Exploring the effect of Professional Noticing in the Context of Algebraic Fractions: A Case Study of National Certificate Vocational L3 Lecturers in a TVET College

A research project presented

By

Salome Nadason

Student Number: 0201103P

Supervised By

Prof Judah Makonye

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DECLARATION

University of the Witwatersrand, Johannesburg

School of Education

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DEDICATION

I dedicate this work in loving memory of my dear mother, Ruby Nadason, who always encouraged me to reach for the stars and pursue my dreams. Mom, I thank you for the immense sacrifices that you have made, for your unconditional love and unending support, that have moulded me into who I am. I love and miss you dearly. Your light and love continue to shine in my life.

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ABSTRACT

Expertise in professional noticing of students mathematical reasoning is an integral component of effective mathematics discourse. What a teacher notices or fails to notice impacts students' performance significantly. Professional noticing of students mathematical thinking, in the mathematics classroom, involves attending to, interpreting, and deciding how to respond to students reasoning (Jacobs et al., 2010). Based on the significance of the construct of noticing, this study explored the effect of professional noticing on the pedagogical practices of NCV Level 3 (L3) lecturers in relation to the simplification of algebraic fractions. Specifically, this study sought to answer the following questions: how do NCV L3 lecturers *attend* to students mathematical reasoning in relation to simplifying algebraic fractions; how do NCV L3 lecturers *interpret* students mathematical reasoning in relation to simplifying algebraic fractions; and what course of action do NCV L3 lecturers take based on students mathematical reasoning in relation to algebraic fractions.

To answer these research questions, a qualitative research design, particularly a case study approach was used. Data were collected in the form of student written responses to a pre-test, field notes, and audio recorded lessons, in the topic of algebraic fractions. Participants were purposively and conveniently recruited. A total of two NCV L3 lecturers and their respective classes, from TVET College B, consented to participate in the study. Before the data collection process, a professional noticing development program (workshop), facilitated by the researcher was conducted. Lecturer participants were requested to analyse the written responses of students by providing a detailed explanation on what they noticed about students reasoning in relation to the simplification of algebraic fractions. Furthermore, the audio recorded lessons were transcribed by the researcher, and transcripts were read several times to accurately determine the degree of noticing displayed by the participants. The three strands of professional noticing and the levels of sophistication framework served as analytical tools to determine the extent to which lecturers attended to, interpreted, and responded to students mathematical reasoning. Verbatim quotations were used to highlight the noticing skills of the participants.

The findings of the study suggest that the lecturers displayed predominantly low (0) and medium (1) levels of noticing when analysing the written responses of students. In terms of noticing in action, participant lecturers were observed demonstrating primarily low (0) levels

of noticing. The study also revealed that participants found it challenging to *notice* students reasoning *during* the teaching of algebraic fractions.

In conclusion, since noticing can be learned (Jacobs et al., 2010), to support lecturers in noticing students mathematical reasoning, it is recommended that TVET colleges provide training programs and workshops that explicitly focus on the learning and implementation of professional noticing. Furthermore, the study suggests that there is a need for the conceptualisation of noticing from a perspective that appreciates and integrates the socio-cultural realities of the students as well as the realistic setting of the classroom, since factors such as students background knowledge, classroom culture, and resources may affect the efficiency of noticing. Therefore, it is advisable that adaptations be made to acknowledge and accommodate the diversity and contextual circumstances of our South African classrooms to cultivate a more inclusive approach to noticing.

***Key words**: TVET, mathematics, professional noticing, South Africa.*

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CHAPTER 1: BACKGROUND TO THE STUDY

1.1 Introduction

Mathematics underperformance has been of national concern. For example, the 2019 Trends in Mathematics and Science Study (TIMMS) reports that South Africa received an average score of 389 points for Mathematics and continues to be one of the lowest performing countries in the world (Arends et al., 2017). By implication, most South African learners are performing well below the grade appropriate curriculum (see Spaull & Kotze, 2015). In more specific detail, results of the learning trajectory indicate that in certain schools, the average learners are performing at approximately a grade 4.5 level in grade 9 (Spaull & Kotze, 2015). At a tertiary level, the Technical and Vocational Education and Training (TVET) mathematics results are not entirely different from the high school results. Previous studies (see Ngoveni, 2018; Stops, 2022), have identified various factors, including inadequate foundational knowledge, socio-economic conditions, lack of resources and absenteeism that contribute to poor performance. This underperformance has motivated many initiatives to improve mathematics learner outcomes at a national level (Mabena et al., 2021).

Quite significant for this current study, one of the topics that have been identified which contributes to poor learner performance is algebraic fractions (Baidoo, 2019). Fractions are a fundamental component of the primary and high school, and the TVET mathematics curriculum. At TVET institutions, students experience difficulties in learning algebraic fractions. These difficulties experienced may be because algebra has been described as one of the most *abstract* domains of Mathematics (Egodawatte, 2011). Algebra can be considered as a key component of Mathematics and the foundation upon which many other significant mathematical concepts are built. Hence, proficiency in algebra is a prerequisite to understanding many other mathematical domains such as geometry, trigonometry, and calculus (Usiskin, 2004). By implication, students with a deficiency in algebraic reasoning experience many problems within other mathematical domains (Permata et al., 2019a). Highlighting the importance of algebra, Usiskin (2004) suggests that while “the soul of mathematics may lie in geometry, algebra is its heart” (p. 149).

The application of algebra extends beyond the mathematical landscape into many other subject areas and fields of study in the school curriculum including economics, physics, and chemistry. For instance, algebra is useful in solving real life problems in the field of engineering, medicine,

business, and psychology. Moreover, algebraic reasoning offers several epistemological benefits such as logical and critical thinking and enhances problem solving skills that contributes significantly to decision making. Critical thinking ignites curiosity and creativity and invokes a profound desire for inquiry which not only encourages "an appreciation for the beauty and elegance of [algebra]" (Department of Basic Education [DBE], 2011, p. 8) but also arouses an inquiry spirit within other mathematical and nonmathematical realms. Despite its extensive application and usefulness, the topic of algebra evokes much stress and anxiety and is considered a major concern for policy makers, teachers, and students worldwide (Wilkie, 2016). Consequently, students who lack competence in algebra or have not done any algebra related courses are often refused entry into many university programmes (Makonye & Stepwell, 2016). Thus, algebra can be considered a 'gatekeeper' to higher education and employment opportunities (Usiskin, 2004). One aspect of algebra that particularly poses significant challenges for students is the simplification of algebraic fractions (Mhakure & Jacobs, 2014).

The National Certificate Vocational (NCV) Assessment guidelines for Mathematics emphasises that Level three (L3) students should be able to simplify algebraic fractions competently and confidently. However, the simplification of algebraic fractions is a problematic concept that most NCV L3 students struggle to grasp. Studies have identified different types of errors and misconceptions that students display when simplifying algebraic fractions (Baidoo, 2019; Baidoo et al., 2020; Egodawatte, 2011; Makonye & Khanyile, 2015; Makonye & Stepwell, 2016). These studies have also advised that the identified errors and misconceptions need to be incorporated into instructional practices and used as opportunities for learning (Borasi, 1994). Teaching that is aimed at addressing errors and misconceptions require a nuanced appreciation of the mathematical reasoning of the student. Obtaining a comprehensive interpretation of students' mathematical reasoning is a fundamental component of effective teaching and learning of mathematics and may be achieved through the construct of professional noticing (PN). Professional noticing of students' mathematical thinking when simplifying algebraic fractions is a highly responsive and adaptive instructional strategy and is largely informed by the mathematical knowledge of the teacher (Thomas et al., 2017). Against this background, this study sought to explore NCV L3 mathematics lecturers professional noticing of students' mathematical reasoning when simplifying algebraic fractions.

1.2 Problem statement

The simplification of algebraic fractions presents several challenges for many students and contributes significantly to their underachievement in mathematics. Mhakure et al. (2014) suggest that these challenges stem from a poor foundational (procedural and conceptual) knowledge of algebra and fractions. Simplifying algebraic fractions competently depends on both procedural (*knowing how*) and conceptual knowledge (*knowing why*) (Rittle-Johnson & Schneider, 2015). While students have knowledge of the steps or procedures (procedural knowledge) involved in simplifying algebraic fractions, these rules are often learned without understanding the underlying concepts. Thus, the memorisation of these procedures usually results in the misapplication of correct rules as students *know how* but do not *know why* (Rittle-Johnson & Schneider, 2015). Since each algebraic fraction is composed of a rich web of relationships between algebraic concepts and arithmetic fractions, a robust conceptual understanding of these connections is fundamental to simplifying algebraic fractions competently. An inadequate appreciation of the underlying concepts embedded within algebraic fractions often results in students committing several errors and misconceptions. Most of these errors and misconceptions often go unnoticed as some teachers employ traditional, '*talk and chalk*' methods of teaching. This form of classroom discourse is what Lobato et al. (2005) refer to as Initiation-Reply-Evaluation (IRE). IRE forms the basis of '*talk and chalk'*, where evaluation is used to lead students to correct answers. In this approach, teachers ask recall or '*test*' questions; students' responses are then evaluated to determine how closely they are aligned to the information transmitted by the teacher (Brodie, 2007). Davis (1997) defines this as evaluative listening, where the teacher listens for correct or incorrect responses and no attention is given to the rationale associated with students reasoning. Although evaluative listening is important, it is not sufficient for finding out the root cause of continued errors and misconceptions made by students while solving algebraic functions.

As a result, Davis (1997) advises that teachers should listen interpretatively which allows the teacher to initiate mathematical discussions and make sense of students' mathematical reasoning. This form of listening provides opportunities to uncover the errors and misconceptions that students have and allows teachers to resource these misconceptions. A key component of professional noticing involves listening interpretatively to students thinking to accurately attend to and interpret students' mathematical strategies. Teaching algebraic fractions through a professional noticing intervention strategy places high value on students mathematical thinking rather than focusing on right or wrong answers. This learner-centric

approach depicts the teacher as a mediator in the learning process and encourages students to actively participate in their learning (Lobato et al., 2005).

The topic of algebraic expressions involving fractions contributes 20% in the NCV Level three final examination (DHET, 2012). Proficiency in solving algebraic fractions is a prerequisite to the understanding of other mathematical concepts within the NCV curriculum such as geometry, trigonometry, and calculus. Moreover, once completing the NCV program, most of the students proceed to the National Accredited Technical Education Diploma (NATED) program. Within the NATED program, calculus forms the main component of the curriculum. For example, students who are unable to simplify algebraic fractions competently experience several challenges when asked to derive expressions such as; $y = \frac{3+4cosx-sin2x}{cosx}$ or integrate $\int \frac{cos^3 t-1}{cos^2 t} dt$. It is therefore important for teachers to *notice* salient aspects of students' mathematical reasoning when simplifying algebraic fractions. Although professional noticing has been well recognised as an effective strategy to improve mathematics teaching and learning in general, limited studies have researched its importance specifically in algebra (LaRochelle et al., 2019; Walkoe, 2015). Furthermore, there are currently no studies that focus on supporting lecturers professional noticing in the domain of algebraic fractions, particularly within the South African, TVET (NCV) context. It is for this reason that this study aimed to engage with the concept of professional noticing by exploring the professional noticing practices of NCV L3 lecturers in relation to the simplification of algebraic fractions.

1.3 Rationale for the study

The justification for this study relates to the importance of teachers being aware of the errors and misconceptions that students display when simplifying algebraic fractions (Makonye & Khanyile, 2015). It also highlights the need to employ professional noticing as a strategy for *attending* to students' mathematical strategies, *interpreting* students understanding in relation to the strategies and *deciding* how to respond to students reasoning. A few studies have documented the types of errors and misconceptions in the topic of simplification of algebraic fractions and proposed intervention strategies that utilise these errors as opportunities for learning (Makonye & Stepwell, 2016; Makonye, & Khanyile, 2015; Mhakure & Jacobs, 2014; Mulungye et al., 2016; Permata et al., 2019). However, there are no studies that focus explicitly on supporting lecturers in noticing students' mathematical reasoning when simplifying algebraic fractions. LaRochelle et al. (2019) argue that the field lacks research regarding the

extent to which practicing teachers notice students thinking. Since teacher professional noticing is context and domain specific (Walkoe, 2015), generalising results from other mathematical domains or across different levels of schooling is inappropriate (La Rochelle, 2015).

In this context, the current research is significant because it sought to extend prior work on professional noticing by focusing on how NCV Level 3 lecturers *notice* students' mathematical thinking during the simplification of algebraic fractions. Thus, the study contributes to the research gap within the TVET sector and the paucity of research in the explicit use of the construct of professional noticing in algebra, specifically in the simplification of algebraic fractions. Since the problems associated with this topic often filter into other mathematical domains, research that is aimed at improving academic performance within the topic of simplification of algebraic fractions may lead to an overall improvement in mathematics performance.

1.4 Main research objective

The main objective of the study was to explore the professional noticing practices of NCV L3 lecturers in relation to the simplification of algebraic fractions.

Specific objectives

To achieve the above-mentioned broad objective, the following were the specific objectives of this study.

- Investigate how NCV L3 lecturers *attend* to students mathematical reasoning in relation to simplifying algebraic fractions.
- Investigate how NCV L3 lecturers *interpret* students mathematical reasoning in relation to simplifying algebraic fractions.
- Explore courses of action NCV L3 lecturers *take* based on students mathematical reasoning in relation to algebraic fractions.

1.5 Main research question

To what extent do NCV L3 lecturers notice students' mathematical reasoning in relation to the simplification of algebraic fractions?

Sub-questions

- How do NCV L3 lecturers *attend* to students mathematical reasoning in relation to simplifying algebraic fractions?
- How do NCV L3 lecturers *interpret* students mathematical reasoning in relation to simplifying algebraic fractions?
- What course of action do NCV L3 lecturers take based on students mathematical reasoning in relation to algebraic fractions?

1.6 Definition of key terms

To establish the context of this study, the following key terms are defined:

***Algebra*:** refers to a branch of mathematics that can be described as a mathematical language, consisting of various rules and symbols.

***Algebraic fractions*:** refers to an algebraic quotient; it is a fraction that consists of an algebraic expression in its numerator and/or denominator.

***Simplification*:** means reducing the algebraic fraction to its simplest form.

***Professional Noticing*:** refers to paying attention to nuanced features of students mathematical thinking. It is an expertise involving three interrelated components (attending, interpreting, and responding).

***Errors and Misconceptions*:** Errors are a product of misconceptions. Misconceptions symbolise incorrect thinking patterns.

***Pedagogical Practices*:** refers to the teaching strategy or approach employed by the lecturer to facilitate teaching and learning.

