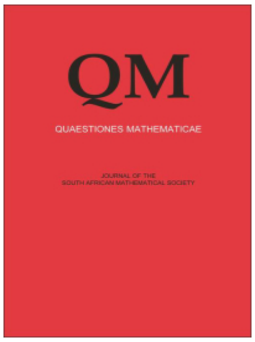

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INTERIOR VERTICES AND EDGES IN INTEGER PARTITIONS

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Abstract. This paper characterises the shape of the Young diagram associated with integer partitions in terms of two newly invented statistics which are defined in the text: namely interior vertices and horizontal edges. Generating functions and associated asymptotic results are developed for both interior vertices and horizontal edges.

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Key words: Partitions,vertices, edges, generating function, asymptotics.

1. Introduction. Partition statistics date back to the 1930s when a paper in nuclear physics appeared on the subject (see [9]). To our knowledge, this is the earliest mention of partition statistics and demonstrates the relevance of the subject in physics, where partition statistics naturally occur. However, it was Erdös and Lehner [4] in 1941 who created the interest in pure mathematics by determining

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the average value for quantities such as the number of parts, the number of distinct part sizes and the size of the largest part in a random partition of n. The current paper studies statistics which are descriptive of the shape of the associated Young diagram and rely on characterising the Young diagram in terms of newly invented concepts interior vertices and interior edges.

An *integer partition* of a positive integer n is a sequence of non-increasing positive integers $\pi_1, \pi_2, \ldots, \pi_m$ (each called a part) which sums to n. i.e. $\sum_{i=1}^{m} \pi_i = n$. The number of integer partitions of n is denoted by $p(n)$; this is called the *partition function* where $p(0) := 1$. Some recent results in relation to other statistics on integer partitions appear in [3, 6, 8, 13, 14, 15].

We represent a partition graphically using a Young diagram, see [2]. Each part of the partition is represented by a column where the height or the number of cells is equal to the size of the part. These columns are bottom justified and the heights are in non-increasing order. A vertex of a partition is a pair of positive integers (n, m) in the Young diagram. An *interior vertex* is one which does not lie on the boundary of the Young diagram. A (horizontal) *edge* of a partition is a segment linking the two lattice points (n, m) and $(n+1, m)$ in the Young diagram. In particular, we call this edge a (horizontal) *d-edge* if it links d interior vertices where $d = 0, 1$ or 2. An illustration of interior vertices and horizontal edges is shown in Figures 1 and 2.

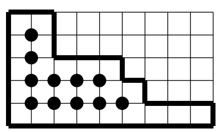

Figure 1: The partition 553332111 and its eleven interior vertices

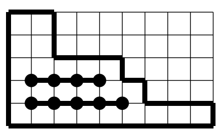

Figure 2: The partition 553332111 and its seven 2-edges

Some recent work on these latter statistics has been done by Mansour in [10] where the statistic of interior vertices in a set partition is studied, and by Mansour and Shabani in [11] where the number of interior vertices and edges in a bargraph were considered (for other examples see [12] and references therein).

The remainder of the paper is organized as follows. In Section 2, we introduce various results that are needed for the rest of the paper. In Section 3, we produce the generating function for the total number of interior vertices for the integer partitions of n and finally in Section 4, we deal separately with the question of the three types of interior horizontal edges.

2. Preliminaries. Let $Int_v(\pi)$ denote the set of interior vertices of a given partition π and we denote its cardinality by $int_v(\pi)$.

We shall state results that will be needed in the following sections:

The well known generating function for all partitions of n, where x tracks the size n is (see [1])

$$P(x) := \prod_{i=1}^{\infty} \frac{1}{1-x^i}.$$

We also need the famous identity of Euler, see page 19 of [1]:

$$1 + \sum_{k=1}^{\infty} t^k \prod_{j=1}^{k} \frac{1}{1-x^j} = \prod_{k=0}^{\infty} \frac{1}{1-tx^k}, \tag{2.1}$$

and we use the following lemma:

LEMMA 2.1.

$$\sum_{j=1}^{\infty} \left(\sum_{i=j+1}^{\infty} \frac{x^i}{1-x^i} \right) x^j \prod_{i=1}^{j} \frac{1}{1-x^i} = P(x) - \sum_{i=1}^{\infty} \frac{x^i}{1-x^i} - 1.$$

Proof. Consider

$$\sum_{j=1}^{\infty} \left(\sum_{i=j+1}^{\infty} \frac{x^i}{1-x^i} \right) x^j \prod_{i=1}^{j} \frac{1}{1-x^i}. \tag{2.2}$$

