GRADE 9 LEARNERS' STRATEGIES AND ERRORS IN SOLVING ARITHMETIC AND ALGEBRAIC LINEAR EQUATIONS

By

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Abstract

This research project reveals strategies used and errors made by grade 9 learners in solving arithmetic and algebraic linear equations in a township school in the West Rand District. A class of forty-five learners were given a written test on arithmetic and algebraic linear equations. Based on the test, five learners were selected for further interviews, and were audio-recorded. Both the test and interviews were analysed and the findings are reported in this report. One of the findings is that learners treat arithmetic and algebraic linear equations differently. This pointed to the second finding that different strategies were employed in solving arithmetic and algebraic equations, and that errors made are variably different, too, between arithmetic and algebraic linear equations. One of the strategies, which were also an error, in solving arithmetic equations is right to left reasoning. In algebraic linear equations the balance method and transposition were the most prevalent strategies, and the most common errors included conjoining (which was not observed in arithmetic equations). Besides the findings above, learners further displayed difficulties with the simplification of algebraic expressions. This on its own hampered their progress into solving the algebraic equations. On considering strategies and errors in solving arithmetic and algebraic linear equations (in this study) it became hard to easily distinguish between them when errors are so prevalent. Recommendations pertaining to solving arithmetic and algebraic linear equations were made.

Declaration

I declare that this research report is my own unaided work. It is being submitted for the degree of Master of Science by coursework at the University of Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

Signed: 

Lethu Gumpo

26 March, 2014.

Dedication

This research report is dedicated to the Gumpo Institution; whose members are the two directors (I and my lovely wife, Deliwe) and the shareholders (our dear children: Nomcebo, Khayalethu, Cebolenkosi, Inhle, Nombuso, Kealeboga and Thandolwenkosi). As a director, I sacrificed so much of the company's time while working on this report and would like to express my sincere thanks to my co-director and shareholders for their patience, tolerance and prayers. No one complained throughout the challenging times that I/we went through. Thank you, lovely family.

Acknowledgements

There are a number of people who assisted me in carrying out this research whom I will not mention by name and state in what ways they contributed to the success of this research report. However, no one whom I interacted with during the course of this study should feel left out; you are all are appreciated for your valuable inputs and contributions.

The time taken to come up with this research report was rather too packed and tense due to many duties that demanded my attention, at the same time. However, the tension was loosened by the following individuals and groups of people who kept my spirit of perseverance and determination revived.

- I am rather running short of words to express my heartfelt thanks, excitement and sincere appreciation to my supervisor, Dr. Craig Pournara. Craig guided me, encouraged me, tolerated me, was patient with me and above all he was diligent in his conduct. Craig did not mince his words when he said that I was waffling, pressure accumulated but eventually, his criticism was nothing but constructive. Yes I feel greatly indebted to him for his diligence and hard-work in guiding me throughout the study.
- A special thanks and commendation to my dear learners who participated in this study. Their participation, co-operation and contribution played a significant role without which this research report would not have succeeded.
- The school and community where the research was conducted contributed significantly too, directly or indirectly.
- My wife, Deliwe, was very inspirational both physically and spiritually. And our kids, also, played their own role because a thought of them would motivate me to work even harder.
- Lastly, knowledge that with God nothing is impossible and recognition of His Power thereof kept me healthy. I neither gave up nor fell sick despite all the pressure that I felt. That is love.

Special thanks to all human contributions and experiences, and sustenance from the Mighty God.

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List of Abbreviations

B, balance method

BEQ, breaking the equation into two parts

C, correct answer

CJN, conjoining

CP, correct answer pairs

DN, double negative

ECE, equations changed into algebraic expressions

INK, interference of new knowledge

IOS, ignoring operation sign

IRB, insertion and removal of brackets

O, other (e.g. *strategy or error*)

PCS, proving correctness of solution

RLR, right to left reasoning

S, slip

T, transposition

TSBN, take sign of a bigger number

TI, trial and improvement

UAQ, unanswered question

1. *Chapter 1*

INTRODUCTION

1.1 Introduction

This research report is on strategies used by grade 9 learners in solving both arithmetic and algebraic equations; and errors that they make in the process of solving these equations. Strategies in solving equations involve a lot of processes that learners undergo in order to come up with solutions to equations. There are many strategies used in solving equations and many errors committed whilst solving the equations. In the process of solving equations individuals come up with wrong solutions due to various reasons; like, simply making a mistake or forgetting what to do as well as inappropriate use of existing knowledge.

1.2 Problem Statement

Understanding and solving first-degree algebraic equations is an important component of school algebra in grades 8 and 9. However, a large number of learners misunderstand linear equations and consistently make procedural and conceptual mistakes, (Araya, Calfucura, Jiménez, Aguirre, Palavicino, Lacourly, Soto-Andrade and Dartnell, 2010). From my experience as a teacher, most grade 9 learners struggle with algebraic equations. The most difficult part in most cases is carrying out procedures. For example, adding or subtracting the same number on both sides of the equation; multiplying or dividing by the same number on both sides and in some cases removing brackets as well as simplifying some terms. This may also be caused to an extent by the fact that an individual learner has his/her own way of doing his/her work regardless of the way the learner may have been taught by the teacher in class. From my experience as a teacher, the problem may lie with both teacher and learner. For example the teacher may assume that what he/she has communicated to the learner has been understood while the learner may have made sense of the work in a different way to what the teacher intended.

A quote by Chile Department of Education (2008) as cited in Araya et al. (2010) states:

> Too many students in high school algebra classes are woefully unprepared for learning even the basics of algebra. The types of errors these students make when attempting to solve algebraic equations reveal that they do not have a firm understanding of many basic principles of arithmetic (e.g., commutativity, distributivity), and many do not even understand the concept of equality. Many students have difficulty grasping the syntax or structure of algebraic expressions and do not understand procedures for transforming equations (e.g., adding or subtracting the same value from both sides of the equation) or why transformations are done the way they are (p. 216).

This is true, for even, in this study such errors as have been mentioned in the quote have been encountered. In the South African context, the poor results of grade 9 learners in 2012 in the

"Annual National Assessment" (ANA), (Department of Basic Education (DBE), 2012a), is an indicator of the need to determine the strategies that learners use as they solve equations. It is worth noting that there were, in the ANA examination, eighteen problems to do with algebraic expressions, algebraic linear equations and other equations (fractional, exponential, and quadratic) most of which culminated in solving linear equations (e.g. questions 1, 2 and 5). For example, in part ***2.5.4*** once 32 has been raised to the power 5 with base 2, then the exponents $x + 1$ and 5 are equated; thus the exponential equation has culminated in a linear equation. The importance of learners' strategies in solving algebraic linear equations cannot be underestimated. So, this research report reports on the findings based on these claims with a main focus on strategies that grade 9 learners use in solving linear equations, as well as errors that emanate therefrom.

1.3 Rationale

The focus of my study is on investigating learners' mathematical strategies and errors when dealing with simple arithmetic and linear equations of the form: $a + \square = b$; $c = \square - \square$; $\boldsymbol{x} + d = e$; $f\boldsymbol{y} + g = h\boldsymbol{y} + i$, where $\square$ and letters $\boldsymbol{x}$ and $\boldsymbol{y}$ represent numbers to be found to make the equations true and all other letters stand for constants. For example, $3 + \square = 7$; $8 = \square - \square$; $\boldsymbol{x} + 9 = 10$; $5\boldsymbol{y} + 4 = 2\boldsymbol{y} + 6$. So the boxes and letters $\boldsymbol{x}$ and $\boldsymbol{y}$ are variables whereas the rest are parameters. Equations with boxes are arithmetic whereas those with letters $\boldsymbol{x}$ and $\boldsymbol{y}$ are algebraic, (see section 2.4). I included arithmetic equations to investigate whether the grade 9 learners would show similar kinds of strategies and interpretations on both arithmetic and algebraic linear equations, since learners encountered arithmetic equations in primary school. In primary school, before introducing algebraic symbols and terms, boxes are used for missing numbers. In this study arithmetic equations are considered as introductory, so to speak. From my experience as a mathematics teacher I am aware that algebra plays an important role in the school mathematics, and further. For example, calculus requires high levels of algebraic skills.

1.4 Curriculum focus

On reviewing the curriculum document (Senior Phase: Department of Basic Education (DBE), 2012b), grade 9 learners are expected to deal with algebraic expressions such as: $\boldsymbol{x + y + z}$ and $\boldsymbol{x + x + x}$. After the knowledge of algebraic expressions learners are required to solve simple equations by: inspection, trial and improvement, as well as determine the numerical value of an expression by substitution (DBE, 2012b). Additive and multiplicative inverses are also included in the solution of equations. According to the curriculum document, algebraic manipulation work is expected to be covered from grade 7 through to grade 9.

1.5 Critical Questions

The following are the research questions that will enable me to gain deeper insight into learners' strategies in solving these equations as well as noting errors they make (in the process of solving the equations). The two questions that guide the study are:

1. What strategies do grade 9 learners use in solving arithmetic and algebraic linear equations?
2. What errors do grade 9 learners make when they solve arithmetic and algebraic linear equations?

When solving an equation the main aim is to 'isolate' the unknown variable in the equation (which can be any letter of the alphabet), and be equated to a number or simplified expression. The process involved in isolating this variable involves a lot of processes and the knowledge of what to do when and how. If a learner, successfully and correctly, isolates the variable then s/he can be said to have strategically solved the equation. So I define a strategy as a method or plan chosen and executed to bring about a desired goal or outcome. Contrary to this I define an error as a failure to strategically bring about a desired outcome using a particular strategy. In short an error is a deviation from accuracy or correctness brought about either by a careless mistake like forgetfulness, a slip or a misconception, (see section 2.7). I would like to draw the reader's attention to errors as being influenced by previous knowledge. Errors must not be taken for simple mistakes or slips, but are "the symptoms of the underlying conceptual structures that are the cause of errors", (Olivier, 1989 p. 3). More literature relating to these strategies and errors will be considered and discussed in chapter two.

1.6 Summary

In this chapter I have introduced the research study, focusing on the problem statement, rationale, curriculum focus and critical research questions. Below is a brief overview of the rest of work that is contained in this research report:

(1) Chapter 2: literature review and an outline of the theoretical and conceptual frameworks;

(2) Chapter 3: focused on the research design and methodology;

(3) Chapter 4: focused mainly on the data analysis of arithmetic equations;

(4) Chapter 5: focused mainly on the data analysis of algebraic linear equations; and

(5) Chapter 6: focused on the big findings of the research, the insights and possible recommendations emanating from the study

2. ***Chapter 2***

THEORETICAL AND CONCEPTUAL FRAMEWORKS and THE LITERATURE REVIEW

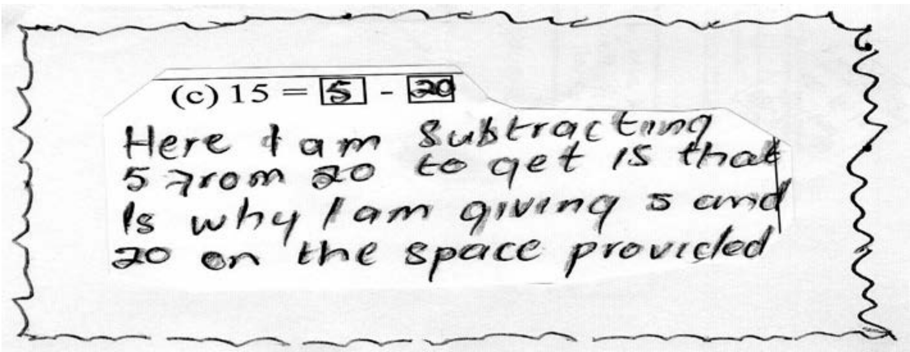
2.1 Introduction

In this chapter the theoretical framework underpinning this study, and the conceptual framework used in the analysis of data are outlined. Further, I discuss the literature relating to arithmetic and algebra linear equations; strategies used in solving these equations; typical errors involved when solving these equations and some possible misconceptions leading to errors.

2.2 Theoretical framework

I draw on the constructivist perspective on learning where Olivier (1989) cites Piaget (1970) and Skemp (1979), that a constructivist perspective on learning assumes that concepts are not taken directly from experience, but that a person's ability to learn and what he learns from an experience depend on the quality of the ideas that he is able to bring to that experience. So in the process of solving linear equations, despite what learners know and have been taught, they will still bring their own ideas depending on the sense they make of the equations. Olivier, 1989, says that sometimes when learners provide wrong solutions, they may be answering a different question. For example, in this study, one learner answered the question $15 = \square - \square$ as shown in figure 2.1 below.

Figure 2.1: Learner's response to **$15 = \square - \square$**



The explanation the learner provided confirms that there is right to left reasoning towards the number 15. The orders in which numbers are written do matter, for example 5 – 20 and 20 – 5 will result in -15 and 15 respectively. Essien and Setati (2006) have alluded to the fact that order has a bearing in the way learners treat their work in relation to the positioning of the equal sign. The example they gave was that of learners refusing to accept that the mathematical sentence of the form $\square = 3 + 5$. To the learners the mathematical sentence was only correct if it was written in the form $5 + 3 = \square$ where the movement is now from left to right, unlike the former where the movement is from right to left. The idea behind this issue of equality will be discussed fully later when dealing with typical errors in solving equations. In a similar manner one may say that the learner changed the order of 15 = $\square$ - $\square$ into $\square$ - $\square$ =15 resulting in 20 – 5 = 15.

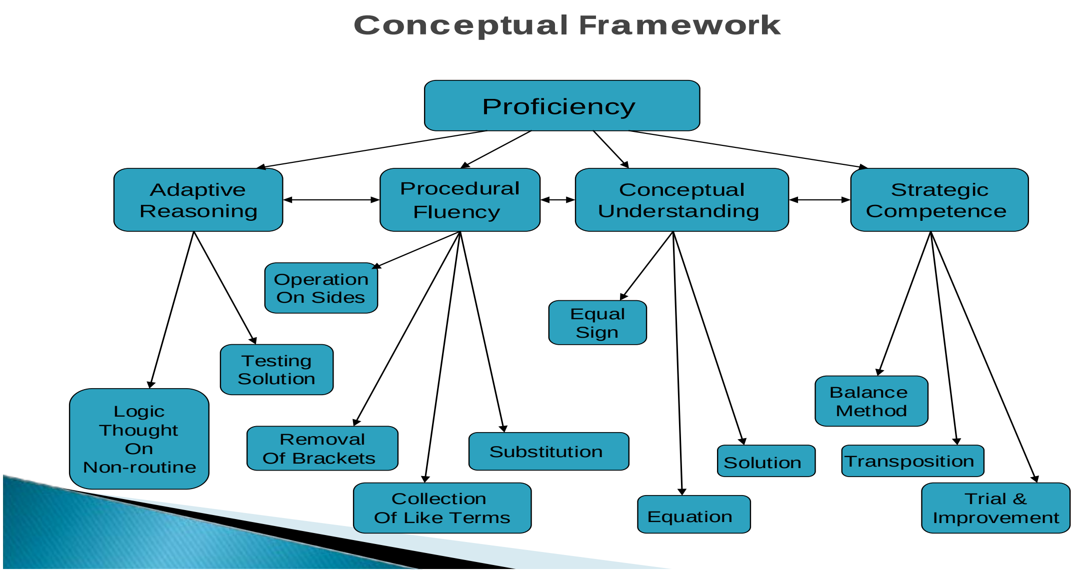
Referring back to the constructivist's perspective, it is pointed out that although instruction clearly affects what learners learn, it does not determine it, because the learner is an active participant in the construction of his/her own knowledge. Keeping this theory in mind, all that learners have written and said has been taken into consideration when analysing the data, because sometimes one may be tempted to focus on what is expected forgetting that learners are making sense of equations based on the experiences they bring.

2.3 Conceptual Framework

I use mathematical proficiency as my conceptual framework (Kilpatrick, Swafford & Findell, 2001). According to Kilpatrick, et al. (2001) mathematical proficiency consists of five intertwined strands namely: conceptual understanding; procedural fluency; strategic competency; adaptive reasoning and productive disposition. However, I have purposely left out productive disposition in my study, for I felt that it can be of less use in my data analysis. Productive disposition is defined by Kilpatrick, et al (2001) as "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy", (p. 116). Hence I left it because it diverged from my focus which was on strategies and errors. Figure 2.2 below is a diagrammatic representation of my conceptual framework. A learner's proficiency is determined by the extent to which the learner demonstrates efficiency in the four strands that are in focus in the study. The strands are not hierarchical, but they inform each other, and the arrows between the strands indicate that they are all intertwined. The more an individual displays competence in several strands, the more proficient s/he is said to be. Conceptual understanding occurs when the learner has an understanding of the meaning of basic concepts, symbols, and key words. These include the structure of an equation, i.e. the meaning of the equal sign, and what is meant by a solution. Procedural fluency has to do with all the processes that a learner performs when solving an equation. These include grouping/collection of like terms on either side of the equation, substitution, removal of brackets and skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. Strategic competence means that a learner has a specific strategy that he/she uses in solving the equation, like balance method, transposition and others. A point to be noted is that one may consider these strategies as procedures since one may be following some rules since as mentioned previously, these strands are intertwined.

Learners may come up with other ways of solving the equations not covered by literature, and these strategies will still be considered. Lastly, adaptive reasoning has to do with what a learner does when faced with a situation to solve an equation as well as when having solved the equation, e.g. does the solution have any significance. I can call it a stage of decision-making before, during and after solving the equation.

Figure 2.2: Conceptual framework



Using the conceptual framework in figure 2.2, if a learner was to solve an equation: $\mathbf{3}\boldsymbol{x} + \mathbf{5} = \boldsymbol{x} + \mathbf{9}$, and writes '$\mathbf{3}\boldsymbol{x} - x + \mathbf{5} = \boldsymbol{x} - x + \mathbf{9}$', according to me the 'conceptual understanding', 'strategic competency' and 'procedural fluency' strands are all included. Before embarking on any strategy there has to be some conceptual understanding, i.e. if one is not guessing. Based on this understanding follows the appropriate strategy to be used which is accompanied by some appropriate procedures. For example, let us consider the problem $3\boldsymbol{x} + 4 = 19$. The first step may be to subtract 4 from both sides giving rise to $3\boldsymbol{x} = 15$. Secondly, knowledge of multiplicative inverse may lead to one dividing both sides by 3 leading to the final expression $\boldsymbol{x} = 5$, which happen to be the final step and solution to the problem. So this would be a typical example that indicates that the learner understands the meaning of the equal sign in the context of an equation (conceptual understanding), and selects the appropriate methods (strategic competency) and uses the correct procedures (procedural fluency) and all this accomplished because of careful thinking and knowing that $3 \times 5 + 4 = 15 + 4 = 19$ (adaptive reasoning).

2.4 Literature review

Kieran (1981) worked with 12- to 14-year-olds who were first asked the meaning of the equal sign followed by a request of an example showing the use of the equal sign, and it is said that most of them described the equal sign in terms of the answer and limited themselves to examples involving an operation on the left side and the result on the right; like $3 + 5 = 8$ or $3 \times 4 = 12$. Later, the learners were introduced to the use of the equal sign to include multiple operations on both sides through constructing arithmetic equalities, initially with one operation on each side, e.g. $2 \times 6 = 4 \times 3$ (the same operation), and then $2 \times 6 = 10 + 2$ (different operations) until constructing equalities with two operations on each side ($7 \times 2 - 3 = 5 \times 2 + 1$), and then to multiple operations on each side, e.g., $7 \times 2 + 3 - 2 = 5 \times 2 - 1 + 6$. Kieran (1981) then gave "the name *arithmetic identities* to all of the above in order to reserve the term *equation* for use in the algebraic sense" and also "the reason for extending the notion of the equal sign to include multiple operations on both sides was to provide a foundation for the later construction of meaning for non-trivial algebraic equations (which have multiple operations on both sides)" (p. 321). After hiding 2 in the expression: $7 \times 2 - 3 = 5 \times 2 + 1$ first with a finger, then with a box ($7 \times \square - 3 = 5 \times 2 + 1$) and finally, with a letter ($7 \times \boldsymbol{a} - 3 = 5 \times 2 + 1$), Kieran then defined an equation as an arithmetic identity with a hidden number.

In this study I specifically define equations of the form: $\square + 5 = 21$ arithmetic and $\boldsymbol{x} + 5 = 21$ algebraic, where the difference is simply the use of either a box or a letter. In algebra, an equation is defined as a proposition asserting the equality of two quantities, and expressed by the sign '=' between them, (Webster, 1828). Algebraic equations emanate from algebraic expressions, like $2\boldsymbol{x} + 4 - \boldsymbol{x}$ or $4\boldsymbol{y} - 6$ which use letters of the alphabet, hence $4\boldsymbol{y} - 6 = 4$ is an algebraic linear equation. By contrast, expressions like $3 - 5 + 4$ or $13 + 3$ are simple arithmetic expressions or number sentences, but $13 + \square = 14$ is an arithmetic equation. Behr, Erlwanger and Nichols (1980) referred to expressions like $\square = 2 + 3$ and $3 + 2 = 2 + 3$ as sentences about number relationships indicating the sameness of two sets of objects. In particular: $\square + 5 = 21$ as an arithmetic equation requires the learner to find the value that goes into the box that when added to 5 it will give the value 21. In the same manner $\boldsymbol{x} + 5 = 21$ as an algebraic equation, is a problem that requires the learner to find the value of $\boldsymbol{x}$, that when added to 5 will give the value 21. In both cases the value found is called the solution of the equation.

2.5 Shift from arithmetic to algebra

Herscovics and Linchevski (1994), cited in Carraher, Schliemann, Brizuela & Earnest (2006), proposed the existence of a cognitive gap between arithmetic and algebra, and maintained that it can be characterized as the learners' inability to operate spontaneously with (or on) the unknown. They said that although they recognized that young children routinely solve problems containing unknowns, like 5 +? = 8, they argued that students solve such problems without having to represent and operate on the unknowns. Instead, they maintain that the students simply use counting procedures or the inverse operation to produce a result. Again they queried the idea that students are often introduced to algebra through first-order equations of the form: $a\boldsymbol{x} + b = c\boldsymbol{x} + d$ or $a\boldsymbol{x} + b = d$, because this introduces far too many new issues at once. This introduction also encourages students to view variables as having a single value. I believe that the converse is true if the algebraic expressions have been introduced preceding equations. For example, given an expression like $\boldsymbol{x} +2$ and asked to find its value, the learner can understand that many values can be found depending on the value of $\boldsymbol{x}$, e.g. if $\boldsymbol{x} = 2$; the value is 4, if $\boldsymbol{x} = 10$; the value is 12, if $\boldsymbol{x} = -7$; the value is -5, and so on.

According to Filloy and Sutherland (1996) and Schmidt and Bednarz (1997) cited in Dooren, Verschaffel, and Onghena (2002), when students move from primary to secondary school, the acquisition of an algebraic way of mathematical reasoning and problem solving is one of the most important mathematical learning tasks. Many a time there is a notable gap between primary and secondary mathematics in as far as algebra is concerned.

On a different note Reed (1999) and Schmidt (1994) cited in Dooren et al. (2002) stated that students on entering secondary school, are introduced to an algebraic way of thinking and problem solving, using symbols representing unknown quantities to write equations. And operating on these equations leads to the identification of the unknowns and ultimately to the answer of the problem. In the earlier years before secondary school level, learners do interact with equations in the form of numbers in boxes. For example a question may be asked to find the number that goes into the box if $\mathbf{7} + \square = 15$. The shift from boxes to letters (which takes place at secondary level) needs to be paid attention to for learners to transcend the gap between arithmetic and algebra more smoothly.

Kuchemann (1981) asserts that it is very common for learners to provide numerical values to letters before manipulating them in any way. Kuchemann, further, maintains that the assigning of different meanings to letters used as variables by learners and the assigning of letters to numbers affects the problem difficulty. The reason for this may be attributed to the nature of mathematics that learners deal with in primary school, among which is the use of boxes for numbers that was mentioned previously. One of the most likely reasons for this may be lack of clarity on dealing with simplification of algebraic expressions in introductory algebra. There is a tendency for assigning real objects to letters used by teachers attempting to introduce reality in the learning so that learners may understand. For example in the simplification of algebraic expressions like: $c + c + c =$? The teacher might say 'one chicken plus one chicken plus one chicken gives three chickens (abbreviated, $3c$)'. This relates a letter with the number 1, such that whenever a variable is used with a coefficient of 1, the learner would recall the number 1 instead of considering the variable as is. In the case of solving the equation if the learner assigns a value to the unknown it may change the problem altogether. For example given an equation: $2\boldsymbol{x} = 2$ or $\boldsymbol{x} + 3 = -2$, one may say $\boldsymbol{x} = 2$ or $\boldsymbol{x} = 1$ respectively. The reasoning for the values provided being simply that $2\boldsymbol{x}$ means two '$\boldsymbol{x}$'s and $\boldsymbol{x}$ means one '$\boldsymbol{x}$', in other words the solution is basically given in terms of the coefficient of $\boldsymbol{x}$. Algebraic competence before equations is somehow being emphasized. In other words it is highly likely that poor performance in equations may link to a lack of understanding of the meaning of letters in algebraic expression. Dooren et al. (2002), claim that algebraic competence is essential, especially, considering that a part of the solution process in linear equations consists of manipulating letters for numbers. So, the understanding of the meanings of the use of letters to represent numbers is vital. For example, learners should understand that $3\boldsymbol{x}$ means 3 times $\boldsymbol{x}$. So if $\boldsymbol{x} = 3$; $3\boldsymbol{x}$ must result in 9 not 33.

There is a great need for a link between primary and secondary treatment of both algebra and equations. According to Schmidt and Bednarz (1997) cited in Dooren et al. (2002), learners in primary school have developed concepts, techniques, and habits that are regarded as arithmetical and as a result find difficulties in dealing with algebraic way of thinking that is encountered in secondary school. What the authors are emphasising is that primary school teachers should develop in students a rich knowledge base consisting of several fundamental mathematical concepts, including the symmetrical meaning of the equal sign (=). Furthermore, secondary school teachers should firstly have a good understanding of the mathematical histories of students entering secondary education. For example, they mention that the equal sign in arithmetic tasks at the primary school level often has the meaning of being merely a *results sign* whereas in algebraic equations the meaning of this sign is fully symmetric and transitive.

2.6 Strategies used in solving linear equations

The literature has identified a number of methods for solving algebraic linear equations which is one and the same thing as *strategies* I am dealing with in this study. These are, also, referred to as strategic competency in the framework. Kuchemann (1981) says that children frequently tackle mathematical problems with methods that have little or nothing to do with what they have been taught. This reasoning points out that in a study like this one, learners' strategies and their understanding have to be carefully considered and taken into account in data analysis. Besides the documented strategies that are mathematically correct, learners come up with their own, based on their prior knowledge or experience, which may be different from the known strategies. This research only focuses on simple linear equations where one or more of the strategies discussed below are to be expected in the solution of linear equations.

(a) ***The balance method (BM)***

The balance method uses the scale concept of adding or subtracting equal amounts on both sides of the equation so as to remain with balancing amounts on both sides. For example, considering an equation like: $2\boldsymbol{x} + 3 = 7$, subtracting 3 from both sides will reduce the equation to $2\boldsymbol{x} = 4$ and dividing by 2 on both sides will reduce the equation to $\boldsymbol{x} = 2$. The balance method as advocated by Kieran and Drijvers (2006) seems appropriate for the initial teaching of linear equations, in which the balance as a mental model of equivalence and the choice of appropriate solving strategy steps, are the main learning goals. However, the method is taken up further as a balance in which case the task is to simplify or reduce an equation to the form $\boldsymbol{x} = \ldots$, while maintaining the equilibrium of both sides of the equation. Vlassis, (2002), cited in Kieran & Drijvers, (2006), agrees that the use of the balance model can certainly be of help to students to understand the notion of equation.

Araya et al. (2010) advocate the use of analogies as a strategy in comparison to the traditional symbolic one, where analogies use models and the traditional methods use letters for numbers. They maintain that the use of analogies as a strategy in solving algebraic equation has an important impact on students. For example, a typical example of an analogy is when 'a two-pan balance is used for the equals sign, a box for a variable, candies for numbers, and guessing the number of candies inside a box' is used (Araya et al., 2010). The notion of a variable is represented by a box or container containing an unknown number of candies. For example, in $2\boldsymbol{x} + 3 = 7$ one may take $2\boldsymbol{x}$ as two boxes with some candies inside plus three free candies outside on one side of the pan weighing the same as seven candies on the other side of the pan.

