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Research Article

Fundamental Transformations to Inhomogeneous Parabolic PDEs

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The present paper is devoted to inhomogeneous $(1+1)$ parabolic equations. In particular, we find a mapping which transforms the equations into a target equation. We prove a necessary condition which allows for the inhomogeneity to vanish. Consequently, we provide a concise presentation on a method of integrability of these equations. The theory is illustrated by several examples.

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1. Introduction

A myriad of physical processes are described by inhomogeneous equations. The inhomogeneous term often represents a source or forcing term for external actions on a process. In the heat transfer equation, for example, the inhomogeneous version contains a heat source. It is an area of great interest to develop strategies to solve such equations.

Lie [1], the pioneer of symmetry transformations, performed a complete characterisation of second-order equations or parabolic equations. He identified so called "canonical forms" of these equations. Subsequently, there exists a desire for transforming a solution of a given PDE into a solution of a different PDE. Ideally, in one direction, one may search for a one-to-one mapping between the infinitesimal symmetries of a given equation to a target equation. This would require their Lie algebras to be isomorphic. Such a mapping must be invertible and provide a means of linking the solutions of a given PDE to a desired PDE. An invaluable PDE in this regard is in fact the heat equation, so much so, that many works have been devoted to reducing parabolic equations to it [2–5]. The algebraic properties of the linear heat equation make it a superior target equation. The properties worth mentioning are its fundamental solution [6], convolution and Lie algebra.

Parabolic PDEs, such as the heat equation, play a central role in mathematical modelling due to their ability to describe time-dependent processes that involve diffusion, heat flow, and other forms of gradual change. Parabolic PDEs appear in every science, in modelling heat conduction [7], population dynamics [8], option pricing [9] and tumours [10]. We recall an important feature of a parabolic equation of the following form:

$$\frac{\partial}{\partial t}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t)+b(x,t)\frac{\partial}{\partial x}u(x,t)+c(x,t)u(x,t)=0. \tag{1}$$

This equation admits a linear equivalence transformation, $u(x,t)=\omega(x,t)e^{-\gamma(x,t)}$, if and only if its coefficients satisfy the following condition:

$$b_x-b_{xx}+b_t+2c_x=0, \tag{2}$$

which then guarantees the solvability for γ [11, 12].

The aim of this study is to formalise an approach to a general class of parabolic PDEs that admit an inhomogeneous function. We map this general class to certain versions of the heat equation, revealing hidden structures. In a nonlinear situation, our transformations reduce a nonlinear problem to a linear one. The transformations we present encompass a unified solution technique to facilitate the analysis of potent PDEs in the literature. This approach may impact the existing numerical methods as well. Some cases where numerical methods were applied on models relevant to our study include some important option pricing studies [13, 14].

Unlike the existing works in the literature, which typically assume homogeneous equations and moreover, are aimed at specific equations; our study introduces a flexible framework that can accommodate inhomogeneous equations. More importantly, this study has broader applicability to a wide range of PDEs, allowing for novel theoretical insights and computational strategies that distinguish it from the existing techniques.

For example, several interesting studies have involved variable transformations to solve important parabolic equations [15–17]. However, none of these works have proposed a method to deal with a generalised equation admitting an inhomogeneous term. These shortcomings have motivated the development of our new approach. In addition, our study addresses a significant connection between linear parabolic equations and a generalised class of nonlinear equations.

This paper is organised into several sections. In Section 2, we explore an inhomogeneous nonlinear equation with constant coefficients and prove how it may be transformed to a simple linear equation. We describe how its general solution may be obtained as well as its Cauchy problem. Furthermore, in Section 3, we propose a comprehensive transformation method to solve a family of inhomogeneous linear equations. Examples are provided in all cases. Section 4 offers a summary of the study and expected future developments.

2. An Inhomogeneous Nonlinear Equation

In this section, we show how transformations serve as a bridge between complicated models and the heat equation. Fundamental solutions to difficult inhomogeneous PDEs are found through an intricate framework. We demonstrate several practical applications of the approach.

