satisfies (C') if and only if $\psi_u(x) \in \Delta$. $\psi_u^{-1}(\Delta)$ is a neat submanifold of $\mathcal{S}$ (see theorem 2.15).

We say ψ_u is *transversal* to Δ if, for each s with $\psi_u(s) = y \in \Delta$, the image of $D\psi_u(s)$ and $T_y(\Delta)$ as subspaces of $T_y(\mathcal{S})$ span $T_y(\mathcal{S})$.

$\psi_u \pitchfork \Delta$ is a condition on the second order derivative of the utility function and hence a restriction on the curvature of the indifference surfaces. The transversality requirement can therefore be considered as a generalisation of the requirement that the preferences of the agents give rise to C^1 demand functions. As we shall see in the following proposition, the notion of transversality is not very restrictive.

Proposition 3.2. Let $Y = \{u \in (U)^m | \psi_u \pitchfork \Delta\}$. Then, Y is open and dense in $(U)^m$ and contains the set of m-tuples of utilities which define C^1 demand functions.

Proof. Openness and denseness are a direct consequence of the Thom Transversality Theorem 2.23 (set $C^1(\bar{P}, \mathbb{R})^m$ equal to $C^1(\bar{P}, \mathbb{R}^m)$ and work with $J^1(\bar{P}, \mathbb{R}^m)$).

To verify the last part of the proof, we say that a utility function $u : \bar{P} \to \mathbb{R}$ *defines a demand function* $f : \bar{S} \times \mathbb{R}^+ \to \bar{P}$ if there is a well defined function

$$f(p, w) = \text{maximum } x \text{ of } u \text{ on the set } \{y \in \bar{P} | p(y) = w\}.$$

In particular, the map $\phi : \bar{P} \to \bar{S} \times \mathbb{R}^+$ defined by

$$\phi(x) = \{\lambda Du(x), \lambda Du(x)(x)\}$$

must be locally invertible if u defines a C^1 demand function, that is, the derivative $D\phi(x)$ of ϕ at each point must be a linear isomorphism.

Under these conditions the composition

$$\mathbb{R}^l \xrightarrow{D\phi} T_{(p,w)}(\bar{S} \times \mathbb{R}^+) \xrightarrow{\text{projection}} T_p(\bar{S})$$

is surjective.

If this is true for each u_i, it follows that ψ_u is a regular map (surjective derivative everywhere). This implies ψ_u is transversal to Δ.

Therefore, if u_i defines a C^1 demand function, then ψ_u is transversal to Δ. □

Remark 3.3. Suppose $Du(x) \in P$ for all $x \in \bar{P}$. Then a necessary and sufficient condition on u in order that it defines a C^1 demand function at $x \in \bar{P}$ is that the second derivative $D^2u(x)$ restricted to $\ker(Du(X))$ is negative definite (as a symmetric bilinear form).

Proposition 3.4. $\dim(\psi_u^{-1}(\Delta)) = m + l - 1$.

Proof. Theorem 2.15 (and its proof) says that if $\psi_u \pitchfork \Delta$ then $\psi_u^{-1}(\Delta)$ is a submanifold of $\mathcal{S}$ with codimension $\psi_u^{-1}(\Delta) =$ codimension Δ. □

Let $\Gamma = \{(r, x, p) \in (\bar{P})^m \times \mathcal{S} | (x, p) \in \Sigma_r\}$. Then Γ consists of all the initial allocations and their possible attainable states.

Proposition 3.5. The following conditions hold :

1. Γ is a submanifold of $(\bar{P})^m \times \mathcal{S}$ of dimension $2ml - m$; and
2. the projection $\pi_{\mathcal{S}} : (\bar{P})^m \times \mathcal{S} \to \mathcal{S}$ restricted to Γ is a regular map (that is, at every point of Γ, the derivative is surjective).

Proof. Consider the map $\phi : (\bar{P})^m \times \mathcal{S} \to \mathbb{R}^l \times \mathbb{R}^{m-1}$ defined by

$$\phi(r, x, p) = \left(\sum x_i - \sum r_i; p(x_1) - p(r_1); \dots ; p(x_{m-1}) - p(r_{m-1})\right).$$

We claim that ϕ is a regular map. Its derivative at (r, x, p) is given by

$$D\phi(r, x, p)(\bar{r}, \bar{x}, \bar{p}) = \Big(\sum \bar{x}_i - \sum \bar{r}_i; \bar{p}(x_1 - r_1) + p(\bar{x}_1 - \bar{r}_1); \dots ;$$
$$\bar{p}(x_{m-1} - r_{m-1}) + p(\bar{x}_{m-1} - \bar{r}_{m-1})\Big).$$

This has the following meaning : $(\bar{r}, \bar{x}, \bar{p})$ is a tangent vector of $(\bar{P})^m \times \mathcal{S}$ at (r, x, p). Thus $\bar{r} = (\bar{r}_1, \dots, \bar{r}_m)$ can be interpreted as an element of $(\mathbb{R}^l)^m$, $\bar{r}_i \in \mathbb{R}^l$, similarly, $\bar{x}$ is an element of $(\mathbb{R}^l)^m$ and $\bar{p}$ is a vector in $\mathbb{R}^l$ perpendicular to p.

