THE INVESTIGATION OF FILTER SANDS

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THE INVESTIGATION OF FILTER SANDS

AND THE ESTABLISHMENT OF A CRITERION FOR THE CONDITION THAT A SAND IS A FILTER IN ITSELF.

A THESIS FOR THE DEGREE OF MASTER OF SCIENCE

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WITWATERSRAND

BY

G.E. BLIGHT, B.Sc.(Eng.)

Thesis

DECLARATION:

I hereby declare that this thesis:

(a) is entirely my own work
and (b) has not been submitted as a thesis for a Master's degree at any other university.

SIGNED: G. E. Blight

DATE: 22ND JANUARY, 1958.

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SYNOPSIS:

Comparatively little is known about the behaviour of filter sands when subjected to the action of percolating water.

The magnitude of changes of grading due to percolating water and the manner in which these changes take place are described and criteria are suggested for the selection of sands which will not undergo extensive grading changes.

Changes of grading due to physical and chemical weathering of sand particles are briefly described together with a series of tests on model sand filters which was designed to test the validity of existing filter design criteria under severe conditions.

Information is given about the hydraulic behaviour of sands undergoing grading changes and variable hydraulic properties are correlated with changes in grain structure and grading.

CHAPTER I

INTRODUCTION TO THE THESIS

This thesis is concerned mainly with the study of the type of sand normally used in "soil engineering filters", that is to say filters incorporated in drainage systems, for the relief of pore pressures in earth dams, highway embankments and other masses of earth fill; drains designed to reduce hydrostatic uplift pressures beneath basement floors and reservoirs; and filters designed to protect earth dams, embankments, slopes and other soil structures from both internal and external erosion.

Historical:

Although Soil Mechanics is one of the newest branches of Civil Engineering, the use of filters for drainage is very old and the principles of their use have been appreciated for many years.

Records of the design of drainage systems for earth dams go back to 1902 when the 123 ft. high Tabeaud Earth Dam in California was constructed and provided with a complete internal drainage system.

Rational methods of design for drainage filters were first applied in the 1920's when Karl von Terzaghi employed a weighted relief filter in the repair of concrete aprons of overflow dams in the Austrian Alps. The design criteria for these and subsequent filters have been based on considerations regarding the stability and permeability of both the foundation and filter layers.

Terzaghi's criteria for the design of layered filters were extensively investigated on the behalf of the United States War Department by G.E. Bertram, in 1938, who used a series of tests on model filters composed of layers of single sized sands.

Further investigations were carried out during the 1940's by the U.S. War Dept. These investigations took the form of a series of full scale field tests on the Sardis Dam and were used to provide data for the design of both under-drains and filter well points.

The latest investigation for which results have been published was carried out by K.P. Karpoff of the United States Bureau of Reclamation and lead to the establishment of a further series of filter design criteria for filters constructed of graded natural sands and also lead to a tentative criterion for filters composed of crusher sands.

Summary of Present Filter Design Criteria

Existing data for the design of filters can be divided into two sections:

(i) Terzaghi's criteria extended by the U.S. War Dept.
(ii) The U.S. Bureau of Reclamation criteria.

(i) THE TERZAGHI CRITERIA:

The criterion to be satisfied if no foundation material is to wash through into the filter layers is:

$$\frac{D_{15}\text{ FILTER}}{D_{85}\text{ FOUNDATION}} < 4 \text{ to } 5$$

The criterion which ensures that the filter layer is many times more pervious than the foundation is:

$$\frac{D_{15}\text{ FILTER}}{D_{15}\text{ FOUNDATION}} > 4 \text{ to } 5$$

To prevent the washing of particles from the adjacent filter layers into the collecting channels and pipes the criterion is:

PIPE OR SCREEN OPENING $d \leq D_{60}$ of the adjacent filter layer

(ii) THE U.S.B.R. CRITERIA:

The combined criteria for prevention of leaching and adequate increase in permeability from layer to layer are:

(a) For graded natural subrounded materials:

$$5 < \frac{D_{50}\text{ FILTER}}{D_{50}\text{ FOUNDATION}} < 10$$

(b) For graded crushed rock filters:

$$6 < \frac{D_{50}\text{ FILTER}}{D_{50}\text{ FOUNDATION}} < 18$$

The Need for Further Investigation

When faced with the practical problem of the design of a filter it becomes apparent that existing knowledge of the behaviour of sands and soils when water flows through them is very sketchy indeed. It is true that a number of empirical rules for the design of filters are available which have been found to give good results for most of the time. These design rules, however, are all purely empirical and one can never really be certain that they will work in the case in hand. The following experience can be quoted as an example: It was decided to use a well graded crushed quartzite in a drainage filter of a slimes dam on the Orange Free State gold fields. The very flat sloping grading curve of the crusher sand bridged very well between the fine silt size particles of the slimes and the coarse crushed rocks of the filter core. The filter as a whole satisfied the Terzaghi design criteria. Before the installation of the filter, a model filter was tested and it was found that the crusher sand did not behave as a filter within itself - the fine particles rapidly washed out leaving a skeleton of coarser grains which was unable to retain the slimes foundation. Sands which exhibit this property of being unable to retain or filter their own fines have been named "SANDS WHICH ARE NOT FILTERS WITHIN THEMSELVES".

It is occurrences like this that point to the need for accumulating further knowledge about filter sands. There are enough empirical rules. What is now required is knowledge as to why and how filters and sands operate and behave as they do when water percolates through them.

Any investigation into filters and filter sands should embrace not only the behaviour of the grain structure of the filter or sand, but also the hydraulic behaviour of the filter. A filter is not just a collection of mineral particles arranged in some geometrical order. It is an hydraulic flow system operating on and being operated by the grain structure forming its flow channels. It should be investigated as such.

