Group Theoretical and Compatibility Approaches to Some Nonlinear PDEs Arising in the Study of Non-Newtonian Fluid Mechanics

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ABSTRACT

This thesis is primarily concerned with the analysis of some nonlinear problems arising in the study of non-Newtonian fluid mechanics by employing group theoretic and compatibility approaches.

It is well known now that many manufacturing processes in industry involve non-Newtonian fluids. Examples of such fluids include polymer solutions and melts, paints, blood, ketchup, pharmaceuticals and many others. The mathematical and physical behaviour of non-Newtonian fluids is intermediate between that of purely viscous fluid and that of a perfectly elastic solid. These fluids cannot be described by the classical Navier–Stokes theory. Striking manifestations of non-Newtonian fluids have been observed experimentally such as the Weissenberg or rod-climbing effect, extrudate swell or vortex growth in a contraction flow. Due to diverse physical structure of non-Newtonian fluids, many constitutive equations have been developed mainly under the classification of differential type, rate type and integral type. Amongst the many non-Newtonian fluid models, the fluids of differential type have received much attention in order to explain features such as normal stress effects, rod climbing, shear thinning and shear thickening.

Most physical phenomena dealing with the study of non-Newtonian fluids are modelled in the form of nonlinear partial differential equations (PDEs). It is easier to solve a linear problem due to its extensive study as well due to

the availability of high performance digital computers. However, finding exact solutions of nonlinear problems can be most of the time very difficult. In particular, obtaining an exact analytic solution of a given nonlinear problem is often more complicated as compared to that of a numerical solution, despite the availability of high performance supercomputers and software which provide efficient ways to perform high quality symbolic computations. It should be said that results obtained by numerical methods may give discontinuous points of a curve when plotted. Though numerical and analytic methods for solving nonlinear PDEs have limitations, at the same time they have their advantages too. Engineers, physicists and mathematicians are actively developing non-linear analytical techniques, and one such method which is known for systematically searching for exact solutions of differential equations is the group theoretic approach for differential equations.

The group theoretic methods, most notably since the 60s of the last century, are widely applied to the study of differential equations and aid in the constructing of their exact solutions. This fertile approach was inaugurated by the great Norwegian mathematician Marius Sophus Lie. The main advantage of such methods is that they can successfully be applied to non-linear differential equations. The symmetries of a differential equation are those continuous groups of transformations under which the differential equation remains invariant, that is, a symmetry group maps any solution to another solution. Generally, applications of Lie groups to differential equations include: reduction of order for ODEs, mapping solutions to other solutions, reduction of the number of independent variables of PDEs, construction of group invariant solutions (also for boundary value problems), construction of conservation laws and many other applications.

In recent years, interest in the non-classical symmetry method has also intensified. In this method, one attaches a side condition and then derives a set of determining equations for the functional dependencies. This method, which is a natural extension of the Lie symmetry theory, may lead to new exact solutions of nonlinear PDEs. We believe that this aspect deserves further elucidation, which is also one of the subjects of the presented work.

In this thesis, we have developed the mathematical modelling and the exact solutions to some nonlinear problems arising in the study of non-Newtonian fluid mechanics. Both classical and the non-classical symmetry analysis have been carried out for the flow models under investigation. In our analysis, the primary focus has been on the study of the physical and mathematical structure of third and fourth grade non-Newtonian fluid models. Throughout, we first perform the mathematical modelling of the physical models under consideration and then employ the classical Lie group analysis to obtain all possible Lie point symmetries of the underlying model equations. After obtaining the symmetries, we invoke these to construct the group invariant solutions for each of the problems under investigation. We have also used the non-classical symmetry method on a particular model of a third grade fluid in Chapter 4. In Chapters 3 and 4, the flow structure of third grade fluid models are investigated in detail and some new exact solutions have been obtained whereas in Chapters 5-7 we examine the physical behaviour of a fourth grade fluid in different types of flow situation.

During the second half of this study, we introduce the concept of compatibility. In recent years, the ansatz method has been used to construct exact solutions of nonlinear equations arising in the study of Newtonian and non-Newtonian fluids. In the ansatz method different forms of the solution are assumed and different techniques are used to develop analytical results. In this work, we present a unified treatment to classify exact solutions of models of Newtonian and non-Newtonian fluids which are solved using different techniques. We develop a general compatibility approach and then apply it to several existing studies. We construct some new exact solutions and also reconstruct some existing exact solutions of nonlinear problems for Newtonian and non-Newtonian fluids using the concept of compatibility. Some of these models were previously solved by the ansatz approach and not in a systematic way. We first give a rigorous definition of compatibility and then developed a general compatibility test for a higher order ordinary differential equation (up to fifth-order) to be compatible with a first order ordinary differential equation. This can naturally be extended to higher order with existing computer codes. Several examples from the literature are presented to which the compatibility approach is applied and new and existing exact solutions are deduced. This work has been carried out in Chapter 8.

LIST OF PAPERS FROM THE THESIS

1. **Taha Aziz**, F.M. Mahomed and A. Aziz, Group invariant solutions for the unsteady MHD flow of a third grade fluid in a porous medium, International Journal of Non-Linear Mechanics, 47 (2012) 792-798.

2. **Taha Aziz** and F.M. Mahomed, Closed-form solutions for a nonlinear partial differential equation arising in the study of a fourth grade fluid model, Journal of Applied Mathematics, Article ID 931587, Volume 2012, doi:10.1155/2012/931587.

3. **Taha Aziz** and F.M. Mahomed, Reductions and solutions for the unsteady flow of a fourth grade fluid on a porous plate, Applied Mathematics and Computation, 219 (2013) 9187-9195.

4. **Taha Aziz**, F.M. Mahomed, Muhammad Ayub and D.P. Mason, Non-linear time-dependent flow models of third grade fluids: A conditional symmetry approach, International Journal of Non-Linear Mechanics, 54 (2013) 55-65.

5. **Taha Aziz,** A. Fatima and F.M. Mahomed, Shock Wave Solution for a Nonlinear Partial Differential Equation Arising in the Study of a Non-Newtonian Fourth Grade Fluid Model, Mathematical Problems in Engineering, Volume 2013, Article ID 573170, 5 pages.

6. **Taha Aziz**, R.J. Moitsheki and F.M. Mahomed, Shock wave solution for some nonlinear PDEs arising in the study of a third grade fluid model, Mathematical Problems in Engineering, Volume 2013, Article ID 602902, 6 Pages, 2013. Volume 2013.

7. **Taha Aziz**, F.M. Mahomed, R. Ali and A. Shahzad, Travelling wave solutions for the unsteady flow of a third grade fluid induced due to impulsive motion of flat porous plate embedded in a porous medium, Cambridge-Journal of Mechanics, 30 (2014) 527-535.

8. **Taha Aziz** and F.M. Mahomed, A note on the solutions of some nonlinear equations arising in third grade fluid flows: an exact approach, The Scientific World Journal, Volume 2014, Article ID 109128, 7 pages, Volume 2014.

9. **Taha Aziz** and Asim Aziz, MHD flow of a third grade fluid in a porous half space with plate suction or injection: An Analytical Approach, Applied Mathematics and Computation, 218 (2012) 10443-10453.

10. **Taha Aziz,** A. Fatima, C. M. Khalique and F.M. Mahomed, Prandtl's Boundary Layer Equation for Two-Dimensional Flow: Exact Solutions via the Simplest Equation Method, Mathematical Problems in Engineering, Volume 2013, Article ID 724385, 5 pages.

11. **Taha Aziz**, F.M. Mahomed and D.P. Mason, A unified compatibility method for exact solutions of nonlinear flow models of Newtonian and non-Newtonian fluids, International Journal of Nonlinear Mechanics, (2015), doi:10.1016/j.ijnonlinmec.2015.01.003.

12. **Taha Aziz**, A.B. Magan and F. M. Mahomed, Invariant solutions for the unsteady magneto hydrodynamics (MHD) flow of a fourth-grade fluid induced due to the impulsive motion of a flat porous plate, Brazilian Journal of Physics, 45 (2015) 120–131.

13. **Taha Aziz** and F.M. Mahomed, Group theoretical analysis and invariant solutions for unsteady flow of a fourth grade fluid through a porous space, Applied Mathematics and Computation, 2014 (submitted).

DECLARATION

I declare that this is my own, unaided work except where due references have been made. It is being submitted for the Degree of PhD to the University of the Witwatersrand, Johannesburg, South Africa. It has not been submitted before for any other degree or examination to any other University.

Taha Aziz

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(Signature of candidate)

__________________day of _____________________20______________in____________

DEDICATION

To my parents, my wife Aeeman, my daughter Ayaat, my brother Dr. Asim, my father-in-law Dr. Mir Zaman, my mother-in-law and my mentor Prof Fazal M. Mahomed

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I begin with the name of almighty ALLAH, who bestowed countless blessings upon me to fulfil the requirements for this thesis. Over and above everything else, it gives the greatest satisfaction as I offer my humblest words to thank the almighty ALLAH and millions of Darood-o-Salaam on the Holy Prophet Hazrat Muhammed (P.B.U.H) who is always a torch of guidance for humanity.

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Contents

Chapter 1

Aims, Objectives and Outline

1.1 Introduction

During the past several decades, the study of nonlinear problems dealing with flow models of non-Newtonian fluids has gained prodigious attention. This interest is due to several important applications in engineering and industry such as aerodynamic heating, electrostatic precipitation, petroleum industry, reactive polymer flows in heterogeneous porous media, electrochemical generation of elemental bromine in porous electrode systems, manufacture of intumescent paints for fire safety applications, extraction of crude oil from petroleum products, synthetic fibers, paper production and so forth. Many materials such as polymers solutions, drilling muds, blood, paint, certain oil and greases and many other emulsion are classified as non-Newtonian. Due to the complex physical nature of non-Newtonian fluids there is no single model available in the literature which describes all the properties of non-Newtonian fluids. Therefore, in order to mathematically describe the properties of non-Newtonian fluids, several constitutive equations have been proposed. The flow properties of non-Newtonian fluids are quite different from those of the Newtonian fluids. Therefore, in practical applications one cannot replace the behavior of non-Newtonian fluids with Newtonian fluids and it is very important to analyze the flow behavior of non-Newtonian fluids in order to obtain a thorough understanding and improve the utilization in various manufactures.

The most interesting and important task that we need to address when dealing with the flow problems of non-Newtonian fluids is that the governing equations of these models are of higher order and much more complicated as compared with Navier-Stokes equations. Such fluids have been modelled by constitutive equations which vary greatly in complexity. The resulting nonlinear equations are not easy to solve analytically and the mathematical analysis of such equations would present a particular challenge for scientists and engineers. As a result of nonlinearity, the exact (closed-form) solution of these problems are very difficult to obtain. Advancement in the computational techniques are making it easier to construct exact solutions of linear problems. However, it is still very difficult to solve nonlinear problems analytically. Several studies $[1-3]$ are available in the literature dealing with the flow of non-Newtonian fluids in which researchers have utilized the numerical approach to obtain solutions of nonlinear governing equations. Apart from numerical methods, several other techniques and methods have been developed recently to construct solutions of non-Newtonian fluid flow problems. Some of these useful methods are the variational iteration method, Adomian decomposition method, homotopy perturbation method, homotopy analysis method, semi-inverse variational method and symmetry method. Despite all these methods, exact (closed-form) solutions of non-Newtonian fluid flow problems are still rare in the literature and it is not surprising that new exact (closed-form) solutions are most welcome provided they correspond to physically realistic situations. These exact solutions if available, facilitate the verification of complex numerical codes and are also helpful in stability analysis. Consequently, the exact solutions of the non-Newtonian fluid flow problems are physically important.

Amongst the many models which have been used to describe the physical flow behavior of non-Newtonian fluids, the fluids of differential type have received special attention, see for example [4] for a complete discussion of the relevant issues. One of the widely accepted models amongst non-Newtonian fluids is the class of Rivlin-Ericksen fluids of differential type [5]. Rivlin-Ericksen fluids of differential type have acquired the special status in order to describe the several non-standard features such as normal stress effects, rod climbing, shear thinning and shear thickening [6]. Literature surveys point out that much focus has been given to the flow problems of a non-Newtonian second grade fluid model [3, 7, 8]. A second grade fluid model is the simplest subclass of differential type fluids for which one can reasonably hope to establish an analytic result. In most of the flow aspects, the governing equations for a second grade fluid are linear. Although a second grade fluid model for steady flows is used to predict the normal stress differences, it does not correspond to shear thinning or thickening if the shear viscosity is assumed to be constant. Therefore some experiments may be better described by a third grade fluid [9, 10]. The mathematical model of a third grade fluid represents a more realistic description of the behavior of non-Newtonian fluids. A third grade fluid model represents a further attempt towards the study of the flow structure of non-Newtonian fluids. Therefore, a third grade fluid model is considered in this research project. This model is known to capture the non-Newtonian effects such as shear thinning or shear thickening as well as normal stress. The governing equations for the third grade fluid model are nonlinear and much more complicated than those of Newtonian fluids. They require additional boundary conditions to obtain a physically meaningful solution. This issue has been discussed in detail by Rajagopal [11, 12] and Rajagopal and Kaloni [13]. Fosdick and Rajagopal [14] made a complete thermodynamical analysis of a third grade fluid and derived the restriction on the stress constitutive equation. They investigated some stability characteristics of third grade fluids and showed that they exhibit features different from those of Newtonian and second grade fluids.

Another very important subclass of the differential type fluids is the fourth grade [15, 16] non-Newtonian fluid model. This model is the most generalized model among the differential type fluids. This model is known to capture most of the non-Newtonian flow properties at one time. With this in mind we have considered a fourth grade fluid model in this study for generality. In general, the governing equations for the flow problems of fourth grade fluids are up to fifth-order nonlinear equations. That is why, up to now, very limited studies are reported in the literature, up to now, dealing with the flow problems of a fourth grade fluid and these investigations further narrow down when we speak about the exact solutions of these problems. However, there are few investigations available in the literature in which researchers have utilized various approaches to construct solutions of fourth grade fluid flow problems. For example, in [17] Wang and Wu have tackled the problem for the unsteady flow of fourth grade fluid due to an oscillating plate using numerical methods. The study of peristaltic transport of a fourth grade fluid in an inclined asymmetric channel under the consideration of long wavelength was made by Haroun [18]. Hayat et al. [$19-21$], studied the fourth grade fluid problems in different types of flow situations by using the homotopy analysis method. Recently, the problem of rotation and initial stress on the peristaltic flow of an incompressible fourth grade fluid in asymmetric channel with magnetic field and heat transfer was treated by Abd-Alla et al. [22]. Despite all of these works in recent years, the exact closed-form solutions for the problems dealing with the flow of fourth grade fluids are still rare in the literature.

Lie theory of differential equations (see e.g. [23, 24]) was inaugurated and utilized in the solution of differential equations by the Norwegian mathematician Marius Sophus Lie in the 1870s. The main motive of the Lie symmetry analysis formulated by Lie is to find one or several parameters local continuous transformation groups leaving the equations invariant. Thus the symmetry method for differential equations provide a rigorous way to classify them according to their invariance properties in order to effects reduction and solution of the equations. This allows us to obtain group invariant and partially invariant solutions of differential equations in a tractable manner. The Lie symmetry approach has thus become of great importance in the theory and applications of differential equations and widely applied by several authors to solve difficult nonlinear problems particularly dealing with the flows of non-Newtonian fluids [$25-29$]. The conditional symmetry or nonclassical symmetry approach has its origin in the work of Bluman and Cole [30]. In recent years, interest in the conditional symmetry approach has increased. Information on the nonclassical method and related topics can be found in [$31-34$]. This method has also been used to obtain new exact solutions of a number of interesting nonlinear partial differential equations $[35 - 40]$. There are equations arising in applications that do not admit classical symmetries but have conditional symmetries. Thus this method is useful in constructing exact solutions. The concept of conditional/non-classical symmetry has not been used to find conditionally-invariant solutions of non-Newtonian fluid flow problems. This is the first time that the conditional symmetry approach has been employed to tackle nonlinear problems dealing with the flow models of non-Newtonian third grade fluids. We believe that these deserve further importance in tackling non-Newtonian fluid flow problems. A summery of the nonclassical symmetry method is also presented in chapter 2.

It is often difficult to obtain exact solutions of a higher order nonlinear differential equation. For this reason, many researchers have assumed a form of the exact solution by trial and error. In recent years, the ansatz method has been used to construct exact solutions of nonlinear equations arising in the study of Newtonian and non-Newtonian fluids. In the ansatz method different forms of the solution are assumed and different techniques are used to develop analytical results. In this work, we have also presented a unified treatment to classify exact solutions of models of Newtonian and non-Newtonian fluids which are solved using different techniques. We develop a general compatibility approach and then apply it to several existing studies. We construct some new exact solutions and also reconstruct some existing exact solutions of nonlinear problems for Newtonian and non-Newtonian fluids using the concept of compatibility. Some of these models were previously solved by the ansatz approach but not in a systematic way.

1.2 Aims and objectives

The main objectives that we have achieved during this research is summarized as follows:

- During this research, we have tackled nonlinear problems dealing with the flow models of third and fourth grade non-Newtonian fluids flow problems in different types of flow geometries. We have also taken into account the other different interesting phenomena like flow in a porous medium, flow in the presence of magnetic field etc.

• Our first step was to perform the mathematical modelling of the different flow problems that we have considered.

• Secondly, we have computed all possible symmetries of the model equations both with the classical and non-classical symmetry methods.

• Thirdly, we derived the group invariant solutions and symmetry solutions of the nonlinear partial differential equations arising in the study of underlying non-Newtonian fluid flow models. We have also obtained the numerical solutions where necessary and compared the analytical results with the numerical results.

• During the second half of this research project, we have presented a unified treatment to classify exact solutions of models of Newtonian and non-Newtonian fluids which are solved using different techniques. We have developed a general compatibility approach and then applied it to several existing studies. We have constructed some new exact solutions and also reconstructed some existing exact solutions of nonlinear problems for Newtonian and non-Newtonian fluids using the concept of compatibility. Some of these models were previously solved by the ansatz approach but not in a systematic way.

• After performing all this work we have been able to provide systematic analytical techniques to solve a wide range of nonlinear problems of non-Newtonian fluid mechanics.

1.3 Outline

A detail outline of the thesis is as follows.

In Chapter 2, some basic definitions and concept of fluid mechanics are presented. Moreover, the group theoretic approach for differential equations is also discussed in detail.

The aim of Chapter 3 is to provide the study which deals with the modeling and solution of an unsteady flow of a third grade fluid over an infinite rigid plate within a porous medium. The fluid is electrically conducting in the presence of uniform applied magnetic field in the normal direction to the flow. The flow is generated due to an arbitrary velocity of the plate. By employing Lie group theory, symmetries of the equation are founded and the original third-order partial differential equation is reduced to a third-order ordinary differential equations. The reduced ordinary differential equations are then solved analytically and numerically. The manner in which various physical parameters effects the structure of velocity is discussed with the aid of several graphs. The results presented in Chapters 3 are published in **"[41] International Journal of Non-Linear Mechanics, 47 (2012) 792–798 ".**

In Chapter 4, we have provided a systematic mechanism to generate exact solutions for some nonlinear problems which describe the time-dependent behavior of a third fluid model. We have considered three different types of flow problems dealing with the flow of third grade fluid in a porous medium. In each case, a complete non-classical symmetry analysis has been carried out for the governing nonlinear partial differential equations to construct new classes of conditionally invariant solutions. This is the first time in the literature that this type of approach has been utilized to construct exact solutions for non-Newtonian fluid models. The findings of this chapter have been published in **"[42] International Journal of Non-Linear Mechanics, 54 (2013) 55–65".**

The purpose of Chapter 5 is to venture further in the regime of fourth grade fluid. We have investigated the time-dependent flow of an incompressible fourth grade fluid over a flat rigid plate. The motion is caused due to the motion of the plate in its own plane (xz-plane) with an arbitrary velocity $V(t)$. Reductions and exact solutions of the governing nonlinear PDE for the unidirectional flow of a fourth grade fluid are established using the classical and the conditional symmetry approaches. Finally, the influence of physically applicable parameters of the flow model are studied through several graphs and appropriate conclusions are drawn. The contents of this chapter have been published in

"[43] Journal of Applied Mathematics, Volume (2012), Article ID 931587,16 pages".

In Chapter 6, the unsteady unidirectional flow of an incompressible fourth grade fluid bounded by a porous plate with suction or injection is examined. The disturbance is caused due to the motion of the plate in its own plane with an arbitrary velocity $V(t)$. The governing nonlinear partial differential equation along with suitable boundary and initial conditions, is solved analytically and numerically. To the best of our knowledge, no attempt has been made up to now to obtain the exact analytical solution for the unsteady flow of fourth grade fluid over the porous plate with suitable boundary conditions. A physical interpretation of the obtained results is shown through graphs. The results presented in this chapter are published in **"[44] Applied Mathematics and Computation, 219 (2013) 9187-9195".**

Chapter 7 deals with the modelling and solutions of an incompressible unsteady flow of a fourth grade fluid in a porous half space. The flow has been generated due to motion of the flat plate in its own plane with an impulsive velocity. The governing nonlinear partial differential equation with corresponding boundary conditions has been solved exactly by employing Lie group theoretic approach. All possible Lie point symmetries have been calculated for the modeled fifth-order nonlinear partial differential equation. Some new classes of group invariant solutions have been developed. The influence of various emerging parameters on the flow characteristics has been studied and discussed through graphical analysis of the obtained results. This is the first time in literature that a complete Lie classical symmetry analysis has been carried out for any problem dealing with the flow of a fourth grade fluid model. The contents of this chapter have been written up in the form of a paper and submitted for publication in **"[45] Applied Mathematics and Computation, 2014 (submitted)".**

In Chapter 8, we have developed an efficient compatibility criterion for solving higher order nonlinear ordinary differential equations subject to lower order ordinary differential equations. The compatibility approach enabled us to construct the exact solutions of higher order nonlinear ordinary differential equations subject to lower order ordinary differential equations. This approach was illustrated by applying it to various nonlinear flow problems taken from the literature of Newtonian and non-Newtonian fluid dynamics. We have constructed exact (closed-form) solutions of these problems using the compatibility approach and found some new solutions of these problems. We have also unified the various ansatz based approaches for finding exact solutions for some of the flow models. The compatibility condition generally had physical significance which depended on the problem considered. In some problems it gave the speed of propagation of the travelling wave. The findings of this work have been published in **"[46] International Journal of Non-Linear Mechanics, 2015, doi:10.1016/j.ijnonlinmec.2015.01.003".**

Finally, some concluding remarks and future works are summarized in Chapter 9.

Chapter 2

Fluid Dynamics and Symmetry Approaches

2.1 Introduction

In this chapter, we briefly present some basic definitions and useful concepts of fluid mechanics. These definitions can be found in many standard books of fluid mechanics. Furthermore, the main aspects of the Lie symmetry method for differential equations are also discussed with some words on conditional symmetries.

2.2 Compressible and incompressible flows

A flow in which the variation of the mass per unit volume (density) within the flow is considered constant is known as an incompressible flow. Generally, all liquids are treated as incompressible fluids. On the contrary, flows which are characterized by a varying density are said to be compressible. Gases are normally considered to be compressible. The mathematical equation that describes the incompressibility property of the fluid is given by

$$\frac{d\rho}{dt} = 0, \tag{2.1}$$

where d/dt is the total derivative defined by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}.\boldsymbol{\nabla}, \tag{2.2}$$

where $\mathbf{V}$ represents the velocity of the flow.

2.3 Unsteady and steady flows

A flow in which fluid properties at each point in the flow field do not depend upon time is called a steady flow. For such a flow

$$\frac{\partial f}{\partial t} = 0, \tag{2.3}$$

in which f represents any fluid property. Flow property includes velocity, density and temperature etc. A flow in which fluid properties depend upon time is called an unsteady flow. For such a flow,

$$\frac{\partial f}{\partial t} \neq 0. \tag{2.4}$$

2.4 Laminar and turbulent flows

Laminar flow is one in which each fluid particle has a definite path. In such flow, the paths of fluid particles do not intersect each other. In turbulent flow the paths of fluid particles may intersect each other.

2.5 Rotational and irrotational flows

A flow in which fluid particles rotate about its axis is said to be a rotational flow. For a rotational flow the curl of the velocity vector does not vanish. Mathematically,

$$\boldsymbol{\nabla} \times \mathbf{V} \neq 0, \tag{2.5}$$

where $\mathbf{V}$ is a velocity vector. On the other hand, a flow in which fluid particles do not rotate about its axis is said to be an irrotational flow, For an irrotational flow, the curl of the velocity vector vanishes. Mathematically,

$$\boldsymbol{\nabla} \times \mathbf{V} = 0. \tag{2.6}$$

2.6 Basic equations

2.6.1 The continuity equation

The continuity equation is the mathematical expression of the law of conservation of mass and is given by

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{V}) = 0, \tag{2.7}$$

where ρ is the density and $\mathbf{V}$ the velocity vector.

The continuity equation for steady state fluid becomes

$$\operatorname{div}(\rho \mathbf{V}) = 0. \tag{2.8}$$

For an incompressible fluid the density of any particle is invariable and so the continuity equation for an incompressible fluid becomes

$$\operatorname{div} \mathbf{V} = 0. \tag{2.9}$$

2.6.2 The momentum equation

The momentum equation is the mathematical expression of the law of conservation of linear momentum and is given by

$$\rho \frac{d\mathbf{V}}{dt} = \operatorname{div} \mathbf{T} + \rho \mathbf{b}, \tag{2.10}$$

in which $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{b}$ the body force per unit mass and d/dt the material time derivative defined in Eq. (2.2). The Cauchy stress tensor $\mathbf{T}$ is

$$\mathbf{T} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}, \tag{2.11}$$

and σ_{ii} $(i = x, y, z)$ and τ_{ij} $(j = x, y, z)$ are the normal and shear stresses respectively.

In scaler form, Eq. (2.10) gives

$$\rho \frac{du}{dt} = -\frac{\partial \rho}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho b_x, \tag{2.12a}$$

$$\rho \frac{dv}{dt} = -\frac{\partial \rho}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho b_y, \tag{2.12b}$$

$$\rho \frac{dw}{dt} = -\frac{\partial \rho}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho b_z. \tag{2.12c}$$

2.7 Non-Newtonian fluids

Fluids such as water, air and honey are described as Newtonian fluids. These fluids are essentially modelled by the Navier-Stoke equations, which generally describe a linear relation between the stress and the strain rate. On the other hand, there is a large number of fluids that do not fall in the category of Newtonian fluids. Common examples like mayonnaise, toothpaste, egg whites, liquid soaps and multi grade engine oils are non-Newtonian. A distinguishing feature of many non-Newtonian fluids is that they exhibit both viscous and elastic properties and the relationship between the stress and the strain rate is non-linear. Contrary to Newtonian fluids, there is not a single model that can describe the behavior of all the non-Newtonian fluids. For this reason many models have been proposed and among those models, the fluids of differential type and rate type have acquired attention.

If we denote $\mathbf{T}$ the Cauchy stress tensor of a given fluid, it can be written in general as

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{S}, \tag{2.13}$$

where p denotes the indeterminate part of the stress due to the constraint of incompressibility, $\mathbf{I}$ is the identity tensor and $\mathbf{S}$ is the extra stress tensor. We recall that for a non-Newtonian fluid, the extra stress tensor is related to the viscosity of the fluid, which is a nonlinear function of the symmetric part of the velocity gradient.

2.8 Lie symmetry analysis for differential equations

In this section, we present some basics of Lie symmetry methods for solving differential equations.

2.8.1 Symmetry transformations of differential equations

A transformation under which a differential equation remains invariant (unchanged) is called a symmetry transformation of the differential equation.

Consider a k^{th}-order ($k \geq 1$) system of differential equations

$$F^{\sigma}(x, u, u_{(1)}, ..., u_{(k)}) = 0, \quad \sigma = 1, ..., m, \tag{2.14}$$

where $u = (u^1, ..., u^m)$ called the dependent variable, is a function of the independent variable $x = (x^1, ..., x^n)$ and $u_{(1)}$, $u_{(2)}$ up to $u_{(k)}$ are the collection of all first-, second-, up to k^{th}-order derivatives of u.

A transformation of the variable x and u, viz.

$$\bar{x}^i = f^i(x, u), \quad \bar{u}^{\alpha} = g^{\alpha}(x, u), \quad i = 1, ..., n; \ \alpha = 1, ..., m, \tag{2.15}$$

is called a symmetry transformation of the system (2.14) if (2.14) is form-invariant in the new variables $\bar{x}$ and $\bar{u}$, that is

$$F^{\sigma}(\bar{x}, \bar{u}, \bar{u}_{(1)}, ..., \bar{u}_{(k)}) = 0, \ \ \sigma = 1, ..., m, \tag{2.16}$$

whenever

$$F^{\sigma}(x, u, u_{(1)}, ..., u_{(k)}) = 0, \ \ \sigma = 1, ..., m. \tag{1.17}$$

For example, the first-order Abel equation of the second kind

$$f\frac{df}{dy} = y^3 + yf \tag{2.18}$$

has symmetry transformations

$$\bar{y} = ay, \quad \bar{f} = a^2 f, \ \ a \in \mathbb{R}^+. \tag{2.19}$$

2.8.2 One-parameter Lie groups

A set G of transformations

$$T_a: \quad \bar{x}^i = f^i(x, u, a), \quad \bar{u}^{\alpha} = g^{\alpha}(x, u, a), \quad i = 1, ..., n; \ \ \alpha = 1, ..., m, \tag{2.20}$$

where a is a real parameter, is a continuous one-parameter Lie group of transformations if the following properties are satisfied

(i) Closure: If T_a, T_b in G and $a, b \in \mathbb{R}$, then

$$T_b T_a = T_c \in G, \quad c \in \mathbb{R}. \tag{2.21}$$

(ii) Identity: $T_0 \in G$ such that

$$T_0 T_a = T_a T_0 = T_a \tag{2.22}$$

for and $a \in \mathbb{R}$ and $T_a \in G$. T_0 is the identity.

(iii) Inverses: For $T_a \in G$, $a \in \mathbb{R}$, there exists $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathbb{R}^+$ such that

$$T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0 \tag{2.23}$$

For example, the set of transformations

$$T_a : \ \bar{x} = ax, \ \bar{u} = au, \ a \in \mathbb{R}^+ \tag{2.24}$$

forms a one-parameter Lie group.

2.8.3 Infinitesimal transformations and generator

According to Lie's theory, the construction of a one-parameter group G is equivalent to the determination of the corresponding infinitesimal transformations

$$\bar{x}^i \approx x^i + a\xi^i(x,u), \quad \bar{u}^\alpha \approx u^\alpha + a\eta^\alpha(x,u). \tag{2.25}$$

One can introduce the symbol X of the infinitesimal transformations by writing (2.25) as

$$\bar{x}^i \approx (1 + aX)x^i, \quad \bar{u}^\alpha \approx (1 + aX)u^\alpha, \tag{2.26}$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{2.27}$$

is known as the infinitesimal operator or generator of the group G.

2.8.4 Lie symmetry method for partial differential equations

Here we discuss the classical Lie symmetry method to obtain all possible symmetries of a system of partial differential equations.

Let us consider a p^{th} order system of partial differential equations in n independent variables $x = (x_1, ..., x_n)$ and m dependent variable $u = (u_1, ..., u_m)$, viz.

$$E(x, u, u_{(1)}, ..., u_{(p)}) = 0, \tag{2.28}$$

where $u_{(k)}, 1 \leq k \leq p$, denotes the set of all k^{th} order derivative of u, with respect to the independent variables defined by

$$u^{\alpha}_{(k)} = \left\{ \frac{\partial^k u}{\partial x_{i_1}, ..., x_{i_k}} \right\}, \tag{2.29}$$

with

$$1 \leq i_1, i_2, ..., i_k \leq n. \tag{2.30}$$

For finding the symmetries of Eq. (2.28), we first construct the group of invertible transformations depending on the real parameter a, that leaves Eq. (2.28) invariant, namely

$$\bar{x}_1 = f^1(x, u, a), ..., \bar{x}_n = f^n(x, u, a), \quad \bar{u}^{\alpha} = g^{\alpha}(x, u, a). \tag{2.31}$$

The above transformations have the closure property, are associative, admit inverses and identity transformation and are said to form a one-parameter group.

Since a is a small parameter, the transformations (2.31) can be expanded in terms of a series expansion as

$$\begin{aligned} \bar{x}_1 &= x_1 + a\xi_1(x, u) + O(a^2), ..., \bar{x}_n = x_n + a\xi_n(x, u) + O(a^2) \\ \bar{u}_1 &= u_1 + a\eta_1(x, u) + O(a^2), ..., \bar{u}_m = u_m + a\eta_n(x, u) + O(a^2) \end{aligned} \tag{2.32}$$

The transformations (2.32) are the infinitesimal transformations and the finite transformations are found by solving the Lie equations

$$\xi_1(\bar{x},\bar{u}) = \frac{d\bar{x}_1}{da}, ..., \xi_n(\bar{x},\bar{u}) = \frac{d\bar{x}_n}{da}, \eta(\bar{x},\bar{u}) = \frac{d\bar{u}}{da}, \tag{2.33}$$

with the initial conditions

$$\begin{aligned} \bar{x}_1(\bar{x},\bar{u},a) \quad &|_{\,a=0} = x_1, ..., \bar{x}_n(\bar{x},\bar{u},a) \mid_{a=0} = x_n, \\ \bar{u}_1(\bar{x},\bar{u},a) \quad &|_{\,a=0} = u_1, ..., \bar{u}_m(\bar{x},\bar{u},a) \mid_{a=0} = u_m \end{aligned} \tag{2.34}$$

where $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$ and $\bar{u} = (\bar{u}_1, ..., \bar{u}_m)$. The transformations (2.31) can be denoted by the Lie symmetry generator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{2.35}$$

where the functions $\xi^i(i = 1, ..., n)$ and $\eta^\alpha(\alpha = 1, ..., m)$ are the coefficient functions of the operator X.

The operator (2.35) is a symmetry generator of Eq. (2.28) if

$$X^{[p]}E \mid_{E=0} = 0, \tag{2.36}$$

where $X^{[p]}$ represents the p^{th} prolongation of the operator X and is given by

$$
\begin{aligned}
X^{[1]} &= X + \sum_{i=1}^{n} \zeta_{x_i}^{\alpha} \frac{\partial}{\partial u_{x_i}}, && (2.37)\\
X^{[2]} &= X^{[1]} + \sum_{i=1}^{n} \sum_{j=1}^{n} \zeta_{x_i x_j}^{\alpha} \frac{\partial^2}{\partial u_{x_i} u_{x_j}}, && (2.38)\\
&.\\
&.\\
&.\\
X^{[p]} &= X^{[1]} + ... + X^{[p-1]} + \sum_{i_1=1}^{n} ... \sum_{i_p=1}^{n} \zeta_{x_{i_1}...x_{i_p}}^{\alpha} \frac{\partial^p}{\partial u_{x_{i_1}} ... u_{x_p}}, && (2.39)
\end{aligned}
$$

with

$$u_{x_i} = \frac{\partial u}{\partial x_i}, \quad u_{x_{i_1}...x_{i_k}} = \frac{\partial^k u}{\partial x_{i_1} ... \partial x_{i_k}}. \tag{2.40}$$

In the above equations, the additional coefficient functions satisfy the following relations

$$\zeta_{x_i}^{\alpha} = D_{x_i}(\eta) - \sum_{j=1}^{n} u_{x_j} D_{x_i}(\xi^j), \tag{2.41}$$

$$
\begin{aligned}
\zeta_{x_i x_j}^{\alpha} &= D_{x_j}(\eta^{x_i}) - \sum_{k=1}^{n} u_{x_i x_k} D_{x_j}(\xi^k),\\
&.\\
&.\\
&.\\
\zeta_{x_{i_1}...x_{i_p}}^{\alpha} &= D_{x_{i_p}}(\eta^{x_{i_1}...x_{i_{p-1}}}) - \sum_{j=1}^{n} u_{x_{i_1}...x_{i_{p-1}} x_j} D_{x_{i_p}}(\xi^j), && (2.42)
\end{aligned}
$$

where D_{x_i} denotes the total derivative operator and is given by

$$D_{x_i} = \frac{\partial}{\partial x_i} + u_{x_i} \frac{\partial}{\partial u} + \sum_{j=1}^{n} u_{x_i x_j} \frac{\partial}{\partial x_j} + ... \tag{2.43}$$

The determining equation (2.36) results in a polynomial in terms of the derivatives of the dependent variable u. After separation of Eq. (2.36) with respect to the partial derivatives of u and their powers, one obtains an overdetermined system of linear homogeneous partial differential equations for the coefficient functions ξ^{i}'s and η^{α}'s. By solving the overdetermined system one has the following cases:

• There is no symmetry, which means that the Lie point symmetry ξ^i and η^α are all zero.

• The point symmetry has $r \neq 0$ arbitrary constants; in this case we obtain an $r-$dimensional Lie algebra of point symmetries.

• The point symmetry admits some finite number of arbitrary constants and arbitrary functions, in which case we obtain an infinite-dimensional Lie algebra.

2.8.5 Example on the Lie symmetry method

Here we illustrate the use of the Lie symmetry method on the well-known Born-Infeld quation [47] given by

$$(1-u_t^2)u_{xx} + 2u_x u_t u_{xt} - (1+u_x^2)u_{tt} = 0. \tag{2.44}$$

We seek for an operator of the form

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{2.45}$$

The operator given in Eq. (2.45) is a symmetry generator of Eq. (2.44) if

$$X^{[2]}(1-u_t^2)u_{xx} + 2u_x u_t u_{xt} - (1+u_x^2)u_{tt}) \mid_{\text{Eq. (2.44)}=0} = 0. \tag{2.46}$$

The second prolongation in this case is

$$X^{[2]} = X + \zeta^x \frac{\partial}{\partial u_x} + \zeta^t \frac{\partial}{\partial u_t} + \zeta^{xx} \frac{\partial}{\partial u_{xx}} + \zeta^{xt} \frac{\partial}{\partial u_{xt}} + \zeta^{tt} \frac{\partial}{\partial u_{tt}}, \tag{2.47}$$

with the coefficients

$$\begin{aligned}
\zeta^x &= D_x(\zeta - \xi^1 u_x - \xi^2 u_t) + \xi^1 u_{xx} + \xi^2 u_{xt}, & (2.48)\\
\zeta^t &= D_t(\zeta - \xi^1 u_x - \xi^2 u_t) + \xi^1 u_{xt} + \xi^2 u_{tt}, & (2.49)\\
\zeta^{xx} &= D_x^2(\zeta - \xi^1 u_x - \xi^2 u_t) + \xi^1 u_{xxx} + \xi^2 u_{xxt}, & (2.50)\\
\zeta^{tt} &= D_t^2(\zeta - \xi^1 u_x - \xi^2 u_t) + \xi^1 u_{xxt} + \xi^2 u_{xtt}, & (2.51)\\
\zeta^{xt} &= D_x D_t(\zeta - \xi^1 u_x - \xi^2 u_t) + \xi^1 u_{xtt} + \xi^2 u_{ttt}. &
\end{aligned}$$

where the operators D_x and D_t denote the total derivatives defined in Eq. (2.43). Using the invariance condition, i.e., applying the second prolongation $X^{[2]}$ to Eq. (2.44), leads to an overdetermined system of linear homogenous system of partial differential equations given by

$$\begin{aligned}
\xi^2_{xx} &= 0, \quad \xi^2_{xu} = 0, \quad \xi^2_{tt} = 0, \quad \xi^2_{\mathrm{uu}} = 0,\\
\xi^1_x - \xi^2_t &= 0, \quad \xi^1_t - \xi^2_x = 0,\\
\xi^1_u + \eta_x &= 0, \quad \xi^2_t - \eta_u = 0,\\
\xi^2_u - \eta_t &= 0, \quad \xi^2_{tu} + \eta_{xx} = 0.
\end{aligned} \tag{2.52}$$

By solving the system (2.52), we obtain a four-dimensional Lie symmetry algebra generated by the operators

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \ X_2 = \frac{\partial}{\partial x}, \ X_3 = \frac{\partial}{\partial u}, \\
X_4 &= t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}, \\
X_5 &= -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}, \\
X_6 &= u\frac{\partial}{\partial t} + t\frac{\partial}{\partial u}, \\
X_7 &= x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}. \qquad (2.53)
\end{aligned}$$

2.8.6 Group invariant solution

Here we state the use of the symmetry for finding special exact solutions to differential equations. For a symmetry operator of the form

$$X = \xi^1(x,u)\frac{\partial}{\partial x_1} + ... + \xi^n(x,u)\frac{\partial}{\partial x_n} + \eta(x,u)\frac{\partial}{\partial f}, \qquad (2.54)$$

the group invariant solution $f = f(x)$ is obtained from $X(f - f(x)) \mid_{f=f(x)} = 0$ and is deduced by solving the corresponding characteristic system which is defined by

$$\frac{dx_1}{\xi^1(x,f)} = ... = \frac{dx_n}{\xi^n(x,f)} = \frac{df}{\eta(x,f)}. \qquad (2.55)$$

2.8.7 Travelling wave solutions

Travelling wave solutions are the one special kind of group invariant solutions which are invariant under a linear combination of time-translation and space-translation generators. These solutions are of two types depending upon the value of the wave speed (say m).

Let X_1 and X_2 be time-translation and space-translation symmetry generators respectively. Then the solution corresponding to the generator

$$X = X_1 + mX_2, \quad (m > 0), \tag{2.56}$$

would represent a forward wave-front type travelling wave solution. In this case, the waves are propagating away from the plate. On the other hand, the solution corresponding to the generator

$$X = X_1 - mX_2, \quad (m > 0), \tag{2.57}$$

would represent a backward wave-front type travelling wave solution. In this case the waves are propagating towards the plate.

2.8.8 Non-classical symmetry method for partial differential equations

Here we present a brief version of the nonclassical symmetry method for partial differential equations. In recent years, interest in the conditional/non-classical symmetry approach has increased. There are equations arising in applications that do not admit classical symmetries but have conditional symmetries. Thus this method is useful in constructing exact solutions.

We begin by considering a k^{th} order partial differential equation

$$G(x, u, u^{(1)}, ..., u^{(k)}) = 0, \tag{2.58}$$

in n independent variables $x = (x_1, ..., x_n)$ and one dependent variable u, with $u^{(k)}$ denoting the derivatives of the u with respect to the x up to order k defined by

$$u^{(k)} = \left[\frac{\partial^k u}{\partial x_{i_1} ... x_{i_k}} \right], \tag{2.59}$$

with

$$1 \leq i_1, ..., i_k \leq n. \tag{2.60}$$

Suppose that $\mathbf{X}$ is a vector field of dependent and independent variables:

$$\mathbf{X} = \xi^1(x,u)\frac{\partial}{\partial x_1} + ... + \xi^n(x,u)\frac{\partial}{\partial x_n} + \eta(x,u)\frac{\partial}{\partial u}. \tag{2.61}$$

The functions ξ^i and η are the coefficient functions of the vector field $\mathbf{X}$.

Suppose that the vector field $\mathbf{X}$ is the non-classical symmetry generator of (2.58), then the solution

$$u = f(x_1, x_2, ..., x_n) \tag{2.62}$$

of equation (2.58) is an invariant solution of (2.58) under a one-parameter subgroup generated by $\mathbf{X}$ if the condition

$$\Phi(x,u) = \eta(x,u) - \sum_{i=1}^{n} \xi^i(x,u)\frac{\partial u^\alpha}{\partial x_i} = 0, \tag{2.63}$$

holds together with equation (2.58). The condition given in equation (2.63) is known as an *invariant surface condition.* Thus the invariant solution of (2.58) is obtained by solving the invariant surface condition (2.63) together with equation (2.58).

For equations (2.58) and (2.63) to be compatible, the k^{th} prolongation $\mathbf{X}^{[k]}$ of the generator $\mathbf{X}$ must be tangent to the intersection of G and the surface Φ, that is

$$\mathbf{X}^{[k]}(G) \mid_{G\cap\Phi} = 0. \tag{2.64}$$

If the equation (2.63) is satisfied, then the operator $\mathbf{X}$ is called a non-classical infinitesimal symmetry of the k^{th} order partial differential equation (2.58).

For the case of two independent variables, t and y, two cases arise, viz when $\xi^1 \neq 0$ and $\xi^1 = 0$.

When $\xi^1 \neq 0$ the operator is

$$\mathbf{X} = \frac{\partial}{\partial t} + \xi^2(t, y, u)\frac{\partial}{\partial y} + \eta(t, y, u)\frac{\partial}{\partial u}, \tag{2.65}$$

and thus

$$\Phi = u_t - \eta + \xi^2 u_y = 0, \tag{2.66}$$

is the invariant surface condition.

When $\xi^1 = 0$ the operator is

$$\mathbf{X} = \frac{\partial}{\partial y} + \eta(t, y, u)\frac{\partial}{\partial u}, \tag{2.67}$$

and hence

$$\Phi = u_y - \eta = 0. \tag{2.68}$$

is the invariant surface condition.

2.8.9 Example on the non-classical symmetry method

We illustrate the use of the non-classical symmetry method on the well-known heat equation

$$u_t = u_{xx}. \tag{2.69}$$

Consider the infinitesimal operator

$$X = \xi(t, x, u)\frac{\partial}{\partial x} + \tau(t, x, u)\frac{\partial}{\partial t} + \eta(t, x, u)\frac{\partial}{\partial u}. \tag{2.70}$$

The invariant surface condition is

$$\phi(t, x, u) = \eta(t, x, u) - \xi(t, x, u)\frac{\partial u}{\partial x} - \tau(t, x, u)\frac{\partial u}{\partial t} = 0. \tag{2.71}$$

One can assume without loss of generality that $\tau = 1$, so that Eq. (2.71) takes the form

$$\phi(t,x,u) = \eta(t,x,u) - \xi(t,x,u)\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0. \tag{2.72}$$

The nonclassical symmetries determining equations are:

$$X^{[2]}u_t - u_{xx}|_{\text{Eq.}(2.69)=0,\phi=0} = 0, \tag{2.73}$$

where $X^{[2]}$ is the usual third prolongation of operator X.

Applying the method to the heat PDE (2.69) with $\tau = 1$ yields

$$\begin{aligned} \xi &= \xi(x,t), & (2.74)\\ \eta &= A(x,t)u + B(x,t), & (2.75)\end{aligned}$$

where

$$\begin{aligned} A_t + 2A\xi_x - A_{xx} &= 0, & (2.76a)\\ B_t + 2B\xi_x - B_{xx} &= 0, & (2.76b)\\ \xi_t + 2\xi\xi_x - \xi_{xx} + 2A_x &= 0. & (2.76c)\end{aligned}$$

The solution of system (2.76) give the following nonclassical infinitesimals

$$\begin{aligned} \xi &= -\frac{w_t v - w v_t}{w_x v - w v_x}, & (2.77)\\ \tau &= 1, & (2.78)\\ \eta &= \frac{v_t w_x - v_x w_t}{w_x v - w v_x}(u - u_0) + \xi f_x + f_t. & (2.79)\end{aligned}$$

where w, v and f satisfy the heat equation.