***Public TVET College*:** refers to a government owned TVET College. In this study the TVET College is situated in the Gauteng province of South Africa and is named TVET College B.

***Case Study*:** refers to the research method employed in the study to facilitate a detailed exploration of the phenomena.

1.7 Layout of the study

This research report comprises of six chapters, organised as follows:

Chapter one

Chapter one presents the background of the study, which includes an introduction to the study; the statement of the problem that the study aims to explore; the rationale for the study as well as the research objectives; the research questions that the study seeks to answer; key terms within the study are explained; and lastly a detailed layout of the study is presented.

Chapter two

This chapter provides a review of the extant literature in relation to the study objectives.
The literature review is outlined as follows:

- Errors and Misconceptions
- Errors and misconceptions in the simplification of algebraic fractions
- Professional noticing and its impact on teaching and learning
- Theoretical perspectives and conceptualisations of noticing
- Teaching interventions by means of a professional learning community.

The literature review is followed by the theoretical and analytical framework that underpins the study.

Chapter three

Chapter three discusses the methodology of the study. It begins with an introduction; research paradigm; the research design employed which was a qualitative case study approach; research site and sample; the methods of data collection as well as data analysis; the quality criteria; and ethical considerations.

Chapter four

In this chapter, the data analysis and findings are presented according to the research questions.

Chapter five

This chapter presents a discussion of the key findings of the study and relates the findings to extant literature as well as to the theoretical framework.

Chapter six

This research report is concluded in chapter 6. The chapter begins with an integrated conclusion, thereafter a reflection of the framework on professional noticing is presented; followed by limitations of the study; and finally, suggestions for TVET colleges, NCV mathematics lecturers as well as for researchers are provided.

CHAPTER 2: LITERATURE REVIEW

2.1 Introduction

In recent times, the notion of professional noticing has gained momentum with many researchers offering different perspectives and conceptualisations of the construct. However, despite these diverse views, a common theme has been identified. Previous studies have noted that central to the idea of professional noticing is being sensitive to students mathematical reasoning (see Brodie, 2011, 2014; Dindyal et al., 2021; Frenzel et al., 2019; Jacobs et al., 2010). Professional noticing of students mathematical reasoning illuminates' conceptions and misconceptions; and presents the opportunity to resource errors to advance conceptual understanding and make learning meaningful.

In this chapter, I begin with a generalised account of errors and misconceptions, followed by the errors and misconceptions which relate to the simplification of algebraic fractions and finally, a detailed description of professional noting is presented. I also provide a detailed discussion of the theoretical framework used in accordance with the objectives of this study. To conclude, I present a summary of the key ideas and identified gaps which forge the relevance of this current study.

2.2 Errors and misconceptions

The contribution of a student to the process of learning is characterised by the errors and misconceptions they commit (Nesher, 1987). Although an integral component of the learning process, errors and misconceptions occur across contexts and present several challenges in accomplishing the desired mathematical outcomes (Nesher, 1987). Errors are differentiated from mistakes, blunders, or slip-ups, which occur due to negligence and are not associated with conceptual or cognitive schemas. When pointed out, mistakes or slip-ups, can be readily rectified by the students themselves (Nesher, 1987).

Errors, however, are often a product of misconceptions (Brodie, 2014). Misconceptions signify "a line of thinking that causes errors, all resulting from an incorrect underlying premise, rather than sporadic, unconnected, and non-systematic" (Nesher, 1987, p. 35). Since misconceptions are actively constructed by the students themselves and are usually deeply entrenched within

conceptual schemas, they can be resilient to instruction (Baidoo, 2019). Worth noting is that errors are rarely taught and are not a consequence of failure on the part of the student or teacher (Brodie, 2014). Due to the conceptual nature of misconceptions and the psychological processes involved, it is important that teachers are cognisant of the way that students grasp mathematical concepts (Smith et al., 1993). Successful concept formation involves resourcing errors as opportunities to stimulate rich mathematical inquiry rather than perceiving errors as indicative of a lack of intelligence or failure and a call for punishment (Borasi, 1994). The way a teacher deals with errors can either facilitate or impede students' learning. A robust Pedagogical Content knowledge (PCK) is necessary to support learning and accurately interpret students' mathematical reasoning to uncover the source of the error and resource the error accordingly (Ball et al., 2008).

2.2.1 Errors and misconceptions in the simplification of algebraic fractions

The errors that students commit when simplifying algebraic fractions stem from a weak foundational knowledge of common fractions (Baidoo et al., 2020; Mhakure, 2014). Mhakure (2014) claim that the multifaceted nature of fractions poses several challenges and is often regarded as the one of most problematic concepts of the curriculum. Efficient understanding of the multifaceted fraction construct, in its entirety, requires an appreciation of each of the interrelated sub-constructs. According to Mhakure et al. (2014), this involves an understanding of the following: *part-whole* (this refers to the idea of apportioning an object into smaller equal parts), *ratio* (a way of representing equivalent fractions), *operator* (important in understanding the multiplication operations involving fractions), *measure* (competency to order fractions on a number line), and *quotient* (the idea that fractions can be presented as a division operation). The failure of students to conceptualise the interrelated sub-constructs of fractions often leads to several difficulties in the learning of algebraic fractions.

Although categorised differently, many of the errors are in fact similar and may be attributed to a deficiency in relational understanding of both normal fractions and algebra (Makonye & Stepwell, 2016). In a study by Makonye and Khanyile (2015), eight categories of errors were identified during a pre-probing test, namely: cancellation errors, confusing the factors, no recognition of the common factor, unable to factorise a trinomial, lowest common denominator error, careless or random error, correct answer obtained by using incorrect mathematical rule, and dropping the denominator. Some of these errors were also recorded by Makonye and Stepwell (2016) and were categorised according to the typographical framework presented by

Figueras et al. (2008). The errors were classified as conjoining errors, incorrect cancelling, incomplete cancelling, de-fractionalisation, like term error and addition and subtraction of the numerator. Similarly, the document analysis of 10 scripts performed by Mhakure et al. (2014) revealed cancellation errors, equation errors, performing a mistaken operation error and grouping like terms error. Most of the errors identified in extant literature stem from a poor conceptual understanding and an over-reliance on rote-learning of procedures. Baidoo et al. (2020) argue that ignoring errors and misconceptions is detrimental to students' performance in mathematics and will likely reinforce the incorrect thinking patterns. As a result, Mhakure et al. (2014) advice that a thorough analysis of students mathematical reasoning is required to accurately address the deficiencies associated with algebraic fractions. Understanding students' thinking when simplifying algebraic fractions can be achieved through the construct of professional noticing (Jacobs et al., 2010).

2.3 Professional noticing and its impact on teaching and learning

> The range of what we think and do is limited by what we fail to notice. And because we fail to notice that we fail to notice, there is little we can do to change until we notice how failing to notice shapes our thoughts and deeds. (Goleman, 1985, p. 24).

Noticing is part of our everyday lives, whether consciously or unconsciously different aspects of the environment evoke our attention. Sherin et al., (2011) state that "…in everyday language noticing is a term used to indicate the act of observing or recognising something…" (p. xxv). We see things differently, and usually construct our own perception of reality. As a result, noticing is not objective but rather subject to the way an individual processes and makes sense of a particular situation (Jacobs et al., 2010). Professional noticing, however, differs from everyday noticing, as it is purposeful and distinctive and focused on achieving certain professional goals (Dindyal et al., 2021). Embedded within every profession are nuances that its members become sensitised to. To be effective, an individual must develop the capacity to be able to "read" his/her professional context (Rooney & Boud, 2019). For instance, doctors need to pay attention to certain symptoms that may not manifest physically or be visible to the naked eye. In business, professionals need to be able to "read" the market or notice significant phenomenon to make predications or contingency plans (Rooney & Boud, 2019). Thus, paying attention to salient features is fundamental and failure to do so often results in inadequate outcomes or may, in some instances, lead to detrimental ramifications. Worth noting, is the

distinction between inner and outer noticing. Outer noticing focuses on the physical features or behaviourism of students, for example, noticing how a student is dressed or that he/she is fidgety, or that the student is not sitting properly. Inner noticing, on the other hand involves paying attention to the students mathematical reasoning and being aware of the underlying conceptions and misconceptions (Wessels, 2018). In the classroom context, professional noticing is not a mindless activity, or a consequence of merely being perceptive. Instead, it is a highly specialised cognitive activity that is directed at achieving a distinct purpose (Dindyal et al., 2021). Although, noticing may be an implied characteristic of teaching, it is important to create an explicit pedagogy of noticing (Rooney & Boud, 2019). Purposeful noticing within the mathematics classroom involves paying attention to salient features of students' mathematical reasoning. This is often an arduous task since mathematics classrooms are often buzzing with considerable volumes of sensory inputs (Dindyal et al., 2021). As a result, teachers require a robust mathematical knowledge to distinguish *nuances* from *'noise'* by accurately sifting out significant aspects of students thinking to advance the cognitive development of the student.

Dindyal et al. (2021) argue that what a teacher notices or fails to notice often determines the instructional trajectory of the lesson. Several factors, for example, educational philosophies, culture, beliefs, values, and experiences often informs the lens through which a teacher views the classroom (Jacobs et al., 2010). Proficient professional noticing is a key competency of effective mathematical discourse that provides a precise understanding into the cognitive processes and rationale that underpin students' mathematical ideas (Wessels, 2018). Therefore, it is fundamental for teachers to *learn* to notice in specialised ways. Learning to notice intentionally enables teachers to act in a novel and responsive manner rather than in an automated, mechanistic way (Dindyal et al., 2021). Although an integral component of effective pedagogy, responsive or ambitious teaching is challenging as a large component of teaching is unplanned or improvisational (Thomas et al., 2015). In-the-moment decisions vary from preplanned responses as teachers are expected to continuously analyse and interpret students thinking and make connections to what they already know about the mathematical development of the student (Franke et al., 2007). Working with students' mathematical thinking, Brodie (2011) shows how learning is influenced by teacher-learner interaction. The analysis of four secondary school teachers depicts a shift from traditional teaching practices towards a more reform-oriented approach. By encouraging student participation and

contributions and explicitly engaging with the mathematical reasoning, teachers are more likely to understand the reasons behind the student errors. Although the teachers in the study showed a change in their pedagogical practice, they encountered several challenges and therefore need to be supported with appropriate intervention strategies (Brodie, 2011). Engaging epistemologically with students' mathematical ideas is a difficult task as teachers need to be wary of either favouring the mathematics at the expense of student ideas or neglecting the mathematics completely. Furthermore, allowing student contributions to determine the lesson trajectory goes against conventional power relations, whereby the teacher is the source of knowledge and learners are empty vessels that need to be filled with knowledge (Brodie, 2011, 2017; Franke et al., 2007). Changing the image of mathematics education of what constitutes knowledge and whose knowledge is acceptable, disputes the entire perception of the purpose of schooling and how knowledge is acquired (Brodie, 2011). Professional noticing of students' mathematical ideas explicates students reasoning and engenders a more learner-centred approach (Brodie, 2011). This promotes learning through inquiry and confronts the often 'taken-for granted truths'; in this approach a disparate relation between knowledge and power (Brodie, 2011) emerges which positions the teacher and student as co-authors of knowledge construction.

2.4 Theoretical perspectives and conceptualisations of noticing

Over the years, a variety of perspectives and conceptualisations of teacher noticing have been suggested. A systematic literature review of teacher noticing (König et al., 2022; Santagata et al., 2021) proposes four key theoretical perspectives, namely: 1) a cognitive-psychological perspective considers the cognitive process involved as teachers observe and make sense of students mathematical reasoning, 2) a socio-cultural perspective describes noticing in terms of its socially situated nature, 3) a discipline-specific perspective characterises noticing as a discipline and emphasises the need for teachers to be aware and sensitive to students thinking and 4) an expertise-related perspective considers the difference in the noticing capacity between experts and novices and illuminates the skills that need to be developed in novice teachers to assist them in noticing more sophisticatedly. Within each perspective different conceptualisations of the construct emerge. Although diverse theoretical perspectives and conceptualisations of professional noticing exist; several scholars (van Es et al., 2022; Amador, 2022; Dindyal et al., 2021a; Jacobs et al., 2010; Sherin et al., 2011; Thomas et al., 2015) agree that key features of professional noticing involve being able to attend to the most significant

aspects of students reasoning and make sense of or interpret those aspects to make the most appropriate decisions.

In this research, professional noticing of students' mathematical reasoning is described as the synchronised enactment of three interrelated skills (Jacobs et al., 2010); *attending* to students' mathematical strategies, *interpreting* the reasoning reflected in the strategies and *responding* according to these understandings. Attending, interpreting, and responding to students mathematical reasoning enables teachers to detect erroneous patterns of thinking and use the errors as "springboards for inquiry" (Borasi, 1994).

Attending to the mathematically significant details of students' strategies is an important skill as it is likely to inform the teachers instruction (Jacobs et al., 2010). Attending to students' thinking goes beyond the superficial aspects of the solution strategy. Rather, it entails an extensive unpacking of the students' thinking to appreciate the conceptions and misconceptions. This may involve paying attention to the way a student counts, or a specific mode of representation selected, or the type of approach used during problem solving. The extent to which a teacher can extract the mathematical essence of a students' strategy directly informs his/her ability to accurately interpret and respond appropriately. Interpreting students understanding involves the ability to provide rich explanations that is consistent with the significant aspects of students understanding as well as research on students' mathematical development (Jacobs et al., 2010) . This entails the ability to distinguish between strategies and explain what the student did and did not do, as well as acknowledge strategies that would've been more appropriate in solving the problem. Interpreting involves going beyond offering generalisations toward a more individualised, precise account of students' mathematical rationale (Gallagher et al., 2022). Davis (1997) argues that the way a teacher listens influences the teachers' ability to accurately interpret students understanding and draw valid inferences about students reasoning. Thus, listening is an integral component of mathematics teaching and learning and plays an important role in noticing.