Theorem 1. *The $(1+1)$ inhomogeneous nonlinear equation*

$$\frac{\partial u(x,t)}{\partial t} - A\frac{\partial^2}{\partial x^2}u(x,t) - B\frac{\partial}{\partial x}u(x,t) - C\left(\frac{\partial}{\partial x}u(x,t)\right)^2 - D = E(t), \tag{3}$$

where $A \neq 0, B, C \neq 0$ and D are arbitrary constants, is transformed to the classical homogeneous heat equation

$$\omega_\tau - \omega_{yy} = 0, \tag{4}$$

under the invertible transformations.

a. $A > 0$, let

$$y = \frac{x}{\sqrt{A}}, \quad \tau = t, \tag{5}$$

$$u(x,t) = \frac{A}{C}\left(\ln\omega(y,\tau) - \frac{By}{2\sqrt{A}} - F(\tau)\right), \tag{6}$$

on condition that $F(\tau)$ satisfies the relation

$$E(\tau) - \frac{-4((d/d\tau)F(\tau))A - 4CD + B^2}{4C} = 0. \tag{7}$$

b. $A < 0$, let $A = -\widetilde{A}$, where $\widetilde{A} > 0$ and

$$y = \frac{x}{\sqrt{\widetilde{A}}}, \quad \tau = -t, \tag{8}$$

$$u(x,t) = -\frac{\widetilde{A}}{C}\left(\ln\omega(y,\tau) + \frac{By}{2\sqrt{\widetilde{A}}} - F(\tau)\right), \tag{9}$$

on condition that $F(\tau)$ satisfies the relation

$$E(\tau) - \frac{-4((d/d\tau)F(\tau))\widetilde{A} - 4CD + B^2}{4C} = 0. \tag{10}$$

Proof 1. Applying the Hopf–Cole transformation [18, 19] for $A > 0$,

$$u = \frac{A}{C}\ln v(x,t), \tag{11}$$

converts (3) to

$$\frac{\partial v(x,t)}{\partial t} - A\frac{\partial^2}{\partial x^2}v(x,t) - B\frac{\partial}{\partial x}v(x,t) - \frac{C}{A}v(x,t)(D + E(t)) = 0. \tag{12}$$

Equation (12) is a linear parabolic equation, and by application of the transformation (5), we found

$$\frac{\partial}{\partial t}v(y,\tau)-\frac{\partial^2}{\partial y^2}v(y,\tau)-B\frac{\partial}{\partial y}v(y,\tau)\frac{1}{\sqrt{A}}-\frac{Cv(y,\tau)(D+E(\tau))}{A}=0, \tag{13}$$

Taking a reversible transformation of the dependent variable $v(y,\tau)$,

$$v(y,\tau)=\omega(y,\tau)e^{-(By/(2\sqrt{A}))-F(\tau)}, \tag{14}$$

subject to (7), which transforms (13) to the homogeneous heat equation (4). Finally, the pair of transformations (11) and (14) produces (6), a solution to the general inhomogeneous equation (3).

For $A<0$, we apply the Hopf–Cole transformation $u=-(\widetilde{A}/C)\ln v(x,t)$, which converts (3) to

$$\frac{\partial v(x,t)}{\partial t}+\widetilde{A}\frac{\partial^2}{\partial x^2}v(x,t)-B\frac{\partial}{\partial x}v(x,t)+\frac{C}{A}v(x,t)(D+E(t))=0. \tag{15}$$

Analogously, the proof follows as above, but with $v(y,\tau)=\omega(y,\tau)e^{(By/2\sqrt{A})-F(\tau)}$. □

It is common to combine D with $E(t)$; however, many PDEs that fall under this equation depict both a constant term and a separate inhomogeneous term (for instance, Example 1 below). Our method allows one to analyse such equations, regardless if D is combined with $E(t)$ or not. We note that Theorem 1 gives us a method to solve equations of the type (3). We present some examples on its utility.

Example 1. Consider a type of the inhomogeneous Hamilton–Jacobi–Bellman equation [20, 21] to model mean-variance hedging, an essential component in finance that plays an important role in risk management. The form of the equation that we consider has $A>0, B, C$ and D as arbitrary nonzero constants in (3), and with $E(t)=t^2$. Thus, by condition (7), we find that

$$F(\tau)=-\frac{\tau^3C}{3A}-\frac{CDt}{A}+\frac{B^2\tau}{4A}+C_1, \tag{16}$$

for C_1 as any constant. Hence, by (14), the transformation for the dependent variable is

$$v(y,\tau)=\omega(y,\tau)e^{-(By/2)(1/\sqrt{A})+\left(\tau^3C/3A\right)+(CD\tau/A)-\left(B^2\tau/4A\right)-C_1}. \tag{17}$$