The expression $\sum_{i=j+1}^{\infty} \frac{x^i}{1-x^i}$ represents the largest part size as multiple copies of a part of size i where $i \geq j+1$. The second largest part size is j and $\prod_{i=1}^{j} \frac{1}{1-x^i}$ is the generating function for all parts less than or equal to j. The full expression (2.2) is therefore the generating function for all partitions decomposed into at least two part sizes i (where $i > j$) and j, for all possible j. Thus, (2.2) is the generating function for partitions with at least two parts sizes; i.e., with generating function

$$P(x) - \sum_{i=1}^{\infty} \frac{x^i}{1-x^i} - 1.$$

□

In a paper by Grabner, Knopfmacher and Wagner, [7], an asymptotic study of a general class of generating functions of the form $P(x)F(x)$ was undertaken, with numerous applications given to various partition statistics. We make use of two of these asymptotic results in the current paper:

PROPOSITION 2.2. *Let* $f(x) = P(x)\sum_{j=1}^{\infty}\frac{x^j}{1-x^j}$ *then*

$$\begin{aligned}
&\frac{1}{P(n)}[x^n]f(x)\\
&\quad=\frac{\sqrt{6n}}{2\pi}\left(\log n + 2\gamma - \log\frac{\pi^2}{6}\right) + \frac{3}{2\pi^2}\log n + \frac{1}{4} + \frac{3}{2\pi^2}\left(1 + 2\gamma - \log\frac{\pi^2}{6}\right)\\
&\qquad\qquad + O\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

PROPOSITION 2.3. *Let* $g(x) = \frac{x}{1-x}P(x)$ *then*

$$\frac{1}{P(n)}[x^n]g(x) = \frac{\sqrt{6n}}{\pi} + \frac{6-\pi^2}{2\pi^2} + O\left(\frac{1}{\sqrt{n}}\right).$$

3. Interior vertices. Define $P(n,k)$ to be the set of partitions of n such that the largest part is k. Let $V_k(x;q)$ be the generating function for the number of partitions in $P(n,k)$ according to the number of interior vertices, namely,

$$V_k(x;q) = \sum_{n\geq k}\sum_{\pi\in P(n,k)} x^n q^{int_v(\pi)}.$$

From the definition, we have

$$V_k(x;q) = x^k + x^k q^{k-1}V_k(x;q) + x^k q^{k-2}V_{k-1}(x;q) + \cdots + x^k V_1(x;q),$$

from which we deduce

$$V_k(x;q) - xV_{k-1}(x;q) = x^k q^{k-1}V_k(x;q).$$

Thus,

$$V_k(x;q) = \frac{x}{1-x^k q^{k-1}}V_{k-1}(x;q)$$

with $V_0(x;q) = 1$. Hence,

$$V_k(x;q) = \frac{x^k}{\prod_{j=1}^{k}(1-x^j q^{j-1})}.$$

This leads to the following result.

THEOREM 3.1. *The generating function*

$$V(x,y;q) = \sum_{k\geq 0}\sum_{n\geq k}\sum_{\pi\in P(n,k)} x^n y^k q^{int_v(\pi)}$$

for the number of partitions of n according to value of the largest part, number of parts and to the number of interior vertices respectively is given by

$$V(x,y;q)=\sum_{k\geq 0}\frac{x^k y^k}{\prod_{j=1}^{k}(1-x^j q^{j-1})},$$

where x,y,q respectively track the three statistics.

Theorem 3.1 yields

$$\begin{aligned}V'(x)&=\frac{d}{dq}V(x,1;q)\mid_{q=1}=\sum_{k\geq 0}\frac{x^k}{\prod_{j=1}^{k}(1-x^j)}\sum_{j=1}^{k}\frac{(j-1)x^j}{1-x^j}\\&=\underbrace{x\frac{d}{dx}\sum_{k\geq 0}\frac{x^k}{\prod_{j=1}^{k}(1-x^j)}}_{A}-\underbrace{\sum_{k\geq 1}\frac{x^k}{\prod_{j=1}^{k}(1-x^j)}\sum_{j=1}^{k}\frac{1}{1-x^j}}_{B}\end{aligned}$$

We consider the two terms A and B separately.