Davis (1997) proposes three forms of listening: evaluative, interpretative, and hermeneutic listening. Evaluative listening involves listening for a specific answer; the teacher expects the student to respond by providing the 'correct' answer. If the student does not respond the way the teacher requires then the teacher will often funnel or guide the student into delivering the desired result (Davis, 1997). In the study by Davis (1997), Wendy (teacher) asked the student to justify their reasoning *mathematically*; this suggests that she wanted the student to explain

the answer in relation to what was taught in class. In this sense, no appreciation is given to the student reasoning. Evaluative listening often involves "listening *for* something, rather than listening *to* the student" (Davis, 1997, p. 359). This form of listening is superficial as the teacher is not really interested in what the student is saying nor is she/he willing to decipher the intuitive reasoning of the student. Instead, the teacher is only focused on the mathematical rigor contained within the responses. By listening evaluatively, teachers seldom deviate from the course of the lesson causing them to miss out on significant learning opportunities (Davis, 1997). Interpretative listening, on the other hand, involves listening intentionally and deliberately and allows the teacher to access the subjective sense made during problem solving which provides a robust account of students mathematical reasoning. This form of listening moves beyond passively hearing or receiving students' responses and considers the implicit, tacit, or unarticulated facets of student thinking. Interpretative listening involves listening constructively and subjectively (Davis, 1997). That is, the teacher listens purposefully to intuit the sense making process of the student. Thus, interpretative listening goes beyond listening to *assess* but rather to *access* the idiosyncratic reasoning of the student (Davis, 1997). Hermeneutic listening creates a dispersed or collective authority within the classroom. Davis (1997) suggests that a teacher who listens hermeneutically becomes a participant in the learning process; learning occurs through participation and negotiation. By implication, "hermeneutic listening demands the willingness to interrogate the taken for granted and the prejudices that frame our perceptions and action" (Davis, 1997, p. 370). Since noticing involves understanding and interpreting the implicit often invisible articulations of student reasoning, Davis' (1997) notion of interpretative listening is best suited for the purpose of this research.

Listening is a key component of effective teaching. The way a teacher listens is often dictated by their epistemological assumptions or personal perception of the nature of mathematics (Ernest, 1995). While it is important to give heed to the issues of knowing and knowledge it is also important to recognise the ontological considerations of knowledge. That is, how students think and rationalise a problem situation. Acknowledging and appreciating student reasoning allows for the emergence of mathematical identities (Brodie, 2014). Establishing a mathematical identity allows students to position themselves as capable and competent participants within the mathematics classroom. Mathematics education has both moral and ethical ramifications (David, 1996). Thus, supporting students in establishing a mathematical identity and seeing themselves as doers of mathematics is one of the responsibilities of teaching. Davis (1997) advice that "listening engenders a particular response-ability" (p. 373)

and assists teachers to gain access into the sense making process of the student to respond deliberately and purposefully rather than habitually.

The teachers' choice of responses is often influenced by the extent to which he/she attended to and interpreted students reasoning. The strength of making connections between students' mathematical reasoning and the common conceptions or misconceptions encountered frequently determines the teacher's response after analysing students' tactics (Hine & Lesseig, 2021). Since each student demonstrates a unique understanding of the mathematical concepts they have learned and because students develop their own knowledge, the teacher's response must be tailored to account for the many views on knowledge acquisition (Jacobs et al., 2010). It is crucial that the teacher's responses are particularly based on their interpretation of the students' comprehension rather than on how they personally perceive the circumstances. Effective responses offer various possibilities to improve students' comprehension by enhancing their existing mathematical knowledge. Responding may involve probing students to justify or explain their reasoning, which advances conceptual understanding; asking students to compare various approaches, which advances strategic competence; or encouraging the use of various representational modalities, which advances adaptive reasoning (Kilpatrick, 2001). Furthermore, giving precedence to students' thinking and individual ways of working allows students to appreciate mathematics as sensible and worthwhile, thus promoting a productive disposition. In this light, mathematics is viewed as a corpus of knowledge that is dynamic and flexible rather than a rigid set of pre-established truths (Ernest, 1995). Although discussed individually, the three strands of professional noticing are interwoven and occur harmonically in an instructional scene. *Figure 1* below depicts the interrelated component skills of professional noticing:

Figure 1: The interrelated component skills of professional noticing. Adopted from Gallagher (2022).

In their study, Thomas et al (2017), indicate how a pre-service teacher focused on the salient aspects of the child's counting strategy. The teacher *noticed* not only what the child did "…started to count by 1…", but also what the child did not do "…He did NOT start at 7 and count up" (Thomas et al., 2017, p. 17). Furthermore, the teacher's mathematical knowledge informed the precise interpretation "I think the child's understanding of mathematics is either figurative or perpetual" and the teachers understanding of research on children's mathematical reasoning advised that "… [the child] still needs that visual to touch feeling of using his fingers" (Thomas et al., 2017, p. 17). Due to the accuracy and specificity in attending and interpreting, the consequential decision was tailored to the specific needs of the child "I would cover both sets of objects and see how well he responded when he was unable to actually count the objects" (Thomas et al., 2017, p. 17). The decision made by the teacher is not imposed, the teacher does not provide the student with a strategy but rather supports the child in selecting his own approach. Responding in this manner assists in advancing conceptual understanding.

Similarly, in their study, Jacobs and her colleagues (2010) indicate how preservice teachers attend to, interpret, and respond to the written activity "Todd has 6 bags of M&Ms. Each bag has 43 M&Ms. How many M&Ms does Todd have?" (p. 178) The mathematical thinking of three children, Alexis, Cassandra, and Josie were analysed. Each child offered unique strategies in solving the problem; in response, one of the teachers proposed the following (Jacobs et al., 2010, p. 188).

> "For Alexis, I would use more round numbers to see if she could use a more efficient strategy, like 6 bags of 50".
>
> "For Cassandra, I would try similar numbers again to see if she could perform her calculations without error. Maybe 8 bags of 48 - 8 would leave even numbers of bags to add up".
>
> "For Josie, I'd go with a larger number of M&Ms and more bags. Maybe she would try a different strategy. 13 bags of 77. I'd be interested to see if she'd use the same strategy and if so, how would she break the numbers up".

By reflecting on each child's strategy, the recommendations proposed were personalised and informed by drawing on research of children's mathematical thinking and in-depth knowledge of mathematics (Jacobs et al., 2010). Thus, the authors (Jacobs et al, 2010) advise that proficient professional noticing foregrounds pedagogical practices that consider "…both the details of the specific child's strategies and the research on children's mathematical development" (p. 172) Since each child's level of conceptual understanding differs; pedagogical approaches need to

be dynamic, flexible and consist of a co-ordinated continuous cycle of attending, interpreting, and responding to a plethora of discrete phenomenon. This form of instruction is highly adaptive and responsive and leads to long-term learning. Students are encouraged to participate in a process of problem solving and teachers mediate this process through "*process-help assistance"* instead of "*product-help assistance"* (Hine & Lesseig, 2021, p. 110). "*Process-help assistance"* provides students with the opportunity to engage in a productive struggle instead of focusing on a right or wrong answer. In this light, the *process* of problem solving is given higher value than obtaining a correct or incorrect answer (Hine & Lesseig, 2021). This form of student- centred instructional practice promotes a more optimistic attitude in the mathematics classroom and advances conceptual understanding, strategic competence, adaptive reasoning, and productive disposition (Kilpatrick, 2001).

Figure 2 below provides a detailed representation of responsive and adaptive teaching (Gallagher et al., 2022)

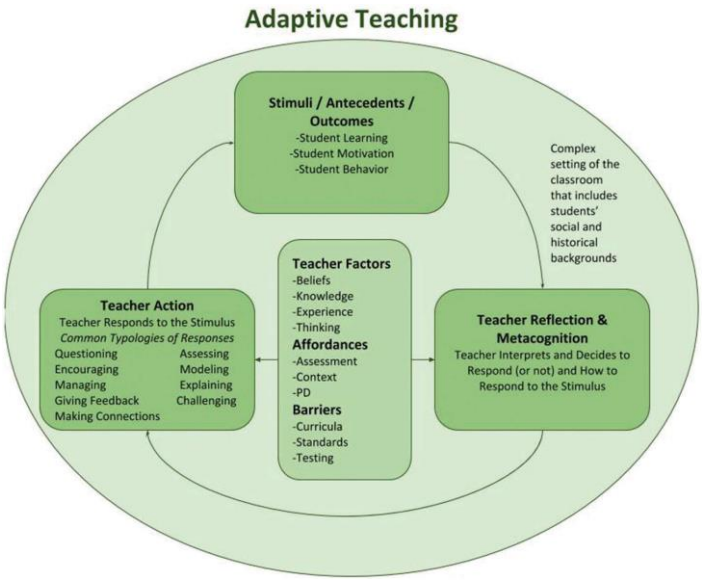


Figure 2: A detailed representation of responsive and adaptive teaching. Adopted from Gallagher et al. 2022

Promoting mathematical proficiency is one of the main goals of mathematics education and fundamental to the holistic mathematical advancement of the student. Supporting students in becoming proficient entails noticing their mathematical reasoning purposefully and mindfully. Paying attention to nuances and navigating the complex perpetual field (Dindyal et al., 2021) buzzing with high volumes of sensory input is a laborious task. Furthermore, the ability to disregard "[the] preferences or biases that inevitably shape noticing" (Dindyal et al., 2021, p.

xxi) and focus on professional goals rather than personal goals involves a particular expertise. Amador (2022) suggests that teachers need to be trained to sift out salient aspects of student reasoning "…because the skill to notice is not inherent, but something that can be learned, noticing should be taught so teachers can improve their ability to respond to students" (p. 4). Jacobs et al. (2010) indicate that teacher development programs support teachers in developing the expertise to notice. Accordingly, Brodie (2014) suggests the idea of a professional learning community as a form of teacher learning, where teachers work together to understand student reasoning and appreciate errors and misconceptions as reasonable.

2.5 Teaching interventions by means of a professional learning community (PLC)

Professional learning communities (PLCs) act as a "safe haven" to support teachers in improving their practice (Brodie, 2014). Being part of a learning community encourages teachers to reflect on their practice, identify their weaknesses and strengths and explore new ideas with each other. Through collaboration teachers can seek solutions to any problems and explore new approaches to improving current practices (Brodie, 2014). Since noticing is an "activity that can be learned as well as an activity that promotes learning" (Rooney & Boud, 2019, p. 3), the professional learning community creates a safe environment where teachers can openly share their concerns and challenges, admit their weakness, and learn from each other. Teachers in the community share a common goal, the mathematical wellbeing of the student. Being vulnerable and open to learning usually takes an individual out of his/her comfort zone and often assists in developing the capacity to notice. "Without a well-developed capacity to notice, [teachers may not] operate successfully within the world of practice" (Rooney & Boud, 2019, p. 2). The professional learning community may assist in developing teaching interventions that focus on noticing student reasoning and acknowledging errors as a key aspect of learning. As teachers learn to appreciate students reasoning and recognise that each student's knowledge construction differs, teachers are more likely to stray away from reverting to their default position and responding in a mechanistic, habitual manner (Brodie, 2014; Rooney & Boud, 2019; Sherin et al., 2011). Results of the study conducted by Goldsmith and Seago (as cited in Amador, 2022) indicate that teachers who participated in a professional development program were able to offer more robust interpretations of students mathematical reasoning than those teachers that were not part of the program. The development program supported teachers in transitioning from an evaluative interpretation of students' artefacts to

providing a more in-depth account. Similarly, Brodie (2014) reveals how a professional learning community served as a mechanism in supporting teachers to transition from identifying an error, to interpreting the error and finally engaging with the error. These shifts in teaching practices also led to growth in their own mathematical knowledge. Teaching interventions can assist teachers in moving away from imposing learning goals to establishing learning goals as the session goes along in connection to the students' mathematical understanding. The teacher must also be aware of each student's progress throughout the class because students' grasp of mathematical concepts varies. Therefore, the teacher and student might set mathematical goals and work toward achieving them based on how the students understanding develops (Dindyal et al., 2021; Sherin et al., 2011).Thus, teaching interventions that are dynamic, adaptable, and contingent upon students understanding are fundamental to ensuring mathematical proficiency.

2.6 Theoretical framework

This study was framed by a socio-cultural perspective of learning and draws on the different mathematical knowledge domains to accurately analyse students mathematical reasoning when simplifying algebraic fractions (see Ball et al., 2008). In the subsequent sub-sections, I provide detailed discussions on a socio-cultural perspective followed by a discussion on mathematical knowledge for teaching. In addition, I include an explanation on the need to combine a socio-cultural perspective with the mathematical knowledge domains to form the study's theoretical framework. I also highlight the relevance of these theories to my current study.

2.6.1 A socio-cultural perspective

Vygotsky (1978) notes that an aspect of development and instruction is the zone of proximal development (ZPD). Within this domain, children do not have the cognitive ability to complete tasks on their own and need clear instructions from their teachers. Using signs, tools and artefacts, the teacher, or a peer, mediates the learning process (McLeod, 2018). According to Vygotsky (1978), the zone of proximal development consists of the concepts and skills that are in the process of being developed during guided instruction. Fundamental to the notion of PN is the opportunity for teachers to accurately identify each student's ZPD. As the teacher attends to students' strategies and accurately interprets students reasoning, the teacher can work within the students ZPD and offer appropriate responses that will provide opportunities for students to develop their understandings. According to Wertsch (1984) conflict arises as the students'

understanding of the problem situation (*situation definition)* contrasts to that of the teacher and through the construct of professional noticing as an intervention strategy or tool (*semiotic mediation*), *inter-subjectivity* is achieved where the student rejects their current erroneous patterns of thinking and develops new ways of reasoning. Vygotsky (1993) advocates for instructional practices that are geared above the current cognitive capabilities of students to establish cognitive doubt, "…since the purpose of education was not to create harmony within the individual, but to provide a mechanism for acculturating the individual into a particular form of thought" (Cockcroft, 2009, p. 475). Accordingly, he introduces the idea of *scientific concepts* and *everyday concepts.* Everyday concepts are concepts that have developed in the individual without any direct instruction, whereas scientific concepts are 'new' concepts that are communicated to the child through explicit instruction (Cockcroft, 2009). By using PN as an intervention tool, the students' mathematical reasoning is thoroughly examined bringing to the fore the amateur concepts (misconceptions). By means of conflict resolution and highly adaptable responses within the ZPD, teachers can assist students in the transition between amateur concepts to scientific concepts. This form of instruction produces a robust understanding of simplification of algebraic fractions. The extent to which a teacher can accurately examine students thinking and assist students in moving from everyday concepts to scientific concepts is largely informed by the mathematical knowledge of the teacher (Hill et al., 2005).