This thesis is designed to investigate the behaviour of filters from a new angle. The object of the thesis is to find out something at least of how filters operate, why filters operate; how and why filters fail to operate successfully. To achieve this aim a number of separate investigations have been carried out which may at first appear entirely unrelated but which, when considered as a whole, give an insight into how and why filters and filter sands behave as they do.

The Field Covered by this Thesis

Broadly speaking this thesis has investigated two aspects of the behaviour of filters and filter sands:

(i) The first section inquires into the changes of grading which filter sands undergo due to various causes. The three causes of changes of grading which are investigated are: seepage forces, chemical decomposition and physical weathering. It includes studies of the mechanism, causes and effects of grading changes on the properties of the sands themselves and on filters incorporating the sands. Criteria are developed for the condition that a sand should behave as a filter within itself.

The special apparatus developed for the experimental investigation is described and commented on.

The effects of internal grading changes on the operation of layered protective filters is described.

(ii) The second section inquires into the effect which the grading changes described in (i) have on the hydraulic behaviour of filter sands and a number of semi-empirical expressions describing the behaviour accompanying grading changes are derived.

CHAPTER II

A STATISTICAL INVESTIGATION OF GRADING CURVES AND CHANGES OF GRADING DUE TO THE ACTION OF PERCOLATING WATER

General:

The particle size distribution curve or grading curve is the most convenient and in many ways the best way of describing a cohesionless soil such as a filter sand. The grading curve gives in condensed and pictorial form complete information about the particular constitution of the sand. Before going on to describe the testing procedure and apparatus used for the experimental work, consider the possible sources of variation and errors in grading curves. A thorough knowledge of the errors to which any testing procedure is liable and the ability to analyse and judge their relative importance are indispensable for reliable results. Statistical methods enable one to get an excellent idea of the accuracy of a set of figures and to estimate the true value of a numerical conclusion. All existing filter design criteria are based on the use of a grading curve.

The Accuracy of any Grading Curve:

There are at least three possible sources of variation which may affect the accuracy with which a grading curve can be determined:

(i) Errors in weighing.
(ii) Errors due to slight differences in sieving technique from sample to sample.
(iii) Errors due to natural variation of the sand from sample to sample.

These errors are "confounded", i.e. they have a combined effect rather than separate effects, and special tests had to be performed to determine the effects of individual sources of variation.

Of these three sources of variation the third was found to be the most serious. It is obvious that even in a very homogeneous mass of sand natural variation will exist from sample to sample. This variation does not present a very big problem when the samples available for sieve analysis are large. However, in this case, owing to the general small size of the samples used, a large natural variation could be expected.

The investigation of these three sources of error will be described later.

The Grading Curve in General:

Statistically, the grading curve may be analysed as follows:

A sample of sand is weighed and has weight W ....(1)

The sand will contain particles of various sizes and it is postulated that:

$W_{(x)}$ = Weight of sand in a sample with particle size $\leqq x$ ....(2)

($W_{(x)}$ is further assumed to be a continuous and differentiable function).

The sand is then sieved through a succession of sieves with openings of size

$$x_k > x_{k-1} > x_{k-2} \ldots\ldots\ldots\ldots > x_1$$

On the sieve of size x_k, sand of weight which will be denoted by

$W - w_k = \triangle w_k$ is retained ....(3)

Thus $w_k = W - \triangle w_k$ is an estimate of true $W_{(x_k)}$ for this sample of sand.

On the next sieve of size x_{k-1}, sand of weight

$\triangle w_{k-1} = w_k - w_{k-1}$ is retained ....(4)

Thus $w_{k-1} = w_k - \triangle w_{k-1} = W - \triangle w_k - \triangle w_{k-1}$ ....(5)
is an estimate of true $W_{(x_{k-1})}$ for this sample of sand.

Continuing in this way a series of 'measures' are obtained which are arranged as follows:

TABLE I

Size Class Interval	Observed Wts. Retained	Observed Wt. Less Than	
$0 - x_1$	w_1	Less than: x_1	w_1
$x_1 - x_2$	$\triangle w_1$	x_2	w_2
$x_2 - x_3$	$\triangle w_2$	x_3	w_3
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
$x_{k-1} - x_k$	$\triangle w_{k-1}$	x_k	w_k
$> x_k$	$\triangle w_k$	∞	W
	W		

The basic observations in any sieving test are now taken to be $w_1, w_2, w_3, \ldots\ldots\ldots w_k$.

So far, concern has been shown for finding the true $W_{(x)}$ for one sample of sand. But successive samples from nominally the same sand are expected to differ a little one from another. What is really required is the 'true' $W_{(x)}$ for the body of sand from which the samples are drawn. This will be denoted by $\overline{W}_{(x)}$ when it is desired to distinguish clearly between $\overline{W}_{(x)}$ and $W_{(x)}$ for a particular sample.

The problem now resolves itself into two:

A: The problem of choosing a mathematical form for $\overline{W}_{(x)}$ and estimating the constants in it.

B: If this cannot be done an attempt can be made to define certain parameters in $\overline{W}_{(x)}$.

This can be done even though the exact form of $\overline{W}_{(x)}$ is not known and the parameters can then be estimated.