To show that ϕ is surjective, it is sufficient to solve the following system of equations for a given $a \in \mathbb{R}^l$ and real numbers $b_1, \dots, b_{m-1}$

$$\sum x_i - \sum r_i = a$$
$$\bar{p}(x_i - r_i) + p(\bar{x}_i - \bar{r}_i) = b_i \quad i = 1, \dots, m-1.$$

For this, set $\bar{p}$ and $\bar{x}_i = 0$ for all i. Then let $\bar{r}_i$ satisfy $p(\bar{r}_i) = b_i$ for $i = 1, \ldots, m-1$ (which we can do since $p \neq 0$) and

$$\bar{r}_m = \sum_{i=1}^{m-1} \bar{r}_i.$$

Thus ϕ is regular and $\Gamma = \phi^{-1}(0)$ is a submanifold of the required dimension.

To prove the second part of the proposition, we note that the tangent space $T_{(r,x,p)}(\Gamma) \subset T_{(r,x,p)}((\bar{P})^m \times \mathcal{S})$ may be described as

$$\Big\{(\bar{r}, \bar{x}, \bar{p}) \in (\mathbb{R}^l)^m \times (\mathbb{R}^l)^m \times S_p \Big| \sum x_i = \sum r_i;$$
$$\bar{p}(x_i - r_i) + p(\bar{x}_i - \bar{r}_i) = 0 \text{ for } i = 1, \ldots, m-1\Big\}$$

where $S_p = \{\bar{p} \in \mathbb{R}^l \mid \bar{p} \perp p\}$. To see $\pi_{\mathcal{S}}|\Gamma : \Gamma \to \mathcal{S}$ is regular, it suffices to show that given (r, x, p), $\bar{x}$ and $\bar{p}$, the following can be solved for $\bar{r}$

$$\sum \bar{x}_i = \sum \bar{r}_i$$
$$\bar{p}(x_i - r_i) + p(\bar{x}_i - \bar{r}_i) = 0 \quad i = 1, \ldots, m-1.$$

This can be done by choosing $\bar{r}_i, i = 1, \ldots, m-1$ such that $p(\bar{r}_i) = p(\bar{x}_i) - \bar{p}(x_i - r_i)$ (which can be done since $p \neq 0$) and then letting

$$\bar{r}_m = \sum_{i=1}^{m} \bar{x}_i - \sum_{i=1}^{m-1} \bar{r}_i$$

and the proposition is proved. □

Corollary 3.6. Let $u \in Y$. Then $Z_u = \{(r, x, p) \in \Gamma \mid \psi_u(x, p) \in \Delta\}$ is a submanifold of Γ of dimension ml.

Proof. By proposition 3.2, ψ_u is transversal to Δ. The composition $\Gamma \hookrightarrow (\bar{P})^m \times \mathcal{S} \xrightarrow{\pi_{\mathcal{S}}} \mathcal{S}$ is regular by 3.5. It follws that the composition

$$\Gamma \hookrightarrow (\bar{P})^m \times \mathcal{S} \xrightarrow{\pi_{\mathcal{S}}} \mathcal{S} \xrightarrow{\psi_u} (S)^{m+1}$$

is transversal to Δ. Now by proposition 3.4 we are done. □

Definition 3.7. An economy (r, u) will be called *regular* provided

1. $u \in Y$
2. π_Q restricted to Z_u has r as a regular value where $\pi_Q : (\bar{P})^m \times \mathcal{S} \to (\bar{P})^m$ is the projection.

3.5. **Existence of Equilibria.** To prove the existance of economic equilibria we make two assumption which will not be required for the stability section hereafter.

The *boundary condition* is a condition on the pair $(r, u) \in \bar{P} \times U$

BC: For each $x \in \partial\bar{P}$, $Du(x)(r - x) > 0$

The boundary condition has the effect of making the indifference surfaces become parallel to the boundary as they near infinity.

Next, say that $u \in U$ satisfies the *monotinicity hypothesis* if

MH: $Du(x) \in \bar{P}$ for all $x \in \bar{P}$.

In what follows, this could be weakened to

MH': There is an open half plane H in $\mathbb{R}^l$ and $Du(x) \in H \quad \forall x \in \bar{P}$.

Let $\mathcal{F} = \{(r,u) \in (\bar{P})^m \times (U)^m \mid$ for each i, u_i satisfies (MH) and (r_i, u_i) satisfies (BC) $\}$. Here, $r = (r_1, \ldots, r_m)$ and $u = (u_1, \ldots, u_m)$. No assumption of convexity on the utility functions is needed. If, however, one assumes enough convexity such that the utility functions define C^1 demand functions, we show that the extended price equilibria are the classical ones. Thus let $\mathcal{F}_c \subset \mathcal{F}$ be the set of all economies which satisfy the additional *convexity hypothesis*

CH: For each $x \in \bar{P}$, the restriction $D^2u_i(X)\mid_{ker Du(x)}$ is negative definite.