2.6.2 Mathematical knowledge for teaching

An accurate interpretation of the students understanding of algebraic fractions requires a robust mathematical knowledge for teaching (MKT). This form of knowledge is distinct from other types of mathematics knowledge used in professions such as physics, engineering, economics, or accounting. According to Ball et al. (2005), "knowing mathematics for teaching demands a kind of depth and detail that goes beyond what is needed to carry out the algorithm reliably (p. 22). Drawing on Shulman (1986) categories of mathematical knowledge, Ball et al. (2005) propose a framework that conceptualises the different categories and sub-categories of knowledge that mathematics teachers need for effective teaching. MKT is categorised into two main domains: subject matter knowledge (SMK) and pedagogical content knowledge (PCK). SMK comprises of 3 dimensions: common content knowledge (CCK) which is not specific to the teaching profession; horizon content knowledge (HCK) which is the appreciation of the interconnectivity and relationships that exist between mathematical topics and concepts taught

in lower grades and subsequent grades (Ball et al., 2008); and specialised content knowledge (SCK) which is a specialised form of mathematics that is exclusive to the work of teaching, and informs the teachers ability to attend to and interpret students strategies (Jacobs et al., 2010).While PCK contains the knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of the content and curriculum (KCC) (Ball et al., 2008). Figure 3 below represents the different domains of mathematical knowledge for teaching.

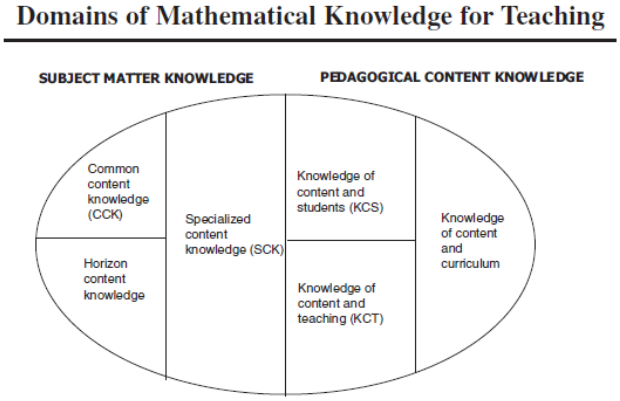


Figure 3: The different domains of mathematical knowledge for teaching. Adopted from Ball et al. (2008)

Within the PCK dimension, Ball et al. (2008) distinguishes between three strands of knowledge; KCS is the knowledge that allows teachers to anticipate the errors that students will commit during problem solving as well as the challenges associated with certain concepts. Thus, KCS is crucial in understanding the mathematical reasoning of students as well the depth of probing necessary to elicit further responses (Ball et al., 2008)

Ball et al. (2008) illustrates how *CCK*, *SCK* and *KCS* complement each other during a learning event:

> recognising a wrong answer is common content knowledge (CCK), whereas sizing up the nature of an error, especially an unfamiliar error, typically requires nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive of specialised content knowledge (SCK). In contrast, familiarity with common errors and deciding which of the several errors students are most likely to make are examples of knowledge of content and students (KCS). (p. 401)

Finally, KCT is a combination of knowledge about mathematics and knowledge about teaching (Ball et al., 2008). It involves the coordination between the mathematics content to be taught

and the instructional options available to teach it effectively. KCT is an important component in responsive teaching as it informs the decisions that teachers make; when to use students' contributions to initiate inquiry, when to provide clarification, when to provide scaffolding or when to pose new examples (Ball et al., 2008). Each domain of MKT plays a pivotal role in attending, interpreting, and responding to students' mathematical reasoning.

2.6.3 Relevance of the theories to this study (A combination of a socio-cultural perspective and MKT)

Mathematics pedagogy is inseparable from the socio-cultural context in which learning occurs. From a social constructivist perspective, learning occurs through social interaction and the lecturer is a mediator, rather than a transmitter of information. Consequently, social constructivism was meaningful to this study as it permits an understanding of the significant role of the social context in appreciating the mathematical reasoning of students in relation to the errors and misconceptions when simplifying algebraic fractions. Moreover, a socio-cultural perspective highlights fundamental aspects of lecturers professional noticing skills. However, social constructivism on its own is insufficient to achieve the main and specific objectives of this study; therefore, it was necessary to consider a theoretical synthesis of MKT and social constructivism. One specific reason for the insufficiency of social constructivism to achieve the objectives of this study in its entirety is because the theory only allows for the pedagogical practices of the lecturers to be located, theoretically. Also, social constructivism is not specific to the subject of mathematics. On the other hand, MKT provides a framework that explicitly and specifically considers the varied domains of mathematical knowledge that are necessary for the proficient teaching and learning of mathematics. MKT offers a more meaningful, structured perspective or explicit description of the domains of mathematical knowledge required to actively interpret students reasoning and decide upon the most appropriate response (see for instance, Thomas et al., 2017). Thus, MKT is germane to understanding the extent to which participants attend to, interpret, and respond to students. Together, both social constructivism and MKT provide a nuanced theoretical framework which assists in providing an in-depth understanding of professional noticing within the TVET NCV L3 mathematics classroom.

2.7 The study's analytical framework

This study was framed analytically by drawing on the three interrelated skills of professional noticing as conceptualised by (Jacobs et al., 2010). These skills include *attending* to students' mathematical ideas, *interpreting* students understanding, and deciding how to *respond* based on students' ideas and mathematical understanding. To analyse the extent to which lecturers *notice* students mathematical reasoning, the levels of sophistication framework was used (see Table 1). This framework was useful because it assisted in characterising, explicitly, the different degrees of attending, interpreting, and responding to students mathematical reasoning. This was valuable in alerting teachers to the depth and detail required to proficiently notice students thinking and effectively support students learning. Table 1 below depicts the different levels of sophistication in attending, interpreting, and responding.

2.7.1 The three strands of professional noticing

Attending: Attending refers to the appreciation of salient mathematical details as reflected in students' strategies (Jacobs et al., 2010). This involves the teachers' ability to offer detailed descriptions of the mathematical thinking of the student during problem solving.

Interpreting: Interpreting refers to the degree to which the teachers' understanding is consistent with the mathematical aspects of students' strategies and the research on students' mathematical development. This is evident when the teacher makes sense of students' tactics and provides robust explanations on what the students *did* and *did not* understand (Jacobs et al., 2010). The teacher offers a precise evaluation of the students reasoning rather than a generalised account; this is an important aspect of interpretation.

Responding: The decisions that inform teachers responses stem from what they have learned about students understanding from a particular situation as well the research on students' mathematical development (Jacobs et al., 2010). Efficient responses are customised according to each student's understanding of the problem and seek to create opportunities for each student to advance their current understanding.

Table 1: Levels of Sophistication in noticing students' mathematical reasoning (Ergene & Mine, 2021, p. 57).

Categories	Description	Examples
Attending		
Low	Described students' strategies as correct or incorrect	Cenk's strategy is incorrect because she gave wrong answer. Pelin got the correct answer.
Medium	Provided general description of both students' strategies	By matching sides of the shapes, Hale thought that the perimeters were equal to each other. Fidan knew that the straight segment in the 1st shape was smaller than the total of the two segments in the 2nd shape.
Robust	Provided detailed description of both students' strategies	Suzan thought that shape consisted of unit squares and the perimeter of the shape decreased as the piece of unit squares were removed, that is, since the area of the shape decreased, the perimeter of it also decreased. Melih thought that the boundary limiting the closed shape was the perimeter. He made numerical operations by counting unit-lengths which were outward-facing edges of the squares as 1 cm in the shapes and found the perimeters correct.
Interpreting		
Low	Provided general interpretation of both students' understanding	I thought Melih got the point. In a similar example, he could reason in the same way and solve such a problem, but Suzan did not seem to understand much.
Medium	Provided general interpretation of one student's understanding and detailed interpretation of another student's understanding	I thought that Sema had fully understood perimeter and her reasoning was measurement based. She did operations using numerical measurements. On the other hand, Jale's reasoning was non-measurement because she did nothing for measurement reasoning. It was holistic because she compared perimeters by looking at the whole shapes. She had a misconception regarding the relationship between perimeter and area. That is, she thought that since the areas of the shapes decreased by cutting pieces, their perimeters also decreased and cutting a larger piece led to a much bigger decrease in the area as well as in perimeter. However, in this case, even though a piece was cut, the lengths of new side(s) were formed after the cut were considered while calculating the perimeter.
Robust	Provided detailed interpretation of both students' understanding Differentiated two students' understanding as well as explained students' misconceptions/ Errors	Actually, they both knew what perimeter meant. Both had measurement reasoning since they used the number of unit length and counted by repeating it. However, while measuring the perimeter of the shapes, Meral did not take equal units. She did not take a fixed unit. She measured repeatedly in different units. She did not understand that she had to take a single unit. That is why she was measuring perimeters wrong. Hence, she was in the incorrect unit iteration level. On the other hand, Poyraz knew what the unit was. He knew what the unit length was. He understood that while measuring the perimeter, the units had to be equal in length and had to be repeated consecutively along the shapes. He made sense of the fact that equal and same units had to be used in measuring perimeter. Hence, Poyraz was at the level of correct unit iteration.
Responding		
Low	Provided general suggestion	I put emphasis on overcoming this misconception by giving the student some more explanatory instructions.
Medium	Provided specific undetailed suggestion	I could start with a simple example composed of unit squares, but in a one-way without such curves. I asked Cenk to find the perimeter. Then, I progressed from simple to complex. So, I could get rid of his misconception.
Robust	Provided specific detailed suggestion	Meral assumed that for instance, 7 units and 1 unit were equal in length. To help her, I could bring something whose length was 7 cm such as a 7 cm piece of wire, and a 1 cm piece of wire. I asked her "Did you think these two were equal in length?" I expected that she would say no, the wire 7 cm in length was longer. I asked her "If 7 units and 1 unit were not the same, could we think of them as the same unit?" Then, I could ask her "Could you consider again which units you took when calculating the perimeter?" In this way, she could recognize that she considered unequal units as equal.

2.7.2 Relevance of the analytical framework to the study

In terms of gaining insight into students' thinking, the notion of professional noticing as conceptualised by Jacobs et al., (2010) was relevant to this study as it served as an intervention strategy that assisted in understanding students reasoning of algebraic fractions. Additionally, the levels of sophistication framework made explicit the level of the lecturers *noticing* skills of students' mathematical reasoning when simplifying algebraic fractions. To accurately discern errors and misconceptions and *notice* students' thinking requires teachers to have a sound mathematical knowledge for teaching, as a result the MKT framework served as a useful tool.

2.8 Conclusion

In this chapter, I discussed the main concepts and ideas underlying the overall and specific objectives of this study. Firstly, a generalised description of errors and misconceptions were presented, thereafter the types of errors and misconceptions displayed by students, specifically, within the topic of algebraic fractions were considered. This was followed by conceptualising the construct of professional noticing as three interrelated strands: attending to nuanced features of students' strategies, interpreting students reasoning, and deciding how to respond based on the understanding thereof. From the review of extant literature, it became apparent that there must be an engagement with the effect of professional noticing on the pedagogical practices of NCV L3 lecturers in relation to the simplification of algebraic fractions. This chapter also outlined the study's theoretical (a combination of social constructivism and MKT) and analytical (the three strands of professional noticing and levels of sophistication) framework, and the relevance of these frameworks to the study. The next chapter provides a detailed discussion of the research methods employed in the study.

CHAPTER 3: RESEARCH METHODS

3.1 Introduction

In this chapter, I describe in detail, the research methods employed in this study. I begin by explaining the research paradigm and research design, followed by an account of the research site and study participants. I also include the sampling methods, as well as the data collection methods and methods for data analysis. Finally, issues around trustworthiness, and ethical considerations of the study are discussed.

3.2 Research paradigm - Interpretivism

The research paradigm forms the foundation or backbone of the research project. Depending on the different point of view and beliefs, a research paradigm may be either positivist, post-positivist, interpretivist, or constructivist, pragmatic or transformative (Scott & Morrison, 2005). Different paradigms are associated with different ways of knowing and making sense of the world. This study was guided by an interpretivist paradigm. Interpretivism extends the boundaries of an objective account of events toward a more subjective view of knowledge (Creswell et al., 2007; Scott & Morrison, 2005). A subjective interpretation of reality assists in establishing an insider's view of the different ways of knowing and informs the socio-cultural perspective that is relevant to this research. Interpretivism affords the opportunity to understand the reasons behind students' conceptions and misconceptions and offers a more in-depth comprehension of the thoughts, beliefs, and motivations in relation to students' mathematical thinking. Furthermore, the interpretivist paradigm provides insight into the intentions and objectives that are inextricably linked to students reasoning. Interpretivism in the mathematics classroom allows for an authentic account of the practice and an interrogation of the conditions of meaning between students and teachers. This study used a case study approach to gain an understanding of the extent to which teachers noticed students reasoning in relation to algebraic fractions.

3.3 Research design

This study employed a qualitative research design, particularly a case study approach was used. Yin (2013) suggests that a qualitative case study approach is particularly useful when there is a need to achieve a robust understanding of the phenomenon to be studied within its specific natural context. Creswell et al. (2007) states that the case study approach, by using several sources of data, will enable a detailed, deep, contextual understanding of the phenomenon. A qualitative case study approach will allow for a close connection to the lecturers' understanding

of students mathematical reasoning in relation to the simplification of algebraic fractions. This approach will also strengthen the reliability of the study by providing a rich collection of data in line with the objectives of this study (Yin, 2013). Multiple data sources, such as, students written artefacts, and observations of teaching were used to give a detailed description of the research questions. Moreover, issues relating to the challenges and the dire need for training and development in terms of in-the moment noticing emerged.

3.4 Case study

A case study of TVET College B in Gauteng, South Africa was used to gain an in-depth understanding of the extent to which lecturers notice students mathematical reasoning in relation to simplification of algebraic fractions. A case study method is used when the contextual circumstances are crucial to the phenomenon of the study (Creswell et al., 2007; Yin, 2013). Since the teaching and learning of mathematics, specifically the understanding of students reasoning, is inseparable from the context, and considering that the research took place within a single TVET college in a specific area, the case study was an appropriate method for the purpose of this research.

3.5 Research site and sample

This study was conducted at a public TVET College B[1] located in Daveyton; a township situated within the Gauteng province. TVET College B was purposefully selected as the lecturers in this college often participate in training and development programs. Etikan and Musa (2016) state that "the idea behind purposive sampling is to concentrate on people with particular characteristics who will better be able to assist with the relevant research" (p. 3). The target population for this study was all NCV L3 lecturers. For this study, the sample size consisted of two NCV L3 lecturers and their respective classes. These lecturers were conveniently selected; access to these lecturers was both practical and feasible (Etikan & Musa, 2016). The NCV L3 classroom is comprised of approximately 30 mixed-ability students. Students in NCV L3 are made up of high school dropouts, or students who received a condoned pass in L2 while others were promoted to L3 having met the minimum requirements in L2. The two lecturers that were chosen both have tertiary mathematics qualifications, with mathematics as a major, and both lecturers are considered subject specialists (having obtained a class average of greater than 60%, every year, for the past 10 years).

[1] The name TVET College B is a pseudonym for the TVET College that was the research site.