(b) ***The cover up method (C)***

Kieran and Drijvers (2006) consider another strategy called the 'cover up strategy'. The main *conceptual aspect* is to *cover up* certain part of the equation when dealing with complex equations, e.g. like one involving fractions and square roots. I use the equation $2\boldsymbol{x} + 3 = 7$ for demonstration purposes. One will have to cover up $2\boldsymbol{x}$. Now the idea is to think of a single number that can be added to 3 so that the equation is made true. When the number has been thought to be 4, then uncovering $2\boldsymbol{x}$ one will automatically have to equate it to 4, i.e. $2\boldsymbol{x} = 4$. Covering $\boldsymbol{x}$ in $2\boldsymbol{x}$ leads to 2 times a number = 4 which that number happens to be 2, then $\boldsymbol{x} = 2$. The advantage of this method is that it may reduce some errors involved in the processes and procedures involved in solving equations.

(c) ***Trial and improvement method (TI)***

Trial and error is another method that is commonly used, and it is included as one of the strategies in solving equations in the curriculum document (DBE, 2012). As an example let us consider the equation $2\boldsymbol{x} + 3 = 13$. To find what $\boldsymbol{x}$ represents the learner is likely to try out different values by substituting in the equation until the left hand side is equal to the right the right hand side. When $\boldsymbol{x} = 4$: $2 \text{ x } 4 = 8$; and $8 + 3 = 11$, but $11 \neq 13$. Another value is tried until the one that when substituted makes both sides of the equation equal (i.e. the correct value). The idea is that the chosen numbers/values for trial are chosen in some kind of systematic way and substituted into the equation, hence the term trial and improvement. The method is authentic in its own right and sometimes may be referred to as an intelligent guess rather than simple guess work.

(d) ***Transposition (T)***

In algebra, transposition is defined as the bringing of any term of an equation to the other side, (Webster, 1828). For example, if given that $a + b = c$, and changes are made such that $a = c - b$, then b is said to be transposed; or in $\boldsymbol{x}^2 + 3\boldsymbol{x} = 10$ changes are made such that $\boldsymbol{x}^2 + 3\boldsymbol{x} - 10 = 0$, then 10 is said to be transposed. This is a method that we use frequently in solving equations both linear and non-linear as in the examples given. Transposition is much faster and less prone to many errors that are incurred by other methods.

2.7 Misconceptions, errors and slips in algebra

"The notion of a misconception denotes a line of thinking that causes a series of errors all resulting from an incorrect underlying premise, rather than sporadic, unconnected and non-systematic errors", (Nesher, 1987 p. 35). According to Olivier (1989) misconceptions are the results of the underlying beliefs and principles in the cognitive structure of the individual leading to systematic conceptual errors. Araya et al. (2010) alluded to the fact that whilst learners tried different strategies to solve algebraic equations the main difficulties were some of the key misconceptions that are observable when students solve equations. In other words errors arise because of concepts that have been used or applied inappropriately. Finally, Olivier (1989) defines errors as wrong answers that arise in one's work because of planning and are systematic in nature in that they are regularly applied under the same circumstances, and slips are wrong answers due to processing. It must be emphasised that although errors and slips are both wrong answers the difference is that slips may be made by both experts and novices. Further, slips are easily detected and corrected. Ginsburg (1977) cited in Olivier (1989), says that errors are objectively illogical and wrong whereas, from on the learner's perspective, they may make sense. For example in solving equations like $2\boldsymbol{x} + 3 = 7$ and $2\boldsymbol{x} - 3 = 7$; if a learner transposes the numbers on the left hand side in each case and provides the following results: $2\boldsymbol{x} = 7 + 3$ and $2\boldsymbol{x} = 7 - 3$ respectively, then the learner has made an error which is consistent. There is a notable misconception, in which, the underlying premise may be related to simplification of algebraic expressions where there is grouping of like terms. When collecting like term, operational signs before the terms do not change. To shed light I provide another example to clarify what happens in simplification of algebraic expressions. Consider $5 - \boldsymbol{x} + 3$ and $5 + \boldsymbol{x} - 3$ to be simplified. $5 - \boldsymbol{x} + 3$ simplifies to $5 + 3 - \boldsymbol{x}$ since 3 is added and $\boldsymbol{x}$ is subtracted, finally leading to $8 - \boldsymbol{x}$. The same applies to the second, $5 + \boldsymbol{x} - 3$ simplifies to $5 - 3 + \boldsymbol{x}$ since 3 is subtracted and $\boldsymbol{x}$ is added, finally leading to $2 + \boldsymbol{x}$. If this correct procedure in algebraic expression has been applied in equations as is, during transposition it is a misconception leading to erroneous solutions in as far as equations are concerned. In my study, the test items I have designed, will enable me to determine if a learner makes an error, what the nature of that error is, i.e. with proper choice of test items I am able to conclude whether the error is systematic or not. For example if the same error is repeated in similar problems then it is highly likely that it is systematic otherwise it is not.

2.8 Some errors that influence the solving of equations

This section focuses on errors that have serious impact on learners' performance as they solved equations. Listed below are the major errors that I will discuss in chapters 4 and 5 during data analysis.

- Use of the equal sign.
- Right-to-left reasoning.
- Conjoining.
- Operation signs ignored or not taken into cognisance, i.e. ignoring operation sign.
- Errors due to commutative and distributive properties.
- Interference of new knowledge.

It is important to note that some of the errors that have been mentioned above appear in both arithmetic and algebraic equations while others do not. However, the tendency is that those appearing in arithmetic are likely to appear in algebraic solutions as well but less likely that errors in algebraic tasks will appear in arithmetic tasks.

(a) ***Use of the equal sign***

The way the learners understand and use the equal sign (=) has a bearing in the result of their work. According to Stacey and MacGregor (1997) they call the results sign a make or give sign, which could be used to link parts of a calculation. Again, Stacey and MacGregor (1997), mention that students' interpretation of equations can be greatly influenced by prior expressions in arithmetic; and they provide an example of $3 + 5 = 8 \times 7 = 56 \div 2 = 28$. When considering the start and end of the expression $3 + 5 = 8 \times 7 = 56 \div 2 = 28$; we note that $3 + 5 \neq 28$. What is evident here is that the equal sign is used in a restricted sense, and when it comes to equations this use of it (=) is an obstacle to understanding equations. Essien and Setati (2006) in their study with the grades 8 and 9 learners (in one of the schools in Johannesburg) pointed out that the most dominant interpretation of the equal sign is as a do-something, unidirectional symbol. They pointed out that learners see the equal sign as a tool for writing the answer rather than as a relational symbol to compare quantities. Learners consider the symbol, =, as a 'do something signal' **(**Behr, Erlwanger and Nichols, 1980; Kieran 1981**).**

For example, in the calculation: $3 + 5 = 8$, the '=' has a *results* in, *makes* or *gives* meaning. There is a difference between the meanings of the equal signs in: $3 + 5 = 8$ and $2\boldsymbol{x} + 3 = 13$. The first expression is an equation resulting from simplifying the expression $3 + 5$, and the second is an equation that requires one to solve for $\boldsymbol{x}$. In regards to this the notion of and understanding equality or the meaning of an equal sign has a role to play in the solution of equations. As has already been mentioned earlier; the equal sign in arithmetic tasks at the primary school level often has the meaning of being merely a results sign whereas in algebraic equations the meaning of this sign is fully symmetric and transitive. The importance of proper interpretation of the equal sign cannot be overemphasised. Essien and Setati (2006) asserted that the equal sign is a tool (or *a relational symbol*) without which the learners' mathematical explanations or solutions are meaningless. According to my understanding of what Essien and Setati are putting across is that for one to say that the solution of the equation $3\boldsymbol{x} + 4 = 19$ is 5, then the implication is that the individual has a full comprehension of the meaning of the equal sign and used that knowledge efficiently to come up with the solution.

(b) ***Right-to-left reasoning (RLR)***

Right-to-left reasoning is a situation involving subtraction where a learner does the subtraction from right to left, rather than from left to right, as is the convention. For example, 2 – 7 = -5 but using right to left reasoning a learner will get a result of 5 simply by subtracting 2 from 7. According to Gallardo and Rojano (1990), Gallardo (2002), Gallardo and Hernandez (2005) and Vlassis (2008) this is an exhibition of some inhibitory mechanism as the learner is faced with a negative solution. When such situations arise, usually learners tend to look for a solution in the positive domain. In the example: 3 + 2 – 13 = 5 – 13 = 13 – 5 = 8, we see right to left reasoning. 5 – 13 is changed to 13 – 5 because the learner is not in a position to accept that a bigger number can be subtracted from a smaller number or simply because a difference between 5 and 13 is 8 because '–' is taken to mean difference.

(c) ***Conjoining (CJN)***

I have noticed in practice as a teacher that unfamiliar situations that learners come across are, to an extent, the cause of changes in their strategies which eventually lead to some errors. For example, one learner simplified $3\boldsymbol{x} + 5$ to $8\boldsymbol{x}$ and $\boldsymbol{x}^2 + 4 + 2$ to $\boldsymbol{x}^2 6$, see figure 5.4, and the expressions 8x and $\boldsymbol{x}^2 6$ are the results of conjoining. Other examples of conjoining are in observed in the following expressions $\boldsymbol{x} + 3 = 4\boldsymbol{x}$ and this one where the negative has been presumably ignored $2\boldsymbol{x} + \boldsymbol{y} – 3\boldsymbol{w} = 6\boldsymbol{xyw}$. Gallardo and Rojano (1990) brought to light the fact that conjoining arises because of the tendency to close the expression to a single value or solution as seen in the given examples; $8\boldsymbol{x}$, $\boldsymbol{x}^2 6$, $4\boldsymbol{x}$ and $6\boldsymbol{xyw}$. Gallardo (2002) says that the focus is simply on addition as if there is no sign at all as observed in $2\boldsymbol{x} + \boldsymbol{y} – 3\boldsymbol{w} = 6\boldsymbol{xyw}$. Stacey and McGregor (1997) alluded to the fact that in cases where learners bring their own experiences of dealing with letters as representatives of other things it is up to the educators to discern and do the best to ensure that they are appropriately assisted into sorting out which of the symbols promote algebraic learning. In this study I will show that conjoining as an error is more common in algebraic equations than in arithmetic equations.

(d) ***Operation signs ignored or not taken into cognisance (IOS)***

There is likelihood that learners do make errors some time because the order of the operation signs has been mixed up. For example, we have seen above that $2\boldsymbol{x} + \boldsymbol{y} – 3\boldsymbol{w}$ results in $6\boldsymbol{xyw}$ where presumably the learner just added coefficients of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{w}$ ignoring the '–' before the 3, hence 2 + 1 + 3 = 6. Vlassis (2004) says that mix-up of operational signs is very common in expressions that have many terms. Further, Vlassis refers to it as confusion in selection of the minus sign. Gallardo and Hernandez (2005), also concur by stating that students have difficulties with addition and subtraction of signed expressions and natural numbers. Kieran (1985) cited in Araya et al (2010) articulates that there is failure to correctly order the operations by learners, which represents a failure to correctly add and subtract numbers and do divisions on both sides of the equation. One of the examples given by Vlassis is that when solving $\boldsymbol{x} + 4 = 2$, the learner does not accept that it is possible to find the value of $\boldsymbol{x}$ as being negative. What emanates thereafter is that the problem is changed in one way or the other resulting in the change in operations. $17 = \square$ **- 6** is an example of an equation where the operation sign may be ignored for a solution like 11 as a number that goes into the box. 11 and 6 are erroneously added when the subtraction sign has been ignored. The same may be said for $4 - \boldsymbol{x} = 21$ when the solution is given as 17. Vlassis emphasizes that errors are increased with the introduction of negative numbers.

(e) ***Errors due to commutative and distributive properties***

Applications of commutative ($\boldsymbol{a} + 3 = 3 + \boldsymbol{a}$) and distributive ($2(\boldsymbol{a} + 1) = (2 \text{ x } \boldsymbol{a}) + (2 \text{ x } 1)$) properties are cited as one source of errors in learners' work (Siegler, 1998 cited in Araya et al, 2010). Commutative property can affect the transposition process in equations since the transposed term changes its sign. For example, suppose $\boldsymbol{a} + 3 = 5$ is to be solved and the learner is aware that he/she is to transpose or group like terms. Then in the process of grouping like terms he/she takes 3 as is and adds it to 5 on the other side giving rise to $\boldsymbol{a} = 5 + 3$. Commutative has been wrongly used though on its own right it is good. Distributive property has an effect on the solution of an equation if learners cannot remove brackets properly, see figures 5.3 – 5.13. None of the learners were able to use the distributive property; hence like in the case of commutativity this, also, will lead to wrong answers.

(f) ***Interference of new knowledge (INK)***

Olivier (1989) as well as Stacey and MacGregor (1997) talk of interference that is brought about by new strategies, concepts and algorithms that tend to be confused and substituted for each other. For example in dealing with algebraic expressions it is common practice that addition and subtraction of like terms is taught before exponents. Later when exponents are introduced learners tend to confuse expressions like $\boldsymbol{x} + \boldsymbol{x} + \boldsymbol{x}$ and $\boldsymbol{x} \text{ x } \boldsymbol{x} \text{ x } \boldsymbol{x}$, where $\boldsymbol{x} + \boldsymbol{x} + \boldsymbol{x}$ simplifies to $3\boldsymbol{x}$ and $\boldsymbol{x} \text{ x } \boldsymbol{x} \text{ x } \boldsymbol{x}$ simplifies to $\boldsymbol{x}^3$. The tendency, then, is that when expressions like $\boldsymbol{x} + \boldsymbol{x} + \boldsymbol{x}$ are revisited or come across later some learners would think of the newly learnt concept on exponents, hence making errors in their manipulations based on the new knowledge.

2.9 Summary

Based on the constructivist theory all learners' work is to be valued and every detail to be taken into consideration when analysing data. On considering the literature reviewed, there is more emphasis on algebra. One of the key issues in the treatment of algebra is that if learners are not competent in working with algebraic expressions, they are likely to face difficulties in dealing with linear equations. Literature points to the fact that errors may arise due to some misconceptions related to previous correct knowledge. Also, according to literature the transition from arithmetic to algebra has an impact on the performance of learners, and as such there is a need to bridge appropriately learners' experiences associated with their arithmetic work to algebraic work. There are many strategies that can be used by learners to solve equations and many errors that may emanate due to misconceptions. Finally, it is important to note that there is a very thin line in distinguishing between some errors and strategies in the process of solving equations due to the fact that some of the errors are results of strategies used, for example right to left reasoning. I return to this issue in the analysis.

3. *Chapter 3*

THE RESEARCH DESIGN and METHODOLOGY

3.1 Introduction

This chapter focuses on the research study and methodology – the sample, the area where data was collected, how data was collected, as well as issues on reliability and validity on both data collection and instruments. The study consisted of two phases: (1) a class of grade 9 learners was given a test to write; and then (2) a selected number of learners out of the class were interviewed. The two phases were designed in such a way that the second phase was informed by the first phase. Therefore there was a need for sufficient time for phase one to be completed before phase two commenced.

3.2 Study design and sample

The data was collected by means of a test and interviews. Both the test and interviews were the means to answering the research questions. The test was the main data source before interviews were conducted. The interviews were audio-recorded for further analysis at own pace as well as ensuring enough time for transcription and accuracy in the process thereof.

(a) ***The sample***

The study was conducted in a township school situated on the outskirts of Randfontein. The majority of learners came from the township in which the school was situated, with a large number also coming from a nearby township highly populated with zinc shacks. The school has an enrolment of about 1 400. It was a no-fee school, as the learners come from poor families. English was used as a medium of instruction, though learners consist of a mix of se-Tswana, isiXhosa and isiZulu speaking learners. The study was conducted with a class of 45 grade 9 learners constituting 27 girls and 18 boys. Their ages ranged from fifteen to seventeen years. The selection of the class was purposive. I, initially, had taught the class in their first term in grade 9, before some changes were made. Finally, interviews were conducted with only five learners. Learners to be interviewed were selected based on their performance in the test. After analyzing the test scripts, five learners of different abilities were purposefully selected so as to have a more representative sample to be interviewed.

(b) ***Piloting***

Test and interview questions were piloted to check that learners could make sense of the questions, and to eliminate misleading questions. The learners used for the pilot study were not the same learners as those who wrote the final version of the test or those who were interviewed. I piloted my research instruments so as to eliminate ambiguous items that may not be clear to the participants. It is imperative to note that learners have their own way of interpreting what they were being asked which may lead to some slight variation.

(i) Piloting the test questions

Ten grade 9 learners were given the initial test to write. I analysed the test and found that some questions were not of much help in as far as answering to the research questions and giving of the information sourced. The test questions initially consisted of three sections but were ultimately reduced to two. For example, the questions that required true or false responses from learners and could not clearly show the strategy used or error made. The problems are given below where learners were expected to respond by writing either the word **true** or **false** to indicate the correctness of the statement:

1. $4 + 2 + 7 = 9 – 1 + 3$ _________________
2. $2 \times 4 = 16 \div 2$ ______________________

The other test items that were removed were fractional equations (e.g. $\frac{x}{2} + \frac{2}{3} = \frac{3}{4}$) because of the multiple steps involved. The research interest was on analysing the basic simple strategies as defined in chapter 1. The scope of the study expanded through the inclusion of fractional equations when some problems that emanated dealt with fractions per se and not with equations.

(ii) Piloting the interview questions

Piloting the interviews also ensured that the time expected per learner during the actual interviews was established. This ensured that the tasks were properly constructed so as to elicit the kind of information I needed to focus on. The interview questions were further streamlined after analysing the test questions, i.e. those similar to the test questions removed. Those questions, whose errors were seemingly repeated, were left out to avoid repetition during the interview as well as minimizing time.

(c) ***Instruments for test***

The test questions and the appropriate instructions for learners are found in ***Appendix: A***. Section 1 questions focus on arithmetic equations where learners are expected to find numbers in boxes. In section 1, learners are required to find the solution of equations in an implicit way. What I mean is that the learner cannot explicitly tell that he/she is solving an equation. The strategies employed in solving equations may be used indirectly or without mathematical reasoning/explanation (this is my assumption and may not be true). Compared to section 2, section 2 is the final stage that the learner comes up and explicitly tells what he/she has been doing all along. Strategies used will come up clearly. Section 1 questions and their justifications thereof are outlined in table 3.1 below.

Table 3.1: Section 1 Test-Questions

Section 1: arithmetic equations		
Section 1	**Test questions**	**Justification**
Testing equivalence: Learners' concept of 'equality'.	**(a) 7 + ☐ = 15**	Learners are required to identify the correct number that goes into the box. It may be easier to infer the correct number that goes into the box because of addition. To ensure that guess work is avoided more related problems are given to test equivalence.
	(b) 7 - ☐ = 15	The change that has been effected is the operation subtraction. In (a) the operation was addition. Still, learners are required to identify the correct number that goes into the box.
	(c) ☐ - 7 = -2	The problem in (b) shows that a small number subtracts another number and the result is a bigger number. In (c) 7 is subtracted from a number and the result is negative.
	(d) ☐ + 3 = 8 - ☐	This equation could be approached in at least 2 different ways. If learners allow the boxes to have different values, then this equation has infinitely many solutions where the sum of the numbers in the boxes is 5. If the learner assumes both boxes must have the same value, then there is a single solution (x=2,5).
	(e) 17 = ☐ - 6	The questions, (a), (b) and (c) have the box on the left-hand side. In this problem the result is given first, and then it seeks to find the number that 6 is subtracted from to give 17.
	(f) 13 = ☐ - ☐	This problem foregrounds the equal sign as an equivalence relation. There are infinitely many combinations that can satisfy the subtraction of one number from the other to give 13.
	(g) ☐ + 35 = 47 - ☐	This is like (d) above, however the values of the two numbers have been made larger. Through this I wanted to see whether larger numbers have impact on the strategy used or error made.

Section 2 questions focus on algebraic linear equations. Though the questions in the two sections are different, the most important point to note is that the questions are structurally similar. Section 1 questions (a) – (e) and (g) are similar to the corresponding section 2 questions. Question (f) may be considered the odd one out in the sense that if both boxes were replaced by the same letter (for example: 13 = $\boldsymbol{x} - \boldsymbol{x}$) the equation would have no solution. Consequently question (f) in section 2 was designed with a different structure. In the rest of the questions, if boxes are replaced by letters they can be solved algebraically. In most cases there is only one solution, although (g) which is ☐ + 3 = 8 **-** ☐**,** is an exception as can be seen in the illustration that follows. Suppose a learner puts 5 in the first box, i.e. on the left hand side, then the left hand side will be equal to 8 after simplification. The number that goes into the box on the right hand side must be 0 so that the result of simplifying the right hand side will give the same value as that on the left hand side, which is 8. In other words to maintain equality on both sides of the equation the first number determines the second in each case. It does not matter which side of the equation the number is first chosen, e.g. if on the right hand side one chose to put 4 in the box then on the left hand side the only number that can be chosen is 1. If the boxes are replaced by a letter, like $\boldsymbol{x}$, then the equation becomes $\boldsymbol{x} + 3 = 8 - \boldsymbol{x}$. After transposition the problem will be like: $\boldsymbol{x} + \boldsymbol{x} = 8 - 3$, which simplifies to $2\boldsymbol{x} = 5$. In essence sections 1 and 2 questions are not different except for the difference in variables used, that is, boxes and letters for numbers. Table 3.2 gives section 2 questions and their justification.

Table 3.2: Section 2 Test-Questions

Section 2: algebraic linear equations		
Section 2	**Test questions**	**Justification**
Solving linear equations	**(a) x + 5 = 21**	This is a simple algebraic linear equation which can be solved by trial and error method.
	(b) 2x – 3 = 11	This is a more challenging problem than (a). It involves subtraction and requires a multiplicative inverse since the coefficient of x is 2.
	(c) x + 3(x – 2) = 6	This further stretches the question difficulty by including removal of brackets and there is more than one term with an unknown. It is more demanding than both (a) and (b).
	(d) x – 5(x – 1) = 3	Whilst (c) has an operation addition just before the bracket, here there is a subtraction. I wanted to see the impact of the "minus".
	(e) 3x + 5 = x + 9	This equation has the unknown on both sides. The research (get a ref) suggests that this form of equation cannot be solved arithmetically by inspection.
	(f) 3x + 5 = x + 3 + 2(x + 1)	This equation has *infinitely* many solutions. A lot of errors are expected in this problem as some learners may find it strange when they try to simplify as they may get 0 = 0.
	(g) x + 2 – (-2) = x - 4	This problem has no solution. From the face value it is impossible that 4 added to a number (x) can be the same as 4 subtracted from the same number (x). It is included to investigate the learners' adaptive reasoning in responding to such an example.

(d) ***Instruments for interview***

After analysis of the test, I selected only some of the questions for the interview. Some questions were modified, while others were eliminated completely to focus on key issues that had emerged from the analysis and that were in line with answering the research questions. The interview questions are found in ***Appendix: B***.

A comparison between test and interview questions is outlined in the tables 3.3 and 3.4 that follow, where table 3.3 gives comparison of section 1 questions and table 3.4 gives comparison of section 2 questions. Learners did not have difficulties with test question: 7 + ☐ = 15, however the first item: 5 + ☐ = 13 was included in the interview with the intention of setting the learner being interviewed at ease and as a way of encouragement for the learner to answer to the other three items. In the test, many learners provided correct answers for items (b) and (c), but (d) was really a challenge. Therefor it was necessary to probe learners' reasoning in the interview.

Table 3.3: Section 1 test and interview questions compared

Test Items	**Interview Items**
(a). 7 + ☐ = 15	(a). 5 + ☐ = 13
(b). 7 - ☐ = 15	(b). 5 - ☐ = 11
(c). 13 = ☐ - ☐	(c). 15 = ☐ - ☐
(d). ☐ + 3 = 8 - ☐	(d). 19 - ☐ = 10 + ☐

The same comparison of test and interview questions was done with section 2 equations.

Table 3.4: Section 2 test and interview questions compared

Test Items	Interview Items
(a). $x + 5 = 21$	(a). $4 + x = 23$
(b). $9 - x = 13$	(b). $4 - x = 21$
(c). $2x + 3(x - 2) = 6$	(c). $3(x + 2) = 18$
(d). $3x + 5 = x + 9$	(d). $3x + 2 = 8 + x$
(e). $x + 2 - (-2) = x - 4$	(e). $x + 3 = x - 3$

It is important to note that the interview questions were modified to make them simpler than the test questions. For example, in table 3.4 question (a), (b) and (d) were not changed much from the test questions. Question (c) was made simpler by leaving out the first term. Question (e) was modified significantly by ensuring that the expression 2 – (-2) was simplified into a single number. See table 3.4 below for the changes effected from test to interview question items.

3.3 Reliability

Reliability in my study involves all that has transpired in conducting the entire research process, that is, how data was collected and how data was analyzed and interpreted. According to Punch (2009), reliability has two aspects. The first being consistency over time (which is said to be expressed in the question that if the same instruments were given to the same subjects under the same conditions, but at different times, then to what extent would they get the same results?). The other is the internal consistency (and in this case the question concerns the harmony in which the items are consistent with each other in terms of working in the same direction). In terms of consistency over time, and in the context of this study, what has to be considered is that the human mind cannot be predicted but depends on the individual. So it is imperative to know here that I am dealing with human subjects and the information obtained at any one particular time will largely depend on the individuals in what they will be thinking and their depth of understanding of the content knowledge.

In terms of internal consistency, I would return to my research instruments and maintain that all the questions designed are simple linear equations. Also, cognisance must be taken of how the interview questions compare to test questions (see tables 3.3 and 3.4 above). Leedy and Ormrod (2010) defined reliability as the consistency with which measuring instruments give certain results with the entity being measured constant. In my study the instruments are designed for grade 9 and so therefore have to be used with the grade 9's. During the writing of the test learners were monitored so that they could not discuss their work, and this ensured that the work produced belonged to individuals and not a group. Adding to the reliability of data analyzed is the availability of both learners' scripts and interview recordings, should there be need for further analysis by someone else.

3.4 Validity

Besides taking all precautions for the research instruments to be reliable, I have also ensured that the instruments used are not complicated, but measure that which they are to intended to measure; for example, equations requiring complicated multiple steps have been avoided completely. Punch (2009) together with Leedy and Ormrod (2010), define validity as an indicator of the extent to which an instrument measures what it is supposed to measure. Again, the question which seems to be central is: how are we to know that the measuring instrument measures that which it is supposed to measure? Piloting of the research instruments prior to data collection point to the fact that the instruments were valid. During the pilot the objective was to validate the instruments so that the data collected is accurate for what is being sought. The learners' responses pointed to what they knew both verbally and in writing, and the analysis is based on what the learners wrote and said. To me this is another dimension that may be considered in as far as validity of data and the analysis is concerned. The instruments used were valid considering their appropriateness to the grade as well as being piloted before use in the actual study.

3.5 Ethical issues and Practical steps taken in conducting the research

McMillan and Schumacher (2010) together with Leedy and Ormrod (2010) maintain that wherever human beings are involved, we have to look closely at the issue of ethical implications of what is proposed to be done. The guidelines that have to be taken into account include policies regarding informed consent, deception, confidentiality, anonymity, privacy as well as protection from harm. The first thing I did (after written consent and clearance by both the University of the Witwatersrand (Protocol no. 2013ECE121M) and Gauteng Department of Education (D2014/111)) was to write letters to the principal of the school, seeking permission to carry out the study (in his school) and informing him about the research. The participants were invited to participate in this study, and were made fully aware of the intentions of the research and the implications of their participation in that they were in no way obliged or forced to participate in written form. Further, letters were written to the parents informing them about the study as well as asking their permission to allow their children to partake in the study (as the learners were under the age of eighteen).

Confidentiality of their identity was assured and that the results of their participation will be used for this study only. The invitation to participate in the research had clearly stated that the participants were to write a test first and thereafter be interviewed. Although learners had agreed to be interviewed, after writing the test, several were reluctant to be interviewed. Also, one potential participant (who had consented to participate in the interviews) declined to be interviewed because the parent did not give consent. It is normal when it comes to being recorded or taking photos that some people naturally do not like the idea. The parent's rights were respected, and the learner did not participate. It appeared that they were not keen to be interviewed because the test had been difficult for them and they feared being asked similar questions. I acted ethically in that I did not force anyone to be interviewed. I explained to them that interviews would be based on some written work, and I would ask them to explain what they had written. After this five learners agreed to be interview without being coerced. Finally, all the participants' rights were observed and no information was divulged publicly to anyone except for the writing of this report.