That is, u_i locally defines a C^1 demand function (see remark 3.3).

Stated formally, the main existence theorem is

Theorem A (Smale [25]). The following conditions hold :

1. An economy (r,u) in $\mathcal{F}$ has an extended price equilibrium.
2. An economy in $\mathcal{F}_c$ has a classical price equilibrium (and in fact, every extended price equilibrium is classical and conversely).
3. If in addition $(r,u) \in \mathcal{F}$ is regular, then there is an odd number of price equilibria (x,p)

The extended price equilibria of the theorem have the property that all of the x_i are in the interior P of $\bar{P}$. (See Dierker [7] for a related article relating to demand funcitons.)

We begin with a preliminary lemma. In this section on existence we work in the $C^2_w(\bar{P}, \mathbb{R})$ topology[10] . Since $\mathbb{R}^n$ is a metric space, the weak topology coincides with the compact open topology (see §2.2.1).

Proposition 3.8. The following conditions hold :

1. $\mathcal{F}$ is arcwise connected.
2. If $(r,u) \in \mathcal{F}$ and (x,p) is an extended price equilibrium for the economy $(r,u), x = (x_i, \ldots x_m)$ then there is no $x_i \in \partial\bar{P}$.
3. There is a regular economy $(\bar{r}, \bar{u}) \in \mathcal{F}$ with precisely one extended equilibrium.

Proof. Let $(r,u), (r',u') \in \mathcal{F}$. We construct an arc in $\mathcal{F}$ connecting (r,u) to (r,u') and another connecting (r,u') to (r',u'). Let $r = (r_1, \ldots, r_m), u = (u_1, \ldots, u_m)$ etc. For each $0 \le t \le 1$ and for each $i = 1, \ldots, m$ let $u_{i,t} = tu_i + (1-t)u_i'$. Then $u_{i,t}$ is in $C^2(\bar{P}, \mathbb{R})$ for each t. It is also continuous in t [11] . For each $x \in \bar{P}$,

$$Du_{i,t}(x) = tDu_i(x) + (1-t)Du_i'(x),$$

thus $Du_{i,t}(x) \neq 0$ using the monotinicity hypothesis (MH). So $u_i \in U$. At the same time we see that $u_{i,t}$ itself satisfies the monotinicity hypothesis. We can check directly

[10] By linearity this is defined in terms of neighbourhoods of zero. Let $K_q = \{x \in P \mid \|x\| \le q\}, q = 1, 2, \ldots$. A neighbourhood $N_{\epsilon,q}$ of 0 is defined by some $\epsilon > 0$ and some $q = 1, 2, \ldots$ as follows. $g \in N_{\epsilon,q}$ if, for all $x \in K_q$, the C^2 distance to $g(x)$ is less than ϵ.

that the boundary condition (BC) is true for $(r_i, u_{i,t})$. Thus we have constructed an arc (r, u_t) in $\mathcal{F}$ from (r, u) to (r, u')

Define $r_{i,t} = tr_i + (1-t)r'_i$. Again (BC) is satisfied for each $(r_{i,t}, u'_i)$ and we have found an arc between (r, u') and (r', u'). Thus $\mathcal{F}$ is arcwise connected.

Consider the second part of the theorem. If $(x, p) \in E_{ex}(r, u)$, then for each $i = 1, \ldots, m$, $p(r_i - x_i) = 0$ and hence $Du_i(x_i)(r_i - x_i) = 0$. Therefore, by (BC), $x_i \notin \partial \bar{P}$. This proves the second part.

Finally, define $u^* : \bar{P} \to \mathbb{R}$ by

$$u^*(x) = \prod_{i=1}^{l} (x^i + 1)$$

and $r^* \in \bar{P}$ by $r^* = (l, \ldots, l)$. Then we claim (r^*, u^*) satisfies (BC) and u^* satisfies (MH). Differentiating gives

$$Du^*(x) = \left(\prod_{j \neq 1} (x^j + 1); \ldots; \prod_{j \neq l} (x^j + 1) \right).$$

Clearly, $Du^*(x) \in \bar{P}$ for each $x \in \bar{P}$ (in fact, $Du^*(x) \in P$) and so (MH) is satisfied. To check (BC), observe

$$\frac{Du^*(x)(r^* - x)}{u^*(x)} = \frac{l - x^1}{x^1 + 1} + \cdots + \frac{l - x^l}{x^l + 1};$$

thus it is sufficient to show

$$\sum_{i=1}^{l} \frac{l}{x^i + 1} > \sum_{i=1}^{l} \frac{x^i}{x^i + 1},$$

provided $x \in \bar{P}$ or at least one of the x^i is zero. Suppose $x^k = 0$. Then

$$\sum_{i=1}^{l} \frac{l}{x^i + 1} > l > \sum_{i=1}^{l} \frac{x^i}{x^i + 1}.$$

Now choose $(\bar{r}, \bar{u}) \in \mathcal{E}$ with $\bar{r} = (r_1, \ldots, r_m) = (r^*, \ldots, r^*)$ and $\bar{u} = (u_1, \ldots, u_m) = (u^*, \ldots, u^*)$. Then it is easy to see that the economy $(\bar{r}, \bar{u})$ has a unique price equilibrium (x, p) with $x = r$, $p = (1/\sqrt{l}, \ldots, 1/\sqrt{l})$ and is regular. □