Chapter 3

Group Invariant Solutions for the Unsteady Magnetohydrodynamic Flow of a Third Grade Fluid in a Porous Medium

3.1 Introduction

The purpose of this chapter is to study the time-dependent flow of an incompressible non-Newtonian fluid over an infinite rigid plate. The flow is induced due to the arbitrary velocity $V(t)$ of the plate. The fluid occupies the porous half space $y > 0$ and is also electrically conducting in the presence of a constant applied magnetic field in the transverse direction to the flow. Such flows have applications in transpiration cooling, gaseous diffusion, boundary layer control and so forth. Analytical solutions of the governing non-linear partial differential equation for the unidirectional flow of a third grade fluid are established using the symmetry approach. We construct three types of analytical solutions by employing the Lie symmetry method and the better solution from the physical point of view is shown to be the non-travelling wave solution. We also present numerical solutions of the governing PDE and the reduced ODE and compare with the analytical results. Finally, the influence of emerging parameters are studied through several graphs with emphasis on the study of the effects of the magnetic field and non-Newtonian fluid parameters.

An outline of the chapter is as follows. In Section 3.2, the mathematical modelling is carried out for the flow problem under investigation. In Section 3.3, all possible Lie point symmetries of the modelled partial differential equation are calculated. The group invariant solutions are presented in sections 3.4. Finally, the graphical discussion and the conclusions are summarized in Sections 3.5 and 3.6 respectively.

3.2 Mathematical formulation and constitutive relations

The constitutive relation for an incompressible and thermodynamically compatible third grade fluid [41] has the form

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1. \tag{3.1}$$

In the above relation, $\mathbf{T}$ is the Cauchy stress tensor, p the pressure, $\mathbf{I}$ the identity tensor, μ the dynamic viscosity, α_1, α_2 and β_3 are the material constants, and $\mathbf{A}_i$ $(i = 1-3)$ are the Rivlin-Ericksen tensors which are defined through the following equations:

$$\begin{aligned} \mathbf{A}_1 &= (\mathrm{grad}\,\mathbf{V}) + (\mathrm{grad}\,\mathbf{V})^T, && (3.2) \\ \mathbf{A}_n &= \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(\mathrm{grad}\,\mathbf{V}) + (\mathrm{grad}\,\mathbf{V})^T\mathbf{A}_{n-1};\ (n>1), && (3.3) \end{aligned}$$

where $\mathbf{V}$ denotes the velocity field, d/dt is the material time derivative defined by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}.\boldsymbol{\nabla}), \tag{3.4}$$

in which $\boldsymbol{\nabla}$ is the gradient operator.

Here we consider a Cartesian coordinate system $OXYZ$ with the y-axis in the vertical upward direction and x-axis parallel to the rigid plate at $y = 0$. The flow of an incompressible and electrically conducting fluid of third grade is bounded by an infinite rigid plate. The fluid occupies the porous half space $y > 0$. The plate is infinite in the XZ-plane and therefore all the physical quantities except the pressure depend on y only. Thus, for the flow under consideration, we seek a velocity of the form

$$\mathbf{V} = [u(y,t), 0, 0]. \tag{3.5}$$

The above definition of velocity identically satisfies the law of conservation of mass for incompressible fluid, viz.

$$\operatorname{div}\mathbf{V} = \mathbf{0}. \tag{3.6}$$

The unsteady motion through a porous medium in the presence of magnetic field is governed by

$$\rho\frac{d\mathbf{V}}{dt} = \operatorname{div}\mathbf{T} + \mathbf{J}\times\mathbf{B} + \mathbf{R}, \tag{3.7}$$

where

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V}\times\mathbf{B}), \tag{3.8}$$

in which ρ is the fluid density, σ the electrical conductivity, $\mathbf{J}$ the current density, $\mathbf{B}$ the magnetic induction so that $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ ($\mathbf{B}_0$ and $\mathbf{b}$ are the applied and induced magnetic fields respectively), $\mathbf{R}$ the Darcy's resistance in the porous medium and $\mathbf{E}$ the imposed electric field.

In this study we consider the fluid to be electrically conducting in the presence of an imposed uniform magnetic field in the transverse direction to the flow. In the small magnetic Reynold's number approximation, in which the induced magnetic field can be ignored, the effect of the imposed electric field and induced magnetic field are negligible and only the effects of the applied magnetic field $\mathbf{B}_0$ is applicable. In this case the current density in Eq. (3.8) becomes

$$\mathbf{J} = \sigma(\mathbf{V} \times \mathbf{B}_0). \tag{3.9}$$

Thus the magnetic force on the right hand side of Eq. (3.7) is

$$\mathbf{J} \times \mathbf{B} = -\sigma B_0^2 \mathbf{V}. \tag{3.10}$$

Following reference [48], the constitutive relationship between the pressure drop and the velocity for the unidirectional flow of a third grade fluid is

$$\frac{\partial p}{\partial x} = -\frac{\phi}{\kappa}\left[\mu + \alpha_1 \frac{\partial}{\partial t} + 2\beta_3 \left(\frac{\partial u}{\partial y}\right)^2\right] u, \tag{3.11}$$

where κ is the permeability and ϕ the porosity of the porous medium.
The pressure gradient in Eq. (3.11) is regarded as a measure of the flow resistance in the bulk of the porous medium. If R_x is a measure of the flow resistance due to the porous medium in the $x-$direction, then R_x through Eq. (3.11) is given by

$$R_x = -\frac{\phi}{\kappa}\left[\mu + \alpha_1 \frac{\partial}{\partial t} + 2\beta_3 \left(\frac{\partial u}{\partial y}\right)^2\right] u. \tag{3.12}$$

Using Eq. (3.5), having in mind Eqs. (3.1), (3.2), (3.3), (3.7), (3.10) and (3.12), one obtains the following governing equation in the absence of the modified pressure gradient

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + 6\beta_3 \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \frac{\phi}{\kappa}\left[\mu + \alpha_1 \frac{\partial}{\partial t} + 2\beta_3 \left(\frac{\partial u}{\partial y}\right)^2\right] u - \sigma B_0^2 u. \tag{3.13}$$

The relevant initial and boundary conditions are

$$
\begin{aligned}
u(y,0) &= g(y), \qquad y>0, &(3.14)\\
u(0,t) &= u_0 V(t), \quad t>0, &(3.15)\\
u(\infty,t) &= 0, \qquad t>0, &(3.16)
\end{aligned}
$$

where u_0 is the reference velocity, $g(y)$ and $V(t)$ are as yet undetermined. These are specified via, the Lie group approach.

We introduce the non-dimensional quantities

$$\bar{u} = \frac{u}{u_0}, \quad \bar{y} = \frac{u_0 y}{\nu}, \quad \bar{t} = \frac{u_0^2 t}{\nu}, \quad \bar{\alpha}_1 = \frac{\alpha_1 u_0^2}{\rho\nu^2}, \quad \bar{\beta}_3 = \frac{2\beta_3 u_0^4}{\rho\nu^3}, \quad \bar{\phi} = \frac{\phi\nu^2}{\kappa u_0^2}, \ \bar{M}^2 = \frac{\sigma B_0^2 \nu}{\rho u_0^2}. \tag{3.17}$$

Eq. (3.13) and the corresponding initial and the boundary conditions then take the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + 3\beta_3 \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \phi\left[u + \alpha_1 \frac{\partial u}{\partial t} + \beta_3 u \left(\frac{\partial u}{\partial y}\right)^2\right] - M^2 u. \tag{3.18}$$

$$
\begin{aligned}
u(y,0) &= g(y), \quad y>0, &(3.19)\\
u(0,t) &= V(t), \quad t>0, &(3.20)\\
u(\infty,t) &= 0, \qquad t>0. &(3.21)
\end{aligned}
$$

For simplicity we drop the bars of the non-dimensional quantities. We rewrite Eq. (3.18) as

$$\frac{\partial u}{\partial t} = \mu_* \frac{\partial^2 u}{\partial y^2} + \alpha_* \frac{\partial^3 u}{\partial y^2 \partial t} + \beta \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \beta_* u \left(\frac{\partial u}{\partial y}\right)^2 - \phi_* u - M_*^2 u\ , \tag{3.22}$$

where

$$\begin{aligned}
\mu_* &= \frac{\mu}{1+\alpha_1\phi}, \quad \alpha_* = \frac{\alpha_1}{1+\alpha_1\phi}, \quad \beta = \frac{3\beta_3}{1+\alpha_1\phi}, \\
\beta_* &= \frac{\beta_3\phi}{1+\alpha_1\phi}, \quad \phi_* = \frac{\phi}{1+\alpha_1\phi}, \quad M_*^2 = \frac{M^2}{1+\alpha_1\phi}.
\end{aligned} \tag{3.23}$$

Eq. (3.22) is solved subject to conditions (3.19) – (3.21).

3.3 Lie symmetry analysis

In this section, we briefly discuss how to determine Lie point symmetry generators admitted by Eq. (3.22). We use these generators to solve Eq. (3.22) analytically subject to the conditions (3.19) – (3.21).

We look for transformations of the independent variables t, y and the dependent variable u of the form (see the previous chapter on symmetries)

$$\bar{t} = \bar{t}(t, y, u, \epsilon), \;\; \bar{y} = \bar{y}(t, y, u, \epsilon), \;\; \bar{u} = \bar{u}(t, y, u, \epsilon), \tag{3.24}$$

which constitue a group where ϵ is the group parameter such that Eq. (3.22) is left invariant. From Lie's theory, the transformations in Eq. (3.24) are obtained in terms of the infinitesimal transformations

$$\bar{t} \simeq t + \epsilon\tau(t, y, u), \;\; \bar{y} \simeq y + \epsilon\xi(t, y, u), \;\; \bar{u} \simeq u + \epsilon\eta(t, y, u), \tag{3.25}$$

or the operator

$$X = \tau(t, y, u)\frac{\partial}{\partial t} + \xi(t, y, u)\frac{\partial}{\partial y} + \eta(t, y, u)\frac{\partial}{\partial u}, \tag{3.26}$$

which is a generator of the Lie point symmetry of Eq. (3.22) if the following determining equation holds, viz

$$X^{[3]}[u_t - \mu_* u_{yy} - \alpha_* u_{yyt} - \beta(u_y)^2 u_{yy} + \beta_* u(u_y)^2 + \phi_* u + M_*^2 u] \mid_{15} = 0. \tag{3.27}$$

The operator $X^{[3]}$ is the third prolongation of the operator X which is

$$X^{[3]} = X + \zeta^t \frac{\partial}{\partial u_t} + \zeta^y \frac{\partial}{\partial u_y} + \zeta^{yy} \frac{\partial}{\partial u_{yy}} + \zeta^{yyy} \frac{\partial}{\partial u_{yyy}} + \zeta^{tyy} \frac{\partial}{\partial u_{tyy}} \ , \tag{3.28}$$

with

$$\begin{aligned}
\zeta^t &= D_t\eta - u_t D_t\tau - u_y D_t\xi \ , \\
\zeta^y &= D_y\eta - u_t D_y\tau - u_y D_y\xi \ , \\
\zeta^{yy} &= D_y\zeta^y - u_{ty} D_y\tau - u_{yy} D_y\xi \ , \\
\zeta^{yyy} &= D_y\zeta^{yy} - u_{tyy} D_y\tau - u_{yyy} D_y\xi \ , \\
\zeta^{tyy} &= D_t\zeta^{yy} - u_{tyy} D_t\tau - u_{yyy} D_t\xi \ ,
\end{aligned} \tag{3.29}$$

and the total derivative operators

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{ty} \frac{\partial}{\partial u_y} + \ldots \ , \\
D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{yy} \frac{\partial}{\partial u_y} + u_{ty} \frac{\partial}{\partial u_t} + \ldots \ ,
\end{aligned} \tag{3.30}$$

Substituting the expansions of Eq. (3.29) into the symmetry condition (3.27) and separating by powers of the derivatives of u, as τ, ξ, and η are independent of the derivatives of

u, lead to the overdetermined system of linear homogeneous partial differential equations

$$
\begin{aligned}
\tau_u &= 0 \\
\tau_y &= 0 \\
\xi_u &= 0 \\
\xi_y &= 0 \\
\xi_t &= 0 \\
\eta_{uu} &= 0 \\
\eta_y &= 0 \\
\mu_* \tau_t + \alpha_* \eta_{tu} &= 0 \\
M_*^2 \eta + \phi_* \eta + u M_*^2 \tau_t + u \phi_* \tau_t + \eta_t - u M_*^2 \eta_u - u \phi_* \eta_u &= 0 \\
\eta + u \tau_t + u \eta_u &= 0 \\
\tau_t + 2\eta_u &= 0.
\end{aligned} \tag{3.31}
$$

The solution of the above system gives rise to two cases. These are:

Case 1: $\phi_* + M_*^2 \neq \mu_*/\alpha_*$

For this case, we find a two-dimensional Lie algebra generated by

$$
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial y} \ . \tag{3.32}
$$

Case 2: $\phi_* + M_*^2 = \mu_*/\alpha_*$

Here we obtain a three-dimensional Lie algebra generated by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial y}, \\ X_3 &= -\left(\frac{1}{\phi_* + M_*^2}\right) e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial t} + u e^{2(\phi_* + M_*^2)t} \frac{\partial}{\partial u}. \end{aligned} \tag{3.33}$$

3.4 Solutions of the problem

In this section, similarity solutions (group invariant solutions) which satisfy the initial and boundary conditions (3.19) – (3.21) corresponding to the above symmetries for Eq. (3.22) are constructed.

3.4.1 Travelling wave solutions

Travelling wave solutions are special kind of group invariant solutions which are invariant under a linear combination of the time-translation and the space-translation generators. We search for an invariant solution under the operator

$$X = X_1 + cX_2 \ , \ \Rightarrow X = \frac{\partial}{\partial t} + c\frac{\partial}{\partial y} \ , \tag{3.34}$$

which denotes forward wave-front type travelling wave solutions with constant wave speed c. The characteristic system of Eq. (3.34) is

$$\frac{dy}{c} = \frac{dt}{1} = \frac{du}{0}. \tag{3.35}$$

Solving Eq. (3.35), the invariant is given as

$$u(y,t) = F(x_1), \quad where \ \ x_1 = y - ct \ . \tag{3.36}$$

Using Eq. (3.36) into Eq. (3.22) results in a third-order ordinary differential for $F(x_1)$, viz.

$$-c\frac{dF}{dx_1} = \mu_*\frac{d^2F}{dx_1^2} - \alpha_* c\frac{d^3F}{dx_1^3} + \beta\left(\frac{dF}{dx_1}\right)^2\frac{d^2F}{dx_1^2} - \beta_* F\left(\frac{dF}{dx_1}\right)^2 - (\phi_* + M_*^2)F, \quad (3.37)$$

and the above equation admits the exact solution of the form (which we require to be zero at infinity due to the second boundary condition)

$$F(x_1) = A\exp\left(\frac{\sqrt{\beta_*}}{-\sqrt{\beta}}x_1\right), \quad (3.38)$$

provided that

$$\sqrt{\frac{\beta_*}{\beta}}\left[c\alpha_*\frac{\beta_*}{\beta} - c\right] + \left[\frac{\beta_*}{\beta}\mu_* - (\phi_* + M_*^2)\right] = 0. \quad (3.39)$$

Thus the above solution can be written as

$$u(y,t) = \exp\left(\frac{\sqrt{\beta_*}(y - ct)}{-\sqrt{\beta}}\right). \quad (3.40)$$

We note that this solution satisfies the boundary condition (3.19), (3.20) and (3.21) with

$$V(t) = \exp\left(\frac{c\sqrt{\beta_*}}{\sqrt{\beta}}t\right) \quad \text{and} \quad g(y) = \exp\left(-\frac{\sqrt{\beta_*}}{\sqrt{\beta}}y\right). \quad (3.41)$$

We remark here that $V(t)$ and $g(y)$ depend on physical parameters of the flow. This solution in Eq. (3.40) is plotted in Figures 3.1 and 3.2 for various values of the emerging parameters which satisfy the condition (3.39).

Moreover, we can also find group invariant solutions corresponding to the operator X_1 and X_3, which provide other meaningful solutions as well.

3.4.2 Group-invariant solutions corresponding to X_3

The operator X_3 is given as

$$X_3 = -\left(\frac{1}{\phi_* + M_*^2}\right) e^{2(\phi_*+M_*^2)t}\frac{\partial}{\partial t} + ue^{2(\phi_*+M_*^2)t}\frac{\partial}{\partial u}. \tag{3.42}$$

The characteristic system of Eq. (3.42) is

$$\frac{dt}{-\left(\frac{1}{\phi_*+M_*^2}\right)} = \frac{du}{u} = \frac{dy}{0}. \tag{3.43}$$

By solving the above system, the invariant is given by

$$u(y,t) = w(y)\exp[-(\phi_* + M_*^2)t], \tag{3.44}$$

where $w(y)$ as yet is an undetermined function of y.

Substituting Eq. (3.44) into Eq. (3.22) yields the linear second-order ordinary differential equation

$$\frac{d^2w}{dy^2} - \frac{\beta_*}{\beta}w = 0, \tag{3.45}$$

Using conditions (3.20) and (3.21), one can write the boundary conditions for Eq. (3.45) as

$$w(0) = 1\ ,\ w(l) = 0\ , l \to \infty\ , \tag{3.46}$$

where

$$V(t) = \exp[-(\phi_* + M_*^2)t]. \tag{3.47}$$

We solve Eq. (3.45) subject to the boundary conditions given in Eq. (3.46) for positive β_*/β. We obtain

$$w(y) = \exp\left(-\frac{\sqrt{\beta_*}}{\sqrt{\beta}}y\right). \tag{3.48}$$

Substituting this $w(y)$ in Eq. (3.44), we deduce the solution for $u(y,t)$ in the form

$$u(y,t) = \exp\left[-\{\left(\phi_* + M_*^2\right)t + \sqrt{\frac{\beta_*}{\beta}}y\}\right]. \tag{3.49}$$

We observe that this solution (3.49) satisfies the boundary conditions (3.19), (3.20) and (3.21) with

$$V(t) = \exp[-(\phi_* + M_*^2)t] \text{ and } g(y) = \left(-\sqrt{\frac{\beta_*}{\beta}}y\right) \tag{3.50}$$

Again $V(t)$ and $g(y)$ depend on physical parameters of the flow. This solution in Eq. (3.49) is plotted in Figures 3.3 – 3.6 for various values of the emerging parameters.

3.4.3 Group-invariant solutions corresponding to X_1

The time translation generator X_1 is given by

$$X_1 = \frac{\partial}{\partial t}. \tag{3.51}$$

The invariant solution admitted by X_1 is the steady-state solution

$$u(y,t) = G(y). \tag{3.52}$$

Introducing Eq. (3.52) into Eq. (3.22) yields the third-order ordinary differential equation for $G(y)$, viz.

$$\mu_* \frac{d^2G}{dy^2} + \beta\left(\frac{dG}{dy}\right)^2 \frac{d^2G}{dy^2} - \beta_* G\left(\frac{dG}{dy}\right)^2 - \phi_* G - M_*^2 G = 0, \tag{3.53}$$

with the boundary conditions

$$\begin{aligned} G(0) &= v_0, \\ G(y) &= 0, \quad y \to \infty, \end{aligned} \tag{3.54}$$

where l can take a sufficiently large value with $V = v_0$ a constant. The above equation (3.53) admits the exact solution of the form (which we also require to be zero at infinity due to the second boundary condition)

$$G(y) = v_0 \exp\left(-\frac{\sqrt{\beta_*}}{\sqrt{\beta}} y\right), \tag{3.55}$$

provided that

$$\frac{\beta_*}{\beta}\mu_* - (\phi_* + M_*^2) = 0. \tag{3.56}$$

We also solve the above boundary value problem numerically using the Mathematica solver **NDSolve**. This solution is plotted in Figure 3.7.

3.4.4 Numerical Solution

We present the numerical solution of PDE (3.22) subject to the initial and boundary conditions:

$$\begin{aligned} u(0,t) &= V(t) = e^{-t}, \quad u(\infty,t) = 0, \quad t > 0, \\ u(y,0) &= g(y) = e^{-y}, \quad y > 0. \end{aligned} \tag{3.57}$$

This solution is plotted in Figure 3.8 using the Mathematica® solver **NDSolve.**

Remark 1 *The group invariant solution* (3.49) *corresponding to* X_3 *contains the magnetic field term and shows the effects of magnetic field on the physical system directly whereas the travelling wave solution* (3.40) *and the group invariant solution* (3.55) *are the solutions provided that the conditions along which these solutions are valid must hold. These solutions does not directly contain the term which is responsible for showing the effects of the magnetic field. Thus these solutions are valid for the particular value of the magnetic field parameter. To emphasize, we say that the group invariant solution of the non-travelling type generated by* X_3 *is better than the other two group invariant solutions* *in the sense that is best represents the physics of the problem considered.*

3.5 Graphical results and discussion

In this section, the velocity profiles obtained in the previous sections are plotted through several graphs. By the group invariants the partial differential equation is transformed into an ordinary differential equation which is numerically integrated and presented here. The velocity profiles versus y are illustrated in Figures $3.1-3.8$ for various values of the emerging parameters.

The travelling wave solutions over an infinite rigid plate are depicted in Figures 3.1 and 3.2 for various values of the time t and wave speed c. These solutions do not predict the effects of the magnetic field on the flow. It is noted that the effect of both time t and wave speed c is seen to increase the velocity profile.

To see the effects of magnetic field on the flow, the group invariant solution (3.49) is plotted in Figures $3.3-3.6$ for different values of the emerging parameters. Figure 3 shows the velocity profiles for various values of the time t at fixed values of other parameters. As expected, the velocity profile will decay with time. In order to examine the effects of the magnetic field, the velocity profiles versus y for various values of the magnetic field parameter M_* are depicted in Figure 3.4. It shows that the velocity decreases when the magnitude of the magnetic field is increased. The effects of the fluid parameters β and β_* in the presence of magnetic field are shown in Figures 3.5 and 3.6. These figures reveal that β and β_* have opposite roles on the velocity even in the presence of the magnetic field. From these figures, it is noted that velocity increases for large values of β whereas it decreases for increasing β_*.

The numerical solutions of ordinary differential equation (3.53) subject to boundary conditions (3.54) are shown in Figure 3.7 for different values of the magnetic field parameter M_*. It is observed that the effects of M_* on the velocity profiles are the same as noted previously for the analytical solutions. Finally in Figure 3.8 we have plotted the numerical solution of the governing PDE (3.22) for small variations of time, and it is observed that the velocity decreases as time increases, which is the same observation made previously for the analytical results.

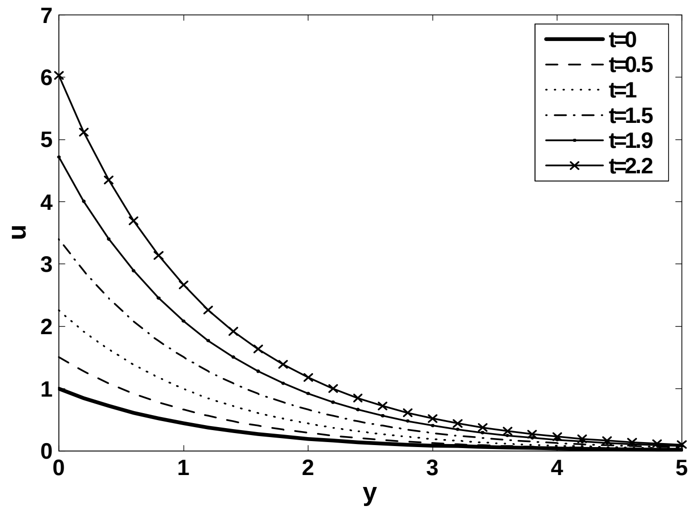

Figure 3.1: The diagram of the velocity field (3.40) for varying values of t when $\beta = 1.5, \beta_* = c = 1$ are fixed.

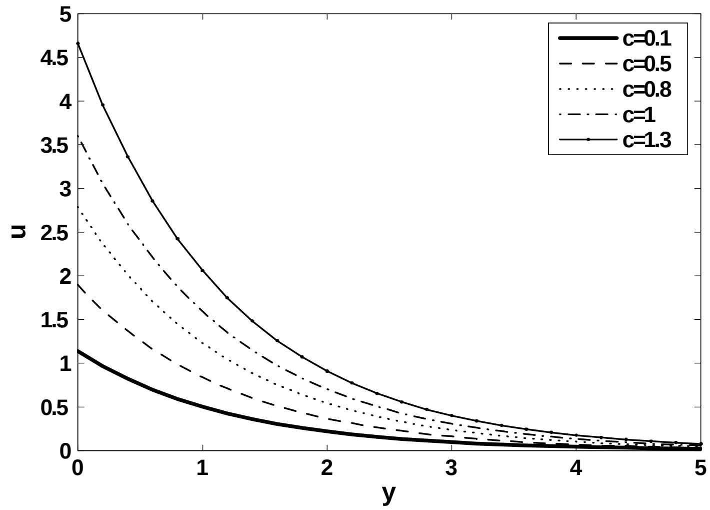

Figure 3.2: The diagram of the velocity field (3.40) for varying values of c when $\beta = 1.5, \beta_* = 1, t = \pi/2$ are fixed.

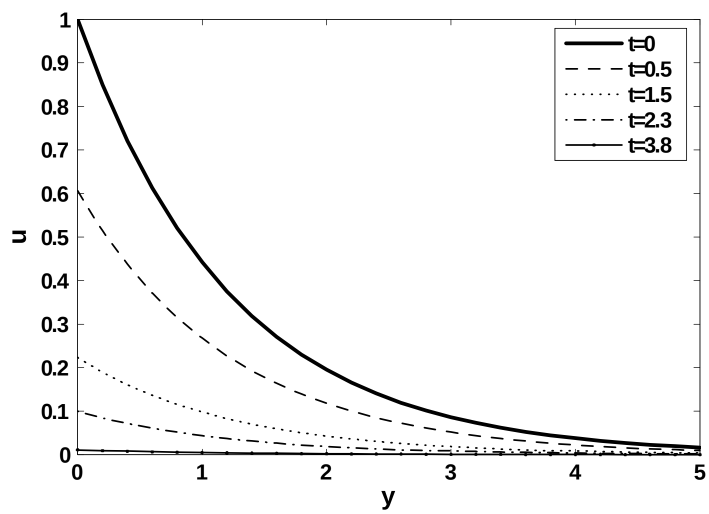

Figure 3.3: The diagram of the velocity field (3.49) for varying values of t when $\beta = 1.5, \beta_* = 1, M_* = 0.5, \phi_* = 1$ are fixed.

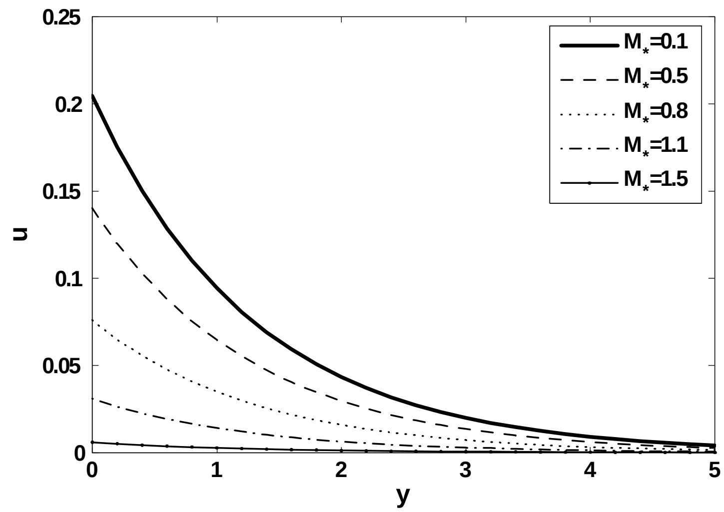

Figure 3.4: The diagram of the velocity field (3.49) for varying values of M_* when $\beta = 2.5, \beta_* = 1.5, \phi_* = 1, t = \pi/2$ are fixed.

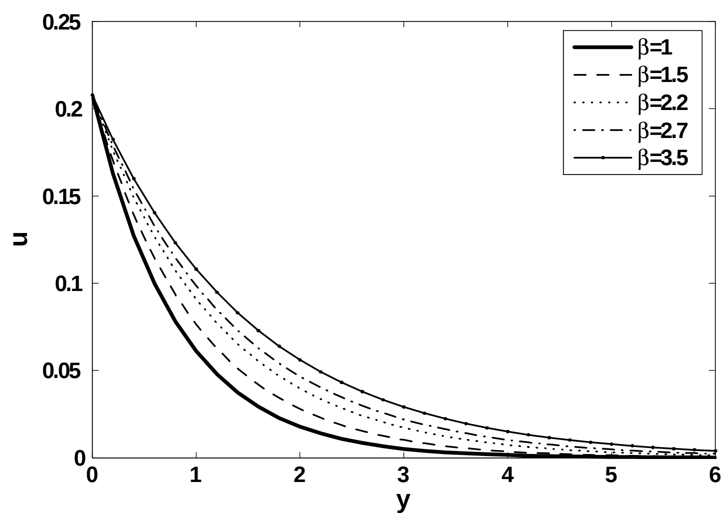

Figure 3.5: The diagram of the velocity field (3.49) for varying values of β when $\beta_* = 1.5, M_* = 0.5, \phi_* = 1, t = \pi/2$ are fixed.

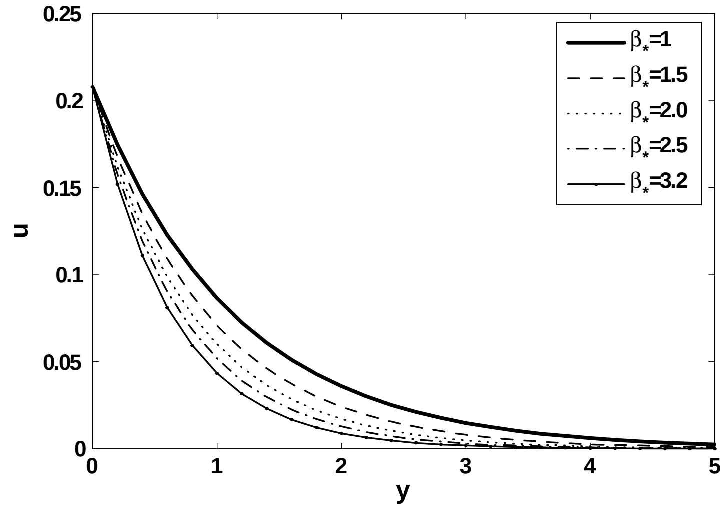

Figure 3.6: The diagram of the velocity field (3.49) for varying values of β_* when $\beta = 1.5, M_* = 0.5, \phi_* = 1, t = \pi/2$ are fixed.

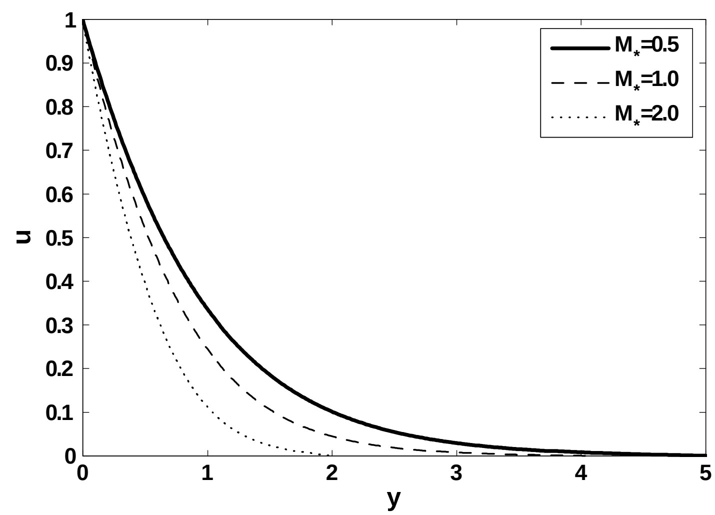

Figure 3.7: Numerical solution of (3.53) subject to conditions (3.54) for varying values of M_* when $\beta = 1.5, \mu_* = \phi_* = 0.5, \beta_* = 1$ are fixed.

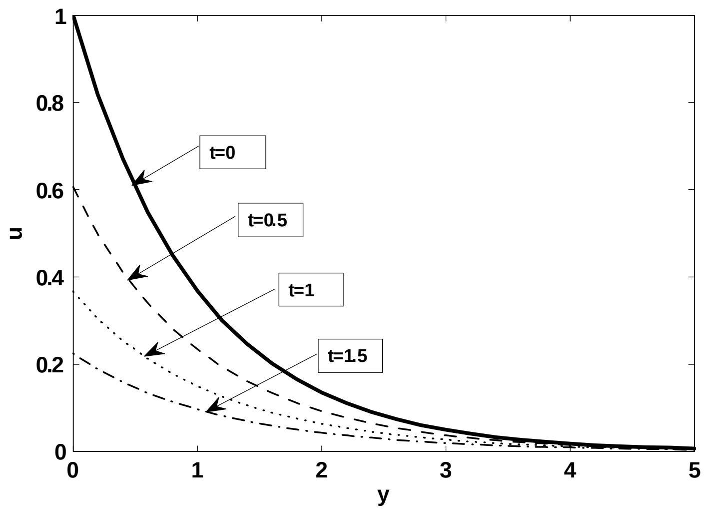

Figure 3.8: Numerical solution of PDE (3.22) subject to conditions (3.55) for varying values of t when $\alpha_* = 2, \mu_* = 2.5, \beta = 1.5, \beta_* = 2.6, \phi_* = 0.8, M_* = 0.5$ are fixed.

3.6 Conclusions

In this chapter, we have considered the unsteady magnetohydrodynamic flow of third grade fluid over an infinite moved rigid plate. The fluid occupied the porous half space $y > 0$. The governing nonlinear problem that comprised the balance laws of mass and momentum was modelled. We performed a complete Lie group analysis on the governing nonlinear PDE. It was found that the principal Lie algebra was two-dimensional which was then extended to three dimensions for the restrictive cases of the parameters. We obtained three types of group invariant solutions with the help of these symmetries. In Remark 1, a detailed discussion was included regarding the better solution in the sense of imposing conditions from a physical point of view of the problem considered. This was shown to be the non-travelling wave group invariant solution (3.49). Finally, we presented the numerical solution of the governing PDE with a suitable choice of boundary and initial conditions. The way in which various physical parameters affect the behavior of the velocity was analyzed with the help of several graphs.

Chapter 4

Nonlinear Time-Dependent Flow Models of Third Grade Fluids: A Conditional Symmetry Approach

4.1 Introduction

The present chapter continues the research which was carried out in [49, 41, 50]. In Hayat et al. [49], some classically invariant solutions were obtained for unsteady flow of a third grade fluid in a porous medium. In [41], the analysis of [49] was extended by taking into account the magnetohydrodynamic (MHD) nature of the fluid. We also utilized the Lie classical symmetry approach to construct a new class of group invariant solutions for the governing nonlinear partial differential equation. Aziz and Aziz [50] recently extended the flow model of [41] by including suction and injection at the boundary of the flow. The same group theoretic approach has been used to obtain closed-form invariant solutions of the nonlinear boundary-value problem. In this work, we revisit these three flow problems.

In this communication some nonlinear flow problems dealing with the unsteady flow of a third grade fluid in porous half-space are analyzed. A new class of closed-form conditionally invariant solutions for these flow models are constructed by using the conditional or non-classical symmetry approach. All possible non-classical symmetries of the model equations are obtained and various new conditionally invariant solutions have been constructed. The solutions are valid for a half-space and also satisfy the physically relevant initial and the boundary conditions.

4.2 Problem 1: Unsteady flow of a third grade fluid over a flat rigid plate with porous medium

Hayat et al. [49] solved the time-dependent problem for the flow of a third grade fluid in a porous half-space. The governing nonlinear partial differential equation in [49], with a slight change of notation, is given by

$$\rho\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial y^2}+\alpha_1\frac{\partial^3 u}{\partial y^2\partial t}+6\beta_3\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}-\frac{\phi}{\kappa}\left[\mu+\alpha_1\frac{\partial}{\partial t}+2\beta_3\left(\frac{\partial u}{\partial y}\right)^2\right]u, \tag{4.1}$$

where $u(y,t)$ is the velocity component, ρ is the density, μ the coefficient of viscosity, α_1 and β_3 are the material constants (for details on these material constants and the conditions that are satisfied by these constants, the reader is referred to [14]), ϕ the porosity and κ the permeability of the porous medium.

In order to solve the above Eq. (4.1), the relevant time and space dependent velocity boundary conditions are specified as follows:

$$u(0,t) = u_0V(t), \quad t>0, \tag{4.2}$$

$$u(\infty,t) = 0, \quad t>0, \tag{4.3}$$

$$u(y,0) = g(y), \quad y>0, \tag{4.4}$$

where u_0 is the reference velocity. The first boundary condition (4.2) is the no-slip condition and the second boundary condition (4.3) says that the main stream velocity is zero. This is not a restrictive assumption since we can always measure velocity relative to the main stream. The initial condition (4.4) indicates that the fluid is initially moving with some non-uniform velocity $g(y)$.

In [49], the governing problem was solved without transforming the problem into dimensionless form. Here we first non-dimensionalize the problem and then obtain solutions. We define the non-dimensional quantities as

$$\bar{u} = \frac{u}{u_0}, \quad \bar{y} = \frac{u_0 y}{\nu}, \quad \bar{t} = \frac{u_0^2 t}{\nu}, \quad \bar{\alpha} = \frac{\alpha_1 u_0^2}{\rho \nu^2}, \quad \bar{\beta} = \frac{2\beta_3 u_0^4}{\rho \nu^3}, \quad \frac{1}{\bar{K}} = \frac{\phi \nu^2}{\kappa u_0^2}. \tag{4.5}$$

Eq. (4.1) and the corresponding initial and the boundary conditions take the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \alpha \frac{\partial^3 u}{\partial y^2 \partial t} + 3\beta \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \frac{1}{K}\left[u + \alpha \frac{\partial u}{\partial t} + \beta u \left(\frac{\partial u}{\partial y}\right)^2\right], \tag{4.6}$$

$$u(0,t) = V(t), \quad t > 0, \tag{4.7}$$

$$u(y,t) \to 0 \ \text{ as } \ y \to \infty, \quad t > 0, \tag{4.8}$$

$$u(y,0) = g(y), \quad y > 0, \tag{4.9}$$

where $V(0) = g(0)$. For simplicity we suppress the bars of the non-dimensional quantities. We rewrite Eq. (4.6) as

$$\frac{\partial u}{\partial t} = \mu_* \frac{\partial^2 u}{\partial y^2} + \alpha_* \frac{\partial^3 u}{\partial y^2 \partial t} + \beta \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \beta_* u \left(\frac{\partial u}{\partial y}\right)^2 - \frac{1}{K_*} u, \tag{4.10}$$

where

$$\begin{aligned} \mu_* &= \frac{1}{(1+\alpha/K)}, \quad \alpha_* = \frac{\alpha}{(1+\alpha/K)}, \quad \beta = \frac{3\beta}{(1+\alpha/K)}, \\ \beta_* &= \frac{\beta/K}{(1+\alpha/K)}, \quad \frac{1}{K_*} = \frac{1/K}{(1+\alpha/K)}. \end{aligned} \tag{4.11}$$

We solve Eq. (4.10) subject to the conditions (4.7) – (4.9).

4.2.1 Non-classical symmetry analysis

Here we present the complete non-classical symmetry analysis of PDE (4.10) and develop some invariant solutions of problem 1.

Consider the infinitesimal operator

$$\chi = \xi^1(t,y,u)\frac{\partial}{\partial t} + \xi^2(t,y,u)\frac{\partial}{\partial y} + \eta(t,y,u)\frac{\partial}{\partial u}. \tag{4.12}$$

The invariant surface condition is

$$\phi(t,y,u) = \eta(t,y,u) - \xi^1(t,y,u)\frac{\partial u}{\partial t} - \xi^2(t,y,u)\frac{\partial u}{\partial y} = 0. \tag{4.13}$$

The nonclassical symmetries determining equations are:

$$\chi^{[3]}\text{Eq.}(4.10)|_{\text{Eq.}(4.10)=0,\phi=0} = 0, \tag{4.14}$$

where $\chi^{[3]}$ is the usual third prolongation of operator χ.

Here we find the determining equations for two different cases:

Case 1: When $\xi^1 = 1$ and $\xi^2 \neq 0$

In this case, we obtain the following set of determining equations (which have been generated manually)

$$
\begin{aligned}
\xi^2_u &= 0,\\
2\beta\xi^2_{yu} - \beta\eta_{\mathrm{uu}} &= 0,\\
\eta_y &= 0,\\
\xi^2_{yy} &= 0,\\
\xi^2_y &= 0,\\
\beta_*\eta - \frac{\beta_* u\xi^2_t}{\xi^2} + \beta_* u\eta_u &= 0,\\
\frac{\mu_*\xi^2_t}{\xi^2} - \alpha_*\eta_{tu} + \frac{\alpha_*\xi^2_t\eta_u}{\xi^2} &= 0,\\
\frac{\beta\xi^2_t}{\xi^2} + 2\beta\xi^2_y - 2\beta\eta_u &= 0,\\
\frac{\eta}{K_*} - \frac{u\xi^2_t}{K_*\xi^2} - \frac{\eta\xi^2_t}{\xi^2} + \eta_t - \frac{u\eta_u}{K_*} &= 0.
\end{aligned}
\tag{4.15}
$$

We solve the determining equations (4.15) for two cases:

Case I: $\left(\mu_* - \frac{\alpha_*}{K_*}\right) \neq 0$

For this case, we deduce

$$
\chi = \frac{\partial}{\partial t} + c\frac{\partial}{\partial y},
\tag{4.16}
$$

where c is any arbitrary constant. The above symmetry is a combination of translations in time and space.

Case II: $\left(\mu_* - \frac{\alpha_*}{K_*}\right) = 0$

In this case, we obtain a two-dimensional symmetry algebra generated by

$$
\chi_1 = \left[\exp\left(-\frac{2}{K_*}t\right) - K_*\right]\frac{\partial}{\partial t} + u\frac{\partial}{\partial u},
\tag{4.17}
$$

$$
\chi_2 = -K_*\frac{\partial}{\partial t} + \exp\left(-\frac{2}{K_*}t\right)\frac{\partial}{\partial y} + u\frac{\partial}{\partial u},
\tag{4.18}
$$

Remark 2 *The symmetries given in Eqs. (4.17) and (4.18) are the linear combination* *of classical point symmetries of PDE (4.10) also found in* [49]*, which imply that the symmetries (4.17) and (4.18) are also classical point symmetries. Thus no extra non-classical symmetry is obtained for the case when* $\xi^1 = 1$ *and* $\xi^2 \neq 0$.

Now we consider the second case:

Case 2: When $\xi^1 = 1$ and $\xi^2 = 0$

For this case, we obtain the following set of determining equations

$$\begin{aligned} \eta_{uu} &= 0, \\ \eta_y &= 0, \\ \eta - u\eta_u &= 0, \\ \mu_*\eta + \mu_* K_* \eta_t - \alpha_* u\eta_{tu} - \alpha_* K_* \eta\eta_{tu} - \mu_* u\eta_u + \alpha_*\eta\eta_u + \alpha_* K_* \eta_t\eta_u - \alpha_* u\eta_u^2 &= 0. \end{aligned} \tag{4.19}$$

Again we solve the determining equations (4.19) for two-cases:

Case I: $\left(\mu_* - \frac{\alpha_*}{K_*}\right) \neq 0$

For this case, we find

$$\chi = \frac{\partial}{\partial t} + \theta u \frac{\partial}{\partial u}, \tag{4.20}$$

where θ is a constant to be determined.

Case II: $\left(\mu_* - \frac{\alpha_*}{K_*}\right) = 0$

For this case, we get

$$\chi = \frac{\partial}{\partial t} + \lambda(t) u \frac{\partial}{\partial u}, \tag{4.21}$$

where $\lambda(t)$ is any function of time t. Thus infinite many nonclassical symmetry generators are obtained in this case for any arbitrary value of function $\lambda(t)$.

Remark 3 *The symmetries given in Eqs. (4.20) and (4.21) are the only nonclassical symmetries of PDE (4.10).*

4.2.2 Conditionally invariant solutions

Here we first present the conditionally invariant solution corresponding to the non-classical symmetry generator

$$\chi = \frac{\partial}{\partial t} + \theta u \frac{\partial}{\partial u}. \tag{4.22}$$

The characteristic corresponding to Eq. (4.22) is

$$\frac{dt}{1} = \frac{dy}{0} = \frac{du}{\theta u}. \tag{4.23}$$

By solving Eq. (4.23), the conditionally-invariant solution is given as

$$u(y,t) = \exp(\theta t) F(y), \tag{4.24}$$

where $F(y)$ is as yet an undetermined function of y.

By the substitution of Eq. (4.24) into Eq. (4.10) yields a second-order ordinary differential equation in $F(y)$, given by

$$\left(\frac{dF}{dy}\right)^2 \left[\frac{d^2F}{dy^2} - \frac{\beta_*}{\beta} F\right] = 0, \tag{4.25}$$

and the relation for θ

$$-\theta F + \mu_* \frac{d^2F}{dy^2} + \alpha_* \theta \frac{d^2F}{dy^2} - \frac{1}{K_*} F = 0. \tag{4.26}$$

The reduced ODE (4.25) is solved subject to the boundary conditions

$$F(0) = 1, \quad F(y) \to 0 \quad \text{as} \quad y \to \infty. \tag{4.27}$$

We solve Eq. (4.25) subject to the boundary conditions (4.27) for two cases.

Case 1: $\left(\frac{dF}{dy}\right)^2 = 0$ and $\frac{d^2F}{dy^2} - \frac{\beta_*}{\beta} F \neq 0$

In this case we have

$$\frac{dF}{dy} = 0. \tag{4.28}$$

Thus the solution of (4.25) subject to the boundary conditions (4.27) is

$$F(y) = C, \text{ where } C \text{ is constant.} \tag{4.29}$$

Using Eq. (4.29) in Eq. (4.26), we deduce $\theta = -\frac{1}{K_*}$

In this case, the solution for $u(y,t)$ is

$$u(t) = \exp\left(-\frac{1}{K_*}t\right). \tag{4.30}$$

The solution for u is independent of y. The velocity decreases exponentially with time at a rate which depends only on K and α. Expressed in dimensional variables, (4.31) is

$$u(t) = u_0 \exp\left[-\frac{t}{\left(\frac{\kappa}{\phi\nu} + \frac{\alpha_1}{\mu}\right)}\right]. \tag{4.31}$$

The fluid velocity decreases as the porosity increases and increases as the permeability of the porous medium increases.

Case 2: $\frac{d^2F}{dy^2} - \frac{\beta_*}{\beta}F = 0$ and $\left(\frac{dF}{dy}\right)^2 \neq 0$

In this case the solution for $F(y)$ is written as

$$F(y) = a\exp\left(\sqrt{\frac{\beta_*}{\beta}}y\right) + b\exp\left(-\sqrt{\frac{\beta_*}{\beta}}y\right). \tag{4.32}$$

By applying the boundary conditions (4.27), we have $a = 0$ (for a bounded solution) and $b = 1$ (by using first the boundary condition). Thus the solution for $F(y)$ is

$$F(y) = \exp\left(-\sqrt{\frac{\beta_*}{\beta}}y\right). \tag{4.33}$$

Substituting $F(y)$ into Eq. (4.26), we obtain the value for θ in the form

$$\theta = -\left(\frac{\beta - \mu_* \beta_* K_*}{K_* \beta - \alpha_* K_* \beta_*}\right). \tag{4.34}$$

Finally, the solution for $u(y,t)$ is

$$u(y,t) = \exp\left[-\left\{\left(\frac{\beta - \mu_* \beta_* K_*}{K_* \beta - \alpha_* K_* \beta_*}\right) t + \sqrt{\frac{\beta_*}{\beta}} y\right\}\right]. \tag{4.35}$$

Note that the above solution satisfies the boundary and initial conditions with

$$V(t) = \exp\left[-\left(\frac{\beta - \mu_* \beta_* K_*}{K_* \beta - \alpha_* K_* \beta_*}\right) t\right], \tag{4.36}$$

$$g(y) = \exp\left[-\sqrt{\frac{\beta_*}{\beta}} y\right], \tag{4.37}$$

where $V(t)$ and $g(y)$ depend on the physical parameters of the flow model. The solution (4.35) is the new exact closed-form solution of the physical model considered in [49]. The physical behavior of the various flow parameters involved in the solution expression (4.35) is described in Figures 4.1 $-$ 4.4.