3.6 Summary

The piloting of both the test questions and interviews played a vital role in enabling me to streamline both the test and interview question items, eliminating those items that were likely to cause confusion to learners and adjusting the interview questions into a more focussed scope. The next chapter focuses on data analysis of arithmetic equations.

4. *Chapter 4*

THE DATA ANALYSIS 1: *Strategies and errors in solving arithmetic equations*

4.1 Introduction

This Chapter 4 focuses on strategies used and errors made by learners in solving arithmetic equations for both test and interviews. Chapter 5 will also focus on strategies and errors relating to algebraic equations for both test and interviews too. It is of interest to note that there is a difference in strategies used in solving arithmetic equations compared to what was expected in as far as literature is concerned. A lot of strategies are more related to the errors made. In other words errors are the results of those particular strategies used. Some strategies and errors that have been discussed in the literature are used and committed by learners. However there are some strategies that have emerged, directly related to learners' reasoning as observed from their test answers and my understanding of their interview responses.

4.2 Codes used

In section 2.8 in the literature review I highlighted some of the errors from literature and coded them according to what the literature say as well as according to the most prevalent learners strategies used or errors made. Some of those strategies and errors will be discussed in this chapter, e.g. right to left reasoning (RLR) and ignoring the operation sign (IOS). Following are some of the codes that will be used in my data analysis in this chapter, and their explanations.

- Slip (S), that is, a mistake that can easily be corrected.
- Taking sign of a bigger number (TSBN) is used where the error is due to incorrectly assigning to the solution the sign of the addend with the bigger absolute value. More appropriately, this principle is used when operating on two numbers with different signs. For example consider the 2 examples: -7 + 4 and 4 + (-7). In the first problem the difference between 7 and 4 is 3, and taking the sign of 7, the solution becomes -3. In comparison 4 + (-7) gives the same solution as 4 – 7, which is -3. The error occurs in 4 – 7 when 7 is taken to be negative and both 4 and 7 are positive because the '–' the operation subtraction.
- Proving correctness of solution (PCS). In an equation like $\square + 3 = 8 - \square$, the numbers in the boxes are taken to be 5 on the left hand side and 3 on the right hand side, since 5 + 3 gives 8 and 8 – 3 gives 5 (the point is to prove the correctness of 5). Nothing like this has been referred to in the literature.

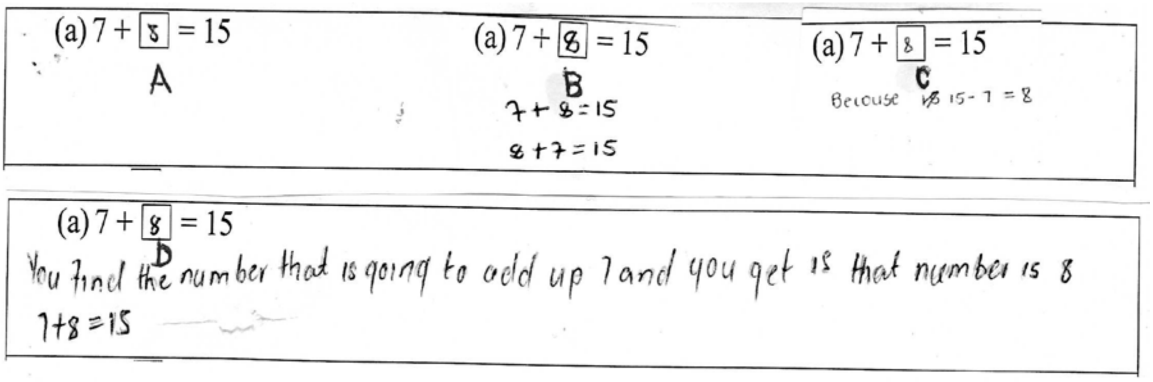
- Breaking the equation into two parts (BEQ). In this case $\square + 3 = 8 - \square$ is broken into two distinct equations with the values in the boxes being found independently of each other. *The numbers in the boxes were found one at a time independently of each other. In other words the equations were treated as two different equations instead of one.* For example the two solutions are 5 and 5, and the two equations are $\square + 3 = 8$ and $3 = 8 - \square$. Behr, Erlwanger and Nichols (1980) as well as Essien and Setati (2006) have demonstrated this observation as well, although their cases are not exactly the same. Behr et al believe that learners do not see sameness of two sets of objects given such sentences as $4 + 5 = 5 + 4$. The tendency when given such will be to write $4 + 5 = 9$. Essien and Setati on the other hand provide an example like $13 + 5 = \square + 5$ to find the missing number in the box. As a result the answer is immediately written after the equal sign. For example, 18 will be written in the box.
- The code "other" (O) is used when I could not find a reasonable explanation of the learners' thinking in the case of arithmetic equations (O).
- Correct answer (C).
- Correct pairs of answers (CP). For example, in $\square + 3 = 8 - \square$ if 5 is to be inserted in the box on the left hand side then 0 has to be inserted in the other on the right hand side, giving the correct pair of answers (5 ; 0).
- Unanswered question (UAQ) where learners did not write any solution to an equation.

4.3 General analysis of the test

After the learners had written the test, their scripts were considered, taking into account the following: (1) the learners' ability to follow instructions and (2) the learner answers, whether they were correct or incorrect answers for each of the question items. I found that only twelve learners had followed the instructions to the extent of demonstrating the correctness of their answers either through substitution or by explanations. The rest of the learners simply put a number in the box which they thought was the desired number, but did not explain their solutions.

In figure 4.1, Learner **A**, just wrote the answer and did not explain anything. Learner **B** showed with emphasis that the answer is 8 by using commutative properties of the addition of numbers. Learner **C** showed that if 7 is subtracted from 15 the answer is 8. This is equivalent to the method of transposition, though it is not certain that the learner was transposing or that it is the method the learner used. Learner **D** explained verbally that the number that adds to 7 was to be found so that this number and seven added up to 15. The strategy used can be trial and improvement, or any other (the learner did not specify). See figure 4.1 below for the learner responses.

Figure 4.1: Learners A, B, C and D responses to the test question (a) in section 1



Learner **A** wrote 8 in the box, but did not explain his/her solution. Learner **B** wrote two number sentences, probably as a way of emphasising that the 8 in the box is true and that 7 + 8 is the same as 8 + 7, and both add up to 15. Learner **C** gives a reason that 15 minus 7 is equal to 8. Learner **D** explains and demonstrates his/her explanation (as can be seen). The first learner is one of the many who were simply writing missing numbers in boxes without explanations or any form of trying to convince the reader of their solutions, and the other three are part of the few who provided some evidence of their reasoning.

On considering whether the learners' responses corresponded with the expected answers of the various items, I was able to determine how the questions were answered, whether the answers were correct or incorrect. Pertaining to the way the learners responded to each of the question items, I was interested in those answers that were not expected and at the same time seemed to be similar or the same for the majority of learners. The incorrect answers point to some errors emanating from some knowledge that is misused in one way or the other. The questions that most learners did not answer correctly are (d) and (g). See the total correct answers row (**T/C/answers**) on Table 4.1. Note that for questions (d), (f) and (g) there are no precise values that can be prescribed. These questions have infinitely many solutions, and it also depends on the way individual views them. They do not have precise answers, except on special conditions (which are not prescribed). Table 4.1 shows part of the initial analysis of the test. Note that: (a), (b) … (g) are question numbers; G901, G902 … G945 are learner codes and C1, C2 … in the last row indicate columns 1 to 8. Complete data can be viewed in ***Appendix: C*** initial data tables. From the table, the first row (columns C2-C8) provides the question numbers of the questions in section 1 that learners answered; followed by the corresponding responses in row two. Rows three to forty-seven are learner responses to the different question items, while the last row gives the total number of learners with the correct answers as expected (in particular columns C2, C3, C4 and C6) or the combinations that make the statement (equation) true. From the results in the table, it is evident that all the learners provided the correct answer of 8. In addition, question (a) was a simple addition-problem, and did not expect learners to make such errors as they would have done when dealing with subtraction problems. See Table 4.1 below.

Table 4.1: Learners' answers to test section 1 questions

Question Item	(a)	(b)	(c)	(d)	(e)	(f)	(g)
C	*8*	*-8*	*5*	CP_1	*23*	CP_2	CP_3
G901	8	22	9	5 ; 5	12	20 ; 7	12 ; 12
G902	8	-8	5	3 ; 2	24	15 ; 2	2 ; 10
G903	8	22	9	5 ; 3	12	20 ; 7	12 ; 12
G904	8	22	5	5 ; 5	23	20 ; 7	12 ; 12
…	…	…	…	…	…	…	…
G943	8	21	5	5 ; 0	25	14 ; 1	9 ; 4
G944	8	22	5	5 ; 3	23	20 ; 7	12 ; 12
G945	8	-8	5	5 ; 5	-11	2 ; 15	12 ; 12
T/C/answers	**45**	**24**	**27**	**3**	**18**	**31**	**2**
C1	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**

Questions (b) to (g) indicate many variations in the answers provided by learners. Moreover, there are some answers that are wrong but seem to be common in a number of learners' responses. This becomes a point of interest when considering that Nesher (1987) maintained that misconception is a line of thinking that causes a series of errors all resulting from an incorrect underlying premise. For example in (b) there are 15 answers that were all the same. The response of 22 is definitely 22 is not a slip but the result of working from right to left, referred to as right to left reasoning (RLR), (Gallardo and Rojano, 1990; Gallardo and Hernandez, 2005 and Vlassis, 2008).

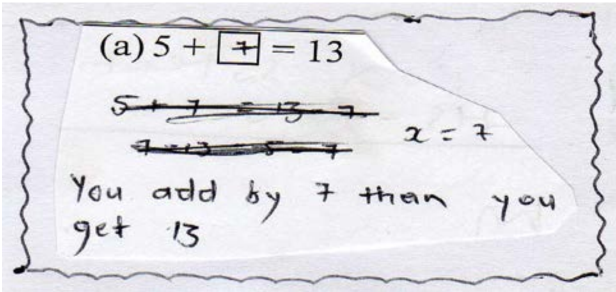
4.4 Question by question analysis

This part of the analysis considers how the learners performed in each question, paying special attention to those answers that are wrong so that the possible likely source of those errors might be accounted for. Should there be need for further analysis or clarity the reader is referred to initial data tables in ***Appendix: C***, tables 1 to 12

(a) ***Test and interview questions:*** $7 + \square = 15$ and $5 + \square = 13$

All learners obtained correct responses to question (a) in the test. In the interview Kison made a mistake by mentioning that the number that goes into the box is 7. I interviewed him on how he obtained 7 and these are some of his responses to my probing: "*…this side it means that 5 plus the missing number here is equals to 13. I go to said 5 plus this missing number I subtract 13 by 5. So I got that 7.*" Following his reasoning he transposed 5 and subtracted it from 13 to obtain 7. The strategy employed is correct, however the answer obtained is not. Another issue to consider is that some of these learners are not so fluent in English. They cannot express themselves clearly. This is demonstrated by for example figure 4.2 below where Kison wrote his reasoning and argument that 7 is the correct answer to his work. He says 'you add by 7 than you get 13' meaning that you add five and seven to get thirteen.

Figure 4.2: Kison's reasoning on 5 + ☐ = 13



Kison first wrote something and cancelled it. I believe that it was part of his working in finding the number in the box, and after finding this number, he then wrote it. Note, also, that no calculators were allowed; hence some simple mistakes were made by learners in some of their work. Table 4.2 below provides a summary of this analysis of the test and interview questions for questions (a).

Table 4.2: Error summary for both test and interview questions: 1

Test and Interview questions	**Noted Common Errors**		
	C	**S**	**Total number of learners**
a) 7 + ☐ = 15	45	nil	45
a) 5 + ☐ = 13	4	1	5

(b) ***Test and interview questions:*** 7 - ☐ = 15 and 5 - ☐ = 11

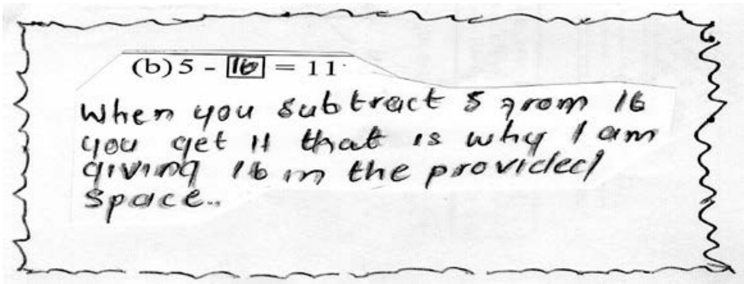
Table 4.3 below provides a summary of the answers that learners gave for test and interview questions (b). In the test, 25 learners had correct answers (C), which also happens to be the most common in this question. Seventeen of these learners most likely worked out their problems from left to right (RLR) because of their solution of 22; and 4 of the learners responses indicate that they most likely ignored the operation sign (IOS), (Vlassis, 2004; Kieran (1985) cited in Araya et al, 2010). The most common error is RLR, which from another *point of view* can be taken as a strategy that was used by learners although it was not expected. It is no surprise that learners could come up with their own strategies as literature has alluded to the fact that whilst learners try different strategies to solve equations, there are some difficulties due to some misconceptions that are observable when learners solve the equations (Araya et al, 2010). The two errors are also noticeable in the interview questions. Table 4.3 provides the summary analysis of the two questions.

Table 4.3: Error summary for both test and interview questions: 2

Test and Interview questions	Noted Common Errors			
	C	RLR	IOS	Total number of learners
(b) 7 - ☐ = 15	24	17	4	45
(b) 5 - ☐ = 11	1	2	2	5

The following are some of the learner responses during the interviews as well as their written explanations. Figure 4.3 gives Kitos' original work and his justification.

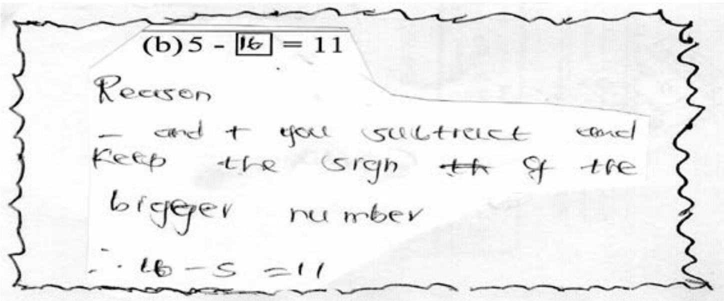
Figure 4.3: Kitos' reasoning on 5 - ☐ = 11



When he was interviewed Kitos mentioned that five and the number he placed in the box must give eleven. He explained that sixteen minus five is equal to eleven (i.e. 16 – 5 = 11) and that was why the number he placed in the box must be sixteen. What I have written in brackets is different from what we see in the extract above. When I further questioned that what he wrote (5 – 16) and what he was saying (16 – 5) was different, he admitted that he was a bit confused. When he realised that I persisted, implying that there was an error in his work, he said that he was a bit confused. He accepted that he was wrong.

Vicky mentioned that sixteen is bigger than five, indicating that the operation subtraction has nothing to do with the position of numbers. The smaller number is subtracted from the bigger number as long as one has identified the bigger number. Finally the reverse of five subtract sixteen (i.e. 16 – 5) prevails. During the interview, Vicky explained that she first added five and eleven to get sixteen then had to put sixteen into the box. After she had done that, she saw that sixteen and five simplified to eleven: "*Then I putted that sixteen here; and subtracted five and sixteen, which gives me eleven*". I persisted in my probing to determine what she did in order to determine precisely how she simplified the expression: 5 - 16. Through demonstration, she agreed that she subtracted from right to left, although she did not use the terminology left to right. Realising that she probably did not do it correctly, she said "*I subtracted sixteen and five, which gives me eleven. I didn't use the signs when I subtracted*". I inferred that the implication of not 'signing' a number implies that the expression was reinterpreted to change the subtrahend into the minuend. Vicky found the difference between sixteen and five regardless of order. I inferred this because when I questioned what Vicky meant by not using the signs, she just said *"Yes. I only subtracted"*. This last statement was repeated at some other point in time when she said *"… I only did the subtraction"*. Figure 4.4 shows Vicky's work.

Figure 4.4: Vicky's reasoning on 5 - □ = 11



From the figures and what learners said I assumed that right to left reasoning is a common error when faced with a negative solution.

Two learners Miriam and Violet put six as their solutions into the box. The subtraction symbol was ignored right from the beginning. After substituting 6 into the box she added up five and six to get eleven. The most visible errors that the learners made are right to left reasoning and to a lesser extent ignoring the operation sign (of subtraction).

(c) ***Test question:*** □ - 7 = -2

Variations in the answers to □ - 7 = -2 have increased compared to those of (b). Answers classified as *other* (O) are observed whereas in the other two questions above there was none. Again I notice that there are more negatives in (c) than in (b), and that has an implication in contributing to the errors made, e.g. Gallardo and Hernandez (2005) have alluded to the fact that students have difficulties with addition and subtraction of signed expressions and natural numbers. The correct answer was the most common. In this question, the most likely answers that could give a difference of 2 constituted 33.3% (made up of answers like ±9 and -5) compared to 62.2% of the correct answers and only 4.5% (±4 and 10) cannot be accounted for. The new error that is seen in this question is what I call 'taking the sign of a bigger number' (TSBN). Learners have quite often during the interviews, spoken of taking the sign of a bigger number whenever a negative is involved in arithmetic, for example taking -9 as a number in the box thus obtaining -9 – 7 = -2. The implication of this is that we as teachers sometimes mention to the learners that when subtracting any two numbers, consider their difference and take the sign of the bigger number. This statement has its limits, and it works for some problems (like 5 – 2 = 3 and 2 – 5 = -3) and does not work for some (like -9 – 7 = -16) because usually the rule is applied when the numbers have opposite signs. Another problem is the issue of double negatives. For example learners mention that a negative and negative gives a positive; like negative nine and negative seven gives a positive nine and positive seven. Then the difference between these two numbers is now two. However, the bigger number is nine whose sign is negative and finally the solution is negative two. In short, we as teachers are quite often contributing to learners' misconceptions when our intentions are to assist learners in succeeding.

Table 4.4: Error summary for test question: 1

Question Item	Noted Common Errors				
	C	RLR	IOS/TSBN	O	Total number of learners
(c) □ - 7 = -2	27	8	5	5	45

(d) ***Test and interview questions:*** □ + 3 = 8 - □; □ + 35 = 47 - □ and 19 - □ = 10 + □

From learners' responses to all three questions, it was evident they were not treated as one equation. The most common answers result from providing two pairs of values for each equation, for example, 5 *and* 5 and 5 *and* 3for □ + 3 = 8 - □; 12 *and* 12 and 12 *and* 35 for □ + 35 = 47 - □ and then 9 *and* 9 for 19 - □ = 10 + □. The pairs 5 *and* 5; 12 *and* 12 and 9 *and* 9 are the results of breaking the equations into two different equations that are then solved independently, whereas the pairs 5 *and* 3 and 12 *and* 35 are the result of providing the means of proving the correctness of the first value in the solution. In the test questions only three learners had the correct answers for (d) and only two for (g). The two who obtained correct answers to (g) are two of the same learners who got correct answers in (d). This shows consistence in working. Comparing (d) and (g), there is evidence that the use of large numbers or small numbers did not contribute to the slips made by learners. The numbers may be small but the effect is the same. The ideas used, however, in (d) are similar to the ones in (g). In other words learners' reasoning did not change. Having said this, the issue lies with the two newly discovered errors, namely breaking the equations into two (BEQ) and proving correctness of solution (PCS). Literature has alluded to the challenges that learners experience when faced with problems having equality statements with two plus signs (Behr et al, 1980), the need for learners to see a unique result before the operations on numbers mean anything (Kieran, 1981) or the regard of the equal sign as a signal to compute what precedes it (Essien and Setati, 2006).

Analysis of test question (d) before learners were interviewed indicated the following. At first glance the combination of 5 and 5 is most likely the result of learners possibly thinking of a number (the same number) that can be put into each of the boxes. Then the solutions with the number 5 happen to give the numbers 3 and 8 in each case, which are found on both sides of the equation. On further analysis, the combination 5 and 5 is possibly the result of the learners breaking the equation into two parts and then solving them separately. For example, the two are: $\square + 3 = 8$ and $3 = 8 - \square$, and in each case the number in the box is 5. This is the reason for coding it as BEQ i.e. for breaking the equation into two parts and solving them separately. The other new code is PCS which represents "proving correctness of solution" of the number in the first box. Considering the first equation with answer 5 in the box, substituting 3 into the other box gives the same answer 5 proving that 5 is the correct choice of answer, or vice-versa. This coding also works well for equation (g).

Considering 5 *and* 5 the likely reason for such an answer would be that the learner may have thought of a number (the same number) that can be put into each of the boxes. On one side, the 5 would give the value of the number on the other side of the expression, e.g. the left-hand side will result in the value of 8 on the right-hand side (i.e. $5 + 3 = 8$). If the 5 is put in the box on the right-hand side then the value is 3 on the left-hand side (i.e. $8 – 5 = 3$). Both 8 and 3 are visible on either side of the expression. So the equation seems to have been broken into two separate equations like: $\square + 3 = 8$ and $8 - \square = 3$. This being the case, the paired values 5 *and* 5 makes sense as a solution to $\square + 3 = 8 - \square$. Substituting the pair of values (5 *and* 5) leads to $5 + 3 = 8 – 5 = 3$. This is typical of what Stacey and MacGregor (1997) have maintained that students' interpretation of equations can be greatly influenced by prior expressions in arithmetic. The idea of solving an equation may be there, but the results indicate that it is now a different equation being solved (not the original one), where the use of the equal sign as would normally be used in solving equations is no longer visible. The other pairs of numbers that may indicate the same reasoning are also 5 *and* -5. An interesting combination of values is 5 *and* 3. Ten learners provide this solution to their problems. It is possible their reasoning, from $\square + 3 = 8 - \square$, was as follows: the number in the box added to three gives the result eight; and three subtracted from eight provides the same number five, that has been found. In symbols that means $\underline{5} + 3 = 8 – \underline{3} = \underline{5}$. Taking the extreme values it would be the first number is equal to the last number. 5 *and* 2 cannot be accounted for except that it is a slip for 5 *and* 3.

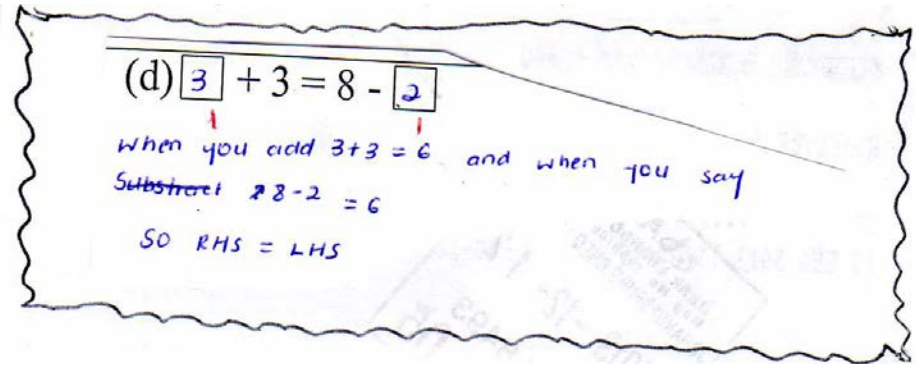
An equivalence view of the equal sign is visible in the solution 8 *and* 3, where the only error is that of ignoring the operation subtraction/sign (IOS) on the right-hand side. Substituting the solution (8 *and* 3) on the left-hand side provides: 8 + 3 = 11 and on the right-hand side: 8 – 3. Without taking the subtraction into cognisance but only considering 8 and 3, this will still provide the same result eleven (11) and hence 8 + 3 = 8 – 3 = 11 makes sense. The same can be said of 5 *and* 16. The equality of both sides can be maintained if it is assumed that on the right-hand side this learner performed the common error of right to left reasoning. For example, the left-hand side is not that much of a problem since the addition is straight forward. On the right-hand side, however, 8 – 16 may have been mistaken for 16 – 8 which is equal to 8, hence 5 + 3 = 8 – 16 = 8. Lastly, I found no plausible reason for the solution 5 *and* 11. BEQ was the most common error and most unexpected method that learners used. Before considering the learner responses in the interviews, table 4.5 gives a summary of the results of the test questions and interview question.

Table 4.5: Error summary for both test and interview questions: 3

Test and Interview questions	Noted Common Errors								
	C	S	RLR	IOS	BEQ	PCS	UAQ	O	Totals
(d) ☐ + 3 = 8 - ☐	3	1	1	1	27	10	nil	2	**45**
(g) ☐ + 35 = 47 - ☐	2	3	nil	1	30	4	2	3	**45**
(d) 19 - ☐ = 10 + ☐	nil	nil	nil	nil	5	nil	nil	nil	**5**

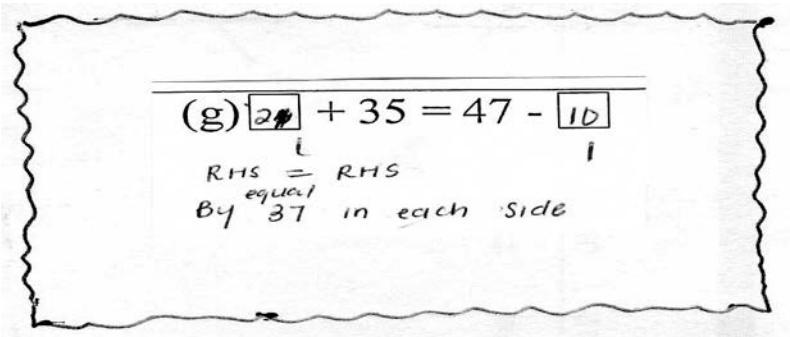
From the table, it is evident that five of the learners interviewed used the same method of breaking the equation. Their explanations will be reviewed shortly. However before this is done, I would like to discuss one particular learner (the parents did not consent that the learner be interviewed) whom I had really wanted to interview. This learner indicated that the two sides of the expression must add up to the same value after the numbers in the boxes have been found. After finding the numbers for each of the boxes on either side of the equation the learner explains explicitly, one at a time, the meaning of the numbers in the boxes. For example, the learner writes that 3 + 3 = 6 and 8 – 2 = 6, then consciously concludes that one is the left hand side (LHS) and the other the right hand side (RHS) and both are equal. Consistency is displayed in (g) by the same learner. See figures 4.5 and 4.6 for this learner's work.

Figure 4.5: Learner's reasoning on $\square + 3 = 8 - \square$



No one can mistake this presentation above for something else. The explanations are clear despite the fact that we see the three and two substituted in the equation. There might be a number of reasons for how this learner obtained the values three and two, for example, trial and improvement or any other method. Most importantly the learner displays full understanding that an equation has two sides and those sides must remain at balance once the solution has been found.

Figure 4.6: Learner's reasoning on $\square + 35 = 47 - \square$

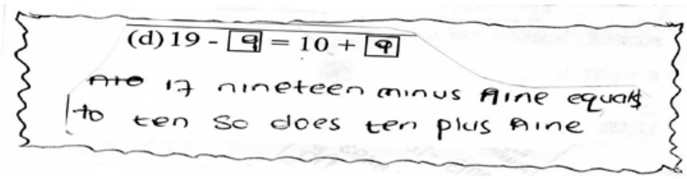


The conceptual understanding of this learner is that these expressions have two sides separated by an equal sign; and the two sides must be equal.

The following are the findings of the learners who were interviewed.