The Semi-Logarithmic Grading Curve

The semi-logarithmic grading curve in which particle sizes are plotted to a logarithmic scale against the percentage finer than that size to a natural scale is the most familiar and widely used form of grading curve. Suitable parameters can be defined and estimated as follows:

Put $z = \log x$ and rewrite table I as:

Size Class Interval	Observed Proportion of Wt. Retained		Observed Proportion of Wt. Less than z
$-\infty$ to z_1	$w_1/W = p_1$	Less than: z_1	$w_1/W \quad p_1$
z_1 to z_2	$\triangle w_1/W = \triangle p_2$	z_2	$w_2/W = p_2$
. .	.	.	. .
. .	.	.	. .
. .	.	.	. .
. .	.	.	. .
z_{k-1} to z_k	$\triangle w_{k-1}/W = \triangle p_{k-1}$	z_k	$w_k/W = p_k$
over z_k	$\triangle w_k/W = \triangle p_k$	∞	1

If the assumption that W is a known constant is correct (as it is for all practical purposes), then the new variables p_1, p_2, p_3, ... etc. remain independent within each sample and independent from sample to sample.

$W_{(x)}$ transforms into $W_{(z)}$, (say) ....(6)

Define

$$P_{(z)} = \frac{W_{(z)}}{W} \qquad \text{i.e.} \quad \overline{P}_{(z)} = \frac{\overline{W}_{(z)}}{W} \quad \text{.....................(7)}$$

The values of p_i are 'observations' of $P_{(z_i)}$ and

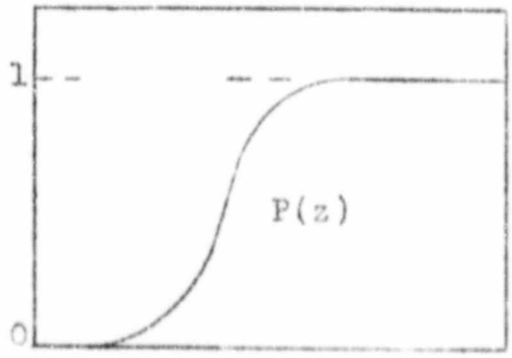
$$\frac{p_{1i} + p_{2i} + p_{3i} + \ldots\ldots}{k} \quad \text{(the mean from k samples)}$$

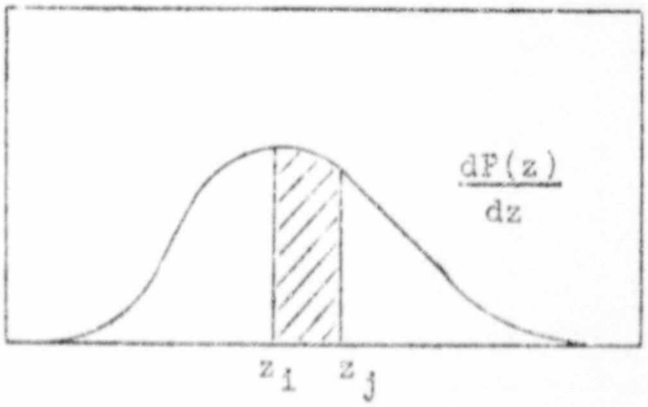
is an observation of $\overline{P}_{(z_i)}$.

The practical effect of the transformation $z = \log x$ is in many cases that $P_{(z)}$ is a reasonably symmetrical 'sigmoid' or 'S-shaped' curve, while $\frac{dP_{(z)}}{dz}$ becomes a fairly symmetrical 'bell-shaped' curve which has the property that:

$$\int_{z_i}^{z_j} d\,P_{(z)}\,dz = P_{(z_j)} - P_{(z_i)}$$

$$= \text{Proportion of weight between 'sizes' } z_i \text{ and } z_j \text{ ....(8)}$$





Although the exact mathematical form of $P'_{(z)} = \frac{dP_{(z)}}{dz}$ is unknown, the curve possesses "position" and "scatter".

Possible measures of these are:

$$\xi = \int_{-\infty}^{+\infty} z\, P'_{(z)}\, dz$$

$$\sigma_z^{\,2} = \int_{-\infty}^{+\infty} (z-\xi)^2\, P'_{(z)} dz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

These can be estimated independent of the exact form of $P_{(z)}$.

As an estimate of ξ, $\hat{\xi}$ can be used,

where $\hat{\xi} = p_1\xi_1 + \triangle p_1\xi_2 + \triangle p_2\xi_3 + \ldots\ldots$

$$+ \triangle p_{k-1}\xi_k + \triangle p_k\xi_{k+1} \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots (10)$$

In which the ξ_i ($i = 1$ to $k+1$) are pre-assigned values of z, one within each class interval (these are conventionally taken as the nominal particle sizes themselves).

If $E(\hat{\xi})$ is the 'expected value' or 'true mean value' of $\hat{\xi}$, then:

$$E_{(\hat{\xi})} = \xi_1 \int_{-\infty}^{z_1} P'_{(z)} dz + \xi_2 \int_{z_1}^{z_2} P'_{(z)} dz + \ldots$$

$$+ \xi_{k+1} \int_{z_k}^{\infty} P'_{(z)} dz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

If the ξ_i were properly chosen, then from the mean value theorems this would be:

$$\int_{-\infty}^{z_1} z\, P'_{(z)} dz + \int_{z_1}^{z_2} z\, P'_{(z)} dz + \ldots + \int_{z_{k+1}}^{\infty} z\, P'_{(z)} dz$$

$$= \int_{-\infty}^{+\infty} z\, P'_{(z)} dz = \xi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Unfortunately the correct ξ_i are unknown and whatever set is chosen will give an $E_{(\hat{\xi})}$ which is only slightly biassed. i.e.

$$E_{(\hat{\xi})} = \xi + \text{slight biass}$$

If the same ξ_i is always used, the biass should be of the same order from one sample to another.