In order to study the effect of time on the velocity distribution, we have plotted solution (4.35) in Figure 4.1. We see that with the increase in t, the velocity approaches a steady- state. This figure depicts that velocity decreases as time increases. Clearly, the unsteady behavior of velocity field is observed for $0 \leqslant t \leqslant 2.5$. For $t > 2.5$, the velocity profile remains the same. In other words, we can say that the steady-state behavior of the velocity is achieved for $t > 2.5$. The influence of the porosity of the porous medium K_* on the flow problem 1 is illustrated in Figure 4.2. As anticipated, with an increase of the porosity of the porous medium causes an decrease in drag force and hence the flow velocity increases. In Figures 4.3 and 4.4, we see the physical behavior of the third grade fluid parameters β and β_* on the conditional symmetry solution (4.35). These figures reveal that β and β_* have opposite roles on the velocity field. From these figures, it is noticed that velocity increases for large values of β showing the shear thinning behavior of the third grade fluid whereas it decreases for increasing β_* which shows the shear thickening behavior of the fluid.

Now we have

$$\frac{\beta - \mu_*\beta_*K_*}{K_*\beta - \alpha_*K_*\beta_*} = \frac{2}{3K + 2\alpha} > 0, \quad \frac{\beta_*}{\beta} = \frac{1}{3K}, \tag{4.38}$$

Making use of Eq. (4.38) into Eq. (4.35), the solution for $u(y,t)$ can be expressed as

$$u(y,t) = \exp\left[-\left\{\frac{2t}{3K + 2\alpha} + \frac{y}{\sqrt{3K}}\right\}\right]. \tag{4.39}$$

Expressed in dimensional form interms of the original variable, (4.39) is

$$u(y,t) = \exp\left[-\left\{\frac{2t}{\frac{3\kappa}{\phi\nu} + \frac{2\alpha_1}{\rho\nu}} + \sqrt{\frac{\phi}{3\kappa}}y\right\}\right]. \tag{4.40}$$

We observe that $u(y,t)$ always decreases as t and y increases. The decrease is greater for large porosity and smaller for large permeability.

Now we consider the invariant solution under the non-classical symmetry operator

$$\chi = \frac{\partial}{\partial t} + \lambda(t)u\frac{\partial}{\partial u}. \tag{4.41}$$

By the solution of corresponding characteristics system of above symmetry generator, the invariant solution is written as

$$u(y,t) = \exp\left[\gamma(t)\right]G(y), \quad \text{where} \quad \gamma(t) = \int \lambda(t)dt, \tag{4.42}$$

where $G(y)$ is an unknown function to be found. Making use of Eq. (4.42) into PDE (4.10), we obtain a second-order linear ODE for $G(y)$:

$$\frac{d^2G}{dy^2} - \frac{1}{K_*\mu_*}G = 0, \quad \text{where} \quad \beta = K_*\mu_*\beta_*. \tag{4.43}$$

The transformed boundary conditions are:

$$\begin{aligned} G(0) &= 1, \\ G(y) &\to 0 \text{ as } y \to \infty. \end{aligned} \tag{4.44}$$

The solution of reduced ODE (4.43) subject to the boundary conditions (4.44) is

$$G(y) = \exp\left(-\sqrt{\frac{1}{K_*\mu_*}}y\right). \tag{4.45}$$

With the use of Eq. (4.45) into Eq. (4.42), the solution for $u(y,t)$ (for any arbitrary function $\lambda(t)$) is written as

$$u(y,t) = \exp\left[\gamma(t) - \sqrt{\frac{1}{K_*\mu_*}}y\right] \quad \text{where} \quad \gamma(t) = \int \lambda(t)dt, \tag{4.46}$$

with

$$V(t) = \exp\left[\gamma(t)\right] \quad \text{and} \quad g(y) = \exp\left(-\sqrt{\frac{1}{K_*\mu_*}}y\right). \tag{4.47}$$

Finally, we consider the invariant solution corresponding to operator

$$\chi = a_1\chi + a_2\chi_1 + a_3\chi_3, \text{ where } a_i \ (i = 1, 2, 3) \text{ are constants.} \tag{4.48}$$

where χ, χ_1 and χ_2 are given in Eqs. (4.16), (4.17) and (4.18). It is worth to notice that the invariant solutions without the linear combination of χ, χ_1 and χ_2 have already been reported in [49]. Here we are considering another new invariant solution which has not been found in [49].

The invariant solution corresponding to operator (4.48) is

$$u(y,t) = \left[a_1 + a_3 \exp\left(\frac{2}{K_*}t\right)\right]^{\frac{-1}{2}} h(y), \text{ with } \mu_* = \frac{\alpha_*}{K_*}, \tag{4.49}$$

where $h(y)$ is an undetermined function of y. With the use of Eq. (4.49) into Eq. (4.10) leads a second order ODE in $h(y)$, viz.

$$\mu_* a_1 \frac{d^2h}{dy^2} + \beta \left(\frac{dh}{dy}\right)^2 \frac{d^2h}{dy^2} - \beta_* h \left(\frac{dh}{dy}\right)^2 - \frac{a_1}{K_*} h = 0, \tag{4.50}$$

and the invariant boundary conditions are:

$$\begin{aligned} h(0) &= 1, \\ h(y) &\to 0 \text{ as } y \to \infty. \end{aligned} \tag{4.51}$$

If $a_1 = 0$, we get the same invariant solution found in [49]. However, if $a_1 \neq 0$, then take $a_1 = 1$. Then the ODE (4.50) becomes

$$\mu_* \frac{d^2h}{dy^2} + \beta \left(\frac{dh}{dy}\right)^2 \frac{d^2h}{dy^2} - \beta_* h \left(\frac{dh}{dy}\right)^2 - \frac{h}{K_*} = 0, \tag{4.52}$$

The above equation admits the exact solution of the form (which we require to be zero at infinity due to the second boundary condition)

$$h(y) = A \exp\left(-\frac{\sqrt{\beta_*}}{\sqrt{\beta}} y\right), \tag{4.53}$$

provided that

$$\mu_* \left(\frac{\beta_*}{\beta}\right) - \frac{1}{K_*} = 0, \tag{4.54}$$

Finally, the solution for $u(y,t)$, given in Eq. (4.49), becomes

$$u(y,t) = \left[a_3 \exp\left(\frac{2}{K_*} t\right)\right]^{\frac{-1}{2}} \exp\left(-\frac{\sqrt{\beta_*}}{\sqrt{\beta}} y\right) \tag{4.55}$$

Remark 4 *Here we have obtained three new invariant solutions of the flow model* [49]*. The invariant solution corresponding to the non-classical symmetry generator given in Eq. (4.35) is better then the other two invariant solutions given in Eqs. (4.46) and (4.55)* *in the sense that it best represents the physics of the problem under investigation.*

4.3 Problem 2: Unsteady magnetohydrodynamic (MHD) flow of a third grade fluid in porous medium

In this problem we extend the previous flow model of third grade fluid by considering the fluid to be electrically conducting under the influence of a uniform magnetic field applied transversely to the flow.

The unsteady MHD flow of a third grade fluid in a porous half-space (in a non-dimensionalized form) is governed by [41]

$$\frac{\partial u}{\partial t} = \mu_* \frac{\partial^2 u}{\partial y^2} + \alpha_* \frac{\partial^3 u}{\partial y^2 \partial t} + \beta \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \beta_* u \left(\frac{\partial u}{\partial y}\right)^2 - \left(\frac{1}{K_*} + M_*^2\right) u, \tag{4.56}$$

where μ_*, α_*, β, β_*, and $1/K_*$ are defined in Eq. (4.11) and

$$M_*^2 = \frac{M^2}{(1 + \alpha/K)}. \tag{4.57}$$

Now we have to solve Eq. (4.56) subject to the boundary conditions (4.7) – (4.9).

4.3.1 Non-classical symmetry analysis

Here we compute all possible non-classical symmetry generators of PDE (4.56) and develop some classically invariant solution of problem 2. We find the determining equations for two different cases.

Case 1: When $\xi^1 = 1$ and $\xi^2 \neq 0$

In this case, we obtain the following set of determining equations (which have been generated manually)

$$
\begin{aligned}
\xi^2_u &= 0,\\
2\beta\xi^2_{yu} - \beta\eta_{\,\mathrm{uu}} &= 0,\\
\eta_y &= 0,\\
\xi^2_{yy} &= 0,\\
\xi^2_y &= 0,\\
\beta_*\eta\xi^2 - \beta_* u\xi^2_t + \beta_* u\eta_u\xi^2 &= 0,\\
\mu_*\xi^2_t - \alpha_*\eta_{tu}\xi^2 + \alpha_*\xi^2_t\eta_u &= 0,\\
\beta\xi^2_t + 2\beta\xi^2_y\xi^2 - 2\beta\eta_u &= 0,\\
\frac{\eta}{K_*} + M_*^2\eta - \frac{u\xi^2_t}{K_*\xi^2} - \frac{M_*^2 u\xi^2_t}{\xi^2} - \frac{\eta\xi^2_t}{\xi^2} + \eta_t - \frac{u\eta_u}{K_*} - M_*^2 u\eta_u &= 0.
\end{aligned}
\tag{4.58}
$$

We solve the determining equations (4.58) for two cases:

Case I: $\left[\mu_* - \left(\frac{1}{K_*} + M_*^2\right)\alpha_*\right] \neq 0$

For this case, we get

$$
\chi = \frac{\partial}{\partial t} + m\frac{\partial}{\partial y}, \tag{4.59}
$$

where m is any arbitrary constant. The above symmetry is a combination of translations in time and space variables.

Case II: $\left[\mu_* - \left(\frac{1}{K_*} + M_*^2\right)\alpha_*\right] = 0$

In this case, we obtain a two-dimensional symmetry algebra generated by operators

$$
\chi_1 = \left[\exp\left(-\left(\frac{1}{K_*} + M_*^2\right)2t\right) - \left(K_* + M_*^2\right)\right]\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \tag{4.60}
$$

$$
\chi_2 = -\left(K_* + M_*^2\right)\frac{\partial}{\partial t} + \exp\left(-\left(\frac{1}{K_*} + M_*^2\right)2t\right)\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}. \tag{4.61}
$$

Remark 5 *The symmetries given in Eqs. (4.60) and (4.61) are the linear combination of classical point symmetries of PDE (4.56) also found in* [41]*. Therefore, the symmetries (4.60) and (4.61) are also classical point symmetries. Thus there is no non-classical* *symmetry obtained for the case when* $\xi^1 = 1$ *and* $\xi^2 \neq 0$.

Now we consider the second case:

Case 2: When $\xi^1 = 1$ and $\xi^2 = 0$

For this case, we obtain the following set of determining equations

$$\eta_{uu} = 0, \tag{4.62}$$

$$\eta_y = 0, \tag{4.63}$$

$$\eta - u\eta_u = 0, \tag{4.64}$$

$$\begin{aligned} 0 = {} & \mu_*\eta + \mu_* K_* M_*^2 \eta + \mu_* K_* \eta_t - \alpha_* u \eta_{tu} - \alpha_* K_* M_*^2 u \eta_{tu} \\ & -\alpha_* K_* \eta \eta_{tu} - \mu_* u \eta_u - \mu_* K_* M_*^2 u \eta_u + \alpha_* \eta \eta_u \\ & +\alpha_* K_* M_*^2 \eta \eta_u + \alpha_* K_* \eta_t \eta_u - \alpha_* u \eta_u^2 - \alpha_* K_* M_*^2 u \eta_u^2. \end{aligned} \tag{4.65}$$

Again we solve the determining equations (4.62)-(4.65) for two-cases:

Case I: $\left[\mu_* - \left(\frac{1}{K_*} + M_*^2\right)\alpha_*\right] \neq 0$

For this case, we deduce

$$\chi = \frac{\partial}{\partial t} + \lambda u \frac{\partial}{\partial u}, \tag{4.66}$$

where λ is a constant to be determined.

Case II: $\left[\mu_* - \left(\frac{1}{K_*} + M_*^2\right)\alpha_*\right] = 0$

For this case, we obtain

$$\chi = \frac{\partial}{\partial t} + \delta(t) u \frac{\partial}{\partial u}, \tag{4.67}$$

where $\delta(t)$ is any function of time t. Thus infinite many nonclassical symmetry generators obtained in this case for any arbitrary value of function $\delta(t)$.

4.3.2 Conditionally invariant solutions

We first consider the conditionally invariant solution corresponding to the non-classical symmetry generator

$$\chi = \frac{\partial}{\partial t} + \lambda u \frac{\partial}{\partial u}. \tag{4.68}$$

where λ is a parameter to be determine. The invariant solution corresponding to (4.68) is

$$u(y,t) = \exp(\lambda t) F_1(y), \tag{4.69}$$

where $F_1(y)$ is an undetermined function of y. Substituting Eq. (4.69) in Eq. (4.56) leads to a second-order ordinary differential equation in $F_1(y)$, given by

$$\left(\frac{dF_1}{dy}\right)^2 \left[\frac{d^2F_1}{dy^2} - \frac{\beta_*}{\beta} F_1\right] = 0, \tag{4.70}$$

and the relation for λ

$$-\lambda F_1 + (\mu_* + \alpha_* \lambda) \frac{d^2F_1}{dy^2} - \left(\frac{1}{K_*} + M_*^2\right) F_1 = 0 \tag{4.71}$$

The reduced ODE (4.70) is solved subject to the boundary conditions

$$\begin{aligned} F_1(0) &= 1, \\ F_1(y) &\to 0 \text{ as } y \to \infty, \end{aligned} \tag{4.72}$$

for two cases.

Case 1: $\left(\frac{dF_1}{dy}\right)^2 = 0$ and $\frac{d^2F_1}{dy^2} - \frac{\beta_*}{\beta} F_1 \neq 0$

In this case we obtain a time dependent solution

$$u(t) = C \exp(\lambda t), \tag{4.73}$$

where

$$\lambda = -\left(\frac{1}{K_*} + M_*^2\right). \tag{4.74}$$

The solution can be expressed as

$$u(y,t) = \exp\left[-\left(\frac{1+KM^2}{K+\alpha}\right)t\right], \tag{4.75}$$

and in dimensional form

$$u(y,t) = u_0 \exp\left[-\frac{\left(1+\frac{\kappa u_0^2 M^2}{\phi\nu}\right)}{\left(\frac{\kappa}{\phi\nu}+\frac{\alpha_1}{\rho\nu}\right)}t\right], \tag{4.76}$$

The effects of the magnetic field transverse to the fluid flow is to cause the fluid velocity to decay more rapidly with time. If we set $M_*^2 = 0$, we recover the time-dependent solution (4.31) of the previous problem.

Case 2: $\frac{d^2F_1}{dy^2} - \frac{\beta_*}{\beta}F_1 = 0$ and $\left(\frac{dF_1}{dy}\right)^2 \neq 0$

In this case the bounded solution for $F_1(y)$ is written as

$$F_1(y) = \exp\left(-\sqrt{\frac{\beta_*}{\beta}}y\right). \tag{4.77}$$

Substituting $F_1(y)$ into Eq. (4.71), we deduce the value for λ in the form

$$\lambda = -\left(\frac{\beta + K_*\beta M_*^2 - \mu_*\beta_*K_*}{K_*\beta - \alpha_*K_*\beta_*}\right). \tag{4.78}$$

Using the values of $F_1(y)$ and λ in Eq. (4.69), the solution for $u(y,t)$ is written as

$$u(y,t) = \exp\left[-\left\{\left(\frac{\beta + K_*\beta M_*^2 - \mu_*\beta_*K_*}{K_*\beta - \alpha_*K_*\beta_*}\right)t + \left(\sqrt{\frac{\beta_*}{\beta}}\right)y\right\}\right]. \tag{4.79}$$

The solution (4.79) satisfies the boundary and initial conditions with

$$V(t) = \exp\left[-\left(\frac{\beta + K_*\beta M_*^2 - \mu_*\beta_* K_*}{K_*\beta - \alpha_* K_*\beta_*}\right) t\right], \tag{4.80}$$

$$g(y) = \exp\left[-\left(\sqrt{\frac{\beta_*}{\beta}}\right) y\right], \tag{4.81}$$

where $V(t)$ and $g(y)$ depend on the physical parameters of the flow model. The solution given in Eq. (4.79) is again the new solution of the flow problem considered in [41]. Note that, if there is no magnetic field ($M_* = 0$), we recover the solution (4.35) in the Problem 1. To study the behavior of the magnetic field of the flow problem 2, the conditional symmetry solution (4.79) is plotted for the varying values of magnetic field parameter M_* in Figure 4.5. From the plot, it is clearly observed that fluid velocity decreases with an increase in the magnetic field strength. Thus, the graphical behavior of the magnetic parameter is opposite to the porosity parameter, that is what we expected physically.

Now

$$\frac{\beta + K_*\beta M_*^2 - \mu_*\beta_* K_*}{K_*\beta - \alpha_* K_*\beta_*} = \frac{2 + 3KM^2}{3K + 2\alpha} > 0, \tag{4.82}$$

and the solution (4.79) can be expressed as

$$u(y,t) = \exp\left[-\left\{\left(\frac{2 + 3KM^2}{3K + 2\alpha}\right) t + \frac{y}{\sqrt{3K}}\right\}\right]. \tag{4.83}$$

Expressed in dimensional form (4.83) becomes

$$u(y,t) = u_0 \exp\left[-\left\{\left(\frac{2 + \frac{3\kappa u_0^2 M^2}{\phi\nu^2}}{\frac{3\kappa}{\phi\nu} + \frac{2\alpha_1}{\rho\nu}}\right) t + \left(\sqrt{\frac{\phi}{3\kappa}}\right) y\right\}\right]. \tag{4.84}$$

In the dimensional form of the solution, the magnetic field does not affect the dependence of $u(y,t)$ on y. It increase the rate-of-decrease of $u(y,t)$ with time.

Now we consider the conditionally invariant solution under the non-classical symmetry operator

$$\chi = \frac{\partial}{\partial t} + \delta(t)u\frac{\partial}{\partial u}. \tag{4.85}$$

Solving of corresponding Langrangian system of the above symmetry generator, the invariant solution is written as

$$u(y,t) = \exp\left[\zeta(t)\right] G_1(y), \quad \text{where} \;\; \zeta(t) = \int \delta(t)dt, \tag{4.86}$$

where $G_1(y)$ is an unknown function to be found. With the insertion of Eq. (4.86) into PDE (4.56), we obtain a second-order linear ODE for $G_1(y)$:

$$\frac{d^2G_1}{dy^2} - \left(\frac{1}{K_*\mu_*} + \frac{M_*^2}{\mu_*}\right) G_1 = 0, \quad \text{where} \;\; \frac{\beta}{\beta_*} = \left(\frac{1}{K_*} + M_*^2\right)\mu_*. \tag{4.87}$$

The transformed boundary conditions are:

$$\begin{aligned} G_1(0) &= 1, \\ G_1(y) &\to 0 \text{ as } y \to \infty. \end{aligned} \tag{4.88}$$

The solution of reduced ODE (4.87) is solved subject to the boundary conditions (4.88) is

$$G_1(y) = \exp\left(-\sqrt{\left(\frac{1}{K_*\mu_*} + \frac{M_*^2}{\mu_*}\right)} y\right). \tag{4.89}$$

Making use of Eq. (4.89) into Eq. (4.86), the solution for $u(y,t)$ (for any arbitrary function $\delta(t)$) is written as

$$u(y,t) = \exp\left[\zeta(t) - \sqrt{\left(\frac{1}{K_*\mu_*} + \frac{M_*^2}{\mu_*}\right)} y\right] \quad \text{where} \;\; \zeta(t) = \int \delta(t)dt, \tag{4.90}$$

The solution (4.90) satisfies the boundary and initial conditions with

$$
\begin{aligned}
V(t) &= \exp\left[\zeta(t)\right], && (4.91)\\
g(y) &= \exp\left(-\sqrt{\left(\frac{1}{K_*\mu_*}+\frac{M_*^2}{\mu_*}\right)}y\right). && (4.92)
\end{aligned}
$$

Note that for $M_* = 0$, we recover the solution (4.46) of the previous problem.

Finally, we consider the invariant solution corresponding to operator

$$
\chi = b_1\chi + b_2\chi_1 + b_3\chi_3, \text{ where } b_j \ (j = 1, 2, 3) \text{ are constants.} \tag{4.93}
$$

where χ, χ_1 and χ_2 are given in Eqs. (4.59), (4.60) and (4.61). The invariant solutions without the linear combination of χ, χ_1 and χ_2 have already been investigated in [41]. Here we present another new invariant solution which has not been reported in [41].

The invariant solution corresponding to operator (4.93) is

$$
u(y,t) = \left[b_1 + b_3 \exp\left(\frac{1}{K_*} + M_*^2\right)2t\right]^{\frac{-1}{2}} h_1(y), \text{ with } \frac{\mu_*}{\alpha_*} = \left(\frac{1}{K_*} + M_*^2\right), \tag{4.94}
$$

where $h_1(y)$ is an undetermined function of y. Using Eq. (4.94) into Eq. (4.56) leads a second order ODE in $h_1(y)$:

$$
\mu_* b_1 \frac{d^2 h_1}{dy^2} + \beta\left(\frac{dh_1}{dy}\right)^2 \frac{d^2 h_1}{dy^2} - \beta_* h_1 \left(\frac{dh_1}{dy}\right)^2 - \left(\frac{1}{K_*} + M_*^2\right) b_1 h_1 = 0, \tag{4.95}
$$

with

$$
\begin{aligned}
h_1(0) &= 1, && (4.96)\\
h_1(y) &\to 0 \text{ as } y \to \infty.
\end{aligned}
$$

If $b_1 = 0$, we get the same invariant solution found in [41]. However, if $b_1 \neq 0$, then take

$b_1 = 1$. Then the ODE (4.95) becomes

$$\mu_* \frac{d^2 h_1}{dy^2} + \beta \left(\frac{dh_1}{dy} \right)^2 \frac{d^2 h_1}{dy^2} - \beta_* h_1 \left(\frac{dh_1}{dy} \right)^2 - \left(\frac{1}{K_*} + M_*^2 \right) h_1 = 0, \tag{4.97}$$

The above equation admits the exact solution of the form (which we require to be zero at infinity due to the second boundary condition)

$$h_1(y) = A \exp \left(-\frac{\sqrt{\beta_*}}{\sqrt{\beta}} y \right), \tag{4.98}$$

provided that

$$\mu_* \left(\frac{\beta_*}{\beta} \right) - \left(\frac{1}{K_*} + M_*^2 \right) = 0, \tag{4.99}$$

Finally, the solution for $u(y, t)$ is written as

$$u(y, t) = \left[b_3 \exp \left(\frac{1}{K_*} + M_*^2 \right) 2t \right]^{\frac{-1}{2}} \exp \left(-\frac{\sqrt{\beta_*}}{\sqrt{\beta}} y \right) \tag{4.100}$$

Remark 6 *We have constructed three new invariant solutions of flow model* [41]*. The invariant solution corresponding to non-classical symmetry generator given in Eq. (4.79) is better then the other two invariant solutions given in Eqs. (90) and (100) due to the fact that it best represents the physics of the problem considered.*

4.4 Problem 3: Unsteady MHD flow of third grade fluid in a porous medium with plate suction/injection

This model is an extension of the previous two problems with combined effects of plate suction/injection and the MHD nature of the fluid.

The time-dependent magnetohydrodynamic flow of a third grade in a porous half-space with plate suction/injection is governed by [50]

$$\begin{aligned}\frac{\partial u}{\partial t} &= \mu_*\frac{\partial^2 u}{\partial y^2}+\alpha_*\frac{\partial^3 u}{\partial y^2\partial t}-\alpha_* W\frac{\partial^3 u}{\partial y^3}+\beta\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}-\beta_* u\left(\frac{\partial u}{\partial y}\right)^2\\ &\quad +W\frac{\partial u}{\partial y}-\left(\frac{1}{K_*}+M_*^2\right)u, \end{aligned} \tag{4.101}$$

where $W > 0$ denotes the suction velocity and $W < 0$ indicates blowing velocity with μ_*, α_*, β, β_*, $1/K_*$ and M_*^2 are defined in Eqs. (4.11) and (4.30). The velocity W is dimensionless and given by $W = \frac{W_0}{u_0}$. The boundary conditions remain the same as given in Eqs. (4.7) – (4.9).

By performing the non-classical symmetry analysis of PDE (4.101), the only conditional symmetry generator calculated for the PDE (4.101) is given by

$$\frac{\partial u}{\partial t} = \epsilon u, \tag{4.102}$$

where ϵ is a constant to be found. The corresponding operator is

$$\chi = \frac{\partial}{\partial t}+\epsilon u\frac{\partial}{\partial u}. \tag{4.103}$$

Again the invariant solution corresponding to (4.103) is

$$u(y,t) = \exp(\epsilon t)F_2(y), \tag{4.104}$$

where $F_2(y)$ is an undetermined function of y. Substituting Eq. (4.104) in Eq. (4.101) leads to a second-order linear ODE in $F_2(y)$, given by

$$\left(\frac{dF_2}{dy}\right)^2\left[\frac{d^2F_2}{dy^2}-\frac{\beta_*}{\beta}F_2\right] = 0, \tag{4.105}$$

and the relation for ϵ

$$-\epsilon F_2 + (\mu_* + \alpha_* \epsilon) \frac{d^2 F_2}{dy^2} - \alpha_* W \frac{d^3 F_2}{dy^3} + W \frac{dF_2}{dy} - \left(\frac{1}{K_*} + M_*^2 \right) F_2 = 0. \tag{4.106}$$

Again the reduced ODE (4.105) is solved subject to the boundary conditions

$$\begin{aligned} F_2(0) &= 1, \\ F_2(y) &\rightarrow 0 \text{ as } y \rightarrow \infty, \end{aligned} \tag{4.107}$$

for two cases. For the first case we obtain the same time-dependent solution given in Eq. (4.74) with λ replaced by ϵ where

$$\epsilon = -\left(\frac{1}{K_*} + M_*^2 \right). \tag{4.108}$$

For the second case,

$$\frac{d^2 F}{dy^2} - \frac{\beta_*}{\beta} F = 0 \text{ and } \left(\frac{dF}{dy} \right)^2 \neq 0. \tag{4.109}$$

The bounded solution for $F_2(y)$ is deduced as

$$F_2(y) = \exp\left(-\sqrt{\frac{\beta_*}{\beta}} y \right). \tag{4.110}$$

Inserting $F_2(y)$ into Eq. (4.106), we find the value for ϵ in the form

$$\epsilon = -\left(\frac{1}{\beta - \alpha_* \beta_*} \right) \left[\frac{\beta}{K_*} + \beta M_*^2 - \mu_* \beta_* + W \sqrt{\frac{\beta_*}{\beta}} (\beta - \alpha_* \beta_*) \right], \tag{4.111}$$

where $\beta \neq \alpha_* \beta_*$. Thus the final solution for $u(y,t)$ is written as

$$u(y,t) = \exp\left[-\left\{ \left(\frac{1}{\beta - \alpha_* \beta_*} \right) \left(\frac{\beta}{K_*} + \beta M_*^2 - \mu_* \beta_* + W \sqrt{\frac{\beta_*}{\beta}} (\beta - \alpha_* \beta_*) \right) t + \sqrt{\frac{\beta_*}{\beta}} y \right\} \right], \tag{4.112}$$

provided that $\beta \neq \alpha_* \beta_*$. Note that the above solution satisfies the boundary and initial conditions for the particular values of the arbitrary functions $V(t)$ and $g(y)$, viz.

$$V(t) = \exp\left[-\left(\frac{1}{\beta - \alpha_*\beta_*}\right)\left(\frac{\beta}{K_*} + \beta M_*^2 - \mu_*\beta_* + W\sqrt{\frac{\beta_*}{\beta}}\left(\beta - \alpha_*\beta_*\right)\right)t\right] \quad (4.113)$$

$$g(y) = \exp(-\sqrt{\frac{\beta_*}{\beta}}y). \quad (4.114)$$

The conditional symmetry solution (4.112) is again the new exact solution of this problem. Note that if we set $M_* = W = 0$ in solution (4.112), we can easily deduce the conditional symmetry solutions of the previous two models. The graphical behavior of solution (4.112) is shown. For $W > 0$ suction occurs and $W < 0$ corresponds to injection or blowing. It is observed that with the increase in the suction parameter the velocity decreases so does the boundary layer thickness. This is quite in accordance with the fact that suction causes a reduction in the boundary layer thickness, whereas the effect of increasing injection are quite opposite to that of suction. So from these figures it is evident that our conditional symmetry solution (4.112) is valid for both suction and blowing.

Now we have

$$\left(\frac{1}{\beta - \alpha_*\beta_*}\right)\left(\frac{\beta}{K_*} + \beta M_*^2 - \mu_*\beta_* + W\sqrt{\frac{\beta_*}{\beta}}\left(\beta - \alpha_*\beta_*\right)\right) = \frac{2+3KM^2}{3K+2\alpha} + \frac{W}{\sqrt{3K}}. \quad (4.115)$$

Making use of Eq. (4.115) into Eq. (4.112), the solution $u(y,t)$ can be written as

$$u(y,t) = \exp\left[-\left\{\left(\frac{2+3KM^2}{3K+2\alpha} + \frac{W}{\sqrt{3K}}\right)t + \frac{y}{\sqrt{3K}}\right\}\right]. \quad (4.116)$$

The suction/injection does not affect the dependence on y of $u(y,t)$ but it does significantly affect its dependence on time t. For suction at the plate, $W > 0$, suction increases the rate of decrease of $u(y,t)$ with time. For injection or blowing at the plate $W < 0$ and we have $W = -|W|$. There are three cases to consider. Define the injection velocity w_I

by

$$w_I = \sqrt{3K}\left(\frac{2+3KM^2}{3K+2\alpha}\right). \tag{4.117}$$

For weak injection for which $|W| < w_I$, the velocity $u(y,t)$ still decreases with time but at a reduced rate. When $|W| = w_I$, the velocity $u = u(y)$ which is independent of time. For strong fluid injection at the base, $|W| > w_I$ and the fluid velocity $u(y,t)$ increases with time. The magnetic field increases w_I and the injection velocity has to be greater to make $u(y,t)$ to increase with time.

Expressed in dimensional form the solution (4.116) is written as

$$u(y,t) = u_0 \exp\left[-\left\{\left(\frac{2+\frac{3\kappa u_0^2 M^2}{\phi\nu^2}}{\frac{3\kappa}{\phi\nu}+\frac{2\alpha_1}{\rho\nu}} + \frac{W_0}{\sqrt{\frac{3\kappa}{\phi}}}\right)t + \left(\sqrt{\frac{\phi}{3\kappa}}\right)y\right\}\right], \tag{4.118}$$

and the critical injection velocity is

$$w_I = -\left(\frac{3\kappa}{\phi}\right)^{\frac{1}{2}}\left(\frac{2+\frac{3\kappa u_0^2 M^2}{\phi\nu^2}}{\frac{3\kappa}{\phi\nu}+\frac{2\alpha_1}{\rho\nu}}\right). \tag{4.119}$$

From (4.118) the velocity profile at $t = 0$ is given by

$$u(y,0) = u_0 \exp\left[-\left(\sqrt{\frac{\phi}{3\kappa}}\right)y\right]. \tag{4.120}$$

It depends on the ratio $\frac{\phi}{\kappa}$ of the porous material and is independent of the fluid parameters ν, α and β.

4.5 Graphical analysis of solutions

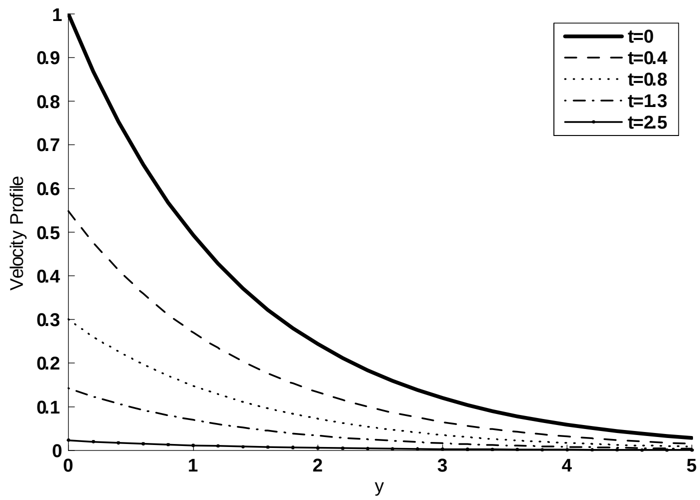

Figure 4.1: Profile of solution (4.35) for the varying values of time t, when $\beta = 1$, $\beta_* = 0.5$, $\alpha_* = 1$, $\mu_* = 1$, $K_* = 0.8$ are fixed.

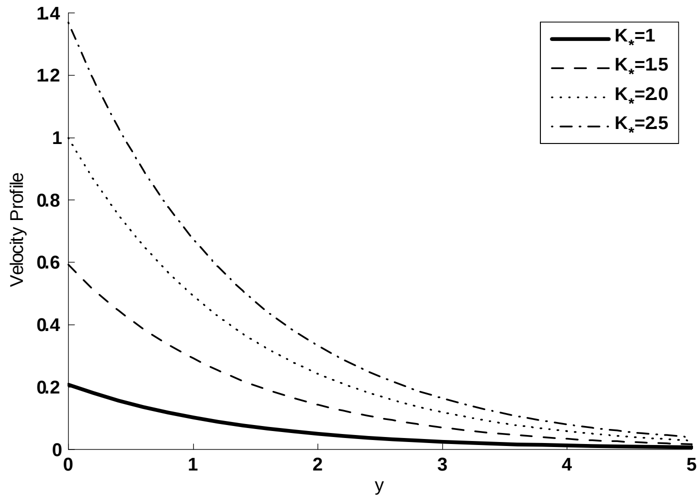

Figure 4.2: Profile of solution (4.35) for the varying values of porosity parameter K_*, when $\beta = 1$, $\beta_* = 0.5$, $\alpha_* = 1$, $\mu_* = 1$, $t = \pi/2$ are fixed.

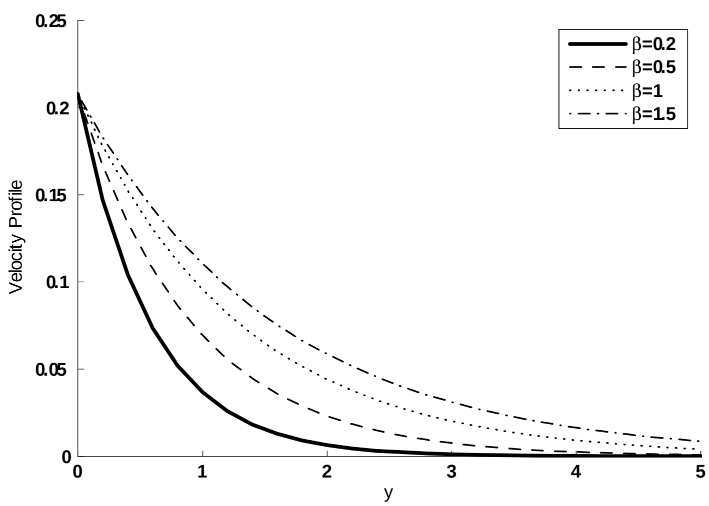

Figure 4.3: Profile of solution (4.35) for the varying values of third grade parameter β, when $K_* = 1$, $\beta_* = 0.6$, $\alpha_* = 1$, $\mu_* = 1$, $t = \pi/2$ are fixed.

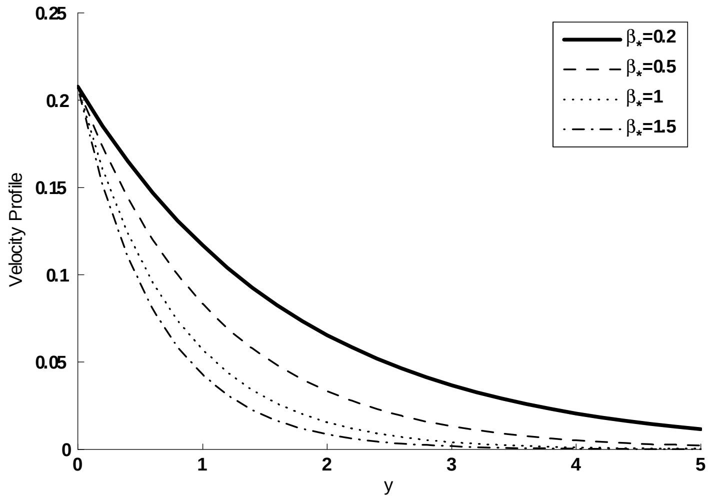

Figure 4.4: Profile of solution (4.35) for the varying values of third grade parameter β_*, when $K_* = 1$, $\beta = 0.6$, $\alpha_* = 1$, $\mu_* = 1$, $t = \pi/2$ are fixed.

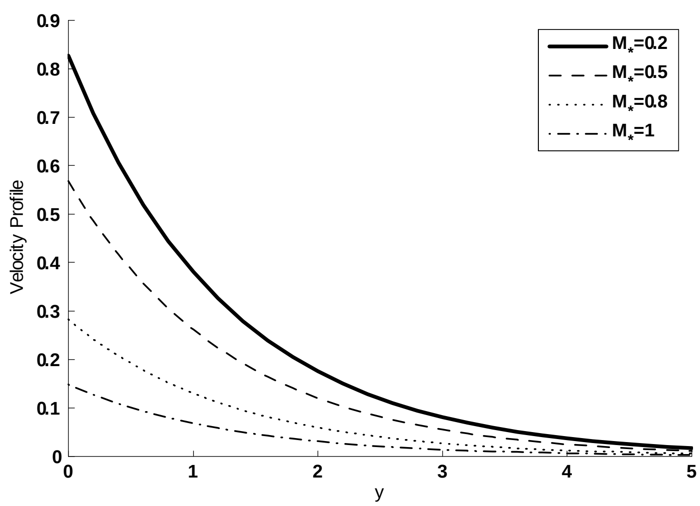

Figure 4.5: Profile of solution (4.79) for the varying values of magnetic field parameter M_*, when $K_* = 1.5$, $\beta = 2.5$, $\beta_* = 1.5$, $\alpha_* = 0.2$, $\mu_* = 1$, $t = \pi/2$ are fixed.

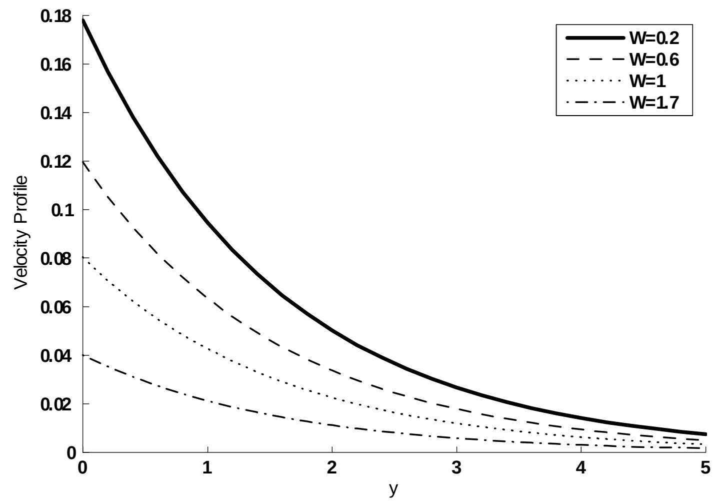

Figure 4.6: Profile of solution (4.112) for the varying values of suction parameter $W > 0$, when $K_* = 3$, $\beta = 2.5$, $\beta_* = 1$, $\alpha_* = 0.1$, $\mu_* = 1$, $M_* = 1$, $t = \pi/2$ are fixed.

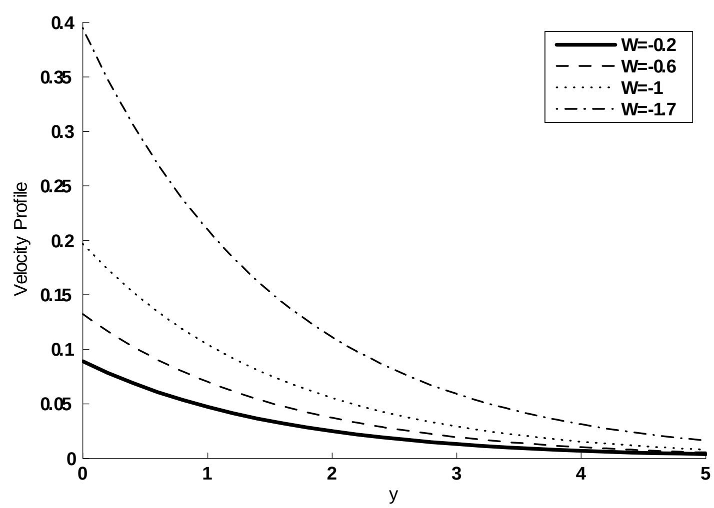

Figure 4.7: Profile of solution (4.112) for the varying values of injection parameter $W < 0$, when $K_* = 1$, $\beta_* = 1$, $\alpha_* = 0.1$, $\mu_* = 1$, $M_* = 1$, $t = \pi/2$ are fixed.

4.6 Conclusions

In this chapter, we have provided a systematic mechanism to generate exact solutions for some nonlinear problems [49, 41, 50] which describe the time-dependent behavior of a third grade non-Newtonian fluid model in porous medium. In each case, a complete non-classical symmetry analysis has been performed for the governing PDEs to construct new classes of conditionally invariant solutions. This is the first time in the literature that this type of approach has been utilized to construct closed-form solutions for non-Newtonian fluid models. The flow models considered in this study are prototype ones but the method that we have employed can be used to investigate a wide range of nonlinear problems in non-Newtonian fluid mechanics. This work can be extended in a number of ways. The non-classical method can be used to obtain exact solutions of nonlinear problems dealing with the flow models of other non-Newtonian fluids and also the conditional symmetry approach can be used to reduce certain nonlinear partial differential equations arising in the study of non-Newtonian fluid models to ordinary differential equations and then numerical investigation of the reduced ODEs can be made for specific nonlinear problems. Thus, the novelty of the presented work comes from the sense that it gave us a new trend to solve nonlinear boundary value problems that arise not only in the field of non-Newtonian fluid mechanics but also in the other fields of science and engineering.

Finally, from the obtained results we can make the following important conclusions:

• The spatial dependence of u is determined only by the ratio of the permeability of the medium to the porosity of the medium. It is independent of the properties of the fluid and it does not depend on the magnetic field or the suction/injection.

• The dependence of u on time is determined by the permeability and porosity of the medium and also by the properties of the fluid and the magnetic field and suction/injection velocity at the base. It depend on the fluid through the parameter β of a second grade fluid.

• The initial velocity w_I for the injection of fluid at the base was found. It is defined by (4.117) and depends on the permeability and porosity of the medium as well as the properties of the fluid. The magnitude of w_I increases as the magnitude of the magnetic field increases. If the injection velocity at the base equals w_I then the solution for u does not depend on time. If the magnitude of the injection velocity exceeds $|w_I|$ then u will increase with time at any position of y.

Chapter 5

Closed-Form Solutions for a Nonlinear Partial Differential Equation Arising in the Study of a Fourth Grade Fluid Model

5.1 Introduction

The unsteady unidirectional flow of an incompressible fourth grade fluid bounded by a suddenly moved rigid plate is studied in this chapter. The governing nonlinear higher order partial differential equation for this flow in a semi-infinite domain is modelled. Translational symmetries in variables t and y are employed to construct two different classes of closed-form travelling wave solutions of the model equation. A conditional symmetry solution of the model equation is also obtained. The physical behavior and the properties of various interesting flow parameters on the structure of the velocity are presented and discussed. In particular, the significance of the rheological effects are mentioned.

The chapter is organized in the following fashion. In Sections 5.2 and 5.3, the problem formulation is carried out for the flow model under investigation. Travelling wave and the conditional symmetry solutions of the modelled equation are presented in sections 5.4 and 5.5 respectively. The physical discussion of the solutions is made in Section 5.6. Some concluding remarks are given in Section 5.7.

5.2 Fundamental equations

The basic equations governing the time-dependent flow of an incompressible fluid are the continuity equation and the momentum equation, viz.

$$\operatorname{div}\mathbf{V} = \mathbf{0}, \qquad \rho\frac{d\mathbf{V}}{dt} = \operatorname{div}\mathbf{T}, \tag{5.1}$$

where $\mathbf{V}$ is the velocity vector, ρ the density of the fluid, d/dt the material time derivative and $\mathbf{T}$ the Cauchy stress tensor. For a fourth grade fluid the Cauchy stress tensor satisfies the constitutive equations [43]

$$\mathbf{T} = -p\mathbf{I} + \sum_{j=1}^{n}\mathbf{S}_j \quad \text{with} \quad n = 4, \tag{5.2}$$

where p is the pressure, $\mathbf{I}$ the identity tensor and $\mathbf{S}_j$ the extra stress tensor

$$\mathbf{S}_1 = \mu\mathbf{A}_1, \tag{5.3}$$

$$\mathbf{S}_2 = \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{5.4}$$

$$\mathbf{S}_3 = \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_1, \tag{5.5}$$

$$\begin{aligned}\mathbf{S}_4 = {} & \gamma_1\mathbf{A}_4 + \gamma_2(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3) + \gamma_3\mathbf{A}_2^2 + \gamma_4(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2) \\ & \gamma_5(\mathrm{tr}\mathbf{A}_2)\mathbf{A}_2 + \gamma_6(\mathrm{tr}\mathbf{A}_2)\mathbf{A}_1^2 + \left[\gamma_7\mathrm{tr}\mathbf{A}_3 + \gamma_8\mathrm{tr}(\mathbf{A}_2\mathbf{A}_1)\right]\mathbf{A}_1.\end{aligned} \tag{5.6}$$

Here μ is the dynamic viscosity, α_i $(i = 1, 2)$, β_i $(i = 1, 2, 3)$ and γ_i $(i = 1, 2, ..., 8)$ are material constants. The Kinematical tensors $\mathbf{A}_1$ to $\mathbf{A}_4$ are the Rivlin-Ericksen tensors defined by

$$\mathbf{A}_1 = (\operatorname{grad}\mathbf{V}) + (\operatorname{grad}\mathbf{V})^T, \tag{5.7}$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(\operatorname{grad}\mathbf{V}) + (\operatorname{grad}\mathbf{V})^T\mathbf{A}_{n-1};\ (n>1) \tag{5.8}$$

in which grad is the gradient operator.