Now to solve (3), we invoke the transformation (11) to obtain

$$u(x,t)=\frac{A}{C}\ln\left(\omega(y,\tau)e^{-(By/2)(1/\sqrt{A})+\left(C\tau^3/3A\right)+(CD\tau/A)-\left(B^2\tau/4A\right)-C_1}\right), \tag{18}$$

where $\omega(y,\tau)$ is any solution to the heat equation. Suppose we adopt the heat kernel,

$$\Psi(y,\tau):=\begin{cases}\dfrac{1}{\sqrt{4\pi\tau}}e^{-\left(y^2/4\tau\right)}, & (y\in\mathbb{R},\tau>0),\\[2ex] 0, & (y\in\mathbb{R},\tau<0),\end{cases} \tag{19}$$

which is a solution to the heat equation. This gives (18) as an explicit solution for (3), viz.,

$$u(x,t)=-\frac{A}{2C}\left(2\ln(2)+\ln(\pi)-2\ln\left(\frac{1}{\sqrt{t}}e^{-(1/12)\left(\left(-4Ct^4+3B^2t^2-12CDt^2+12AC_1t+6Btx+3x^2\right)/(At)\right)}\right)\right). \tag{20}$$

Example 2. Suppose that one desires an initial value solution of (3), say the initial condition $u(x,0)=f(x)$. Then, the transformed initial condition for equation (4) is $\omega(y,0)=\exp(C/Af(y)+\gamma(y,0))\equiv g(y)$.

This Cauchy problem for the heat equation,

$$\begin{cases}\omega_\tau-\omega_{yy}=0, & \mathbb{R}\times(0,\infty),\\ \omega(0,\tau)=g(y), & \text{on }\mathbb{R}\times\{\tau=0\},\end{cases} \tag{21}$$

admits the convolution as a solution [22].

$$\omega(y,\tau)=\int_{\mathbb{R}}\Psi(y-\eta,\tau)g(\eta)d\eta\quad(y\in\mathbb{R},\tau>0), \tag{22}$$

that is also a solution of the heat equation (4).

For example, let $f(x)=2x$, so that $g(y)=e^{(2yC/\sqrt{A})+(By/2\sqrt{A})+C_1}$. Solving the convolution integral gives us

$$\omega(y,\tau)=\frac{1}{2}e^{(1/4A)\left(B^2\tau+8BC\tau+16C^2\tau+2\sqrt{A}By+8\sqrt{A}Cy+4AC_1\right)}\cdot\left(2\sqrt{(B\tau+4C\tau+\sqrt{A}y)^2}+\sqrt{(B\tau+4C\tau+\sqrt{A}y)^2}\times\operatorname{erf}\left(\frac{1}{2}(B\tau+4C\tau+\sqrt{A}y)\frac{1}{\sqrt{A}}\frac{1}{\sqrt{\tau}}\right)-(B\tau+4C\tau+\sqrt{A}y)\times\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{1}{A\tau}(B\tau+4C\tau+\sqrt{A}y)^2}\right)\right)\frac{1}{\sqrt{(B\tau+4C\tau+\sqrt{A}y)^2}}. \tag{23}$$

Finally, the solution of the nonlinear equation (3) is

$$u(x,t)=\frac{A}{C}\ln\left(\frac{1}{2}e^{\left(B^2t+8BCt+16C^2t+4AC_1+2Bx+8Cx\right)/(4A)}\left(2\sqrt{(Bt+4Ct+x)^2}+\sqrt{(Bt+4Ct+x)^2}\operatorname{erf}\left(\frac{Bt+4Ct+x}{2}\frac{1}{\sqrt{A}}\frac{1}{\sqrt{t}}\right)-(Bt+4Ct+x)\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{(Bt+4Ct+x)^2}{At}}\right)\right)e^{-(Bx/2A)+\left(Ct^3/3A\right)+(CDt/A)-\left(B^2t/4A\right)-C_1}\times\frac{1}{\sqrt{(Bt+4Ct+x)^2}}\right). \tag{24}$$

3. Inhomogeneous Linear Equations

In this section, we establish mappings for a general inhomogeneous linear equation to the inhomogeneous heat equation. A special case of constant coefficients is also presented along with detailed examples.