As an estimate of σ^2, $\hat{\sigma}^2$ can be used where:

$$\hat{\sigma}^2 = p_1\varrho_1^{\,2} + \Delta p_1\varrho_2^{\,2} + \ldots + \Delta p_{k-1}\varrho_k^{\,2} + \Delta p_k\varrho_{k+1}^{\,2} - (\hat{\varrho})^2 \ldots (13)$$

Then

$$E_{(\hat{\sigma}^2)} = \varrho_1^{\,2}\int_{-\infty}^{z_1} P'_{(z)}dz + \varrho_2^{\,2}\int_{z_1}^{z_2} P'_{(z)}dz + \ldots.$$

$$\ldots + \varrho_{k+1}^{\,2}\int_{z_k}^{\infty} P'_{(z)}dz - \varrho^2$$

$$= \int_{-\infty}^{+\infty} z^2 P'_{(z)}dz - \varrho^2$$

$$E_{(\hat{\sigma}^2)} = \int_{-\infty}^{+\infty} (z-\varrho)^2 \; P'_{(z)}dz \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

The Special Case of the 'Normal' Grading Curve

The so-called Gaussian Normal Curve or Probability Curve holds a very important place in statistics and a great deal of very useful theory has been evolved on the assumption that the observations being considered follow the Normal pattern or "have a normal distribution".

It is found that a large proportion of naturally occurring sands have gradings, which when plotted on a semi-logarithmic basis give very close approximations to a curve representing the integral of the Normal Curve function. Most crushed products also have gradings which have an approximately normal form.

The normal grading curve can be completely described by two parameters. The curve has the form:

$$P_{(z)} = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}\,\sigma_z} \; \epsilon^{-\frac{1}{2\sigma_z^2}(z-\varrho)^2} dz \; \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

The corresponding curve estimated from any sample would be:

$$P_{(z)} = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}\,\hat{\sigma}_z} \; \epsilon^{-\frac{1}{2\hat{\sigma}_z^{\,2}}(z-\hat{\varrho})^2} dz \; \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Where σ_z, $\hat{\sigma}_z$, ϱ, $\hat{\varrho}$ have the significance of the previous paragraphs.

If it is reasonable to assume a normal particle size distribution, standard statistical theory can be applied to the observations if required.

Even if the form of an individual grading curve does not approximate very closely to the Normal form, however, statistical theory based on the assumption of a normal distribution can still be of considerable use. It can be proved mathematically that even if a number of sets of observations of a certain statistical variable are only approximately normal, the distribution of the means of those observations is very much nearer normality. The work which follows will be concerned mainly with the means and standard deviations of particle size analysis observations.

Significant Differences Between Grading Curves

In Chapter IV an investigation into the grading changes undergone by a soil sample due to the percolation of water is described in some detail.

The basic problem which had to be solved was as follows:

Two grading curves were available after testing - one representing the particle size distribution of the sand in its original state, the other representing the particle size distribution of the sand after being subjected to the action of percolating water. In practically every case there was a visual difference between the two curves. This difference was possibly due to two main causes:

(i) The difference in gradings could have been a real difference brought about by internal migration of sand particles within the sample caused by the action of the percolating water.

(ii) The difference in gradings could have been apparent only - as the result of random causes of variation, viz:

(a) Differences in sieving technique from sample to sample.

(b) Errors in weighing during the particle size analysis.

(c) The result of natural variation within the mass of sand from sample to sample.

What was required was a reliable "rule of decision" which, in the majority of cases, would enable one to decide confidently whether an observed change of grading was significant, real and caused by flow action, or whether it was probable that the change could be due to chance only. Here again the use of standard statistical theory was invoked.

The treatment of differences depends on the following fundamental theorem:

If a statistical variable x has a frequency function (or distribution law) $f_{(x)}$ with a true mean ξ and true variance σ_x^2; and variable y has a frequency function $g_{(y)}$ with true mean η and true variance σ_y^2; and if both x and y are statistically (or stochastically) independent, then $z = ax + by$ has an unknown frequency function with a true mean $a\xi + b\eta$ and true variance $a^2\sigma_x^2 + b^2\sigma_y^2$.

This theorem is entirely independent of the distributions of the individual variables.

The theory is applied as follows:

The separate grading curves are each assumed to represent random samples from Normal "populations". There immediately exist the following possibilities:

(i) The grading curves could represent random samples from the <u>same</u> Normal population.

(ii) They could represent random samples from two normal populations with the <u>same</u> means and <u>different</u> variances.

(iii) They could represent random samples from two entirely different normal populations.

<u>Investigation of Differences between the Means.</u>

Assume that measure x has true mean ξ, true variance σ_x^2 and that measure y has true mean η and true variance σ_y^2 and that these measures are independent. Then it can be proved that:

measure $\bar{x}$ has true mean ξ and true variance σ_x^2/n_1 and
measure $\bar{y}$ has true mean η and true variance σ_y^2/n_2.

Where $\bar{x}$, $\bar{y}$ are the estimated means from the samples and n_1, n_2 are the numbers of observations in each sample.

In addition, the theory suggests that the distributions of $\bar{x}$ and $\bar{y}$ are nearly normal. Applying the fundamental theorem above: the variable $(\bar{y} - \bar{x})$ will have true mean $(\eta - \xi)$ and true variance $\sigma_x^2/n_1 + \sigma_y^2/n_2$ and will be nearly normal since it represents the difference between two nearly normal variables.

Another theorem exists which states that

$$U = \frac{(\bar{y} - \bar{x}) - (\eta - \xi)}{\sqrt{\dfrac{\sigma_x^2}{n_1} + \dfrac{\sigma_y^2}{n_2}}}$$

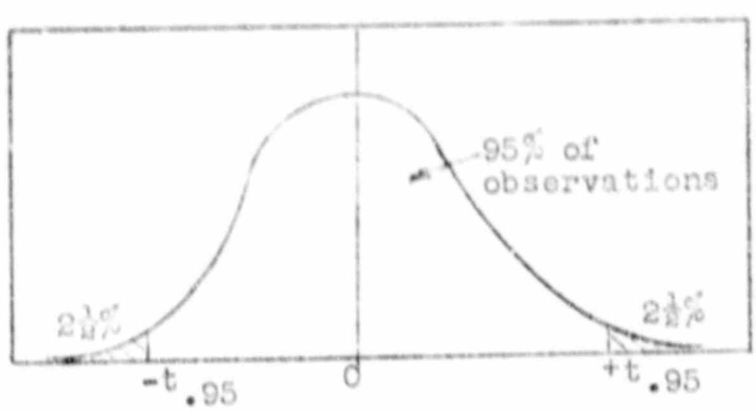
is approximately normal with zero mean and unit standard deviation.