5.3 Flow development

Let an infinite rigid plate occupy the plane $y = 0$ and a fourth grade fluid the half-space $y > 0$. The x-axis and y-axis are chosen parallel and perpendicular to the plate. For $t > 0$, the plate moves in its own plane with arbitrary velocity $V(t)$. By taking the velocity field $(u(y,t), 0, 0)$, the conservation of mass equation is identically satisfied. The governing PDE in u is obtained by substituting Eqs. (5.2) – (5.8) into Eq. (5.1) and rearranging. We deduce the following governing equation in the absence of the modified pressure gradient

$$\begin{aligned}\rho\frac{\partial u}{\partial t} &= \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2\partial t} + \beta_1\frac{\partial^4 u}{\partial y^2\partial t^2} + 6(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + \gamma_1\frac{\partial^5 u}{\partial y^2\partial t^3}\\ &\quad +(6\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5+6\gamma_7+2\gamma_8)\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right].\end{aligned} \tag{5.9}$$

The above equation is subject to the following boundary and initial conditions

$$u(0,t) = U_0V(t), \qquad t>0, \tag{5.10}$$

$$u(\infty,t) = 0, \qquad t>0, \tag{5.11}$$

$$u(y,0) = I(y), \qquad y>0, \tag{5.12}$$

$$\frac{\partial u(y,0)}{\partial t} = J(y), \qquad y>0, \tag{5.13}$$

$$\frac{\partial^2 u(y,0)}{\partial t^2} = K(y), \qquad y>0, \tag{5.14}$$

where U_0 is the reference velocity and $V(t)$, $I(y)$, $J(y)$, $K(y)$ are the unspecified functions. The first boundary condition (5.10) is the no-slip condition and the second boundary condition (5.11) says that the main stream velocity is zero. This is not a restrictive assumption since we can always measure velocity relative to the main stream. The initial condition (5.12) indicate that initially the fluid is moving with some non-uniform velocity $I(y)$. The remaining two initial conditions are the extra two conditions imposed to make the problem well-posed.

We define the dimensionless parameters as

$$\begin{aligned}\bar{u} &= \frac{u}{U_0}, \quad \bar{y} = \frac{U_0 y}{\nu}, \quad \bar{t} = \frac{U_0^2 t}{\nu}, \quad \bar{\alpha} = \frac{\alpha_1 U_0^2}{\rho\nu^2}, \quad \bar{\beta}_1 = \frac{\beta_1 U_0^4}{\rho\nu^3}, \quad \bar{\beta} = \frac{6(\beta_2+\beta_3)U_0^4}{\rho\nu^3}, \\ \bar{\gamma}_1 &= \frac{\gamma_1 U_0^6}{\rho\nu^4}, \quad \bar{\gamma} = (3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)\frac{U_0^6}{\rho\nu^4}. \end{aligned} \tag{5.15}$$

Under these transformations the governing Eq. (5.9) and the corresponding initial and the boundary conditions (5.10) – (5.14) take the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2 \partial t} + \beta_1 \frac{\partial^4 u}{\partial y^2 \partial t^2} + \beta\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_1 \frac{\partial^5 u}{\partial y^2 \partial t^3} + 2\gamma\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y \partial t}\right], \tag{5.16}$$

$$u(0,t) = V(t), \qquad t > 0, \tag{5.17}$$

$$u(\infty,t) = 0, \qquad t > 0, \tag{5.18}$$

$$u(y,0) = f(y), \qquad y > 0, \tag{5.19}$$

$$\frac{\partial u(y,0)}{\partial t} = g(y), \qquad y > 0, \tag{5.20}$$

$$\frac{\partial^2 u(y,0)}{\partial t^2} = h(y), \qquad y > 0, \tag{5.21}$$

where $f(y) = I(y)/U_0$, $g(y) = J(y)/U_0$ and $h(y) = K(y)/U_0$. The functions $V(t)$, $f(y)$, $g(y)$ and $h(y)$ are as yet arbitrary. These functions are constrained in the next section when we seek closed-form solutions using the symmetry technique. For simplicity we have neglected the bars in all the non-dimensional quantities. We consider classical and conditional symmetries of Eq. (5.16). Eq. (5.16) only admits translational symmetries in variables t, y and u, thus the travelling wave solutions are investigated for the model equation (5.16).

5.4 Travelling wave solutions

Travelling wave solutions are special kinds of group invariant solutions which are invariant under a linear combination of time-translation and space-translation symmetry generators. It can easily be seen that Eq. (5.16) admits Lie point symmetry generators, $\partial/\partial t$ (time-translation) and $\partial/\partial y$ (space-translation in y), so that we can construct travelling wave solutions for the model equation.

5.4.1 Backward wave-front type travelling wave solutions

Let X_1 and X_2 be time-translation and space-translation symmetry generators respectively. Then the solution corresponding to the generator

$$X = X_1 - cX_2, \quad (c > 0), \tag{5.22}$$

would represent backward wave-front type travelling wave solutions. In this case, the waves are propagating towards the plate. The Langrangian system corresponding to Eq. (5.22) is

$$\frac{dy}{-c} = \frac{dt}{1} = \frac{du}{0}, \tag{5.23}$$

Solving Eq. (5.23), invariant solutions are given by

$$u(y,t) = F(\xi) \quad \text{with} \quad \xi = y + ct. \tag{5.24}$$

Making use of Eq. (5.24) in Eq. (5.16) results in a fifth-order ordinary differential for $F(\xi)$

$$c\frac{dF}{d\xi} = \frac{d^2F}{d\xi^2} + \alpha c\frac{d^3F}{d\xi^3} + \beta_1 c^2\frac{d^4F}{d\xi^4} + \beta\left(\frac{dF}{d\xi}\right)^2\frac{d^2F}{d\xi^2} + \gamma_1 c^3\frac{d^5F}{d\xi^5} + 2\gamma\frac{d}{d\xi}\left[c\left(\frac{dF}{d\xi}\right)^2\frac{d^2F}{d\xi^2}\right]. \tag{5.25}$$

Thus the PDE (5.16) becomes an ODE (5.25) along certain curves in the $y-t$ plane. These curves are called *characteristic curves* or just the characteristics. In order to solve Eq. (5.25) for $F(\xi)$, we assume a solution of the form

$$F(\xi) = A\exp(B\xi), \tag{5.26}$$

where A and B are the constants to be determined. Inserting Eq. (5.26) in Eq. (5.25) we obtain

$$(-cB + B^2 + c\alpha B^3 + \beta_1 c^2 B^4 + \gamma c^3 B^5) + e^{2B\xi}(\beta A^2 B^4 + 6c\gamma A^2 B^5) = 0. \tag{5.27}$$

Separating Eq. (5.27) in powers of e^0 and $e^{2B\xi}$, we find

$$e^0 \quad : \quad -cB + B^2 + c\alpha B^3 + \beta_1 c^2 B^4 + \gamma c^3 B^5 = 0, \tag{5.28}$$

$$e^{2B\xi} \quad : \quad \beta A^2 B^4 + 6c\gamma A^2 B^5 = 0. \tag{5.29}$$

From Eq. (5.29), we deduce

$$B = \frac{-\beta}{6c\gamma}. \tag{5.30}$$

Using the value of B in Eq. (5.28), we obtain

$$\frac{\beta}{6\gamma} + \frac{\beta^2}{(6c\gamma)^2} - \frac{\beta^3\alpha c}{(6c\gamma)^3} + \frac{\beta_1\beta^4 c^2}{(6c\gamma)^4} - \frac{\gamma\beta^5 c^3}{(6c\gamma)^5} = 0. \tag{5.31}$$

Thus the exact solution for $F(\xi)$ (provided the condition (5.31) holds) can be written as

$$F(\xi) = A \exp\left[\frac{-\beta\xi}{6c\gamma}\right]. \tag{5.32}$$

So the exact solution $u(y,t)$ which satisfy the condition (5.31) is

$$u(y,t) = \exp\left[\frac{-\beta(y+ct)}{6c\gamma}\right] \quad \text{with} \quad c > 0. \tag{5.33}$$

The solution (5.33) satisfies the initial and boundary conditions (5.17) – (5.21) for the particular values of the unspecified functions $V(t)$, $f(y)$, $g(y)$ and $h(y)$. Using Eq. (5.33) in Eqs. (5.17) – (5.21) results in

$$\begin{aligned}
u(0,t) &= V(t) = \exp\left(\frac{-\beta ct}{6c\gamma}\right), && (5.34a)\\
u(y,0) &= f(y) = \exp\left(\frac{-\beta y}{6c\gamma}\right), && (5.34b)\\
\frac{\partial u(y,0)}{\partial t} &= g(y) = \frac{-\beta}{6\gamma}\exp\left(\frac{-\beta y}{6c\gamma}\right), && (5.34c)\\
\frac{\partial^2 u(y,0)}{\partial t^2} &= h(y) = \frac{\beta^2}{(6\gamma)^2}\exp\left(\frac{-\beta y}{6c\gamma}\right). && (5.34d)
\end{aligned}$$

Here $V(t)$, $f(y)$, $g(y)$ and $h(y)$ depend on the physical parameters of the flow. The solution (5.33) is plotted in Figures 5.1 – 5.4 for different values of the emerging parameters.

5.4.2 Forward wave-front type travelling wave solutions

We look for invariant solutions under the operator $X_1 + cX_2$ (with $c > 0$) which represent forward wave-front type travelling wave solutions with constant wave speed c. In this case the waves are propagating away from the plate. These are solutions of the form

$$u(y,t) = G(\eta) \quad \text{with} \quad \eta = y - ct. \tag{5.35}$$

Using Eq. (5.35) in Eq. (5.16) results in a fifth-order ordinary differential equation for $G(\eta)$,

$$-c\frac{dG}{d\eta} = \frac{d^2G}{d\eta^2} - \alpha c\frac{d^3G}{d\eta^3} + \beta_1 c^2\frac{d^4G}{d\eta^4} + \beta\left(\frac{dG}{d\eta}\right)^2\frac{d^2G}{d\eta^2} - \gamma_1 c^3\frac{d^5G}{d\eta^5} - 2\gamma\frac{d}{d\eta}\left[c\left(\frac{dG}{d\eta}\right)^2\frac{d^2G}{d\eta^2}\right]. \tag{5.36}$$

Following the same methodology adopted for the backward wave-front type travelling wave solutions, the above equation admits exact solutions of the form

$$G(\eta) = A\exp\left[\frac{\beta\eta}{6c\gamma}\right], \tag{5.37}$$

provided

$$\frac{\beta}{6\gamma} + \frac{\beta^2}{(6c\gamma)^2} - \frac{\beta^3\alpha c}{(6c\gamma)^3} + \frac{\beta_1\beta^4c^2}{(6c\gamma)^4} - \frac{\gamma\beta^5c^3}{(6c\gamma)^5} = 0. \tag{5.38}$$

Thus Eq. (5.16) subject to Eq. (5.38) admits the exact solution

$$u(y,t) = \exp\left[\frac{\beta(y-ct)}{6c\gamma}\right] \quad \text{with} \quad c > 0. \tag{5.39}$$

Note that the solution (5.39) does not satisfy the second boundary condition at infinity but satisfy the rest of the boundary conditions for the particular values of the functions $V(t)$, $f(y)$, $g(y)$ and $h(y)$. The solution (5.39) is the shock wave solution and describe the shock wave behavior of the flow with slope approaching infinity along the characteristic. This solution does not show the physics of the model but does predict the hidden shock wave phenomena in the flow. Some examples of shock waves are moving shock, detonation wave, detached shock, attached shock, recompression shock, shock in a pipe flow, shock waves in rapid granular flows, shock waves in astrophysics and so on. Using Eq. (5.39) in Eqs. (5.17) – (5.21) gives

$$
\begin{aligned}
u(0,t) &= V(t) = \exp\left(\frac{-\beta ct}{6c\gamma}\right), && (5.40a)\\
u(y,0) &= f(y) = \exp\left(\frac{\beta y}{6c\gamma}\right), && (5.40b)\\
\frac{\partial u(y,0)}{\partial t} &= g(y) = \frac{-\beta}{6\gamma}\exp\left(\frac{\beta y}{6c\gamma}\right), && (5.40c)\\
\frac{\partial^2 u(y,0)}{\partial t^2} &= h(y) = \frac{\beta^2}{(6\gamma)^2}\exp\left(\frac{\beta y}{6c\gamma}\right). && (5.40d)
\end{aligned}
$$

5.5 Conditional symmetry solution

We consider the PDE (5.16) with the invariant surface condition

$$u_t = \lambda u, \tag{5.41}$$

where λ is a constant to be found. This corresponds to the operator

$$X = \frac{\partial}{\partial t} + \lambda u \frac{\partial}{\partial u}. \tag{5.42}$$

The invariant solution corresponding to (5.42) is

$$u(y,t) = \exp(\lambda t)F(y), \tag{5.43}$$

where $F(y)$ is an undetermined function of y.

Substituting Eq. (5.43) in Eq. (5.16) leads to a linear second-order ordinary differential equation in $F(y)$, given by

$$\frac{d^2F}{dy^2} - \left(\frac{\lambda}{1+\alpha\lambda+\beta_1\lambda^2+\gamma_1\lambda^3}\right)F = 0, \tag{5.44}$$

and the relation

$$\beta + 6\gamma\lambda = 0. \tag{5.45}$$

From Eq. (5.45), we deduce

$$\lambda = -\frac{\beta}{6\gamma}. \tag{5.46}$$

Using the value of λ in Eq. (5.44), we get

$$\frac{d^2F}{dy^2} - \left\{ \frac{\left(-\frac{\beta}{6\gamma}\right)}{\left(1 - \alpha\frac{\beta}{6\gamma} + \beta_1\left(\frac{\beta}{6\gamma}\right)^2 - \gamma_1\left(\frac{\beta}{6\gamma}\right)^3\right)} \right\} F = 0, \tag{5.47}$$

The reduced ODE (5.47) is solved subject to the boundary conditions

$$\begin{aligned} F(0) &= 1, \\ F(y) &\rightarrow 0 \text{ as } y \rightarrow \infty. \end{aligned} \tag{5.48}$$

Now we solve Eq. (5.47) subject to the boundary conditions (5.48) for three different cases.

Case 1: $\left[\frac{\left(-\frac{\beta}{6\gamma}\right)}{1-\alpha\frac{\beta}{6\gamma}+\beta_1\left(\frac{\beta}{6\gamma}\right)^2-\gamma_1\left(\frac{\beta}{6\gamma}\right)^3}\right] = 0.$

We have

$$\frac{d^2F}{dy^2} = 0. \tag{5.49}$$

Solution of (5.49) subject to the boundary conditions (5.48) is

$$F(y) = \text{constant (say } C) \tag{5.50}$$

In this case, the solution for $u(y,t)$ is

$$u(t) = C\exp\left(-\frac{\beta}{6\gamma}t\right). \tag{5.51}$$

Thus a time-dependent solution is obtained in this case.

Case 2: $\left[\frac{\left(-\frac{\beta}{6\gamma}\right)}{1-\alpha\frac{\beta}{6\gamma}+\beta_1\left(\frac{\beta}{6\gamma}\right)^2-\gamma_1\left(\frac{\beta}{6\gamma}\right)^3}\right] < 0.$

In this case the solution for $F(y)$ is written as

$$\begin{aligned} F(y) &= a\cos\left[\sqrt{\left(\frac{\left(-\frac{\beta}{6\gamma}\right)}{1-\alpha\frac{\beta}{6\gamma}+\beta_1\left(\frac{\beta}{6\gamma}\right)^2-\gamma_1\left(\frac{\beta}{6\gamma}\right)^3}\right)}\,y\right] \\ &\quad +b\sin\left[\sqrt{\left(\frac{\left(-\frac{\beta}{6\gamma}\right)}{1-\alpha\frac{\beta}{6\gamma}+\beta_1\left(\frac{\beta}{6\gamma}\right)^2-\gamma_1\left(\frac{\beta}{6\gamma}\right)^3}\right)}\,y\right]. \end{aligned} \tag{5.52}$$

Since $F(\infty) = 0$, an unbounded is solution is obtained for $F(y)$. Thus the solution for $u(y,t)$ does not exists in this case.

Case 3: $\left[\frac{\left(-\frac{\beta}{6\gamma}\right)}{1-\alpha\frac{\beta}{6\gamma}+\beta_1\left(\frac{\beta}{6\gamma}\right)^2-\gamma_1\left(\frac{\beta}{6\gamma}\right)^3}\right] > 0.$

In this case the solution for $F(y)$ is given by

$$\begin{aligned} F(y) &= a\exp\left[\left\{\sqrt{\left(\frac{\left(-\frac{\beta}{6\gamma}\right)}{1-\alpha\frac{\beta}{6\gamma}+\beta_1\left(\frac{\beta}{6\gamma}\right)^2-\gamma_1\left(\frac{\beta}{6\gamma}\right)^3}\right)}\right\}y\right] \\ &\quad +b\exp\left[-\left\{\sqrt{\left(\frac{\left(-\frac{\beta}{6\gamma}\right)}{1-\alpha\frac{\beta}{6\gamma}+\beta_1\left(\frac{\beta}{6\gamma}\right)^2-\gamma_1\left(\frac{\beta}{6\gamma}\right)^3}\right)}\right\}y\right]. \end{aligned} \tag{5.53}$$

By applying the boundary conditions (5.48), we have $a = 0$ (for a bounded solution) and $b = 1$ (by using first the boundary condition). Thus the solution for $F(y)$ is

$$F(y) = \exp\left[-\left\{\sqrt{\left(\frac{\left(-\frac{\beta}{6\gamma}\right)}{1-\alpha\frac{\beta}{6\gamma}+\beta_1\left(\frac{\beta}{6\gamma}\right)^2-\gamma_1\left(\frac{\beta}{6\gamma}\right)^3}\right)}\right\}y\right], \tag{5.54}$$

Substituting $F(y)$ into Eq. (5.43), we obtain the solution for $u(y,t)$ in the form

$$u(y,t) = \exp\left[-\left\{\left(\frac{\beta}{6\gamma}\right)t + \sqrt{\left(\frac{\left(-\frac{\beta}{6\gamma}\right)}{1-\alpha\frac{\beta}{6\gamma}+\beta_1\left(\frac{\beta}{6\gamma}\right)^2-\gamma_1\left(\frac{\beta}{6\gamma}\right)^3}\right)}\, y\right\}\right]. \quad (5.55)$$

Note that the conditional symmetry solution (5.55) satisfies the initial and the boundary conditions (5.17) – (5.21) with

$$V(t) = \exp[-\left(\frac{\beta}{6\gamma}\right)t], \quad (5.56)$$

$$f(y) = \exp\left(-\sqrt{\left(\frac{\left(-\frac{\beta}{6\gamma}\right)}{1-\alpha\frac{\beta}{6\gamma}+\beta_1\left(\frac{\beta}{6\gamma}\right)^2-\gamma_1\left(\frac{\beta}{6\gamma}\right)^3}\right)}\, y\right), \quad (5.57)$$

$$g(y) = -\left(\frac{\beta}{6\gamma}\right)\exp\left(-\sqrt{\left(\frac{\left(-\frac{\beta}{6\gamma}\right)}{1-\alpha\frac{\beta}{6\gamma}+\beta_1\left(\frac{\beta}{6\gamma}\right)^2-\gamma_1\left(\frac{\beta}{6\gamma}\right)^3}\right)}\, y\right), \quad (5.58)$$

$$h(y) = \left(\frac{\beta}{6\gamma}\right)^2\exp\left(-\sqrt{\left(\frac{\left(-\frac{\beta}{6\gamma}\right)}{1-\alpha\frac{\beta}{6\gamma}+\beta_1\left(\frac{\beta}{6\gamma}\right)^2-\gamma_1\left(\frac{\beta}{6\gamma}\right)^3}\right)}\, y\right), \quad (5.59)$$

where $V(t)$, $f(y)$, $g(y)$ and $h(y)$ depend on physical parameters of the flow. The graphical behavior of solution (5.55) is shown in Figure 5.5.

Remark 7 *The backward wave-front type travelling wave closed-form solution* (5.33) *and the conditional symmetry solution* (5.55) *best represent the physics of the problem considered in the sense that these solutions satisfy all the initial and the boundary conditions and also show the effects of different emerging parameters of the flow problem given in Figures* 5.1 – 5.5. *The forward wave-front type travelling wave solution* (5.39) *does not satisfy the second boundary condition at infinity. As a consequence it does not show the behavior of the physical model. But this solution does show the shockwave behavior of the flow. To emphasize, we say that the forward wave-front type travelling wave solution* *is actually a shockwave solution with slope approaching infinity along the characteristic* $\eta = y - ct$, *as shown in Figure* 5.6.

5.6 Physical interpretation of the results

In order to explain and understand the physical structure of the flow problem, various graphs are plotted in Figure $5.1 - 5.6$.

Figure 5.1 shows the influence of time t on the backward wave-front type travelling wave solution (5.33). This figure depicts that velocity decreases as time increases. Clearly, the variation of velocity is observed for $0 \leqslant t \leqslant 5$. For $t > 5$, the velocity profile remains the same. In other words, we can say that the steady-state behavior of the velocity is achieved for $t > 5$.

Figure 5.2 shows the effects of the wave speed c on the velocity profile. It is clearly observed that with the increase of wave speed c the velocity profile is increasing. So in this way, we can remark that both time t and the wave speed c have opposite effects on the backward wave-front type travelling wave solution (5.33) of the governing model.

Figures 5.3 and 5.4 have been plotted to see the influence of the third grade parameter β and fourth grade parameter γ on the backward wave-front type travelling wave solution (5.33) of the flow problem. These figures reveal that both β and γ have an opposite behavior on the structure of the velocity, that is, with an increase in the third grade parameter β, the velocity profile decreases showing the shear thickening behavior of the fluid whereas the velocity field increases for increasing values of the fourth grade parameter γ, which shows the shear thinning behavior of the model. This is in accordance with the fact that a fourth grade fluid model predicts both shear thickening and the shear thinning properties of the flow.

In Figure 5.5 the conditional symmetry solution (5.55) is plotted against the increasing values of time t. The behavior of time on the conditional symmetry solution is the same as observed previously for the backward wave-front type travelling wave solution. That is, with the increase in time t the velocity decreases. However, in this case the steady-state behavior of the velocity is achieved quicker as compared to the backward wave-front type travelling wave solution. The variation of velocity is observed for $0 \leqslant t \leqslant 0.7$. For $t > 0.7$, the steady-state behavior of the velocity is observed.

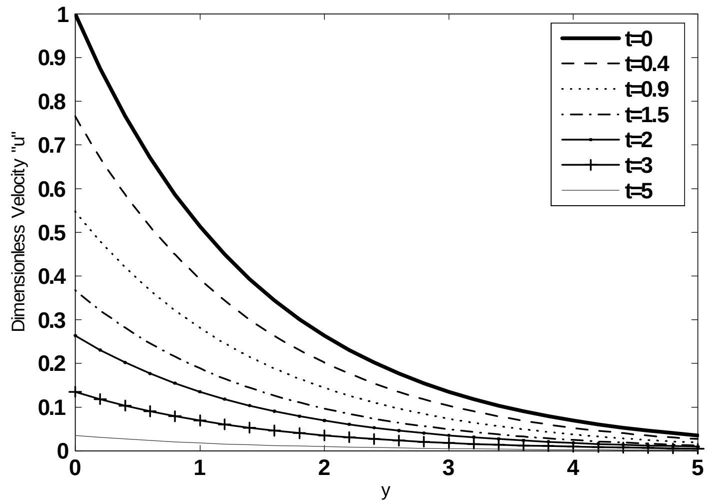

Figure 5.1: Backward wave-front type travelling wave solution (5.33) varying t when $\gamma = 0.5$, $\beta = 2$, $c = 1$ are fixed.

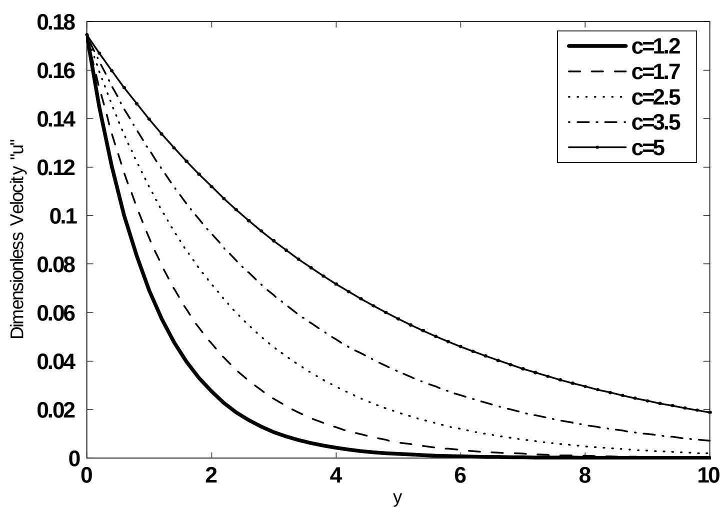

Figure 5.2: Backward wave-front type travelling wave solution (5.33) varying c when

$\gamma = 0.6$, $\beta = 4$, $t = \pi/2$ are fixed.

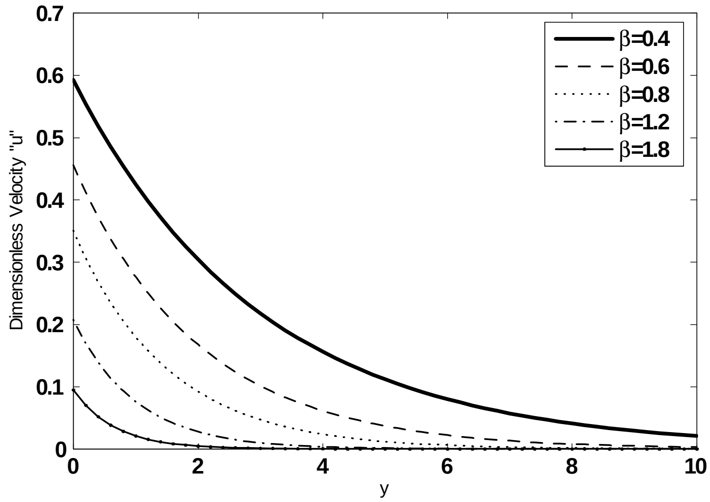

Figure 5.3: Backward wave-front type travelling wave solution (5.33) varying β when $\gamma = 0.2$, $c = 1$, $t = \pi/2$ are fixed.

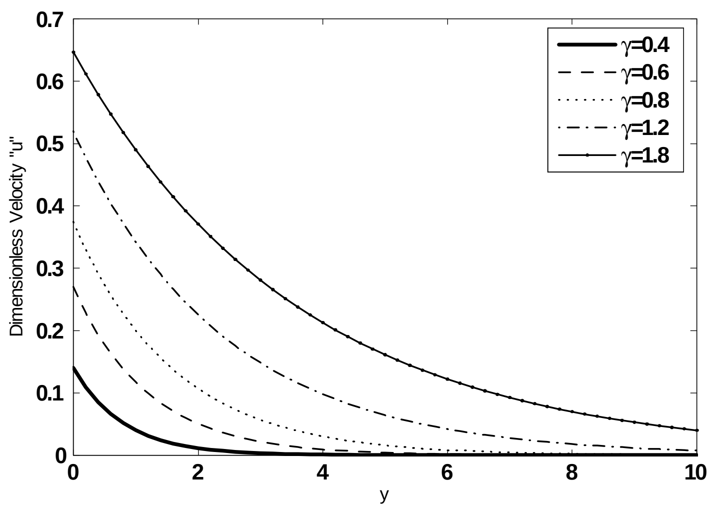

Figure 5.4: Backward wave-front type travelling wave solution (5.33) varying γ when $\beta = 3$, $c = 1$, $t = \pi/2$ are fixed.

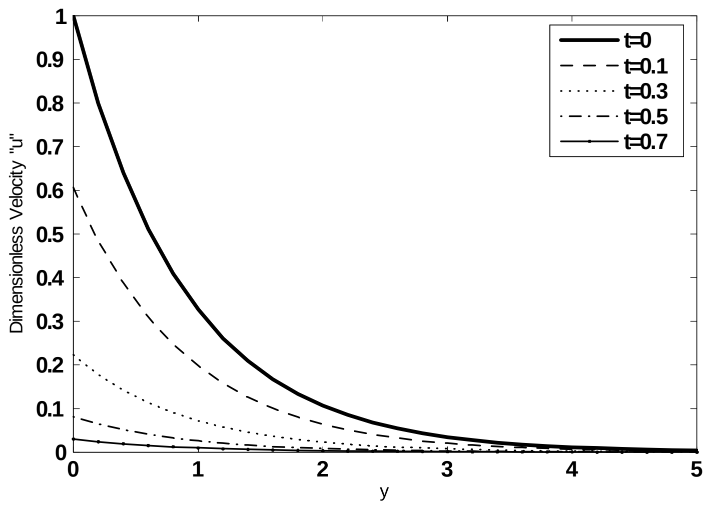

Figure 5.5: Conditional symmetry solution (5.55) varying t when $\gamma = 0.2$, $\gamma_1 = 0.1$, $\beta = 6$, $\beta_1 = 0.5$, $\alpha = 1$ are fixed.

5.7 Conclusions

The present work has been undertaken in order to investigate further the regime of a fourth grade non-Newtonian fluid model. Some reductions and exact (closed-form) solutions for the time-dependent flow of a fourth grade fluid have been established using the symmetry approach. Travelling wave and conditional symmetry solutions are obtained for the governing nonlinear PDE. Both forward and the backward wave-front type travelling wave solutions have been constructed. The better solutions from the physical point of view of the model considered are the conditional symmetry solution and the backward wave-front type travelling wave solution. This issue has also been addressed in detail in Remark 7. Moreover, the work contained herein is theoretical in nature and is a prototype model. The methods used will be helpful for a wide range of nonlinear problems in fluids given the paucity of known exact solutions especially in non-Newtonian fluids.

Chapter 6

Invariant Solutions for the Unsteady Flow of a Fourth Grade Fluid with Plate Suction or Injection

6.1 Introduction

In this chapter, the unsteady unidirectional flow of an incompressible fourth grade fluid bounded by a porous plate is examined. The disturbance is caused due to the motion of the plate in its own plane with an arbitrary velocity $V(t)$. Symmetry reductions are performed to transform the governing nonlinear partial differential equations into ordinary differential equations. The reduced equations are then solved analytically and numerically. The influence of various physical parameters of interest on the velocity profile are shown and discussed through several graphs. A comparison of the present analysis shows excellent agreement between analytical and numerical solutions. To the best of our knowledge, no attempt has been made up to now to obtain the exact analytical solution for the unsteady flow of fourth grade fluid over the porous plate with suitable boundary conditions.

6.2 Problem statement and mathematical modelling

Consider a Cartesian coordinate frame of reference $OXYZ$ with the x-axis along the direction of flow and y-axis in the vertically upward direction. A fourth grade fluid occupies the space $y > 0$ and is in contact with an infinite porous plate at $y = 0$. The motion in the fluid is induced by a jerk to the plate for $t > 0$ (the geometry of the problem is given in Fig. 1).

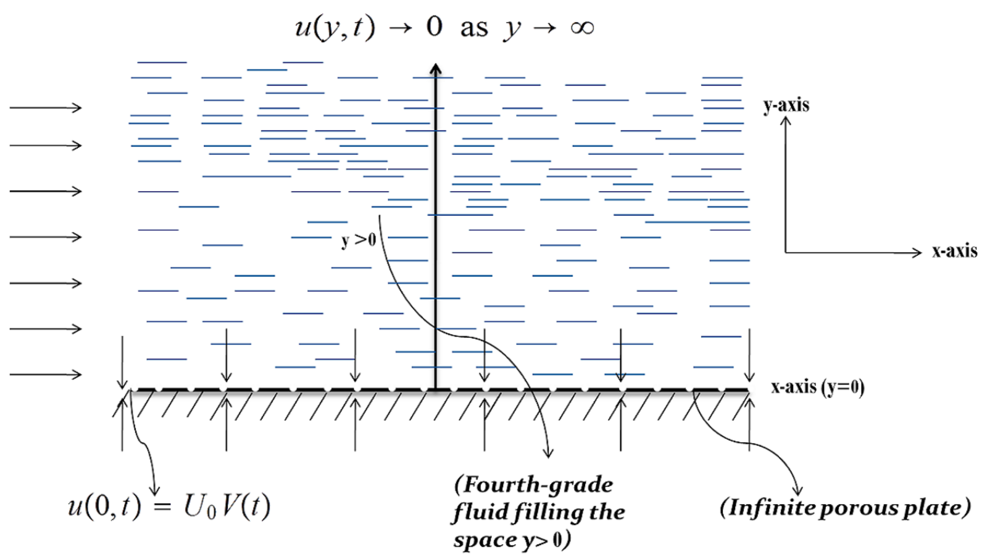

Figure 6.1: Geometry of the physical model and coordinate system.

The unsteady motion of an incompressible fluid in the Cartesian coordinate system is governed by the conservation laws of momentum and mass which are

$$\rho \frac{d\mathbf{V}}{dt} = \operatorname{div} \mathbf{T}, \tag{6.1}$$

$$\operatorname{div} \mathbf{V} = \mathbf{0}. \tag{6.2}$$

In the above equations ρ is the fluid density, $\mathbf{V}$ the velocity vector, d/dt the material time derivative and $\mathbf{T}$ the Cauchy stress tensor.

The Cauchy stress tensor for a fourth grade fluid [44] is given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4 \tag{6.3}$$

where p is the pressure, $\mathbf{I}$ the identity tensor and $\mathbf{S}_i$ $(i = 1, 2, 3, 4)$ the extra stress tensors given by

$$\begin{aligned}
\mathbf{S}_1 &= \mu\mathbf{A}_1, && (6.4)\\
\mathbf{S}_2 &= \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, && (6.5)\\
\mathbf{S}_3 &= \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_1, && (6.6)\\
\mathbf{S}_4 &= \gamma_1\mathbf{A}_4 + \gamma_2(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3) + \gamma_3\mathbf{A}_2^2 + \gamma_4(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2) \\
&\quad \gamma_5(\mathrm{tr}\mathbf{A}_2)\mathbf{A}_2 + \gamma_6(\mathrm{tr}\mathbf{A}_2)\mathbf{A}_1^2 + [\gamma_7\mathrm{tr}\mathbf{A}_3 + \gamma_8\mathrm{tr}(\mathbf{A}_2\mathbf{A}_1)]\,\mathbf{A}_1. && (6.7)
\end{aligned}$$

In Eqs. $(4) - (7)$ μ is the dynamic viscosity, α_i $(i = 1, 2)$, β_i $(i = 1, 2, 3)$ and γ_i $(i = 1, 2, ..., 8)$ are material constants. The kinematical tensors $\mathbf{A}_1$ to $\mathbf{A}_4$ are the Rivlin-Ericksen tensors given by

$$\begin{aligned}
\mathbf{A}_1 &= (\mathrm{grad}\,\mathbf{V}) + (\mathrm{grad}\,\mathbf{V})^T, && (6.8)\\
\mathbf{A}_n &= \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(\mathrm{grad}\,\mathbf{V}) + (\mathrm{grad}\,\mathbf{V})^T\mathbf{A}_{n-1};\ (n > 1), && (6.9)
\end{aligned}$$

in which grad is the gradient operator.

For the flow under investigation, we seek a velocity of the form

$$\mathbf{V} = [u(y,t), -W_0, 0], \tag{6.10}$$

where u is the velocity vector in the x-direction, $W_0 > 0$ is the suction velocity and $W_0 < 0$ corresponds to the injection velocity. The above definition of velocity satisfies the continuity equation given in Eq. (6.2) identically. By substituting Eq. (6.10) in Eq. (6.1), one obtains

$$\rho\frac{\partial u}{\partial t} = \frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xy}}{\partial y}+\frac{\partial T_{xz}}{\partial z}, \quad (6.11)$$

$$0 = \frac{\partial T_{yx}}{\partial x}+\frac{\partial T_{yy}}{\partial y}+\frac{\partial T_{yz}}{\partial z}, \quad (6.12)$$

$$0 = \frac{\partial T_{zx}}{\partial x}+\frac{\partial T_{zy}}{\partial y}+\frac{\partial T_{zz}}{\partial z}. \quad (6.13)$$

On substituting the components of $\mathbf{T}$ from Eqs. (6.3) – (6.9) in Eqs. (6.11) – (6.13) leads to the following equations:

$$\begin{aligned}\rho\left[\frac{\partial u}{\partial t}-W_0\frac{\partial u}{\partial y}\right] =& -\frac{\partial \hat{p}}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}+\alpha_1\left(\frac{\partial^3 u}{\partial y^2\partial t}-W_0\frac{\partial^3 u}{\partial y^3}\right)\\ &+\beta_1\left[\frac{\partial^4 u}{\partial y^2\partial t^2}-2W_0\frac{\partial^4 u}{\partial y^3\partial t}+W_0^2\frac{\partial^4 u}{\partial y^4}\right]+6(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}\\ &+\gamma_1\left[\frac{\partial^5 u}{\partial y^2\partial t^3}-3W_0\frac{\partial^5 u}{\partial y^3\partial t^2}+3W_0^2\frac{\partial^5 u}{\partial y^4\partial t}-W_0^3\frac{\partial^5 u}{\partial y^5}\right]\\ &+2(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right]\\ &-2(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)W_0\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}\right], \quad (6.14)\end{aligned}$$

$$0 = \frac{\partial \hat{p}}{\partial y}, \quad (6.15)$$

$$0 = \frac{\partial \hat{p}}{\partial z}. \quad (6.16)$$

Equations (6.15) and (6.16) show that $\hat{p}$ is not a function of y and z. Hence $\hat{p}$ is a function of x and t. In the absence of a modified pressure gradient the governing equation (6.14) thus becomes

$$
\begin{aligned}
\rho\left[\frac{\partial u}{\partial t} - W_0\frac{\partial u}{\partial y}\right] \quad = \quad & \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\left[\frac{\partial^3 u}{\partial y^2 \partial t} - W_0\frac{\partial^3 u}{\partial y^3}\right] \\
& +\beta_1\left[\frac{\partial^4 u}{\partial y^2 \partial t^2} - 2W_0\frac{\partial^4 u}{\partial y^3 \partial t} + W_0^2\frac{\partial^4 u}{\partial y^4}\right] + 6(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} \\
& +\gamma_1\left[\frac{\partial^5 u}{\partial y^2 \partial t^3} - 3W_0\frac{\partial^5 u}{\partial y^3 \partial t^2} + 3W_0^2\frac{\partial^5 u}{\partial y^4 \partial t} - W_0^3\frac{\partial^5 u}{\partial y^5}\right] \\
& +2(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y \partial t}\right] \\
& -2(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)W_0\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}\right]. \qquad (6.17)
\end{aligned}
$$

The relevant initial and boundary conditions are

$$u(0,t) \quad = \quad U_0 V(t), \quad t>0, \qquad (6.18)$$

$$u(y,t) \quad \rightarrow \quad 0 \text{ as } y\rightarrow\infty,\ t>0, \qquad (6.19)$$

$$u(y,0) \quad = \quad g(y), \qquad y>0, \qquad (6.20)$$

where U_0 is the reference velocity.

On introducing the non-dimensional quantities

$$
\begin{aligned}
\bar{u} &= \frac{u}{U_0},\ \ \bar{y} = \frac{U_0 y}{\nu},\ \ \bar{t} = \frac{U_0^2 t}{\nu},\ \ \bar{W}_0 = \frac{W_0}{U_0},\ \bar{\alpha}_1 = \frac{\alpha_1 U_0^2}{\rho\nu^2}, \\
\bar{\beta}_1 &= \frac{\beta_1 U_0^4}{\rho\nu^3},\ \ \bar{\beta} = \frac{6(\beta_2+\beta_3)U_0^4}{\rho\nu^3},\ \ \bar{\gamma} = \frac{\gamma_1 U_0^6}{\rho\nu^4}, \\
\bar{\Gamma} &= (3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)\frac{U_0^6}{\rho\nu^4}, \qquad (6.21)
\end{aligned}
$$

the corresponding initial and boundary value problem takes the form

$$\begin{aligned}
\frac{\partial u}{\partial t} &= W_0\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial y^2}+\alpha_1\left[\frac{\partial^3 u}{\partial y^2\partial t}-W_0\frac{\partial^3 u}{\partial y^3}\right] \\
&+\beta_1\left[\frac{\partial^4 u}{\partial y^2\partial t^2}-2W_0\frac{\partial^4 u}{\partial y^3\partial t}+W_0^2\frac{\partial^4 u}{\partial y^4}\right]+\beta\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} \\
&+\gamma\left[\frac{\partial^5 u}{\partial y^2\partial t^3}-3W_0\frac{\partial^5 u}{\partial y^3\partial t^2}+3W_0^2\frac{\partial^5 u}{\partial y^4\partial t}-W_0^3\frac{\partial^5 u}{\partial y^5}\right] \\
&+2\Gamma\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right]-2\Gamma W_0\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}\right], \qquad (6.22)
\end{aligned}$$

$$u(0,t) = V(t), \quad t>0, \qquad (6.23)$$

$$u(y,t) \to 0 \text{ as } y\to\infty,\ t>0, \qquad (6.24)$$

$$u(y,0) = g(y), \qquad y>0. \qquad (6.25)$$

For simplicity we have omitted the bars of the non-dimensional quantities.

6.3 Reductions and solutions

It can be seen that Eq. (22) is form invariant under the Lie point symmetry generator $X_1 = \partial/\partial t$ (time translation) and $X_2 = \partial/\partial y$ (space translation). We consider the interesting case of the invariant solution under the operator $X = X_1 - cX_2$ $(c>0)$ which represents wave-front type travelling wave solutions with constant wave speed c. The invariant solution is given by

$$u(y,t) = h(x), \quad \text{where} \quad x = y+ct. \qquad (6.26)$$

Substituting Eq. (6.26) into Eq. (6.22), we deduce the reduced fifth-order ordinary differential equation for $h(x)$, viz.

$$\begin{aligned} c\frac{dh}{dx} &= W_0\frac{dh}{dx} + \frac{d^2h}{dx^2} + \alpha_1\left[c\frac{d^3h}{dx^3} - W_0\frac{d^3h}{dx^3}\right] \\ &+\beta_1\left[c^2\frac{d^4h}{dx^4} - 2W_0c\frac{d^4h}{dx^4} + W_0^2\frac{d^4h}{dx^4}\right] + \beta\left(\frac{dh}{dx}\right)^2\frac{d^2h}{dx^2} \\ &+\gamma\left[c^3\frac{d^5h}{dx^5} - 3c^2W_0\frac{d^5h}{dx^5} + 3cW_0^2\frac{d^5h}{dx^5} - W_0^3\frac{d^5h}{dx^5}\right] \\ &+2\Gamma\frac{d}{dx}\left[c\left(\frac{dh}{dx}\right)^2\frac{d^2h}{dx^2}\right] - 2\Gamma W_0\frac{d}{dx}\left[\left(\frac{dh}{dx}\right)^2\frac{d^2h}{dx^2}\right]. \end{aligned} \tag{6.27}$$

Thus the governing PDE (6.22) reduce to an ODE (6.27) along certain curves in the $y-t$ plane. These curves are called *characteristic curves* or just the characteristics. To solve Eq. (6.27) for $h(x)$, we assume a solution of the form

$$h(x) = A\exp(Bx), \tag{6.28}$$

where A and B are the constants to be determined. Substituting Eq. (6.28) in Eq. (6.27) we obtain

$$\begin{aligned} 0 &= [-(c-W_0)B + B^2 + \alpha_1(c-W_0)B^3 + \beta_1(c-W_0)^2B^4 + \gamma(c-W_0)^3B^5] \\ &+e^{2Bx}[\beta A^2B^4 + 6\Gamma(c-W_0)A^2B^5]. \end{aligned} \tag{6.29}$$

Separating Eq. (6.29) in the exponents of e^0 and e^{2Bx}, we find

$$e^0 \;:\; -(c-W_0)B + B^2 + \alpha_1(c-W_0)B^3 + \beta_1(c-W_0)^2B^4 + \gamma(c-W_0)^3B^5 = 0, \tag{6.30}$$

$$e^{2Bx} \;:\; \beta A^2B^4 + 6\Gamma(c-W_0)A^2B^5 = 0. \tag{6.31}$$

From Eq. (6.31), we deduce

$$B = \frac{-\beta}{6\Gamma(c-W_0)}. \tag{6.32}$$

Using the value of B in Eq. (6.30), we obtain

$$-\frac{\beta}{6\Gamma}=\frac{\beta^2}{(6\Gamma)^2\,(c-W_0)^2}-\frac{\alpha_1\beta^3}{(6\Gamma)^3\,(c-W_0)^2}+\frac{\beta_1\beta^4}{(6\Gamma)^4\,(c-W_0)^2}-\frac{\gamma\beta^5}{(6\Gamma)^5\,(c-W_0)^2}. \tag{6.33}$$

Thus the exact solution for $h(x)$ (provided the condition (6.33) holds) can be written as

$$h(x)=a\exp\left[\frac{-\beta x}{6\Gamma(c-W_0)}\right] \quad \text{with } c>W_0. \tag{6.34}$$

So the exact solution $u(y,t)$ which satisfy the condition (6.33) is

$$u(y,t)=\exp\left[\frac{-\beta(y+ct)}{6\Gamma(c-W_0)}\right] \quad \text{with } c>W_0. \tag{6.35}$$

We note that this solution satisfies the initial and the boundary conditions (6.23), (6.24) and (6.25) with

$$V(t)=\exp\left(-\frac{\beta ct}{6\Gamma(c-W_0)}\right) \quad \text{and} \quad g(y)=\exp\left(-\frac{\beta y}{6\Gamma(c-W_0)}\right), \tag{6.36}$$

where $V(t)$ and $g(y)$ depend on the physical parameters of the flow.
The solution in Eq. (6.35) is plotted in Figures 6.2−6.7 for various values of the emerging physical parameters of the flow model.