For term A, we have the following:

$$A=x\frac{d}{dx}\sum_{k\geq 0}\frac{x^k}{\prod_{j=1}^{k}(1-x^j)}=\sum_{k\geq 0}\frac{x^k}{\prod_{j=1}^{k}(1-x^j)}\sum_{j=1}^{k}\frac{1+(j-1)x^j}{1-x^j}.$$

We simplify term B as follows:

$$\begin{aligned}B&=\sum_{k\geq 1}x^k\prod_{j=1}^{k}\frac{1}{1-x^j}\sum_{j=1}^{k}\left(\frac{1-x^j}{1-x^j}+\frac{x^j}{1-x^j}\right)\\&=\sum_{k\geq 1}x^k\prod_{j=1}^{k}\frac{1}{1-x^j}\left[k+\sum_{j=1}^{\infty}\frac{x^j}{1-x^j}-\sum_{j=k+1}^{\infty}\frac{x^j}{1-x^j}\right]\\&=\underbrace{\sum_{k\geq 1}kx^k\prod_{j=1}^{k}\frac{1}{1-x^j}}_{C}+\underbrace{[P(x)-1]\sum_{j=1}^{\infty}\frac{x^j}{1-x^j}}_{D}-\underbrace{\sum_{k\geq 1}\sum_{j=k+1}\frac{x^j}{1-x^j}x^k\prod_{j=1}^{k}\frac{1}{1-x^j}}_{E}.\end{aligned}$$

In D we used the well known decomposition of partitions according to their largest part, i.e.,

$$P(x)-1=\sum_{k\geq 1}x^k\prod_{j=1}^{k}\frac{1}{1-x^j}.$$

For term E we just use Lemma 2.1. Thus

$$E=P(x)-1-\sum_{i=1}^{\infty}\frac{x^i}{1-x^i}.$$

For term C, we use the Euler identity (2.1) which we differentiate with respect to t to obtain:

$$\sum_{k=1}^{\infty} kt^{k-1} \prod_{j=1}^{k} \frac{1}{1-x^j} = \prod_{k=0}^{\infty} \frac{1}{1-tx^k} \sum_{j=0}^{\infty} \frac{x^j}{1-tx^j};$$

then letting $t = x$ we get

$$\sum_{k=1}^{\infty} kx^{k-1} \prod_{j=1}^{k} \frac{1}{1-x^j} = \prod_{k=0}^{\infty} \frac{1}{1-x^{k+1}} \sum_{j=0}^{\infty} \frac{x^j}{1-x^{j+1}}. \tag{3.1}$$

Thus

$$C = \prod_{k=0}^{\infty} \frac{1}{1-x^{k+1}} \sum_{j=0}^{\infty} \frac{x^{j+1}}{1-x^{j+1}} = P(x) \sum_{j=1}^{\infty} \frac{x^j}{1-x^j}. \tag{3.2}$$

So we have

$$\begin{aligned} B &= P(x) \sum_{j=1}^{\infty} \frac{x^j}{1-x^j} + [P(x)-1] \sum_{j=1}^{\infty} \frac{x^j}{1-x^j} - P(x) + 1 + \sum_{i=1}^{\infty} \frac{x^i}{1-x^i} \\ &= 1 + P(x) \left(2 \sum_{i=1}^{\infty} \frac{x^i}{1-x^i} - 1 \right). \end{aligned}$$

So, altogether

$$V'(x) = x \frac{d}{dx} \sum_{k=0}^{\infty} x^k \prod_{j=1}^{k} \frac{1}{1-x^j} - 1 - P(x) \left(2 \sum_{i=1}^{\infty} \frac{x^i}{1-x^i} - 1 \right).$$

We now apply (2.1) to the differentiated term and obtain:

THEOREM 3.2. *The generating function for the total number of interior vertices in partitions of n is*

$$xP'(x) - 1 - P(x) \left(2 \sum_{i=1}^{\infty} \frac{x^i}{1-x^i} - 1 \right). \tag{3.3}$$

The average number of interior vertices in partitions of n is

$$\begin{aligned} &n - \frac{\sqrt{6n}}{\pi} \left(\log n + 2\gamma - \log \frac{\pi^2}{6} \right) - \frac{3}{\pi^2} \log n + \frac{1}{2} \\ &\qquad - \frac{3}{\pi^2} \left(1 + 2\gamma - \log \frac{\pi^2}{6} \right) + O\left(\frac{1}{\sqrt{n}} \right). \end{aligned} \tag{3.4}$$

To prove (3.4) we apply Propositions 2.2 and 2.3 to (3.3). □

4. Horizontal edges. In this section we study the number of d-edges for $d = 0, 1$ or 2 in partitions of n. Let k be the size of the first part. We suppress variables and define $I_k := I_k(x, q_0, q_1, q_2, q_0', q_1')$ to be the generating function for partitions where x tracks the size of the partition, q_0 counts the total number of 0-edges other than those in the first column, q_1 the total number of 1-edges other than those in the first column and q_2 the total number 2-edges. We also use q_0' as opposed to q_0 to track these type of edges which are in the first column of the partition and similarly for q_1'. Clearly there are no 2-edges in the first column, so we do not need q_2'. We also use $I_k^* := I_k^*(x, q_0, q_1, q_2) := I_k(x, q_0, q_1, q_2, q_1, q_2)$ to be the generating function where we replace q_0' by q_1 and q_1' by q_2 in the expression and finally we use $I_k^o(x, q_0, q_1, q_2) := I_k(x, q_0, q_1, q_2, q_0, q_1)$ to be the generating function where we replace q_0' by q_0 and q_1' by q_1 in I_k. It is this latter which tracks the total number of the various d-edges and which ultimately is what we wish to solve for.