However, as the true values of σ_x^2, σ_y^2 are unknown, estimates of the variances $\hat{\sigma}_x^2$, $\hat{\sigma}_y^2$ must be used in the expression for U giving:

$$t = \frac{(\bar{y} - \bar{x}) - (\eta - \xi)}{\sqrt{\dfrac{\hat{\sigma}_x^2}{n_1} + \dfrac{\hat{\sigma}_y^2}{n_2}}}$$

This variable can still be regarded as nearly normal with zero mean and unit standard deviation provided n_1 and n_2 are large.

Let $t_{.95}$ represent the value of t such that, on the average, 95% of values of t lie between $\pm t_{.95}$



i.e. $\dfrac{(\bar{y} - \bar{x}) - (\eta - \xi)}{\sqrt{\dfrac{\hat{\sigma}_x^2}{n_1} + \dfrac{\hat{\sigma}_y^2}{n_2}}}$ lies between $\pm\, t_{.95}$ on about 95 times in 100.

or $(\bar{y} - \bar{x})$ lies between $(\eta - \xi) \pm t_{.95}\sqrt{\dfrac{\hat{\sigma}_x^2}{n_1} + \dfrac{\hat{\sigma}_y^2}{n_2}}$

on about 95 times in 100.

Whenever this statement is true, then it is true that $(\eta - \xi)$ lies between $(\bar{y} - \bar{x}) \pm t_{.95}\sqrt{\dfrac{\hat{\sigma}_x^2}{n_1} + \dfrac{\hat{\sigma}_y^2}{n_2}}$ with 95% confidence.

If $(\eta - \xi)$ has a value significantly different from zero, it can be concluded that, with 95% confidence, the two variables have different means, and vice-versa.

Values of t for any desired confidence limits are tabulated in any standard statistical text book.

Investigation of Differences between the Standard Deviations

In comparing variances, it proves most convenient to study their ratio $\hat{\sigma}_x^2/\hat{\sigma}_y^2$ and decide whether this ratio differs significantly from unity.

In the treatment of variances two more statistical laws of distribution are introduced:

(i) The CHI-SQUARE DISTRIBUTION with f Degrees of Freedom which is defined by:

$$f_{(z)} = C_f\, z^{f/2-1}\, e^{-z/2} \qquad \text{for } z \geqslant 0$$

$$f_{(z)} \equiv 0 \qquad \text{for } z < 0.$$

This yields a one parameter family of curves of parameter f termed "f Degrees of Freedom".

(ii) The VARIANCE RATIO DISTRIBUTION which is defined as follows:

If τ_1^2 has a chi-square distribution with f_1 degrees of freedom, and

τ_2^2 has a chi-square distribution with f_2 degrees of freedom, and if τ_1^2 and τ_2^2 are independent, then

$v^2 = \tau_1^2/f_1 \,/\, \tau_2^2/f_2$ has a variance ratio distribution with f_1 and f_2 degrees of freedom

$$f_{(v^2)} = \frac{C\,(v^2)^{f_1/2 - 1}}{(f_1 + f_2 v^2)^{f_1 + f_2}} \qquad \text{for } 0 \leqq v^2 < \infty$$

The application is as follows:

It can be proved that $(n_1 - 1)\dfrac{\hat{\sigma}_x^{\,2}}{\sigma_x^{\,2}}$ has a chi-square distribution with $(n_1 - 1)$ degrees of freedom and $(n_2 - 1)\dfrac{\hat{\sigma}_y^{\,2}}{\sigma_y^{\,2}}$ has a chi-square distribution with $(n_2 - 1)$ degrees of freedom.

So that $v^2 = \dfrac{\hat{\sigma}_x^{\,2}}{\sigma_x^{\,2}} \Big/ \dfrac{\hat{\sigma}_y^{\,2}}{\sigma_y^{\,2}}$ has a variance ratio distribution with $(n_1 - 1)$ and $(n_2 - 1)$ degrees of freedom.

If it is assumed that $\sigma_x^{\,2} = \sigma_y^{\,2}$, then $v^2 = \hat{\sigma}_x^{\,2}/\hat{\sigma}_y^{\,2}$ can be directly observed.

If $\sigma_x^{\,2} = \sigma_y^{\,2}$ then v^2 comes from a "variance ratio population", if v^2 is too large to reasonably belong to this population, then the assumption $\sigma_x^{\,2} = \sigma_y^{\,2}$ is suspect.

Tabulations of the Chi-square and Variance Ratio distributions are readily available.

THE VALIDITY OF THE INITIAL ASSUMPTION OF A NORMAL PARTICLE SIZE DISTRIBUTION

It has already been stated that the statistical theory here used depends directly on the initial assumption of a normal particle size distribution on the semi-logarithmic grading curve plot. It has also been stated that even if the distribution of the original observations were only vaguely normal in form, the distribution of the means of those observations would be much closer to normality. It is on this argument that the treatment of significant differences between grading curves is based.

The closeness (or otherwise) with which the original particle size analysis observations follow the Normal distribution can be tested objectively by the use of a statistical test for "Goodness of Fit" - the CHI-SQUARE TEST.