Moreover, we can find group invariant solutions corresponding to the operator X_1, which provide other meaningful solutions as well. The solution corresponding to X_1 would be the steady state solution. The independent invariants of X_1 are

$$u=H, \quad (H=H(y)). \tag{6.37}$$

Inserting Eq. (6.37) into Eq. (6.22), we obtain

$$W_0\frac{dH}{dy}+\frac{d^2H}{dy^2}-\alpha_1 W_0\frac{d^3H}{dy^3}+\beta_1 W_0^2\frac{d^4H}{dy^4}+\beta\left(\frac{dH}{dy}\right)^2\frac{d^2H}{dy^2}$$
$$-\gamma W_0^3\frac{d^5H}{dy^5}-2\Gamma W_0\frac{d}{dy}\left[\left(\frac{dH}{dy}\right)^2\frac{d^2H}{dy^2}\right]=0. \tag{6.38}$$

The transformed boundary conditions are

$$\begin{aligned} H(0) &= l_1, \text{ where } l_1 \text{ is a constant} \qquad (6.39)\\ H(y) &= 0 \quad \text{as} \quad y\to\infty. \end{aligned}$$

It can also be seen that Eq. (6.38) admits the solution

$$u=H(y)=\exp\left(\frac{\beta y}{3\Gamma W_0}\right), \tag{6.40}$$

provided that

$$-\frac{\beta}{3\Gamma}=\frac{\beta^2}{(3\Gamma)^2(W_0)^2}-\frac{\alpha_1\beta^3}{(3\Gamma)^3(W_0)^2}+\frac{\beta_1\beta^4}{(3\Gamma)^4(W_0)^2}-\frac{\gamma\beta^5}{(3\Gamma)^5(W_0)^2}. \tag{6.41}$$

Note that the solution given in Eq. (6.40) is valid only for injection ($W_0<0$), for the case of suction ($W_0>0$) the solution does not satisfy the second boundary condition at infinity and places a restriction on the physical parameters. One can also solve the reduced equation (6.38) subject to the boundary conditions (6.39) numerically. This solution is plotted in Figure 6.8 using the Mathematica® solver **NDSolve.**

Finally, we note that further reductions of Eq. (6.27) are possible. Eq. (6.27) admits the Lie point generator $G=\partial/\partial x$. The corresponding invariants of G are:

$$m=h,\quad h'=n \quad \text{with} \quad n=n(m). \tag{6.42}$$

Under these invariants Eq. (6.27) reduces to a fourth-order ODE in n, given as

$$\begin{aligned}(c-W_0)n &= \frac{dn}{dm}+\alpha_1(c-W_0)\frac{d^2n}{dm^2}+\beta_1(c-W_0)^2\frac{d^3n}{dm^3}+\beta n^2\frac{dn}{dm}\\ &\quad+\gamma(c-W_0)^3\frac{d^4n}{dm^4}+2\Gamma(c-W_0)\left[2n\left(\frac{dn}{dm}\right)^2+n^2\frac{d^2n}{dm^2}\right]. \qquad (6.43)\end{aligned}$$

Eq. (6.43) now admits Lie point symmetry generator $L=\partial/\partial m$, the independent invariants of L are:

$$p=n, \quad n'=q \quad \text{with} \quad q=q(p). \qquad (6.44)$$

Under these invariants Eq. (6.43) reduces to a third-order ODE in q,

$$\begin{aligned}(c-W_0)p &= q+\alpha_1(c-W_0)q\frac{dq}{dp}+\beta_1(c-W_0)^2\left[q\left(\frac{dq}{dp}\right)^2+q^2\frac{d^2q}{dp^2}\right]\\ &\quad+\beta p^2q+\gamma(c-W_0)^3\left[q\left(\frac{dq}{dp}\right)^3+4q^2\frac{d^2q}{dp^2}\frac{dq}{dp}+q^3\frac{d^3q}{dp^3}\right]\\ &\quad+2\Gamma(c-W_0)\left[2pq^2+p^2q\frac{dq}{dp}\right]. \qquad (6.45)\end{aligned}$$

The above Eq. (6.45) does not generate any Lie point symmetries. These further reductions are not useful in this analysis as difficulties arises when transforming solutions of Eq. (6.45) back into the original variables.

6.4 Results and discussion

In order to see the physical aspects of the problem under investigation, the influence of the various emerging physical parameters on the velocity field is analyzed in this section.

- Figure 6.2 show the effects of time t on the velocity profile. This figure depicts that velocity decreases for large values of time. Clearly, the variation of velocity is observed for $0 \leqslant t \leqslant 4$. For $t > 4$, the velocity profile remains the same. In other words, we can say that the steady-state behavior is achieved for $t > 4$.

• In Figure 6.3, the effects of wave speed c on the velocity profile has been shown. It is clearly observed that with the increase of wave speed c the velocity profile is increasing. So in this way, we can remark that both time t and the wave speed c have opposite effects on the analytical solution (6.35) of the model equation.

• Figures 6.4 and 6.5 show the variation of the velocity field with suction ($W_0 > 0$) and injection ($W_0 < 0$) parameters respectively. It is observed from Figure 6.4 that the velocity field decreases as the boundary layer thickness with an increase in suction parameter $W_0 > 0$. This behavior of the velocity field is expected physically that suction causes a reduction in the boundary layer thickness. Figure 6.5 is displayed for the variation of the injection velocity. The effects of the increasing injection would be quite opposite to those of suction. So from these figures, it has been observed that our analytical solution (6.35) remains stable both for the case of suction and injection provided that $c > W_0$.

• In order to see the influence of the third grade parameter β and fourth grade parameter Γ on the flow problem, Figures 6.6 and 6.7 have been plotted. These figures reveal that both β and Γ have opposite effects on the structure of the velocity, i.e. with an increase in the parameter β, the velocity profile is decreases showing the shear thickening behavior of the fluid whereas the velocity field increases for increasing values of Γ, which shows the shear thinning behavior of the fluid.

• The numerical solution of the reduced ODE (6.38) subject to the boundary conditions (6.39) is plotted in Figure 6.8 for varying values of W_0. It is observed that the numerical solution shows the same behavior as we have observed for the analytical solution.

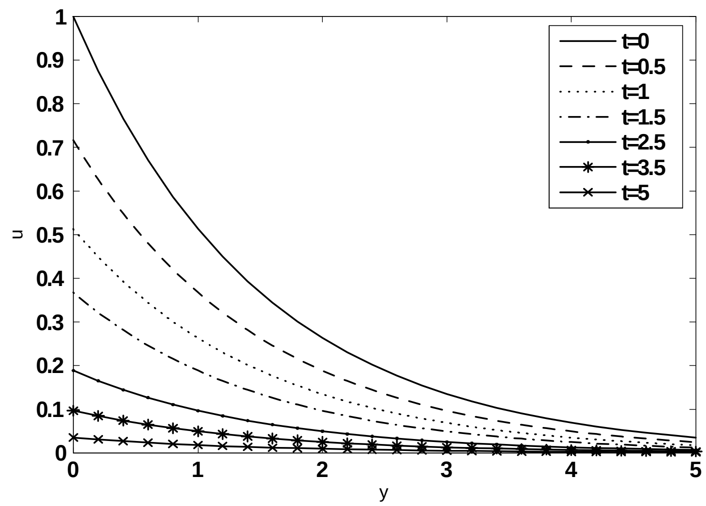

Figure 6.2: Plot of the analytical solution (6.35) for varying values of t, when $\beta = 2$, $\Gamma = 1$, $c = 1$, $W_0 = 0.5$ are fixed.

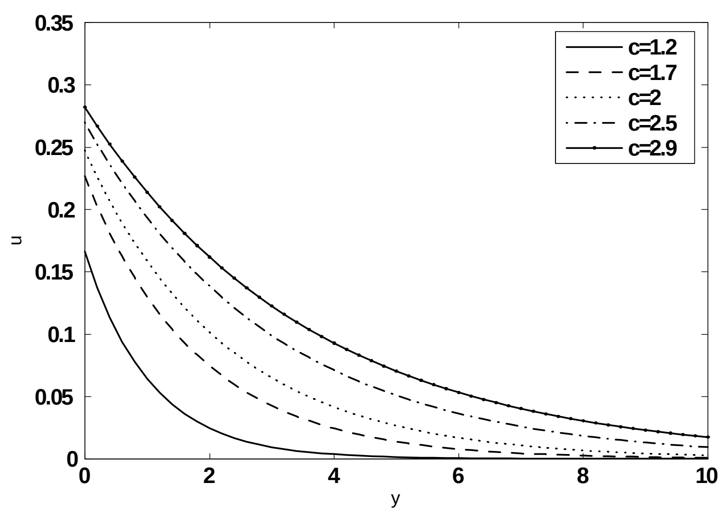

Figure 6.3: Plot of the analytical solution (6.35) for varying values of c, when $\beta = 2$, $\Gamma = 1$, $t = \pi/2$, $W_0 = 0.5$ are fixed.

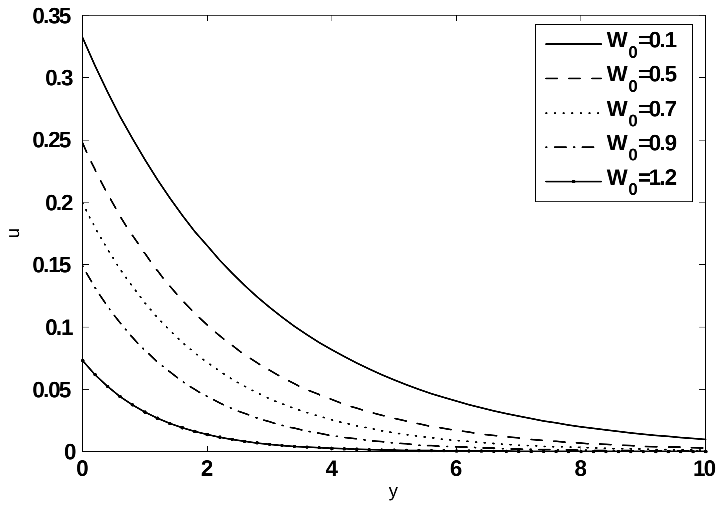

Figure 6.4: Plot of the analytical solution (6.35) for varying values of $W_0 > 0$, when $\beta = 4$, $\Gamma = 1$, $t = \pi/2$, $c = 2$ are fixed.

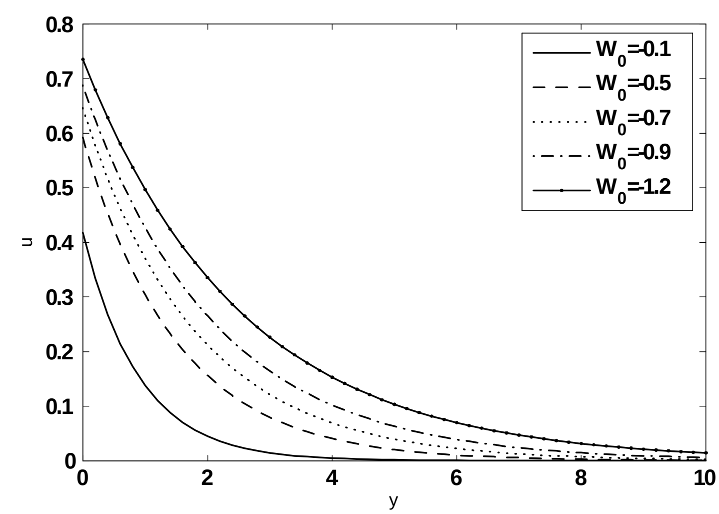

Figure 6.5: Plot of the analytical solution (6.35) for varying values of $W_0 < 0$, when $\beta = 4$, $\Gamma = 1$, $t = \pi/2$, $c = 0.5$ are fixed.

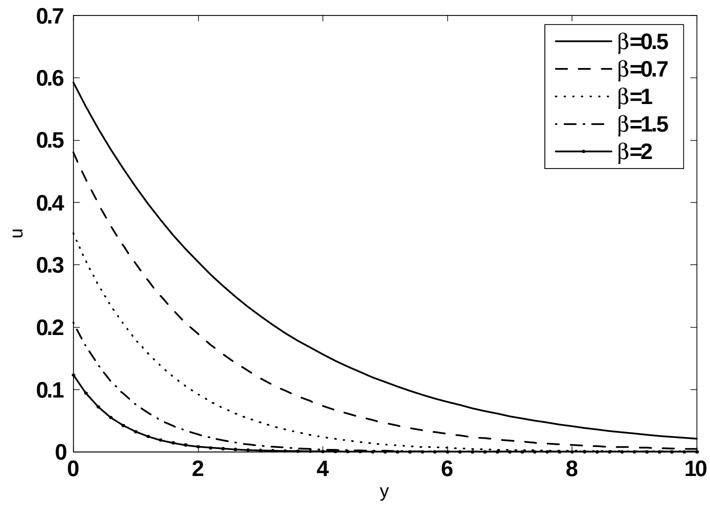

Figure 6.6: Plot of the analytical solution (6.35) for varying values of β, when $c = 1$, $\Gamma = 0.5$, $t = \pi/2$, $W_0 = 0.5$ are fixed.

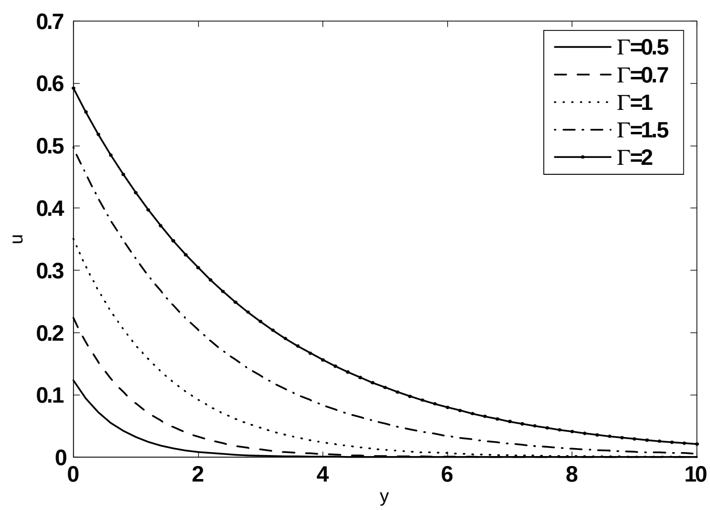

Figure 6.7: Plot of the analytical solution (6.35) for varying values of Γ, when $c = 1$, $\beta = 2$, $t = \pi/2$, $W_0 = 0.5$ are fixed.

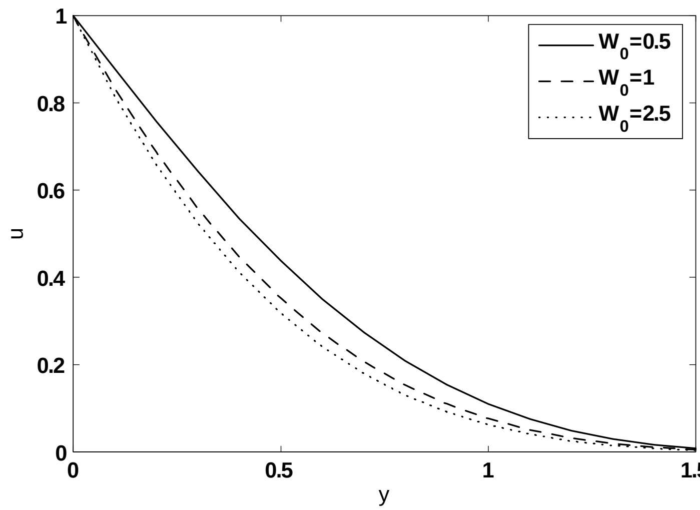

Figure 6.8: Numerical solution of the ODE (6.38) subject to the boundary conditions (6.39) for varying values of W_0, when $\alpha_1 = 0.2$, $\beta_1 = 1$, $\beta = 2$, $\gamma = 1$, $\Gamma = 0.1$ are fixed.

6.5 Conclusions

In this chapter, we have modelled the unidirectional flow of a fourth grade fluid over an infinite moved porous plate. Translational type symmetries have been employed to perform reductions of the governing nonlinear model partial differential equation to different ordinary differential equations. The reduced equations are then solved analytically as well as numerically. It is seen that the numerical solution has the same behavior as the analytical solution. The method of solution that we have adopted is also prosperous for tackling wide range of nonlinear problems in non-Newtonian fluid mechanics.

Chapter 7

Group Theoretical Analysis and Invariant Solutions for Time-Dependent Flow of a Fourth Grade Fluid Through a Porous Space

7.1 Introduction

The present chapter aims to discuss the unsteady flow of an incompressible non-Newtonian fourth grade fluid. The flow is caused by arbitrary plate velocity $V(t)$. The fluid also occupies the porous half space $y > 0$ over an infinite flat plate. Lie group theory is utilized to obtain all possible point symmetries of the governing nonlinear partial differential equation. One-dimensional optimal system of subalgebras are also obtained for the modelled equation. With the use of symmetries and the corresponding subalgebras, some new classes of group invariant solutions have been developed for the modelled problem. The influence of various emerging parameters on the flow characteristics has been studied and discussed through several graphs.

7.2 Formulation of the problem

Introduce the Cartesian coordinate system $OXYZ$ with the x-axis along the direction of flow and the y-axis perpendicular to it. Consider the unsteady unidirectional flow of an incompressible fourth grade fluid. The fluid occupies the porous half space $y > 0$ over an infinite flat rigid plate. The flow is generated due to the translation of the plate in its own place with in impulsive velocity $V(t)$. Since the plate is infinite in $XZ-$plane, therefore all the physical quantities except the pressure depends on y only. For the flow model under consideration, we seek a velocity of the form

$$\mathbf{V} = [u(y,t), 0, 0], \tag{7.1}$$

where u denotes the velocity of the fluid in $x-$direction. It should be noted that Eq. (7.1) satisfies the law of conservation of mass for incompressible fluid, i.e.

$$\operatorname{div} \mathbf{V} = 0. \tag{7.2}$$

The unsteady motion of an incompressible non-Newtonian fluid through a porous medium is governed by

$$\rho \frac{d\mathbf{V}}{dt} = \operatorname{div} \mathbf{T} + \mathbf{R}, \tag{7.3}$$

in which ρ is the fluid density, d/dt the material time derivative, $\mathbf{T}$ the Cauchy stress tensor and $\mathbf{R}$ the Darcy′s resistance due to porous medium.

The Cauchy stress tensor $\mathbf{T}$ for an incompressible fourth grade fluid is

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4, \tag{7.4}$$

where p is the pressure, $\mathbf{I}$ the identity tensor and the extra stress tensors $\mathbf{S}_1 - \mathbf{S}_4$ are given by

$$
\begin{aligned}
\mathbf{S}_1 &= \mu \mathbf{A}_1, && (7.5)\\
\mathbf{S}_2 &= \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, && (7.6)\\
\mathbf{S}_3 &= \beta_1 \mathbf{A}_3 + \beta_2 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \beta_3 (\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_1, && (7.7)\\
\mathbf{S}_4 &= \gamma_1 \mathbf{A}_4 + \gamma_2 (\mathbf{A}_3 \mathbf{A}_1 + \mathbf{A}_1 \mathbf{A}_3) + \gamma_3 \mathbf{A}_2^2 + \gamma_4 (\mathbf{A}_2 \mathbf{A}_1^2 + \mathbf{A}_1^2 \mathbf{A}_2) \\
&\quad \gamma_5 (\mathrm{tr}\mathbf{A}_2)\mathbf{A}_2 + \gamma_6 (\mathrm{tr}\mathbf{A}_2)\mathbf{A}_1^2 + \left[\gamma_7 \mathrm{tr}\mathbf{A}_3 + \gamma_8 \mathrm{tr}(\mathbf{A}_2 \mathbf{A}_1)\right] \mathbf{A}_1. && (7.8)
\end{aligned}
$$

Here μ is the dynamic viscosity, α_i $(i = 1, 2)$, β_j $(j = 1, 2, 3)$ and γ_k $(k = 1, 2, ..., 8)$ are the material parameters. The Rivlin-Ericksen tensors $\mathbf{A}_1$ to $\mathbf{A}_4$ are defined by

$$
\begin{aligned}
\mathbf{A}_1 &= (\boldsymbol{\nabla}\mathbf{V}) + (\boldsymbol{\nabla}\mathbf{V})^T, && (7.9)\\
\mathbf{A}_n &= \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(\boldsymbol{\nabla}\mathbf{V}) \ + (\boldsymbol{\nabla}\mathbf{V})^T \mathbf{A}_{n-1}; \ (n > 1), && (7.10)
\end{aligned}
$$

in which $\boldsymbol{\nabla}$ is the gradient operator.

The substitution of Eq. (7.1) into Eqs. (7.4)-(7.8) yields the following non-zero components of the Cauchy stress tensor

$$\begin{aligned}
T_{xx} &= -p+\alpha_2\left(\frac{\partial u}{\partial y}\right)^2+2\beta_2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}+2\gamma_2\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y\partial t^2}+\gamma_3\left(\frac{\partial^2 u}{\partial y\partial t}\right)^2 \\
&\quad +2\gamma_6\left(\frac{\partial u}{\partial y}\right)^4, \qquad (7.11)\\
T_{xy} &= T_{yx}=\mu\frac{\partial u}{\partial y}+\alpha_1\frac{\partial^2 u}{\partial y\partial t}+\beta_1\frac{\partial^3 u}{\partial y\partial t^2}+2(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^3+\gamma_1\frac{\partial^4 u}{\partial y\partial t^3} \\
&\quad +2(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}, \qquad (7.12)\\
T_{yy} &= -p+(2\alpha_1+\alpha_2)\left(\frac{\partial u}{\partial y}\right)^2+(6\beta_1+\beta_2)\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t^2}+(4\gamma_4+4\gamma_5+2\gamma_6)\left(\frac{\partial u}{\partial y}\right)^4 \\
&\quad +\gamma_1\left[6\left(\frac{\partial^2 u}{\partial y\partial t}\right)^2+8\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y\partial t^2}\right]+2\gamma_2\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial y\partial t^2} \\
&\quad +\gamma_3\left[\left(\frac{\partial^2 u}{\partial y\partial t}\right)^2+4\left(\frac{\partial u}{\partial y}\right)^4\right]. \qquad (7.13)
\end{aligned}$$

To incorporate the effects of the pores on the velocity field, we have made use of generalized Darcy's law consistent with the thermodynamics stability of the flow. This law relates the pressure drop induced by the fractional drag and velocity and ignores the boundary effects of the flow. Since the pressure gradient can be interpreted as a measure of the flow resistance in the bulk of the porous medium and Darcy's resistance $\mathbf{R}$ is the measure of the resistance to the flow in the porous media. According to Darcy's law, we have

$$\mathbf{R}=\boldsymbol{\nabla}\mathbf{p}=-\frac{\phi}{\kappa}(\text{Apparent Viscosity})\mathbf{V}. \qquad (7.14)$$

where ϕ is the porosity and κ the permeability of the porous medium. For different non-Newtonian fluid flows the apparent viscosity is different. By making use of Eq. (12), the apparent viscosity for unsteady unidirectional flow of a fourth grade fluid over a rigid plate is calculated as

$$\text{Apparent Viscosity} = \frac{\text{Shear Stress}}{\text{Shear Rate}} = \mu + \alpha_1 \frac{\partial}{\partial t} + \beta_1 \frac{\partial^2}{\partial t^2} + 2(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y} \right)^2 + \gamma_1 \frac{\partial^3}{\partial t^3}$$
$$+2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8) \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y \partial t}. \tag{7.15}$$

With the use of Eq. (7.15) into Eq. (7.14), the x-component of $\mathbf{R}$ for the unidirectional flow over a rigid plate is given by

$$R_x = -\frac{\phi}{\kappa}[\mu + \alpha_1 \frac{\partial}{\partial t} + \beta_1 \frac{\partial^2}{\partial t^2} + 2(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y} \right)^2 + \gamma_1 \frac{\partial^3}{\partial t^3}$$
$$+2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8) \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y \partial t}]u. \tag{7.16}$$

Finally by making use of Eq. (7.1), Eq. (7.16) and Eqs. (7.11) – (7.13) into Eq. (7.3), one deduces the following governing equation in the absence of the modified pressure gradient

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + \beta_1 \frac{\partial^4 u}{\partial y^2 \partial t^2} + 6(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2}$$
$$+\gamma_1 \frac{\partial^5 u}{\partial y^2 \partial t^3} + 2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8) \frac{\partial}{\partial y} [\left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y \partial t}]$$
$$-[\mu u + \alpha_1 \frac{\partial u}{\partial t} + \beta_1 \frac{\partial^2 u}{\partial t^2} + 2(\beta_2 + \beta_3) u \left(\frac{\partial u}{\partial y} \right)^2 + \gamma_1 \frac{\partial^3 u}{\partial t^3}$$
$$+2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8) u \left(\frac{\partial u}{\partial y} \right) \left(\frac{\partial^2 u}{\partial y \partial t} \right)] \frac{\varphi}{\kappa}. \tag{7.17}$$

In order to solve Eq. (7.17), the relevant boundary and initial conditions are:

$$
\begin{aligned}
u(0,t) &= U_0 V(t), \qquad t>0, && (7.18)\\
u(\infty,t) &= 0, \qquad\quad t>0, && (7.19)\\
u(y,0) &= I(y), \qquad y>0, && (7.20)\\
\frac{\partial u(y,0)}{\partial t} &= J(y), \qquad y>0, && (7.21)\\
\frac{\partial^2 u(y,0)}{\partial t^2} &= K(y), \quad y>0, && (7.22)
\end{aligned}
$$

where U_0 is the reference velocity and $V(t)$, $I(y)$, $J(y)$, $K(y)$ are yet functions to be determined. The first boundary condition (7.18) is the no-slip condition and the second boundary condition (7.19) says that the main stream velocity is zero. This is not a restrictive assumption since we can always measure velocity relative to the main stream. The initial condition (7.20) indicate that initially the fluid is moving with some non-uniform velocity $I(y)$. The remaining two initial conditions are the extra two conditions imposed to make the problem well-posed.

We introduce the following dimensionless quantities

$$
\begin{aligned}
\bar{u} &= \frac{u}{U_0}, \quad \bar{y} = \frac{U_0 y}{\nu}, \quad \bar{t} = \frac{U_0^2 t}{\nu}, \quad \bar{\alpha} = \frac{\alpha_1 U_0^2}{\rho\nu^2}, \quad \bar{\beta}_1 = \frac{\beta_1 U_0^4}{\rho\nu^3}, \quad \bar{\beta} = \frac{2(\beta_2+\beta_3)U_0^4}{\rho\nu^3},\\
\bar{\gamma} &= \frac{\gamma_1 U_0^6}{\rho\nu^4}, \quad \bar{\Gamma} = (3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)\frac{U_0^6}{\rho\nu^4}, \quad \bar{\varphi} = \frac{\varphi\nu^2}{\kappa U_0^2}. \qquad (7.23)
\end{aligned}
$$

Under these transformations the governing equation (7) and the corresponding initial and the boundary conditions (7.18) – (7.22) take the form

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2\partial t} + \beta_1\frac{\partial^4 u}{\partial y^2\partial t^2} + 3\beta\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + \gamma\frac{\partial^5 u}{\partial y^2\partial t^3} + 2\Gamma\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right]\\
&\quad - \left[u + \alpha\frac{\partial u}{\partial t} + \beta_1\frac{\partial^2 u}{\partial t^2} + \beta u\left(\frac{\partial u}{\partial y}\right)^2 + \gamma\frac{\partial^3 u}{\partial t^3} + 2\Gamma u\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y\partial t}\right)\right]\varphi, \qquad (7.24)
\end{aligned}
$$

$$
\begin{aligned}
u(0,t) &= V(t), \qquad t>0, && (7.25)\\
u(\infty,t) &= 0, \qquad t>0, && (7.26)\\
u(y,0) &= f(y), \qquad y>0, && (7.27)\\
\frac{\partial u(y,0)}{\partial t} &= g(y), \qquad y>0, && (7.28)\\
\frac{\partial^2 u(y,0)}{\partial t^2} &= h(y), \qquad y>0, && (7.29)
\end{aligned}
$$

where $f(y)=I(y)/U_0$, $g(y)=J(y)/U_0$ and $h(y)=K(y)/U_0$. The functions $V(t)$, $f(y)$, $g(y)$ and $h(y)$ are as yet arbitrary.

For simplicity we have neglected the bars in all the non-dimensional quantities. Eq. (7.24) can be rewritten as

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \mu_*\frac{\partial^2 u}{\partial y^2}+\alpha_*\frac{\partial^3 u}{\partial y^2\partial t}+\beta\frac{\partial^4 u}{\partial y^2\partial t^2}+\beta_*\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}+\gamma_*\frac{\partial^5 u}{\partial y^2\partial t^3}+2\Gamma_*\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right]\\
&\quad -\kappa_1 u-\kappa_2\frac{\partial^2 u}{\partial t^2}-\kappa_3 u\left(\frac{\partial u}{\partial y}\right)^2-\kappa_4\frac{\partial^3 u}{\partial t^3}-\kappa_5 u\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y\partial t}\right), \qquad (7.30)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_* &= \frac{1}{(1+\alpha\varphi)}, \quad \alpha_*=\frac{\alpha}{(1+\alpha\varphi)}, \quad \beta=\frac{\beta_1}{(1+\alpha\varphi)},\\
\beta_* &= \frac{3\beta}{(1+\alpha\varphi)}, \quad \gamma_*=\frac{\gamma}{(1+\alpha\varphi)}, \quad \Gamma_*=\frac{\Gamma}{1+\alpha_1\phi},\\
\kappa_1 &= \frac{\varphi}{(1+\alpha\varphi)}, \quad \kappa_2=\frac{\beta_1\varphi}{(1+\alpha\varphi)}, \quad \kappa_3=\frac{\beta\varphi}{1+\alpha_1\phi},\\
\kappa_4 &= \frac{\gamma\varphi}{(1+\alpha\varphi)}, \quad \kappa_5=\frac{2\Gamma\varphi}{(1+\alpha\varphi)}. \qquad (7.31)
\end{aligned}
$$

7.3 Classical Lie symmetry analysis

In this section, we discuss how to determine Lie point symmetry generators admitted by PDE (7.30). We use these generators to obtain group invariant solutions.

We now look for transformations of the independent variables t, y and the dependent variable u of the form

$$\bar{t} = \bar{t}(t, y, u, \epsilon),\ \ \bar{y} = \bar{y}(t, y, u, \epsilon),\ \ \bar{u} = \bar{u}(t, y, u, \epsilon), \tag{7.32}$$

which from a group, where ϵ is the group parameter, and leave Eq. (7.30) invariant. From Lie's theory (see also chapter 3), the transformations in Eq. (7.32) are obtained in terms of the infinitesimal transformations

$$\bar{t} \simeq t + \epsilon\tau(t, y, u),\ \ \bar{y} \simeq y + \epsilon\xi(t, y, u),\ \ \bar{u} \simeq u + \epsilon\eta(t, y, u), \tag{7.33}$$

or the operator

$$X = \tau(t, y, u)\frac{\partial}{\partial t} + \xi(t, y, u)\frac{\partial}{\partial y} + \eta(t, y, u)\frac{\partial}{\partial u}, \tag{7.34}$$

which is a generator of Lie point symmetry of Eq. (7.30) if the following determining equation holds, viz.

$$X^{[5]}\,[\text{Eq. (7.30)}]\,|_{\text{Eq. (7.30)}=0} = 0. \tag{35}$$

The operator $X^{[5]}$ is the fifth prolongation of the operator X which is

$$X^{[5]} = X + \zeta^1\partial u_t + \zeta^2\partial u_y + \cdots + \zeta^{i_1 i_2\cdots i_5}\partial u_{i_1 i_2\cdots i_5}, \tag{7.36}$$

where $i_j,\ \ j = 1, 2, \cdots 5$, refer to the independent variables t, y.
and

$$\zeta^i \;=\; D_i(\eta) - (D_i\xi^j)u_j, \tag{7.37}$$

$$\zeta^{i_1 i_2\cdots i_k} \;=\; D_{i_k}(\eta_{i_1 i_2\cdots i_{k-1}}) - (D_{i_k}\xi^j)u_{i_1 i_2\cdots i_{k-1}j} \tag{7.38}$$

and the total derivative operators are

$$D_t \;=\; \partial_t + u_t\partial u + u_{tt}\partial u_t + u_{ty}\partial u_y + \cdots, \tag{7.39}$$

$$D_y \;=\; \partial_y + u_y\partial u + u_{yy}\partial u_y + u_{ty}\partial u_t + \cdots, \tag{7.40}$$

Substituting the expansions of Eq. (7.37) and Eq. (7.38) into the symmetry condition (7.35) and separating them by powers of the derivatives of u, as τ, ξ, and η are independent of the derivatives of u, lead to an over-determined system of linear homogeneous partial differential equations.

$$
\begin{aligned}
\tau_u &= \tau_y = 0, \\
\xi_u &= \xi_y = \xi_t = 0, \\
\eta_{uu} &= \eta_y = 0, \\
3\mu_*\tau_t + \alpha_*\eta_{tu} + \beta\eta_{ttu} + \gamma_*\eta_{ttut} &= 0, \\
-2\alpha_*\tau_t + \beta\tau_{tt} - 2\beta\eta_{tu} + \gamma_*\tau_{ttt} - 3\gamma_*\eta_{ttu} &= 0, \\
2\tau_t - \kappa_2\tau_{tt} + 2\kappa_2\eta_{tu} - \kappa_4\tau_{ttt} + 3\kappa_4\eta_{ttu} &= 0, \\
\kappa_1\eta + 3\kappa_1 u\tau_t + \eta_t - \kappa_1 u\eta_u + \kappa_2\eta_{tt} + \kappa_4\eta_{ttt} &= 0, \\
-\beta\tau_t + 3\gamma_*\tau_{tt} - 3\gamma_*\eta_{tu} &= 0, \\
\kappa_2\tau_t - 3\kappa_4\tau_{tt} + 3\kappa_4\eta_{tu} &= 0, \\
\kappa_3\eta + 3\kappa_3 u\tau_t + \kappa_3 u\eta_u + \kappa_5 u\eta_{tu} &= 0, \\
-3\beta_*\tau_t - 2\beta_*\eta_u - 3\Gamma_*\eta_{tu} &= 0, \\
\eta + 2u\tau_t + u\eta_u &= 0, \\
\tau_t + \eta_u &= 0. \qquad (7.41)
\end{aligned}
$$

By solving the above system (7.41), the symmetry Lie algebra of PDE (7.30) is three-dimensional and spanned by the generators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial y}, \qquad (7.42)$$

$$X_3 = -\exp\left(\frac{\kappa_3 t}{\kappa_5}\right)\frac{\partial}{\partial t} + \left(\frac{\kappa_3}{\kappa_5}\right)\exp\left(\frac{\kappa_3 t}{\kappa_5}\right)u\frac{\partial}{\partial u}, \qquad (7.43)$$

provided that

$$
\begin{aligned}
3\Gamma_* \left(\frac{\kappa_3}{\kappa_5}\right) - \beta_* &= 0, && (7.44)\\
6\kappa_4 \left(\frac{\kappa_3}{\kappa_5}\right) - \kappa_2 &= 0, && (7.45)\\
6\gamma_* \left(\frac{\kappa_3}{\kappa_5}\right) - \beta &= 0, && (7.46)\\
\kappa_4 \left(\frac{\kappa_3}{\kappa_5}\right)^3 + \kappa_2 \left(\frac{\kappa_3}{\kappa_5}\right)^2 + \left(\frac{\kappa_3}{\kappa_5}\right) - 3\kappa_1 &= 0, && (7.47)\\
4\kappa_4 \left(\frac{\kappa_3}{\kappa_5}\right)^2 + 3\kappa_2 \left(\frac{\kappa_3}{\kappa_5}\right) - 2 &= 0, && (7.48)\\
4\gamma_* \left(\frac{\kappa_3}{\kappa_5}\right)^2 + 3\beta \left(\frac{\kappa_3}{\kappa_5}\right) - 2\alpha_* &= 0, && (7.49)\\
\gamma_* \left(\frac{\kappa_3}{\kappa_5}\right)^3 + \beta \left(\frac{\kappa_3}{\kappa_5}\right)^2 + \alpha_* \left(\frac{\kappa_3}{\kappa_5}\right) - 3\mu_* &= 0. && (7.50)
\end{aligned}
$$

Here X_1 is translation in time, X_2 is translation in space and X_3 has path curves which are equivalent to a combination of translations in t and scaling in u.

7.4 Optimal system

In order to perform the symmetry reductions of Eq. (7.30) and to construct new classes of group invariant solutions in a systematic manner, we construct one-dimensional optimal system of subalgebras for the PDE (7.30). The method employed for constructing an optimal system of one-dimensional subalgebras is that given in [24]. This approach is based upon taking a general vector field from the Lie algebra and reducing it to its simplest equivalent form by applying carefully chosen adjoint transformations.

The Lie series utilized is given by

$$
\mathrm{Ad}(\exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] + \frac{1}{2}\varepsilon^2[X_i,[X_i,X_j]] - ..., \tag{7.51}
$$

where ε is a real constant and $[X_i, X_j]$ is the commutator bracket defined as

$$[X_i, X_j] = X_i(X_j) - X_j(X_i). \tag{7.52}$$

The commutation relations of the Lie point symmetries of PDE (7.30) and the adjoint representations of the symmetry group of (7.30) on its Lie algebra are given in Table 1 and Table 2 respectively. The Table 1 and Table 2 are then used to construct an optimal system of one-dimensional subalgebras for the symmetries of governing PDE (7.30).

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$\frac{\kappa_3}{\kappa_5}X_3$
X_2	0	0	0
X_3	$-\frac{\kappa_3}{\kappa_5}X_3$	0	0

Table 1: Commutator table of the Lie algebra of PDE (7.30).

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$\exp(-\varepsilon\frac{\kappa_3}{\kappa_5})X_3$
X_2	X_1	X_2	X_3
X_3	$X_1 + \varepsilon\frac{\kappa_3}{\kappa_5}X_3$	X_2	X_3

Table 2: Adjoint representation of the Lie algebra of PDE (7.30).

From Table 1 and 2, we find a one-dimensional optimal system of subalgebras corresponding to the above Lie algebra of generators. These are

$$\chi_1 = \langle\ X_3\rangle = \langle -\exp\left(\frac{\kappa_3 t}{\kappa_5}\right)\frac{\partial}{\partial t} + \left(\frac{\kappa_3}{\kappa_5}\right)\exp\left(\frac{\kappa_3 t}{\kappa_5}\right)u\frac{\partial}{\partial u}\rangle, \tag{7.53}$$

$$\chi_2 = \langle X_3 -\ X_2\rangle = \langle -\exp\left(\frac{\kappa_3 t}{\kappa_5}\right)\frac{\partial}{\partial t} + \left(\frac{\kappa_3}{\kappa_5}\right)\exp\left(\frac{\kappa_3 t}{\kappa_5}\right)u\frac{\partial}{\partial u} - \frac{\partial}{\partial y}\rangle, \tag{7.54}$$

$$\chi_3 = \langle X_3 +\ X_2\rangle = \langle -\exp\left(\frac{\kappa_3 t}{\kappa_5}\right)\frac{\partial}{\partial t} + \left(\frac{\kappa_3}{\kappa_5}\right)\exp\left(\frac{\kappa_3 t}{\kappa_5}\right)u\frac{\partial}{\partial u} + \frac{\partial}{\partial y}\rangle, \tag{7.55}$$

$$\chi_4 = \langle X_2\rangle = \langle\frac{\partial}{\partial y}\rangle, \tag{7.56}$$

$$\chi_5 = \langle X_1 + mX_2\rangle = \langle\frac{\partial}{\partial t} + m\frac{\partial}{\partial y}\rangle, \quad \text{where } m \text{ is an arbitrary constant.} \tag{7.57}$$

Here we have three nontrivial and non-translational generators in the above optimal system (7.53)-(7.57). We find the group invariant solutions of the above optimal system of subalgebras. The group invariant approach for the subalgebras $\langle X_3 \pm\ X_2\rangle$ and $\langle X_2\rangle$ is not meaningful for the solution of problem considered as they do not leave the boundary conditions invariant.

7.5 Group invariant solutions

In this section we use the one-dimensional optimal system of subalgebras calculated above to construct the group invariant solutions. The group invariant solutions are obtained by solving the Langrangian equations for the invariants of the generators that give physically meaningful exact solutions.

7.5.1 Invariant solution via subgroup generated by $\chi_1 = \langle X_3\rangle$

We consider the invariant solutions under

$$\chi_1 = -\exp\left(\frac{\kappa_3 t}{\kappa_5}\right)\frac{\partial}{\partial t} + \left(\frac{\kappa_3}{\kappa_5}\right)\exp\left(\frac{\kappa_3 t}{\kappa_5}\right)u\frac{\partial}{\partial u}. \tag{7.58}$$

By solving the corresponding Langrangian system of Eq. (7.58), the invariant solution form is given as

$$u(y,t) = F(y)\exp\left(-\frac{\kappa_3}{\kappa_5}t\right), \tag{7.59}$$

where $F(y)$ is as yet an undetermined function. Now inserting Eq. (7.59) into Eq. (7.30) yields a second-order linear ODE in $F(y)$, viz.

$$\frac{d^2F}{dy^2} - \left[\frac{\left(\kappa_1 - \frac{\kappa_2\beta_*^2}{9\Gamma_*^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}\right]F = 0, \text{ with } \mu_* \neq \frac{\beta^3}{36\gamma_*^2}. \tag{7.60}$$

The reduced ODE (7.60) is solved subject to the boundary conditions

$$F(0) = 1, \quad F(y) \to 0 \text{ as } y \to \infty, \tag{7.61}$$

provided that

$$V(t) = \exp\left(-\left(\frac{\kappa_3}{\kappa_5}\right)t\right). \tag{7.62}$$

We solve Eq. (7.60) subject to the boundary conditions (7.61) for the three different cases.

Case 1: $\frac{\left(\kappa_1 - \frac{\kappa_2\beta_*^2}{9\Gamma_*^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)} = 0.$

Then we have

$$\frac{d^2F}{dy^2} = 0. \tag{7.63}$$

The solution of (7.63) subject to the boundary conditions (7.61) is

$$F(y) = 1. \tag{7.64}$$

In this case, the solution for $u(y,t)$ is written as

$$u(t) = \exp\left(-\frac{\kappa_3}{\kappa_5}t\right). \tag{7.65}$$

Thus a time-dependent solution is obtained in this case. The velocity decreases exponentially with time at a rate which depends only on κ_3 and κ_5. From Eq. (7.65) it is clear that the fluid velocity decreases as κ_3 increases and increases as the parameter κ_5 increases. Now we have

$$\frac{\kappa_3}{\kappa_5} = \frac{\beta}{2\Gamma}, \tag{7.66}$$

Making use of Eq. (7.66) into Eq. (7.65), the time-dependent solution $u(t)$ can be expressed in dimensionless variables as

$$u(t) = \exp\left(-\frac{\beta}{2\Gamma}t\right). \tag{7.67}$$

Case 2: $\dfrac{\left(\kappa_1 - \frac{\kappa_2\beta_*^2}{9\Gamma_*^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)} < 0.$

In this case the solution for $F(y)$ is written as

$$F(y) = a\cos\left[\sqrt{-\frac{\left(\kappa_1 - \frac{\kappa_2\beta_*^2}{9\Gamma_*^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}}\,y\right] + b\sin\left[\sqrt{-\frac{\left(\kappa_1 - \frac{\kappa_2\beta_*^2}{9\Gamma_*^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}}\,y\right]. \tag{7.68}$$

The solution given in Eq. (7.68) is not a meaningful solution as $F(y) \to \infty$ when $y \to \infty$. Therefore, the boundary condition $F(\infty) = 0$ is not satisfied in this case. Thus the solution for $u(y, t)$ does not exist in this case.

Case 3: $\dfrac{\left(\kappa_1 - \frac{\kappa_2\beta_*^2}{9\Gamma_*^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)} > 0.$

Here the solution for $F(y)$ becomes

$$F(y) = a\exp\left[\left\{\sqrt{\frac{\left(\kappa_1 - \frac{\kappa_2\beta_*^2}{9\Gamma_*^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}}\right\}y\right] + b\exp\left[-\left\{\sqrt{\frac{\left(\kappa_1 - \frac{\kappa_2\beta_*^2}{9\Gamma_*^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}}\right\}y\right]. \tag{7.69}$$

By applying the boundary conditions (7.61), we have $a = 0$ (for a bounded solution) and $b = 1$ (by using first the boundary condition). Thus the solution for $F(y)$ is

$$F(y) = \exp\left[-\left\{\sqrt{\frac{\left(\kappa_1 - \frac{\kappa_2\beta_*^2}{9\Gamma_*^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}}\right\} y\right]. \tag{7.70}$$

Substituting $F(y)$ into Eq. (7.59) results in the solution for $u(y,t)$ of the form

$$u(y,t) = \exp\left[-\left\{\left(\frac{\kappa_3}{\kappa_5}\right) t + \sqrt{\frac{\left(\kappa_1 - \frac{\kappa_2\beta_*^2}{9\Gamma_*^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}} y\right\}\right]. \tag{7.71}$$

We observe that this solution (7.71) satisfies the initial and the boundary conditions (7.25) – (7.29) with

$$\begin{aligned}
u(0,t) &= V(t) = \exp\left(-\left(\frac{\kappa_3}{\kappa_5}\right) t\right), && (7.72)\\
u(y,0) &= f(y) = \exp\left(-\sqrt{\frac{\left(\kappa_1 - \frac{\kappa_2\beta_*^2}{9\Gamma_*^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}} y\right), && (7.73)\\
\frac{\partial u(y,0)}{\partial t} &= g(y) = -\left(\frac{\kappa_3}{\kappa_5}\right) \exp\left(-\sqrt{\frac{\left(\kappa_1 - \frac{\kappa_2\beta_*^2}{9\Gamma_*^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}} y\right), && (7.74)\\
\frac{\partial^2 u(y,0)}{\partial t^2} &= h(y) = \left(\frac{\kappa_3}{\kappa_5}\right)^2 \exp\left(-\sqrt{\frac{\left(\kappa_1 - \frac{\kappa_2\beta_*^2}{9\Gamma_*^2}\right)}{\left(\mu_* - \frac{\beta^3}{36\gamma_*^2}\right)}} y\right), && (7.75)
\end{aligned}$$

where $V(t)$, $f(y)$, $g(y)$ and $h(y)$ depend on physical parameters of the flow. The boundary condition (7.72) is the no-slip condition at $y = 0$. The initial velocity $V(0)$ of the rigid plate can be prescribed but its velocity for $t > 0$, $V(t)$, cannot be arbitrary and is given by (7.73). Similarly the initial velocity profile $u(y,0) = f(y)$ cannot be arbitrary and is given by (7.74).

This solution in Eq. (7.71) is plotted in Figures 7.1 – 7.9 for various values of the emerging parameters.

7.5.2 Invariant solution via subgroup generated by $\chi_5 = \langle X_1 + mX_2 \rangle$ with $m = 0$

Here we consider the invariant solutions under

$$\chi_5 = \frac{\partial}{\partial t}. \tag{7.76}$$

The invariant solution admitted by χ_5 is the steady-state solution

$$u(y,t) = H(y). \tag{7.77}$$

Introducing Eq. (7.77) into Eq. (7.30) yields the second-order ordinary differential equation for $H(y)$, viz.

$$\mu_* \frac{d^2H}{dy^2} + \beta_* \left(\frac{dH}{dy}\right)^2 \frac{d^2H}{dy^2} - \kappa_3 H \left(\frac{dH}{dy}\right)^2 - \kappa_1 H = 0, \tag{7.78}$$

with the boundary conditions

$$\begin{aligned} H(0) &= 1, \\ H(y) &\to 0, \quad y \to \infty, \end{aligned} \tag{7.79}$$

In order to solve Eq. (7.78) for $H(y)$, we assume a solution of the form

$$H(y) = p\exp(qy), \tag{7.80}$$

where p and q are the constants to be determined. With the substitution of Eq. (7.80) into Eq. (7.78) results in

$$(\mu_* q^2 - \kappa_1) + e^{2qy}(\beta_* p^2 q^4 - \kappa_3 p^2 q^2) = 0. \tag{7.81}$$

Separating Eq. (7.81) in the powers of e^0 and e^{2qy}, we deduce

$$e^0 \quad : \quad \mu_* q^2 - \kappa_1 = 0, \tag{7.82}$$

$$e^{2qy} \quad : \quad \beta_* p^2 q^4 - \kappa_3 p^2 q^2 = 0. \tag{7.83}$$

From Eq. (7.83), we find

$$q = \pm\sqrt{\frac{\kappa_3}{\beta_*}}. \tag{7.84}$$

We choose

$$q = \sqrt{\frac{\kappa_3}{\beta_*}}, \tag{7.85}$$

so that our solution would satisfy the second boundary condition at infinity. Using the value of q in Eq. (7.82), we obtain

$$\mu_* \left(\frac{\kappa_3}{\beta_*}\right) - \kappa_1 = 0. \tag{7.86}$$

Thus the solution for $H(y)$ (which satisfies the condition (7.86)) is written as

$$u = H(y) = \exp\left(-\sqrt{\frac{\kappa_3}{\beta_*}}y\right), \tag{7.87}$$

7.5.3 Invariant solution via subgroup generated by $\chi_5 = \langle X_1 + mX_2\rangle$ with $m > 0$

We now search for the invariant solution under the operator

$$\chi_5 = \langle X_1 + mX_2\rangle = \frac{\partial}{\partial t} + m\frac{\partial}{\partial y}\ , \ (m > 0) \tag{7.88}$$

which denotes wave-front type travelling wave solutions with constant wave speed m. The invariant is given as

$$u(y,t) = G(\eta), \quad \text{where} \;\; \eta = y - mt \; . \tag{7.89}$$

Using Eq. (7.89) into Eq. (7.30) results in a fifth-order ordinary differential equation for $G(\eta)$,

$$\begin{aligned} 0 \;=\; & m\frac{dG}{d\eta} + \mu_*\frac{d^2G}{d\eta^2} - \alpha_* m\frac{d^3G}{d\eta^3} + \beta m^2\frac{d^4G}{d\eta^4} + \beta_*\left(\frac{dG}{d\eta}\right)^2\frac{d^2G}{d\eta^2} \\ & -\gamma_* m^3\frac{d^5G}{d\eta^5} - 2\Gamma_* m\left(\frac{dG}{d\eta}\right)^2\frac{d^3G}{d\eta^3} - 4\Gamma_* m\frac{dG}{d\eta}\left(\frac{d^2G}{d\eta^2}\right)^2 \\ & -\kappa_1 G - \kappa_2 m^2\frac{d^2G}{d\eta^2} - \kappa_3 G\left(\frac{dG}{d\eta}\right)^2 + \kappa_4 m^3\frac{d^3G}{d\eta^3} \\ & +\kappa_5 mG\left(\frac{dG}{d\eta}\right)\frac{d^2G}{d\eta^2}. \end{aligned} \tag{7.90}$$

with the transformed boundary conditions

$$\begin{aligned} G(0) &= l_1, \;\text{where } l_1 \text{ is a constant,} \\ G(\eta) &\rightarrow 0 \;\;\text{as}\;\; \eta\rightarrow\infty, \\ \frac{d^nG}{d\eta^n} &\rightarrow 0 \;\;\text{as}\;\; \eta\rightarrow\infty, \quad \text{for} \;\; n = 1,2,3. \end{aligned} \tag{7.91}$$

The above boundary value problem is nonlinear and complicated in nature and cannot be solved exactly.