3.5.1 Socio-demographic characteristics of the study participants

The study participants consisted of two male lecturers teaching Mathematics NCV L3 in a public TVET College B, and their respective classes. The age range for both lecturers was from 40-50 years old. One of the lecturers has been teaching NCV L3 mathematics for 8 years, while the other lecturer has been teaching mathematics for over 12 years (but NCV L3 for 9 years). Both lecturers have a Post Graduate Certificate of Education (PGCE). Table 2 illustrates a summary of the socio-demographic characteristics of the lecturers who participated in the study.

Table 2: Socio-demographic Characteristics of Lecturers

Description	**Number of Participants**
Gender	
Male	2
Age group	
40-50	2
Teaching experience	
1-10 years	1
10-21 years	1
Experience teaching NCV L3 Mathematics	
1-10 years	2

Ten Mathematics NCV L3 students from TVET College B were willing to write the pre-test; this included six males and four females. They ranged from the age of 19-21 years old. Table 3 presents a summary of the socio-demographic characteristics of the students.

Table 3: Socio-demographic Characteristics of Students

Description	**Number of Participants**
Gender	
Male	6
Female	4
Age group	
19-21	10

3.6 Data collection

This study was conducted on the topic of simplification of algebraic fractions. Data were collected by means of students' responses to a pre-test and lesson observations. Before the data

collection process, the researcher conducted a professional noticing development program (workshop) with the lecturer participants. The participants were introduced to the construct of professional noticing and it's explicit use in relation to algebraic fractions. This was done by providing the participants with a variety of articles relating to professional noticing. The articles were chosen based on their relevance to this study (Dindyal et al., 2021; Ergene & Mine, 2021; Jacobs et al., 2010; König et al., 2022; Sherin et al., 2011; Thomas et al., 2015). The participants were asked to read the articles and thereafter discussions were held relating to published literature included. Lecturers engaged with the readings and were given the opportunity to ask questions. The researcher facilitated the discussions. The workshop was held for 4 days, and each discussion was approximately 45 minutes to an hour long. The workshop served as an opportunity for the participants to use errors and misconceptions as a tool to access learners mathematical reasoning. Additionally, the workshop provided opportunities for lecturers to interrogate their own pedagogical practices in terms of appreciating errors and misconceptions as resources for mathematical growth rather than problems that must be avoided. During the workshop, lecturers were encouraged to share their strengths and weaknesses in engaging with the challenges that learners face with a view of improving their current practices.

3.6.1 Pre-test

The pre-test was used to identify the errors and misconceptions that the students committed in relation to the simplification of algebraic fractions. The test consisted of 6 questions and students were given 60 minutes to simplify the algebraic fraction and provide an explanation in relation to their reasoning. The pre-test was written under strict test conditions. The questions in the pre-test were collaboratively selected by the lecturers and the researcher and were based on the requirements as set out in the NCV L3 curriculum guidelines (see Appendix A for pre-test questions). At the time of data collection only 10 students were available and willing to write the test. The lecturers indicated that absenteeism remains a major problem that they experience in NCV L3 mathematics classrooms; the number of students in attendance was thus not surprising to the lecturers. The purpose of the pre-test was to explore the extent to which lecturers noticed learners mathematical reasoning, rather than the categorisation of errors and misconceptions. Each lecturer was asked to analyse the students' written responses by providing a detailed explanation on what they noticed about students reasoning. Due to College

commitments and time constraints on the part of the lecturers, only 6 written responses were analysed.

3.6.2 Observations

Each lecturer was observed teaching a session on algebraic fractions. While one of the lecturers spent about an hour teaching, the other lecturer taught for approximately 20 minutes before his lecture was interrupted; all students were moved to a common area for a health and wellness training. Both teaching sessions observed were audio-recorded; the students indicated that they were not comfortable being video-recorded. The audio recordings assisted in precisely capturing the manner in which lecturers made sense of learners mathematical reasoning, thus providing insight into the pedagogical practices of the lecturers. The recordings were fundamental to the findings of the study and were reported verbatim in the analysis section. The observation of lecturers enabled the researcher to explore the extent to which lecturers attend to, interpret, and respond to students' mathematical reasoning while teaching.

3.7 Data analysis

Data analysis is a meaning making process that involves interpreting, understanding, and consolidating data to answer the research questions (Merriam & Tisdell, 2016). In this study, the data from the written responses were analysed to identify the types of errors and misconceptions that students commit and the reasons thereof. The levels of Sophistication framework (Ergene & Mine, 2021) was used to gauge the extent to which the lecturers attend to, interpret, and respond to students reasoning in the written responses as well as during the teaching episodes. Since the main focus of the study was to explore the manner in which lecturers attended to, interpreted, and responded to students mathematical reasoning, emphasis was placed on the extent to which lecturers notice learner errors and misconceptions, rather than the categorisation of the errors and misconceptions committed. In addition, data from the observations were audio-recorded and transcribed verbatim. The transcripts were read several times to establish an accurate understanding and identify different patterns or codes. Data were analysed using a thematic comparative method of analysis (Glaser, 1965). This was an iterative process that involved analysing, comparing, and abstracting data, to obtain meaningful relationships and themes. The findings in this study are supported and reported using verbatim quotations from the observations made, the scripts analysed, and by relevant extant literature.

3.8 Data Quality criteria

Within qualitative research, Merriam and Tisdell (2016) note that there are "strategies for establishing the authenticity and trustworthiness of a study – strategies based on worldviews and questions congruent with the philosophical assumptions underlying this perspective" (p. 239). For the purpose of this study, these strategies include triangulation, a rich-thick description, and peer-review. These strategies are pertinent in considering the issues relating to qualitative research.

3.8.1 Triangulation

Triangulation is a procedure for validity, it is a systematic process where the researcher seeks convergence by using a variety of data sources to identify common themes (Creswell & Miller, 2000). Triangulation may be achieved by using different methods. In this study, lesson observations and document analysis of students' scripts, allowed for the collaboration of evidence. Multiple data sources provide an extensive account of the professional noticing capacity of lecturers through the emergence of common themes and categories.

3.8.2 Rich, thick description

A rich, thick description offers robust, thorough information and allows for credibility in the study. A rich thick description moves beyond simply reporting facts, instead it provides a copious, interrelated detailed account. In terms of this study, a rich thick description was achieved by including detailed narratives in support of understanding the noticing skills of participants. Additionally, a rich thick description allows for the consideration of the applicability or transferability of the study to other contexts (Creswell & Miller, 2000). In terms of this study, since it presents a single case study, transferability is contingent on situations that are comparable or similar to this study.

3.8.3 Peer review

Peer review or peer debriefing is a process of review that is done by an external reviewer to ensure that all the procedures and process of the study have been carried out efficiently and all protocols have been adhered to. Peer review is carried out by an individual who is familiar with the phenomenon that was researched (Creswell & Miller, 2000; Merriam & Tisdell, 2016). The aim of peer debriefing is to provide the researcher with support, and to challenge the researcher

by playing devil's advocate, and it opens opportunities for critical thinking in relation to the methods employed and interpretations thereof (Lincoln & Guba, 1985 as cited in Creswell & Miller, 2000). In this study, peer debriefing sessions were conducted with my supervisor to discuss the preliminary findings, data analysis and emerging themes.

3.9 Ethical considerations

The research process began by obtaining approval, to undertake this study, from the Department of Higher Education and Training (DHET) as well as from Wits School of Education Non-Medical Human Research Ethics Committee (Protocol number: 2023ECE013; see Appendix B). Once ethical clearance was obtained, the principal of TVET College B was approached, and consent was given to use TVET College B as the research site. Thereafter, the lecturer participants were invited to participate in the study. Worth noting is that the principal and study participants were invited before the commencement of data collection. The aim and purpose of the study was carefully and extensively explained to the lecturer participants; all the information was included in the written information letter that was handed out to each participant. Furthermore, informed written consent was obtained from each participant. Anonymity and confidentiality were maintained throughout the research process. Anonymity was maintained by using pseudonyms and not disclosing any form of personal identification. Confidentiality and privacy were maintained by conducting the professional noticing intervention program in a safe, supportive environment. In terms of accessibility of data, the audio-recordings and transcripts of the lessons were stored in a password protected computer and only accessible by the researcher.

3.10 Conclusion

In this chapter, I outlined the research methods employed for the purpose of the study. Specifically, the research paradigm, research design and research site were explained in detail. Also, I described the study population, criteria for the selection of study participants, as well as the methods used for data collection and data analysis. Furthermore, included in this chapter were the techniques for ensuring the dependability of the study. Finally, ethical considerations, such as ethical clearance, confidentiality and anonymity were also considered in this chapter. These methods employed were in alignment with the overall and specific objectives of the study. More so, the methods align with the theoretical framework used. The next chapter presents the findings and results of the data collected.

CHAPTER FOUR: FINDINGS OF THE STUDY

4.1 Introduction

In this chapter, the findings and results of the data collected are presented. To reiterate, the overall objective of the study was to explore the professional noticing practices of NCV L3 lecturers in relation to the simplification of algebraic fractions. Along this line, in this chapter, I present the key findings of the study according to the research questions.

4.2 Presentation of results

From the 10 students who wrote the pre-test, 3 scripts were randomly selected by each lecturer participant. For each of the 6 questions contained in the pre-test, participants were prompted to provide a detailed description of what each student did in each question; what they learned about the students' understanding; and what suggestions (questions or problems) they will pose to each student (Simpson & Haltiwanger, 2017). This resulted in 18 responses per participant (lecturer A analysed student 3, 4, and 7 while lecturer B analysed student 5, 6, and 8; results are presented respectively). However, I have included within this chapter, 3 responses per lecturer, relating to each skill of noticing (see Appendix E for the complete analysis). Furthermore, the data from the lesson observations were also categorised according to the three strands of professional noticing (attending, interpreting, and responding) (Jacobs et al., 2010). Thereafter, drawing on the levels of sophistication framework (Ergene & Mine, 2021), each component or category was coded as low (0), medium (1), or robust (2) (see table 1), depending on the extent to which the lecturer uncovered significant details in students' responses (attending); provided pertinent information about students' understanding (interpreting); and whether the lecturers provided general or specific suggestions in response to students' understanding (responding). Students' reasoning in relation to the simplification of algebraic fractions are illustrated in table 4 below, followed by the analysis in terms of the research questions.

4.2.1 Table presentation of student written responses to pre-test questions

Table 4: Student responses to each pre-test questions

Lecturer A: Student 3 script

Figure 4- Q1 & Q2 respectively

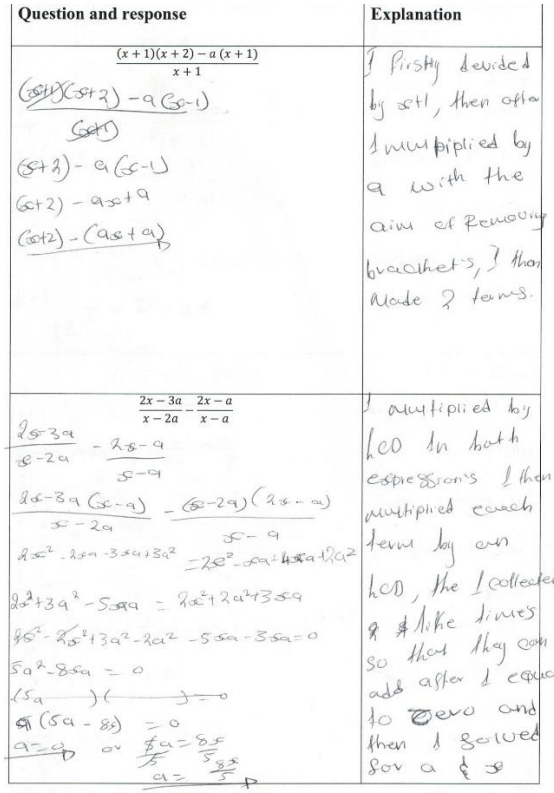


Figure 5- Q3 & Q4 respectively

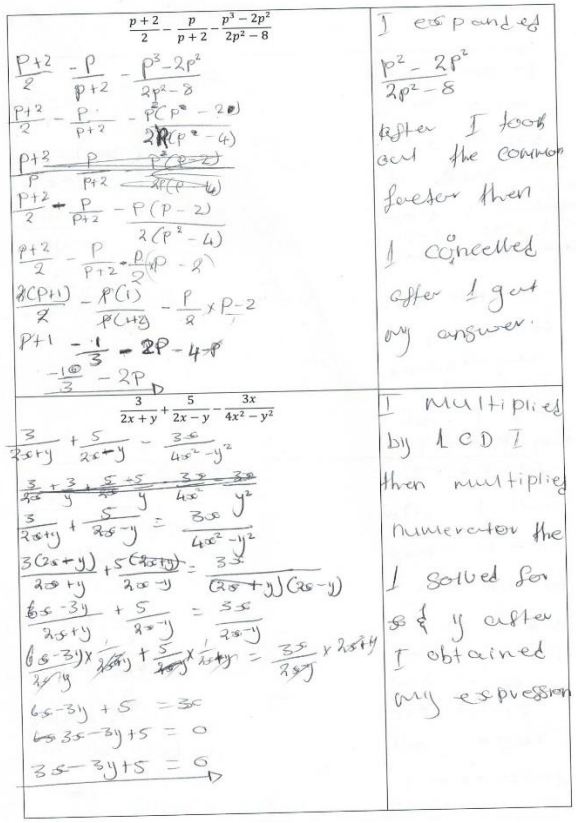
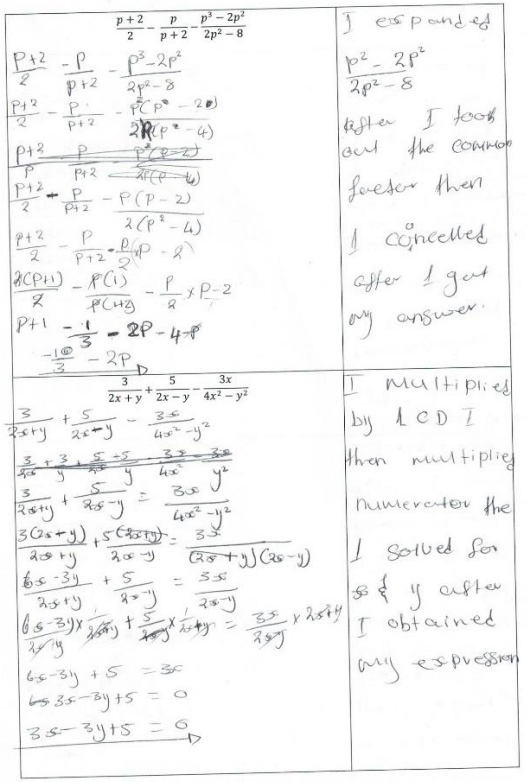