Theorem 2. *The (1 + 1) inhomogeneous linear equation*

$$\frac{\partial u(x,t)}{\partial t}-A(x,t)\frac{\partial^2}{\partial x^2}u(x,t)-B(x,t)\frac{\partial}{\partial x}u(x,t)-C(x,t)u(x,t)=F(x,t), \tag{25}$$

is transformed to the inhomogeneous heat equation

$$\omega_\tau-\omega_{yy}=F(y,\tau)e^{\gamma(y,\tau)}, \tag{26}$$

under the invertible transformations

$$y=\int\frac{1}{\sqrt{|A(x,t)|}}dx,\quad \tau=\pm t, \tag{27}$$

followed by

$$u(x,t)=\omega(y,\tau)e^{-\gamma(y,\tau)}, \tag{28}$$

where γ satisfies the relations

$$\gamma_y=-\frac{1}{2}\overline{B}(y,\tau),\gamma_\tau=\frac{1}{4}\overline{B}(y,\tau)^2-\frac{1}{2}\overline{B}_y(y,\tau)+\overline{C}(y,\tau). \tag{29}$$

Proof 2. The transformation of (25) with (27) yields

$$\frac{\partial}{\partial\tau}u(y,\tau)-\frac{\partial^2}{\partial y^2}u(y,\tau)+\overline{B}(y,\tau)\frac{\partial}{\partial y}u(y,\tau)+\overline{C}(y,\tau)u(y,\tau)=F(y,\tau), \tag{30}$$

where

$$\overline{B}\ (y,\tau) = \frac{1}{2\sqrt{A(x,t)}}\Big(-\int A_{tt}(x,t)(A(x,t))^{-(3/2)}\mathrm{d}x\sqrt{A(x,t)} + A_x(x,t) - 2B(x,t)\Big), \overline{C}(y,\tau) = -C(x,t). \tag{31}$$

Applying (28), we find that

$$\big(\omega_\tau - \omega_{yy} + \big(\overline{B} + 2\gamma_y\big)\omega_y + \big(\gamma_{yy} - \gamma_y^2 - \gamma_\tau - \overline{B}\gamma_y + \overline{C}\big)\omega\big)e^{-\gamma(y,\tau)} = F(y,\tau). \tag{32}$$

On the LHS, we desire the conditions $\overline{B} + 2\gamma_y = 0$ and $\gamma_{yy} - \gamma_y^2 - \gamma_\tau - \overline{B}\gamma_y + \overline{C} = 0$. Substituting the first condition and its partial derivative in y into the second equation gives a system that γ must satisfy, viz., (29).

Hence, equation (32) becomes $(\omega_\tau - \omega_{yy})e^{-\gamma(y,\tau)} = F(y,\tau)$, and the result follows. □

Remark 1. The compatibility condition (2) must be satisfied and for $A(x,t) < 0$, choose $\tau = -t$.

Example 3. Consider the equation of a brain tumour [23], with (25) having $A(x,t) = \lambda, \lambda > 0$, $B(x,t) = (2\lambda)/x$, and $C(x,t) = \kappa - t$. Applying Theorem 2 with $\tau = t$, $y = x/\sqrt{\lambda}$ gives $\overline{B} = -(2/y), \overline{C} = t - \kappa$ and $\gamma = \ln(y) + (\tau^2/2) - \kappa\tau + C_1$. The tumour equation is then solved easily with the help of (28) and solving the inhomogeneous heat equation (26) as follows. If we let the inhomogeneous term be $F(x,t) = x + t$, then we are required to solve

$$\omega_\tau - \omega_{yy} = (y\sqrt{\lambda} + \tau)\exp\bigg(\ln(y) + \frac{\tau^2}{2} - \kappa\tau + C_1\bigg). \tag{33}$$

By Duhamel's principle, this equation is solved by

$$\omega(y,\tau) = \int_0^\tau \int_{\mathbb{R}} \Psi(y-m,\tau-s)F(m,s)e^{\gamma(m,s)}dmds, \tag{34}$$