7.6 Graphical results and discussion

In order to examine and understand the behavior of various interesting physical parameters of the flow problem, the group invariant solution (7.71) is plotted in Figures 7.1 – 7.9.

Figure 7.1 is plotted to show the influence of time t on the invariant solution (7.71). This figure depicts that velocity decreases as time increases. Clearly, the variation of velocity is observed for $0 \leqslant t \leqslant 3$. For $t > 3$, the velocity profile remains the same. In other words, we can say that the steady-state behavior of the velocity is achieved for $t > 3$.

In Figures 7.2 and 7.3, the variation of the parameters κ_1 and κ_2 on the velocity profile is shown. Clearly from the figures it is observed that with an increase in κ_1 the velocity field decreases whereas it increases for increasing κ_2.

Figures 7.4 and 7.5 are plotted to describe the influence of fluid parameter β and β_* on the velocity field. From both of these figures, it is quite obvious that both these parameters have an opposite behavior on the velocity field. That is, with an increase in parameter β, the velocity profile decreases and with an increase in parameter β_* the velocity profile increases.

The effects of the parameters κ_3 and κ_5 on the velocity profile (7.71) are presented in Figures 7.6 and 7.7. Figure 7.6 depicts that the velocity decreases with an increase in κ_3. The effect of κ_5 is opposite to that of κ_3 and is given in Figure 7.7.

Finally Figures 7.8 and 7.9 have been plotted to see the influence of the fourth grade parameter γ_* and Γ_* on the solution (7.71) of the flow problem. These figures reveal that both γ_* and Γ_* have an opposite role on the structure of the velocity, that is, with an increase in the parameter Γ_*, the velocity profile decreases showing the shear thickening behavior of the fluid. On the other hand the velocity field increases for increasing values of the fourth grade parameter γ_*, which shows the shear thinning behavior of the model. This is in accordance with the fact that a fourth grade fluid model predicts both shear thickening and the shear thinning properties of the flow.

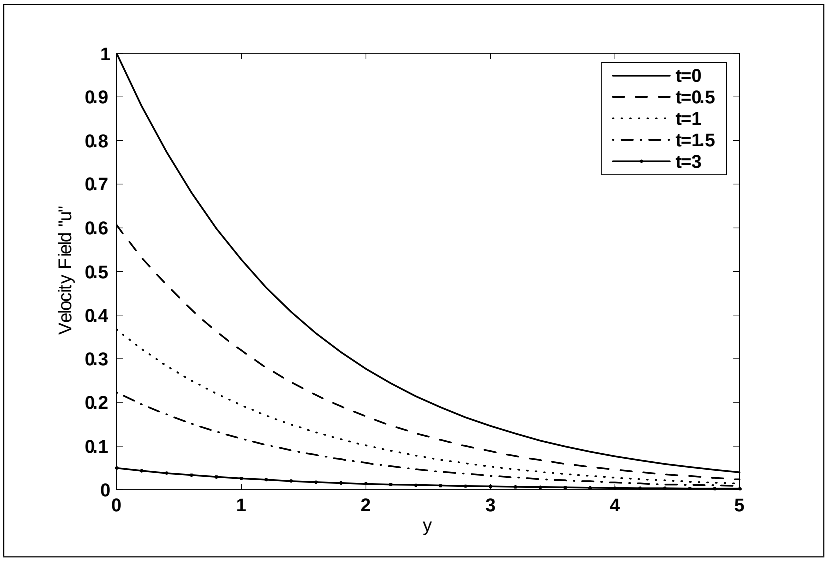

Figure 7.1: Analytical solution (7.71) varying t, with $\kappa_1 = 1.4$, $\kappa_2 = 1$, $\kappa_3 = 1.5$, $\kappa_5 = 1.5$, $\beta_* = 1.5$, $\beta = 1$, $\gamma_* = 1$, $\Gamma_* = 0.5$ and $\mu_* = 1$ are fixed.

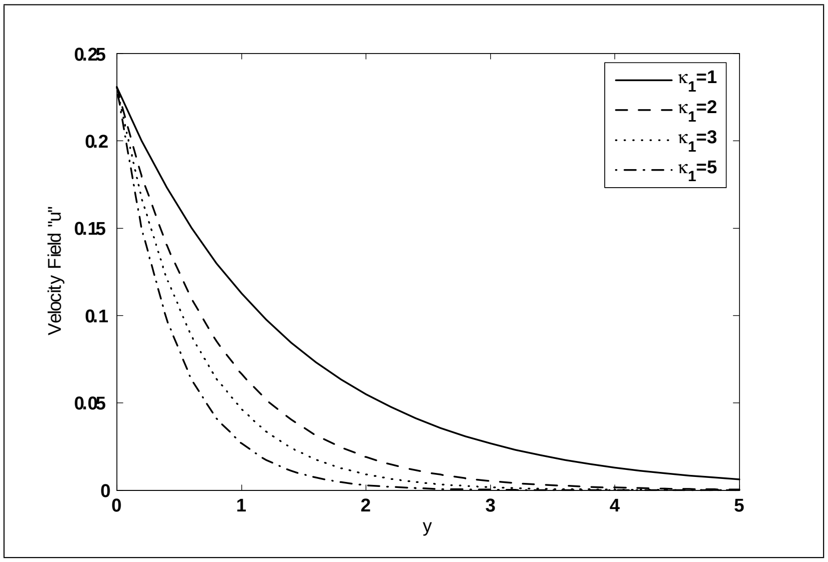

Figure 7.2: Analytical solution (7.71) varying κ_1, with $\kappa_2 = 0.5$, $\kappa_3 = 1.4$, $\kappa_5 = 1.5$, $\beta_* = 1.5$, $\beta = 1$, $\gamma_* = 1$, $\Gamma_* = 0.5$, $\mu_* = 1$ and $t = \pi/2$ are fixed.

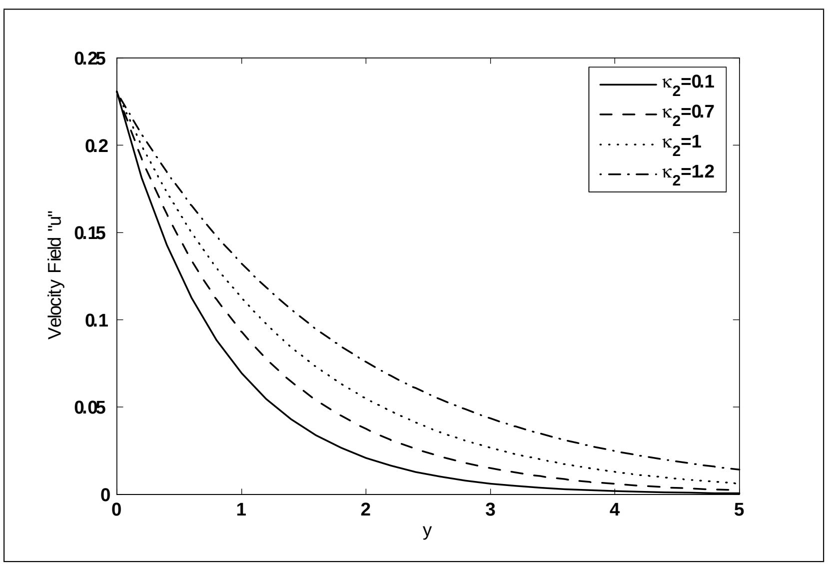

Figure 7.3: Analytical solution (7.71) varying κ_2, with $\kappa_1 = 1.5$, $\kappa_3 = 1.4$, $\kappa_5 = 1.5$, $\beta_* = 1.5$, $\beta = 1$, $\gamma_* = 1$, $\Gamma_* = 0.5$, $\mu_* = 1$ and $t = \pi/2$ are fixed.

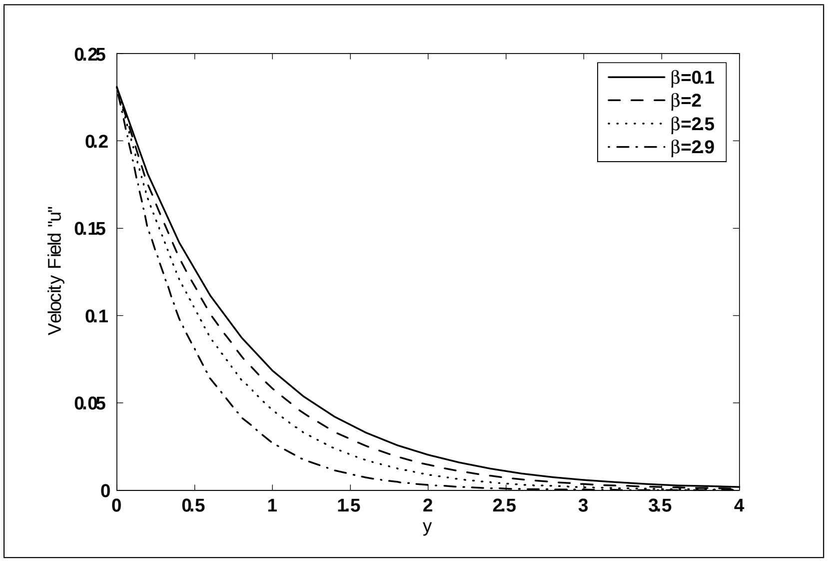

Figure 7.4: Analytical solution (7.71) varying β, with $\kappa_1 = 1.5$, $\kappa_2 = 0.5$, $\kappa_3 = 1.4$, $\kappa_5 = 1.5$, $\beta_* = 1$, $\gamma_* = 1$, $\Gamma_* = 1.5$, $\mu_* = 1$ and $t = \pi/2$ are fixed.

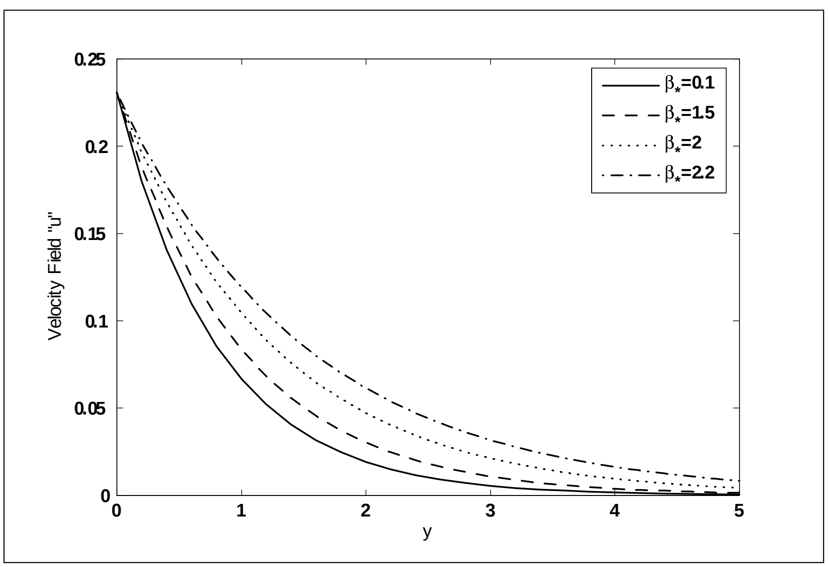

Figure 7.5: Analytical solution (7.71) varying β_*, with $\kappa_1 = 1.5$, $\kappa_2 = 0.5$, $\kappa_3 = 1.4$, $\kappa_5 = 1.5$, $\beta = 1$, $\gamma_* = 1$, $\Gamma_* = 0.5$, $\mu_* = 1$ and $t = \pi/2$ are fixed.

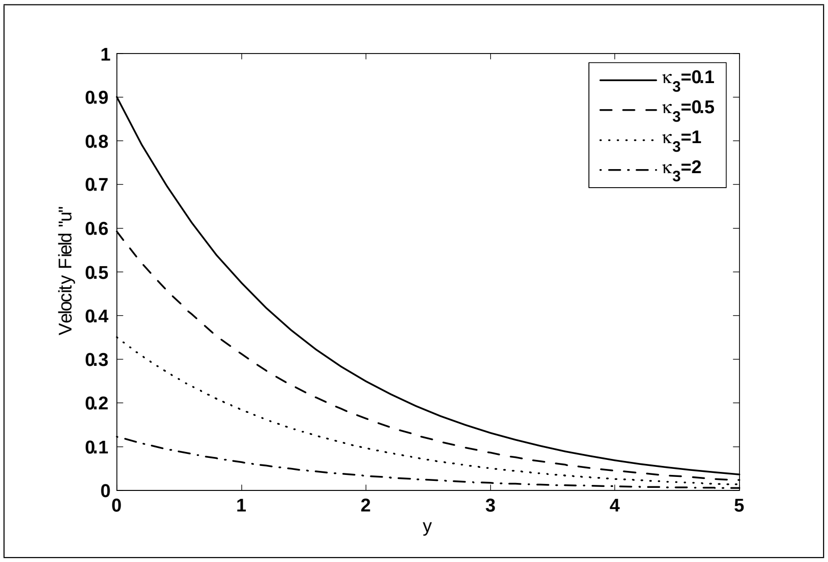

Figure 7.6: Analytical solution (7.71) varying κ_3, with $\kappa_2 = 1$, $\kappa_1 = 1.4$, $\kappa_5 = 1.5$, $\beta_* = 1.5$, $\beta = 1$, $\gamma_* = 1$, $\Gamma_* = 0.5$, $\mu_* = 1$ and $t = \pi/2$ are fixed.

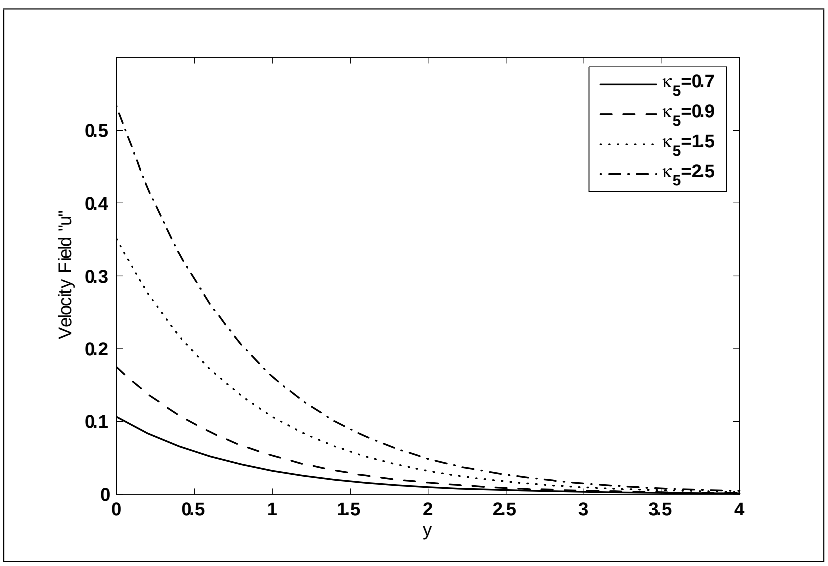

Figure 7.7: Analytical solution (7.71) varying κ_5, with $\kappa_2 = 1$, $\kappa_1 = 1.4$, $\kappa_3 = 1$, $\beta_* = 1.5$, $\beta = 1.5$, $\gamma_* = 1$, $\Gamma_* = 1.5$, $\mu_* = 1$ and $t = \pi/2$ are fixed.

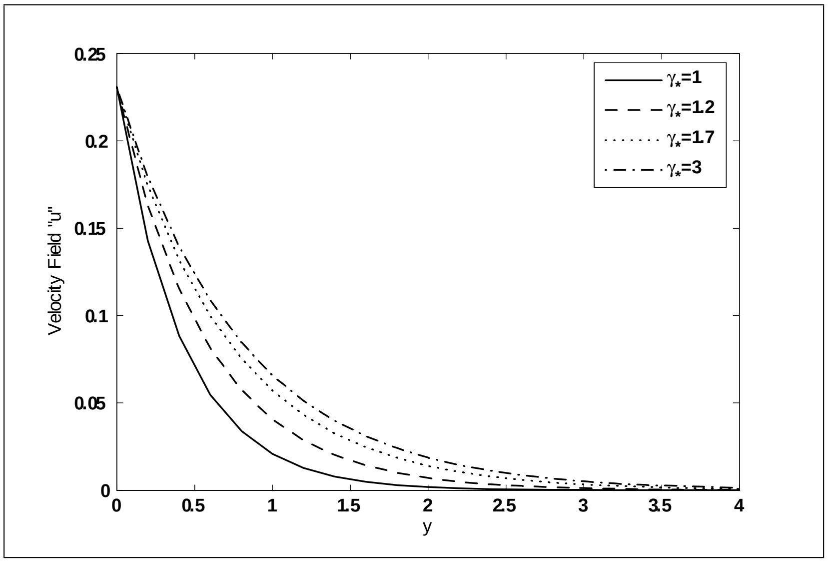

Figure 7.8: Analytical solution (7.71) varying γ_*, with $\kappa_1 = 1.5$, $\kappa_2 = 1.5$, $\kappa_3 = 1.4$, $\kappa_5 = 1.5$, $\beta_* = 1$, $\beta = 3$, $\Gamma_* = 2.5$, $\mu_* = 1$ and $t = \pi/2$ are fixed.

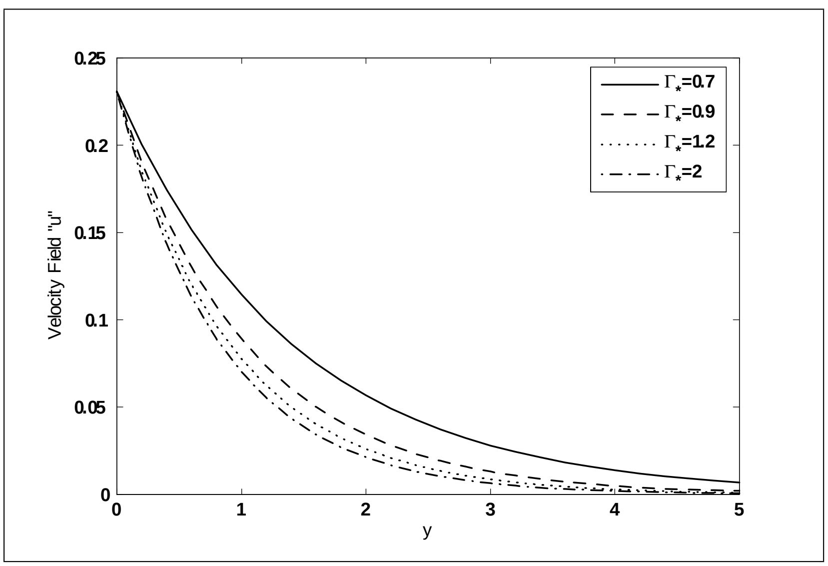

Figure 7.9: Analytical solution (7.71) varying Γ_*, with $\kappa_1 = 1.5$, $\kappa_2 = 1.5$, $\kappa_3 = 1.4$, $\kappa_5 = 0.5$, $\beta_* = 3$, $\beta = 1$, $\gamma_* = 1$, $\mu_* = 1$ and $t = \pi/2$ are fixed.

7.7 Conclusions

The present study aims to discuss the flow problem of an incompressible fourth grade fluid through a porous space solved earlier by Hayat et al. [51]. The authors in [51] have provided the travelling wave solutions and the conditional symmetry solution for the problem dealing with the unsteady flow of a fourth grade fluid over a moved rigid plate. The flow is caused by arbitrary plate velocity $V(t)$. The fluid occupies the porous half space $y > 0$. The model equation in [51] is incorrect as it contains an extra factor u in the second last and the fourth last terms on the right hand side of the equation. As a consequence of this error, the exact closed-form solutions constructed in [51] need revisiting. Due to these facts the physical behavior of these solutions is unrealistic. In the present study, we focus on this problem once again. We have presented the correct form of the governing nonlinear partial differential equation. We have found all possible Lie point symmetries of the model equation. Then we obtained one-dimensional optimal systems of subalgebras which give all possibilities for classifying meaningful solutions in using the Lie group approach. With the help of these symmetries, a new class of group invariant solutions have been reported. The manner in which various flow parameters affect the structure of the velocity has been observed through the aid of several graphs.

The physical mechanism in the problem is diffusion. Fluid velocity generated by the no-slip boundary condition when the plate is impulsively set in motion diffuses in the direction towards the axis of the flow. This causes the velocity profiles to flatten out and the shear stress across the medium to steadily decrease and vanish as $t \to \infty$. The model has some features in common with Stokes$'$ first problem for flow induced in a half-space of viscous fluid when a plate is impulsively set in motion. However, the present study can be described as a generalized Stokes$'$ flow for which the plate is impulsively set into motion with some time-dependent velocity $V(t)$ which can not be prescribed arbitrary but depends on the physical parameters of the flow.

The study considered is prototype one and theoretical in nature but the methodology adopted is going to be very useful to study wide range of complicated nonlinear problems not only in non-Newtonian fluid mechanics but also in the other fields of science and engineering.

Chapter 8

A Unified Compatibility Approach to Construct Exact Solutions of Nonlinear Flow Models of Newtonian and Non-Newtonian Fluids

8.1 Introduction

In this chapter, the compatibility criteria are established for higher order ordinary differential equations to be compatible with lower order ordinary differential equations. Necessary and sufficient compatibility conditions are derived which can be used to construct exact solutions of higher order ordinary differential equations. Using this approach new exact solutions of nonlinear flow problems arising in the study of Newtonian and non-Newtonian fluids are derived. The ansatz approach for obtaining exact solutions for nonlinear flow models of Newtonian and non-Newtonian fluids is unified with the application of the compatibility criteria.

8.2 Compatibility approach

It is often difficult to obtain exact solutions of a higher order nonlinear differential equation. For this reason, many researchers have assumed a form of the exact solution by trial and error. We provide a general compatibility criteria for many of these higher order ordinary differential equations arising in the study of Newtonian and non-Newtonian fluid models. This compatibility approach leads to some new exact solutions of these models and is also helpful in reproducing some existing solutions.

Firstly, we present a precise definition of compatibility:

Definition 8 *Consider the* n*th-order ordinary differential equation*

$$y^{(n)} = P(x, y, y^{(1)}, y^{(2)}, ..., y^{(n-1)}), \quad n \geq 2, \tag{8.1}$$

where x *is the independent variable,* y *the dependent variable and* $y^{(1)}, y^{(2)}, ..., y^{(n)}$ *denote the first, second,* $...,$ n*th derivative of* y *with respect to* x*. If every solution of the* m*th-order ordinary differential equation*

$$y^{(m)} = Q(x, y, y^{(1)}, y^{(2)}, ..., y^{(m-1)}), \quad m < n, \tag{8.2}$$

is also a solution of the n*th-order ordinary differential equation (8.1), then the* m*th-order ordinary differential equation (2) is said to be compatible with the* n*th-order ordinary differential equation (1).*

We will consider the compatibility of a nth-order ($n \geq 2$) ordinary differential equation with a first order ordinary differential equation so that $m = 1$.

8.2.1 Compatibility criterion for a fifth order ordinary differential equation

Here we develop a compatibility criterion or compatibility test for a fifth order ordinary differential equation to be compatible with a first order ordinary differential equation.

Let us consider a fifth-order ordinary differential equation in one independent variable x and one dependent variable y,

$$F(x, y, y^{(1)}, y^{(2)}, ..., y^{(5)}) = 0, \tag{8.3}$$

and a first order ordinary differential equation

$$E(x, y, y^{(1)}) = 0, \tag{8.4}$$

such that

$$J = \frac{\partial[E, F]}{\partial[y^{(1)}, y^{(2)}, ..., y^{(5)}]} \neq 0. \tag{8.5}$$

Then, we can solve for the highest derivatives as

$$y^{(5)} = f(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}), \tag{8.6}$$

and

$$y^{(1)} = e(x, y), \tag{8.7}$$

where f and e are smooth and continuously differentiable functions of x, y and, in the case of f, the derivatives of y.

Now Eq. (8.6) depends on $y^{(1)}, ..., y^{(5)}$ which are obtained by differentiating Eq. (8.7). This gives

$$y^{(2)} = e_x + ee_y, \tag{8.8}$$

$$y^{(3)} = e_{xx} + 2ee_{xy} + e^2 e_{yy} + e_x e_y + ee_y^2, \tag{8.9}$$

$$
\begin{aligned}
y^{(4)} &= e_{xxx} + 3ee_{xxy} + 3e^2e_{xyy} + e^3e_{yyy} + 3ee_{yy}(e_x + ee_y) \\
&\quad +3e_{xy}(e_x + ee_y) + e_y(e_{xx} + 2ee_{xy} + e^2e_{yy} + e_xe_y + ee_y^2)
\end{aligned}
\tag{8.10}
$$

$$
\begin{aligned}
y^{(5)} &= e_{xxxx} + 4ee_{xxxy} + 6e^2e_{xxyy} + 4e^3e_{xyyy} + e^4e_{yyyy} + 6e^2e_{yyy}\{e_x + ee_y\} \\
&\quad +12ee_{xyy}\{e_x + ee_y\} + 4ee_{yy}\{e_{xx} + 2ee_{xy} + e^2e_{yy} + e_xe_y + ee_y^2\} \\
&\quad +5e_{xxy}\{e_x + ee_y\} + 4e_{xy}\{e_{xx} + 2ee_{xy} + e^2e_{yy} + e_xe_y + ee_y^2\} \\
&\quad +e_y[e_{xxx} + 3ee_{xxy} + 3e^2e_{xyy} + e^3e_{yyy} + 3ee_{yy}\{e_x + ee_y\} \\
&\quad +3e_{xy}\{e_x + ee_y\} + e_y\{e_{xx} + 2ee_{xy} + e^2e_{yy} + e_xe_y + ee_y^2\}] + 3e^2e_{yy}.
\end{aligned}
\tag{8.11}
$$

By equating the right hand side of Eq. (8.6) with Eq. (8.11), we obtain

$$
\begin{aligned}
& f[x, y, e, (e_x + ee_y), \left(e_{xx} + 2ee_{xy} + e^2e_{yy} + e_xe_y + ee_y^2\right), \\
& (e_{xxx} + 3ee_{xxy} + 3e^2e_{xyy} + e^3e_{yyy} + 3ee_{yy}(e_x + ee_y) \\
& +3e_{xy}(e_x + ee_y) + e_y(e_{xx} + 2ee_{xy} + e^2e_{yy} + e_xe_y + ee_y^2))] \\
= \; & e_{xxxx} + 4ee_{xxxy} + 6e^2e_{xxyy} + 4e^3e_{xyyy} + e^4e_{yyyy} + 6e^2e_{yyy}\{e_x + ee_y\} \\
& +12ee_{xyy}\{e_x + ee_y\} + 4ee_{yy}\{e_{xx} + 2ee_{xy} + e^2e_{yy} + e_xe_y + ee_y^2\} \\
& +5e_{xxy}\{e_x + ee_y\} + 4e_{xy}\{e_{xx} + 2ee_{xy} + e^2e_{yy} + e_xe_y + ee_y^2\} \\
& +e_y[e_{xxx} + 3ee_{xxy} + 3e^2e_{xyy} + e^3e_{yyy} + 3ee_{yy}\{e_x + ee_y\} \\
& +3e_{xy}\{e_x + ee_y\} + e_y\{e_{xx} + 2ee_{xy} + e^2e_{yy} + e_xe_y + ee_y^2\}] \\
& +3e^2e_{yy},
\end{aligned}
\tag{8.12}
$$

which gives the general compatibility criterion or compatibility test for a fifth order ordinary differential equation to be compatible with a first order ordinary differential equation.

8.2.2 Compatibility criterion for lower order ordinary differential equations

Equations (8.8), (8.9) and (8.10) can be used to derive compatibility criteria for lower order ordinary differential equations.

For a fourth order ordinary differential equation to be compatible with a first order ordinary differential equation, the compatibility test is

$$\begin{aligned} & f[x, y, e, (e_x + ee_y), (e_{xx} + 2ee_{xy} + e^2 e_{yy} + e_x e_y + ee_y^2)] \\ = \; & e_{xxx} + 3ee_{xxy} + 3e^2 e_{xyy} + e^3 e_{yyy} + 3ee_{yy}\{e_x + ee_y\} \\ & +3e_{xy}\{e_x + ee_y\} + e_y\{e_{xx} + 2ee_{xy} + e^2 e_{yy} + e_x e_y + ee_y^2\}. \end{aligned} \tag{8.14}$$

For a third order ordinary differential equation to be compatible with a first order ordinary differential equation, the compatibility criterion takes the form

$$f[x, y, e, (e_x + ee_y)] = e_{xx} + 2ee_{xy} + e^2 e_{yy} + e_x e_y + ee_y^2. \tag{8.15}$$

Finally, for a second order ordinary differential equation to be compatible with a first order ordinary differential equation, the compatibility test becomes

$$f[x, y, e] = e_x + ee_y. \tag{8.16}$$

8.3 Applications of the compatibility approach

In order to demonstrate the application and utility of the compatibility method we apply it in this section to various research problems taken from the literature dealing with flow models for Newtonian and non-Newtonian fluids. Using this approach, some new solutions are constructed and some existing solutions are re-derived.

8.3.1 Viscous flow due to a shrinking sheet

Miklavcic and Wang [52] were the first to investigate the viscous flow due to a shrinking sheet under Prandtl's boundary layer assumptions. The existence and uniqueness of the exact solutions are elegantly proved in [52]. Here, we construct a new exact (closed-from) solution of the boundary layer problem of shrinking sheet by employing the compatibility criterion.

Consider a steady laminar flow over a continuously shrinking sheet in a quiescent fluid. The x-axis runs along the shrinking surface in the direction opposite to the sheet motion and the y-axis is perpendicular to it. Let (u, v, w) be the velocity components along the (x, y, z) directions, respectively. The boundary conditions on the shrinking surface are [52]

$$u = -ax, \;\; v = -a(m-1)y, \;\; w = -W, \tag{8.17}$$

and far away from the sheet

$$u \to 0, \quad v \to 0 \;\; \text{as} \;\; z \to \infty, \tag{8.18}$$

where a (> 0) and W are the shrinking constant and the suction velocity.
The stream function and the similarity variable can be posited in the following form

$$\psi = \sqrt{a\nu}\, x f(\eta) \;\; \text{and} \;\; \eta = \sqrt{\frac{a}{\nu}}\, z. \tag{8.19}$$

With these definitions, the velocities are expressed as

$$u = ax\frac{df}{d\eta}, \;\; v = a(m-1)y\frac{df}{d\eta}, \;\; w = -\sqrt{a\nu}\, m f(\eta). \tag{8.20}$$

Using Eqs. (8.17) – (8.20), the governing problem describing the viscous flow due to a shrinking sheet in a non-dimensionalized form, is given by [52]

$$\frac{d^3 f}{d\eta^3} + mf\frac{d^2 f}{d\eta^2} - \left(\frac{df}{d\eta}\right)^2 = 0, \tag{8.21}$$

$$\begin{aligned} f &= s, \quad \frac{df}{d\eta} = -1 \quad \text{at} \quad \eta = 0, \\ \frac{df}{d\eta} &\to 0 \quad \text{as} \quad \eta \to \infty. \end{aligned} \tag{8.22}$$

where $s = W/m\sqrt{a\nu}$ is the suction/injection parameter. Now we construct an exact closed-form solution of the boundary-value problem (8.21) and (8.22) by the use of our compatibility approach. We check the compatibility of the third-order ODE (8.21) with the first-order ODE

$$\frac{df}{d\eta} + \lambda f = \beta\lambda \quad \text{where} \quad \lambda \neq 0, \tag{8.23}$$

with the solution

$$f(\eta) = A\exp(-\lambda\eta) + \beta, \tag{8.24}$$

where A, λ and β are the parameters to be determined.
Using the compatibility test (8.15) (for a third-order ODE to be compatible with the first-order ODE), we obtain

$$\beta\lambda^3 - \lambda^3 f - \beta\lambda^2 mf + \lambda^2 mf^2 - \beta^2\lambda^2 - \lambda^2 f^2 + 2\beta\lambda^2 f = 0. \tag{8.25}$$

Equating the above equation by powers of f, we find

$$f^2 \; : \; \lambda^2(m-1) = 0, \tag{8.26}$$

$$f^1 \; : \; -\lambda^3 - \beta\lambda^2 m + 2\beta\lambda^2 = 0, \tag{8.27}$$

$$f^0 \; : \; \beta\lambda^3 - \beta^2\lambda^2 = 0. \tag{8.28}$$

From Eq. (8.26), we obtain $m = 1$. This shows that the sheet shrinks in the x-direction.

This is also the physical condition of the flow model. Using $m = 1$, Eqs. (8.27) and (8.28) reduce to

$$\lambda^2 - \beta\lambda = 0, \tag{8.29}$$

which is the second compatibility condition for a third-order ODE (8.21) to be compatible with the first-order ODE (8.23). Thus, the solution of Eq. (8.21), subject to Eq. (8.23), can be written as

$$f(\eta) = A\exp(-\lambda\eta) + \beta, \tag{8.30}$$

provided condition (8.29) holds. Now with the use of the boundary conditions (8.22), we deduce

$$\begin{aligned} A &= \frac{1}{\lambda}, & (8.31)\\ \beta &= \left(s - \frac{1}{\lambda}\right). & (8.32) \end{aligned}$$

Utilizing the value of β in Eq. (8.29) results in

$$\lambda = \frac{s \pm \sqrt{s^2 - 4}}{2}. \tag{8.33}$$

Note that the solution domain of this problem is determined by the suction parameter. In order to have a physically meaningful solution we must have $s^2 > 4$. Here we have two solution branches corresponding to the positive and negative signs in equation (8.33), that is

$$\begin{aligned} \lambda_1 &= \frac{s + \sqrt{s^2 - 4}}{2} > 0, & (8.34)\\ \lambda_2 &= \frac{s - \sqrt{s^2 - 4}}{2} > 0. & (8.35) \end{aligned}$$

The solutions corresponding to λ_1 and λ_2 are written as

$$f(\eta) = s - \frac{1}{\left(\frac{s+\sqrt{s^2-4}}{2}\right)} + \frac{1}{\left(\frac{s+\sqrt{s^2-4}}{2}\right)} \exp\left[-\left(\frac{s+\sqrt{s^2-4}}{2}\right)\eta\right], \tag{8.36}$$

$$f(\eta) = s - \frac{1}{\left(\frac{s-\sqrt{s^2-4}}{2}\right)} + \frac{1}{\left(\frac{s-\sqrt{s^2-4}}{2}\right)} \exp\left[-\left(\frac{s-\sqrt{s^2-4}}{2}\right)\eta\right]. \tag{8.37}$$

Thus two independent solutions are obtained in this case corresponding to positive sign branch and negative sign branch with $s^2 > 4$. The solutions (8.36) and (8.37) are the new exact solutions of the flow model [52], which are not available in the literature. This shows the great potential of the compatibility approach for solving higher-order nonlinear equations exactly. We can also extend this solution by incorporating other interesting physical phenomena to this flow model, such as the consideration of stagnation-point flows dealing with Newtonian and non-Newtonian fluids $[53-56]$, heat and mass transfer analysis, power-law variation of the stretching velocity and temperature and magnetic field. We can also apply the compatibility approach to construct exact solutions of other boundary layer problems due to a shrinking sheet dealing with flow models of non-Newtonian fluids [57, 58]. In all the studies $[53-58]$, the coupled systems of nonlinear ordinary differential equations have been solved numerically. Therefore, one can apply the compatibility approach to look for the exact solutions of these type of flow models as well.

8.3.2 MHD viscous flow due to a shrinking sheet

The magnetohydrodynamic (MHD) viscous flow due to a shrinking sheet has been investigated by Fernández [59], Noor et al. [60] and Sajid and Hayat [61]. In [59] numerical and perturbation solutions are obtained for the flow model. Noor et al. [60] have discussed a series solution for the governing problem using the Adomian decomposition method (ADM) coupled with the Padé approximation. The same model has also been analyzed by Sajid and Hayat [61] by using the homotopy analysis method (HAM).

The governing equation describing the MHD viscous flow due to a shrinking sheet in non-dimensionalized form, is given by [59 – 61],

$$\frac{d^3f}{d\eta^3} + mf\frac{d^2f}{d\eta^2} - \left(\frac{df}{d\eta}\right)^2 - M^2\frac{df}{d\eta} = 0, \tag{8.38}$$

which is subject to the boundary conditions

$$f = s, \frac{df}{d\eta} = -1 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \frac{df}{d\eta} \to 0 \quad \text{as} \quad \eta \to \infty, \tag{8.39}$$

where $s = W/m\sqrt{a\nu}$ is the suction/injection parameter, $M = (\sigma B_0^2/\rho a)^{\frac{1}{2}}$ the magnetic field parameter and $a > 0$ the shrinking constant. We now construct the exact (closed-form) solution of the boundary-value problem (8.38) and (8.39) using the compatibility approach.

We check that the third order ordinary differential equation (8.38) is compatible with the first order ordinary differential equation

$$\frac{df}{d\eta} + \lambda f = \beta\lambda, \quad \lambda \neq 0, \tag{8.40}$$

where λ and β are constants. The general solution of (8.40) is

$$f(\eta) = A\exp(-\lambda\eta) + \beta. \tag{8.41}$$

The parameters to be determined are A, λ and β.

Using the compatibility test (8.15) for a third order ordinary differential equation to be compatible with a first order ordinary differential equation, we obtain

$$\beta\lambda^3 - \lambda^3 f - \beta\lambda^2 mf + \lambda^2 mf^2 - \beta^2\lambda^2 - \lambda^2 f^2 + 2\beta\lambda^2 f - M^2\beta\lambda + M^2\lambda f = 0. \tag{8.42}$$

Now separating equation (8.42) in powers of the dependent variable f, we obtain

$$f^2 \ : \ \lambda^2 m - \lambda^2 = 0, \tag{8.43}$$

$$f^1 \ : \ -\lambda^3 - \beta\lambda^2 m + 2\beta\lambda^2 + M^2\lambda = 0, \tag{8.44}$$

$$f^0 \ : \ \beta\lambda^3 - \beta^2\lambda^2 - M^2\beta\lambda = 0. \tag{8.45}$$

From Eq. (8.42), we obtain $m = 1$. This shows that the sheet shrinks in the x-direction, which is also the physical compatibility condition of the flow model. Using $m = 1$, Eqs (8.44) and (8.45) reduce to

$$\lambda^2 - \beta\lambda - M^2 = 0, \tag{8.46}$$

which is the second compatibility condition for the third order ordinary differential equation (8.38) to be compatible with the first order ordinary differential equation (8.40). Thus, the solution of Eq. (8.38), subject to Eq. (8.40), can be written as

$$f(\eta) = A\exp(-\lambda\eta) + \beta, \tag{8.47}$$

provided condition (8.46) is satisfied. Now applying the boundary conditions (8.39), we deduce

$$A = \frac{1}{\lambda}, \tag{8.48}$$

$$\beta = \left(s - \frac{1}{\lambda}\right). \tag{8.49}$$

Substituting (8.49) into (8.46), we obtain

$$\lambda = \frac{s \pm \sqrt{s^2 - 4(1 - M^2)}}{2}. \tag{8.50}$$

We observe that the solution domain of this problem is determined by the suction and magnetic parameters. In order to have a physically meaningful solution we must have

$\lambda > 0$. Thus we arrive at three distinct cases:

Case 1: $0 < M < 1$

For real solutions it is necessary that $s^2 > 4(1-M^2)$. There are two solution branches corresponding to the positive and negative sign in equation (8.50), that is

$$\lambda_1 = \frac{s + \sqrt{s^2 - 4(1 - M^2)}}{2} > 0, \tag{8.51}$$

$$\lambda_2 = \frac{s - \sqrt{s^2 - 4(1 - M^2)}}{2} > 0. \tag{8.52}$$

The solutions corresponding to λ_1 and λ_2 are written as

$$f(\eta) = s - \frac{1}{\lambda_1} + \frac{1}{\lambda_1} \exp\left[-\lambda_1 \eta\right], \tag{8.53}$$

$$f(\eta) = s - \frac{1}{\lambda_2} + \frac{1}{\lambda_2} \exp\left[-\lambda_2 \eta\right]. \tag{8.54}$$

Thus two independent solutions are obtained in this case corresponding to positive sign branch and negative sign branch with $s^2 > 4\left(1 - M^2\right)$.

Case 2: $M = 1$

In this case, we have

$$\lambda = s, \;\; \lambda = 0. \tag{8.55}$$

We do not consider the case $\lambda = 0$. The case $\lambda = s$ gives

$$f(\eta) = s + \frac{1}{s}[\exp\left[(-s\eta) - 1\right]. \tag{8.56}$$

There is one solution obtained in this case with $\lambda = s$ but s must be greater than zero.

Case 3: $M > 1$

In this case, we have

$$\lambda_1 = \frac{s + \sqrt{s^2 + 4(M^2 - 1)}}{2} > 0, \tag{8.57}$$

$$\lambda_2 = \frac{s - \sqrt{s^2 + 4(M^2 - 1)}}{2} < 0. \tag{8.58}$$

Thus only one physically meaningful solution is obtained in this case corresponding to λ_1, given by

$$f(\eta) = s - \frac{1}{\lambda_1} + \frac{1}{\lambda_1} \exp\left[-\lambda_1 \eta\right]. \tag{8.59}$$

The solution domain of this problem is shown in Figure 8.1.

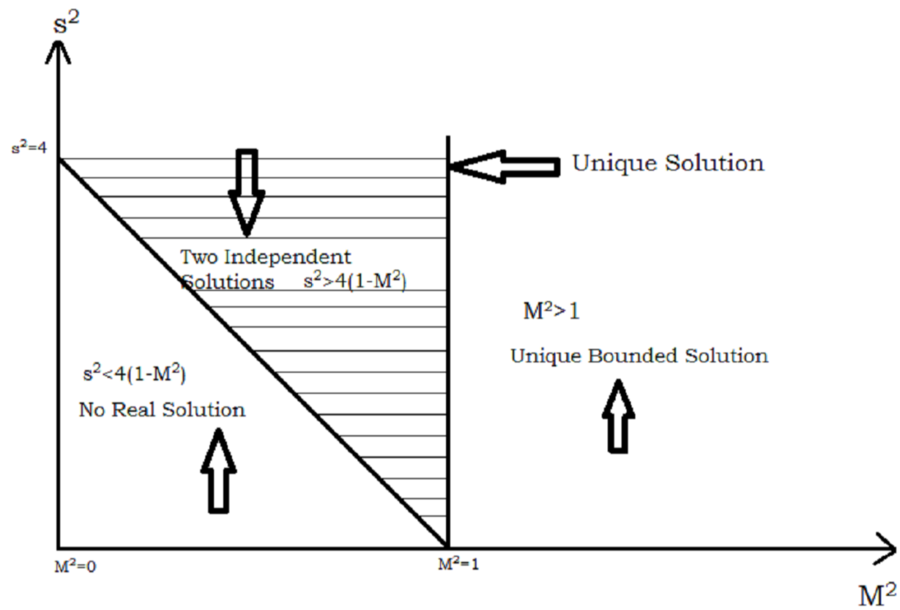

Figure 8.1: The solution domain at different values of M as a function of the mass suction parameter s.

The exact solutions obtained here were reported before by Fang and Zhang [62]. The authors in [62] have employed an ansatz method to construct these solutions. We have shown here that these solutions are actually formulated through the compatibility approach.

8.3.3 Flow due to a stretching sheet/surface

One of the classical application of Prandtl boundary layer equations is the boundary layer flow induced due to a stretching sheet/surface. The motivation of study the flow problems of viscous fluid over a stretching surface stems because of its several important engineering and industrial applications. The flow engendered by such a sheet occurs in cooling and extrusion processes, paper production, the boundary layer along material handling conveyers, polymer processing unit of a chemical plant, metal working process in metallurgy and many others.

Crane [63] obtained the analytical solution for a boundary layer flow of an incompressible viscous fluid over a stretching sheet. Apparently no such simple exact solution exists for a such a flow. Nevertheless some approximate, yet reasonably accurate, solutions have been developed and reported in the literature. We show here that the classical solution, obtained by Crane [63], is actually formulated through compatibility.