Figure 6- Q5 & Q6 respectively



Lecturer A: Student 4's Script

Figure 7- Student 4 - Q1 & Q2 respectively

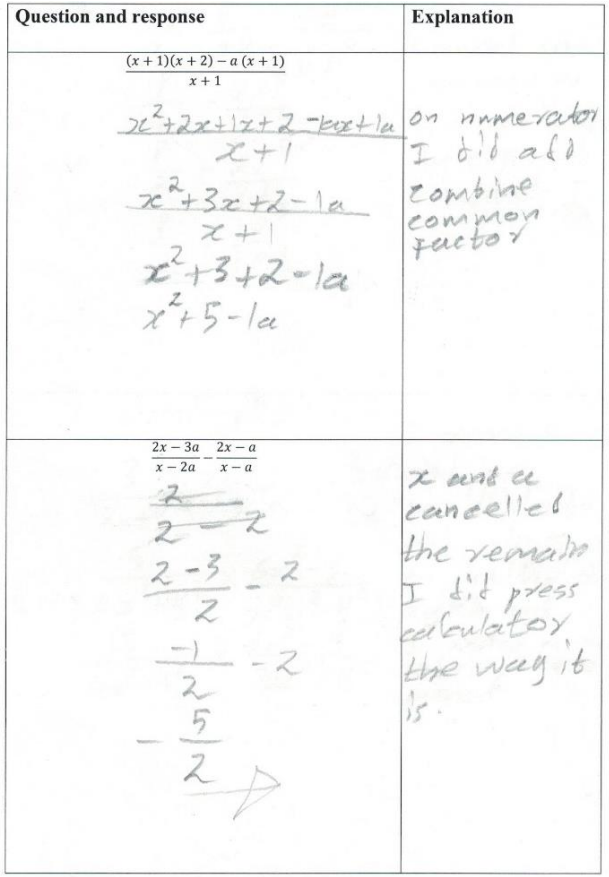


Figure 8- Q3 & Q4 respectively

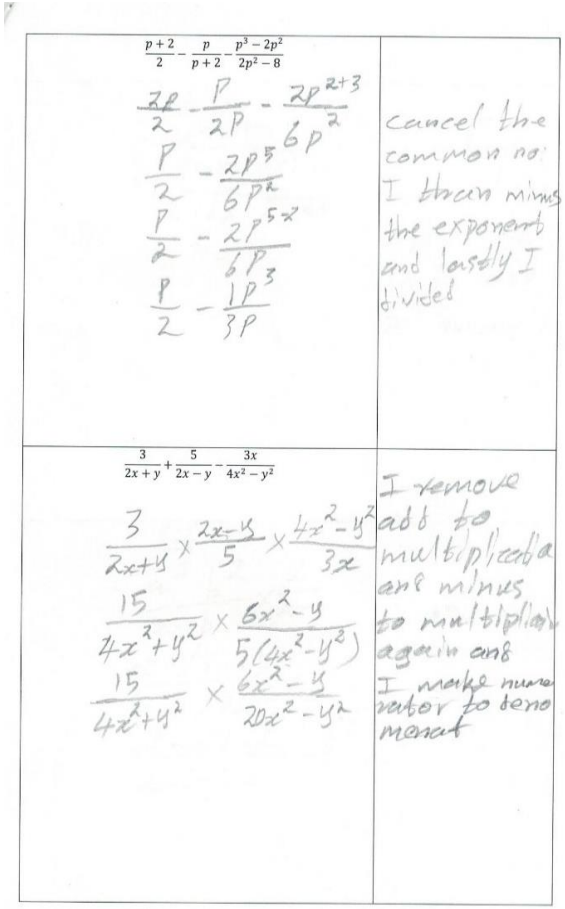
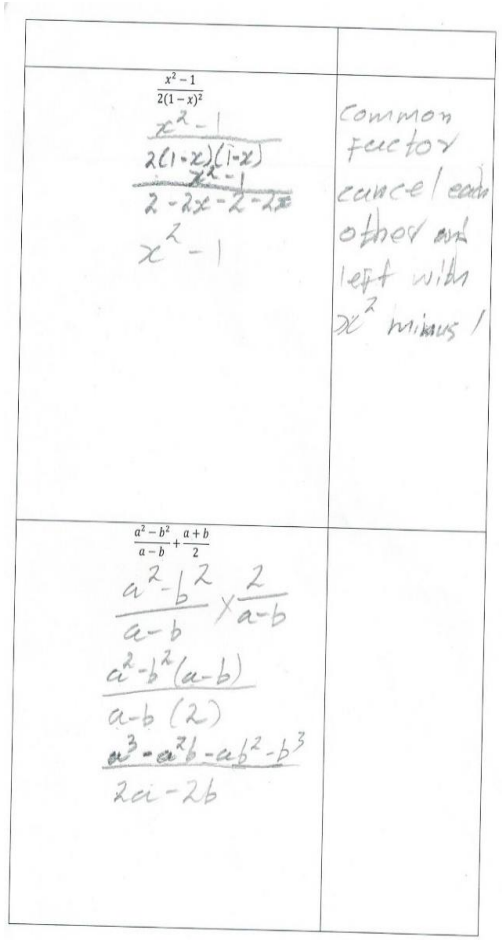


Figure 9- Q5 & Q6 respectively



Lecturer A: Student 7 script

Figure 10- Q1 & Q2 respectively

Question and response	Explanation
$\frac{(x+1)(x+2)-a(x+1)}{x+1}$ $= \frac{(x+1)(x+2)}{(x+1)} - \frac{a(x+1)}{x+1}$ $= x+2 - \frac{ax+a}{x+1}$ $= x+2 - \frac{ax+a}{x+1} \times \frac{(x-1)}{x-1}$ $= x+2 - \frac{(ax+a)(x-1)}{x^2-1}$ $= x+2 - \frac{(ax+a)(x-1)}{(x-1)(x+1)}$ $= x+2 - \frac{(ax+a)}{x+1}$ $= x+2 - \frac{a(x+1)}{x+1}$ $= x+2-a$	
$\frac{2x-3a}{x-2a} - \frac{2x-a}{x-a}$ $= \frac{(x-2a)(2x-a)}{} - \frac{(2x-3a)(x-a)}{}$ $= (2x^2-5ax+a^2) - (2x^2-5ax+3a^2)$ $= a^2+3a^2$ $= 4a^2$	

Figure 11- Q3 & Q4 respectively

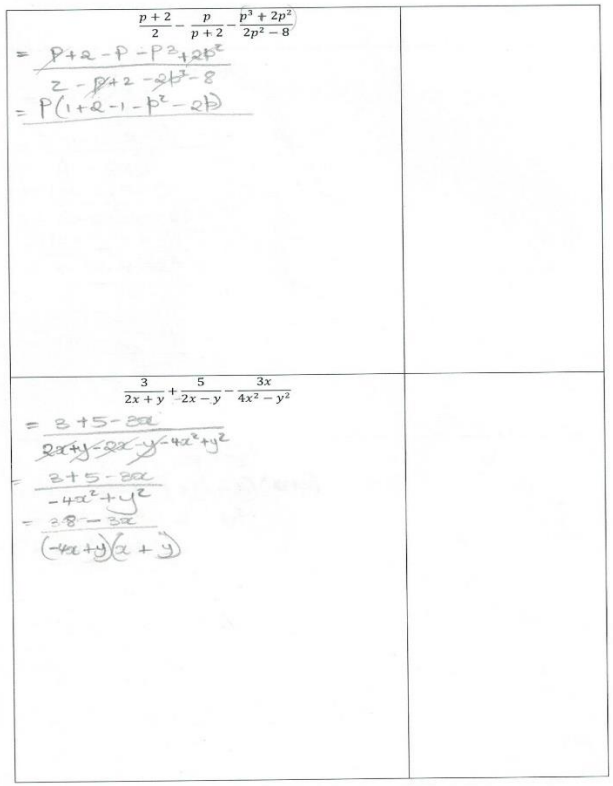
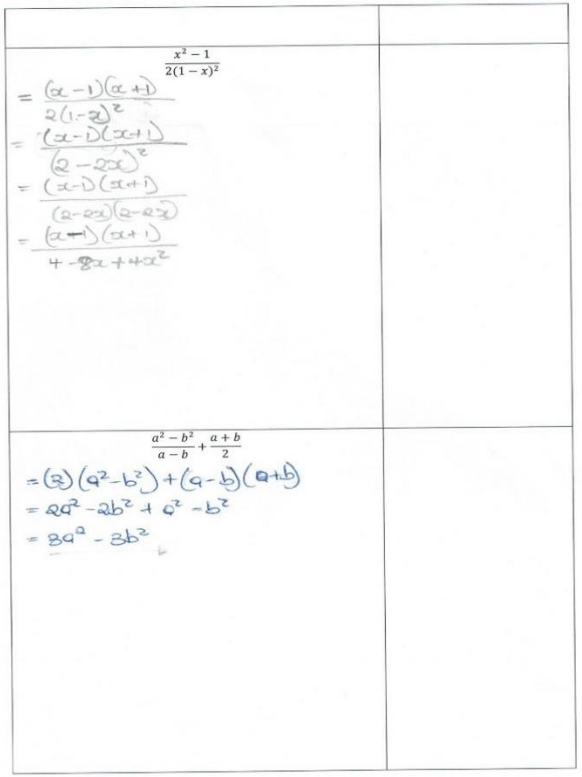


Figure 12- Q5 & Q6 respectively



Lecturer B: Student 5's Script

Figure 13- Q1 & Q2 respectively

Question and response	Explanation
$\frac{(x+1)(x+2)-a(x+1)}{x+1}$ $(x+2)-a(x+1)$ $(a+2)-ax+1$ $a+2-ax+1$ $a-ax+2+1$ $a(1-x)+3$ $(1-x)(a+3)$	
$\frac{2x-3a}{x-2a}-\frac{2x-a}{x-a}$ $\frac{(2x-a)[(1-3)(1)]}{(x-a)[(1-2)-1]}$ $\frac{(2x-a)(2)}{(x-a)}$ $\frac{4x-2a}{x-a}$ $\frac{2(2x-a)}{(x-a)}$	

Figure 14- Q3 & Q4 respectively

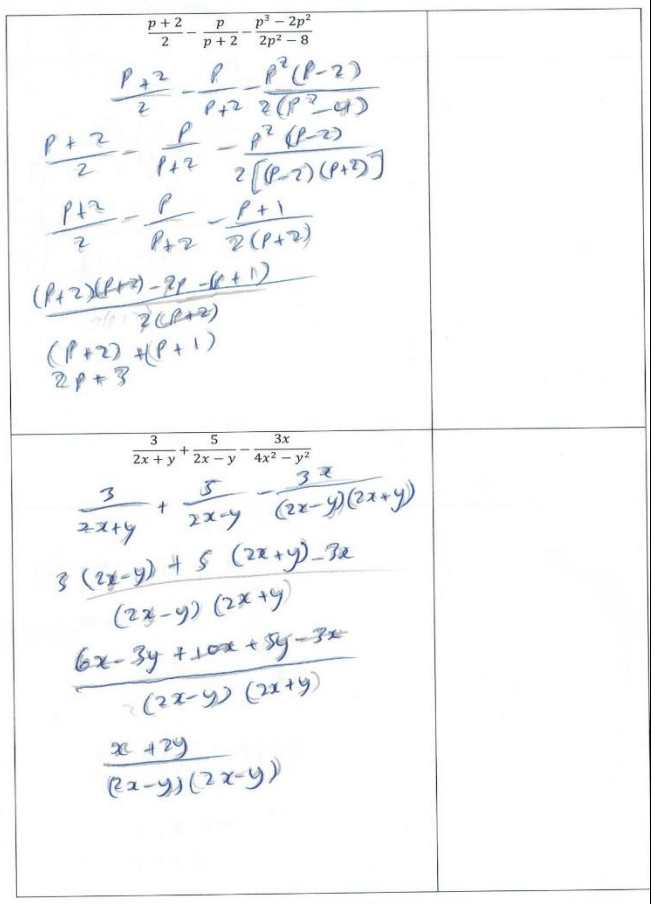
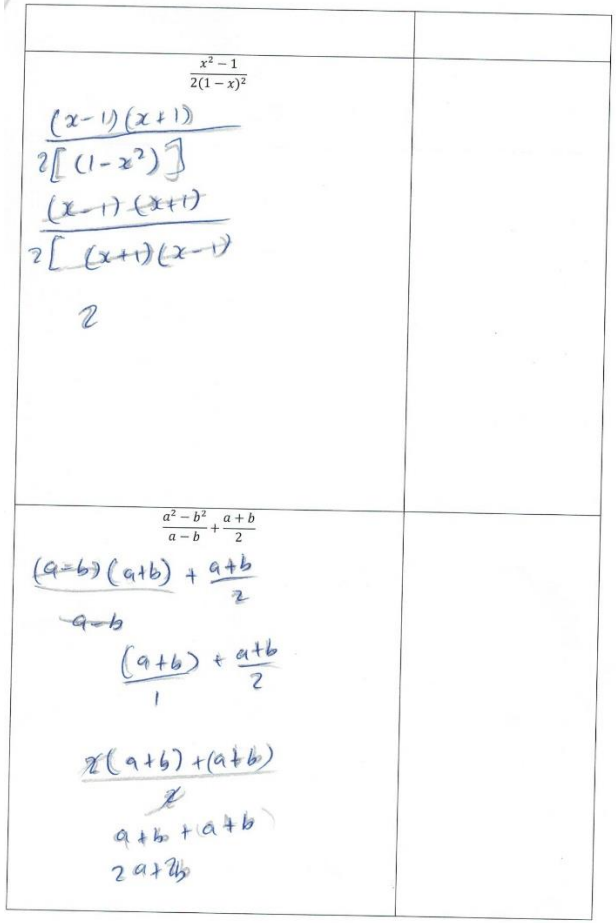
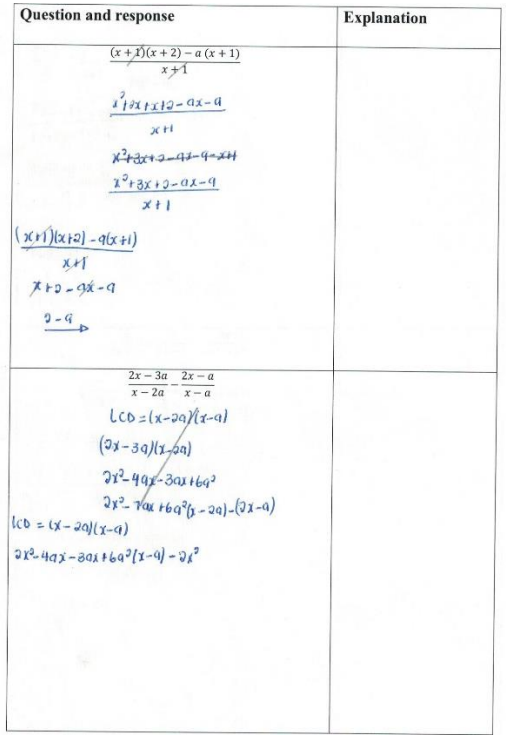