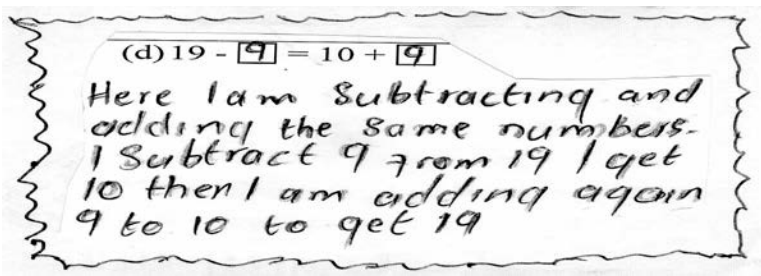
I have already mentioned that in expressions like $19 - \square = 10 + \square$ learners are breaking it into two distinct equations, like (i) $19 - \square = 10$ and (ii) $10 + \square = 19$; and then solve them apiece. Miriam, in figure 4.7, indicates that nineteen minus nine is equal to ten and again ten plus nine equals nineteen. The two sides are worked separately. According to Miriam's explanation the number in the box on the left-hand side must be such that it provides the value of ten on the other side, that is, the right-hand side. A possible explanation is that she has taken the part $19 - \square = 10$, and left out the other box. The statement 'if nineteen minus nine equals to ten so does ten plus nine' possibly means that $19 – 9 = 10$ and $10 + 9 = 19$. In summary Miriam did not show a separation of the equation into two equations but I believe that she did separate the equations but only in her mind. Her initial work before the interview is shown in figure 4.7 below.

Figure 4.7: Miriam's reasoning on 19 - $\square$ = 10 + $\square$



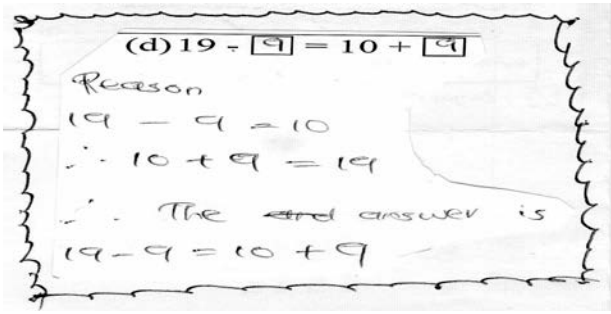
Remaining on the same problem, Kitos' explanation, by considering his sentence construction in figure 4.8., is that the equation has been broken into two different equations He explains that he subtracts nine from nineteen to get ten (19 – 9 = 10). Again he clarifies that he adds nine to ten to get nineteen (9 + 10 = 19). The same ideas that are expressed by Miriam are now evident in Kitos' work. During the interview he repeated his words that were written as is. He obtained the nine by possibly only considering part of the equation: 19 - $\square$ = 10; in which he found the number 9. This is breaking the equation into two different equations or formulating one's own work. There is also evidence from other learners that the equation is broken into two parts which are then solved separately in the form: 19 - $\square$ = 10 *and* 19 = 10 + $\square$. The breaking of the equation may not be literally so, but the fact that the learners did not consider the numbers to make both sides equal, point to the fact that different equations are being solved. These equations are not the original equation; the original has been tampered with. Another point is that learners have their own interpretations. Figure 4.8 indicates Kitos' explanation to how he did his work.

Figure 4.8: Kitos' reasoning on 19 - $\square$ = 10 + $\square$



Vicky working on the same problem, in figure 4.9, shows that she does not have an equivalence view of the equal sign. She writes that 19 – 9 = 10 and conclusively says that 10 + 9 = 19. These number sentences are no doubt equivalent to the equations 19 - $\square$ = 10 and 19 = 10 + $\square$ or 10 + $\square$ = 19. The sense of an equation becomes blurred when considering Vicky's statement in which she explains that the answer is 19 – 9 = 10 + 9 because 19 – 9 ≠ 10 + 9 in as much as 10 ≠ 19. I believe that in this case the learner is taking the equal sign in a unidirectional sense (Essien and Setati, 2006). During the interview I asked Vicky how she got the nines in each of the boxes, and her response was "*I subtracted these two numbers nineteen minus ten is equal to nine. So I putted that nine here because if you subtract nineteen by nine it will give you ten and when you add nine into ten it will give you nineteen*". Vicky agreed that each of the nines in the boxes was there because it made each of the numbers (10 and 19) true when substituted in each of the boxes one at a time. See figure 4.9.

Figure 4.9: Vicky's reasoning on 19 - □ = 10 + □



When a 9 is substituted on the left hand side 10 is the immediate answer after the equal sign. The box on the right is either invisible or ignored or left to be used to when finding the number that will make 19 true.

(e) ***Test question:*** 17 = □ - 6

The types of answers that learners wrote in this question are very varied. The most common response in this equation is the correct answer. However, double negative (DN) is frequently observed as a new error compared to other errors like IOS. Slips were also frequent in this question. Despite the highest number of correct responses, the errors are also more varied. In the errors made, -11 and 11 were most prevalent. The most likely reason for this choice may be the fact that 6 and 11 add up to 17. In other words the question of signs (negative) is not taken into consideration; it is ignored or overlooked completely. For example using -11, the expression becomes -11 – 6 on the right-hand side. This indicates the tendency to consider the idea of a double negative like 11 - -6. The tendency is that two negatives result in a positive (+), thus resulting in learners adding 11 + 6. It was observed that quite often, learners do not differentiate between -11 – 6 and 11 - -6. As long as there are two numbers and two negatives they change the signs to positive. Ignoring the operation sign will result in addition of 11 and 6. This is observed where learners give solutions like 11. Eleven is considered as a solution based on the fact that the operation subtraction has been ignored in the expression 11 – 6, and learners end up with 11 + 6. In this case 11 + 6 will result in 17 as required. The other solutions are attributed to simple mistakes. Table 4.6 provides a summary of the findings.

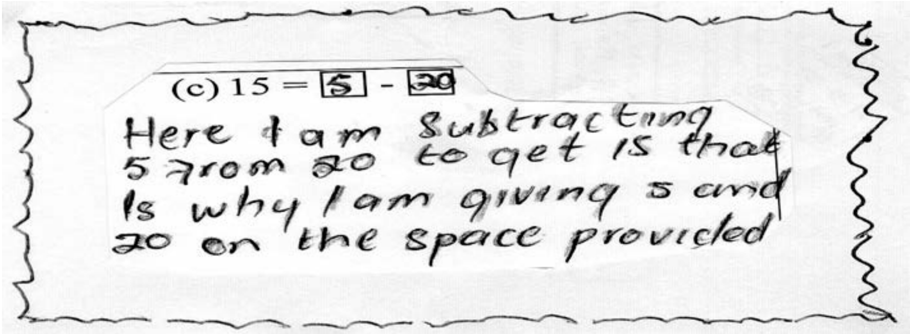
Table 4.6: Error summary for test question: 2

Question Item	Noted Common Errors					
	C	S	IOS	DN	O	Total number of learners
(e) 17 = □ - 6	18	7	13	6	1	45

(f) ***Test and interview questions:*** 13 = □ - □ and 15 = □ - □

Different types of answers were given in this problem. However the errors that learners made are traceable, except for two. There were thirty-one correct ordered pairs of answers, five resulting from right to left reasoning, six from ignoring the operation sign and a negligible number from others. In fact all of the errors have been discussed in the previous problems. The most common pairs of answers, in 13 = □ **-** □**,** are 7 *and* 6, -7 *and* 6 as well as 7 *and* 20. The absolute values of the pairs of answers either have a difference or a sum of thirteen. This sum or difference of 13 results when signs are either ignored or worked from right to left or vice versa. Solutions, for example, that are a result of right to left reasoning are: 7 *and* 20, 0 *and* 13, 2 *and* 15 and the rest are a result of negative signs ignored like: 3 *and* 10, and -8 *and* 5. From the learners' side Kitos consistently worked from left to right even when provided with boxes to put his numbers right. In figure 4.10 Kitos was supposed to simply subtract: twenty subtract five and write: 20 – 5. It appears to me that Kitos is using the subtraction symbol in such a manner that the position of numbers in relation to the symbol does not matter. What seems to matter is that the difference between the numbers does exist and to find the difference you simply subtract the smaller of the two numbers as long the numbers are of the same sign. What I mean is that from the expression 5 – 20, the 5 is taken as a number 5 and the 20 as a number 20 (meaning that they are both positive), and the operation subtraction (-) is seeking for difference between the two numbers such that you subtract the smaller from the bigger resulting in subtracting five from twenty. I could find no other reason that can explain his work except the findings that learners tend to close just after the equal sign (Behr et al 1980). The closing of the solution provides the answer 15, and is as a result of working towards the answer, so to speak. On being interviewed he said exactly what he wrote. Now someone who is listening and someone who is seeing have different perspectives concerning what is happening or being done. I noted that it makes no difference or very little difference to learners as to how they arrange their numbers when it comes to subtraction. See figure 4.10 below. From Kitos' language he is working from right to left because he mentions that he is subtracting 5 from 20 and 5 is on the left side of 20.

Figure 4.10: Kitos' work and justification on 15 = ☐ - ☐



This is not uncommon at all as Behr et al (1980) expounds that when a learner was presented with a number sentence: ☐ = 1 + 2, the learner wrote '3 = 1+ 2'. When asked to read, the learner shouted "2 plus 1 equals 3", (Behr et al, 1980, p. 14). Table 4.7 summarizes the outcome of the test and interview results.

Table 4.7: Error summary for both test and interview questions: 4

Test and Interview questions	Noted Common Errors					
	C	RLR	IOS	DN	O	Total number of learners
f) 13 = ☐ - ☐	31	5	6	1	2	**45**
(d) 15 = ☐ - ☐	2	2	nil	nil	1	**5**

There are two common errors that learners made, that is, right to left reasoning and ignoring of the operation sign.

4.5 Summary

In conclusion the interviews revealed that though the learners did not show any working in their work to qualify the strategy they were using, they did use strategies like transposition and balance method. Trial and error could be observed when the learner could not explain what he/she had done. Common errors that learners committed were those of right to left reasoning and ignoring of the subtraction sign. From the answers that learners have come up with in their work it is also observed that there are a variety of strategies that they use on arithmetic equations. The most difficult part on strategies used to solve these equations is to differentiate them from errors committed. From learners' answers the strategies they used are the ones that lead to errors, for example, RLR, IOS, BEQ, and PCS. As much as these are errors they are also strategies. My argument is that an error results because of a certain strategy that is used which is not appropriate for the solution found. A summary overview of the errors and strategies noted in learners' work is given in summary table 4.8. The test and interview questions are put side by side, first the test then interview, hence the coding T//I in summary table 4.8. They apply also in the interview questions which are part of this big picture.

Table 4.8: Summary on learner responses for both test and interviews (T//I)

Question Items (T//I)	Noted Common Errors			
	RLR	IOS	BEQ	PCS
(a) 7 + ☐ = 15//5 + ☐ = 13	-	-	-	-
(b) 7 - ☐ = 15//5 - ☐ = 11	17//2	4//2	-	-
(c) ☐ - 7= -2	8	5	-	-
(d) ☐ + 3 = 8 - ☐//19 - ☐ = 10 + ☐	1	-	27//5	10
(e) 17 = ☐ - 6	-	13	-	-
(f) 13 = ☐ - ☐//15 = ☐ - ☐	5//2	6	-	-
(g) ☐ + 35 = 47 - ☐	-	-	30	4
Total Frequency	31//4	28//2	57//5	14

RLR, **IOS**, **BEQ** and **PCS** are errors and at the same time are strategies that learners were using in order to solve the arithmetic equations. **BEQ** was not expected at grade 9 level, and at this magnitude. Literature does refer to a particular case where grades 1-6 learners made an error in solving a problem like 9 + 5 = ☐ + 4 (Falkner, Levi and Carpenter (1999) cited in Essien and Setati, 2006). It is maintained that the learners put the answer immediately after the equal sign. They showed no regard to the added number (4). Considering the example given they would put 14 in the box as a solution. **PCS** is a new observation, for there is no mention of it in the literature.

The manner in which learners solved arithmetic equations is different from the way I expected grade 9s to solve them. Their strategies are more erroneous. Whilst one may expect strategies like: balance method, transposition and there is none visible.

5. *Chapter 5*

THE DATA ANALYSIS 2: *Strategies and errors in solving algebraic linear equations*

5.1 Introduction

Chapter 5 focuses on strategies used and errors made by learners in solving algebraic linear equations. The structure of analysis in this chapter is not different to that of chapter four. Test and interview data will be analyzed in parallel. In chapter four the emphasis was more on errors made than on strategies used by learners because the strategies were implicit. In this chapter, there is more emphasis on both strategies used and errors made. The nature of the variables used (letters for numbers) allows the strategy and error to be interpreted simultaneously and more explicitly. The strategies referred to in the literature together with conceptual framework in chapter two will guide my analysis in this chapter.

5.2 Codes used

The following codes listed below are used in this chapter and will be encountered in my data analysis. There are some codes that have been discussed and used in the previous chapters; and I am not going to discuss them again, for example right to left reasoning (RLR), ignoring of the operation sign (IOS) and other strategies and errors (O) which I said earlier on are not referred to in the literature.

- Conjoining (CJN), i.e. bunching together of letters and numbers when being added or subtracted.
- Insertion and removal of brackets (IRB).
- Interference of new knowledge (INK). This happens when old knowledge cannot be remembered but the new knowledge is used instead. Most of the times teaching and learning of mathematics is sequential, and if one cannot relate old and current knowledge there is danger of committing an error of interference.
- Ignoring of the minus sign as an operator (IOS).
- Equations changed into expressions algebraic (ECE). There are a number of cases where learners simply do their own things and instead of solving for the unknown end up simplifying expressions.
- Balance method (B), which is one of the strategies used that is referred to by the literature. For example, given an equation like $x + 3 = 7$ the expectation is that three must be subtracted to remain with x. And to balance the equation, the same value is subtracted on both sides.

- Transposition (T). Using the same example $x + 3 = 7$, three must be taken from the left hand side of the equation to the right so that x remains by itself. In this case three is said to be transposed. The opposite operation is effected since the transposed term will be on the opposite side.
- Trial and improvement (TI). For TI various values of x in $x + 3 = 7$ are substituted in the equation for x, searchingly until the left hand side is equal to the right hand side.

5.3 General analysis in algebraic equations

A lot of issues arose in the analysis of algebraic equations, unlike in the previous chapter where arithmetic equations were analysed. The errors that learners made increased as compared to those in arithmetic equations, e.g. changes of equations into expressions, inability to simplify like terms, conjoining of terms, as well as improper application of the distributive law where brackets are involved. Strategies in solving the equations that are referred to in the literature, like balance method, transposition are now explicit unlike in the previous chapter. Table 5.1 gives a summary and brief explanation for the strategies used by learners in the test questions. For detailed initial information see tables 13 to 26 in ***Appendix: C***. The results reveal that in as much as formal defined strategies like balance method, were used, learners do have their own ways of solving algebraic equations. Learners used recommended strategies (referred to in literature) and other strategies (those that literature is silent about). See table 5.1 below.

Table 5.1: Learner strategies on section 2 questions

Section 2 Item	Strategy Usage B	T	TI	O	Comments
(a) $x + 5 = 21$	**11**	**15**	**15**	**4**	More formal expected methods were used in this question.
(b) $2x - 3 = 11$	**22**	**15**	**3**	**12**	There is an increase in the usage of other strategies in $2x - 3 = 11$ compared to $x + 5 = 21$. Likely cause introduction of subtraction.
(c) $9 - x = 13$	**11**	**16**	**10**	**6**	Less other strategies in $9 - x = 13$ *than in* $2x - 3 = 11$, *and likely cause is the difference in the coefficients of x.*
(d) $2x + 3(x - 2) = 6$	**19**	**8**	**3**	**12**	The introduction of the bracket may be having an influence in this question. More other undefined strategies come in.
(e) $x - 5(x - 1) = 3$	**4**	**10**	**2**	**19**	Formal strategies have diminished greatly.
(f) $3x + 5 = x + 9$	**19**	**15**	**3**	**14**	Expected strategies resurface, but still there is more of other.
(g) $3x + 5 = x + 3 + 2(x + 1)$	**12**	**16**	**1**	**15**	There is still insignificant change from $3x + 5 = x + 9$ *to* $3x + 5 = x + 3 + 2(x + 1)$.
(h) $x + 2 - (-2) = x - 4$	**5**	**15**	**2**	**14**	Introduction of the negative seem to have more influence on strategy used. More than other undefined strategies crop in and fomal expected disappears.
Total usage of a particular strategy	**103**	**110**	**39**	**96**	

When considering the last row of table 5.1, in equations (a) and (c) there is less use of other (O) strategies. In these other strategies, it must be noted that they do not result in correct solutions because they are erroneous on themselves. They are so varied that I did not consider it worth coding them separately but to group them together as other. However it must be noted that errors emanating from these strategies include ECE, IRB among others. These are the only equations with only one variable in the equation, with coefficient of one. Trial and improvement is least used compared to all other strategies in all the equations except (a) and (c).

Table 5.2 highlights a summary of strategies that were used and errors committed in common equations for both the test and interview questions. These, also, are the equations where much attention will be focused in my analysis. In the first questions for both test and interview questions learners did not reveal many errors, but the errors became more with the progression of the questions from the first.

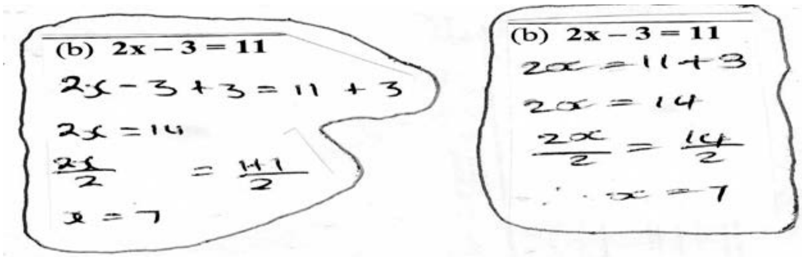
Table 5.2: Learner strategies and errors on comparable test and interview questions

Equation	Strategies	Errors
(a) $x + 5 = 21$ **(T)** (a) $4 + x = 23$ **(I)**	Some used balance method and some used transposition. These were the only two common methods that were used. At least one or two used substitution or trial and improvement. Already I hope you realise that the structure of these arithmetic and algebraic equations are similar, but the methods have differed.	There were no notable errors, except for slips
(b) $9 - x = 13$ **(T)** (b) $4 - x = 21$ **(I)**	Most of the time the time the operation sign was ignored. However even though this was the case some notable strategy was used like transposition or simple substitution like in extract 24.	IOS
(f) $3x + 5 = x + 9$ **(T)** (d) $3x + 2 = 8 + x$ **(I)**	In these equations transposition was the most used strategy, and very few learners simplified this equation to its end in the test.	A mixture of errors was evident in these problems like: CJN, INK as well as changing the equation into an expression.

(a) ***Strategies***

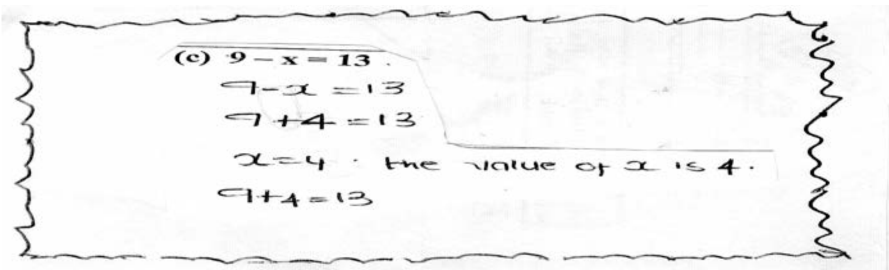
Consider figure 5.1 below, showing how two learners solved for x in the equation: $2x - 3 = 11$. Both learners have four lines of work, and they only differ in the first line of their work; where one uses balance method and the other transposition. Note that in the third line they both use the balance method, when they divided both sides of their equations by 2. In terms of strategy used the first learner used balance method (**B**) and the second used both balance and transposition (**B & T**). Very few learners used only one method and very few used any three or four, the majority used a combination of any two. The use of balance method combined with transposition was the most common used, and then trial and improvement combined with any other method was second. See detailed information in the tables that have been referred to in ***Appendix: C***. What is evident from learners work is that the formal strategies mastered are balance method and transposition.

Figure 5.1: Examples of learners' strategies



In figure 5.2 a combination of a strategy and error are clearly displayed by a learner. This learner has solved the equation and finally arrives at a conclusion that ***x*** is equal to four. Trial and improvement seem to be the strategy used because in line 3 of the equation we see that 4 have been substituted for ***x***. The learner then adds nine and four which equals thirteen. This learner further shows his/her argument for the correctness of the solution. However, there is an error in the learner's solution because the equation states that $9 - \boldsymbol{x} = 13$, but the subtraction operation is nowhere to be seen in the solution process. See figure 5.2.

Figure 5.2: Example of a strategy and an error

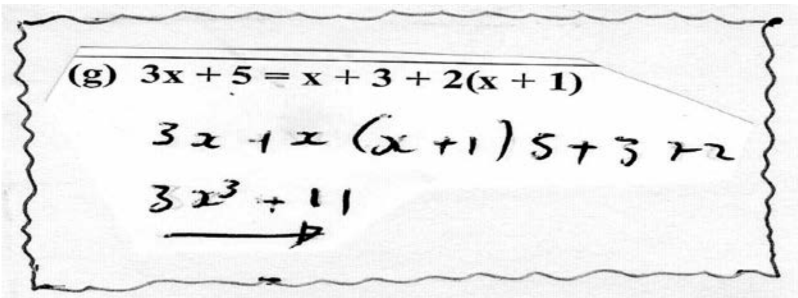


It must be acknowledged that there is an error of ignoring the operation sign in this learner's work. The value of *x* is given as 4 when in actual fact it must be -4.

(b) ***Errors***

Figure 5.3 below shows how seriously errors can affect the solution of equations in algebraic equations compared to arithmetic equations. Right from the onset this learner embarks on separating variables from numbers as all terms in ***x*** are written first and then numbers later. The 5, 3 and 2 are respectively taken to the extreme side of the bracket. The whole process is non-mathematical and secondly the equal sign has been left out implying that the equation has been transformed into some other expression. The sense of solving for ***x*** is not there at all implying that the equivalence view of the equal sign is not present in this learner. Further the learner simplifies like terms and gets the final solution/expression, which is $3\boldsymbol{x}^3 + 11$. This learner seem to be inclined to adding terms together for it is likely that 1 + 5 + 3 + 2 = 11 can be a coincidence. So in like manner $3\boldsymbol{x} + \boldsymbol{x} + \boldsymbol{x} = 3\boldsymbol{x}^3$, which in this case can be termed interference of new knowledge because $3\boldsymbol{x} + \boldsymbol{x} + \boldsymbol{x}$ is supposed to add up to 5x. Three $\boldsymbol{x}$ cubed ($3\boldsymbol{x}^3$) is likely to result from the product of the above and not the sum. When focussing on the strategy used by the learner in this case it can be classified as other (O) or own, for exceptional cases. See the learner's work in figure 5.3 below where the equation has been changed into an expression (ECE).

Figure 5.3: Error in a learner's work



It is possible that the nature of the equation determines the error that the learner makes, because if instead of $\boldsymbol{x}$**'**s there were boxes, the learner would have inserted numbers of some sort. In arithmetic equations, in chapter four above, all given solutions were numbers.

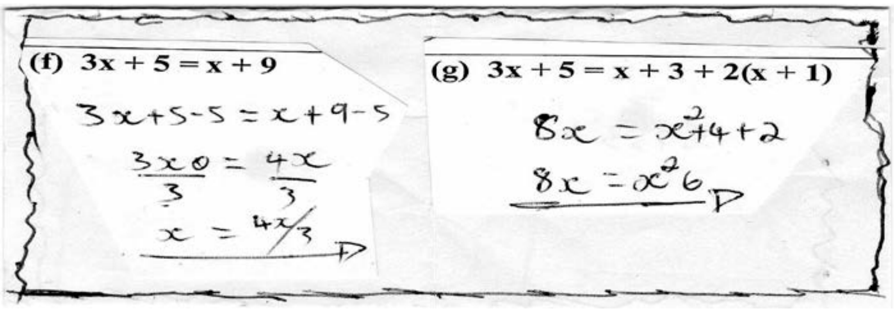
5.4 Errors in learners' work

There are some errors that were noted in arithmetic equations, in chapter four, which are also found in this chapter. CJN and INK were introduced as additional errors that were noted in algebraic equations and are discussed here using learners' extracts to illustrate some of these findings.

(a) ***Conjoining (CJN)***

Conjoining is one of the errors that was made and had a serious effect on the solution of equations. Figure 5.4 shows the impact of CJN in one of the learners' extracts. In (f) $3\boldsymbol{x}0$ and $4\boldsymbol{x}$ are points where the learner conjoined his/her work when simplifying most likely expressions like $3\boldsymbol{x} + 0$ and $\boldsymbol{x} + 4$. See figure 5.4 below. Also, in (g), see $\boldsymbol{x}^26$.

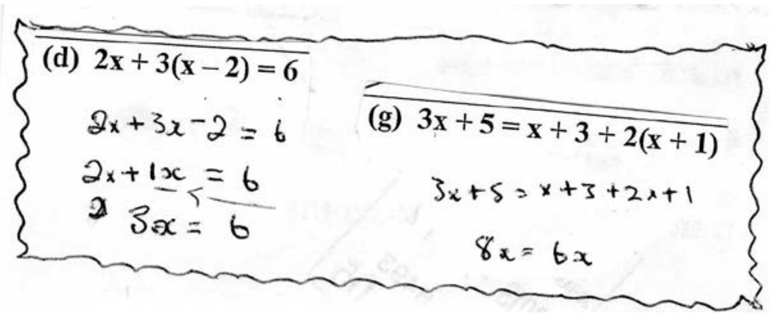
Figure 5.4: Effects of conjoining 1



Considering the solutions to both of the equations above; we see that there is no precise solution to x as it still appears on both sides of the equations. In (f) after simplifying 5 – 5 = 0 (on the left hand side), the 0 together with $3x$ are conjoined into $3x0$. On the right hand side it seems like 9 – 5 resulted in 4 so that x + 4 were then conjoined into $4x$. Similarly in (g), $8x$ resulted from conjoining $3x + 5$ and $x^2 6$ resulted from conjoining $x^2 + 4 + 2$. This figure will be referred to later when discussing other errors that are observed.

Let us have a look at another learner's work on conjoining in figure 5.5. In (d) conjoining seen in the simplification of $3x – 2$, where $1x$ is the result. The learner does not differentiate between the two terms $3x$ and 2. And it is most likely that this learner considered the coefficient of x, i.e. 3 subtract 2 and obtains 1. This 1 was then combined with an x resulting in $1x$ as seen. In (g) the right-hand side of the equation was simplified first in line 1(of the learner's work). In line two $8x$ and $6x$ are both conjoined expressions from the line immediately above. Again, as in the first example above, both learners do not seem to be finding the value of x but simply simplifying the expressions on either side of the equations.

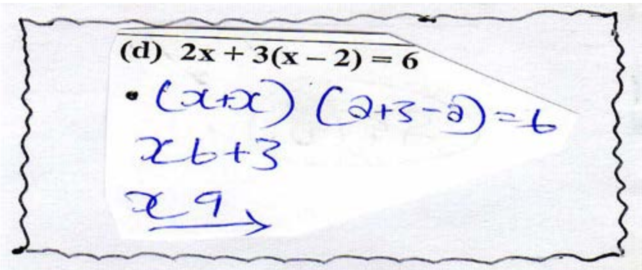
Figure 5.5: Effects of conjoining 2



(b) ***Brackets: Insertion and removal of brackets (IRB)***

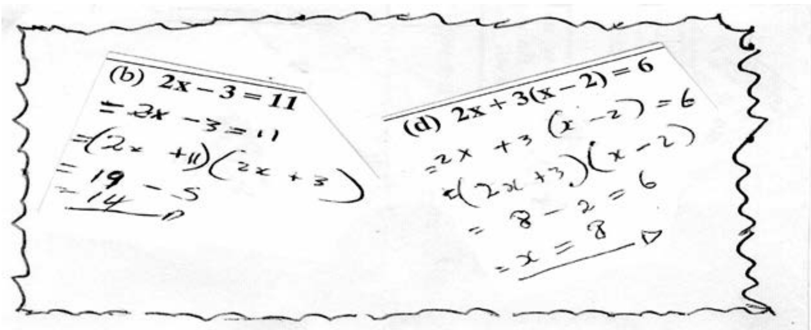
There was a lot of evidence of learners struggling with brackets. Some learners would introduce brackets in their work whilst others faced problems of removing them. Besides other things, introduction of brackets alters the structure of expressions (Gallardo and Rojano, 1990); and many a time when learners introduces these brackets do not take cognisance of the signs outside the brackets (Vlassis, 2004). Below are some examples from the learners' work in which learners introduced some brackets in their work. In figure 5.6 there is both bracketing and conjoining. The work was made more difficult by introducing these brackets, which led to errors. Further, the introduction of brackets complicated the equation to the extent that it was changed and the learner ended up no longer solving for x. For example, the learner separated letters from numbers and grouped them separately, i.e. in line 1. In line 2, my assumption is that (x + x) was simplified to x and (2 + 3 – 2) to 3, then 6 from the right hand side was conjoined with the x resulting in x6 + 3. Eventually, the expression, **x**9 is the result of adding x6 and 3, with the 9 being a result of adding 6 and 3 then conjoining the result with x. See the learner's work in figure 5.6 below.