Consider the boundary layer flow of viscous (Newtonian) fluid induced by a stretching sheet which moves in its own plane with a velocity at each point which is proportional to the distance from the origin, thus

$$\text{velocity of sheet} = V_0 x, \tag{8.60}$$

where V_0 is a constant. Defining the stream function

$$\psi(x, y) = xF(y). \tag{8.61}$$

The velocity components in the x and y directions, $u(x, y)$ and $v(x, y)$, are related to the stream function $\psi(x, y)$ as

$$u(x,y) = \frac{\partial \psi}{\partial y}, \quad v(x,y) = -\frac{\partial \psi}{\partial x}. \tag{8.62}$$

The governing boundary value problem for boundary layer flow of a viscous fluid due to a stretching sheet is governed by [63]

$$\nu \frac{d^3F}{dy^3} + F\frac{d^2F}{dy^2} - \left(\frac{dF}{dy}\right)^2 = 0, \tag{8.63}$$

subject to the boundary conditions

$$F(0) = 0, \quad \frac{dF}{dy}(0) = V_0, \quad \frac{dF}{dy} \to 0 \quad \text{as} \quad y \to \infty. \tag{8.64}$$

We now check that the third-order ODE (8.63) is compatible with the first-order ODE

$$\frac{dF}{dy} - \alpha F = V_0 \quad \text{where} \quad \alpha \neq 0, \tag{8.65}$$

with solution

$$F(y) = A\exp(\alpha y) - \frac{V_0}{\alpha}, \tag{8.66}$$

where A and α are the constants to be found from the compatibility test.
Using the compatibility test given in Eq. (8.10), for a third-order ODE to be compatible with a first-order ODE, we obtain

$$\nu\alpha^2 V_0 - \nu\alpha^3 F + V_0\alpha F - V_0^2 + 2\alpha V_0 F = 0. \tag{8.67}$$

Now equating Eq. (8.67) by powers of F, we determine

$$F^1 \quad : \quad -\nu\alpha^3 - V_0\alpha + 2\alpha V_0 = 0, \tag{8.68}$$

$$F^0 \quad : \quad \nu\alpha^2 V_0 - V_0^2 = 0. \tag{8.69}$$

From Eqs. (8.68) and (8.69), we deduce

$$\alpha = \pm\sqrt{\frac{V_0}{\nu}}, \tag{8.70}$$

and we choose

$$\alpha = -\sqrt{\frac{V_0}{\nu}}, \tag{8.71}$$

so that our solution satisfy the boundary condition at infinity in Eq. (8.64). Thus, the solution of the third-order ODE (8.63) compatible with the first-order ODE (8.65), is written as

$$F(y) = A\exp\left(-\sqrt{\frac{V_0}{\nu}}y\right) + \sqrt{\nu V_0}. \tag{8.72}$$

Now applying the boundary conditions (8.64), we arrive at

$$A = -\sqrt{\nu V_0}. \tag{8.73}$$

Thus, the exact closed-form solution for $F(y)$ takes the form

$$F(y) = -\sqrt{\nu V_0}\exp\left(-\sqrt{\frac{V_0}{\nu}}y\right) + \sqrt{\nu V_0}. \tag{8.74}$$

The corresponding stream function and the velocity components are:

$$\psi(x,y) = x\left[-\sqrt{\nu V_0}\exp\left(-\sqrt{\frac{V_0}{\nu}}y\right) + \sqrt{\nu V_0}\right], \tag{8.75}$$

$$u(x,y) = xV_0\exp\left(-\sqrt{\frac{V_0}{\nu}}y\right), \tag{8.76}$$

$$v(x,y) = \sqrt{\nu V_0}\left[1 - \exp\left(-\sqrt{\frac{V_0}{\nu}}y\right)\right]. \tag{8.77}$$

The solutions (8.74)−(8.77) are exactly the same as obtained by Crane [63]. Here, we have shown that these solutions actually arise from the compatibility test. We can also extend this solution by including various other aspects to this problem, such a consideration of heat and mass transfer [64, 65], power-law variation of the stretching velocity and temperature and magnetic field [66, 67]. We can also apply the compatibility approach to construct exact solutions of other boundary layer problems due to a stretching sheet dealing with flow models of non-Newtonian fluids [68 − 70].

8.3.4 MHD viscous flow due to a stretching sheet

The model of the boundary layer flow of an incompressible viscous fluid over a non-linear stretching sheet was discussed by Ghotbi [71] and very recently by Fathizadeh et al. [72]. In [71], the homotopy analysis method (HAM) was applied to obtain the solution of the governing model and in [72], the homotopy perturbation method (HPM) was used on the same flow model. Here, we present the exact solution of that model by using the compatibility method. The dimensionless flow problem describing the (MHD) viscous flow due to a stretching sheet, is given by [71, 72]:

$$\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} - \beta\left(\frac{df}{d\eta}\right)^2 - M\frac{df}{d\eta} = 0, \tag{8.78}$$

$$f(0) = 0, \quad \frac{df}{d\eta}(0) = 1, \quad \frac{df}{d\eta}(\infty) = 0, \tag{8.79}$$

where

$$\beta = \frac{2n}{n+1} \quad \text{and} \quad M = \frac{2\sigma B_0^2}{\rho c(1+n)}. \tag{8.80}$$

We check that the third order ordinary differential equation (8.78) is compatible with the first order ordinary differential equation

$$\frac{df}{d\eta} + \alpha f = 1 \quad \text{where} \quad \alpha \neq 0, \tag{8.81}$$

with solution

$$f(\eta) = A\exp(-\alpha\eta) + \frac{1}{\alpha}, \tag{8.82}$$

where A and α are the constants to be found from the compatibility test.

Using the compatibility test given in Eq. (8.15), for a third order ordinary differential equation to be compatible with a first order ordinary differential equation, we obtain

$$\alpha^2 - \alpha^3 f - \alpha f + \alpha^2 f^2 - \beta - \beta\alpha^2 f^2 + 2\beta\alpha f - M + M\alpha f = 0. \tag{8.83}$$

Separating Eq. (8.83) by powers of f, we find

$$f^2 \;:\; \alpha^2(1-\beta) = 0, \tag{8.84}$$

$$f^1 \;:\; -\alpha^3 - \alpha + 2\beta\alpha + M\alpha = 0, \tag{8.85}$$

$$f^0 \;:\; \alpha^2 - \beta - M = 0. \tag{8.86}$$

From Eq. (8.84) we obtain $\beta = 1$ which is the physical constraint of the flow model. From Eqs. (8.85) and (8.86) we deduce

$$\alpha = \pm\sqrt{1+M}. \tag{8.87}$$

Now using the boundary conditions (8.79), we deduce from (8.82) that

$$\alpha = +(1+M)^{\frac{1}{2}}, \quad A = -\frac{1}{(1+M)^{\frac{1}{2}}}. \tag{8.88}$$

Thus, the exact closed-form solution for $f(\eta)$ takes the form

$$f(\eta) = \frac{1}{(1+M)^{\frac{1}{2}}}\left[1 - \exp[-(1+M)^{\frac{1}{2}}\eta]\right]. \tag{8.89}$$

The solution (8.89) is the new exact solution of the flow model [71, 72] when $\beta = 1$. For viscous flow over a nonlinearly stretching sheet without the effects of magnetic field

($M = 0$), discussed numerically by Vajravelu [73], we can deduce the exact solution when $\beta = 1$ as

$$f(\eta) = [1 - \exp(-\eta)]. \tag{8.90}$$

8.3.5 MHD flow of a Sisko fluid in porous medium with plate suction/injection

Khan et al. [74] investigated the steady flow of a Sisko fluid over an infinite porous plate. The flow is produced by the motion of the plate with velocity U_0. The fluid occupies the porous half space $y > 0$ and is also electrically conducting in the presence of a constant applied magnetic field in the transverse direction to the flow. The governing second-order nonlinear ordinary differential equation was solved using the homotopy analysis method. Here, we construct the exact solution of the same flow model by using the compatibility approach. The dimensionless boundary value problem that describes the MHD flow of a Sisko fluid in a porous medium is given by [74]

$$\frac{d^2 f}{d\eta^2} + nb^* \frac{d^2 f}{d\eta^2}\left(\frac{df}{d\eta}\right)^{n-1} + V_0^* \frac{df}{d\eta} - \frac{b^*}{K} f \left(\frac{df}{d\eta}\right)^{n-1} - \left(\frac{1}{K} + M^2\right) f = 0, \tag{8.91}$$

$$f = 1 \quad \text{at} \quad \eta = 0, \quad f \to 0 \quad \text{as} \quad \eta \to \infty, \tag{8.92}$$

where

$$b^* = \frac{b}{a}\left(\frac{U_0^2}{\nu}\right)^{n-1}, \quad V_0^* = \frac{V_0 \mu}{a U_0}, \quad \frac{1}{K} = \frac{\varphi \nu^2}{\kappa U_0^2}, \quad M^2 = \frac{\sigma B_0^2 \nu^2}{a U_0^2}. \tag{8.93}$$

In Eq. (8.93), b^* is the Sisko fluid parameter, $V_0^* \gtrless 0$ the suction/injection parameter, $\frac{1}{K}$ the porosity parameter and M the magnetic field parameter.

We now obtain the condition for the ordinary differential equation (8.91) to be compatible with the first order ordinary differential equation

$$\frac{df}{d\eta} + \theta f = 0 \quad \text{where} \quad \theta \neq 0, \tag{8.94}$$

with solution

$$f(\eta) = A\exp(-\theta\eta), \tag{8.95}$$

where A and θ are the parameters to be determined. Using the compatibility test given in Eq. (8.16) for a second-order ordinary differential equation to be compatible with a first order ordinary differential equation, we obtain

$$\theta^2 f + nb^*\theta^2 f(-\theta f)^{n-1} - V_0^*\theta f + \alpha^2 f^2 - \frac{b^*}{K} f(-\theta f)^{n-1} - \left(\frac{1}{K} + M^2\right) f = 0. \tag{8.96}$$

Separating Eq. (8.96) by powers of f we obtain

$$f^n \quad : \quad b^*\left(n\theta^2 - \frac{1}{K}\right) = 0, \tag{8.97}$$

$$f^1 \quad : \quad \theta^2 - V_0^*\theta - \left(\frac{1}{K} + M^2\right) = 0. \tag{8.98}$$

From Eq. (8.97), we deduce

$$\theta = \pm\frac{1}{\sqrt{nK}}. \tag{8.99}$$

We choose $\theta = +1/\sqrt{nK}$ so that (8.94) satisfies the second boundary condition in (8.92). Equation (8.98) can be expressed in the form

$$M^2 K + \frac{V_0^* K^{\frac{1}{2}}}{\sqrt{n}} + \frac{(n-1)}{n} = 0, \tag{8.100}$$

which is the required compatibility condition on the physical parameters of the flow model for a second-order ordinary differential equation (8.91) to be compatible with the first order ordinary differential equation (8.94). From the first boundary condition in (8.92), $A = 1$. Thus, the solution for $f(\eta)$ which satisfies the compatibility condition (8.100) is

$$f(\eta) = \exp\left[-\frac{1}{\sqrt{nK}}\eta\right]. \tag{8.101}$$

The compatibility condition (8.100) can be regarded as a quadratic equation for $K^{\frac{1}{2}}$. The solution of (8.100) for $(nK)^{\frac{1}{2}}$ is

$$(nK)^{\frac{1}{2}} = \frac{1}{2M^2}\left[-V_0^* \pm (V_0^{*2} + 4(1-n)M^2)^{\frac{1}{2}}\right]. \tag{8.102}$$

For $0 < n \leqslant 1$ the solution is always real while for $n > 1$ it is real provided

$$V_0^{*2} > 4(n-1)M^2. \tag{8.103}$$

For $V_0^* > 0$ which describes suction, there is one unique positive real solution which exists only for $0 < n < 1$ and can be expressed as

$$(nK)^{\frac{1}{2}} = \frac{V_0^*}{2M^2}\left[\left(1 + \frac{4(1-n)M^2}{V_0^{*2}}\right)^{\frac{1}{2}} - 1\right]. \tag{8.104}$$

For $V_0^* < 0$, which describes injection, there is one unique positive real solution which exists only for $0 < n < 1$, given by

$$(nK)^{\frac{1}{2}} = \frac{-V_0^*}{2M^2}\left[1 + \left(1 + \frac{4(1-n)M^2}{V_0^{*2}}\right)^{\frac{1}{2}}\right], \tag{8.105}$$

while for $n \geqslant 1$ there are two positive real solutions provided (8.103) is satisfied

$$(nK)^{\frac{1}{2}} = \frac{-V_0^*}{2M^2}\left[1 \pm \left(1 - \frac{4(1-n)M^2}{V_0^{*2}}\right)^{\frac{1}{2}}\right], \tag{8.106}$$

and the solution is not unique.

For the flow model without the effects of plate suction/injection [75] for which $V_0^* = 0$, the exact solution remains formally the same and is given by Eq. (8.101). However, the compatibility condition (8.100) becomes

$$\frac{1}{nK} - \left(\frac{1}{K} + M^2\right) = 0, \tag{8.107}$$

and therefore

$$M^2 = \frac{1}{K}\left(\frac{1-n}{n}\right) > 0 \quad \text{provided} \quad 0 < n < 1. \tag{8.108}$$

Thus, the solution of the flow model [75] is valid only for $0 < n < 1$, that is, only for pseudoplastic behavior.

8.3.6 Steady flow of a third grade fluid in porous half-space

The nonlinear steady flow model of a third grade fluid in porous half-space was first analyzed by Hayat et al. [48] by using the homotopy analysis method. Later the same problem was investigated again by Faiz [76] and the asymptotic analysis was carried out for the governing model. Here, we construct the exact closed-form solution of that model using the compatibility criterion. The dimensionless problem is given as [48, 76]

$$\frac{d^2f}{dz^2} + b_1\frac{d^2f}{dz^2}\left(\frac{df}{dz}\right)^2 - b_2f\left(\frac{df}{dz}\right)^2 - cf = 0, \tag{8.109}$$

$$f(0) = 1 \quad \text{and} \quad f \to 0 \quad \text{as} \;\; z \to \infty, \tag{8.110}$$

where

$$b_1 = \frac{6\beta_3 V_0^4}{\mu\nu^2}, \quad b_2 = \frac{2\beta_3\varphi V_0^2}{\kappa\mu}, \quad c = \frac{\varphi\nu^2}{\kappa V_0^2}, \tag{8.111}$$

are the third grade and porosity parameters respectively.

We have to check the compatibility of equation (8.109) with the first order ordinary differential equation

$$\frac{df}{dz} + \epsilon f = 0 \quad \text{where} \quad \epsilon \neq 0, \tag{8.112}$$

with solution

$$f(z) = A\exp(-\epsilon z), \tag{8.113}$$

where A and ϵ are the parameters to be determined. Using the compatibility test given by Eq. (8.16) for a second-order ordinary differential equation to be compatible with a first order ordinary differential equation, we obtain

$$\epsilon^2 f + b_1\epsilon^4 f^3 - b_2\epsilon^2 f^3 - cf = 0. \tag{8.114}$$

Separating Eq. (8.114) by powers of f, we obtain

$$f^3 \quad : \quad \epsilon^2(b_1\epsilon^2 - b_2) = 0, \tag{8.115}$$

$$f^1 \quad : \quad \epsilon^2 - c = 0. \tag{8.116}$$

From Eq. (8.115), we deduce

$$\epsilon = \pm\sqrt{\frac{b_2}{b_1}}. \tag{8.117}$$

Substituting (8.117) into (8.116) gives

$$\frac{b_2}{b_1} - c = 0. \tag{8.118}$$

By imposing the boundary conditions (8.110), we obtain

$$A = 1, \quad \epsilon = +\sqrt{\frac{b_2}{b_1}}. \tag{8.119}$$

Thus the solution for the boundary value problem (8.109) and (8.110) is

$$f(z) = \exp\left[-\sqrt{\frac{b_2}{b_1}}z\right], \tag{8.120}$$

provided the parameters b_1, b_2, and c satisfy the compatibility condition (8.118).

8.3.7 Unsteady flow of a third grade fluid

Hayat et al. [49], discussed the unsteady incompressible flow of a third grade fluid in a porous medium. The flow is generated by the motion of a plate in its own plane with an arbitrary velocity $V(t)$. A Lie symmetry analysis was performed to reduce the governing nonlinear third order partial differential equations to ordinary differential equations. The reduced equations were then solved analytically and numerically. In [49], the travelling wave solution of the governing model was constructed with the use of the ansatz method. Here, we reproduce this solution with the aid of the compatibility approach. The governing nonlinear partial differential equation in [49] is

$$\frac{\partial u}{\partial t} = \mu_* \frac{\partial^2 u}{\partial y^2} + \alpha \frac{\partial^3 u}{\partial y^2 \partial t} + \gamma_1 \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} - \gamma_2 u \left(\frac{\partial u}{\partial y} \right)^2 - \phi_1 u, \tag{8.121}$$

where u is the velocity component parallel to the plate, μ_* is the coefficient of viscosity, α, γ_1 and γ_2 are the material constants and ϕ_1 is the porosity of the porous medium.

The relevant time and space dependent velocity boundary conditions are specified as follows:

$$u(0,t) = V(t), \quad t > 0, \tag{8.122}$$

$$u(y,t) \to 0 \text{ as } y \to \infty, \quad t > 0, \tag{8.123}$$

$$u(y,0) = g(y), \quad y > 0, \tag{8.124}$$

where

$$g(0) = V(0). \tag{8.125}$$

The first boundary condition (8.122) is the no-slip condition and the second boundary condition (8.123) states that the main stream velocity is zero. This is not a restrictive assumption since we can always measure velocity relative to the main stream. The initial condition (8.124) indicates that the fluid is initially moving with some non-uniform velocity $g(y)$.

Using the classical Lie method of infinitesimal transformations [49], the similarity solution for equation (8.121) is found to be

$$u(y,t) = U(x_1) \quad \text{where} \quad x_1 = y + ct, \tag{8.126}$$

which represents a wave front type travelling wave propagating in the negative $y-$direction with constant speed c. The substitution of Eq. (8.126) in Eq. (8.121) yields a third order ordinary differential equation for $U(x_1)$:

$$c\frac{dU}{dx_1} - \mu_*\frac{d^2U}{dx_1^2} - \alpha c\frac{d^3U}{dx_1^3} - \gamma_1\left(\frac{dU}{dx_1}\right)^2\frac{d^2U}{dx_1^2} + \gamma_2 U\left(\frac{dU}{dx_1}\right)^2 + \phi_1 U = 0, \tag{8.127}$$

with transformed boundary conditions

$$U(0) = l_1, \quad U(x_1) \to 0 \quad \text{as} \quad x_1 \to \infty, \tag{8.128}$$

where l_1 is a constant. We now solve the reduced Eq. (8.127) subject to the boundary conditions (8.128) using the approach of compatibility. We check the compatibility of Eq. (8.127) with the first order ordinary differential equation

$$\frac{dU}{dx_1} + \varepsilon U = 0 \quad \text{where} \quad \varepsilon \neq 0, \tag{8.129}$$

with solution

$$U(x_1) = A\exp(-\varepsilon x_1), \tag{8.130}$$

where A and ε are the parameters to be determined from the compatibility condition. Using the compatibility test given in Eq. (8.15) for a third order ordinary differential equation to be compatible with a first order ordinary differential equation, we obtain

$$\phi_1 U + c\varepsilon U + \gamma_2\varepsilon^2 U^3 - \gamma_1\varepsilon^4 U^3 - \mu_*\varepsilon^2 U + \alpha c\varepsilon^3 U = 0. \tag{8.131}$$

Separating Eq. (8.131) by powers of U we find

$$U \quad : \quad \phi_1 + c\varepsilon - \mu_*\varepsilon^2 + \alpha c\varepsilon^3 = 0, \tag{8.132}$$

$$U^3 \quad : \quad \left(\gamma_2 - \gamma_1\varepsilon^2\right)\varepsilon^2 = 0. \tag{8.133}$$

From Eq. (8.133), we find

$$\varepsilon = \pm\sqrt{\frac{\gamma_2}{\gamma_1}}, \tag{8.134}$$

and choose

$$\varepsilon = +\sqrt{\frac{\gamma_2}{\gamma_1}}, \tag{8.135}$$

so that solution (8.130) satisfies the boundary condition at infinity. By substituting (8.135) into Eq. (8.132), we obtain

$$\phi_1 + c\left(\frac{\gamma_2}{\gamma_1}\right)^{\frac{1}{2}} - \mu_*\frac{\gamma_2}{\gamma_1} + c\alpha\left(\frac{\gamma_2}{\gamma_1}\right)^{\frac{3}{2}} = 0. \tag{8.136}$$

Thus, the exact solution for $U(x_1)$, provided the compatibility condition (8.136) holds, is

$$U(x_1) = l_1 \exp\left(-\sqrt{\frac{\gamma_2}{\gamma_1}}\, x_1\right). \tag{8.137}$$

Hence the exact solution $u(y,t)$ which satisfies the compatibility condition (8.136) is

$$u(y,t) = \exp\left(-\sqrt{\frac{\gamma_2}{\gamma_1}}(y+ct)\right). \tag{8.138}$$

We note that the solution (8.138) satisfies the boundary and initial conditions (8.122), (8.123) and (8.124) with

$$V(t) = \exp\left(-\sqrt{\frac{\gamma_2}{\gamma_1}}ct\right) \quad \text{and} \quad g(y) = \exp\left(-\sqrt{\frac{\gamma_2}{\gamma_1}}y\right). \tag{8.139}$$

The condition $V(0) = g(0)$ is clearly satisfied by (8.139). We see that $V(t)$ and $g(y)$ depend on the physical parameters of the flow. The solution given by Eq. (8.138) is exactly the same as obtained in [49] using the ansatz method.

We observe that the compatibility condition (8.136) gives the speed c of the travelling wave,

$$c = \frac{\left[\mu_* \frac{\gamma_2}{\gamma_1} - \phi_1\right]}{\left(\frac{\gamma_2}{\gamma_1}\right)^{\frac{1}{2}} \left[1 + \alpha \frac{\gamma_2}{\gamma_1}\right]}. \tag{8.140}$$

For a wave travelling away from the plate it is necessary that $c < 0$ and therefore that

$$\phi_1 > \mu_* \frac{\gamma_2}{\gamma_1}. \tag{8.141}$$

8.3.8 Unsteady MHD flow of a third grade fluid with suction/blowing

Aziz and Aziz [50] very recently extended the flow model of [49] by including a magnetic field and suction/injection in the flow. The same group theoretic approach was used to obtain closed-form invariant solutions of the nonlinear boundary-value problem. Here, we reproduce the exact travelling wave solution obtained in [50] by using the compatibility approach. The governing initial boundary value problem in dimensionless form is given as [50]

$$\frac{\partial u}{\partial t} = \mu_* \frac{\partial^2 u}{\partial y^2} + \alpha_* \frac{\partial^3 u}{\partial y^2 \partial t} - \alpha_* W_0 \frac{\partial^3 u}{\partial y^3} + \gamma \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \gamma_* u \left(\frac{\partial u}{\partial y}\right)^2 + W_0 \frac{\partial u}{\partial y} - (\phi_* + M_*^2) u, \tag{8.142}$$

where $W_0 > 0$ indicates suction velocity and $W_0 < 0$ corresponds to blowing or injection velocity. The relevant boundary and initial conditions are specified as follows:

$$u(0,t) = V(t), \quad t > 0, \tag{8.143}$$

$$u(y,t) \to 0 \text{ as } y \to \infty, \quad t > 0, \tag{8.144}$$

$$u(y,0) = f(y), \quad y > 0, \tag{8.145}$$

where

$$f(0) = V(0). \tag{8.146}$$

Using the classical Lie theory of infinitesimal transformations [50], the invariant solution for equation (8.142) is written as

$$u(y,t) = G(x), \quad \text{where} \quad x = y + ct. \tag{8.147}$$

Substituting Eq. (8.147) into Eq. (8.142) results in a third order ordinary differential equation for $G(x)$:

$$\begin{aligned} c\frac{dG}{dx} &= \mu_*\frac{d^2G}{dx^2} + c\alpha_*\frac{d^3G}{dx^3} - \alpha_* W_0\frac{d^3G}{dx^3} + \gamma\left(\frac{dG}{dx}\right)^2\frac{d^2G}{dx^2} \\ &\quad -\gamma_* G\left(\frac{dG}{dx}\right)^2 + W_0\frac{dG}{dx} - (\phi_* + M_*^2)G, \end{aligned} \tag{8.148}$$

with the invariant boundary conditions

$$G(0) = l, \tag{8.149}$$

$$G(x) \rightarrow 0 \ \text{as} \ x \rightarrow \infty, \tag{8.150}$$

where l is a constant. We now obtain the exact solution of the reduced ordinary differential equation (8.148) using compatibility. We check the compatibility of the third order ordinary differential equation with the first order ordinary differential equation

$$\frac{dG}{dx} + \delta G = 0 \quad \text{where} \quad \delta \neq 0, \tag{8.151}$$

with solution

$$G(x) = A\exp(-\delta x), \tag{8.152}$$

where A and δ are the parameters to be determined from the compatibility criterion. Using the general compatibility test (8.15), we obtain

$$-c\delta G + \mu_*\delta^2 G - \alpha_* c\delta^3 G - \alpha_* W_0\delta^3 G + \gamma\delta^4 G^3 - \gamma_*\delta^2 G^3 - W_0\delta G - (\phi_* + M_*^2)G = 0. \tag{8.153}$$

By the separation of Eq. (8.153) by powers of the dependent variable G, we obtain

$$G \quad : \quad -c\delta + \mu_*\delta^2 - \alpha_* c\delta^3 - \alpha_* W_0\delta^3 - W_0\delta - (\phi_* + M_*^2) = 0, \tag{8.154}$$

$$G^3 \quad : \quad \left(\gamma\delta^2 - \gamma_*\right)\delta^2 = 0. \tag{8.155}$$

From Eq. (8.155), we obtain

$$\delta = \pm\sqrt{\frac{\gamma_*}{\gamma}}. \tag{8.156}$$

From the boundary condition (8.150) as $x \to \infty$ we see that we must take the positive sign in (8.156):

$$\delta = +\sqrt{\frac{\gamma_*}{\gamma}}. \tag{8.157}$$

Substitution of Eq. (8.157) into Eq. (8.154) gives

$$-(c+W_0)\left(\frac{\gamma_*}{\gamma}\right)^{\frac{1}{2}} + \mu_*\frac{\gamma_*}{\gamma} - \alpha_*(c+W_0)\left(\frac{\gamma_*}{\gamma}\right)^{\frac{3}{2}} - (\phi_* + M_*^2) = 0. \tag{8.158}$$

Thus imposing the boundary condition (8.149) at $x = 0$, the solution of the third order ordinary differential equation (8.148) subject to the first order ordinary differential equation (8.151) is

$$G(x) = l\exp\left[-\sqrt{\frac{\gamma_*}{\gamma}}x\right], \tag{8.159}$$

provided condition (8.158) is satisfied. Finally, the exact solution $u(y,t)$, which satisfies the compatibility condition (8.158), is

$$u(y,t) = \exp\left[-\sqrt{\frac{\gamma_*}{\gamma}}(y+ct)\right]. \tag{8.160}$$

The solution (8.160) is the same as that found in [50] by employing the ansatz technique. From the solution (8.160), we can also recover the exact travelling wave solution of the flow model without the effects of suction or blowing [41]. Thus, for the problem considered in [41], the exact solution is the same as given in Eq. (8.160) but the result of imposing the compatibility condition on the physical parameters is changed to

$$-c\left(\frac{\gamma_*}{\gamma}\right)^{\frac{1}{2}}+\mu_*\frac{\gamma_*}{\gamma}-\alpha_* c\left(\frac{\gamma_*}{\gamma}\right)^{\frac{3}{2}}-(\phi_*+M_*^2)=0. \tag{8.161}$$

The physical significance of the compatibility conditions (8.158) and (8.161) is that they give the speed of the travelling wave. From condition (8.158), we deduce that

$$c+W_0=\frac{\left[\mu_*\frac{\gamma_*}{\gamma}-(\phi_*+M_*^2)\right]}{\left(\frac{\gamma_*}{\gamma}\right)^{\frac{1}{2}}\left[1+\alpha_*\frac{\gamma_*}{\gamma}\right]}, \tag{8.162}$$

while (8.161) gives (8.162) with $W_0=0$. From (8.147), for a wave travelling away from the plate, it is necessary that $c<0$ and therefore

$$\phi_*+M_*^2>\left(\frac{\gamma_*}{\gamma}\right)^{\frac{1}{2}}\left[\mu_*\left(\frac{\gamma_*}{\gamma}\right)^{\frac{1}{2}}+\left(1+\alpha_*\frac{\gamma_*}{\gamma}\right)W_0\right]. \tag{8.163}$$

8.3.9 Hydromagnetic flow of a second grade fluid over a stretching surface

Vajravelu and Rollins [77], carried out the study of analyzing the flow characteristics in an electrically conducting second grade fluid permeated by a uniform magnetic field over a stretching surface. The governing fourth-order nonlinear ordinary differential equation is solved numerically. We present here the exact (closed form) solution of that problem using the compatibility approach. The problem describing the MHD flow of a second grade fluid over a stretching sheet [77] is given by

$$\left(\frac{df}{d\eta}\right)^2-f\frac{d^2f}{d\eta^2}-\frac{d^3f}{d\eta^3}+M\frac{df}{d\eta}-\lambda_1\left[2\frac{df}{d\eta}\frac{d^3f}{d\eta^3}-\left(\frac{d^2f}{d\eta^2}\right)^2-f\frac{d^4f}{d\eta^4}\right]=0, \tag{8.164}$$

$$
\begin{aligned}
f(\eta) &= R, \quad \frac{df}{d\eta} = 1 \quad \text{at} \quad \eta = 0, \\
\frac{df}{d\eta} &\to 0, \quad \frac{d^2 f}{d\eta^2} \to 0 \quad \text{as} \quad \eta \to \infty,
\end{aligned} \tag{8.165}
$$

where $\lambda_1 = \lambda B/\rho\nu$ is the non-Newtonian parameter, $M = \sigma B_0^2/\rho B$ the magnetic parameter and $R = v_0/(B\nu)^{\frac{1}{2}}$ the suction parameter.

We now check that the fourth-order ordinary differential equation (8.164) is compatible with the first order ordinary differential equation

$$
\frac{df}{d\eta} + \beta f = a\beta, \tag{8.166}
$$

where $\beta \neq 0$. The solution of (8.166) is

$$
f(\eta) = a + b\exp(-\beta\eta), \tag{8.167}
$$

where a, b and β are the parameters to be determined. Using the compatibility test given in Eq. (8.14), we obtain

$$
a\beta + M - \beta^2 - \lambda_1 a\beta^3 = 0. \tag{8.168}
$$

Utilizing the boundary conditions (8.165), we deduce

$$
a = R + \frac{1}{\beta}, \tag{8.169}
$$

$$
b = -\frac{1}{\beta}. \tag{8.170}
$$

Now substituting (8.169) for a into Eq. (8.168), we obtain

$$
H(\beta) = \lambda_1 R\beta^3 + (1 + \lambda_1)\beta^2 - R\beta - (1 + M) = 0. \tag{8.171}
$$

Finally, the solution for $f(\eta)$ is written as

$$f(\eta) = R + \frac{1}{\beta} - \frac{1}{\beta}\exp[-\beta\eta], \tag{8.172}$$

where β is the positive real root of the cubic equation (8.171). From Descartes rule of signs, $H(\beta) = 0$ has not more than one positive real root and since $H(\beta) \to \infty$ as $\beta \to \infty$ and $H(0) = -(1 + M) < 0$ it follows that $H(\beta) = 0$ has exactly one positive real root.

The one positive real root of (8.171) can be found approximately for small R by using a regular perturbation expansion. Let

$$\beta = \beta_0 + R\beta_1 + O(R^2), \quad \text{as} \quad R \to 0. \tag{8.173}$$

Substituting the perturbation expansion (8.173) into Eq. (8.171) results in

$$\lambda_1 R\beta_0^3 + \beta_0^2 + 2\beta_0\beta_1 R + \lambda_1\beta_0^2 + 2\beta_0\beta_1\lambda_1 R - \beta_0 R - (1 + M) + O(R^2) = 0. \tag{8.174}$$

Equating to zero the terms of $O(1)$ and $O(R)$ in (8.174), we obtain

$$O(1) \quad : \quad (1 + \lambda_1)\beta_0^2 - (1 + M) = 0, \tag{8.175}$$

$$O(R) \quad : \quad 2(1 + \lambda_1)\beta_0\beta_1 + \lambda_1\beta_0^3 - \beta_0 = 0. \tag{8.176}$$

Solving Eqs. (8.175) and (8.176), we deduce

$$\beta = \sqrt{\frac{1 + M}{1 + \lambda_1}} + \frac{(1 - \lambda_1 M)}{2(1 + \lambda_1)^2} R + O(R^2), \tag{8.177}$$

which is the required positive real root of β provided M is not too large. Using this value of β in Eq. (8.172), we obtain

$$f(\eta) = R + \frac{1}{\left(\sqrt{\frac{1+M}{1+\lambda_1}} + \frac{(1-\lambda_1 M)}{2(1+\lambda_1)^2} R\right)} \left[1 - \exp\left[-\left(\sqrt{\frac{1+M}{1+\lambda_1}} + \frac{(1-\lambda_1 M)}{2(1+\lambda_1)^2} R\right)\right]\eta\right]. \tag{8.178}$$

The exact solution for the flow model without the effects of the magnetic field ($M = 0$) and suction ($R = 0$) discussed by Vajravelu and Roper [78], can be deduced from the perturbation solution (8.178). For when $M = R = 0$, (8.178) reduces to

$$f(\eta) = \sqrt{1+\lambda_1}\left(1 - \exp\left[-\frac{1}{\sqrt{1+\lambda_1}}\eta\right]\right). \tag{8.179}$$

8.3.10 Stokes$'$ problem for a fourth grade fluid

The unsteady unidirectional flow of an incompressible fourth grade fluid bounded by a suddenly moved rigid plate was studied by Hayat et al. [79]. The governing nonlinear higher order partial differential equation for this flow in a semi-infinite domain was modelled. An exact travelling wave solution of the model equation was obtained by using an ansatz method.

We construct the solution of this model by the use of the compatibility approach. The governing partial differential equation modelled in [79], with a slight change of notation, is given by

$$\begin{aligned} \rho\frac{\partial u}{\partial t} &= \mu\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2 \partial t} + \beta\frac{\partial^4 u}{\partial y^2 \partial t^2} + \beta_*\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + \gamma\frac{\partial^5 u}{\partial y^2 \partial t^3} \\ &\quad + \Gamma\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y \partial t}\right], \end{aligned} \tag{8.180}$$

where

$$\beta_* = 6(\beta_2 + \beta_3), \quad \Gamma = 2(3\gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + 3\gamma_7 + \gamma_8). \tag{8.181}$$

Using the classical Lie symmetry method, the invariant solution for equation (8.180) is written as [79]

$$u(y,t) = v(x), \quad \text{where} \quad x = y - ct. \tag{8.182}$$

Substituting Eq. (8.182) into Eq. (8.180) results in a fifth-order ordinary differential equation for $v(x)$:

$$\begin{aligned} -c\rho\frac{dv}{dx} &= \mu\frac{d^2v}{dx^2} - \alpha c\frac{d^3v}{dx^3} + \beta c^2\frac{d^4v}{dx^4} + \beta_*\left(\frac{dv}{dx}\right)^2\frac{d^2v}{dx^2} - \gamma c^3\frac{d^5v}{dx^5} \\ &\quad -c\Gamma\frac{d}{dx}\left[\left(\frac{dv}{dx}\right)^2\frac{d^2v}{dx^2}\right]. \end{aligned} \tag{8.183}$$

We now check that the fifth-order ordinary differential equation (8.183) is compatible with the first order ordinary differential equation

$$\frac{dv}{dx} + \theta v = 0 \;\; \text{with} \quad \theta \neq 0, \tag{8.184}$$

which has solution

$$v(x) = A\exp(-\theta x), \tag{8.185}$$

where A and θ are the parameters to be determined. Using the compatibility test for a fifth-order ordinary differential equation to be compatible with the first order ordinary differential equation given in Eq. (8.12), we obtain

$$0 = \left[-c\rho\theta + \mu\theta^2 + c\alpha\theta^3 + \beta c^2\theta^4 + \gamma c^3\theta^5\right] v + \left[\beta_*\theta^4 + 3c\Gamma\theta^5\right] v^3. \tag{8.186}$$

Separating Eq. (8.186) by the powers v^1 and v^3, we find

$$v^1 \;:\; -c\rho\theta + \mu\theta^2 + c\alpha\theta^3 + \beta c^2\theta^4 + \gamma c^3\theta^5 = 0, \tag{8.187}$$

$$v^3 \;:\; \beta_*\theta^4 + 3c\Gamma\theta^5 = 0. \tag{8.188}$$

From Eq. (188), we deduce

$$\theta = -\frac{\beta_*}{3c\Gamma}. \tag{8.189}$$

Substituting θ given by Eq. (8.189) into Eq. (8.187), we obtain

$$\frac{\rho\beta_*}{3\Gamma}+\frac{\mu\left(\beta_*\right)^2}{c^2\left(3\Gamma\right)^2}-\frac{\alpha\left(\beta_*\right)^3}{c^2\left(3\Gamma\right)^3}+\frac{\beta_1\left(\beta_*\right)^4}{c^2\left(3\Gamma\right)^4}-\frac{\gamma\left(\beta_*\right)^5}{c^2\left(3\Gamma\right)^5}=0. \tag{8.190}$$

Eq. (8.190) is the compatibility condition for a fifth-order ordinary differential equation (8.183) to be compatible with the first order ordinary differential equation (8.184). Thus the solution for $v(x)$ can be written as

$$v(x)=A\exp\left[\left(\frac{\beta_*}{3c\Gamma}\right)x\right], \tag{8.191}$$

and therefore the solution for $u(y,t)$ is

$$u(y,t)=A\exp\left[\left(\frac{\beta_*}{3c\Gamma}\right)(y-ct)\right]. \tag{8.192}$$

The compatibility condition (8.190) can be written as

$$c^2=\frac{\frac{\beta_*}{3\Gamma}\left[\gamma\left(\frac{\beta_*}{3\Gamma}\right)^3-\beta_1\left(\frac{\beta_*}{3\Gamma}\right)^2+\alpha\left(\frac{\beta_*}{3\Gamma}\right)-\mu\right]}{\rho}, \tag{8.193}$$

which determines the wave speed c. The wave speed will be real if

$$\gamma\left(\frac{\beta_*}{3\Gamma}\right)^3-\beta_1\left(\frac{\beta_*}{3\Gamma}\right)^2+\alpha\left(\frac{\beta_*}{3\Gamma}\right)-\mu>0. \tag{8.194}$$

In [79], the authors did not impose initial and boundary conditions on the solution. Here, we observe that the solution (8.192) does satisfies the boundary and initial conditions

$$
\begin{aligned}
u(0,t) &= U(t), && (8.195a)\\
u(y,0) &= P(y), && (8.195b)\\
\frac{\partial u(y,0)}{\partial t} &= Q(y), && (8.195c)\\
\frac{\partial^2 u(y,0)}{\partial t^2} &= R(y), && (8.195d)
\end{aligned}
$$

where

$$
\begin{aligned}
U(t) &= A\exp\left(-\frac{\beta_*}{3\Gamma}t\right), && (8.196a)\\
P(y) &= A\exp\left(\frac{\beta_*}{3c\Gamma}y\right), && (8.196b)\\
Q(y) &= -\left(\frac{\beta_*}{3c\Gamma}\right)P(y), && (8.196c)\\
R(y) &= \left(\frac{\beta_*}{3c\Gamma}\right)^2 P(y), && (8.196d)
\end{aligned}
$$

with

$$A = U(0) = P(0). \tag{8.197}$$

The boundary condition (8.195a) is the no-slip condition at $y = 0$. The initial velocity $U(0)$ of the rigid plate can be prescribed but its velocity for $t > 0$, $U(t)$, cannot be arbitrary and is given by (8.196a). Similarly the initial velocity profile $u(y,0)$ cannot be arbitrary and is given by (8.196b).

8.3.11 Unsteady magnetohydrodynamic flow of a fourth grade fluid

The unsteady magnetohydrodynamic flow of an incompressible electrically conducting fluid over a moving plate was investigated by Hayat et al. [80]. Translational type symmetries were employed to reduce the governing fifth-order nonlinear partial differential equation to a fifth order ordinary differential equation. The reduced ordinary differential equation was solved exactly by the use of an ansatz approach. Here, we present the exact solution of the reduced higher order ordinary differential equation by using the compatibility criterion. The governing partial differential equation investigated in [80], is given by

$$\begin{aligned}\rho\frac{\partial u}{\partial t} &= \mu\frac{\partial^2 u}{\partial y^2}+\alpha_1\frac{\partial^3 u}{\partial y^2\partial t}+\beta_1\frac{\partial^4 u}{\partial y^2\partial t^2}+\beta\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}+\gamma_1\frac{\partial^5 u}{\partial y^2\partial t^3}\\ &+\gamma\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y\partial t}\right]-\rho MH^2u, \qquad (8.198)\end{aligned}$$

where $\beta=6(\beta_2+\beta_3)$ is the third grade parameter and $\gamma=2(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)$ is the fourth grade fluid parameter, respectively. The appropriate boundary and initial conditions are

$$u(0,t) = u_0V(t), \quad t>0, \qquad (8.199)$$

$$u(\infty,t) = 0, \quad t>0, \qquad (8.200)$$

$$u(0,y) = g(y), \quad y>0, \qquad (8.201)$$

where u_0 is the reference velocity, $V(0)=1$ and $g(0)=u_0$.

The invariant solution specified in [80] is given as

$$u(y,t)=F(x_1), \quad \text{where} \quad x_1=y+ct. \qquad (8.202)$$

Substituting Eq. (8.202) into Eq. (8.198) results in a fifth-order ordinary differential equation for $F(x_1)$ given by

$$\begin{aligned}c\rho\frac{dF}{dx_1} &= \mu\frac{d^2F}{dx_1^2}+\alpha_1c\frac{d^3F}{dx_1^3}+\beta_1c^2\frac{d^4F}{dx_1^4}+\beta\left(\frac{dF}{dx_1}\right)^2\frac{d^2F}{dx_1^2}+\gamma_1c^3\frac{d^5F}{dx_1^5}\\ &+c\gamma\frac{d}{dx_1}\left[\left(\frac{dF}{dx_1}\right)^2\frac{d^2F}{dx_1^2}\right]-\rho MH^2F(x_1). \qquad (8.203)\end{aligned}$$

The transformed boundary conditions are

$$
\begin{aligned}
F(0) &= u_0, \\
F(x_1) &\to 0 \quad \text{as} \quad x_1 \to \infty.
\end{aligned}
\tag{8.204}
$$

The velocity of the plate, $V(t)$, and the initial profile , $g(y)$, which occur in Eqs. (8.199) and (8.201) cannot be prescribed arbitrary and are determined by the solution.

We now check the compatibility of a fifth-order ordinary differential equation (8.203) subject a first order ordinary differential equation

$$\frac{dF}{dx_1} + \lambda F = 0 \quad \text{with} \quad \lambda \neq 0, \tag{8.205}$$

which has solution

$$F(x_1) = A\exp(-\lambda x), \tag{8.206}$$

where A and λ are the parameters to be determined. Using the compatibility criterion given in Eq. (8.12) for a fifth-order ordinary differential equation to be compatible with the first order ordinary differential equation, we obtain

$$(c\rho\lambda + \mu\lambda^2 - \alpha_1 c\lambda^3 + \beta_1 c^2\lambda^4 - \gamma_1 c^3\lambda^5 - \rho M H^2)F + (\beta\lambda^4 - 3c\gamma\lambda^5)F^3 = 0. \tag{8.207}$$

Separating Eq. (8.207) by powers of F and F^3, we find

$$F \quad : \quad c\rho\lambda + \mu\lambda^2 - \alpha_1 c\lambda^3 + \beta_1 c^2\lambda^4 - \gamma_1 c^3\lambda^5 - \rho M H^2 = 0, \tag{8.208}$$

$$F^3 \quad : \quad \beta\lambda^4 - 3c\gamma\lambda^5 = 0. \tag{8.209}$$

From Eq. (8.209), we deduce

$$\lambda = \frac{\beta}{3c\gamma}. \tag{8.210}$$

Using this value of λ in Eq. (8.208), we find that

$$\frac{\rho\beta}{3\gamma}+\frac{\mu\beta^2}{c^2\left(3\gamma\right)^2}-\frac{\beta^3\alpha_1}{c^2\left(3\gamma\right)^3}+\frac{\beta_1\beta^4}{c^2\left(3\gamma\right)^4}-\frac{\gamma_1\beta^5}{c^2\left(3\gamma\right)^5}-\rho MH^2=0. \tag{8.211}$$

Thus the exact solution for $F(x_1)$ (provided the compatibility condition (8.211) holds) is written as

$$F(\xi)=u_0\exp\left[\frac{-\beta}{3c\gamma}x_1\right], \tag{8.212}$$

Finally, the exact solution $u(y,t)$ which satisfy the condition (8.211) is

$$u(y,t)=u_0\exp\left[\frac{-\beta(y+ct)}{3c\gamma}\right]\quad\text{with}\quad c<0, \tag{8.213}$$

The solution (8.213) is the same as the solution obtained in [] using the ansatz approach.

The compatibility condition (8.211) can be written as

$$c^2=\frac{\left(\frac{\beta_*}{3\gamma}\right)^2\left[\gamma_1\left(\frac{\beta_*}{3\gamma}\right)^3-\beta_1\left(\frac{\beta_*}{3\gamma}\right)^2+\alpha_1\left(\frac{\beta_*}{3\gamma}\right)^1-\mu\right]}{\left(\frac{\rho\beta}{3\gamma}-\rho MH^2\right)}, \tag{8.214}$$

provided

$$\rho MH^2\neq\frac{\rho\beta}{3\gamma}. \tag{8.215}$$

Eq. (8.214) reduces to Eq. (8.193) for $H=0$. Equation (8.214) determines the wave speed c. The range of the values of β for which c is real depends not only on the zeros of the cubic polynomial in β in the numerator of (8.214) but also on the sign of the denominator.

8.4 Conclusions

In this chapter, we have developed an efficient compatibility criterion for solving higher order nonlinear ordinary differential equations subject to lower order ordinary differential equations. The compatibility approach enabled us to construct the exact solutions of higher order nonlinear ordinary differential equations subject to lower order ordinary differential equations. This approach was illustrated by applying it to various nonlinear flow problems taken from the literature of Newtonian and non-Newtonian fluid dynamics. We have constructed exact (closed-form) solutions of these problems using the compatibility approach and found some new solutions of these problems. We have also unified the various ansatz based approaches for finding exact solutions for some of the flow models.

We emphasize one of the central points which is a consequence of our systematic approach. Other approaches such Adomian, Homotopy and similar methods have been applied numerously. However their rigorous application is unclear and the solutions they produce are not dictated by the physical problem at hand. The compatibility method as advocated here is a more systematic, rigorous and general method. The side conditions reflect the physical significance of the problem and can be chosen to a large extent in order to obtain meaningful special solutions. In some of the physical models it gave the speed of propagation of the travelling wave. Thus the side conditions represent a particular type of constraint on a physical system which when imposed makes the physical system compatible. In other words, these insights of the results are very important to state explicitly. Thus from a physical point of view there is a way to choose the side conditions on mechanical and physical grounds and this approach is fruitful as amply shown in this work. Also for the case of classical problem of steady viscous flow, the compatibility and group approach shows the non-uniqueness of solutions for high Reynolds number flows.

The method developed in this study is a contribution to the derivation and unification of exact solutions of nonlinear ordinary differential equations in fluids. To the best of our knowledge, the concept of compatibility and generalized groups has not been used before in fluid mechanics to investigate exact solutions of higher order ordinary differential equations subject to lower order ordinary differential equations. In this work, we have applied this approach only to nonlinear problems of Newtonian and non-Newtonian fluid dynamics thus achieving unification of the many ad hoc approaches utilized to date.

There is the potential to apply this technique to nonlinear problems in other areas such as solid mechanics, wave mechanics and general relativity.

Refinement and extension of this compatibility approach could be investigated. We have established the compatibility of higher order ordinary differential equations, from second-order up to fifth-order, subject to linear first order ordinary differential equation with known exponential type of exact solution and known symmetry group. We could consider the compatibility of higher order ordinary differential equations subject to other differential equations. This may lead to new exact solutions in other areas of science and engineering.