The test is as follows:

An hypothesis is set up as to the law of distribution and its mathematical form. The parameters in the chosen form are then estimated and it is desired to know whether the result is plausible or not.

Suppose that in n statistically independent observations certain events occur $f_1, f_2, f_3 \ldots\ldots\ f_k$ times, where $\sum f_i = n$

The chosen hypothesis suggests that the respective probabilities are $p_1, p_2, p_3 \ldots\ldots\ p_n$, where $\sum p = 1$.

Hence in n experiments the events can be expected to occur $np_1, np_2, np_3 \ldots\ldots\ np_k$ times

Then, according to the theory,

$$\frac{(f_1 - np_1)^2}{np_1} + \frac{(f_2 - np_2)^2}{np_2} + \ldots\ldots + \frac{(f_k - np_k)^2}{np_k} = \chi^2$$

has, approximately, a chi-square distribution. The application of the test is best illustrated by example.

In order to illustrate the practical application of the above theory, a complete set of typical calculations will be found in the appendix to this chapter. The calculations are for a typical soil used in the experimental work - Crusher Sand 'A' (for further details see the Sample Index), and comprise calculations for:

(a) Sample means and standard deviations.
(b) Tests for significant differences between the means and variances.
(c) A Chi-square goodness of fit test.

THE INVESTIGATION OF SOURCES OF RANDOM VARIATION IN GRADING CURVES

Random variation in grading curves, as previously explained, may arise due to any combination of three possible causes:

(i) Errors in weighing during the particle size analysis.
(ii) Errors due to slight differences in sieving technique from sample to sample.
(iii) The effects of natural variation of the sand from sample to sample.

(i) The Effect of Weighing Errors.

The basic observations taken in any sieving analysis are the weights $w_1, w_2, w_3 \ldots\ldots w_k$, where w_i is the observed weight of sand of particle size less than x_i. (see Table I). It would appear that the errors in these weighings are statistically independent - a point which is made use of subsequently.

Suppose that Δw_k contains a positive error e_k, i.e. too much sand has been retained on the first sieve. w_k will then have a negative error $-e_k$.

Then Δw_{k-1} will automatically have $-e_k$ as part of its error due to lack of the sand which ought to be there, but has been accidentally retained on the preceding sieve. Δw_{k-1} will also contain other errors e_{k-1}, say, due to faulty sieving technique, etc.

$$\text{Then: } w_{k-1} = W - \Delta w_k - \Delta w_{k-1}$$

$$= W - \Delta W_k - e_k - \Delta W_{k-1} + e_k - e_{k-1}$$

$$w_{k-1} = W(x_{k-1}) - e_{k-1}$$

This shows that this estimate has an error which is independent of the previous error.

The assumption that the errors in the measures are statistically independent is vital in what follows in the third part of this section.

(ii) The Effect of Sieving Technique

In order to discover what effect sieving technique had on the results of a sieve analysis, a special test was carried out. A sample of Jukskei River sand was sieved through a series of sieves in the normal manner using a "Ro-Tap" mechanical sieve shaker and a sieving time of 5 minutes.

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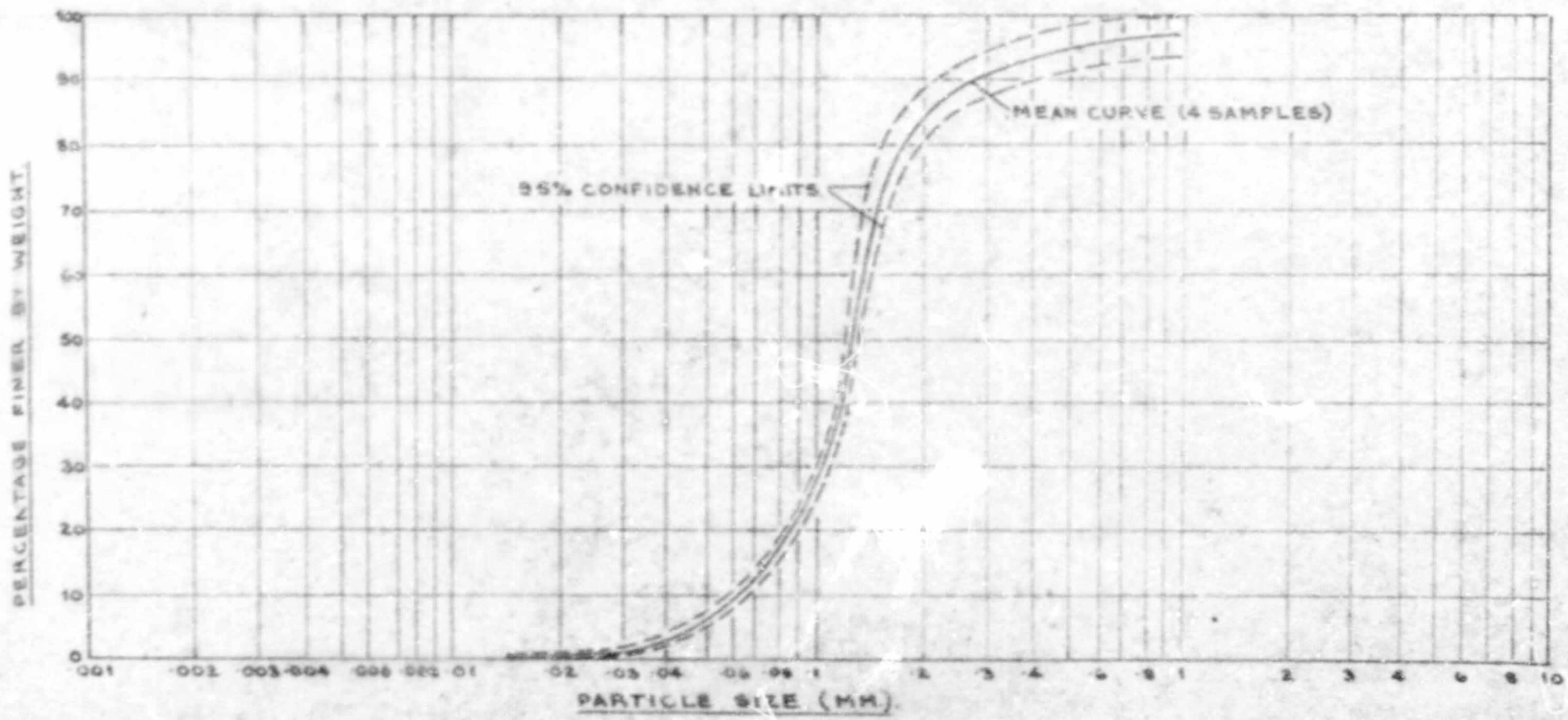
OBSERVED NATURAL VARIATION OF A RIVER SAND.