Figure 5.6: Insertion of brackets 1



Another example is in figure 5.7 where the learner introduces the brackets. What actual happens is that once brackets are introduced in one's work the work becomes more complicated even for the learner to continue in a systematic way. See figure 5.7.

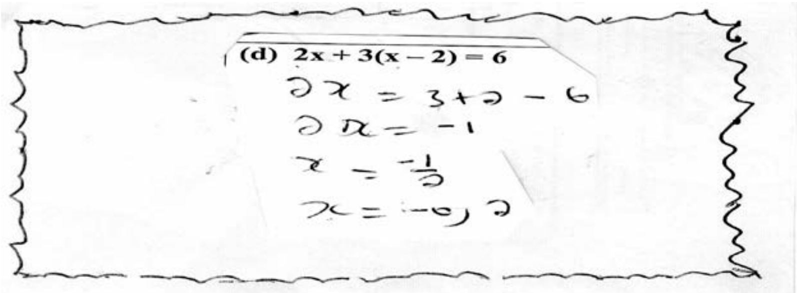
Figure 5.7: Insertion of brackets 2



This is the work for the same learner and it seems to be consistent in as far as insertion of brackets is concerned.

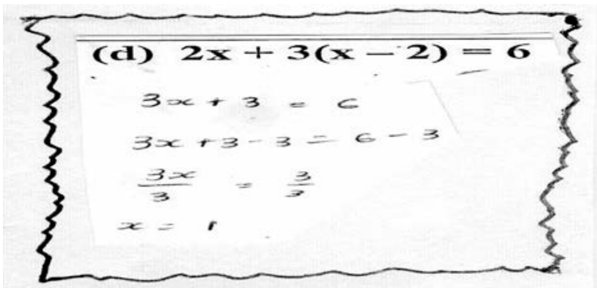
Lastly, the issue of removing brackets led many to produce erroneous solutions just because the brackets could not be removed properly. Figures 5.8 and 5.9 below are typical examples where learners fail to apply the distributive law (Siegler, 1998 cited in Araya et al, 2010). Learners are seen to be struggling and having difficulties with the removal of brackets. For example in figure 5.8, it seems like this learner simply took 3 and 2 to the other side and removed the brackets. There are no mathematically accepted methods of removing brackets observed. The solution to this equation is arrived at by taking into cognisance proper strategies and procedures, but points to the fact arithmetic skills are lacking. Half as a decimal is 0.5 not 0.2.

Figure 5.8: Removing brackets 1



The example in figure 5.9 is quite interesting in that we now see 3***x*** and 3, without knowing precisely how it was done. Through inference one may say that ***x*** – 2 was conjoined into an x which further added to 2***x*** resulting in 3***x***, and the three outside the bracket was left alone. These are just possibilities and there is no learner to explain how it was solved. The procedures thereafter are mathematically correct, but the problem occurred with the removal of the bracket. See figure 5.9 below.

Figure 5.9: Removing brackets 2

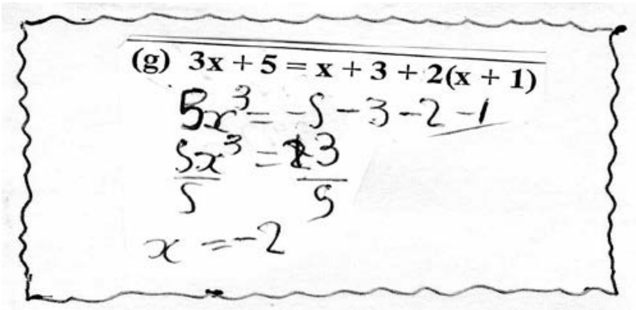


(c) ***Interference of new knowledge (INK)***

New knowledge has been found to be influential in errors that learners make. Stacey and McGregor (1997)'s findings conclude that learners bring in new knowledge to their work and it destabilises the old one which would not be secure. This is likely to happen if the learners have not grasped the basic concepts well enough. The errors made include exponents and I would like to divide them into two groups. The first will be for those equations with brackets and the other for those without the brackets. Figures 5.10 to 5.13 have brackets and all learners display the error of writing the exponents with the variables.

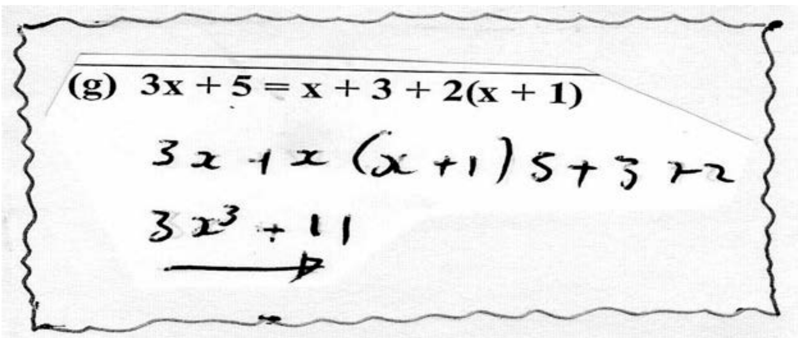
In figure 5.10 I had to follow closely what this learner was doing. This was but the beginning because as you would notice in (g) there are three terms in x and the result is that now there is a cubic term eventually. The issue here is that this learner is aware of multiplication of letters for numbers. For example adding x and x simplifies to $2x$ but multiplying an x and x simplifies to x^2, so is $x + x + x = 3x$ but multiplying the three xs together results in x^3. To me now there is some confusion of addition of algebraic terms together with multiplication of algebraic of like terms. See the learner's work.

Figure 5.10: Error of exponents (INK) 1



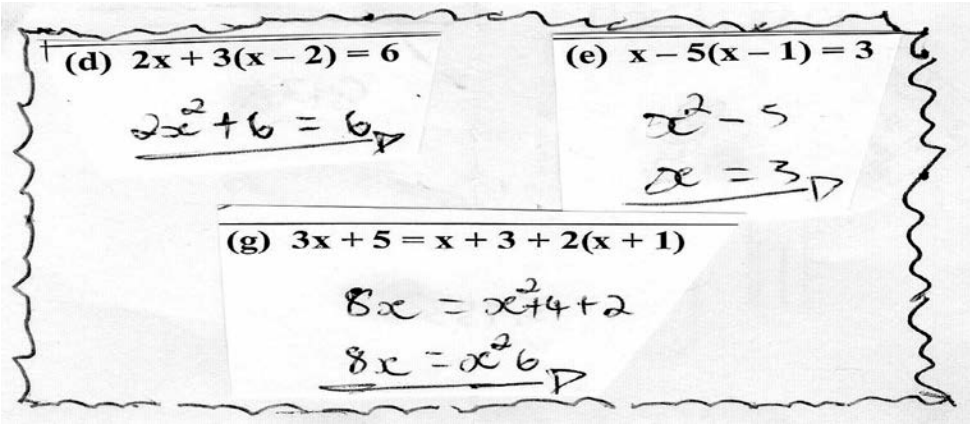
The learner in figure 5.11 is no longer solving an equation from the work displayed. Again there are three terms with a variable x. All the terms in x seem to have been multiplied and all numbers added together. The three terms in x eventually are cubed whereas all the numerical terms add up to eleven. What happens with brackets has already been addressed in the section on brackets, as the trend is the same. See the learner's work in fig. 5.11.

Figure 5.11: Error of exponents (INK) 2



In figure 5.12, all the extracts have been grouped together and reflect the work of one learner. This learner squared or multiplied the x's most likely because of the brackets in these problems. Consider (g) and you will observe that the squaring has been done only on the right-hand side, where the bracket is. Interestingly, again, is the fact that this learner is not concentrating on solving the equations but on simplifying the expressions on each of the sides separately and just leaving them in their simplified form. See figure 5.12 below.

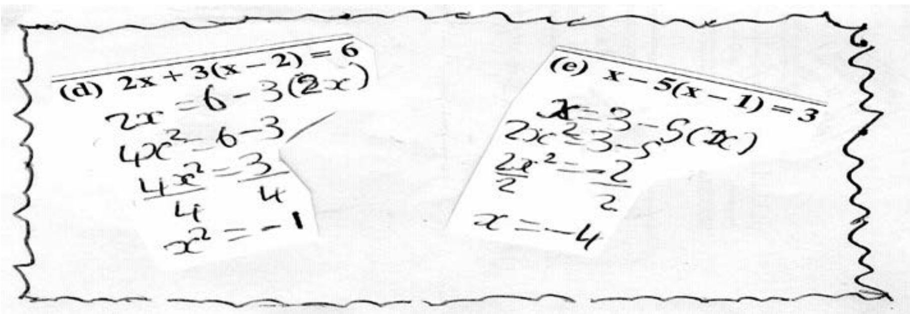
Figure 5.12: Error of exponents (INK) 3



In equations (d) and (g) the learner simplified the expressions on each of the sides and did not further try to express x as being equal to something as was done in (e).

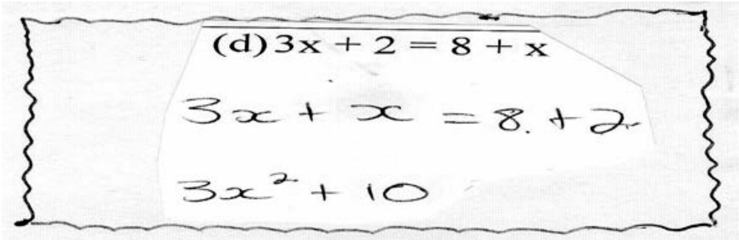
In figure 5.13 the same learner has solved both equations (d) and (e). First we can see that the variable $\boldsymbol{x}$ is on the same side of the equations. However, on removing the brackets one of the $\boldsymbol{x}$**'**s is transposed to the other side. This is consistently done in both equations. On the second line of the learner's work, when like terms are grouped or collected, is where there is this issue of exponents. The learner most probably, multiplied or added these terms. The likelihood of being added is high because $2\boldsymbol{x} + 2\boldsymbol{x}$ will result in four and $\boldsymbol{x} + \boldsymbol{x}$ will also result in two, unlike when being multiplied the consistency breaks. In the final solution for (d) it seems like the fours divide, leaving $\boldsymbol{x}^2$. In (e), the two in the denominator divides or cancels both the coefficient and exponent of $\boldsymbol{x}^2$. This is what learners are displaying. See figure 5.13.

Figure 5.13: Error of exponents (INK) 4



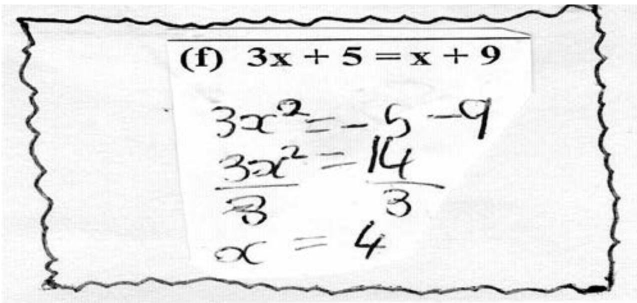
In figure 5.14 and 5.15 a different story is written. There are no brackets to indicate that the exponents are due to multiplication. The only possible cause for the error is the new knowledge on exponents. For the learner in figure 5.14, it is possible that the learner mistook the operation plus for multiplication in error. There are other errors here too, like the law of commutativity because of the grouping of like terms on opposite sides of the equation. Again the equation ceases to exist. See the learner's work figure 5.14 below.

Figure 5.14: Error of exponents (INK) 5



In figure 5.15 the learner seems to have transposed like terms to one side with numbers on the right and variables on the left. Again the concentration is on the x's and not on the coefficients; hence the term $3x^2$ was obtained. On the right hand side in the first line negatives have been introduced, but in the second line there is a positive fourteen. It is likely here too, that the error of double negative has been made. The fact that there is a negative five and subtract nine is reason enough for the learner to say double negatives give a positive. Finally, the solution is given as shown in figure 5.15 below.

Figure 5.15: Error of exponents (INK) 6



(d) ***Ignoring of the minus sign as an operator (IOS)***

Ignoring of the operation sign is the most common error in algebraic equations as compared to right to left reasoning in arithmetic equations. Refer to figure 5.2; there is no indication of a mistake that the learner was saying the value of ***x*** is 4. So to get the value, 13, the operation subtraction was ignored or treated as a positive. From these extracts it is clear that learners make a lot of errors in their work due to some misconceptions as well as individual learner perception.

5.5 Question by question analysis

This part considers how the learners performed in each question, paying special attention to the strategies used and the proficiency of learners in their work: which I will consider to indicate the item difficulty in each case. Errors made will also be analyzed in this section. Extracts from learners' work will be used as examples. However before getting to the question by question analysis *per se* I would like to clarify the issue of strategies and errors by considering some examples from learners' work.

(a) ***Test and interview questions:*** $x + 5 = 21$ and $4 + x = 23$

In this question all three known strategies (balance method, transposition and trial and improvement) were used almost an equal number of times. Learners could subtract 5 from both sides of the equation, simplify and end up with the desired solution. Evidence also shows that learners could easily transpose 5 into the other side, where the number 21 is, and proceed to simplify. As many learners, as those who used transposition, concluded the value of x by first substituting that value to show that it adds to the value on the right-hand side. It is also important to note that a learner is able to use more than one method in solving one problem. There are four cases in which learners used other strategies other than the known ones. In terms of proficiency, 38 learners were deemed to be proficient.

Table 5.3: Summary for questions: $x + 5 = 21$ and $4 + x = 23$

	Test Item				Interview Item			
	$x + 5 = 21$				$4 + x = 23$			
	B	T	TI	O	B	T	TI	O
Total	11	15	15	4	1	4	2	1

(i) *Errors in interviewed learners' work, and learner explanations*

There were no notable errors in the first question; from both the test and interview items. Consistency in the test and interview solutions shows that learners understood their work.

(b) ***Test question:*** $2x - 3 = 11$

The use of the trial and improvement strategy declined greatly in this question, and other learner methods increased. This question posed challenges to the learners who opted for their own methods, which I found difficult to follow. However, a large number used the formal methods of balance and transposition, especially balance method, see table 5.2 below. Compared to $x + 5 = 21$ above, there are more learners who used the balance method in this question than in the first. As for transposition 15 learners still used it as their strategy. The proficiency of learners dropped from 38 to 25. This problem posed some challenges to the learners and many did not solve for x. See summary table 5.4.

Table 5.4: Summary for $2x - 3 = 11$

	Test Item			
	$2x - 3 = 11$			
	B	T	TI	O
Total	22	15	3	12

(c) ***Test and interview questions:*** $9 - x = 13$ and $4 - x = 21$

There is a drop in using other methods, a drop in using the balance method, an increase in trial and improvement method and another increase in transposition. Transposition seems to be the most favoured strategy among all learners. Learners' proficiency decreased by one. In the equation $2x - 3 = 11$, there is only one learner who demonstrated total proficiency and there were two learners in the equation $9 - x = 13$. See the summary in table 5.5.

Table 5.5: Summary for questions: $9 - x = 13$ and $4 - x = 21$

	Test Item				Interview Item			
	9 - x = 13				4 - x = 21			
	B	T	TI	O	B	T	TI	O
Total	11	16	10	6	1	4	2	1

The most used strategy by the interviewed learners is transposition.

(ii) *Errors in interviewed learners' work, and learner explanations*

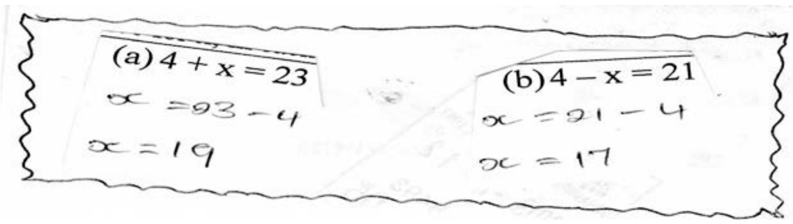
The range of errors for the five interviewed learners were: ignoring of the operation sign when it is negative (evident in Vicky's work), right to left reasoning (the second part of Violet's work and first part of Kitos' work), other errors emanating from learners' perception and understanding (including slips) and leaving work in un-simplified form (Miriam's solutions). The errors that we observe here are similar to those of section 1 questions. The difference that we see between the sets of questions, the second and the first, can be attributed to the simple reason that the latter has a subtraction sign. Learners, that I observed, seem to have a fear of subtraction and negative signs, in general. The last table, table 18, compares the third pair of questions. This question does not have a subtraction sign but it was the question in which learners performed the worst in so much so that learners had to abandon formal strategies and ended up using their own (as constructivists) which suited them best. When asked to defend their work they could not. However, it has to be noted that pressure increased when they were faced with equations with negatives. Table 5.6 below gives the solution and error analysis of the test and interview equations.

Table 5.6: Error analysis 1

	Test Question (T)	Interview Question (I)	General Comments
	9 – x = 13	*4 – x = 21*	
Vicky	x = 4	x = 17	IOS: in both (Test & Interview)
Violet	x = 13x	x = 25	RLR: in Interview
Kison	2x = 14	x = -17	O: did not solve for x in Test
Kitos	x = 22	x = 4	RLR: in (Test) & lack of sound reasoning (Interview)
Miriam	-x = 4	-x = 17	Did not simplify for x in both (Test & Interview)

The following are Vicky's ideas on how she solved the equation $4 - x = 21$. The equation is set side by side with $4 + x = 23$ that Vicky answered in the test (as shown in figure 5.16). Vicky said, *"So, what I did is, I took all the variables to, to my left hand side, and the numbers which are the constants to my right hand side. So, I putted the x this side and* *twenty-three this side because for it to be positive when it comes to the right hand side it would be a negative. Then I subtracted and keep the sign of the bigger number which is nineteen*". The method used is transposition, which is gauged from the first sentence. In the second sentence the only number moved is four, although she talks of twenty-three, which seems to be a slip because the problem has the number 23 and not four. The point I want to draw our attention to is that the sign did not change but four is simply being subtracted on the other side since it was being added from the side it is transported from. But what is the learner saying? She is saying that the sign has changed. I asked Vicky about the second problem and she further said: "*Even here I also did the same. I putted x here, twenty-one will remain as twenty-one, and then because four is positive when it comes to the right hand side it will be a negative. Then I subtract them because, because a positive and a negative subtracted give the sign of a bigger number which is …*"

Figure 5.16: Vicky's ideas



On transposing the four the (-***x***) became ***x***, and if Vicky had checked for her solution by substituting back into the equation most probably she would have realised that the value seventeen does not satisfy the equation. However, this is not surprising as we have seen that many learners just ignore the negative. I determined in the third problem that the learners have a problem when working with many terms. They easily gave up when they saw that the problem was long and there were a mixture of operations which gives them a problem especially when it has to do with subtraction.

(d) ***Test and interview questions:*** $2x + 3(x - 2) = 6$ and $3(x + 2) = 18$

There was a decline in the use of both transposition and trial and improvement methods. Balance method was increased, so was learners' own methods. The most probable reasons for these changes were the introduction of brackets. I suspect that the focus of some learners changed and they were more intent on removing brackets which was a challenge judging from the learners' performance as well as in non-formal strategies. See table 5.7

Table 5.7: Summary for questions: $2x + 3(x - 2) = 6$ and $3(x + 2) = 18$

	Test Item				Interview Item			
	2x + 3(x - 2) = 6				3(x + 2) = 18			
	B	T	TI	O	B	T	TI	O
Total	19	8	3	12	2	4	2	1

(e) ***Test question:*** $x - 5(x - 1) = 3$

All the recommended methods of solving equations have diminished in this problem. What increased was the learners' own ways of solving equations. Nineteen learners used their own methods; some of which I found difficult to understand. Learner proficiency decreased drastically in this equation. The subtraction involved must be the one that has an impact on the performance as a whole. There are only four learners who were rated as proficient in this problem. See table 5.8 for summary.

Table 5.8: Summary for $x - 5(x - 1) = 3$

	Test Item			
	x - 5(x - 1) = 3			
	B	T	TI	O
Total	4	10	2	19

This summary clearly shows that learners do have a lot of problems with the subtraction of terms.

(f) ***Test and interview questions:*** $3x + 5 = x + 9$ and $3x + 2 = 8 + x$

The formal methods were regained, trial and improvement was reduced to three learners. Learners' own methods were also reduced as they became more formal. There was improvement in the proficiency of learners. Consider the summary table 5.9 for this information.

Table 5.9: Summary for questions: $3x + 5 = x + 9$ and $3x + 2 = 8 + x$

	Test Item				Interview Item			
	3x + 5 = x + 9				*3x + 2 = 8 + x*			
	B	T	TI	O	B	T	TI	O
Total	19	15	3	14	2	5	1	1

(iii) *Errors in interviewed learners' work, and learner explanations*

The equation $3x + 2 = 8 + x$ was a challenge to the learners. Asking the learners questions to express themselves seemed to make the situation worse as it seemed like I was condemning them for what they had done. Violet's test and interview work was different. When asked to explain her work she said that she collected like terms to one side, and that she just put like terms together. The strategy of transposing did not seem to be of use to her for she was just collecting like terms the way she would treat simple algebraic expressions. Mathematically, she used the commutative law of addition and grouped terms as in algebraic expressions. When I asked about the squared term, she said that she added the ***x'***s together.

Kison's test and interview work was also different. Kison when asked to explain his work, mentioned that the first solution ($x = 1$) is for x on the right-hand side and the second ($x = 9$) is for x on the left-hand side. He assigned x the value 1 on the right hand side, and then added the 1 to 8 giving 9. Then he attributed this sum to the value of $\boldsymbol{x}$ on the left hand side. I further asked him how that was possible, because that was something new to me. He could not explain explicitly and he admitted that he was confused. He said that the problem confused him, in other words he was saying that he did not know what to do.

Kitos confessed that x's confuse him and he does not know what to do. When I asked him to explain to me what he did, he said that he collected like terms together on one side and numbers on the other. From there he simplified them. Asked about where the x-squared came from, he replied that he added three $\boldsymbol{x}$ and $\boldsymbol{x}$ (e.g. $3\mathbf{x} + \mathbf{x}$). He did not multiply but added the x's together. I gave him another situation and asked what he would do if he was to add two x and x ($2\boldsymbol{x} + \boldsymbol{x}$). He said that he was going to get the answer two x-squared ($2\boldsymbol{x}^2$). What I noticed is that, besides other errors, Kitos could not differentiate between multiplication and addition of like terms in algebra.

Miriam gave up and said it was confusing. She then took all numbers to one side leaving x on the other side. I asked about the other x and she said that she left the other x because they were just the same. In other words she is saying that she was solving for x and all she needed was to simplify numbers and that the simplification was equal to x. For example from '$3\boldsymbol{x} + 2 = 8 + \boldsymbol{x}$' she held one $\boldsymbol{x}$ constant and transposed 3 an 2 such that she had $\boldsymbol{x} = 8 - 3 - 2 = 3$. Table 5.10 provides a summary of the solutions and errors in both the test and interview questions.

Table 5.10: Error analysis 2

	Test Question	***Interview Question***	
	3x + 5 = x + 9	***3x + 2 = 8 + x***	**General Comments**
Vicky	$\frac{3x}{3} = \frac{4}{3}$	$\frac{4x}{4} = \frac{6}{4}$	**Left un-simplified in both cases**
Violet	**x = 4.8**	**3x² + 10**	**Slip in (Test) & IKN in (Interview)**
Kison	**17x²**	**x=1 x=9**	**INK in (Test) & Lack of reasoning in (Interview)**
Kitos	**3x+x=5+9**	**3x² = 10**	**Commutative property in (Test) & INK in (Interview)**
Miriam	**x=4.8**	**x=3**	**Slip in (Test) & C in (Interview) through TI method**

(g) ***Test question:*** $3x + 5 = \boldsymbol{x} + 3 + 2(\boldsymbol{x} + 1)$

One would expect that the performance should be similar to the first problem in (a), since it is only addition of terms. Formal methods (balance and transposition) balance with learners' own strategies with the exception of trial and improvement which is negligible. This problem, also, has some brackets which we have seen has a big impact by the performance displayed by learners. And further, despite being an equation, this problem has many terms; and according to Vlassis (2004) confusion is easy in expressions with too many terms ending with the mixing up of operational signs. I therefore, agree with what Doreen et al (2002) say in that algebraic competence cannot be avoided. What I understand is that for one to be able to solve an equation strategically with no errors made, one has to be flexible in dealing with algebraic expressions. For example one must be able to simplify algebraic expressions without difficulty so that the main focus is in solving equations rather than simplifying expressions. Concerning proficiency there was a drop in $3\boldsymbol{x} + 5 = \boldsymbol{x} + 3 + 2(\boldsymbol{x} + 1)$ compared to $3\boldsymbol{x} + 5 = \boldsymbol{x} + 9$ fifteen learners were considered proficient and in this case only ten are said to be proficient.

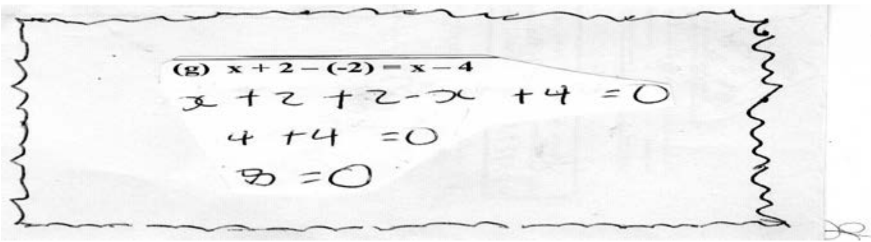
Table 5.11: Summary for $3\boldsymbol{x} + 5 = \boldsymbol{x} + 3 + 2(\boldsymbol{x} + 1)$

	Test Item			
	3x + 5 = x + 3 + 2(x + 1)			
	B	T	TI	O
Total	12	16	1	15

(h) ***Test and interview questions:*** $\boldsymbol{x} + 2 - (-2) = \boldsymbol{x} - 4$ and $\boldsymbol{x} + 3 = \boldsymbol{x} - 3$

In general learners did not do well in this problem. Some learners would work it out in the first few lines. When writing the value of ***x,*** they lose the variable and end up with some numbers. For example, see figure 5.17 below where the learner simplified the ***x'***s to zero and remained with 2 + 2 + 4 on the left hand side. Further, this was simplified to 4 + 4, which eventually resulted in 8. At this stage the learner just left the solution at 8 = 0.

Figure 5.17: Learner's work on $\boldsymbol{x} + 2 - (-2) = \boldsymbol{x} - 4$



The learner simplified the left-hand side correctly into x + 2 + 2 and transposed all the terms from the right-hand side to the left fluently. On further simplification we can see that x subtracted x correctly and the x was no more, leaving the addition of 2 and 2 and 4 which added up to 8. Now lastly, 8 is equal to 0, though mathematically eight and zero are not equal. Learner proficiency was low in this equation as well. Table 5.12 is a summary of learners' performance in this question, as shown below.

Table 5.12: Summary for questions: $x + 2 - (-2) = x - 4$ and $x + 3 = x - 3$

	Test Item				Interview Item			
	x + 2 - (-2) = x - 4				x + 3 = x - 3			
	B	T	TI	O	B	T	TI	O
Total	5	15	2	14	2	3	2	1

Comparing $x + 2 - (-2) = x - 4$ and $x + 3 = x - 3$ note that $x + 3 = x - 3$ is a simplified version of a similar equation because $x + 2 - (-2) = x - 4$ simplifies to $x + 4 = x - 4$.

5.6 Ideas from interviewed learners

One of the questions that the interviewees were asked was to define an equation. There were mixed ideas depending on the learner's efficiency in expressing him/herself. Some learners could not express themselves clearly though. Before considering some points from learners let us consider the summary in the table below.