Chapter 9

Concluding Remarks and Future Work

9.1 Non-Newtonian fluid flow models

The dynamics and mechanics of non-Newtonian fluid flow problems have been an important area of interest in the recent few years. The study of non-Newtonian fluids involves the modelling of materials with dense molecular structure such as polymer solutions, slurries, blood, and paints. These material exhibit both viscous behavior like liquids and elastic behavior like solids. The flow phenomena of non-Newtonian fluids occur in a variety of industrial and technological applications. Some of them are the extraction of crude oil from petroleum products, oil and gas well drilling, food stuff, extrusion of molten plastic, paper and textile industry, and so on. The flow properties of non-Newtonian fluids are quite different from those of the Newtonian fluids. Therefore, in practical applications, one cannot replace the behavior of non-Newtonian fluids with Newtonian fluids and it is very important to analyze and understand the flow behavior of non-Newtonian fluids in order to obtain a thorough understanding and improve utilization in various manufactures. The model equations of the problem dealing with the flow of non-Newtonian fluids are higher order nonlinear and complex in nature. Several methods have been developed in the recent years to obtain the solutions of these sort of flow problems. Some of these techniques are variational iteration method, Adomian decomposition method, homotopy perturbation method, homotopy analysis method, semi-inverse variational method. But all these techniques are limited in the search for the exact (closed-form) solutions of the non-Newtonian fluid flow problems.

The purpose of this thesis was to develop the mathematical modelling and the exact solutions to some nonlinear problems arising in the study of non-Newtonian fluid mechanics. In our analysis, we considered the differential type fluid models of third and fourth grade. A third grade fluid model represents a further attempt towards the study of the flow behavior of non-Newtonian fluids. This model is known to capture the non-Newtonian affects such as shear thinning or shear thickening as well as normal stress. A Fourth grade fluid model is another very important subclass of the differential type fluids. This model is the most generalized model among the differential type fluids. This model is known to capture most of the non-Newtonian flow properties at one time. With these facts in mind, the studies related to the unsteady flow models of third and fourth grade fluids have been taken into consideration in Chapters $3-7$. In Chapters 3 and 4, the flow properties of third grade fluid model was investigated in detail and in Chapters $5-7$, we examined the physical behavior of fourth grade fluid in different types of flow situation.

9.2 Group theoretical analysis and invariant solutions

A symmetry of a differential equation is an invertible transformation that leaves the underlying equation invariant. We employed the Lie's algorithm to find symmetries that depend continuously on one-parameter transformations that constitute a group. In most of the fluid flow problems that we considered in this thesis, the governing equations were complex, so generating the determining equations of such problems turns out to be tedious. Fortunately, now there are several symbolic computation packages (Mathematica, Maple,...) available which are very helpful in finding the determining equations for any differential equation. In this work, we utilized the program YaLie [81], written in Mathematica for the generation and manipulation of the determining equations for symmetries, for the fourth grade problem considered in Chapter 7. Throughout this research, we first performed the mathematical modelling of the physical situation under consideration and then employed the complete classical Lie group analysis on the underlying models. After obtaining the symmetries, we utilized these to construct the group invariant solutions for each of the problem under investigation. We also employed the conditional symmetry approach on some particular models of third grade fluids in Chapter 4. Many equations arising in applications do not admit Lie point symmetries but have conditional symmetries. Thus this method is also powerful in obtaining exact solutions of such problems. In some situations if we were unable to construct the analytical results, we tried to reduce the underlying model with the help of symmetries and then solved the reduced model numerically using some of the powerful numerical solvers like **NDSolve** in Mathematica® and **PDEPE** in MatLab®. In both the analytical and numerical approaches, emphasis was given to study the physical behavior and properties of non-Newtonian fluids.

9.3 Compatibility approach for differential equations

Notwithstanding the above mentioned techniques, we also studied the concept of compatibility. During the second half of this thesis, the compatibility criteria were established for higher order ordinary differential equations to be compatible with lower order ordinary differential equations. Necessary and sufficient compatibility conditions were derived which can be used to construct exact solutions of higher order ordinary differential equations. Using this approach new exact solutions of nonlinear flow problems arising in the study of Newtonian and non-Newtonian fluids were derived. The ansatz approach for obtaining exact solutions for nonlinear flow models of Newtonian and non-Newtonian fluids were also unified with the application of the compatibility criteria.

It is often difficult to obtain exact solutions of a higher order nonlinear differential equation. For this reason, many researchers have assumed a form of the exact solution by trial and error. We provided a general compatibility criteria for many of these higher order ordinary differential equations arising in the study of Newtonian and non-Newtonian fluid models. This compatibility approach leads to some new exact solutions of these models and are also helpful in reproducing some existing solutions.

9.4 Future work

We conclude here by mentioning some further works.

Most of the work considered in this thesis is new and the presented analysis give us insights and observations in understanding the physical behavior of non-Newtonian fluids. In our analysis, we have only considered unidirectional flow which makes the nonlinear part of the inertia $(v.\nabla).v$ to vanish. What we would like to do in future work is to consider multidimensional flows in which one have to solve the system of ODEs and PDEs. Also in some models, the induced magnetic field was neglected. What would be interesting is to see how this affects the model once the above are taken into consideration.

The work considered in Chapter 4 regarding non-classical symmetry analysis can also be extended in a number of ways. The non-classical method can be used to obtain exact solutions of nonlinear problems dealing with the flow models of other non-Newtonian fluids and also the conditional symmetry approach can be used to reduce certain non-linear partial differential equations arising in the study of non-Newtonian fluid models to ordinary differential equations and then numerical investigation of the reduced ODEs can be made for specific nonlinear problems.

The compatibility method developed in Chapter 8 is a contribution to the derivation of exact solutions of nonlinear ordinary differential equations in fluids. To the best of our knowledge, the concept of compatibility has not been used before in fluid mechanics to investigate exact solutions of higher order ordinary differential equations subject to lower order ordinary differential equations. In this thesis, we applied the compatibility approach only to nonlinear problems of Newtonian and non-Newtonian fluid dynamics. There is the potential to apply this technique to nonlinear problems in other areas such as solid mechanics, wave mechanics and general relativity. Refinement and extension of this compatibility approach could also be investigated. We established the compatibility of higher order ordinary differential equations, from second-order up to fifth-order, subject to linear first order ordinary differential equations with known exponential type of exact solution. We could consider the compatibility of higher order ordinary differential equations subject to other differential equations. This may lead to new exact solutions in other areas of science and engineering.

Bibliography

[1] A. Nejat, A. Jalali and M. Sharbatdar, A Newton–Krylov finite volume algorithm for the power-law non-Newtonian fluid flow using pseudo-compressibility technique, J. Non-Newtonian Fluid Mech. 166 (2011) 1158 − 1172.

[2] R. Keslerová and K. Kozel, Numerical modelling of incompressible flows for Newtonian and non-Newtonian fluids, Math. Comput. Simul. 80 (2010) 1783 − 1794.

[3] C.M. Chen, F. Liu, I. Turner and V. Anh, Numerical methods with fourth-order spatial accuracy for variable-order nonlinear Stokes' first problem for a heated generalized second grade fluid, Comput. Math. Appl. 62 (2011) 971 − 986.

[4] J.E. Dunn and K. R. Rajagopal, Fluids of differential type: Critical review and thermodynamics analysis, Int. J. Eng. Sci. 33 (1995) 689 − 729.

[5] R. S. Rivlin and J. L. Ericksen, Stress deformation relation for isotropic materials, J. Ration. Mech. Anal. 4 (1955) 323 − 329.

[6] W.R. Schowalter, Mechanics of non-Newtonian fluids, Pergamon, New York. (1978).

[7] M. Emin Erdogen and C. Erdem Imrak, On unsteady unidirectional flows of a second grade fluid, Int. J. Non-Linear Mech. 40 (2005) 1238 − 1251.

[8] R.L. Fosdick and K.R. Rajagopal, Anomalous features in the model of second-order fluids, Arch. Ration. Mech. Anal. 70 (1979) 145 − 152.

[9] P. D. Ariel, Flow of a third grade fluid through a porous flat channel, Int. J. Eng. Sci. 41 (2003) 1267 – 1285.

[10] B. Sahoo, Hiemenz flow and heat transfer of a third grade fluid, Comm. Nonlinear Sci. Numer. Simul. 14 (2009) 811 – 826.

[11] K.R. Rajagopal, Boundedness and uniqueness of fluids of the differential type, Acta Ciencia Indica. 18 (1) (1982) 1 – 11.

[12] K.R. Rajagopal, On boundary conditions for fluids of the differential type. Navier-Stokes Equations and Related Nonlinear Problems, Plenum Press, New York (1995) 273 – 278.

[13] K.R. Rajagopal and P.N. Kaloni, Some remarks on boundary conditions for fluids of differential type, in: G.A.C. Graham, S.K. Malik, (Eds.) Continum Mechanics and Applications. Hemisphere, New York, 1989, pp. 935 – 942.

[14] R.L. Fosdick and K.R. Rajagopal, Thermodynamics and stability of fluids of third grade, Proc. R. Soc. Lond. A. 339 (1980) 351 – 377.

[15] T. Hayat, A.H. Kara and E. Momoniat, The unsteady flow of a fourth-grade fluid past a porous plate, Math. Comp. Model. 41 (2005) 1347 – 1353.

[16] S. Nadeem and M. Ali, Analytical solutions for pipe flow of a fourth grade fluid with Reynold and Vogel's models of viscosities, Commun. Nonlinear. Sci. Numer. Simulat. 14 (2009) 2073 – 2090.

[17] Y. Wang and W. Wu, Unsteady flow of a fourth-grade fluid due to an oscillating plate, Int. J. Non-Linear Mech. 42 (2007) 432 – 441.

[18] M.H. Haroun, Non-linear peristaltic flow of a fourth grade fluid in an inclined asymmetric channel, Comput. Mater. Sci. 39 (2007) 324 – 333.

[19] M. Sajid, T. Hayat and S. Asghar, On the analytic solution of the steady flow of a fourth grade fluid, Physics Letters A. 355 (2006) 18 – 26.

[20] T. Hayat, M. Sajid and M. Ayub, On explicit analytic solution for MHD pipe flow of a fourth grade fluid, Comm. Nonlinear Sci. Numer. Simul. 13 (2008) 745 – 751.

[21] S. Nadeem, T. Hayat, S. Abbasbandy and M. Ali, Effects of partial slip on a fourth-grade fluid with variable viscosity: An analytic solution, Non-Linear Anal: Real World Appl. 11 (2010) 856 – 868.

[22] A.M. Abd-Alla, S.M. Abo-Dahab and H.D. El-Shahrany, Effects of rotation and initial stress on peristaltic transport of fourth grade fluid with heat transfer and induced magnetic field, J. Magnetism Magnetic Mater. 349 (2014) 268 – 280.

[23] L.V. Ovsiannikov, Group Analysis of Differential Equations, Academic, New York, 1982.

[24] P.J. Olver, Application of Lie Groups to Differential Equations, Springer, New York, 1986.

[25] C. Wafo-Soh, Invariant solutions of the unidirectional flow of an electrically charged power law non-Newtonian fluid over a flat plate in presence of a transverse magnetic field, Comm. Nonlinear Sci. Numer. Simul. 10 (2005) 537 – 548.

[26] P.Y. Picard, Some exact solutions of the ideal MHD equations through symmetry reduction method, J. Math. Anal. Appl. 337 (2008) 360 – 385.

[27] M. Pakdemirli, M. Yürüsoy and A. Küçükbursa, Symmetry groups of boundary layer equations of a class of non-Newtonian fluids, Int. J. Non-Linear Mech. 31 (1996) 267 – 276.

[28] M. Yürüsoy, Similarity solutions for creeping flow and heat transfer in second grade fluids, Int. J. Non-Linear Mech. 39 (2004) 665 – 672.

[29] M. Yürüsoy, Unsteady boundary layer flow of power-law fluid on stretching sheet surface, Int. J. Eng. Sci. 44/5 – 6 (2006) 325 – 332.

[30] G.W. Bluman and J.D. Cole, The general similarity solution of the heat equation, J. Math. Mech. 18 (1969) 1025 – 1042.

[31] E. Pucci and G. Saccomandi, Evolution equations, invariant surface conditions and functional separation of variables, Physica D: Nonlinear Phenomena. 139 (2000) 28 – 47.

[32] P.A. Clarkson, Nonclassical symmetry reductions of the Boussinesq equation, Chaos Solitons & Fractals. 5 (1995) 2261 – 2301.

[33] P.J. Olver and E.M. Vorob'ev, Nonclassical and conditional symmetries, in: N.H. Ibragimov (Ed.), CRC Handbook of Lie Group Analysis, vol. 3, CRC Press, Boca Raton, 1994.

[34] E. Pucci and G. Saccomandi, On the weak symmetry groups of partial differential equations, J. Math. Anal. Appl. 163 (1992) 588 – 598.

[35] E. Pucci, Similarity reductions of partial differential equations, J. Phys. A. 25 (1992) 2631 – 2640.

[36] P.A. Clarkson, Nonclassical symmetry reductions of nonlinear partial differential equations, Math. Comput. Model. 18 (1993) 45 – 68.

[37] W.I. Fushchich, Conditional symmetry of equations of nonlinear mathematical physics, Ukrain. Math. Zh. 43 (1991) 1456 – 1470.

[38] D.J. Arrigo and J.M. Hill, Nonclassical symmetries for nonlinear diffusion and absorption, Stud. Appl. Math. 94 (1995) 21 – 39.

[39] S. Spichak and V. Stognii, Conditional symmetry and exact solutions of the Kramers equation, Nonlinear Math. Phys. 2 (1997) 450 – 454.

[40] A.F. Barannyk and Yu.D. Moskalenko, Conditional symmetry and exact solution of the multidimensional nonlinear d'Alembert equation, Nonlinear Math. Phys. 3 (1996) 336 – 340.

[41] T. Aziz, F.M. Mahomed and A. Aziz, Group invariant solutions for the unsteady MHD flow of a third grade fluid in a porous medium, Int. J. Non-Linear Mech. 47 (2012) 792 – 798.

[42] Taha Aziz, F.M. Mahomed, Muhammad Ayub and D.P. Mason, Non-linear time-dependent flow models of third grade fluids: A conditional symmetry approach, Int. J. Non-Linear Mech. 54 (2013) 55 – 65.

[43] Taha Aziz and F.M. Mahomed, Closed-Form Solutions for a Nonlinear Partial Differential Equation Arising in the Study of a Fourth Grade Fluid Model, J. Appl. Math. Volume (2012), Article ID 931587, 16 pages.

[44] Taha Aziz and F.M. Mahomed, Reductions and solutions for the unsteady flow of a fourth grade fluid on a porous plate, Appl. Math. Comput. 219 (2013) 9187 – 9195.

[45] Taha Aziz and F.M. Mahomed, Group theoretical analysis and invariant solutions for unsteady flow of a fourth grade fluid through a porous space, Appl. Math. Comput. 2014 (submitted).

[46] Taha Aziz, F.M. Mahomed and D.P. Mason, A unified compatibility method for exact solutions of non-linear flow models of Newtonian and non-Newtonian fluids, Int. J. Non-Linear Mech. (2015) doi:10.1016/j.ijnonlinmec.2015.01.003.

[47] M. Born and L. Infeld, Foundation of the new field theory, Proc. Roy. Soc. A. 144 (1934) 425 – 451.

[48] T. Hayat, F. Shahzad and M. Ayub, Analytical solution for the steady flow of the third grade fluid in a porous half space, Appl. Math. Model. 31 (2007) 2424 – 2432.

[49] T. Hayat, H.M. Mamboundou and F.M. Mahomed, Unsteady solutions in a third grade fluid filling the porous space, Math. Prob. Eng. $Vol.$ 2008, $ArticleID$ 139560, 13 $pages$, 2008.

[50] A. Aziz and T. Aziz, MHD flow of a third grade fluid in a porous half-space with plate suction or injection: An analytical approach, Appl. Math. Comput., 218 (2012) 10443 – 10453.

[51] T. Hayat, H. M. Mamboundou and F. M. Mahomed, A note on some solutions for the flow of a fourth grade fluid in a porous space, Non-Linear Anal: Real World Appl. 10 (2009) 368 – 374.

[52] M. Miklavcic and C.Y. Wang, Viscous flow due to a shrinking sheet, Quart. Appl. Math. 64 (2006) 283 – 290.

[53] C.Y. Wang, Stagnation flow towards a shrinking sheet, Int. J. Non-linear Mech. 43 (2008) 377 – 382.

[54] A. Ishak, Y.Y. Lok and I. Pop, Stagnation-point flow over a shrinking sheet in a micropolar fluid, Chem. Eng. Comm. 197 (2010) 1417 – 1427.

[55] N. Bachok, A. Ishak and I. Pop, Boundary layer stagnation-point flow toward a stretching/shrinking sheet in a nanofluid, ASME J. Heat Transf. $ArticleID$ 054501, 5 $pages$, 2013.

[56] N.A. Yacob and A. Ishak, Stagnation point flow towards a stretching/shrinking sheet in a micropolar fluid with a convective surface boundary condition, Can. J. Chem. Eng. 90 (2012) 621 – 626.

[57] A. Ishak, Y.Y. Lok and I. Pop, Non-Newtonian power-law fluid flow past a shrinking sheet with suction, Chem. Eng. Comm. 199 (2012) 142 – 150.

[58] N. Bachok, A. Ishak and I. Pop, Unsteady boundary-layer flow of a nanofluid over a permeable stretching/shrinking sheet, Int. J. Heat Mass Transf. 55 (2012) 2102 – 2109.

[59] F.M. Fernández, On a perturbation treatment of a model for MHD viscous flow, Appl. Math. Comput. 217 (2010) 2307 – 2310.

[60] N.F.M. Noor, S.A. Kechil and I. Hashim, Simple non-perturbative solution for MHD viscous flow due to a shrinking sheet, Comm. Nonlinear Sci. Numer. Simulat. 15 (2010) 144 – 148.

[61] M. Sajid and T. Hayat, The application of homotopy analysis method for MHD viscous flow due to a shrinking sheet, Chaos, Solitons Fractals. 39 (2009) 1317 – 1323.

[62] T. Fang and J. Zhang, Closed-form exact solutions of MHD viscous flow over a shrinking sheet, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 2853 – 2857.

[63] L.J. Crane, Flow past stretching plate, Z. Angew. Math. Phys. 21 (1970) 645 – 647.

[64] L.J. Grubka and K.M. Bobba, Heat transfer characteristics of a continuous, stretching surface with variable temperature, ASME J. Heat Transf. 107 (1985) 248 – 250.

[65] A. Ishak, R. Nazar and I. Pop, Heat transfer over an unsteady stretching permeable surface with prescribed wall temperature, Nonlinear Analysis: Real World Appl. 10 (2009) 2909 – 2913.

[66] A. Ishak, R. Nazar and I. Pop, Hydromagnetic flow and heat transfer adjacent to a stretching vertical sheet, Heat Mass Transf. 44 (2008) 921 – 927.

[67] M.-I. Char, Heat transfer in a hydromagnetic flow over a stretching sheet, Heat Mass Transf. 29 (1994) 495 – 500.

[68] B. Sahoo, Effects of slip on sheet-driven flow and heat transfer of a non-Newtonian fluid past a stretching sheet, Comput. Math. Appl. 61 (2011) 1442 – 1456.

[69] F. Labropulu, D. Li and I. Pop, Non-orthogonal stagnation-point flow towards a stretching surface in a non-Newtonian fluid with heat transfer, Int. J. Thermal Sci. 49 (2010) 1042 – 1050.

[70] M.S. Abel, P.S. Datti and N. Mahesha, Flow and heat transfer in a power-law fluid over a stretching sheet with variable thermal conductivity and non-uniform heat source, Int. J. Heat Mass Transf. 52 (2009) 2902 – 2913.

[71] A.R. Ghotbi, Homotopy analysis method for solving the MHD flow over a non-linear stretching sheet, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 2653 – 2663.

[72] M. Fathizadeh, M. Madani, Y. Khan, N. Faraz, A. Yıldırım and S. Tutkun, An effective modification of the homotopy perturbation method for MHD viscous flow over a stretching sheet, Journal of King Saud University – Science (2011), *doi* : 10.1016/*j.jksus*.2011.08.003.

[73] K. Vajravelu, Viscous flow over a nonlinearly stretching sheet, Appl. Math. Comput. 124 (2001) 281 – 288.

[74] M. Khan, Z. Abbas and T. Hayat, Analytic solution for flow of Sisko fluid through a porous medium, Transp. Porous Med. 71 (2008) 23 – 37.

[75] T. Hayat, H. M. Mamboundou, F.M. Mahomed and M. Khan, Reduction and solutions for MHD flow of a Sisko fluid in a porous medium, J. Porous Media. 12(7) (2009) 695 – 714.

[76] F. Ahmad, A simple analytical solution for the steady flow of a third grade fluid in a porous half space, Comm. Nonlinear Sci. Numer. Simulat. 14 (2009) 2848 – 2852.

[77] K. Vajravelu and D. Rollins, Hydromagnetic flow of a second grade fluid over a stretching sheet, Appl. Math. Comput. 148 (2004) 783 – 791.

[78] K. Vajravelu and T. Roper, Flow and heat transfer in a second grade fluid over a stretching sheet, Int. J. Nonlinear Mech. 34 (1999) 1031 – 1036.

[79] T. Hayat, A.H. Kara and E. Momoniat, Travelling wave solutions to Stokes' problem for a fourth grade fluid, Appl. Math. Model. 33 (2009) 1613 – 1619.

[80] T. Hayat, H. Mambili-Mamboundou, C.M. Khalique and F.M. Mahomed, Effect of magnetic field on the flow of a fourth order fluid, Nonlinear Analysis: Real World Appl. 10 (2009) 3413 – 3419.

[81] J. M. Diaz, Short guide to YaLie: Yet another Lie Mathematica package for Lie Symmetries, http:/library.wolfram.com/infocentre/MatheSource/4231/YaLie.ps.

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MHD flow of a third grade fluid in a porous half space with plate suction or injection: An analytical approach

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ABSTRACT

The present work deals with the modeling and solution of the unsteady flow of an incompressible third grade fluid over a porous plate within a porous medium. The flow is generated due to an arbitrary velocity of the porous plate. The fluid is electrically conducting in the presence of a uniform magnetic field applied transversely to the flow. Lie group theory is employed to find symmetries of the modeled equation. These symmetries have been applied to transform the original third order partial differential equation into third order ordinary differential equations. These third order ordinary differential equations are then solved analytically and numerically. The manner in which various emerging parameters have an effect on the structure of the velocity is discussed with the help of several graphs.

1. Introduction

Fluid flow through a porous media is a common phenomena in nature and in many fields of science and engineering. Important flow phenomenons include transport of water in living plants, trees and fertilizers or wastes in soil. In industry fluid flow through porous media has attracted much attention due to its importance in several technological processes, for example, filtration, catalysis, chromatography, petroleum exploration and recovery, cooling of electronic equipment, etc.

In recent years non-Newtonian fluid flow models have been studied extensively because of their relevance to many engineering and industrial applications, particularly in the extraction of crude oil from petroleum products, synthetic fibers, food stuffs, etc. The non-Newtonian models exhibit a nonlinear relationship between stress and rate of strain. A special differential type fluid, namely the third grade fluid model, is considered in this study. This model is known to capture the non-Newtonian affects such as shear thinning or shear thickening as well as normal stresses. The third grade fluid model is thermodynamically analyzed in detail by Fosdick and Rajagopal [1]. Some recent contributions dealing with the flows of third grade fluid are [2–5]. Few authors have dealt with the flows of a third grade fluid over a porous plate [6] or within a porous medium [7]. The governing equations for non-Newtonain fluid flow models are higher order nonlinear partial differential equations, whose analysis would present a particular challenge for researchers and scientists. Due to the nonlinear terms in partial differential equations the exact solutions are difficult to be obtained for the flow of non-Newtonian fluids. Some recent studies in finding the analytical solutions for such problems can be found in [8–13]. To the best of our knowledge no attempt is available in the literature where the unsteady MHD flow of a third grade fluid over a porous plate within a porous medium is discussed.

The purpose of present investigation is to discuss the analytical solutions for the unsteady MHD flow of a third grade fluid over a porous plate within a porous medium. The flow is induced due to arbitrary velocity $V(t)$ of the plate. The fluid is

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TRAVELLING WAVE SOLUTIONS FOR THE UNSTEADY FLOW OF A THIRD GRADE FLUID INDUCED DUE TO IMPULSIVE MOTION OF FLAT POROUS PLATE EMBEDDED IN A POROUS MEDIUM

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ABSTRACT

This work describes the time-dependent flow of an incompressible third grade fluid filling the porous half space over an infinite porous plate. The flow is induced due to the motion of the porous plate in its own plane with an arbitrary velocity $V(t)$. Translational type symmetries are employed to perform the travelling wave reduction into an ordinary differential equation of the governing nonlinear partial differential equation which arises from the laws of mass and momentum. The reduced ordinary differential equation is solved exactly, for a particular case, as well as by using the homotopy analysis method (HAM). The better solution from the physical point of view is argued to be the HAM solution. The essentials features of the various emerging parameters of the flow problem are presented and discussed.

Keywords: Third grade fluid, Suction/Injection, Travelling wave solutions, Homotopy analysis method (HAM).

1. INTRODUCTION

The study of non-Newtonian fluids involves the modelling of materials with dense molecular structure such as geological materials, polymer solutions, slurries, hydrocarbon oils, blood and paints. These material exhibit both viscous behavior like liquids and elastic behavior like solids. Therefore, the understanding of the complex behavior and properties of non-Newtonian fluids is crucial these days. The problems dealing with the flow of non-Newtonian fluids have several technical applications in engineering and industry. Some of them are the extraction of crude oil from petroleum products, oil and gas well drilling, food stuff, extrusion of molten plastic, paper and textile industry and so on. The flow properties of non-Newtonian fluids are quite different from those of the Newtonian fluids. Therefore, in practical applications one cannot replace the behavior of non-Newtonian fluids with Newtonian fluids and it is very important to analyze and understand the flow behavior of non-Newtonian fluids in order to obtain a thorough understanding and improve utilization in various manufactures.

One of the widely accepted models amongst non-Newtonian fluids is the class of Rivlin-Ericksen fluids of differential type [1] Rivlin-Ericksen fluids have secured special attention in order to describe the several non-standard characteristics of non-Newtonian fluids such as rod climbing, shear thinning, shear thickening and normal stress effects. Literature survey witness that much focus has been given to the flow problems of a second grade fluid [2-5]. A second grade fluid model is the simplest subclass of differential type fluids for which one can reasonably hope to establish an analytic result. In most of the flow aspects, the governing equations for a second grade fluid are linear. Although a second grade fluid model for steady flows is used to predict the normal stress differences, it does not correspond to shear thinning or thickening if the shear viscosity is assumed to be constant. Therefore some experiments may be well described by third grade fluid. The mathematical model of a third grade fluid represents a more realistic description of the behavior of non-Newtonian fluids. A third grade fluid model represents a further attempt towards the study of the flow properties of non-Newtonian fluids. Therefore, a third grade fluid model has considered in this study. This model is known to capture the non-Newtonian affects

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Research Article

Prandtl's Boundary Layer Equation for Two-Dimensional Flow: Exact Solutions via the Simplest Equation Method

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The simplest equation method is employed to construct some new exact closed-form solutions of the general Prandtl's boundary layer equation for two-dimensional flow with vanishing or uniform mainstream velocity. We obtain solutions for the case when the simplest equation is the Bernoulli equation or the Riccati equation. Prandtl's boundary layer equation arises in the study of various physical models of fluid dynamics. Thus finding the exact solutions of this equation is of great importance and interest.

1. Introduction

Many scientific and engineering problems and phenomena are modeled by nonlinear differential equations. Therefore, the study of nonlinear differential equations has been an active area of research from the past few decades. Considerable attention has been devoted to the construction of exact solutions of nonlinear equations because of their important role in the study of nonlinear physical models. For nonlinear differential equations, we do not have the freedom to compute exact (closed-form) solutions and for analytical work we have to rely on some approximate analytical or numerical techniques which may be helpful for us to understand the complex physical phenomena involved. The exact solutions of the nonlinear differential equations are of great interest and physically more important. These exact solutions, if reported, facilitate the verification of complex numerical codes and are also helpful in a stability analysis for solving special nonlinear problems. In recent years, much attention has been devoted to the development of several powerful and useful methods for finding exact analytical solutions of nonlinear differential equations. These methods include the powerful Lie group method [1], the sine-cosine method [2], the tanh method [3, 4], the extended tanh-function method [5], the Backlund transformation method [6], the transformed rational function method [7], the (G'/G)-expansion method [8], the exponential function rational expansion method [9], and the Adomian's decomposition method [10].

Prandtl [11] introduced boundary layer theory in 1904 to understand the flow behavior of a viscous fluid near a solid boundary. Prandtl gave the concept of a boundary layer in large Reynolds number flows and derived the boundary layer equations by simplifying the Navier-Stokes equations to yield approximate solutions. Prandtl's boundary layer equations arise in various physical models of fluid mechanics. The equations of the boundary layer theory have been the subject of considerable interest, since they represent an important simplification of the original Navier-Stokes equations. These equations arise in the study of steady flows produced by wall jets, free jets, and liquid jets, the flow past a stretching plate/surface, flow induced due to a shrinking sheet, and so on. These boundary layer equations are usually solved subject to certain boundary conditions depending upon the specific physical model considered. Blasius [12] solved the Prandtl's boundary layer equations for a flat moving plate problem and gave a power series solution of the problem. Falkner and Skan [13] generalized the Blasius boundary layer problem by considering the boundary layer flow over a wedge

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Reductions and solutions for the unsteady flow of a fourth grade fluid on a porous plate

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ABSTRACT

The unsteady unidirectional flow of an incompressible fourth grade fluid over an infinite porous plate is studied. The flow is induced due to the motion of the plate in its own plane with an arbitrary velocity. Symmetry reductions are performed to transform the governing nonlinear partial differential equations into ordinary differential equations. The reduced equations are then solved analytically and numerically. The influence of various physical parameters of interest on the velocity profile are shown and discussed through several graphs. A comparison of the present analysis shows excellent agreement between analytical and numerical solutions.

1. Introduction

Being scientifically appealing and challenging due to their complex physical nature, the interest of investigators in the study of non-Newtonian fluid flow problems has grown considerably during the past few decades. As a result a wide range of applications in engineering and industry has been performed. Amongst the many models which have been used to describe the flow behavior of non-Newtonian fluids, the fluids of differential type have received special attention as well as much controversy, see e.g. [1] for a complete discussion of the relevant issues. Some recent contributions dealing with the flow of differential type fluids are given in Refs. [2–9].

A simplest subclass of differential type fluids is the second grade fluid. Although the second grade model is found to predict the normal stress differences, it does not take into account the shear thinning and shear thickening phenomena due to its constant apparent shear viscosity. For this reason some experiments may be well described through fluids of grade three or four. With this fact in mind we consider a fourth grade fluid in the present analysis for generality. The fourth grade fluid model is incapable of describing relaxation and retardation phenomena. However, this model describes the shear thinning and shear thickening effects. The governing equations for the fourth grade fluid model are non-linear and much more complicated than those of Newtonian fluids, and thus require additional conditions for a unique solution. This issue has been discussed in detail in Refs. [10–12]. Literature survey witness that limited studies are reported in the literature up to now dealing with the flow problems of fourth grade fluid and these studies further become few when we speak about the exact (closed-form) solutions of these problems. These exact solutions if available, facilitate the verification of numerical solvers and are also helpful in stability analysis. Consequently, the exact (closed-form) solutions of the non-Newtonian fluid flow problems are physically more important. Some useful investigations in this direction are made in the studies [13–22].

In this article, the unsteady unidirectional flow of an incompressible fourth grade fluid bounded by a porous plate is examined. The disturbance is caused due to the motion of the plate in its own plane with an arbitrary velocity $V(t)$. The

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Research Article

Shock Wave Solution for a Nonlinear Partial Differential Equation Arising in the Study of a Non-Newtonian Fourth Grade Fluid Model

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This study focuses on obtaining a new class of closed-form shock wave solution also known as soliton solution for a nonlinear partial differential equation which governs the unsteady magnetohydrodynamics (MHD) flow of an incompressible fourth grade fluid model. The travelling wave symmetry formulation of the model leads to a shock wave solution of the problem. The restriction on the physical parameters of the flow problem also falls out naturally in the course of derivation of the solution.

1. Introduction

A shock wave (also called shock front or simply "shock") is a type of propagating disturbance through a media. It actually expresses a sharp discontinuity of the parameters that delineate the media. Unlike solitons, the energy is a conserved quantity and thus remains constant during its propagation. Shock wave dissipates energy relatively quickly with distance. One source of a shock wave is when the supersonic jets fly at a speed that is greater than the speed of sound. This results in the drag force on aircraft with shocks. These waves also appear in various interesting phenomena in real-life situations. For example, solitons appear in the propagation of pulses through optical fibers. Another example is where cnoidal waves appear in shallow water waves although an extremely scarce phenomena.

The dynamics and mechanics of non-Newtonain fluid flow problems have been an important area of interest in the recent few years. The flow phenomena of non-Newtonian fluids occur in a variety of industrial and technological applications. Because of the diverse physical structure and behavior of non-Newtonian fluids, there is no single mathematical expression which describes all the characteristics of non-Newtonian fluids. Due to this fact, several models of non-Newtonian fluids have been proposed. Apart from this fact, the model equations of the problem dealing with the flow of non-Newtonian fluids are higher-order nonlinear and complex in nature. Several methods have been developed in the recent years to obtain the solutions of these sort of flow problems. Some of these techniques are variational iteration method, the Adomian decomposition method, homotopy perturbation method, homotopy analysis method, and semi-inverse variational method. But all these techniques fail to develop the exact (closed-form) solutions of the non-Newtonian fluid flow problems.

One of the simplest classes of non-Newtonian fluid models is the second grade fluid [1]. Although the second grade model is found to predict the normal stress differences, it does not take into account the shear thinning and shear thickening phenomena due to its constant apparent shear viscosity. For this reason, some experiments may be well described through fluids of grade three or four. Very little attention has been given to date to the flows of fourth grade fluid [2]. This model is known as the most generalized model amongst the differential-type non-Newtonian fluid models [3]. The fourth grade fluid model describes most of the non-Newtonian flow properties at one time. This model is known to capture the interesting non-Newtonian flow properties such as shear thinning and shear thickening that many other non-Newtonian models do not show. This model is also capable of

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Research Article

Shock Wave Solutions for Some Nonlinear Flow Models Arising in the Study of a Non-Newtonian Third Grade Fluid

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This study is based upon constructing a new class of closed-form shock wave solutions for some nonlinear problems arising in the study of a third grade fluid model. The Lie symmetry reduction technique has been employed to reduce the governing nonlinear partial differential equations into nonlinear ordinary differential equations. The reduced equations are then solved analytically, and the shock wave solutions are constructed. The conditions on the physical parameters of the flow problems also fall out naturally in the process of the derivation of the solutions.

1. Introduction

A shock wave is a disturbance that propagates through a media. Shock is yielded when a disturbance is made to move through a fluid faster than the speed of sound (the celerity) of the medium. This can occur when a solid object is forced through a fluid. It represents a sharp discontinuity of the parameters that delineate the media. Unlike solutions where the energy is a conserved quantity and thus remains constant during its propagation, shock wave dissipates energy relatively quickly with distance. One source of a shock wave is when the supersonic jets fly at a speed that is greater than the speed of sound. This results in the drag force on aircraft with shocks. These waves also appear in various interesting phenomena in real life situations. For example, solitons appear in the propagation of pulses through optical fibers. Another example is where cnoidal waves appear in shallow water waves, although this is an extremely scarce phenomena. Some interesting communications dealing with the shock wave solutions are found in [1–3].

During the past several decades, the study of the nonlinear problems dealing with the flow of non-Newtonian fluids has attracted considerable attention. This interest is due to several important applications in engineering and industry such as reactive polymer flows in heterogeneous porous media, electrochemical generation of elemental bromine in porous electrode systems, manufacture of intumescent paints for fire safety applications, extraction of crude oil from the petroleum products, synthetic fibers, and paper production [4]. Due to the diverse physical nature of non-Newtonian fluids, there is no single constitutive expression which describe the physical behavior of all non-Newtonian fluid models. Because of this important issue, several models of non-Newtonian fluids have been proposed in the literature. Together with this factor, the mathematical modelling of non-Newtonian incompressible fluid flows gives rise to nonlinear and complicated differential equations. As a consequence of this nonlinearity factor, the exact (closed-form) solutions of these sort of problems are scarce in general. Several techniques and methods have been developed in the recent few years to construct the solutions of these non-Newtonian fluid flow problems. Some of these useful methods are variational iteration method, Adomian decomposition method, homotopy perturbation method, homotopy analysis method, and semi-inverse variational method. Literature survey witnesses that, despite all these methods, the exact (closed-form) solutions of the non-Newtonian fluid flow problems are still rare in the literature, and it is not surprising that new exact (closed-form) solutions are most welcome, provided they correspond to physically realistic situations. Some interesting and useful communications in this area are made in the studies [5–14].

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Non-linear time-dependent flow models of third grade fluids: A conditional symmetry approach

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ABSTRACT

In this communication some non-linear flow problems dealing with the unsteady flow of a third grade fluid in porous half-space are analyzed. A new class of closed-form conditionally invariant solutions for these flow models are constructed by using the conditional or non-classical symmetry approach. All possible non-classical symmetries of the model equations are obtained and various new classically invariant solutions have been constructed. The solutions are valid for a half-space and also satisfy the physically relevant initial and the boundary conditions.

1. Introduction

During the past several decades, the study of non-linear problems dealing with flow models of non-Newtonian fluids has gained prodigious attention. This interest is due to several important applications in engineering and industry such as aerodynamic heating, electrostatic precipitation, petroleum industry, reactive polymer flows in heterogeneous porous media, electrochemical generation of elemental bromine in porous electrode systems, manufacture of intumescent paints for fire safety applications, extraction of crude oil from petroleum products, synthetic fibers, paper production and so forth. Due to the intricate microstructure of non-Newtonian fluids, there is no single constitutive expression available in the literature which describes the physical behavior and properties of all non-Newtonian fluid models. Because of this, several models of non-Newtonian fluids have been proposed. The mathematical modeling of non-Newtonian incompressible fluid flows gives rise to complicated non-linear differential equations. The situation becomes more involved when we consider exact (closed-form) solutions of these problems. Several techniques and methods have been developed recently to construct solutions of non-Newtonian fluid flow problems. Some of these useful methods are the variational iteration method, Adomian decomposition method, homotopy perturbation method, homotopy analysis method, semi-inverse variational method and symmetry method. Despite all these methods, exact (closed-form) solutions of non-Newtonian fluid flow problems are still rare in the literature and it is not surprising that new exact (closed-form) solutions are most welcome provided they correspond to physically realistic situations.

Amongst the many models which have been used to describe the physical flow behavior of non-Newtonian fluids, the fluids of differential type have received special attention as well as much controversy, see for example [1] for a complete discussion of the relevant issues. Rivlin–Ericksen fluids of differential type have secured special attention in order to describe several non-standard characteristics of non-Newtonian fluids such as rod climbing, shear thinning, shear thickening and normal stress effects. Literature surveys point out that much focus has been given to the flow problems of a non-Newtonian second grade fluid model [2]. A second grade fluid model is the simplest subclass of differential type fluids for which one can reasonably hope to establish an analytic result. In most of the flow aspects, the governing equations for a second grade fluid are linear. Although a second grade fluid model for steady flows is used to predict the normal stress differences, it does not correspond to shear thinning or thickening if the shear viscosity is assumed to be constant. Therefore some experiments may be better described by a third grade fluid. The mathematical model of a third grade fluid represents a more realistic description of the behavior of non-Newtonian fluids. A third grade fluid model represents a further attempt toward the study of the flow structure of non-Newtonian fluids. Therefore, a third grade fluid model is considered in this study. This model is known to capture the non-Newtonian effects such as shear

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Research Article

Closed-Form Solutions for a Nonlinear Partial Differential Equation Arising in the Study of a Fourth Grade Fluid Model

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The unsteady unidirectional flow of an incompressible fourth grade fluid bounded by a suddenly moved rigid plate is studied. The governing nonlinear higher order partial differential equation for this flow in a semiinfinite domain is modelled. Translational symmetries in variables t and y are employed to construct two different classes of closed-form travelling wave solutions of the model equation. A conditional symmetry solution of the model equation is also obtained. The physical behavior and the properties of various interesting flow parameters on the structure of the velocity are presented and discussed. In particular, the significance of the rheological effects are mentioned.

1. Introduction

There has been substantial progress in the study of the behavior and properties of viscoelastic fluids over the past couple of years. This progress is due to the fact that these viscoelastic materials are involved in many manufacturing processes in today's industry. Modelling viscoelastic flows is important for understanding and predicting the behavior of processes and thus for designing optimal flow configurations and for selecting operating conditions. Most of the viscoelastic fluids in industry do not adhere to the commonly accepted assumption of a linear relationship between the stress and the rate of strain and thus are characterized as non-Newtonian fluids. As a result of their complex physical structure and diversity in nature, these materials cannot have a single constitutive relation which describes all the properties of viscoelastic fluids. The flow properties of viscoelastic (non-Newtonian) fluids are quite different from those of viscous and Newtonian fluids. Therefore, in practical applications, one cannot replace the behavior of non-Newtonian fluids with Newtonian fluids

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Group invariant solutions for the unsteady MHD flow of a third grade fluid in a porous medium

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ABSTRACT

This work describes the time-dependent flow of an incompressible non-Newtonian fluid over an infinite rigid plate. The flow is induced due to the arbitrary velocity $V(t)$ of the plate. The fluid occupies the porous half space $y>0$ and is also electrically conducting in the presence of a constant applied magnetic field in the transverse direction to the flow. Analytical solutions of the governing non-linear partial differential equation for the unidirectional flow of a third grade fluid are established using the symmetry approach. We construct three types of analytical solutions by employing the Lie symmetry method and the better solution from the physical point of view is shown to be the non-travelling wave solution. We also present numerical solutions of the governing PDE and the reduced ODE and compare with the analytical results. Finally, the influence of emerging parameters are studied through several graphs with emphasis on the study of the effects of the magnetic field and non-Newtonian fluid parameters.

1. Introduction

Recently, non-Newtonian fluid models have been studied extensively due to their relevance to many industrial and engineering applications, particularly in the extraction of crude oil from petroleum products, synthetic fibers, foodstuffs, etc. In order to mathematically describe the properties of non-Newtonian fluids, several constitutive equations have been proposed. One of the widely accepted models in non-Newtonian fluids is the class of Rivlin–Ericksen fluids [1]. Much focus has been given to the simplest subclass of such fluids known as the second grade fluids. Although a second grade fluid predicts normal stress differences, this model is inadequate as it does not explain shear thinning and shear thickening phenomena when viscosity is constant. A special Rivlin–Ericksen fluid, namely, a third grade fluid model is considered in this study. This model is known to capture the non-Newtonian effects such as shear thinning or shear thickening as well as normal stresses.

Some of the important and useful work on the third grade fluid model are as follows: Fosdick and Rajagopal [2] made a complete thermodynamical analysis of third grade fluid and showed the restriction on the stress constitutive equation. Rajagopal [3] investigated some stability characteristics of third grade fluids and showed that they exhibit features different from that of Newtonian and second grade fluids. The flow of third grade fluids between heated parallel plates was treated by Szeri and Rajagopal [4] and flow past an infinite porous flat plate with suction was examined by Rajagopal et al. [5]. Kacou et al. [6] studied the flow of a third grade in a journal bearing and constructed a perturbation solution. Recently, Vajravelu et al. [7] obtained numerical and exact analytical solutions for third grade fluid between rotating cylinders and did a comparison between numerical and analytical results. Akyildiz et al. with Vajravelu in another paper [8] discussed the exact solutions for third grade fluid for bounded and unbounded domain cases and compared those results with numerical ones.

One of the most important issues arises when we take into account the flow problems of non-Newtonian fluids is that the governing equations of these models are higher order non-linear and much more complicated than those of Newtonian fluids whose analysis would present a particular challenge for researchers and scientists. As a result of non-linearity, the exact solutions are sometimes difficult to obtain for the flows of even Newtonian fluids. Such exact solutions further become scarce when non-Newtonian fluids are taken into account. The exact solutions, if available, facilitate the verification of numerical codes and are also helpful in qualitative analysis. Consequently, the exact solutions for the flows of non-Newtonian fluids have much physical importance. Some of the useful recent contributions in this direction are made in the studies [10–15].

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GENERAL AND APPLIED PHYSICS

Invariant Solutions for the Unsteady Magnetohydrodynamics (MHD) Flow of a Fourth-Grade Fluid Induced Due to the Impulsive Motion of a Flat Porous Plate

Taha Aziz · A. B. Magan · F. M. Mahomed

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Abstract An analysis is carried out to study the time-dependent flow of an incompressible electrically conducting fourth-grade fluid over an infinite porous plate. The flow is caused by the motion of the porous plate in its own plane with an impulsive velocity $V(t)$. The governing nonlinear problem is solved by invoking the Lie group theoretic approach and a numerical technique. Travelling wave solutions of the forward and backward type, together with a steady state solution, form the basis of our analytical analysis. Further, the closed-form solutions are also compared against numerical results. The essential features of the embedded parameters are described. In particular, the physical significance of the plate suction/injection and magnetic field is studied.

Keywords Fourth-grade fluid · Lie symmetry approach · MHD flow · Travelling wave solutions · Porous plate

1 Introduction

The scientific appeal of non-Newtonian fluid mechanics has advocated a deeper study of its theory. There has been substantial interest in the steady and unsteady flows of non-Newtonian fluids over the last few decades. One reason for this interest is the wide variety of applications of such flows, both natural and industrial. These applications range from the extraction of crude oil from petroleum products to the polymer industry. Spin coating is a classic example where the coating fluids are typically non-Newtonian. The non-Newtonian fluid models exhibit a nonlinear stress-strain relation or have a non-zero yield stress. Unlike Newtonian fluids, non-Newtonian fluids, in view of their diversity in nature, cannot have a single constitutive relation. One of the widely accepted models amongst non-Newtonian fluids is the class of Rivlin-Eriksen fluids [1]. Rivlin-Eriksen fluids have secured special attention in order to describe the several non-standard features of non-Newtonian fluids such as shear thickening, shear thinning and normal stress effects.

Due to the complex physical structure of non-Newtonian fluids, these fluids are classified according to the properties which characterise them. Literature surveys witness that much focus has been given to the flow problems of a second-grade fluid [2–4]. A second-grade fluid model, despite its simplicity, has drawbacks in that although it can predict the normal stress differences, it fails to recognise the shear thinning and shear thickening properties if the shear viscosity is assumed to be a constant. This impediment is remedied by the third-grade fluid [5–8] and the fourth-grade fluid models. Very little attention has been given in literature to the studies related to the fourth-grade fluid model [9, 10]. Currently, only the fourth-grade fluid seems to define the properties of non-Newtonian flow phenomena most generally. The fourth-grade fluid model is known to capture the interesting non-Newtonian flow properties such as shear thinning and shear thickening that many other non-Newtonian models do not exhibit. This model is also capable of describing the normal stress effects that lead to phenomena like "die-swell" and "rod-climbing" [11]. With these facts in mind, we have considered a fourth-grade fluid model in this study. In general, the governing equations of the problems dealing with the flow of fourth-grade fluids are

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