Proof of Theorem A. We prove the first part of the theorem by a degree argument. For each $(r, u) \in \mathcal{F}$, define $\beta_{(r,u)} : \mathcal{S} \to (S)^{m+1} \times \mathbb{R}^l \times \mathbb{R}^{m-1}$ by

$$\beta_{(r,u)}(x, p) = (\psi_n(x, p); \sum x_i - \sum r_i; p(x_1) - p(r_1); \ldots; \\ p(x_{m-1}) - p(r_{m-1})).$$

Where ψ_u is defined in equation (3.2).

Then, $(x, p) \in E_{ex}(r, u)$ if and only if $\beta_{(r,u)}(x, p) \cap \Delta \times 0 \times 0 \neq \emptyset$. Furthermore, if (r, u) is regular, then $\beta_{(r,u)}$ is transversal to $\Delta \times 0 \times 0$.

[11] This would not be true if we were working in $C^2_s(\bar{P}, \mathbb{R})$.

From the fact that $(r, u) \in \mathcal{F}$, $\beta_{(r,u)}(\partial\mathcal{S}) \cap \Delta \times 0 \times 0 = \emptyset$, and the degree of $\beta_{(r,u)}$ is defined. If (r, u) is regular, the degree is the number of points in $\beta^{-1}_{(r,u)}(\Delta \times 0 \times 0)$, each counted with the appropriate sign. By the third part of the proposition, the degree of $\beta_{(\bar{r},\bar{u})}$ is one: Now the degree is a homotopy invariant, so that the degree of each $\beta_{(r,u)}$ is one if $(r, u) \in \mathcal{F}$. So the first part of the theorem is true (if $E_{ex}(r, u)$ is empty, then (r, u) is a regular economy !).

The third part of the theorem follows similarly.

The second statement is a consequence of the fact that under the hypothesis $(r, u) \in \mathcal{F}_c$, every extended price equilibria is calssical (u_i must be a maximum at x_i on the budget set by the second deritive condition). This proves the theorem. □

3.6. **Finiteness and Stability of the Extended Price Equilibria.** For the remainder of this section we will not require the the boundary condition (BC) nor the monotinicity hypothesis (MH). We shall also work with the strong topology on $C^2(\bar{P}, \mathbb{R})$.

Proposition 3.9. For a regular economy (r, u), the set of extended price equilibria $E_{ex}(r, u)$ is a finite set which moves continuously in (r, u).

$E_{ex}(r, u)$ is called *stable* if it moves continuously in (r, u). We mean here that if (r_0, u_0) is a regular economy, then the extended price equilibria may be labelled $\sigma_i(r_0, u_0), i = 1, \ldots, k$. Furthermore, there is a neighbourhood N of (r_0, u_0) in $(\bar{P})^m \times (U)^m$ and continuous functions $\sigma_i : N \to \mathcal{S}, i = 1, \ldots, k$ such that the $\sigma_i(r, u)$ label precisely the elements of $E_{ex}(r, u)$. Also, the elements of N are regular economies.

Proof. The manifold Z_u has dimension ml, the same as dimension $(\bar{P})^m$. Since r is a regular value of $\pi_Q \mid Z_u$, the inverse image of this map, by theorem 2.14 is a submanifold of dimension zero, that is a discrete set. By the definition, this set is $E_{ex}(r, u)$. Since this set is bounded, it is also finite. Also, by Sard's theorem (2.17), the set of critical values of $\pi_Q \mid Z_u$ is closed nowhere dense and of measure zero. Finally, the implicit function theorem gives the stability property. □

Thus, the subset of regular economies $\mathcal{O}$ is dense in the space $\mathcal{E}$ of all economies. Thus we have shown :

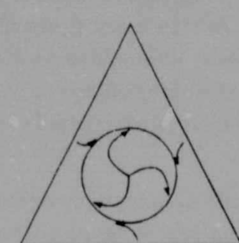
Theorem B (Smale [26]). For an open dense subset $\mathcal{O}$ of economies in $\mathcal{E}$, the set of price equilibria is finite and this set is continuous in the parameters (r, u) of the economies in $\mathcal{O}$.

Remark 3.10. If (r, u) is an economy and $(x, p) \in E_{ex}(r, u)$, then by definition, x_i is a critical point of u_i on the plane $\{y \in \bar{P} | p(y) = p(r_i)\}$. If (r, u) is a regular economy then x_i is a non-degenerate critical point of u_i on this set. (The converse is false, see Smale [25]).