THIS CURVE SHOWS THE LIMITS OF NATURAL VARIATION FOR A COARSE SAND FROM THE JUKSKEI RIVER.
THE LIMITS, CALCULATED WITH 95% CONFIDENCE, ASSUME A NORMAL PARTICLE SIZE DISTRIBUTION.



The sieves were then carefully emptied and cleaned into a large basin, great care being taken to avoid losing any sand. The sand collected was then subjected to a further sieve analysis, the method used being identical with the first. The results and a comparison of these two analyses are shown in the appendix to this chapter. It will be seen that the differences between the two gradings obtained in the successive tests are so small as to be hardly distinguishable on a graph.

Since both the sieving time adopted (five minutes) and the mechanical shaking were standardised for all tests, it can thus be concluded with reasonable confidence that any errors due to sieving technique were negligible. In addition, since these results must also reflect any weighing errors present - as shown up by the difference in the total weights of soil retained in the two sievings - it appears that the sway of weighing errors on the results is also negligible.

(iii) The Effect of Natural Variation of a Sand from Sample to Sample

To discover the extent to which natural variation affects the results of particle size analyses, a special series of four tests was performed. Four samples of approximately the same size of Jukskei River sand (see the Sample Index for details of this sand) were sieved completely independently using the same nest of sieves and identical sieving technique each time. The Jukskei River sand was chosen for these tests as it was considered that in samples of small size the sand would show a higher degree of natural variation than any other sand being used. The reason for this was that the sand contained an appreciable proportion of pebbles over $\frac{3}{8}$ in. in size, several of which weighed up to 10 grammes each. The addition or subtraction of one such pebble from a small sample could quite noticeably affect the resulting particle size curve.

Statistical methods were then applied to the results of these four tests in order to determine the limits within which, on about 95 times in 100 samples, the grading curve for the Jukskei River sand could be expected to fall (provided the constitution of the sand was not drastically altered by some external agency). These limits will be referred to as the "95% confidence limits" for natural variation of the sand.

Due to the large amount of testing required to establish these confidence limits for just one variety of sand, it did not prove practicable to repeat this procedure for all sands used. This series of tests does, however, give a valuable guide to the extent of the random variation due to natural causes which can be expected in a sand taken from a small locality. Naturally, such limited tests can tell nothing about the natural variation to be expected between two samples taken, for example, from the same river bed, but 100 feet apart.

It is considered that for the normal well mixed laboratory bag sample, such a series of tests can give all the information required. Although no similar information is available for the other sands used experimentally, it is reasonable to expect that, as the other sands used were generally finer, they would not exhibit a natural variation outside the limits obtained for the Jukskei River sand.

The method of obtaining the probable limits of variation will now be explained.

The accompanying calculation sheet was prepared for four samples of Jukskei River sand. Certain differences are evident between the samples and these differences are thought to be due to three sources of variation: Weighing errors, sieving technique and natural variation. It has already been shown that the effect of the first two items is negligible and furthermore that the errors in individual measures of the sieving analyses are statistically independent. It is also reasonable to assume that each sieving analysis is statistically independent of the other three. Thus as a result of the four analyses we have four independent values of w_1, four independent values of w_2, etc. If all these assumptions are correct we have an array of four mutually independent observations corresponding to each sieve, each array being independent of any other array of observations.

The problem now reduces to finding the numerical range within which the true mean of each value of w (or p%) lies.

When considering a large number of observations this can be done as follows:

If it is assumed that one is dealing with observations which have a normal distribution, then the form of the frequency function of the population will be:

$$f_{(x)} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left\{\frac{x - \xi}{\sigma}\right\}^2}$$

Where σ represents the standard deviation of the observations, ξ represents the true mean of the observations.

Also: $\sigma^2 = \sum_1^n \frac{(x - \xi)^2}{n}$

For the distribution of the mean $\bar{x}$, it can be shown that

$$f_{(\bar{x})} = \sqrt{\frac{n}{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{n}{2}\left\{\frac{x - \xi}{\sigma}\right\}^2}$$

Also $u = \frac{\bar{x} - \xi}{\sigma/\sqrt{n}}$ is normally distributed with zero mean and unit standard deviation. And it can be shown that the Probability $P\{|U| \leq 1.96\} = 0.95$. i.e. $|U| \leq 1.96$ occurs about 95 times in 100 or $\bar{x}$ lies between $\xi \pm 1.96 \frac{\sigma}{\sqrt{n}}$ about 95 times in 100.

Whenever this is true, then it is true that ξ lies between $\bar{x} \pm 1.96\ \sigma/\sqrt{n}$

This expression gives the 95% confidence limits for ξ, the true sample mean.