Now define

$$I(y, q_0, q_1, q_2, q_0', q_1') := \sum_{k=1}^{\infty} I_k y^k, \quad I^*(y, q_0, q_1, q_2) := \sum_{k=1}^{\infty} I_k^*(y, q_0, q_1, q_2) y^k$$

and

$$I^o(y, q_0, q_1, q_2) := \sum_{k=1}^{\infty} I_k^o(y, q_0, q_1, q_2) y^k.$$

We separate the case where the partition consists of one part only or has more than one part. Thus,

$$I_k = \underbrace{x^k (q_0')^{k+1}}_{\text{1 part only}} + \underbrace{x^k \sum_{s=1}^{k} (q_1')^{s-1} (q_0')^{k-s+2} \left(\frac{q_0}{q_1}\right)^2 I_s^*}_{\text{more than 1 part}} \tag{4.1}$$

Adding a new part of size k replaces the q_0' by q_1 except for the top and bottom edges of the second part, thus the term $\left(\frac{q_0}{q_1}\right)^2$. Multiplying by y^k and summing over all k gives

$$\begin{aligned} I(y, q_0, q_1, q_2, q_0', q_1') &= \sum_{k=1}^{\infty} \left(x^k (q_0')^{k+1} + x^k \sum_{s=1}^{k} (q_1')^{s-1} (q_0')^{k-s+2} \left(\frac{q_0}{q_1}\right)^2 I_s^* \right) y^k \\ &= \frac{xy(q_0')^2}{1 - xyq_0'} + \frac{q_0^2\,(q_0')^2}{q_1' q_1^2 (1 - xyq_0')} I(xyq_1', q_0, q_1, q_2, q_1, q_2). \end{aligned} \tag{4.2}$$

In (4.2), we replace q_0' by q_0 and q_1' by q_1. We preserve the other variables and obtain

$$I(y, q_0, q_1, q_2, q_0, q_1) = \underbrace{\frac{xy{q_0}^2}{1 - xyq_0}}_{a^*(y)} + \underbrace{\frac{q_0^4}{q_1^3 (1 - xyq_0)}}_{b^*(y)} I(xyq_1, q_0, q_1, q_2, q_1, q_2).$$

To iterate this, we need an expression for the I-term on the right. Once again, we apply (4.2) (this time for $q_0{}' = q_1$ and $q_1{}' = q_2$):

$$I(y, q_0, q_1, q_2, q_1, q_2) = \underbrace{\frac{xyq_1^2}{1 - xyq_1}}_{a(y)} + \underbrace{\frac{q_0^2}{q_2(1 - xyq_1)}}_{b(y)} I(xyq_2, q_0, q_1, q_2, q_1, q_2). \tag{4.3}$$

Iterating this infinitely many times yields

$$I(y, q_0, q_1, q_2, q_0, q_1) = a^*(y) + b^*(y)(a(xq_1y) + b(xq_1y)I(yx^2q_1q_2, q_0, q_1, q_2, q_1, q_2)),$$

$$\begin{aligned} b^*(y)b(xq_1y)I(yx^2q_1q_2, q_0, q_1, q_2, q_1, q_2) = b^*(y)b(xq_1y)\big(&a(x^2q_1q_2y) \\ &+b(x^2q_1q_2y)I(yx^2q_1q_2, q_0, q_1, q_2, q_1, q_2)\big) \end{aligned}$$

and so on. Now, adding row by row and then putting $y = 1$ gives

$$\begin{aligned} &I(y, q_0, q_1, q_2, q_0, q_1) \\ &= a^*(y) + b^*(y)\big(a(xq_1y) + b(xq_1y)\big(a(xq_1(xq_2)y) + b(xq_1(xq_2)y)\big((a(xq_1(xq_2)^2y) \\ &\qquad +b(xq_1(xq_2)^2y)(\ldots)\big)\big)\big) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} &I(1, q_0, q_1, q_2, q_0, q_1) \\ &= a^*(1) + b^*(1)\big(a(xq_1) + b(xq_1)\big(a(xq_1(xq_2)) + b(xq_1(xq_2))\big((a(xq_1(xq_2)^2) \\ &\qquad +b(xq_1(xq_2)^2)(\ldots)\big)\big)\big) \end{aligned} \tag{4.5}$$

Here we used the fact that the I terms that involves x^j tends to 0 as j tends to infinity.