Table 5.13: Strategies and proficiency of learners

Section 2 Items	Strategy Usage				Proficiency Yes/No
	B	T	TI	O	
$x + 5 = 21$ (T)	2	3	0	0	5/0
$4 + x = 23$ (I)	0	3	1	1	5/0
$9 - x = 13$ (T)	3	2	0	0	4/1
$4 - x = 21$ (I)	0	3	1	1	2/3
$3x + 5 = x + 9$ (T)	3	5	0	0	3/2
$3x + 2 = 8 + x$ (I)	1	4	0	1	1/4
Frequency	9	20	2	3	

Taking the first two items: $x + 5 = 21$ and $4 + x = 23$ we see that the key strategy which was used is transposition. Balance method was used in the test question but none of it is seen in the interview questions except that one leaner had to employ trial and improvement and another other strategies. The same is true for the other questions. One notable thing for the test questions is that all simply used basically two methods, that is, the balance and transposition only. During the interviews, however, one or two learners became hesitant and had to include these other strategies. Basically, the questions under analysis are not different except for the positioning of the numbers and variables like in the first two on the left-hand side of the equations and the last two on the right-hand side of the equations. In as far as the strands are concerned; it is my observation that all learners have conceptual understanding. They knew what they were to do in the equations in general indicating that when solving for an unknown you will be looking for a value that can be used in place of the variable. Here is what Vicky had to say when I asked her what she understood by an equation, and what was to be done. I quote "*They want you to find the value of x. …, but sometimes they can use different alphabets like y. If they say solve for y they simple means you must find the value of* *y*". For someone with such a concept of what an equation is you would expect not less than a strategically competent learner. From the table we can see that these learners were strategically competent in all the problems except one learner in the last pair of questions. In terms of demonstration and confidence in their work they were more confident in some questions than in others. I, also, observed that procedural fluency was the least in the last two interview questions. For example only one learner demonstrated some procedural fluency in the last question, and that is the same learner who was ranked proficient in that equation. All learners were proficient with the first questions and one or two lacked as they continued with their work. That is the trend seen from the table too. Table 5.14, below, provides a picture of what learners considered to be solutions to their equations.

Table 5.14: Interviewed Learner solutions

	Test Question	***Interview Question***	**Test Question**	***Interview Question***	**Test Question**	***Interview Question***
	$x + 5 = 21$	$4 + x = 23$	$9 - x = 13$	$4 - x = 21$	$3x + 5 = x + 9$	$3x + 2 = 8 + x$
Vicky	$x = 16$	$x = 19$	$x = 4$	$x = 17$	$\frac{3x}{3} = \frac{4}{3}$	$\frac{4x}{4} = \frac{6}{4}$
Violet	$x = 16$	$x = 19$	$x = 13x$	$x = 25$	$x = 4.8$	$3x^2 + 10$
Kison	$x = 16$	$x = 19$	$2x = 14$	$x = -17$	$17x^2$	$x=1$ $x=9$
Kitos	$x = 16$	$x = 19$	$x = 22$	$x = 4$	$3x+x=5+9$	$3x^2 = 10$
Miriam	$x = 16$	$x = 19$	$-x = 4$	$-x = 17$	$x=4.8$	$x=3$
Correct	5	5	0	1	0	1

It is important to note that learners did not have any difficulties with the first pair of equations for the simple reason that it is simple and has one variable. The solution can also be inferred from the fact that it is simple addition. The second pair of equations is like the first pair. They are simple short equations; and have only one variable on one side. Then why are the solutions so different and performed erroneously? The problem lies with the subtraction operation. When asked about their solutions and the strategies, learners did show that whenever a negative sign appeared, they think of a negative and a positive giving a negative, a negative and a negative giving a positive. They, also, go further and talk of the difference between numbers and taking the sign of the bigger number. These are cues that have come to stay in their minds but are sort of mixed up due to lack of understanding of when to use what at what time and when what has happened.

5.7 Summary

In conclusion, learners have a challenge with algebraic expressions, to start with. This challenge is carried over to equations, hence the uncertainty observed in learners during the interviews. Many were hesitant when saying what they knew or answering the questions asked. Strategies that learners use are a mixture of mathematically sound strategies that are found in literature and other strategies due to uncertainty in learners themselves. There are many problems that learners encounter with their algebraic simplification. Learners have displayed in their work a variety of strategies used in solving linear equations, which are both formal and informal. Using the conceptual framework, it has been observed that learners were not consistent in the sense that they were more proficient in some questions and less proficient in others. There was a lot of variation in their work depending on the nature of the question, for example, number of terms involved, operations involved and the variable term whether it is being subtracted or even its position in the equation. Equations with brackets and those with subtraction operators were more problematic for the learners.

Chapter 6

CONCLUSION

6.1 Introduction

The main focus of this study was to investigate strategies and errors in solving arithmetic and algebraic linear equations by grade 9 learners. Two research questions were posed from the outset: (1) What strategies do grade 9 learners use in solving arithmetic and algebraic linear equations? (2) What errors do grade 9 learners make when they solve arithmetic and algebraic linear equations? The study involved a test that was written by 45 learners, and interviews that were conducted with five learners. In this chapter I discuss my findings of the research to answer the above research questions as well as to make recommendations which I think will be beneficial to mathematics teachers as well as the mathematics education research community.

6.2 Findings

The following are the main findings in analysing grade 9 learners' work on arithmetic and algebraic linear equations. There are many strategies that learners used to solve both arithmetic and algebraic equations. In some cases the strategies used are the same, such as trial and improvement in simple cases. In some cases the strategies are different, for example, balance method and transposition were more frequent in algebraic equations than in arithmetic equations. Besides strategies referred to in the literature, there were numerous ways that were displayed by learners in their endeavour to find solutions to the equations. The range of strategies used is also linked to the various errors that learners made in their attempts to come up with the correct solutions to their work. Likewise there were common errors that were made in arithmetic algebraic equations, such as right to left reasoning and ignoring of the operation subtraction. There were also unique errors to both arithmetic equations and algebraic equations. Examples of such were breaking the equation into two parts as well as proving correctness of the solution in arithmetic equations, and several such as in working with algebraic equations such as insertion and removal of brackets, conjoining and changing equations into expressions. More detail is provided in the discussion that follows.

(a) ***Strategies in arithmetic and algebraic equations***

Strategies used by grade 9 learners were different in arithmetic and algebraic linear equations. Arithmetic equations by nature required numerical solutions, which could easily be inferred through trial and improvement method. That is the reason I was able to infer what transpired from the solutions written by learners. In algebraic equations it was the opposite. Some learners were not able to reach the solution stage of the equation, i.e. $\boldsymbol{x}$ = some value. However, the strategy used could be inferred and followed. It appears that the use of a box as a variable had a substantial influence on the method used in arithmetic equations. Similarly, the use of a letter as a variable also led learners to use different approaches. This was observed from the way learners solved these problems in the test and the way those who were interviewed responded to the questions asked. For example when asked to find the number in the box, a learner just wrote that number in the box without evidence of working it out, e.g. $5 + \square = 13$. The same individual given $4 + \boldsymbol{x} = 23$, showed some working either by transposing or by using the balance method. For example, see similar examples in figure 4.1 and figure 5.16a. What was anticipated from the onset is not what learners showed, particularly in arithmetic equations. Based on literature and the conceptual framework the anticipated strategies were trial and improvement, transposition, balance method as well as cover up methods. In arithmetic equations the methods used could not easily be inferred when the answers were correct because the learners simply put the correct answers in the box, though the instructions were specific that they should show how they got their answers. Where the answers were incorrect the method was inferred from the answers given, initially, and later during the interviews learners explained their strategies and reasoning. For algebraic equations the methods were to an extent clear even if the answers given were not correct. The difference in the way arithmetic equations were solved from the way algebraic equations were solved points to a gap that exists between the treatment of arithmetic and algebraic equations by learners. A variety of strategies were evident in arithmetic and algebraic equations when learners solved these equations. This included strategies that were referred to in the literature as well as own (and new) strategies identified during the analysis.

(b) ***Errors in arithmetic and algebraic equations***

The common errors in arithmetic equations, for example, right to left reasoning and ignoring of the operation sign were found to be common also in algebraic equations. Figures 4.3; 4.4 and 4.10 are examples of right to left reasoning in arithmetic equations and figure 5.16b is an example of ignoring the operation sign in algebraic equations. When the learner says the value of $\boldsymbol{x}$ is 17 in the equation $4 - \boldsymbol{x} = 21$, it shows that the learner is saying $4 - 17 = 4 + 17$ thereby ignoring the operation subtraction on the left hand side. Literature alluded to these errors (Gallardo and Rojano, 1990; Gallardo, 2002; Gallardo and Hernandez, 2005; Vlassis, 2004 & 2008). Breaking the equation into two parts was used uniquely for arithmetic equations, none was observed in algebraic equations and many learners made this error in arithmetic equations. Refer to examples on learners' work in figures 4.5 to 4.9. For example, in the equation 19 - $\square$ = 10 + $\square$, the learners broke the equation into two parts to find the number that goes into the box, one at a time. Each of the numbers on either side of the box was maintained as is. The breaking part of it was like: 19 - $\square$ = 10 and either 19 = 10 + $\square$ or 10 + $\square$ = 19. Essien and Setati (2006) alluded to a similar case although theirs was unidirectional in the sense that there was only one box just after the equal sign. I can say that this case is bidirectional because of the two boxes on either side of the equation. In this same equation, another error that was only observed in this particular case was that of accepting the other value in the box, on condition that it proved correctness of the value on the other side. For example 9 was considered to be the number in the box because nineteen subtract 9 resulted in 10 on the right hand side. Also, ten added to 9 resulted in 19 on the left hand side. Because of this, 9 is the number that goes into each of the boxes of in the equation. Common errors on algebraic equations like conjoining, interference of new knowledge, insertion and removal of brackets as well as changing equations into algebraic expressions was not observed in arithmetic equations. A variety of errors were evident in arithmetic and algebraic equations and they included errors referred to in the literature as well as own (and new) errors that were identified.

(c) ***Understanding the equal sign***

Understanding of the equal sign differed among the grade 9 learners, and that had an influence in the solution of equations. Equations were changed into expressions and in some cases there was conjoining on either side of the equation. There are seven cases (by different learners) where learners did such things as changing equations into expressions and/or conjoining, see figures 5.3 – 5.6 and figures 5.11, 5.12 and 5.14. This is a big indicator that the learner did not understand what was implied when saying that two expressions are equal. For example a learner says that 8 = 0, see figure 5.17. In other cases the equal sign was simply left out suggesting that its significance for the learner had ceased. An example of this is when $3\boldsymbol{x} + \boldsymbol{x} = 8 + 2$ is suddenly simplified to $3\boldsymbol{x}^2 + 10$, see figure 5.14. In some cases even if the equal sign was used, it did not connect the two sides of the expression as it should. The use of the equal sign to imply equivalence on both sides was not properly executed. Literature has alluded to the fact that many a time the equal sign is used to link parts of a calculation or influenced by prior expressions in arithmetic or even as a do-something signal (Stacey and MacGregor, 1997; Behr, Erlwanger and Nichols, 1980; Kieran 1981).

6.3 Difficulty of distinguishing between strategy and error

As I reflect on my study, I am forced to challenge my own distinction between a strategy and an error. Trying to separate strategies from errors has been rather difficult. I think that there is a need to take a closer look that these two as they are hard to distinguish when it comes to data analysis. From an expert's view it may be said that learners are making errors but there are times when a learner makes use of a particular strategy which leads to an error. So errors and strategies will be hard to distinguish until the accepted strategies of an expert and the community of mathematics are used. For example in solving the equation like $4 - \boldsymbol{x} = 21$, the learner may argue that if the difference between a number $\boldsymbol{x}$ and 4 is equal to 21 then $\boldsymbol{x}$ is equal to 25. The key word is "difference" and the structure of the problem seems to be ignored. The learner is treating subtraction as difference rather than as "take away" which is not appropriate in this context.

Strategies used in solving arithmetic equations were implicitly implicated with the errors made. For example, breaking an equation into two parts is a strategy employed by the learner to find numbers in each of the boxes on either side of the equal sign. But this strategy does not lead to a correct answer. My observation is that errors and strategies cannot be easily differentiated for the simple reason that these errors are observed to be so due to the nature of strategies that are being used to solve the equations. In other words I am saying that the errors are often a reflection of the strategies used.

6.4 Limitations

The findings in this research report are from just one grade 9 class in a school with ten grade 9 classes, and as such the findings cannot be generalized to all the grade 9 learners in South Africa. To a certain extent they can be limited to the school in which the research was conducted. The study mainly focused on simple arithmetic and algebraic equations and did not focus on equations with fractions. Equations with brackets were also excluded. So, issues pertaining to other aspects other than the focus of the study could not be discussed because they are out of focus. It must be noted, also, that this study was not focusing on the teachers and their teaching methods but on the learners and their work hence the errors that learners made some of the strategies used are attributed to the learners themselves.

In some instances the learners did not respond to questions as required and so I was not able to gain insight into their strategies. If I had included examples of showing the correctness of the solution it would have shed some light into exactly learner strategies especially on algebraic equations. For example, given an equation like $3 + \square = 10$ whose solution is 7 then to show how true the solution was I would write that: 3 + 7 = 10 or 10 – 7 = 3, or 10 – 3 = 7, hence 3 and 7 have to be added to give 10.

To conclude I would like to bring into light that despite all the limitations the study brings into light the same findings that have been found in other bigger and more pronounced researches, i.e. according to literature review and observations made from data analysis.

What I could have done differently or better so that the concern that the problem lay with the learner is not was to have given accompanying examples together with the instructions given to the learner, especially taking into account hints that were brought about by piloting. Examples would have acted as some sort of aid so that the given instruction was more precise. I am sure this would have clarified many issues which most probably I assumed to be clear to the learner whereas that was not so.

6.5 Implications and Recommendations

In this section I would like to bring to light the implications of the findings in the study, of grade 9 strategies used and errors made in solving arithmetic and algebraic linear equations, and suggest recommendations that may lead to the reduction or elimination of problems in similar cases since the research findings have important educational implications. Strategies used in solving arithmetic equations differ to those used in solving algebraic equations. The different situations of strategies used need to be brought into light and the differences in these strategies corrected. For example, an equations like $\square + 3 = 8 - \square$ and $\boldsymbol{x} + 3 = 8 - \boldsymbol{x}$ must not be seen as two different equations that need different strategies because of the variables used, but one must be able to flexibly solve any one of them and come up with same solution. An equation is an equation and the use of different variables must not have an impact in the solution of an equation. Similarities in the strategies used when solving either arithmetic or algebraic linear equations have to be consolidated in cases where the strategy is appropriate and similar. In most cases errors arise due to working with negative numbers. Pertaining to dealing with negative numbers, a thorough job needs to be done so that learners do understand them and are comfortable to work with them as well. Considering the gap that has been observed in the solution of arithmetic and algebraic equations at grade 9 level, I think there is a need to take this into consideration and check in our classrooms if such situations exist more widely, between grades 8 and 9. This will be an opportunity to carry this research forward, at a higher level, in this case focusing at how teachers introduce algebraic linear equations to grade 8 learners bearing in mind that there is a gap to be closed, i.e. between arithmetic and algebraic thinking.

In fact it is a key issue to consider that learners can decide to treat equations differently because of different variables used. For example, given two problems, like: $5 - \square = 11$ and $4 - \boldsymbol{x} = 21$, I do not foresee why one should decide to say that the two problems are different just because one has a box for a variable and the other a letter. The steps advocated by Kieran (1981) in working with 12- to 14-year-olds would prove to be the best method in bringing together differences observed in this study. All necessary precautions were taken to ensure that there is no room for guess work in as far as solving of equations is concerned. The use of variables needs to be addressed such that learners are flexible in working with any given variable in the context of the problem.

Most probably another alternative that I recommend at this point in time would be to think of another research that will focus on finding out if the use of different letters (representations of the unknown value) has an effect on the method used to solve simple linear equations. Same problems using different variables may be given to different groups of students to compare their performance.

6.6 Summary

In this study on grade 9 strategies and errors in solving arithmetic and algebraic linear equations, it is true that the investigation was on learner strategies and errors they made. However, it must not be forgotten that whatever the circumstances may be no student is above his/her teacher. Learners many a time imitate their role models. In short I am saying that educators are, directly or indirectly, involved in these strategies and errors that learner have displayed in solving arithmetic and algebraic linear equations. Whilst interviews were being conducted it was quite an experience to interact with the learners. Some learners when probed for information did not hesitate to mention their teacher by name as the source of their knowledge though in most cases they expressed their own ideas too. We have a challenge as educators to sincerely teach and dig deeper into the veins of knowledge so that when the learners have succeeded in their quest for knowledge we can also be proud partakers.

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APPENDIX: A

Test questions

Section 1

Fill in the missing number(s) in the boxes, and show how true is/ are the number(s) you have filled in, in the space provided under each problem.

(a) $7 + \square = 15$
(b) $7 - \square = 15$
(c) $\square - 7 = -2$
(d) $\square + 3 = 8 - \square$
(e) $17 = \square - 6$
(f) $13 = \square - \square$
(g) $\square + 35 = 47 - \square$

Section 2

Use the spaces provided (under each problem), and solve each of the problems below.

(a) $x + 5 = 21$	(b) $2x - 3 = 11$
(c) $9 - x = 13$	(d) $2x + 3(x - 2) = 6$
(e) $x - 5(x - 1) = 3$	(f) $3x + 5 = x + 9$
(g) $3x + 5 = x + 3 + 2(x + 1)$	(h) $x + 2 - (-2) = x - 4$

APPENDIX: B

Interview questions

Section 1 Fill in the missing number(s) in the box (es), and give reason, in each case, for your choice in the spaces provided under each problem.	
(a) **5 + ☐ = 13**	(b) **5 - ☐ = 11**
(c) **15 = ☐ - ☐**	(d) **19 - ☐ = 10 + ☐**

Section 2 Solve for x, in each of the problems given below.	
(a) **4 + x = 23**	(b) **4 - x = 21**
(c) **3(x + 2) = 18**	(d) **3x + 2 = 8 + x**
(e) **x + 3 = x – 3**	

APPENDIX: C

Initial data tables

Table 1: Test (Section 1)

Item	(a)	(b)	(c)	(d)	(e)	(f)	(g)
C	***8***	***-8***	***5***	***CC_1***	***23***	***CC_2***	***CC_3***
G901	**8**	22	9	5 ; 5	12	**20 ; 7**	12 ; 12
G902	**8**	**-8**	**5**	**3 ; 2**	24	**15 ; 2**	**2 ; 10**
G903	**8**	22	9	5 ; 3	12	**20 ; 7**	12 ; 12
G904	**8**	22	**5**	5 ; 5	**23**	**20 ; 7**	12 ; 12
G905	**8**	8	**5**	5 ; 5	-11	**16 ; 3**	- ; -
G906	**8**	22	**5**	5 ; 3	13	7 ; 20	12 ; 12
G907	**8**	22	**5**	5 ; 5	**23**	**18 ; 5**	12 ; 12
G908	**8**	22	**5**	5 ; 3	**23**	**16 ; 3**	12 ; 12
G909	**8**	**-8**	-9	5 ; 5	11	0 ; 13	12 ; 12
G910	**8**	**-8**	-9	5 ; 5	**23**	**20 ; 7**	13 ; 13
G911	**8**	**-8**	**5**	5 ; 5	**23**	**20 ; 7**	12 ;35
G912	**8**	22	**5**	5 ; 3	**23**	-7 ; 6	12 ; 12
G913	**8**	**-8**	9	5 ; 5	12	7 ; 6	12 ; 12
G914	**8**	22	**5**	5 ; 3	**23**	7 ; 6	12 ; 12
G915	**8**	**-8**	**5**	5 ; 5	22	**7 ; -6**	12 ; 12
G916	**8**	8	**5**	5 ; 11	**23**	7 ; 20	12 ; 12
G917	**8**	**-8**	**5**	5 ; 5	24	**16 ; 3**	12 ; 12
G918	**8**	**-8**	-4	5 ; 5	-10	7 ; 20	12 ;35
G919	**8**	**-8**	-5	5 ; -5	-23	-19 ; -6	12 ; -12
G920	**8**	**-8**	**5**	5 ; 5	11	7 ; 6	12 ; 12
G921	**8**	**-8**	**5**	5 ; 3	11	**15 ; 2**	12 ; 12
G922	**8**	**-8**	**5**	5 ; 3	-11	-8 ; 5	12 ; 12
G923	**8**	**-8**	9	5 ; 3	11	**16 ; 3**	12 ; 12
G924	**8**	**-8**	**5**	5 ; 5	21	**20 ; 7**	12 ; 11
G925	**8**	22	9	5 ; 5	12	**20 ; 7**	12 ; 12
G926	**8**	8	**5**	5 ; 5	11	7 ; 6	12 ; 12
G927	**8**	**-8**	**5**	5 ; 5	**23**	**20 ; 7**	12 ; 12
G928	**8**	**-8**	**5**	5 ; 5	**23**	**20 ; 7**	12 ; 12
G929	**8**	22	**5**	5 ; 5	**23**	**20 ; 7**	12 ; 35
G930	**8**	**-8**	**5**	5 ; 5	-11	**7 ; -6**	12 ; 12
G931	**8**	**-8**	**5**	5 ; 5	**23**	**7 ; -6**	12 ; 12
G932	**8**	8	10	5 ; 3	**23**	3 ; 10	12 ; 12
G933	**8**	22	9	5 ; 5	**23**	**19 ; 6**	12 ; 12
G934	**8**	22	9	5 ; 5	**23**	**19 ; 6**	- ; -
G935	**8**	22	**5**	5 ; 2	11	**20 ; 7**	12 ; 6
G936	**8**	**-8**	9	5 ; 16	25	**17 ; 4**	12 ; 83
G937	**8**	**-8**	-9	8 ; 3	**23**	**20 ; 7**	47 ; 35
G938	**8**	**-8**	-5	5 ; 5	-11	**20 ; 7**	12 ; 35
G939	**8**	22	**5**	5 ; 5	**23**	-7 ; 6	12 ; 12
G940	**8**	22	4	**5 ; 0**	8	**20 ; 7**	**12 ; 0**
G941	**8**	**-8**	-9	5 ; 5	11	**15 ; 2**	12 ; 12
G942	**8**	**-8**	-9	5 ; 5	11	**15 ; 2**	12 ; 12
G943	**8**	21	**5**	**5 ; 0**	25	**14 ; 1**	9 ; 4
G944	**8**	22	**5**	5 ; 3	**23**	**20 ; 7**	12 ; 12
G945	**8**	**-8**	**5**	5 ; 5	-11	2 ; 15	12 ; 12
Total (E.R.)	**45**	**24**	**27**	**3**	**18**	**31**	**2**

Table 2: Section 1

	Test Item	Interview Item			
	(a) 7 + ☐ = 15	(a) 5 + ☐ = 13	Possible Error		Possible Reasons
C	8	8			
G901	8		C		**There are no errors, answers are correct**
G902	8		C		
G903	8		C		
G904	8	8	C	C	
G905	8		C		
G906	8		C		
G907	8		C		
G908	8		C		
G909	8		C		
G910	8		C		
G911	8		C		
G912	8		C		
G913	8		C		
G914	8		C		
G915	8		C		
G916	8		C		
G917	8		C		
G918	8		C		
G919	8		C		
G920	8		C		
G921	8		C		
G922	8		C		
G923	8		C		
G924	8		C		
G925	8	8	C	C	
G926	8		C		
G927	8	7	C	Slip	There is no way 7 and 7 can add to 15.
G928	8	8	C	C	
G929	8		C		
G930	8		C		
G931	8		C		
G932	8		C		
G933	8		C		
G934	8		C		
G935	8		C		
G936	8		C		
G937	8		C		
G938	8		C		
G939	8	8	C	C	
G940	8		C		
G941	8		C		
G942	8		C		
G943	8		C		
G944	8		C		
G945	8		C		
Total	0/45	1/4			

Table 3: Section 1

	Test Item (b) 7 - □ = 15	Interview Item (b) 5 - □ = 11	Possible Error		Possible Reasons
C	-8	-6			
G901	22		RLR		**Right to left reasoning.**
G902	-8		C		Correct
G903	22		RLR		
G904	22	16	RLR	RLR	
G905	8		IOS		Ignoring the operation subtraction
G906	22		RLR		
G907	22		RLR		
G908	22		RLR		
G909	-8		C		
G910	-8		C		
G911	-8		C		
G912	22		RLR		
G913	-8		C		
G914	22		RLR		
G915	-8		C		
G916	8		IOS		
G917	-8		C		
G918	-8		C		
G919	-8		C		
G920	-8		C		
G921	-8		C		
G922	-8		C		
G923	-8		C		
G924	-8		C		
G925	22	6	RLR	IOS	
G926	8		IOS		
G927	-8	-6	C	C	
G928	-8	16	C	RLR	
G929	22		RLR		
G930	-8		C		
G931	-8		C		
G932	8		IOS		
G933	22		RLR		
G934	22		RLR		
G935	22		RLR		
G936	-8		C		
G937	-8		C		
G938	-8		C		
G939	22	6	RLR	IOS	
G940	22		RLR		
G941	-8		C		
G942	-8		C		
G943	21		RLR		
G944	22		RLR		
G945	-8		C		
Total	21/24	4/1			

Table 4: Section 1

	Test Item		
	(c) ☐ - 7 = -2	Possible Error	Possible Reasons
C	5		
G901	9	RLR	
G902	5	C	
G903	9	RLR	
G904	5	C	
G905	5	C	
G906	5	C	
G907	5	C	
G908	5	C	
G909	-9	IOS/TSBN	Ignore operation subtraction/t /Take sign of bigger number
G910	-9	IOS/TSBN	
G911	5	C	
G912	5	C	
G913	9	RLR	
G914	5	C	
G915	5	C	
G916	5	C	
G917	5	C	
G918	-4	O	No way two numbers like 7 and 2 can come with solutions like ±4
G919	-5	O	Through trial and improvement
G920	5	C	
G921	5	C	
G922	5	C	
G923	9	RLR	
G924	5	C	
G925	9	RLR	
G926	5	C	
G927	5	C	
G928	5	C	
G929	5	C	
G930	5	C	
G931	5	C	
G932	10	O	Similar explanation like that for solutions like ±4
G933	9	RLR	
G934	9	RLR	
G935	5	C	
G936	9	RLR	
G937	-9	IOS/TSBN	
G938	-5	O	
G939	5	C	
G940	4	O	
G941	-9	IOS/TSBN	
G942	-9	IOS/TSBN	
G943	5	C	
G944	5	C	
G945	5	C	
Total	18/27		

Table 5: Section 1

	Test Item	Interview Item			
	(d) ☐ + 3 = 8 - ☐	(c) 19 - ☐ = 10 + ☐	Possible Error		Possible Reasons
C	CC_1	CC_2			
G901	5 ; 5		BEQ		**Breaking the equation into two parts**
G902	3 ; 2		CC		Correct pair chosen
G903	5 ; 3		PCS		Proving that chosen number is correct
G904	5 ; 5	9; 9	BEQ	BEQ	
G905	5 ; 5		BEQ		
G906	5 ; 3		PCS		
G907	5 ; 5		BEQ		
G908	5 ; 3		PCS		
G909	5 ; 5		BEQ		
G910	5 ; 5		BEQ		
G911	5 ; 5		BEQ		
G912	5 ; 3		PCS		
G913	5 ; 5		BEQ		
G914	5 ; 3		PCS		
G915	5 ; 5		BEQ		
G916	5 ; 11		O		
G917	5 ; 5		BEQ		
G918	5 ; 5		BEQ		
G919	5 ; -5		Slip		
G920	5 ; 5		BEQ		
G921	5 ; 3		PCS		
G922	5 ; 3		PCS		
G923	5 ; 3		PCS		
G924	5 ; 5		BEQ		
G925	5 ; 5	9; 9	BEQ	BEQ	
G926	5 ; 5		BEQ		
G927	5 ; 5	9; 9	BEQ	BEQ	
G928	5 ; 5	9; 9	BEQ	BEQ	
G929	5 ; 5		BEQ		
G930	5 ; 5		BEQ		
G931	5 ; 5		BEQ		
G932	5 ; 3		PCC		
G933	5 ; 5		BEQ		
G934	5 ; 5		BEQ		
G935	5 ; 2		O		
G936	5 ; 16		RLR		
G937	8 ; 3		IOS		
G938	5 ; 5		BEQ		
G939	5 ; 5	9; 9	BEQ	BEQ	
G940	5 ; 0		CC		
G941	5 ; 5		BEQ		
G942	5 ; 5		BEQ		
G943	5 ; 0		CC		
G944	5 ; 3		PCS		
G945	5 ; 5		BEQ		
Total	42/3	5/0			