4. Scarf's Example

We know that a smooth gradient field on compact manifold must have at least one locally stable equilibrium. More generally, let v be a differentiable vector field whose vectors at each point form an acute angle with the vectors of a gradient vector field w and whose singularities coincide with those of w. Then v need not be a gradient field with respect to any Riemannian metric but still has a locally stable equilibrium. Also v admits no kind of recurrence because "potential energy" decreases over all points except equilibria so, in particular, v had no closed orbits. Such a vector field v would be ideal to model economic equilibria because, for Adam Smith's invisible hand theory to hold, at least one of the price equilibria must be a local attractor.

In this section, however, we outline the construction of an excess demand function f whose behaviour will be far from that of a gradient field[12]. In particular, f will have a unique repelling equilibrium and an attracting closed orbit. f will also stable under perturbations and therefore cannot be dismissed as atypical. This example is due to Scarf [23]. Graphically, it looks as follows :



Scarf's Example

As a consequence of the properties of the utility functions which we shall construct, the vector field f that we discuss will satisfy the following conditions :

- **W:** $p \cdot f(p) = 0$ for every $p \in S$.
- **B:** There is a $k \in \mathbb{R}$ such that for every $p \in S$ each component of $f(p)$ is bounded below by k.
- **D:** If the sequence $(p_j) \in S$ converges to $p_0 \in \partial S$, then $\|f(p_j)\| \to \infty$ (that is, the sum of the components tends to infinity).

Condition (W) is *Walras' law* while (D) is a *desirability* condition. Together (D) and (B) imply that f "points inward" near ∂S. Consider the following proposition :

Proposition 4.1. Assume that f is C^1 and satisfies conditions (W), (B) and (D). If the evolution of the economy is described by the differential equation[13]

$$\dot{p} = \frac{dp}{dt} = f(p), \tag{4.1}$$

then no point $p \in S$ approaches ∂S .

---

[12] Not all forces in economics are conservative !

[13] Solutions of equation (4.1) can be though of as price adjustments where a positive excess demand for some good raises the price of that good.

The results of §5 will be far more general than those presented here and therefore we shall not spend an undue amount of time on the proofs of this sections results. A full proof can be found in Scarf [23].

Proof. The proof consists of the following three observations :

1. no integral curve converges to a point on ∂S;
2. it is impossible to linger around the boundary for a long time; and
3. there is a compact neighbourhood in S around the centre
$$\left(\frac{1}{\sqrt{l}}, \dots, \frac{1}{\sqrt{l}}\right)$$
of S which no integral curve upon entering can leave.

□

For an economy with only two commodities, an excess demand functions f which satisfies (D) must be a gradient field. Therefore Scarf's example will not work in that case. The argument will however make extensive use of the fact that $\dim S = 2$ and will therefore only in the three commodity case (Indeed, Dierker [7] shows that for an odd dimension of S a unique equilibrium cannot be completely unstable. This is due to the fact that the Euler characteristic vanishes on compact manifolds of odd dimension).

The only restriction on the number of agents in the economy we are about to construct is that it be greater that or equal to the number of commodities. For convenience we shall assume there are three agents involved. Agent 1 holds an initial endowment $(r_1^1, r_1^2, r_1^3) \in P$ close to (0,0,1) and has utility function

$$u_1(x^1, x^2, x^3) = -\left(\frac{(\alpha^1)^{a+1}}{(x^1)^a} + \frac{(\alpha^2)^{a+1}}{(x^2)^a} + \frac{(\alpha^3)^{a+1}}{(x^3)^a}\right),$$

where $a > 1$, $b > \frac{a+1}{a-1}$ and $\alpha^i > 0$, $i = 1, 2, 3$ is such that $(\alpha^1, \alpha^2, \alpha^3)$ is close to $(b, 1, 0)$.

This utility function is strictly monotone and defined on the open consumption set P and generates a demand function.

The characteristics from agents 2 and 3 are obtained from those of agent 1 by a cyclic permutation of the indices of the three commodities. By symmetry, $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is a fixed point of f. It is a repelling fixed point as can be seen in the following way. Compute the demand of agent 1 for the case that

$$(\alpha^1, \alpha^2, \alpha^3) = (b, 1, 0) \text{ and } (r_1^1, r_1^2, r_1^3) = (1, 0, 0). \tag{4.2}$$

Use symmetry to obtain the excess demand function and evaluate its Jacobian at $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. The eigenvalues of this Jacobian matrix at the equilibrium consist of a pair of complex conjugates with positive real part. This means that each point near the equilibrium moves along a spiral.

Under assumption 4.2 direct computation yields that

$$\frac{\partial f_2}{\partial p_1} > 0 \text{ on } S. \tag{4.3}$$

Assume that there is another equilibrium $(y^1, y^2, y^3) \in S$. Then, by symmetry, (y^3, y^1, y^2) and (y^2, y^3, y^1) are also equilibria and we can assume, without loss of generality, that $y^1 \geq y^2 \geq y^3$ and $y^1 \neq y^3$. Now 4.3 implies that f_2 increases if commodity 1 becomes more expensive and we get $f_2(y^1, y^2, y^3) \neq 0$ which gives a contradiction. Thus $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ is the only equilibrium for this economy.