Substituting for the expressions for $a^*(1)$ and $b^*(1)$ yields

$$I(1, q_0, q_1, q_2, q_0, q_1) = \frac{xq_0^2}{1 - xq_0} + \frac{q_0^4}{q_1^3(1 - xq_0)} \sum_{j=0}^{\infty} a^*(xq_1(xq_2)^j) \prod_{i=0}^{j-1} b^*(xq_1(xq_2)^i). \tag{4.6}$$

The first few terms of the series expansion gives

$$xq_0^2 + x^2(q_0^3 + q_0^4) + x^3(q_0^4 + q_0^5 + q_0^6) + x^4(q_0^5 + q_0^6 + q_0^4q_1^2 + q_0^7 + q_0^8) + \cdots$$

We illustrate these terms below

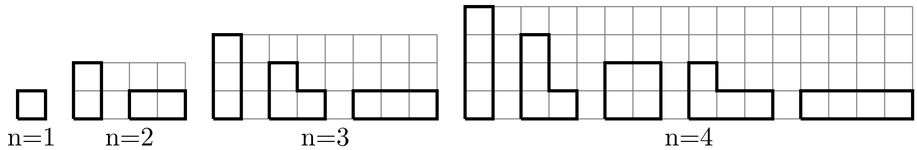

Figure 3: The horizontal edges for $n = 1, 2, 3$ and 4

In the remainder of the paper, when we refer to the generating functions I^o, I, I^*, we suppress all variables except the y variable. For example $I^o(1)$ means we set $y = 1$ while retaining all other variables.

4.1. Number of 0-edges. We are now going to concentrate on the 0-edges, we count them as before with q_0. Since we are not interested at this stage in the other types of horizontal edges, we let $q_1 = q_2 = 1$ and obtain

$$I^o(1)\big|_{q_1=q_2=1} = \frac{xq_0^2}{1-xq_0} + \frac{q_0^4}{1-xq_0}\sum_{j=0}^{\infty} a(x^{j+1})\prod_{i=0}^{j-1} b(x^{i+1}).$$

Since $a(y) = \frac{xyq_1^2}{1-xyq_1}$, and $b(y) = \frac{q_0^2}{q_2(1-xyq_2)}$ with $q_1 = q_2 = 1$ we get

$$\begin{aligned} I^o(1)\big|_{q_1=q_2=1} &= \frac{xq_0^2}{1-xq_0} + \frac{q_0^4}{1-xq_0}\sum_{j=0}^{\infty}\frac{x^{j+2}}{1-x^{j+2}}\prod_{i=0}^{j-1}\frac{q_0^2}{1-x^{i+2}} \\ &= \frac{xq_0^2}{1-xq_0} + \frac{q_0^4}{1-xq_0}\sum_{j=0}^{\infty} x^{j+2}q_0^{2j}\prod_{i=2}^{j+2}\frac{1}{1-x^i}. \end{aligned}$$

We proceed using the usual technique to obtain the total number of horizontal edges by first finding $\frac{\partial I^o(1)}{\partial q_0}\big|_{q_0=1}$. Thus

$$\begin{aligned} \frac{\partial I^o(1)}{\partial q_0}\bigg|_{q_0=1} &= \frac{x(2-x)}{(1-x)^2} + \frac{4-3x}{(1-x)^2}\sum_{j=0}^{\infty} x^{j+2}\prod_{i=2}^{j+2}\frac{1}{1-x^i} + \frac{1}{1-x}\sum_{j=0}^{\infty} 2jx^{j+2}\prod_{i=2}^{j+2}\frac{1}{1-x^i} \\ &= \frac{x(2-x)}{(1-x)^2} + \frac{4-3x}{(1-x)^2}\left[(1-x)P(x)-1\right] + 2\sum_{j=2}^{\infty}(j-2)x^j\prod_{i=1}^{j}\frac{1}{1-x^i} \\ &= \frac{x(2-x)}{(1-x)^2} + \frac{4-3x}{(1-x)^2}\left[(1-x)P(x)-1\right] + 2\sum_{j=2}^{\infty} jx^j\prod_{i=1}^{j}\frac{1}{1-x^i} - 4\sum_{j=2}^{\infty} x^j\prod_{i=1}^{j}\frac{1}{1-x^i}. \end{aligned}$$

In order to find $\sum_{j=2}^{\infty} jx^j\prod_{i=1}^{j}\frac{1}{1-x^i}$ we use the same process as for term C in (3.2), with a change of variable to get

$$\sum_{n=1}^{\infty} nq^{n-1}\prod_{i=1}^{n}\frac{1}{1-q^i} = \prod_{n=0}^{\infty}\frac{1}{1-q^{n+1}}\sum_{i=0}^{\infty}\frac{q^i}{1-q^{i+1}},$$

and therefore

$$\sum_{j=1}^{\infty} jx^j \prod_{i=1}^{n} \frac{1}{1-x^i} = x \prod_{j=0}^{\infty} \frac{1}{1-x^{j+1}} \sum_{i=0}^{\infty} \frac{x^i}{1-x^{i+1}}$$
$$= P(x) \sum_{i=0}^{\infty} \frac{x^{i+1}}{1-x^{i+1}}.$$