Table 6: Section 1

	Test Item		
	(e) 17 = ☐ - 6	Possible Error	Possible Reasons
C	23		
G901	12	IOS	Ignoring the operation sign
G902	24	Slip	
G903	12	IOS	
G904	23	C	Correct
G905	-11	DN	Double negative, i.e. negative and negative gives positive
G906	13	IOS	
G907	23	C	
G908	23	C	
G909	11	IOS	
G910	23	C	
G911	23	C	
G912	23	C	
G913	12	IOS	
G914	23	C	
G915	22	Slip	
G916	23	C	
G917	24	Slip	
G918	-10	DN	It is highly likely that the double negative concept is applied. Then?
G919	-23	Slip	For putting the negative.
G920	11	IOS	
G921	11	IOS	
G922	-11	DN	
G923	11	IOS	
G924	21	Slip	
G925	12	IOS	
G926	11	IOS	
G927	23	C	
G928	23	C	
G929	23	C	
G930	-11	DN	
G931	23	C	
G932	23	C	
G933	23	C	
G934	23	C	
G935	11	IOS	
G936	25	Slip	
G937	23	C	
G938	-11	DN	
G939	23	C	
G940	8	O	
G941	11	IOS	
G942	11	IOS	
G943	25	Slip	
G944	23	C	
G945	-11	DN	
Total	27/18		

Table 7: Section 1

	Test Item	Interview Item			
	(f) 13 = ☐ - ☐	(d) 15 = ☐ - ☐	Possible Error		Possible Reasons
C	CC_3	CC_4			
G901	20 ; 7		C		
G902	15 ; 2		C		
G903	20 ; 7		C		
G904	20 ; 7	20; 5	C	C	
G905	16 ; 3		C		
G906	7 ; 20		RLR		
G907	18 ; 5		C		
G908	16 ; 3		C		
G909	0 ; 13		RLR		
G910	20 ; 7		C		
G911	20 ; 7		C		
G912	-7 ; 6		O		Ignore both -ives or use double –ve principle
G913	7 ; 6		IOS		
G914	7 ; 6		IOS		
G915	7 ; -6		IOS		
G916	7 ; 20		RLR		
G917	16 ; 3		C		
G918	7 ; 20		RLR		
G919	-19 ; -6		O		Ignored negative numbers and treated as +ves
G920	7 ; 6		IOS		
G921	15 ; 2		C		
G922	-8 ; 5		O		Ignore both -ives or use double –ve principle
G923	16 ; 3		C		
G924	20 ; 7		C		
G925	20 ; 7	20; 5	C	C	
G926	7 ; 6		IOS		
G927	20 ; 7	-5; 10	C	O	
G928	20 ; 7	5; 20	C	RLR	
G929	20 ; 7		C		
G930	7 ; -6		C		
G931	7 ; -6		C		
G932	3 ; 10		IOS		
G933	19 ; 6		C		
G934	19 ; 6		C		
G935	20 ; 7		C		
G936	17 ; 4		C		
G937	20 ; 7		C		
G938	20 ; 7		C		
G939	-7 ; 6	15; 30	DN	RLR	**Double negative**
G940	20 ; 7		C		
G941	15 ; 2		C		
G942	15 ; 2		C		
G943	14 ; 1		C		
G944	20 ; 7		C		
G945	2 ; 15		RLR		
Total	14/31	3/2			

Table 8: Section 1

	Test Item	Interview Item			
	(g) ☐ + 35 = 47 - ☐	(c) 19 - ☐ = 10 + ☐	Possible Error		Possible Reasons
C	CC_5	CC_2			
G901	12 ; 12		BEQ		Breaking the equation into two parts
G902	2 ; 10		C		
G903	12 ; 12		BEQ		
G904	12 ; 12	9; 9	BEQ	BEQ	Breaking the equation into two parts
G905	- ; -		UAQ		Unanswered question
G906	12 ; 12		BEQ		
G907	12 ; 12		BEQ		
G908	12 ; 12		BEQ		
G909	12 ; 12		BEQ		
G910	13 ; 13		O		
G911	12 ;35		PCS		Proving the chosen number to be correct
G912	12 ; 12		BEQ		
G913	12 ; 12		BEQ		
G914	12 ; 12		BEQ		
G915	12 ; 12		BEQ		
G916	12 ; 12		BEQ		
G917	12 ; 12		BEQ		
G918	12 ;35		PCS		
G919	12 ; -12		Slip		Highly likely the learner wanted 12; 12
G920	12 ; 12		BEQ		
G921	12 ; 12		BEQ		
G922	12 ; 12		BEQ		
G923	12 ; 12		BEQ		
G924	12 ; 11		Slip		Highly likely the learner wanted 12; 12
G925	12 ; 12	9; 9	BEQ	BEQ	
G926	12 ; 12		BEQ		
G927	12 ; 12	9; 9	BEQ	BEQ	
G928	12 ; 12	9; 9	BEQ	BEQ	
G929	12 ; 35		PCS		
G930	12 ; 12		BEQ		
G931	12 ; 12		BEQ		
G932	12 ; 12		BEQ		
G933	12 ; 12		BEQ		
G934	- ; -		UAQ		
G935	12 ; 6		O		
G936	12 ; 83		O		
G937	47 ; 35		IOS		Ignoring operation subtraction
G938	12 ; 35		PCS		
G939	12 ; 12	9; 9	BEQ	BEQ	
G940	12 ; 0		C		
G941	12 ; 12		BEQ		
G942	12 ; 12		BEQ		
G943	9 ; 4		Slip		Very close to solving an equation
G944	12 ; 12		BEQ		
G945	12 ; 12		BEQ		
Total	41/2	5/0			

Table 9: Comparison between test and interview questions

	Test Qn.	*Interv Qn.*	Test Qn.	*Interv Qn.*	Test Qn.	*Interv Qn.*
	7 + □ = 15	5 + □ = 13	7 - □ = 15	5 - □ = 11	□ + 3 = 8 - □	19 - □ = 10 + □
Vicky	8	8	22	16	5; 5	9; 9
Violet	8	8	22	6	5; 5	9; 9
Kison	8	7	-8	-6	5; 5	9; 9
Kitos	8	8	-8	16	5; 5	9; 9
Miriam	8	8	22	6	5; 5	9; 9
Correct	5	4	2	1	0	0

Table 10: Comparison between test and interview questions

	Test Qn.	*Interv Qn.*	
	7 + □ = 15	5 + □ = 13	General Comments
Vicky	8	8	Correct
Violet	8	8	Correct
Kison	8	7	Slip
Kitos	8	8	Correct
Miriam	8	8	Correct

Table 11: Comparison between test and interview questions

	Test Qn.	*Interv Qn.*	
	7 - □ = 15	5 - □ = 11	General Comments
Vicky	22	16	RLR
Violet	22	6	RLR & IOS
Kison	-8	-6	Correct
Kitos	-8	16	Correct & RLR
Miriam	22	6	RLR & IOS

Table 12: Comparison between test and interview questions

	Test Qn.	*Interv Qn.*	
	□ + 3 = 8 - □	19 - □ = 10 + □	General Comments
Vicky	5; 5	9; 9	Misconception of some sort
Violet	5; 5	9; 9	Misconception of some sort
Kison	5; 5	9; 9	Misconception of some sort
Kitos	5; 5	9; 9	Misconception of some sort
Miriam	5; 5	9; 9	Misconception of some sort

Table 13: Section 2

	Test Item				Interview Item								
	x + 5 = 21				4 + x = 23				Strands				
M/S	**B**	**T**	**TI**	**O**	**B**	**T**	**TI**	**O**	**CU**	**PF**	**AR**	**SC**	**Comment**
G901	0	0	**1**	0					**1**	**1**	0	**1**	Proficient
G902	**1**	0	0	0					**1**	**1**	1	**1**	Very proficient
G903	0	**1**	0	0					**1**	**1**	0	**1**	
G904	0	**1**	0	0	0	**1**	0	0	**1;1**	**1;1**	0;0	**1;1**	
G905	0	**1**	0	0					**1**	**1**	0	**1**	
G906	0	**1**	0	0					**1**	**1**	0	**1**	
G907	0	0	**1**	0					**1**	**1**	0	**1**	
G908	0	0	**1**	0					0	0	0	**1**	Not proficient
G909	0	0	**1**	0					**1**	**1**	0	**1**	
G910	**1**	0	0	0					**1**	**1**	0	**1**	
G911	0	**1**	0	0					**1**	**1**	0	**1**	
G912	0	**1**	0	0					**1**	**1**	0	**1**	
G913	0	0	**1**	0					**1**	**1**	0	**1**	
G914	0	**1**	0	0					**1**	**1**	0	**1**	
G915	**1**	0	0	0					**1**	**1**	0	**1**	
G916	0	0	**1**	0					**1**	**1**	0	**1**	
G917	**1**	0	0	0					**1**	**1**	0	**1**	
G918	**1**	0	0	0					**1**	**1**	0	**1**	
G919	0	0	0	**1**					0	0	0	0	Not proficient
G920	0	0	**1**	0					**1**	**1**	0	**1**	
G921	0	**1**	0	0					**1**	**1**	0	**1**	
G922	**1**	0	0	**1**					**1**	**1**	0	**1**	
G923	0	0	**1**	0					**1**	**1**	0	**1**	
G924	0	0	**1**	0					**1**	**1**	0	**1**	
G925	0	**1**	0	0	0	**1**	0	0	**1;1**	**1;1**	0;0	**1;1**	
G926	0	**1**	0	0					**1**	**1**	0	**1**	
G927	**1**	0	0	0	0	0	0	**1**	**1;1**	**1;0**	0;0	**1;1**	
G928	**1**	0	0	0	0	0	**1**	0	**1;1**	**1;1**	0;0	**1;1**	
G929	0	**1**	0	0					**1**	**1**	0	**1**	
G930	0	**1**	0	0					**1**	**1**	0	**1**	
G931	0	**1**	0	0					**1**	**1**	0	**1**	
G932	**1**	0	0	0					**1**	**1**	0	**1**	
G933	**1**	0	0	0					**1**	**1**	0	**1**	
G934	**1**	0	0	0					**1**	**1**	0	**1**	
G935	0	0	**1**	0					**1**	**1**	0	**1**	
G936	0	0	**1**	0					**1**	**1**	0	**1**	
G937	0	**1**	0	0					**1**	**1**	0	**1**	
G938	0	0	**1**	0					**1**	**1**	0	**1**	
G939	0	**1**	0	0	0	**1**	0	0	**1;1**	**1;1**	0;0	**1;1**	
G940	0	0	0	**1**					0	0	0	0	
G941	0	0	**1**	0					**1**	**1**	0	**1**	
G942	0	0	0	**1**					0	0	0	0	
G943	0	0	**1**	0					**1**	**1**	0	**1**	
G944	0	0	**1**	0					**1**	**1**	0	**1**	
G945	**1**	0	0	0					**1**	**1**	0	**1**	
Total	**11**	**15**	**15**	**4**	**1**	**4**	**2**	**1**	**41;7**	**38;6**	**1;1**	**42;7**	

Table 14

	Test Item					Strands			
	2x - 3 = 11				Strands				
M/S	**B**	**T**	**TI**	**O**	**CU**	**PF**	**AR**	**SC**	**Comment**
G901	1	0	0	0	1	1	0	1	Proficient (p)
G902	1	1	0	0	1	1	1	1	Very proficient (vp)
G903	1	0	0	0	1	1	0	1	
G904	1	1	0	0	1	1	0	1	
G905	1	1	0	0	1	1	0	1	
G906	0	1	0	0	1	1	0	1	
G907	0	0	1	0	1	0	0	0	
G908	1	0	0	0	0	0	0	1	Not proficient (np)
G909	0	0	0	1	1	0	0	1	
G910	0	0	0	1	1	0	0	1	
G911	1	1	0	0	1	1	0	1	
G912	1	1	0	0	1	1	0	1	
G913	0	0	0	1	1	0	0	0	
G914	0	1	0	0	1	0	0	1	
G915	1	0	0	0	1	0	0	1	
G916	0	0	0	1	0	0	0	0	
G917	1	0	0	0	1	1	0	1	
G918	1	0	0	0	1	1	0	1	
G919	0	0	0	0	-	-	-	-	Not proficient at all (npaa)
G920	0	0	0	1	1	0	0	0	
G921	0	1	0	0	0	0	0	1	
G922	1	0	0	0	1	1	0	1	
G923	0	0	0	1	0	0	0	0	
G924	0	0	0	1	0	0	0	0	
G925	1	0	0	0	1	1	0	1	
G926	1	1	0	0	1	1	0	1	
G927	1	0	0	0	1	1	0	1	
G928	1	0	0	0	1	1	0	1	
G929	1	1	0	0	1	1	0	1	
G930	1	1	0	0	1	1	0	1	
G931	1	1	0	0	1	1	0	1	
G932	0	0	0	1	0	0	0	0	
G933	1	0	0	0	1	1	0	1	
G934	0	0	0	0	1	0	0	1	
G935	1	0	0	0	1	1	0	1	
G936	0	0	0	1	1	0	0	0	
G937	0	1	0	0	1	1	0	1	
G938	0	1	0	0	1	1	0	1	
G939	0	1	0	0	1	1	0	1	
G940	0	0	0	1	0	0	0	0	
G941	0	0	0	1	1	0	0	0	
G942	0	0	0	1	0	0	0	0	
G943	0	0	1	0	0	0	0	0	
G944	0	0	1	0	1	1	0	1	
G945	1	0	0	0	1	1	0	1	
Total	22	15	3	12	35	25	1	32	

Table 15

	Test Item				Interview Item								
	9 - x = 13				4 - x = 21				Strands				
M/S	B	T	TI	O	B	T	TI	O	CU	PF	AR	SC	Comment
G901	1	0	0	0					1	1	0	1	P
G902	1	0	0	0					1	1	1	1	VP
G903	1	0	0	0					1	0	0	1	
G904	0	1	0	0	0	1	0	0	1;1	1;1	0;0	1;1	
G905	0	1	0	0					1	1	0	1	
G906	0	1	0	0					1	1	0	1	
G907	0	0	1	0					1	0	0	1	
G908	0	0	1	0					0	0	0	1	
G909	0	0	1	0					1	1	1	1	
G910	0	1	0	0					1	0	0	1	
G911	0	1	0	0					1	1	0	1	
G912	0	1	0	0					1	1	0	1	
G913	0	1	0	0					1	0	0	0	
G914	0	0	1	0					1	0	0	1	
G915	1	0	0	0					0	0	0	1	
G916	0	0	0	1					0	0	0	0	
G917	0	1	0	0					1	1	0	1	
G918	1	0	0	0					1	1	0	1	
G919	0	0	0	1					0	0	0	0	
G920	0	1	0	0					1	0	0	1	
G921	0	1	0	0					1	0	0	1	
G922	1	0	0	0					1	1	0	1	
G923	0	0	0	0					0	0	0	0	
G924	0	0	0	0					1	0	0	0	
G925	1	0	0	0	0	1	0	0	1;1	0;0	0;0	1;1	
G926	0	1	0	0					1	1	0	1	
G927	1	0	0	0	0	0	0	1	1;1	1;0	0;0	1;1	
G928	1	0	0	0	0	0	1	0	1;1	1;0	0;0	1;1	
G929	0	1	0	0					1	0	0	1	Np
G930	0	1	0	0					1	1	0	1	
G931	0	1	0	0					1	1	0	1	
G932	0	0	0	1					0	0	0	0	Np
G933	1	0	0	0					1	1	0	1	
G934	0	0	1	0					1	1	0	1	
G935	0	0	1	0					1	0	0	1	
G936	0	0	0	1					1	0	0	0	
G937	0	1	0	0					1	1	0	1	
G938	0	0	1	0					0	0	0	0	
G939	0	1	0	0	0	1	0	0	1;1	1;1	0;0	1;1	
G940	0	0	0	1					0	0	0	0	
G941	0	0	1	0					1	1	0	1	
G942	0	0	0	1					0	0	0	0	
G943	0	0	1	0					1	1	0	1	
G944	0	0	1	0					1	1	0	1	
G945	1	0	0	0					1	1	0	1	
Total	11	16	10	6	1	4	2	1	36;6	24;4	2;0	35;7	

Table 16

	Test Item				Interview Item					Strands			
	2x + 3(x - 2) = 6				3(x + 2) = 18				Strands				
M/S	**B**	**T**	**TI**	**O**	**B**	**T**	**TI**	**O**	**CU**	**PF**	**AR**	**SC**	**Comment**
G901	1	0	0	0					1	1	0	1	P
G902	1	0	0	0					1	1	0	1	
G903	1	0	0	0					1	0	0	1	Np
G904	1	0	0	0	1	1	0	0	1;1	1;1	0;0	1;1	
G905	1	0	0	0					1	1	0	1	
G906	1	0	0	0					1	1	0	1	
G907	0	0	1	0					1	0	0	1	
G908	0	0	1	0					0	0	0	1	
G909	1	0	0	0					0	0	0	1	
G910	0	0	0	1					0	0	0	0	Np
G911	1	0	0	0					1	1	0	1	
G912	1	1	0	0					1	1	0	1	
G913	0	1	0	0					1	0	0	1	
G914	1	1	0	0					1	1	0	1	
G915	0	0	0	0					1	0	0	0	
G916	0	0	0	1					0	0	0	0	
G917	1	0	0	0					1	0	0	1	
G918	1	0	0	0					1	1	0	1	
G919	0	0	0	0					-	-	-	-	Npaa
G920	0	0	0	1					1	0	0	0	
G921	0	0	0	1					0	0	0	0	
G922	1	0	0	0					1	1	0	1	
G923	0	0	0	0					-	-	-	-	
G924	0	0	0	1					1	0	0	0	
G925	1	0	0	0	0	1	0	0	0;1	0; 1	0;0	1;1	
G926	1	0	0	0					0	0	0	1	
G927	0	0	0	0	0	0	0	1	0;1	0;0	0;0	0;0	
G928	0	0	0	0	0	0	1	0	0;0	0;0	0;0	0;1	
G929	1	1	0	0					1	1	0	1	
G930	0	1	0	0					1	0	0	1	
G931	0	1	0	0					1	1	0	1	
G932	0	0	0	1					0	0	0	0	
G933	0	0	0	0					1	0	0	0	
G934	0	0	1	0					1	0	0	1	
G935	0	0	0	0					1	0	0	1	
G936	0	0	0	1					1	0	0	0	
G937	0	0	0	1					1	0	0	0	
G938	0	0	0	1					0	0	0	0	
G939	1	1	0	0	0	1	0	0	1;1	1;1	0;0	1;1	
G940	0	0	0	1					0	0	0	0	
G941	0	0	0	1					1	0	0	0	
G942	0	0	0	0					0	0	0	0	
G943	0	0	0	0					0	0	0	0	
G944	1	1	0	0					1	1	0	1	
G945	1	0	0	0					1	0	0	1	
Total	19	8	3	12	2	4	2	1	29;5	14;4	00;0	26;6	

Table 17

	Test Item								
	x - 5(x - 1) = 3				Strands				
M/S	B	T	TI	O	CU	PF	AR	SC	Comment
G901	0	0	0	0	1	0	0	0	Np
G902	1	0	0	0	1	0	0	0	Np
G903	0	1	0	0	1	1	0	1	
G904	1	1	0	0	1	1	0	1	
G905	0	0	0	0	1	1	0	1	
G906	0	1	0	0	1	0	0	1	
G907	0	0	1	0	1	0	0	1	
G908	0	0	1	0	0	0	0	1	
G909	0	0	0	1	0	0	0	0	
G910	0	0	0	1	1	0	0	0	
G911	0	0	0	1	1	0	0	1	
G912	0	1	0	0	1	0	0	1	
G913	0	0	0	1	1	0	0	0	
G914	0	1	0	0	0	0	0	1	
G915	0	0	0	0	1	0	0	0	
G916	0	0	0	1	0	0	0	0	
G917	1	0	0	0	1	0	0	1	
G918	0	0	0	1	0	0	0	0	
G919	0	0	0	1	0	0	0	0	
G920	0	0	0	1	1	0	0	0	
G921	0	0	0	1	0	0	0	0	
G922	0	0	0	1	1	0	0	0	
G923	0	0	0	1	0	0	0	0	
G924	0	0	0	1	0	0	0	0	
G925	0	0	0	0	1	0	0	0	
G926	1	0	0	0	1	1	0	1	
G927	0	0	0	0	0	0	0	0	
G928	0	0	0	0	0	0	0	0	
G929	0	1	0	0	1	0	0	1	
G930	0	1	0	0	1	0	0	1	
G931	0	1	0	0	1	0	0	1	
G932	0	0	0	1	0	0	0	0	
G933	0	0	0	0	1	0	0	0	
G934	0	0	0	0	1	0	0	0	
G935	0	0	0	0	1	0	0	0	
G936	0	0	0	1	0	0	0	0	
G937	0	0	0	1	1	0	0	1	
G938	0	0	0	1	0	0	0	0	
G939	0	1	0	0	1	0	0	1	
G940	0	0	0	1	0	0	0	0	
G941	0	0	0	1	1	0	0	0	
G942	0	0	0	1	0	0	0	0	
G943	0	0	0	0	1	0	0	0	
G944	0	1	0	0	1	0	0	1	
G945	0	0	0	0	1	0	0	0	
Total	4	10	2	19	29	4	0	17	

Table 18

	Test Item				Interview Item								
	3x + 5 = x + 9				4 + x = 23				Strands				
M/S	**B**	**T**	**TI**	**O**	**B**	**T**	**TI**	**O**	**CU**	**PF**	**AR**	**SC**	**Comment**
G901	1	1	0	0					1	1	0	1	P
G902	1	0	0	0					1	1	0	1	P
G903	1	0	0	0					1	1	0	1	
G904	1	1	0	0	1	1	0	0	1;1	1;1	0;0	1	
G905	1	1	0	0					1	1	0	1	
G906	1	1	0	0					1	1	0	1	
G907	0	0	1	0					1	0	0	1	
G908	0	0	0	1					0	0	0	0	Np
G909	0	0	0	1					1	0	0	0	
G910	0	0	0	1					1	0	0	0	
G911	1	1	0	0					1	1	0	1	
G912	1	1	0	0					1	0	0	1	
G913	0	0	0	1					1	0	0	0	
G914	0	1	0	0					1	0	0	1	
G915	0	0	0	1					1	0	0	0	
G916	0	0	0	1					0	0	0	0	
G917	1	0	0	0					1	1	0	1	
G918	1	0	0	0					1	0	0	0	
G919	0	0	0	0					-	-	-	-	
G920	0	0	0	0					-	-	-	-	
G921	0	0	0	1					0	0	0	0	
G922	1	0	0	0					1	1	0	1	
G923	0	0	1	0					0	0	0	0	
G924	0	0	0	1					1	0	0	0	
G925	1	1	0	0	0	1	0	0	1;0	1;0	0;0	1;1	
G926	1	0	0	0					1	0	0	1	
G927	0	1	0	0	0	0	0	1	0;1	0;0	0;0	0;0	
G928	0	1	0	0	0	1	0	0	1;1	0;0	0;0	1;1	
G929	1	1	0	0					1	1	0	1	
G930	1	1	0	0					1	1	0	1	
G931	0	1	0	0					1	1	0	1	
G932	0	0	0	1					1	0	0	0	
G933	1	0	0	0					1	0	0	1	
G934	0	0	0	0					0	0	0	0	
G935	0	0	0	0					1	0	0	0	
G936	0	0	0	1					0	0	0	0	
G937	0	1	0	1					1	0	0	1	
G938	0	0	1	0					1	0	0	0	
G939	1	1	0	0	0	1	0	0	1	1;0	0;0	1;1	
G940	0	0	0	1					0	0	0	0	
G941	0	0	0	1					1	0	0	0	
G942	0	0	0	1					0	0	0	0	
G943	0	0	0	0					0	0	0	0	
G944	1	1	0	0					1	1	0	1	
G945	1	0	0	0					1	0	0	0	
Total	19	15	3	14	2	5	1	1	33;5	15;2	00;0	22;6	

Table 19

	Test Item								
	3x + 5 = x + 3 + 2(x + 1)				Strands				
M/S	B	T	TI	O	CU	PF	AR	SC	Comment
G901	1	1	0	0	1	0	0	1	Np
G902	0	0	0	0	0	0	0	0	Npaa
G903	0	1	0	0	1	0	0	0	
G904	1	1	0	0	1	1	0	1	P
G905	0	1	0	0	1	1	0	1	
G906	0	1	0	0	1	1	0	1	
G907	0	0	0	1	1	0	0	0	
G908	0	0	0	1	0	0	0	0	
G909	0	0	0	1	0	0	0	0	
G910	0	0	0	1	1	0	0	0	
G911	1	0	0	0	1	0	0	1	
G912	1	1	0	0	1	0	0	1	
G913	0	1	0	0	1	0	0	1	
G914	1	1	0	0	1	1	0	1	
G915	0	0	0	0	1	0	0	0	
G916	0	0	0	1	0	0	0	0	
G917	1	1	0	0	1	0	0	1	
G918	0	0	0	1	1	0	0	0	
G919	0	0	0	0	-	-	-	-	
G920	0	0	0	1	0	0	0	0	
G921	0	0	0	0	0	0	0	0	
G922	0	0	0	1	1	1	0	1	
G923	0	0	1	0	0	0	0	0	
G924	0	0	0	1	0	0	0	0	
G925	1	1	0	0	1	0	0	1	
G926	1	0	0	0	1	1	0	1	
G927	0	0	0	0	0	0	0	0	
G928	0	1	0	0	1	0	0	1	
G929	0	1	0	0	1	0	0	1	
G930	1	1	0	0	1	0	0	1	
G931	0	1	0	0	1	1	0	1	
G932	0	0	0	1	0	0	0	0	
G933	0	0	0	0	1	0	0	0	
G934	0	0	0	0	0	0	0	0	
G935	0	0	0	0	1	0	0	0	
G936	0	0	0	1	0	0	0	0	
G937	0	0	0	0	1	0	0	1	
G938	0	0	0	1	0	0	0	0	
G939	1	1	0	0	1	1	0	1	
G940	0	0	0	1	0	0	0	0	
G941	0	0	0	1	0	0	0	0	
G942	0	0	0	1	0	0	0	0	
G943	0	0	0	0	0	0	0	0	
G944	1	1	0	0	1	1	0	1	
G945	1	0	0	0	1	1	0	1	
Total	12	16	1	15	27	10	0	20	

Table 20

	Test Item				Interview Item				Strands				
	x + 2 - (-2) = x - 4				x + 3 = x - 3								
M/S	B	T	TI	O	B	T	TI	O	CU	PF	AR	SC	Comment
G901	0	0	0	0					0	0	0	0	Np
G902	0	0	0	0					0	0	0	0	Np
G903	0	1	0	0					0	1	0	1	
G904	0	1	0	0	1	1	0	0	1;1	1;1	0'0	1;1	
G905	0	1	0	0					1	1	0	1	
G906	0	1	0	0					1	1	0	1	
G907	0	0	1	0					1	0	0	0	
G908	0	0	0	0					0	0	0	0	
G909	0	0	0	0					0	0	0	0	
G910	0	0	0	1					1	0	0	0	
G911	0	1	0	0					1	0	0	1	
G912	0	1	0	0					1	0	0	1	
G913	1	0	0	1					1	0	0	1	
G914	0	1	0	0					1	0	0	1	
G915	0	0	0	0					1	1	0	0	
G916	0	0	0	1					0	0	0	0	
G917	1	0	0	0					1	1	0	1	
G918	1	0	0	0					1	0	0	0	
G919	0	0	0	0					-	-	-	-	
G920	0	0	0	1					0	0	0	0	
G921	0	0	0	1					0	0	0	0	
G922	1	0	0	0					1	0	0	0	
G923	0	0	0	1					0	0	0	0	
G924	0	0	0	1					0	0	0	0	
G925	0	1	0	0	0	0	1	0	1;0	0;0	0;0	1;1	
G926	1	1	0	0					1	1	0	1	
G927	0	0	0	0	0	0	0	1	0;1	0	0	0;0	
G928	0	1	0	0	0	1	0	0	1;1	0	0	1;1	
G929	0	1	0	0					1	0	0	1	
G930	0	1	0	0					1	0	0	1	
G931	0	1	0	0					1	0	0	1	
G932	0	0	0	1					0	0	0	0	
G933	1	0	0	0					1	1	0	1	
G934	0	0	0	0					1	0	0	0	
G935	0	0	0	0					1	0	0	0	
G936	0	0	0	1					0	0	0	0	
G937	0	0	0	1					1	0	0	0	
G938	0	0	0	1					0	0	0	0	
G939	0	1	0	0	0	0	0	0	1;-	0;-	0;-	1;-	
G940	0	0	0	1					0	0	0	0	
G941	0	0	0	1					1	0	0	0	
G942	0	0	0	1					0	0	0	0	
G943	0	0	0	0					0	0	0	0	
G944	0	0	1	0					1	0	0	1	
G945	0	1	0	0					1	1	0	1	
Total	5	15	2	14	2	3	2	1	27;3	9;3	00;0	19;4	