Any sufficiently small perturbation of f in the C^1 topology will not alter the nature of the existing equilibrium nor will it introduce a new equilibrium. From the Poincaré-Bendixon theorem we know that a vector field with a unique completely unstable equilibrium has a closed orbit. The Poincaré-Bendixon theorem is however restricted to surfaces and hence we have made essential use of the fact that we have exactly three commodities

This example shows us that mathematical relations which apply to equilibria of a gradient field cannot readily be used in economic theory. One might, however, object to the phenomena in Scarf's example on the grounds that such a dynamic would not occur under a more "realistic" adjustment process. This stance then compels us to examine properties that a vector field on S must posses to be interpreted as an excess demand function. We investigate this question in the following section.

5. The SMD theorem

Clearly, an agent's demand function, being derived from a utility function, cannot be completely arbitrary, for instance it must reflect the transitivity of the preferences. Since we are interested in an aggregate or market excess demand function, f, rather than an individual excess demand we might conjecture that the process of aggregating individual excess demand functions will erase any restrictions on the market excess demand function. Sonnenschein [34] was the first person to make an attack on this problem. In this section we highlight some of the results directed towards resolving this conjecture.

Let Ξ denote the set of continuous tangent vector fields on S and let U be the set of continuous[14], strictly convex utility functions. Let

$$Z : (P)^m \times (U)^m \to \Xi$$

be the map defined by the construction of the excess demand function.

Sonnenschein question is then one about the image of Z in Ξ. We shall proceed by working on $S_\epsilon = \{x \in S | x_i > \epsilon \text{ for all } i\}$, denoting the set of continuous excess demand functions on S_ϵ by Ξ_ϵ.

The SMD theorem is named after the three people who contributed to its statement and proof, which, in our notation is presented in theorem C. Sonnenschein [33] proved that on domains S_ϵ, a dense set of excess demand functions are generated by allowable economies. Mantel [16] proved that any C^1 excess demand function for l commodities is generated by an economy of $2l$ agents. Debreu [5] showed that only l agents and continuous excess demand functions are necessary. We have therefore :

Theorem C (SMD). For $l \geq 2$ and $\epsilon > 0$, the price mapping

$$Z_\epsilon : (P)^m \times (U)^m \to \Xi_\epsilon$$

is surjective if and only if $m \geq l$.

As one might guess, the proof of this theorem requires some technical yet elementary manipulation of excess demand functions. For this reason we only sketch the proof, postponing the details until we have an acceptable generalisation of this result (see Theorem D). Even then, however, the proof will be left as an appendix result.

Proof. f is decomposed into l (consumer) demand functions f_i in the following way : Given $p \in \mathbb{R}^l \backslash \{0\}$, let $\hat{T}_p$ denote hyperplane through the origin perpendicular to p. Let ϵ_i denote the i-th unit vector of $\mathbb{R}^l$ and let $b_i(p)$ denote its projection into $\hat{T}(p)$. Since f is continuous, there exists a positive valued continuous function α such that each component of $f(p) + \alpha(p)p$ is positive for every $p \in S$. Thus

$$f(p) + \alpha(p)p = \sum \beta^i(p)e_i$$

where every β^i is *continuous and strictly positive*. Projecting onto $\hat{T}(p)$ we obtain

$$f(p) = \sum \beta^i(p)b_i(p).$$

[14]The smoothness assumption is not necessary here. The tangency of the level set and the budget plane suffice.

For every i we define

$$f_i(p) = \beta^i(p) b_i(p).$$

Since f satisfies Walras' law, so does each the f_i.

The technical part of the proof is the demonstration that the f_i are generated by utility functions with the required property. In Debreu's paper the f_i are constructed via preferences (not utility functions) under slightly more general conditions. Subsequently Geanakoplos [11] exhibited utility functions for these preferences under the conditions assumed here. □

The implications of this theorem are clear : differential equations of the form

$$\dot{p} = f(p), \tag{5.1}$$

where f is an excess demand function, can generate any behaviour on the "trimmed price simplex" S_ϵ no matter how complex or chaotic. Thus, equations of the form 5.1, which are the natural candidates for describing the behaviour of an economy whose equilibrium points are the zeros of f, cannot do so if we are to believe in Adam Smith's "Invisible Hand".

Remark 5.1. At this stage we note that we could also choose a vector field without a zero since the zero promised by the existence theorem may lie in the in the excised ϵ-region. We shall say more about it in §5.1.

Thus we see that the behaviour exhibited in Scarf's example cannot be blamed on the specification of the price adjustment rule and we have to expect the worst in dimensions high enough to provide space for various "pathological" features.