Thus, we have

$$\sum_{j=2}^{\infty} jx^j \prod_{i=1}^{j} \frac{1}{1-x^i} = P(x) \sum_{i=0}^{\infty} \frac{x^{i+1}}{1-x^{i+1}} - \frac{x}{1-x}.$$

Returning to our derivative, we obtain

$$\left.\frac{\partial I^o(1)}{\partial q_0}\right|_{q_0=1} = \frac{x(2-x)}{(1-x)^2} + \frac{4-3x}{(1-x)^2}[(1-x)P(x)-1]$$
$$+ 2P(x) \sum_{i=0}^{\infty} \frac{x^{i+1}}{1-x^{i+1}} - \frac{2x}{1-x} - 4\sum_{j=2}^{\infty} x^j \prod_{i=1}^{j} \frac{1}{1-x^i}$$
$$= -\frac{x+4}{1-x} - 4\left(P(x) - 1 - \frac{x}{1-x}\right) + \frac{4-3x}{1-x}P(x) + 2P(x)\sum_{i=1}^{\infty} \frac{x^i}{1-x^i}$$
$$= \frac{x}{1-x}\left[P(x)-1\right] + 2P(x)\sum_{i=1}^{\infty} \frac{x^i}{1-x^i}.$$

Hence we have

THEOREM 4.1. *The generating function for the total number of 0-edges in partitions of n is*

$$\frac{x}{1-x}(P(x)-1) + 2P(x)\sum_{i=1}^{\infty} \frac{x^i}{1-x^i}. \tag{4.7}$$

The average number of 0-edges in partitions of n is

$$\frac{\sqrt{6n}}{\pi}\left(\log n + 1 + 2\gamma + \log 6 - 2\log \pi\right) + \frac{3\log n}{\pi^2} + \frac{6+6\gamma+\log 216 - 6\log \pi}{\pi^2} + O\left(\frac{1}{\sqrt{n}}\right). \tag{4.8}$$

We prove (4.8) by applying Propositions 2.2 and 2.3 to (4.7). □

4.2. Number of 1-edges. We return to the equation (4.2)

$$I(1) = \frac{xq_0^2}{1-xq_0} + \frac{q_0^4}{q_1^3(1-xq_0)} \sum_{j=0}^{\infty} a(xq_1(xq_2)^j) \prod_{i=0}^{j-1} b(xq_1(xq_2)^i),$$

and since we are now only interested in the 1-edges, we keep q_1 and discard q_0 and q_2 by setting them both equal to 1. Thus

$$I(1)\big|_{q_0=q_2=1} = \frac{x}{1-x} + \frac{1}{q_1^3(1-x)} \sum_{j=0}^{\infty} a(x^{j+1}q_1) \prod_{i=0}^{j-1} b(x^{i+1}q_1)$$

which after replacing for the expression for $a(y)$ and $b(y)$ yields

$$I(1)\big|_{q_0=q_2=1} = \frac{x}{1-x} + \frac{1}{q_1^3(1-x)} \sum_{j=0}^{\infty} \frac{x^{j+2}q_1^3}{1-x^{j+2}q_1^2} \prod_{i=0}^{j-1} \frac{1}{1-x^{i+2}q_1^2}.$$

As before we differentiate with respect to q_1 and thereafter let $q_1 = 1$ to obtain

$$\begin{aligned}
\frac{\partial I^*(1)}{\partial q_1}\bigg|_{q_1=1} \\
&= \frac{1}{1-x}\left(-3\sum_{j=0}^{\infty} \frac{x^{j+2}}{1-x^{j+2}} \prod_{i=0}^{j-1} \frac{1}{1-x^{i+2}} + \sum_{j=0}^{\infty} \frac{x^{j+2}(3-x^{j+2})}{(1-x^{j+2})^2} \prod_{i=0}^{j-1} \frac{1}{1-x^{i+2}}\right. \\
&\qquad \left. + \sum_{j=0}^{\infty} \frac{x^{j+2}}{1-x^{j+2}} \prod_{i=0}^{j-1} \frac{1}{1-x^{i+2}} \sum_{i=0}^{j-1} \frac{2x^{i+2}}{1-x^{i+2}}\right) \\
&= \frac{1}{1-x}\left(-3\sum_{j=0}^{\infty} x^{j+2} \prod_{i=0}^{j} \frac{1}{1-x^{i+2}} + \sum_{j=0}^{\infty} \frac{x^{j+2}(3-x^{j+2})}{1-x^{j+2}} \prod_{i=0}^{j} \frac{1}{1-x^{i+2}}\right. \\
&\qquad \left. + 2\sum_{j=0}^{\infty} x^{j+2} \prod_{i=0}^{j} \frac{1}{1-x^{i+2}} \sum_{i=0}^{j-1} \frac{x^{i+2}}{1-x^{i+2}}\right).
\end{aligned}$$