Figure 15- Q5 & Q6 respectively



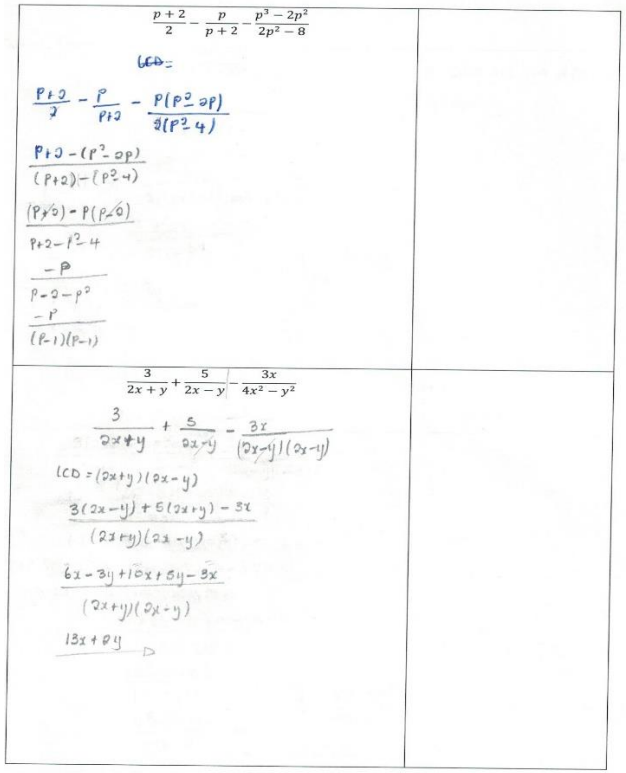
Lecturer B: Student 6's Script

Figure 16- Q1 & Q2 respectively



Question and response	Explanation
$\frac{(x+1)(x+2)-a(x+1)}{x+1}$ $\frac{x^2+2x+x+2-ax-a}{x+1}$ ~~$x^2+2x+x+2-ax-a=x+1$~~ $\frac{x^2+3x+2-ax-a}{x+1}$ $\frac{(x+1)(x+2)-a(x+1)}{x+1}$ $x+2-ax-a$ $2-a$ →	
$\frac{2x-3a}{x-2a}-\frac{2x-a}{x-a}$ LCD $=(x-2a)(x-a)$ $(2x-3a)(x-2a)$ $2x^2-4ax-3ax+6a^2$ $2x^2-7ax+6a^2-(2x-a)$ LCD $=(x-2a)(x-a)$ $2x^2-4ax-3ax+6a^2-(x-a)-2x^2$	

Figure 17- Q3 & Q4 respectively



$\frac{p+2}{2}-\frac{p}{p+2}-\frac{p^3-2p^2}{2p^2-8}$

LCD $=$

$\frac{p+2}{2}-\frac{p}{p+2}-\frac{p(p^2-2p)}{2(p^2-4)}$

$\frac{p+2-(p^2-2p)}{(p+2)-(p^2-4)}$

$\frac{(p+2)-p(p-2)}{p+2-p^2+4}$

$-p$

$p-2-p^2$

$-p$

$(p-1)(p-1)$

$\frac{3}{2x+y}+\frac{5}{2x-y}-\frac{3x}{4x^2-y^2}$

$\frac{3}{2x+y}+\frac{5}{2x-y}-\frac{3x}{(2x-y)(2x-y)}$

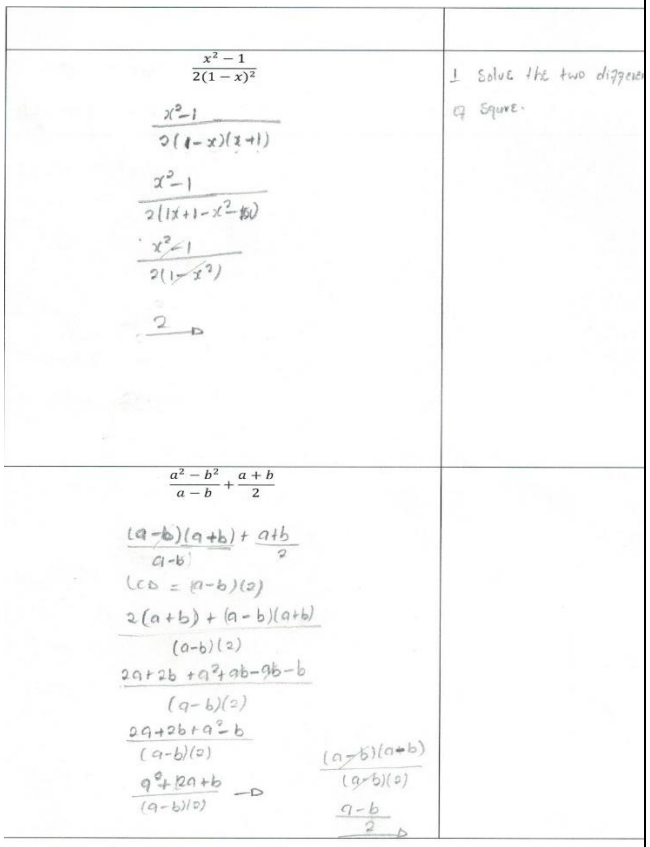
LCD $=(2x+y)(2x-y)$

$\frac{3(2x-y)+5(2x+y)-3x}{(2x+y)(2x-y)}$

$\frac{6x-3y+10x+5y-3x}{(2x+y)(2x-y)}$

$13x+2y$ →

Figure 18- Q5 & Q6 respectively



$\frac{x^2-1}{2(1-x)^2}$

1 Solve the two differen
of Square.

$\frac{x^2-1}{2(1-x)(1-x)}$

$\frac{x^2-1}{2(1x+1-x^2-1x)}$

$\frac{x^2-1}{2(1-x^2)}$

$\frac{2}{}$ →

$\frac{a^2-b^2}{a-b}+\frac{a+b}{2}$

$\frac{(a-b)(a+b)}{a-b}+\frac{a+b}{2}$

LCD $=(a-b)(2)$

$\frac{2(a+b)+(a-b)(a+b)}{(a-b)(2)}$

$\frac{2a+2b+a^2+ab-ab-b}{(a-b)(2)}$

$\frac{2a+2b+a^2-b}{(a-b)(2)}$

$\frac{a^2+2a+b}{(a-b)(2)}$ →

$\frac{(a-b)(a+b)}{(a-b)(2)}$

$\frac{a-b}{2}$ →

Lecturer B: Student 8's Script

Figure 19- Q1 & Q2 respectively

Question and response	Explanation
$\frac{(x+1)(x+2)-a(x+1)}{x+1}$ $\frac{(x+2)(x+1)-a(x+1)}{(x+1)}$ $x+2-a(x+1)$ $2x-a+x$ $2x^2-a$	
$\frac{2x-3a}{x-2a}-\frac{2x-a}{x-a}$ $\frac{x-a}{x-2a}-\frac{2x-a}{x-a}$ $\frac{x^2-a^2}{2x^2-2a^2}$ $\frac{x^2-a^2}{x^2-a^2}$ 1	

Figure 20- Q3 & Q4 respectively

$\frac{p+2}{2}-\frac{p}{p+2}-\frac{p^3-2p^2}{2p^2-8}$

$\frac{p2}{2}-\frac{p}{p2}-\frac{-2p}{2p^2-8}$

$\frac{p2}{2}-\frac{p}{p2}-\frac{-2p}{-6p^2}$

$1p-\frac{1}{2}p-\frac{1}{3}p^3$

$\frac{1}{2}p^2-\frac{1}{6}p^3$

$\frac{1}{6}p^{2-3}$

$\frac{1}{6}p^1$

$\frac{3}{2x+y}+\frac{5}{2x-y}-\frac{3x}{4x^2-y^2}$

$\frac{3}{2x+y}+\frac{5}{2x-y}-\frac{3x}{4x^2-y^2}$

$1,5xy+2,5xy-0,75x^2-y^2$

$4x^2y^2-0,75x^2-y^2$

$3,25xy$

Figure 21- Q5 & Q6 respectively

$\frac{x^2-1}{2(1-x)^2}$

$\frac{x^2-1}{2+1^2-x^2}$

2

$\frac{a^2-b^2}{a-b}+\frac{a+b}{2}$

$\frac{(a+a)-(b+b)}{a-b}+\frac{a+b}{2}$

$\frac{ab+ab-ab+ab}{a-b}+\frac{a+b}{2}$

$\frac{ab^2-ab^2}{a-b}+\frac{a+b}{2}$

$\frac{a-b}{a-b}+\frac{a+b}{2}$

$a-b+\frac{a+b}{2}$

$\frac{a^2+b^2}{2}$

4.2.2 Attending to students' mathematical reasoning in relation to the simplification of algebraic fractions

Under this subsection, I present the lecturers' analysis of each student's scripts selected. This means that the analysis is presented verbatim.

Lecturer A's analysis of test scripts of Student 3, 4 and 7

Student 3's Script (see table 4 for the question and student's written response):

Q1: This student was drawn to the fact that the term $(x + 1)$ *appeared in both the numerator and the denominator and therefore proceeded to cancel them instinctively. The incorrect cancelling also assisted the student to get rid of the fraction, forming a linear expression that most students find easier to work with. The student then mistakenly changed* $(x + 1)$ *to* $(x - 1)$ *and then applied the distributive law, correctly, to remove the brackets. The student then added brackets again, however this was done incorrectly. The student realised that there were no like terms, hence did not proceed further.*

Q4: The student formed an equation by equating the first two terms to the third term. Then cross multiplied each denominator and then correctly identified the denominator of the third term as DOTS (a difference of two squares) and factorised it. He/she then saw the one of the squares $(2x + y)$ *also existed in term 2, on the left-hand side of the equation. This stimulated the student to cancel the* $(2x + y)$ *in both terms. In line 5, he/she multiplied each term on the left side by* $\frac{1}{(2x+y)}$ *and on the right-hand side by* $\frac{(2x+y}{1}$ *to cancel the denominators and form a linear equation. Then he/she made the equation equal to 0 and was left with three unlike terms and could not proceed.*

Q5: The student correctly factorised the numerator as a DOTS, however swopped the two terms within the bracket of the denominator so that he/she could cancel the $(x - 1)$*. Then the student used distributive law correctly in the denominator and couldn't proceed.*

Student 4's Script (see table 4 for the question and student's written response):

Q2: In both terms the student cancelled both variables in the numerator with the same variables in the denominator and was left with a numeric fraction that he/she calculated using a calculator.

Q4: The student changed the addition to multiplication and inverted all three terms. He/she then went back to original question and multiplied the numerators and denominators (incorrectly) of the first two terms, then cross multiplied the second and third terms (incorrectly). Then distributed $5(4x^2 - y^2)$ *incorrectly and did not go further.*

Q6: The student changed the addition to multiplication and inverted the second term, then multiplied the numerator out and the denominator and saw that there was no like terms and ended there.

Student 7's Script (see table 4 for the question and student's written response):

Q1: The student separated the fraction into terms and cancelled the common factor correctly within the first term. In the second term, however, he/she multiplied the bracket out instead of cancelling the common factor, then multiplied by the conjugate of the denominator, then multiplied the denominators of the second term only to factorise it out again and then finally cancel the common factor. The student took a long route but arrived at the correct answer.

Q3: The student wrote down the numerators and denominators of all three terms as a single fraction, then took out p as a common factor in the numerator (although it was not common). He/she ignored the denominator, then just gave up.

Q6: This student ignored the + sign and cross multiplied to get rid of the fraction and then multiplied brackets out and collected like terms to reach a final answer consisting of two unlike terms. He/she didn't continue further by considering 3 as a common factor or recognise the squares as DOTS.

Lecturer B's analysis of test scripts of Student 5, 6 and 8:

Student 5's Script (see table 4 for the question and student's written response):

Q1: The student was drawn to the same term in the numerator and denominator and rushed to cancel instead of factorising first. Then changed the x to a; in the second term he/she distributed the $-a$ into the bracket must made a mistake with one of the signs and then combined like terms. Then took out a as a common factor of the first two terms and then just put brackets around the $a + 3$ so that the result consisted of two brackets.

Q3: In the third term, the student correctly factorised the numerator and denominator and recognised DOTS and correctly cancelled the $(p - 2)$ but then mistakenly wrote p^2 instead of p and when he/she cancelled the common bracket added the 1 instead of multiplying it which led to two terms in the numerator of the third term. He/she correctly identified an LCD and combined the three terms into one fraction, but then was drawn to the common bracket and the common co-efficient of 2 in the numerator and denominator and decided to cancel them. Then added the like terms and ended the solution.

Q4: The student correctly factorised the denominator of the third term using DOTS and found an LCD and correctly formed a single fraction. Then correctly distributed the co-efficient to remove brackets and combined like terms but made a calculation error with the x term, everything else was correct.

Student 6's Script (see table 4 for the question and student's written response)

Q1: This student gave two responses. In the first one, he/she multiplied the numerator out and correctly, added the like terms but saw that there weren't any more like terms and left it. In the second attempt, cancelled the same bracket in the numerator and denominator to get rid of the fraction and then cancelled the x and ax terms as though combining them would give a 0, and remained with two terms.

Q4: The student factorised the denominator of the third term by DOTS and found the LCD and multiplied each term correctly to form one fraction and added/subtracted like terms correctly in the numerator but then ignored the denominator in the final step.

Q6: The student factorised term 1 numerator as DOTS and used an LCD of 2 and $(a-b)$. He/she saw that there was a common bracket of $(a-b)$ but only cancelled the bracket in the numerator not in the denominator which then led to using the LCD of $(a-b)2$. The student then formed a single fraction, carrying the error from previous step and tried to collect like terms which he/she tried to factorise as a trinomial and then cancelled one of the brackets, $(a-b)$, in the numerator with the $(a-b)$ in the denominator.

Student 8's Script (see table 4 for the question and student's written response):

Q1: The student immediately cancelled the same bracket in the numerator of the first term with the denominator. Then multiplied the $-a$ into $(x+1)$ incorrectly, then said $x+2=2x$ and $2x+x=2x^2$ to get an answer of $2x^2-a$.