that is,

$$\begin{aligned}\omega(y,\tau) = {} & \int_0^\tau \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi(\tau-s)}}\, e^{-\left((y-m)^2/4(\tau-s)\right)}(m\sqrt{\lambda} + s)e^{\ln(m) + \left(s^2/2\right) - \kappa s + C_1}dmds \\ & + \frac{1}{2}e^{C_1 - \left(\kappa^2/2\right)}\Bigg(2\Big(e^{1/2(t-\kappa)^2} - e^{1/2\kappa^2}\Big)\bigg(\frac{x}{\sqrt{\lambda}} - 2\sqrt{\lambda}\bigg) \\ & + \sqrt{2}\sqrt{\pi}\bigg(x\kappa\frac{1}{\sqrt{\lambda}} + x^2\frac{1}{\sqrt{\lambda}} + 2(t-\kappa)\sqrt{\lambda}\bigg)\mathrm{erfi}\bigg(\frac{(t-\kappa)\sqrt{2}}{2}\bigg) \\ & + \sqrt{2}\sqrt{\pi}\bigg(x\kappa\frac{1}{\sqrt{\lambda}} + x^2\frac{1}{\sqrt{\lambda}} + 2(t-\kappa)\sqrt{\lambda}\bigg)\mathrm{erfi}\bigg(\frac{\kappa\sqrt{2}}{2}\bigg)\Bigg),\end{aligned} \tag{35}$$

where $\mathrm{erfi}(x) = (2/\sqrt{\pi})\int_0^x e^{t^2}dt$. Hence, by the transformation (28), we have a solution of the tumour *equation as*

$$\begin{aligned}u(x,t) = {} & \frac{1}{2x}\bigg(-2\sqrt{\pi}\bigg(\mathrm{erfi}\bigg(\frac{1}{2}(\kappa-t)\sqrt{2}\bigg) - \mathrm{erfi}\bigg(\frac{1}{2}\kappa\sqrt{2}\bigg)\bigg) \\ & \times\bigg(\frac{1}{2}x^2 + \frac{1}{2}x\kappa + \lambda(t-\kappa)\bigg)\sqrt{2}\,e^{-(1/2)(\kappa-t)^2} - 2(x-2\lambda)\Big(e^{-(1/2)t(-2\kappa+t)} - 1\Big)\bigg).\end{aligned} \tag{36}$$

Lemma 1. *The (1 + 1) inhomogeneous linear equation*

$$\frac{\partial u(x,t)}{\partial t} - A\frac{\partial^2}{\partial x^2}u(x,t) - B\frac{\partial}{\partial x}u(x,t) - Cu(x,t) = F(x,t), \tag{37}$$

where $A \neq 0, B$ and C are arbitrary constants and are transformed to the inhomogeneous heat equation (26) under the invertible transformations (28), for C_1 as any constant, and

a. *$A > 0$, let*

$$y = \frac{x}{\sqrt{A}}, \tau = t, \gamma(y,\tau) = \frac{By}{2}\frac{1}{\sqrt{A}} - C\tau + \frac{B^2\tau}{4A} + C_1, \tag{38}$$

b. *$A < 0$, let $A = -\widetilde{A}$, where $\widetilde{A} > 0$ and*

$$y = \frac{x}{\sqrt{\widetilde{A}}}, \tau = -t, \gamma(y,\tau) = \frac{By}{2}\frac{1}{\sqrt{\widetilde{A}}} - C\tau + \frac{B^2\tau}{4\widetilde{A}} + C_1. \tag{39}$$

Proof 3. The LHS of equation (37) is a linear parabolic equation, and by application of the independent variable transformation of (38) yields

$$\frac{\partial}{\partial \tau}u(y,\tau) - \frac{\partial^2}{\partial y^2}u(y,\tau) - B\frac{\partial}{\partial y}u(y,\tau)\frac{1}{\sqrt{A}} - Cu(y,\tau) = F(y,\tau), \tag{40}$$

for $A > 0$. Thereafter, the transformation (28) of the dependent variable $u(y,t)$ with $\gamma(y,\tau)$ from (38) is applied to (40) to find (26). Similarly, for $A < 0$, we apply the independent variable transformation of (39) to the LHS of (37) to find

$$\frac{\partial}{\partial \tau}u(y,\tau) - \frac{\partial^2}{\partial y^2}u(y,\tau) - B\frac{\partial}{\partial y}u(y,\tau)\frac{1}{\sqrt{\widetilde{A}}} - Cu(y,\tau) = F(y,\tau), \tag{41}$$

and thereafter the transformation (28) of the dependent variable $u(y,t)$ with $\gamma(y,\tau)$ from (39) is applied to (41) to find (26). □

Example 4. Consider the equation describing the diffusional process in a force field of weight [24], that is, equation (37) with $A > 0$, $C = 0$, and let $F(x,t) = x/\sqrt{A}$ for the inhomogeneous function. After applying the transformations (38), one needs to solve $\omega_\tau - \omega_{yy} = ye^{(By/2)(1/\sqrt{A}) + (B^2\tau/4A) + C_1}$.