Note that

$$\begin{aligned}
-\sum_{j=0}^{\infty} x^{j+2} \prod_{i=0}^{j} \frac{1}{1-x^{i+2}} + \sum_{j=0}^{\infty} \frac{x^{j+2}}{1-x^{j+2}} \prod_{i=0}^{j} \frac{1}{1-x^{i+2}} \\
= \sum_{j=0}^{\infty} \frac{x^{j+2}}{1-x^{j+2}} \prod_{i=0}^{j} \frac{1}{1-x^{i+2}} (1-(1-x^{j+2})).
\end{aligned}$$

Thus

(4.9)

$$\begin{aligned}
&\left.\frac{\partial I^*(1)}{\partial q_1}\right|_{q_1=1}\\
&= \frac{1}{1-x}\left(3\sum_{j=0}^{\infty}\frac{x^{2j+4}}{1-x^{j+2}}\prod_{i=0}^{j}\frac{1}{1-x^{i+2}} - \sum_{j=0}^{\infty}\frac{x^{2j+4}}{1-x^{j+2}}\prod_{i=0}^{j}\frac{1}{1-x^{i+2}}\right.\\
&\qquad\left. +2\sum_{j=0}^{\infty}x^{j+2}\prod_{i=0}^{j}\frac{1}{1-x^{i+2}}\sum_{i=0}^{j-1}\frac{x^{i+2}}{1-x^{i+2}}\right)\\
&= \frac{2}{1-x}\left(\sum_{j=0}^{\infty}x^{j+2}\prod_{i=0}^{j}\frac{1}{1-x^{i+2}}\sum_{i=0}^{j}\frac{x^{i+2}}{1-x^{i+2}}\right) + 2\sum_{j=1}^{\infty}x^{j}\prod_{i=1}^{j}\frac{1}{1-x^{i}}\sum_{i=2}^{j}\frac{x^{i}}{1-x^{i}}\\
&= 2\sum_{j=1}^{\infty}x^{j}\prod_{i=1}^{j}\frac{1}{1-x^{i}}\left(\sum_{i=1}^{\infty}\frac{x^{i}}{1-x^{i}} - \frac{x}{1-x} - \sum_{i=j+1}^{\infty}\frac{x^{i}}{1-x^{i}}\right)\\
&= 2(P(x)-1)\sum_{i=1}^{\infty}\frac{x^{i}}{1-x^{i}} - 2(P(x)-1)\frac{x}{1-x} - 2\sum_{j=1}^{\infty}\left(\sum_{i=j+1}^{\infty}\frac{x^{i}}{1-x^{i}}\right)x^{j}\prod_{i=1}^{j}\frac{1}{1-x^{i}}.
\end{aligned}$$

(4.10)

To evaluate (4.10) we need Lemma 2.1 to finally obtain

$$\left.\frac{\partial I^*(1)}{\partial q_1}\right|_{q_1=1} = 2\left[P(x)\sum_{i=1}^{\infty}\frac{x^{i}}{1-x^{i}} - P(x)\frac{1}{1-x} + \frac{1}{1-x}\right].$$

Hence, we have the following theorem:

THEOREM 4.2. *The generating function for the total number of 1-edges in partitions of n is*

$$2\left[P(x)\sum_{i=1}^{\infty}\frac{x^{i}}{1-x^{i}} - P(x)\frac{x}{1-x} - P(x) + \frac{1}{1-x}\right]. \tag{4.11}$$

The average number of 1-edges in partitions of n is

$$\begin{aligned}
&\frac{\sqrt{6n}}{\pi}\left(-2+2\gamma+\log 6+\log n-2\log\pi\right) - \frac{1}{2}\\
&\quad - \frac{3}{\pi^2}\left(1-2\gamma-\log 6-\log n+2\log\pi\right) + O\left(\frac{1}{\sqrt{n}}\right).
\end{aligned} \tag{4.12}$$