5.1. **Characterisation of the Equilibrium Price Set.** The work of Sonnenschein, Mantel and Debreu shows that there are no restrictions on market excess demand functions if no restrictions are placed on agent's preferences. In light of remark 5.1, a natural question to ask is : *is there an excess demand function f of an economy (r, u) on some S_ϵ such that $S_\epsilon \supset f^{-1}(0)$ and $f^{-1}(0)$ is also the set of equilibrium prices for (r, u)?* Mas-Colell [17] extended the previous theorems (under slightly different conditions) in order to provide the first general answer to this question.

Up to here we have defined an economy in terms of consumers (r_i, u_i). Mantel worked with preference functions $\succcurlyeq_i$ rather that the utility functions u_i but all our notions for utility carry over to preferences. In particular, we say an economy $(r_1, \ldots, r_m, \succcurlyeq_1, \ldots, \succcurlyeq_m)$ generates an (aggregate) excess demand function f if, for each agent $(r_i, \succcurlyeq_i)$, there is a well defined *demand function*

$$\mathcal{D}_i(p) = \{x | p(x) = p(r) \text{ and } x \succcurlyeq_i y \ \forall\, y \text{ such that } p(y) = p(r)\}.$$

f is defined, as always, by the sum of the individual demands less the sum of the initial resources.

Theorem D. Let f be an excess demand function on S which satisfies conditions (B) and (D) from the previous section. For any $\epsilon > 0$ there is a μ with $0 < \mu < \epsilon$ and an l consumer exchange economy $(r, \succcurlyeq) = (r_1, \ldots, r_l, \succcurlyeq_1, \ldots, \succcurlyeq_l)$ with each $\succcurlyeq_i$ continuous, strictly convex and monotone, such that

1. $(r, \succcurlyeq)$ generates f on S_μ; and
2. $f^{-1}(0) \subset S_\mu$ is the set of equilibrium prices of $(r, \succcurlyeq)$.

The proof of this result is technical and since it does not shed any extra light on the problem, it will be presented in appendix B.

An immediate corollary of this result, which restores the intent of the SMD theorem (see remark 5.1) is the following result concerning the structure of the equilibrium prices of an economy :

Corollary 5.2. Let A be a compact non-empty subset of S. Then there exists an economy $(r, \succcurlyeq)$ whose agents have continuous, strictly convex and monotone preferences such that A is the equilibrium price set of $(r, \succcurlyeq)$.

Proof. Let $\rho(p, A)$ denote the distance of p from A and let e be the identity in $\mathbb{R}^l$. Then $\rho(p, A) = 0$ if and only if $p \in A$ and $\rho(\cdot, A)$ is continuous. Pick any $\bar{q} \in A$ and define f by

$$f(p) = \rho(p, A)\left((\bar{q}_1/p_1, \bar{q}_2/p_2, \ldots, \bar{q}_l/p_l) - (\bar{q} \cdot e/p \cdot e)e\right).$$

Then it is easy to check that f is an excess demand function satisfying the conditions of the previous theorem and that $f(p) = 0$ if and only if $p \in A$. □

This is clearly a negative result; however, if f is C^1 then more can be said about $f^{-1}(0)$. Prior to this work on market excess demand, Debreu [3] had shown that "most" C^1 market excess demand functions satisfying the condition of theorem D have the property that

$$\det Df(p) \neq 0, \text{ at every } p \in f^{-1}(0).$$

In particular, the inverse function theorem implies that $f^{-1}(0)$ is finite if the above property holds. Indeed, Dierker [6] has shown that for such demand functions

$$\sum_{p \in f^{-1}(0)} \operatorname{sgn} \det Df(p) = (-1)^{l+1}$$

and thus that $f^{-1}(0)$ has an odd number of points. With these considerations in mind, Mas-Colell [17] showed the following :

Theorem 5.3. Suppose $l \geq 3$ and let $A \subset S$ be a finite nonempty set and $i : A \to \{-1, 1\}$ which satisfies $\sum_{p \in A} i(p) = (-1)^{l+1}$. Then there exists an economy $(r, \succcurlyeq)$ whose agents have continuous, monotone and strictly convex preferences such that $(r, \succcurlyeq)$ generates an excess demand function f satisfying

1. $f^{-1}(0) = A$
2. $\sum_{p \in f^{-1}(0)} \operatorname{sgn} \det Df(p) = i(p)$ for each $p \in A$.

We see therefore that the economic assumptions, made on an individual level, hardly restrict the market excess demand. This seriously erodes our faith in any Invisible Hand but since this theory plays such a central role in economics, we wish to be absolutely certain that it is flawed beyond repair before we proceed to recast the foundations of economics.

One way to save Adam Smith's theory would be to say that the equation 5.1 is much to simple to describe the evolution of a "real economy". In this case we tnen must try to construct a market mechanism which would be required to converge to an equilibria from all starting points. We take up this challenge in the following section.

6. The Global Newton Method

The observations of the previous section lead us naturally to consider the following question : *"Are there economic processes which converge to an equilibrium from any initial point ?"* Since economists are interested in computing equilibria, one major goal of general equilibrium theory would be to describe an economic process which is stable and constructive, in the sense that it offers a way of computing these equilibria. We describe here one such process, the *Global Newton Method* (GNM), so named because locally it essentially Newton's method for finding the zeros of a system of non-linear equations and it works globally. Despite the fact that this method is purely mathematical (there was and is no evidence to support the fact that "real economies" behave in this fashion), it gives support to the fact that economies could comply with the Invisible Hand Theory.