By Duhamel's principle, equation (26) will have the following solution:

$$\begin{aligned}\omega(y,\tau) &= \int_0^\tau \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi(\tau - s)}} e^{-\left((y-m)^2/4(\tau-s)\right)} m e^{(Bm/2)(1/\sqrt{A}) + \left(B^2 s/4A\right) + C_1} dm ds \\ &= \frac{\tau(2\sqrt{A}\,y + B\tau)e^{(B(2\sqrt{A}\,y + B\tau)/(4A))C_1}}{2\sqrt{A}}.\end{aligned} \tag{42}$$

Hence, by the transformation (28), we have a solution of the diffusional process in a force field of weight, in original variables, as

$$u(x,t) = \frac{t(Bt + 2x)}{2}\frac{1}{\sqrt{A}}. \tag{43}$$

Example 5. Now suppose one wanted to impose initial data, $u(x,0) = h(x)$, on (37). By transformation (28), this implies the condition $\omega(y,0) = h(y)e^{\gamma(y,0)} \equiv g(y)$, on (26). That is, we solve the Cauchy problem for the inhomogeneous heat equation,

$$\begin{cases} \omega_\tau - \omega_{yy} = F(y,\tau)e^{\gamma(y,\tau)}, & \text{in } \mathbb{R} \times (0,\infty), \\ \omega(0,\tau) = g(y), & \text{on } \mathbb{R} \times \{\tau = 0\}. \end{cases} \tag{44}$$

We can construct that this problem has the following solution:

$$\omega(y,\tau) = \int_{\mathbb{R}} \Psi(y - m,\tau)g(m)dm + \int_0^\tau \int_{\mathbb{R}} \Psi(y - m,\tau - s)F(m,s)e^{\gamma(m,s)} dm ds. \tag{45}$$

For example, let $h(x) = x^2$, then $g(y) = h(y)e^{\gamma(y,0)} = y^2 Ae^{(By/2\sqrt{A}) + C_1}$. Solving the above integral yields

$$\begin{aligned}\omega(y,\tau) &= e^{(B/4A)(B\tau + 2\sqrt{A}\,y) + C_1}\left(B^2\tau^2 + 2\sqrt{A}\,B\tau y + A\left(y^2 + 2\tau\right)\right) \\ &\quad + \frac{\tau}{2}e^{(B/4A)(B\tau + 2\sqrt{A}\,y) + C_1}(B\tau + 2\sqrt{A}\,y)\frac{1}{\sqrt{A}},\end{aligned} \tag{46}$$

so that

$$u(x,t) = \left(2A^{3/2}t + (Bt+x)^2\sqrt{A} + \frac{Bt^2}{2} + tx\right)\frac{1}{\sqrt{A}}, \quad (47)$$

is an initial value solution to (37).

Setting $t = 0$ checks that the solution is subject to the initial condition $h(x) = x^2$.

4. Conclusion and Future Directions

We have performed a treatment of some particular second-order PDEs, by connecting them to the heat equation. We were able to retrieve, via transformations, a solution process for these PDEs with vast applications in the real world, encompassing initial conditions and external influences. It is proven here, that transformations are fundamental tools in mathematics and offer new perspectives on the analysis and problem solving of parabolic problems.

Regarding the exact solutions of inhomogeneous parabolic PDEs that lack smoothness or even exhibit weak singularities, it is possible that conversion to a simpler model may induce various changes in the behaviour of solutions. To account for this problem, one must carefully consider the following. It is best to apply transformations that are smooth and invertible (a diffeomorphism), so that the solution to the transformed PDE will typically preserve original behaviours and features. In general, coordinate changes that are linear are more desirable, while nonlinear transformations should be used with caution. Furthermore, it is a positive outlook that our method involves exact transformations and not approximate transformations, as the latter is less reliable. If one is concerned that badly behaved or nowhere differentiable solutions may be hidden by changing variables, a numerical analysis may assist in considering the validity and accuracy of the resulting solutions.

In future work, we intend to construct a new transformation process of highly nonlinear systems of equations that capture some of the features of parabolic PDEs, having an essential understanding of the models' temporal evolution and spatial effects.

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Disclosure

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The author declares no conflicts of interest.

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