The result (4.12) follows by applying Propositions 2.2 and 2.3 to (4.11). □

4.3. Number of 2-edges. We first consider the total number of horizontal edges, these are made up of the three types tracked by q_0, q_1 and q_2. Thereafter we obtain the generating function for q_2, by subtracting the generating function for q_0 and q_1 from the generating function for the total number of horizontal edges. Since the number of horizontal edges in each column is equal to the size of that column plus one, the generating function counting all the horizontal edges with the variable y is

$$A = \prod_{i=1}^{\infty} \frac{1}{1 - y^{i+1}x^i}.$$

Thus

$$\left.\frac{\partial A}{\partial y}\right|_{y=1} = \prod_{i=1}^{\infty} \frac{1}{1-x^i} \sum_{r=1}^{\infty} \frac{(r+1)x^r}{1-x^r} = P(x) \sum_{r=1}^{\infty} \frac{(r+1)x^r}{1-x^r}.$$

Now consider

$$\begin{aligned} x\frac{d}{dx}P(x) &= xP(x)\sum_{r=1}^{\infty}(1-x^r)\frac{d}{dx}\frac{1}{1-x^r} \\ &= P(x)\sum_{r=1}^{\infty}\frac{rx^r}{1-x^r}. \end{aligned}$$

Hence

$$\left.\frac{\partial A}{\partial y}\right|_{y=1} = x\frac{d}{dx}P(x) + P(x)\sum_{r=1}^{\infty}\frac{x^r}{1-x^r}.$$

Therefore the generating function for q_2 is

$$\begin{aligned} &x\frac{d}{dx}P(x) + P(x)\sum_{r=1}^{\infty}\frac{x^r}{1-x^r} - 2\left(\frac{1}{1-x}(1-P(x)) + P(x)\sum_{i=1}^{\infty}\frac{x^i}{1-x^i}\right) \\ &\qquad - \frac{x}{1-x}(P(x)-1) - 2P(x)\sum_{i=1}^{\infty}\frac{x^i}{1-x^i} \\ &= x\frac{d}{dx}P(x) - 3P(x)\sum_{r=1}^{\infty}\frac{x^r}{1-x^r} + \frac{2-x}{1-x}(P(x)-1). \end{aligned}$$

So we have obtained

THEOREM 4.3. *The generating function for the total number of 2-edges in partitions of n is*

$$xP'(x) - 3P(x)\sum_{i=1}^{\infty}\frac{x^i}{1-x^i} + \frac{2-x}{1-x}(P(x)-1). \tag{4.13}$$

The average number of 2-edges in partitions of n is

$$\begin{aligned} &n - \frac{3\sqrt{3n}}{\sqrt{2}\,\pi}\log n - \frac{\sqrt{3n}}{\sqrt{2}\,\pi}(-2+6\gamma+\log 216 - 6\log\pi) - \frac{9\log n}{2\pi^2} \\ &\qquad + \frac{3}{4} - \frac{3}{2\pi^2} - \frac{9\gamma}{\pi^2} + \frac{9\log\frac{\pi^2}{6}}{2\pi^2} + O\left(\frac{1}{\sqrt{n}}\right). \end{aligned} \tag{4.14}$$

To prove the asymptotic result (4.14), we apply Propositions 2.2 and 2.3 to (4.13) using the fact that

$$\frac{2-x}{1-x}(P(x)-1) = \frac{x}{1-x}(P(x)-1) + 2(P(x)-1).$$

□

REMARK 4.4. It may occur to the reader that we have not also studied the equivalent concept of "vertical" edges. This is simply because all results relating to these are identical to results for horizontal edges by considering the conjugate partitions instead.

REMARK 4.5. One can study the number of points on the boundary as well (instead of interior) of the Young diagram. By Pick's theorem, the number of boundary points is a function of the size of the partition π and the number of interior points:

$$b(\pi) = 2(n - int_v(\pi) + 1).$$

REMARK 4.6. Since also $b(\pi) = 2(\text{largest part}(\pi) + \text{number of parts}(\pi))$, we have by Pick's theorem,

$$int_v(\pi) = n - \text{largest part}(\pi) - \text{number of parts}(\pi) + 1.$$

Now the average value of the largest part and the number of parts are equal and are known to both be given by Proposition 2.2. Using this Proposition and the formula above to recompute the average value of $int_v(\pi)$ does indeed give the asymptotic result of Theorem 3.2.

5. Conclusion. In this paper we have used two new concepts of interior vertices and horizontal d-edges of partitions, recently defined for the bargraph representation of set partitions and integer compositions in [10] and [11]. These concepts provide an alternative way of understanding integer partitions. Already firmly established in the historical studies of these objects is the associated underlying geometrical idea of the Young diagram which is in fact just a particular example of a bargraph. The new concepts are directly associated with this geometry and we therefore believe they provide an alternative approach to problem solving in the context of integer partitions. In the body of the paper, we have provided various generating function and associated asymptotics for interior vertices and horizontal d-edges.

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