As in previous sections, let $\bar{P} = \{(x^1, \ldots, x^l) \in \mathbb{R}^l | x^i \geq 0, \text{ each } i\}$. Interpret $x \in \bar{P}$ as a commodity bundle and $p \in \bar{P}\backslash\{0\}$ as the set of prices of the l goods. The results will be stated and proved in terms of an excess demand function f. Again, conditions which are reasonable from an economic standpoint (and which *can* be derived from microeconomic principals - though this will not be done here) are imposed on f. The following hypotheses on f are needed :

C: $f : \bar{P}\backslash\{0\} \to \mathbb{R}^l$ is continuous;
H: $f(\lambda p) = f(p)$ for $\lambda > 0$;
W: $p \cdot f(p) = 0$;
D: $f(p^i) > 0$ if $p^i = 0$.

Note the new desirability condition[15] (D). Interpreted it as implying that if the i-th good is free there will be a non-negative excess demand for that good.

On the price space S the differential equation

$$\dot{p} = \frac{dp}{dt} = f(p) \tag{6.1}$$

is well defined and the equilibria of this differential equation coincide with the price equilibria for the excess demand function. As we have seen in the previous section, solutions to such equations do not necessarily lead to equilibria.

In this section, a modification of the differential equation 6.1 which depends only on the excess demand function and whose solutions converge to an equilibria under quite general conditions is presented. In sections to come, the economic implications of this process will be examined to see to what extent it resolves certain problems in general equilibrium theory.

A constructive proof using fixed point arguments has been developed by Scarf, the method presented here was developed by Smale [27]. Unlike Scarf's method, it's spirit is in keeping with Walras' idea of studying the existence of equilibria in terms of solving equations. In other words, we know an excess demand function f is a morphism whose source space consists of price systems and whose target space

[15] An alternative proof of the existence of a price equilibrium in these conditions would be to note that condition (D) implies that the vector field f points in on the boundary so that, by the Hopf theorem [18], there is a price system p^* such that $f(p^*) = 0$.

consists of commodity bundles. Thus rather than artificially construct and solve endomorphisms, which would be necessary in the fixed point approach, solutions to $f(p) = 0$ are constructed directly using differentiable calculus.

Theorem E. If the excess demand function satisfies conditions (C), (H), (W) and (D), then there is a price system $p^* \in \bar{S}$ such that $f(p^*) = 0$.

The proof of this theorem is given constructively as the solution to an ordinary differential equation. It is based on the following two theorems, the first of which is a purely mathematical problem of solving a system of n non-linear equations in n variables. The *Global Newton* is the ordinary differential equation and is given by

$$Df(x)\frac{dx}{dt} = -\lambda f(x), \tag{6.2}$$

where $Df(x) = (\partial f_i / \partial x_j)(x)$ and λ is a real number unspecified for the moment. Let D^l denote the disk of radius 1 in $\mathbb{R}^l$.

Theorem 6.1. Let $f : D^l \to \bar{P}$ be a continuous map satisfying the boundary condition

B$_D$: If $x \in \partial D^l$, then $f(x)$ is not of the form μx for $\mu > 0$.

Then there is a $x^* \in D^l$ with $f(x^*) = 0$.

Proof. We prove the theorem initially under a more restrictive boundary condition. Suppose therefore that $f : D^l \to \mathbb{R}^l$ is a C^2 map which satisfies the very strong boundary condition :

SB: $f(x) = -x$ for all $x \in \partial D^l$.

Define the map $g : D^l \backslash E \to S$ by $g(x) = f(x)/\|f(x)\|$ where $E = \{x \in D^l | f(x) = 0\}$ is the solution set. Since g is C^2, Sard's theorem implies that the regular values of g has full measure in S. Let y be such a value. Then by the inverse function theorem, $g^{-1}(y)$ is a 1 dimensional submanifold which must contain $-y$ by the boundary condition (SB). Let V be the component of $g^{-1}(y)$ starting at $-y$. So V is a non-singular arc starting from $-y$ and open at the opposite end. V cannot meet ∂D^l at any point other than $-y$ and because it is non-singular at that point, it intersects ∂D^l once only. Since V is closed in $D^l \backslash E$, its limit points lie in E. In particular E is nonempty and following along V starting from $-y$ one must eventually converge to E.

To further explicate the constructive nature of this solution, it can be shown that V is the solution curve of the Global Newton[16] ordinary differential equation

$$Df(x)\frac{dx}{dt} = -\lambda f(x)$$

where $\lambda = \pm 1$ is chosen according to the sign of the determinant of $Df(x)$ and changes with x. To see this, parametrise V by $x(t)$. By the definition of g, $x(t)$ must satisfy the identity

$$f(x(t)) \equiv y \|f(x(t))\|.$$