Table 21: Commonly Used Methods

	Total					
	B	T	TI	O	Total	Comments
G901	5	2	1	0	8	Bal.; Transp.; Trial & Impr. (**B, T & TI**)
G902	6	1	0	0	7	(B & T)
G903	4	4	0	0	8	(B & T)
G904	5	7	0	0	12	(T & B)
G905	3	6	0	0	9	(T & B)
G906	2	7	0	0	9	(T & B)
G907	0	0	7	1	8	Trial & Impr. & Other (**TI & O**)
G908	1	0	4	2	7	(TI; O & B)
G909	1	0	2	4	7	(O; TI & B)
G910	1	1	0	6	8	(O; B & T)
G911	4	5	0	1	10	(T; B & O)
G912	4	8	0	0	12	(T & B)
G913	1	3	1	4	9	(O; T; B & TI)
G914	2	7	1	0	10	(T; B & TI)
G915	3	0	0	1	4	(B & O)
G916	0	0	1	7	8	(O & TI)
G917	7	2	0	0	9	(B & T)
G918	6	0	0	2	8	(B & O)
G919	0	0	0	3	3	(O)
G920	0	1	1	5	7	(O; T & TI)
G921	0	3	0	4	7	(O & T)
G922	6	0	0	3	9	(B)
G923	0	0	3	3	6	(TI & O)
G924	0	0	1	6	7	(O & TI)
G925	5	4	0	0	9	(B & T)
G926	6	4	0	0	10	(B & T)
G927	3	1	0	0	4	(B & T)
G928	3	3	0	0	6	(B & T)
G929	3	8	0	0	11	(T & B)
G930	3	8	0	0	11	(T & B)
G931	1	8	0	0	9	(T & B)
G932	1	0	0	7	8	(O & B)
G933	5	0	0	0	5	(B)
G934	1	0	2	0	3	(TI & B)
G935	1	0	2	0	3	(TI & B)
G936	0	0	1	7	8	(O & TI)
G937	0	4	0	4	8	(T & O)
G938	0	1	3	4	8	(O; TI & T)
G939	3	8	0	0	11	(T & B)
G940	0	0	0	8	8	(O)
G941	0	0	2	6	8	(O & TI)
G942	0	0	0	7	7	(O)
G943	0	0	3	0	3	(TI)
G944	3	4	4	0	11	(T; TI & B)
G945	6	1	0	0	7	(B & T)
%	30%	31.71%	11.14%	27.14%		

Table 22: Proficiency

	Learner Proficiency								
	$x + 5 = 21$	$2x - 3 = 11$	$9 - x = 13$	$2x + 3(x - 2) = 6$	$x - 5(x - 1) = 3$	$3x + 5 = x + 9$	$3x + 5 = x + 3 + 2(x + 1)$	$x + 2 - (-2) = x - 4$	**Score**
G901	Y	Y	Y	Y	N	Y	N	N	**5**
G902	Y	Y	Y	Y	N	Y	N	N	**5**
G903	Y	Y	N	N	Y	Y	N	N	**4**
G904	Y	Y	Y	Y	Y	Y	Y	Y	**8**
G905	Y	Y	Y	Y	Y	Y	Y	Y	**8**
G906	Y	Y	Y	Y	N	Y	Y	Y	**7**
G907	Y	N	N	N	N	N	N	N	**1**
G908	N	N	N	N	N	N	N	N	**0**
G909	Y	N	Y	N	N	N	N	N	**2**
G910	Y	N	N	N	N	N	N	N	**1**
G911	Y	Y	Y	Y	N	Y	N	N	**5**
G912	Y	Y	Y	Y	N	N	N	N	**4**
G913	Y	N	N	N	N	N	N	N	**1**
G914	Y	N	N	Y	N	N	Y	N	**3**
G915	Y	N	N	N	N	N	N	N	**1**
G916	Y	N	N	N	N	N	N	N	**1**
G917	Y	Y	Y	N	N	Y	N	Y	**5**
G918	Y	Y	Y	Y	N	N	N	N	**4**
G919	N	N	N	N	N	N	N	N	**0**
G920	Y	N	N	N	N	N	N	N	**1**
G921	Y	N	N	N	N	N	N	N	**1**
G922	Y	Y	Y	Y	N	Y	Y	N	**6**
G923	Y	N	N	N	N	N	N	N	**1**
G924	Y	N	N	N	N	N	N	N	**1**
G925	Y	Y	N	N	N	Y	N	N	**3**
G926	Y	Y	Y	N	Y	N	Y	Y	**6**
G927	Y	Y	Y	N	N	N	N	N	**3**
G928	Y	Y	Y	N	N	N	N	N	**3**
G929	Y	Y	N	Y	N	Y	N	N	**4**
G930	Y	Y	Y	N	N	Y	N	N	**4**
G931	Y	Y	Y	Y	N	Y	Y	N	**6**
G932	Y	N	N	N	N	N	N	N	**1**
G933	Y	Y	Y	N	N	N	N	Y	**4**
G934	Y	N	Y	N	N	N	N	N	**2**
G935	Y	Y	N	N	N	N	N	N	**2**
G936	Y	N	N	N	N	N	N	N	**1**
G937	Y	Y	Y	N	N	N	N	N	**3**
G938	Y	Y	N	N	N	N	N	N	**2**
G939	Y	Y	Y	Y	N	Y	Y	N	**6**
G940	N	N	N	N	N	N	N	N	**0**
G941	Y	N	Y	N	N	N	N	N	**2**
G942	N	N	N	N	N	N	N	N	**0**
G943	Y	N	Y	N	N	N	N	N	**2**
G944	Y	Y	Y	Y	N	Y	Y	N	**6**
G945	Y	Y	Y	N	N	N	Y	Y	**5**
Total	**41**	**25**	**24**	**14**	**4**	**15**	**10**	**7**	

Table 23: Methods

	Test Item				Test Item				Test Item				Total				
	x + 5 = 21				9 - x = 13				3x + 5 = x + 9				Strategies				Common
Method	**B**	**T**	**TI**	**O**	**B**	**T**	**TI**	**O**	**B**	**T**	**TI**	**O**	**B**	**T**	**TI**	**O**	**Strategy**
Vicky	0	1	0	0	0	1	0	0	1	1	0	0	1	3	0	0	Transposition
Violet	0	1	0	0	1	0	0	0	1	1	0	0	2	2	0	0	Bal & Trans
Kison	1	0	0	0	1	0	0	0	0	1	0	0	2	1	0	0	Balance Met.
Kitos	1	0	0	0	1	0	0	0	0	1	0	0	2	1	0	0	Balance Met.
Miriam	0	1	0	0	0	1	0	0	1	1	0	0	1	3	0	0	Transposition
Total	2	3	0	0	3	2	0	0	3	5	0	0	8	10	0	0	

Table 24: Methods

	Interview Item				Interview Item				Interview Item				Total				
	4 + x = 23				4 - x = 21				3x + 2 = 8 + x				Strategies				Common
Method	**B**	**T**	**TI**	**O**	**B**	**T**	**TI**	**O**	**B**	**T**	**TI**	**O**	**B**	**T**	**TI**	**O**	**Strategy**
Vicky	0	1	0	0	0	1	0	0	1	1	0	0	1	3	0	0	Transposition
Violet	0	1	0	0	0	1	0	0	0	1	0	0	0	3	0	0	Transposition
Kison	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	3	Other
Kitos	0	0	1	0	0	0	1	0	0	1	0	0	0	1	2	0	Trial & Improve
Miriam	0	1	0	0	0	1	0	0	0	1	0	0	0	3	0	0	Transposition
Total	0	3	1	1	0	3	1	1	1	4	0	1	1	10	2	3	

Table 25: Strands

	x + 5 = 21				9 - x = 13				3x + 5 = x + 9				Proficiency		
	Strands				Strands				Strands						
	CU	**PF**	**AR**	**SC**	**CU**	**PF**	**AR**	**SC**	**CU**	**PF**	**AR**	**SC**	**Qn1**	**Qn1**	**Qn1**
Vicky	1	1	0	1	1	1	0	1	1	1	0	1	**P**	**P**	**P**
Violet	1	1	0	1	1	0	0	1	1	1	0	1	**P**	**NP**	**P**
Kison	1	1	0	1	1	1	0	1	0	0	0	0	**P**	**P**	**NP**
Kitos	1	1	0	1	1	1	0	1	1	0	0	1	**P**	**P**	**NP**
Miriam	1	1	0	1	1	1	0	1	1	1	0	1	**P**	**P**	**P**
Total	5	5	0	5	5	4	0	5	4	3	0	4	5	4	3

Table 26: Strands

	4 + x = 23				4 - x = 21				3x + 2 = 8 + x				Proficiency		
	Strands				Strands				Strands						
	CU	**PF**	**AR**	**SC**	**CU**	**PF**	**AR**	**SC**	**CU**	**PF**	**AR**	**SC**	**Qn1**	**Qn1**	**Qn1**
Vicky	1	1	0	1	1	1	0	1	1	1	0	1	**P**	**P**	P
Violet	1	1	0	1	1	0	0	1	0	0	0	1	**P**	**NP**	NP
Kison	1	0	0	1	1	0	0	1	1	0	0	0	**NP**	**NP**	NP
Kitos	1	1	0	1	1	0	0	1	1	0	0	1	**P**	**NP**	NP
Miriam	1	1	0	1	1	1	0	1	1	0	0	1	**P**	**P**	NP
Total	5	4	0	5	5	2	0	5	4	1	0	4	4	2	1

APPENDIX: D

Interviewed learners' test and interview Learner Scripts

<u>Vicky's</u> <u>Test S1 & S2</u>

✓ M(2)

PARTICIPANT IDENTIFICATION CODE: 1990A

Test
Section 1
Fill in the missing number(s) in the boxes, and show how true is/are the number(s) you have filled in, in the space provided under each problem.

(a) $7 + \boxed{8} = 15$ (1) $15 - 7 = 8$ then $8 + 7 = 15$ (1)
(b) $7 - \boxed{22} = 15$ 0
(c) $\boxed{5} - 7 = -2$ (1)
(d) $\boxed{5} + 3 = 8 - \boxed{5}$ 0
(e) $17 = \boxed{23} - 6$ (1)
(f) $13 = \boxed{20} - \boxed{7}$ (1) (1)
(g) $\boxed{12} + 35 = 47 - \boxed{12}$ 0 TRND

9904

Section 2
Use the spaces provided (under each problem), and solve each of the problems below.

(a) $x + 5 = 21$ $x = 21 - 5$ T $x = 16$ A $\therefore x = 16$	**(b) $2x - 3 = 11$** $2x = 11 + 3$ T $2x = 14$ S $\frac{2x}{2} = \frac{14}{2}$ BM $\therefore x = 7$ A
(c) $9 - x = 13$ $x = 13 - 9$ T $x = 4$ $\therefore x = 4$	**(d) $2x + 3(x - 2) = 6$** $2x + 3x - 6 = 6$ RB $2x + 3x = 6 + 6$ T $5x = 12$ S $\frac{5x}{5} = \frac{12}{5}$ B
(e) $x - 5(x - 1) = 3$ $x - 5x + 5 = 3$ RB $x - 5x = 3 - 5$ T $5x = -2$ S $\frac{5x}{5} = \frac{2}{5}$ B	**(f) $3x + 5 = x + 9$** $3x - x = 9 - 5$ CLT / T $3x = 4$ S $\frac{3x}{3} = \frac{4}{3}$ BM
(g) $3x + 5 = x + 3 + 2(x + 1)$ $3x + 5 = x + 3 + 2x + 2$ RB $3x - x - 2x = 3 + 2 - 5$ CLT / T $5x = 10$ S $\frac{5x}{5} = \frac{10}{5}$ B $\therefore x = 2$	**(g) $x + 2 - (-2) = x - 4$** $x + 4 = x - 4$ S $x + x = -4 + 4$ CLT / T $x = 0$

Vicky's Interview S1 & S2

Interview questions & Participant Code: G904

Section 1
Fill in the missing number(s) in the box(es), and give reason, in each case, for your choice in the spaces provided under each problem.

(a) $5 + \boxed{8} = 13$ 1

Reason:
$13 - 5 = 8$
$\therefore 5 + 8 = 13$

(b) $5 - \boxed{16} = 11$ F4

Reason
− and + you subtract and keep the sign of the bigger number QA
$\therefore 16 - 5 = 11$

(c) $15 = \boxed{20} - \boxed{5}$ 1

Reason
$15 + 5 = 20$
$20 - 5 = 15$
$\therefore 15 = 20 - 5$

(d) $19 - \boxed{9} = 10 + \boxed{9}$ F4

Reason
$19 - 9 = 10$
$\therefore 10 + 9 = 19$
$\therefore$ The ~~and~~ answer is
$19 - 9 = 10 + 9$ QA

2

G904

Section 2
Solve for x, in each of the problems given below.

(a) $4 + x = 23$
$x = 23 - 4$ T
$x = 19$ A

(b) $4 - x = 21$
$x = 21 - 4$
$x = 17$ QA

(c) $3(x + 2) = 18$
$3x + 6 = 18$ RB
$3x = 18 - 6$ F
$3x = 12$ S
$\frac{3x}{3} = \frac{12}{3}$ B
$x = 4$ A

(d) $3x + 2 = 8 + x$
$3x - x = 8 - 2$ T
$4x = 6$ S
$\frac{4x}{4} = \frac{6}{4}$ B
QA

(e) $x + 3 = x - 3$
$x + x = -3 - 3$ T / QA
$2x = -6$ S
$\frac{2x}{2} = \frac{-6}{2}$ B
$x = 3$ QA

3

Violet's Test S1 & S2

✓ m@

PARTICIPANT IDENTIFICATION CODE: 9925

Test
Section 1
Fill in the missing number(s) in the boxes, and show how true is/are the number(s) you have filled in, in the space provided under each problem.

(a) $7 + \boxed{8} = 15$ I
(b) $7 - \boxed{22} = 15$ O
(c) $\boxed{9} - 7 = -2$ O
(d) $\boxed{5} + 3 = 8 - \boxed{5}$ O
(e) $17 = \boxed{12} - 6$ O
(f) $13 = \boxed{20} - \boxed{7}$ I I
(g) $\boxed{12} + 35 = 47 - \boxed{12}$ O TRND

9925

Section 2
Use the spaces provided (under each problem), and solve each of the problems below.

(a) $x + 5 = 21$ $x = 21 + 5$ T $x = 16$ A	(b) $2x - 3 = 11$ $2x - 3 + 3 = 11 + 3$ Bm $2x = 14$ S $\frac{2x}{2} = \frac{14}{2}$ Bm $x = 7$ A
(c) $9 - x = 13$ $9 - x + x = 13 + x$ Bm $x = 13x$ S	(d) $2x + 3(x - 2) = 6$ $2x - 3 + 3 - 2 = 6 + 2$ Oth $2x = 8$ $\frac{2x}{4} = \frac{8}{4}$ B $= 2$
(e) $x - 5(x - 1) = 3$ $-x = 13 - 9$ Oth $-x = 4$	(f) $3x + 5 = x + 9$ $3x = 5 + 9$ T $\frac{3x}{3} = \frac{14}{3}$ S|Bm $x = 4,8$
(g) $3x + 5 = x + 3 + 2(x + 1)$ $3x = 3 + 2 + 1 - 5$ T $\frac{3x}{3} = \frac{1}{3}$ S|Bm $x = 0,3$	(g) $x + 2 - (-2) = x - 4$ $x = 4 - 2 + 2$ T $x = 4$

<u>Violet's</u> <u>Interview S1 & S2</u>

<u>Interview questions & Participant Code: S925</u>

Section 1
Fill in the missing number(s) in the box(es), and give reason, in each case, for your choice in the spaces provided under each problem.

(a) $5 + \boxed{8} = 13$ \	(b) $5 - \boxed{6} = 11$ QA
(c) $15 = \boxed{20} - \boxed{5}$ \ FU	(d) $19 - \boxed{9} = 10 + \boxed{9}$ QA

2

S925

Section 2
Solve for x, in each of the problems given below.

(a) $4 + x = 23$ $x = 23 - 4$ T $x = 19$ A	(b) $4 - x = 21$ $x = 24\ 21 + 4$ T FU $x = 25$ QA
(c) $3(x + 2) = 18$ $3x + 6 = 18$ RB $x = 18 - 9$ T FU $x = 9$ QA	(d) $3x + 2 = 8 + x$ $3x + x = 8 + 2$ T FU $3x^2 + 10$ QA
(e) $x + 3 = x - 3$ $6 + 3 = 6 - 3$ T \ FU	

3

Kison's Test S1 & S2

✓ WG 5

PARTICIPANT IDENTIFICATION CODE: K9927

Test
Section 1
Fill in the missing number(s) in the boxes, and show how true is/are the number(s) you have filled in, in the space provided under each problem.

(a) $7 + \boxed{8} = 15$ You find the number that is going to add by 7 and ~~correct~~ get 15 and is 8 ✓
$7 + 8 = 15$
1

(b) $7 - \boxed{-8} = 15$ You have to ~~Spam~~ Simpfic By -8 ~~an~~ to get 15
1

(c) $\boxed{5} - 7 = -2$ You Simptrit By 5 in to -7 and you will get -2
1

(d) $\boxed{5} + 3 = 8 - \boxed{5}$ You Simptri the number you add and than you get the number you add
0
$5 + 3 = 8 - 5$
$8 = 3$
$8 - 3 = 3 - 3$

(e) $17 = \boxed{23} - 6$ You find the number and than you Simprti by 6 to get 17
$17 = 23 - 6$
1
$23 - 6 = 17$

(f) $13 = \boxed{20} - \boxed{7}$
$20 - 7 = 13$?
1
Cover the number you want ~~to~~ add and the number is must be nfetive number

(g) $\boxed{12} + 35 = 47 - \boxed{12}$ You add by 12 and than you Simprit by the number you have been add
0
TRND

9927

Section 2
Use the spaces provided (under each problem), and solve each of the problems below.

(a) x + 5 = 21 $x = 16$ CUM / T & I $x + 5 - 5 = 21 - 5$ 8 M $x + 0 = 16$ $x = 16$ A	**(b) 2x − 3 = 11** $2x = 14$ CUM $2x - 3 - 3 = 11 + 3$ BM / Er $2x - 0 = 14$ $2x = 14$ S $x = ?$ what is solving for x
(c) 9 − x = 13 $x = 22$ $9 - 9 - x = 13 + 9$ BM / Er $0 - x = 22$ $x = 22$	**(d) 2x + 3(x − 2) = 6** $2x(3x - 6) = 6$ RB $2x + 3x - 0$? $5x$
(e) x − 5(x − 1) = 3 $x^2 - x + 5x - 4$? $x^2 - 4x$ $= 3$	**(f) 3x + 5 = x + 9** $3x^2 + 14$ CT $17x^2$ Conjoining
(g) 3x + 5 = x + 3 + 2(x + 1) $3x + x(x + 1)5 + 3 + 2$ $3x^3 + 11$? (=) cease to exist	**(g) x + 2 − (−2) = x − 4** $x + x - 2(-2) - 4$ $x^2 + 4 - 4$? x^2

Where x appears twice there is written x^2
Where x appears three times there is written x^3.
However in (d) there is ... x appear twice

Kison's Interview S1 & S2

Interview questions & Participant Code: 69927

Section 1
Fill in the missing number(s) in the box(es), and give reason, in each case, for your choice in the spaces provided under each problem.

(a) 5 + $\boxed{7}$ = 13

~~5 + 7 = 13~~ x = 7 QA

You add by 7 than you get 13

(b) 5 - $\boxed{-6}$ = 11

5 - 11 = -6
x = -6 QA

You add by -6 the your answer will be 11

(c) 15 = $\boxed{-5}$ - $\boxed{10}$ 10 QA

You add by -ve number than you will get the +ve number

(d) 19 - $\boxed{9}$ = 10 + $\boxed{9}$ QA

The number you add is the number that you will spireti and get it.

2

9927

Section 2
Solve for x, in each of the problems given below.

(a) 4 + x = 23

x = 17 A FU

(b) 4 − x = 21

x = -17 A FU

(c) 3(x + 2) = 18

x = 6 QA

(d) 3x + 2 = 8 + x

x = 1
x = 9 QA

(e) x + 3 = x − 3

x = 3 QA
x = 6

3

Kitos' Test S1 & S2

MLE

PARTICIPANT IDENTIFICATION CODE: G928 1

Test
Section 1
Fill in the missing number(s) in the boxes, and show how true is/are the number(s) you have filled in, in the space provided under each problem.

(a) 7 + $\boxed{8}$ = 15
You find the number that is going to add up 7 and you get 15 that number is 8
7+8 = 15

(b) 7 - $\boxed{-8}$ = 15
You find the number that is going to subtracted up with 7 and you get 15 that number is −8
7 − −8 = 15

(c) $\boxed{5}$ - 7 = -2
You find the number that you are going to subtract up with 7 and you get −2 so that number is 5
5 − 7 = −2

(d) $\boxed{5}$ + 3 = 8 - $\boxed{5}$
You find the number that you are going to add up with 3 and you get 8 and you are going to find the number that you are going to subtract up with 8 and you get the number is 5, 5+3 = 8−5

(e) 17 = $\boxed{23}$ - 6
You find the number that you are going to subtract up with 6 and you get 17 so that number is 23
23 − 6 = 17

(f) 13 = $\boxed{20}$ - $\boxed{7}$
You find the numbers that are going to subtract both of them and then you get 13 that numbers are 20 and 7
20 − 7 = 13

(g) $\boxed{12}$ + 35 = 47 - $\boxed{12}$
TRND
You find the number that you are going to add up with 35 and you get 47 and that number is 12. And then you find the number that you are going to subtract up with 47 and that number is 12. 12 + 35 = 47 − 12

G928

Section 2
Use the spaces provided (under each problem), and solve each of the problems below.

(a) x + 5 = 21 $x+5-5=21-5$ B $x+0=16$ S $x=16$ A	**(b) 2x − 3 = 11** ~~2x+2=11 2~~ $2x+2-3=11+2$ S $4x-3=13$ $\frac{1x}{1}=\frac{13}{1}$ $x=13$
(c) 9 − x = 13 $9+9-x=13+9$ B $0-x=22$ S $x=22$	**(d) 2x + 3(x − 2) = 6** $2x+x+3-2=6$ RB/Er ~~$2x^2+1=6$~~
(e) x − 5(x − 1) = 3 $x-x+5-1=3$ RB/Er	**(f) 3x + 5 = x + 9** $3x+x=5+9$ CLT/T
(g) 3x + 5 = x + 3 + 2(x + 1) $3x+x\neq x=5+3(2+1)$ CLT/T	**(g) x + 2 − (-2) = x − 4** $x+x-2\neq(-2)-4$ T

The first 3 qns was solving for x. The rest could do nothing to show that the learner understands. Most likely the change of strategy was due to introduction of brackets in (d). Interviews will solve this mystery.

Kitos' Interview's S1 & S2

Interview questions & Participant Code: G928

Section 1

Fill in the missing number(s) in the box(es), and give reason, in each case, for your choice in the spaces provided under each problem.

(a) $5 + \boxed{8} = 13$ 1 When you add up 5 + 8 you get 13 that is why I am adding up with 8	(b) $5 - \boxed{16} = 11$ QA When you subtract 5 from 16 you get 11 that is why I am giving 16 in the provided space.
(c) $15 = \boxed{5} - \boxed{20}$ QA Here I am subtracting 5 from 20 to get 15 that is why I am giving 5 and 20 on the space provided	(d) $19 - \boxed{9} = 10 + \boxed{9}$ Fu Here I am subtracting and adding the same numbers. I subtract 9 from 19 I get 10 then I am adding again 9 to 10 to get 19 why?

G928

Section 2

Solve for x, in each of the problems given below.

(a) $4 + x = 23$ $4 + 19 = 23$ T1 $\therefore x = 19$ A	(b) $4 - x = 21$ $4 - 25 = 21$ T1 $\therefore x = 4$ QA
(c) $3(x + 2) = 18$ $\cancel{3x + 6 = 18}$ $3(4+2) = 18$ T1 $18 + 6 = 18$ QA	(d) $3x + 2 = 8 + x$ $3x + x = 8 + 2$ T $3x^2 = 10$ Fu $2x + x = 2x^2$
(e) $x + 3 = x - 3$ $x + x = 3 - 3$ T $x^2 = 0$ Fu	

3

Miriam's Test S1 & S2

M(7)

PARTICIPANT IDENTIFICATION CODE: [G939]

Test
Section 1
Fill in the missing number(s) in the boxes, and show how true is/are the number(s) you have filled in, in the space provided under each problem.

(a) $7 + \boxed{8} = 15$ 1
(b) $7 - \boxed{22} = 15$ 0
(c) $\boxed{5} - 7 = -2$ 1
(d) $\boxed{5} + 3 = 8 - \boxed{5}$ 0
(e) $17 = \boxed{23} - 6$ 1
(f) $13 = \boxed{7} - \boxed{6}$ 0 INT   *(Likely thought $7 - -6 = 7 + 6 = 13$)
(g) $\boxed{12} + 35 = 47 - \boxed{12}$ 0 TRND

9939

Section 2
Use the spaces provided (under each problem), and solve each of the problems below.

(a) $x + 5 = 21$ $x = 21 - 5$ T $x = 16$ A	(b) $2x - 3 = 11$ $2x = 11 + 3$ T $2x = \frac{13}{2}$ B $x = 6.5$
(c) $9 - x = 13$ $-x = 13 - 9$ T $-x = 4$ S	(d) $2x + 3(x - 2) = 6$ $2x = 6 + 2 - 3$ T $\frac{2x}{2} = \frac{5}{2}$ BM $x = 2,5$
(e) $x - 5(x - 1) = 3$ $x = 3 + 5 + 1$ T with E $x = 9$	(f) $3x + 5 = x + 9$ $3x = 5 + 9$ T $\frac{3x}{3} = \frac{14}{3}$ S / BM $x = 4,8$
(g) $3x + 5 = x + 3 + 2(x + 1)$ $3x = 3 + 2 + 1 - 5$ T $\frac{3x}{3} = \frac{1}{3}$ S / BM $x = 0,3$	(g) $x + 2 - (-2) = x - 4$ $x = 4 - 2 + 2$ T $x = 4$

Miriam's Interview S1 & S2

Interview questions & Participant Code: [G939]

Section 1
Fill in the missing number(s) in the box(es), and give reason, in each case, for your choice in the spaces provided under each problem.

(a) $5 + \boxed{8} = 13$

adding 5 and 8 you will get thirteen

(b) $5 - \boxed{6} = 11$ — FU

five minus six equals to negative eleven

(c) $15 = \boxed{15} - \boxed{30}$ — QA

17 thirty minus fifteen it equals to fifteen so does fifteen times or multiply by two — FU

(d) $19 - \boxed{9} = 10 + \boxed{9}$ — FU

17 nineteen minus nine equals to ten so does ten plus nine ?

2

G939

Section 2
Solve for x, in each of the problems given below.

(a) $4 + x = 23$ — T A

$x = 23 - 4$
$x = 19$

(b) $4 - x = 21$ — T FU

$-x = 21 - 4$
$-x = 17$

(c) $3(x + 2) = 18$ — T/QA

$x = 18 - 3 - 2$
$x = 13$

(d) $3x + 2 = 8 + x$ — T/QA

$x = 8 - 2 - 3$
$x = 3$

(e) $x + 3 = x - 3$ — ? QA