In practice, however σ^2 is unknown and an estimate $\hat{\sigma}$ has to be used. In this case a distribution function is required for $t = \frac{\bar{x} - \xi}{\hat{\sigma}/\sqrt{n}}$ not $u = \frac{\bar{x} - \xi}{\sigma/\sqrt{n}}$

This distribution function is known as STUDENT'S 't' DISTRIBUTION in which $t = \frac{\bar{x} - \xi}{\sqrt{\frac{\sum(x_i - \bar{x})^2}{n(n-1)}}}$

Thus for small samples the confidence limits become:

$$\xi \text{ lies between } \bar{x} \pm t_{.95}\sqrt{\frac{\sum(x_i - \bar{x})^2}{n(n-1)}} \text{ with 95\% confidence.}$$

The factor t depends on the number of observations available (or on the Degrees of Freedom). In general t has (n - 1) degrees of freedom, so that in this case with sets of 4 observations, t has 3 degrees of freedom and a value $t_{(.95,\ 3)} = 3.18$.

The above theory was applied to the results of each sieve in turn to give the confidence limits shown on the accompanying chart and grading.

CONCLUSION:

The use of statistical methods for the estimation of quantities and as a guide to decision making has proved very useful in this work. Statistical methods cannot give an absolute answer to any problem, but they yield an unbiassed estimate of the true answer and indicate the limits of uncertainty of that answer. A knowledge of the limits of uncertainty is very useful in evaluating the reliability of any set of experimental data. In the great majority of cases decisions and estimates can be made without statistics. The statistical analysis merely confirms the decision and estimates its reliability.

CHAPTER III

THE APPARATUS AND TESTING PROCEDURE DEVELOPED FOR THE INVESTIGATION OF FILTER SANDS

DESCRIPTION OF THE APPARATUS USED IN THIS SERIES OF TESTS.

Introduction: The Purposes of the Apparatus:

The apparatus developed to deal with this investigation was designed to provide information about:

(i) The grading changes which take place in sands under the influence of percolating water.

(ii) The hydraulic characteristics of the soil-water flow system via permeability tests.

It was this latter object which required most thought in the design of the apparatus.

All forms of permeability test are notoriously unreliable unless special steps are taken to eliminate, as much as possible, the several "side effects" which may occur. Experience has shown the importance of these side effects to be such that if no steps are taken to eliminate them, they may completely mask the true permeability characteristics of the soil being tested. As far as is known, all these side effects lead to values for the permeability less than the true values. The main effects to be guarded against are:

(a) A low and decreasing Degree of Saturation of the soil sample arising:

 (i) due to the presence of air entrapped within the pores of the sample during the initial saturating process, and

 (ii) due to subsequent progressive segregation of air within the soil voids as a result of air released from solution in the percolating water.

(b) The formation of impervious 'skins' within the filter due to algal growth.

(c) The formation of similar filter 'skins' due to the filtration of impurities in the percolating water.

It was found during the tests that no danger of algal growth arose provided the duration of the filter tests was limited to about 24 hours and the filter system was shaded from direct sunlight during this period.

In order to avoid the formation of 'skins' due to impurities in the water, distilled water was used initially for all tests. However, it was subsequently found that ordinary mains water (as supplied by the Rand Water Board) contained so little matter in suspension that it could be used without apprehension.

The degree of saturation of the soil during the test was controlled (a) by a carefully designed testing apparatus and procedure, and (b) by the use during the initial stages of the test of de-aired distilled water. Incidentally, the lack of oxygen in the water acted as an additional inhibitor of algal growth.

The paragraphs which follow will describe the construction and layout of the Filter Test Apparatus and the Water De-Airing apparatus. Drawings and photographs accompany the description.

Filter Test Apparatus : Test Cylinders.

The samples of sand tested were contained within glass cylinders of 3 in. internal diameter, ⅛ in. wall thickness and 9 in. in length. The cylinders were closed at each end by chromium plated steel end plates of 5 in. overall diameter and ½ in. thickness. The opposite rims of the glass cylinder fitted into circular grooves cut in each end plate, the edges sealing against rubber 'Gaco' rings inserted between the grooves in the metal and the rims of the glass cylinder. The cylinders and end plates were clamped together by means of three ¼ in. diameter steel bolts fitted with washers and wingnuts. Water was fed into and out of the cylinders by means of brass nipples screwed into ½ in. diameter holes tapped into the centres of each end plate.

The sand sample itself was contained between porous plates fitting closely the inside of the glass cylinders and of ½ in. thickness. The porous discs were cut from a ceramic filter tile and offered negligible resistance to flow, but acted as spreaders to the inflowing water and eliminated disturbance of either end of the soil sample due to turbulence either while saturating the sample or during the course of the filter test.

The cylinders were connected to the rest of the hydraulic circuit by means of rubber tubing pushed over the brass nipples and held secure by rubber bands.

Each glass cylinder bore a label giving its identification number, carefully measured cross-sectional area and weight. The test cylinders were supported during testing on a special rack built of 'Dexion' angle.

Filter Test Apparatus : Hydraulic Circuit.

The hydraulic circuit for the filter test apparatus was arranged as shown in the diagram. The initial portion of the test was run on de-aired distilled water contained in a 20-litre carboy with a bottom delivery outlet. The construction of the carboy and its associated connections will be described in detail as part of the water de-airing apparatus. The head used was approximately 9 feet and was supplied via a simple overflow type constant head chamber. This overflow chamber was fed by a trickle of tap water, the overflow discharging into a sink. The delivery from the constant head chamber fed into a large meteorological balloon contained in the de-aired water reservoir and separating the de-aired distilled water from the tap water. This inflowing water displaced equal volumes of de-aired water which was fed into the upper end of the test cylinder.