Q2: The student cross multiplied the denominator of one term with the numerator of the other term but got a binomial rather than a trinomial (did not use foil correctly). Then he/she disregarded the co-efficient of 2 in both terms of the denominator so that he/she was left with the same numerator and denominator that could be cancelled to get an answer of 1.

Q5: The student did not factorise the numerator; in the denominator the student just squared each term in the bracket and then noticed that the numerator and denominator had similar terms and so cancelled them, leaving him/her with an answer of 2 instead of a fraction.

Table 5: An example of the level of Professional Noticing displayed by the lecturers

Attending		
Code	**Description**	**Examples**
Low (0)	Described strategy as incorrect/correct with a level of uncertainty	*The student wrote down the LCD but then did not work with a fraction. He/she multiplied the numerator and denominator of the first term together and then multiplied the $6a^2$ term by the denominator of the second term and then subtracted $2x^2$(not sure why).*
Medium (1)	Provides a general description of the students' strategy	*In term 1, the student combined the two terms of the denominator, then divided the 3 of numerator with the 2 in the denominator; in term 2, the student divided the numerator with 2 in the denominator; in term 3, the student divided the 3 in the numerator with the 4 of the denominators, ignoring the x in the numerator. By doing this the student got rid of the fraction. In line 3, the student added the co-efficient on the first and second term and added the variables incorrectly using incorrect exponential laws. In the last step, the student subtracted the co-efficient of the first two terms and subtracted the exponents of the variables incorrectly.*

4.2.3 Interpreting students' mathematical reasoning in relation to the simplification of algebraic fractions

Under this subsection, I present the lecturers' analysis of each student's scripts selected. This means that the analysis is presented verbatim.

Lecturer A's interpretation of Student 3, 4 and 7 based on the written responses:

Student 3's Script (see table 4 for the question and student's written response):

> ***Q1:*** *This student fell into the trap that most students fall into. As soon as they see something that looks the same in the numerator and denominator, they want to cancel it. The student did not understand that there were two terms in the numerator and that there was a common bracket or common factor that he/she could remove and only then cancel. This student did not know the difference between cancelling factors and cancelling terms. Also, by cancelling the denominator,*

the student no longer had a fraction, making it easier for them to work with. This is another common error that students make, they find it easier to work with a linear expression. This student does not know when to cancel or how to factorise. The student knew how to distribute the negative into the bracket but did not recognise how the sign would change when he/she put the bracket back.

Q2: *This student does not understand how to add or subtract fractions, shows no concept of an LCD. He/she confuses an expression with an equation and does not understand when cross multiplication is possible. He/she knows how to FOIL correctly and combine like terms and move terms correctly to equate to 0 and solve for a correctly. Even though he/she knows this, the thinking in dealing with the fraction is completely wrong. This student finds it easy to work with an equation instead of a fraction so created the equation.*

Q6: *Correctly identified DOTS but it could be a careless error of the sign in the one bracket. No concept of addition/subtraction of fractions or LCD. Again, doesn't understand fractions so gets rid of the denominator. No understanding of how an expression changes when bracketing the terms.*

Student 4's Script (see table 4 for the question and student's written response):

Q2: *No understanding of subtraction of fraction, or concept of LCD. The student was just drawn the same variables in the numerator and denominator. Like in Q1, the student does not know when to cancel. By getting rid of the variables the student was left with an arithmetic fraction which was solved using a calculator, which shows that he/she does not know how to subtract basic arithmetic fractions.*

Q3: *Shows no understanding of how to add/subtract fractions, also no concept of an LCD. Shows no understanding between variables and constants and adding/subtracting of like or unlike terms. Lacks knowledge of basic algebra and arithmetic. Wrong application of exponential rules but knows that when dividing like bases exponents are subtracted.*

Q4: *No understanding of when to invert the fraction, or the context in which this operation is applicable. Does not know how to add/subtract fractions. No recognition of DOTS or the idea of an LCD. Also shows no understanding of cross multiplication when it can or can't be used.*

Student 7's Script (see table 4 for the question and student's written response):

Q1: *The student understands how to separate one term into two fractions again. I find it strange that the student didn't recognise the common bracket in the second term, this could be because it was not bracketed. He/she went about a roundabout before cancelling the common factor. This*

showed an understanding of DOTS, multiplying by conjugate of denominator and correct factorisation eventually.

Q3: *Again, no concept of LCD or addition/subtraction of fractions. Does not know how to factorise correctly. This student did not know what to do so just combined the numerator and denominator and incorrectly cancelled.*

Q4: *As previously, no concept of LCD or how to add/subtract fractions. Just combined numerator and denominator and added/subtracted like terms in the denominator. No consideration for the fraction as a whole but works on separate parts.*

Lecturer B's interpretation of Student 5, 6 and 8 based on written responses:

Student 5's Script (see table 4 for the question and student's written response):

Q2: *Lacks knowledge of basic factorisation, of how to identify a common factor. No understanding of addition or subtraction of fractions, which means there is no understanding of the idea or even importance of an LCD. Looking at how he/she added the integers, shows that he/she doesn't even know basic primary school adding or subtracting of integers. Which is very worrying. The student also does not know when to cancel.*

Q4: *Like in the previous question, the student understands how to do DOTS and how to find the LCD. Also understands how to add and subtract the fractions using an LCD and correctly distributes the co-efficient into the brackets. The student made a careless mistake when adding the* x *terms however did the* y *terms correctly.*

Q6: *Sees the DOTS in term 1 and cancels correctly, and interestingly acknowledges that he/she is left with 1 in the denominator when cancelling, unlike in Q5. He/she also knows how to select an LCD but then does not understand that he/she can't cancel. After incorrect cancelling, understands how to add like terms.*

Student 6's Script (see table 4 for the question and student's written response):

Q1: *Based on the two solutions the student provided, he/she does not understand the idea of factorisation or looking for a common factor and removing it. The student does not understand why he/she can't cancel, showing no acknowledgement of the fraction as a whole. The student understands how to distribute or FOIL brackets but looking at response 2, he/she does not fully understand when to add/subtract like terms.*

Q5: *Just like student 5, does not think about how to match the numerator and denominator to make it possible to cancel. Makes the same mistake of ignoring the 1 in the numerator so gives*

an answer of 2, instead of $\frac{1}{2}$*. No understanding of the concept of a fraction so wants to get rid of the fraction.*

Q6: *Correctly interprets DOTS but does not realise that he/she is left with 1 in the denominator after cancelling the common bracket. Thus, this effects the LCD which effects the entire solution. Understands how to add/subtract like terms. Again, shows no conceptual understanding of the whole fraction.*

Student 8's Script (see table 4 for the question and student's written response):

Q1: *No understanding of a common factor, or factorisation. Does not know how to distribute the* $-a$ *correctly. No understanding of add/subtracting like terms. The student also does not know the difference between addition or multiplication of terms, so he/she said* $x + 2 = 2x$ *and* $2x + x = 2x^2$.

Q2: *No understanding of subtraction of fractions, or the idea of an LCD. Does not understand when not to cross multiply or how to Foil two brackets, he/she forgot about inner terms. The student has no idea when to cancel or not. Due to a lack of knowledge he/she manipulated the fractions to get an answer of 1.*

Q3: *Does not understand how to factorise or identify a common factor. No idea of what an LCD is or its use. Shows no respect for rules of adding/subtracting like terms. He/she does not know how to work with exponents (no adherence to basic exponential rules).*

Table 6: An example of the level of Noticing displayed by the lecturers.

Interpreting		
Categories	**Description**	**Examples**
Low (0)	Provides a general interpretation of the students' understanding	*Shows no consideration for DOTS and does not understand the difference between addition and multiplication of fractions. Does not understand when to invert and multiply. No concept of LCD.*
Medium (1)	Provides a limited interpretation of the students' understanding, often in broad terms	*This student fell into the trap that most students fall into. As soon as they see something that looks the same in the numerator and denominator, they want to cancel it. The student did not understand that there were two terms in the numerator and that there was a common bracket or common factor that he/she could remove and only then cancel. This student did not know the difference between cancelling factors and cancelling terms. Also, by cancelling the denominator, the student no longer had a fraction, making it easier for them to work with. This is another common error that students make, they find it easier to work with a linear expression. This student does not know when to cancel or how to factorise. The student knew how to distribute the negative into the bracket but did not recognise how the sign would change when he/she put the bracket back.*

4.2.4 *Responding to students' mathematical reasoning in relation to the simplification of algebraic fractions*

Under this subsection, I present the lecturers' analysis of each student's scripts selected. This means that the analysis is presented verbatim.

Lecturer A's choice of response to Student 3, 4 and 7 based on written responses

Student 3's Script (see table 4 for the question and student's written response):

This student seemed to make similar mistakes throughout the test. To help this student I would take him/her back to basic factorisation to help identify common factors and then common brackets. I would show the difference between $\frac{(x+1)(x+2)}{(x+1)}$ *and* $\frac{(x+1)(x+2)-a(x+1)}{(x+1)}$*, when you can*

cancel and when you can't cancel. While the student knows how signs change when you remove a bracket, he/she did not consider the change of signs when putting the brackets back. This may be a careless mistake or not, I will have to talk to the student first to find out. The student needs to be taken back to basic addition and subtraction of arithmetic fractions, to help with the concept of an LCD. Then move to algebraic fractions. I would also show the difference between an expression and an equation and when it is possible to solve for a variable and when you need to simplify only and can't solve for a variable in an expression. So, basically, I would give the student examples of algebraic expressions and equations and show when he/she can solve for a certain variable and when it can't be done. He/she also needs to see when he/she can cross multiply and when can't. This student needs assistance with seeing the fraction as a whole and not as separate parts which can be manipulated separately. I will also give questions on division and multiplication of algebraic fractions; this can assist with also showing when to cancel and when not to.

Student 4's Script (see table 4 for the question and student's written response):

To help this student, I would go back to basic arithmetic fractions involving adding, subtracting, multiplying, and dividing. This student needs help with the correction application of exponential rules and the context in which these rules are relevant. Introduce the student to basic factorisation, DOTS and provide questions where cancellation is appropriate and not appropriate. Once the student understands the basic algebra and arithmetic fractions then I can slowly introduce him/her to algebraic fractions. The student may also benefit from working with addition and subtraction and multiplication and division of both arithmetic and eventually algebraic fractions.

Student 7's Script (see table 4 for the question and student's written response):

To help this student I would take him/her back to addition/subtraction of arithmetic fractions. Maybe start with 2 terms than move on to more terms and introduce the concept of an LCD. I would also start with simple problems on factorisation and then move on to factorisation of a common bracket and show how to manipulate the negative or positive sign inside the bracket so that it becomes a common bracket. The student also needs help to identify DOTS. Like the other students, the student needs help to understand the algebraic fraction as a whole, knowing that you can't just add/subtract all the denominators or numerators.

All these students need to be taken back to basic arithmetic fractions, so I'm I can see myself having a class devoted to finding an LCD. Which is unbelievable because these students are expected to know this especially at this level.

Lecturer B's choice of response to Student 5, 6 and 8 based on written responses

Student 5's Script (see table 4 for the question and student's written response):

In assisting the student, I will start off with basic factorisation examples of how to take out a common factor. We will show the difference between when cancelling can be done, using examples of multiplication and division of algebraic fractions then moving to addition and subtraction of fractions. I will use examples to explain how an algebraic fraction is made up, it component parts and that one needs to view it in its entirety and not separately. We will have to look at part-whole relationships so that the student can build conceptual understanding rather than just cancel without thinking.

Student 6's Script (see table 4 for the question and student's written response):

To assist, the student needs to be introduced to basic arithmetic fractions and the concept of a fraction as a whole. There needs to be an awareness of the components parts and how they come together to form a fraction. As with student 5, we will look into part-whole relationships so assist in foundational fraction knowledge. The concept of an LCD and how to use it needs to be worked on. We will work on examples involving correct/incorrect cancelling as well as examples whereby getting rid of fractions seem to be attractive.

Student 8's Script (see table 4 for the question and student's written response):

I honestly do not know how this student managed to get to L3. This student needs help with the basic of the basics. We will start by showing what an algebraic expression is, what is the difference between variables, constants, co-efficient, what are like/unlike terms. How to perform different operations in algebra. Show the student how a fraction is composed, how to work with arithmetic fractions first. Need to show him/her rules of exponents, how to apply the rules. Basically, this student needs to be taken through all of algebra again, build concepts of fractions only then can we move to algebraic fractions. There should be serious intervention. I honestly do not why this was never noticed before.

Table 7: An example of the level of Professional Noticing displayed by the lecturers.

Responding		
Categories	**Description**	**Examples**
Low (0)	Provides general suggestions	*I honestly do not know how this student managed to get to L3. This student needs help with the basic of the basics. We will start by showing what an algebraic expression is, what is the difference between variables, constants, co-efficient, what are like/unlike terms. How to perform different operations in algebra. Show the student how a fraction is composed, how to work with arithmetic fractions first. Need to show him/her rules of exponents, how to apply the rules. Basically, this student needs to be taken through all of algebra again, build concepts of fractions only then can we move to algebraic fractions. There should be serious intervention. I honestly do not why this was never noticed before.*
Medium (1)	Provides specific undetailed suggestions	*This student seemed to make similar mistakes throughout the test. To help this student I would take him/her back to basic factorisation to help identify common factors and then common brackets. I would show the difference between* $\frac{(x+1)(x+2)}{(x+1)}$ *and* $\frac{(x+1)(x+2)-a(x+1)}{(x+1)}$*, when you can cancel and when you can't cancel. While the student knows how signs change when you remove a bracket, he/she did not consider the change of signs when putting the brackets back. This may be a careless mistake or not, I will have to talk to the student first to find out. The student needs to be taken back to basic addition and subtraction of arithmetic fractions, to help with the concept of an LCD. Then move to algebraic fractions. I would also show the difference between an expression and an equation and when it is possible to solve for a variable and when you need to simplify only and can't solve for a variable in an expression. So, basically, I would give the student examples of algebraic expressions and equations and show when he/she can solve for a certain variable and when it can't be done. He/she also needs to see when he/she can cross multiply and when can't. This student needs assistance with seeing the fraction as a whole and not as separate parts which can be manipulated separately. I will also give questions on division and multiplication of*

		algebraic fractions; this can assist with also showing when to cancel and when not to.

4.2.5 'In the moment' noticing (based on researcher field notes 2023)

The lesson observations revealed that both lecturers found it challenging to *attend*, *interpret*, and *respond* to students in the moment, while teaching.

For example, while lecturer A was attending to a student's strategy involving question 2, the following were observed:

> *"You said that this a is the same as this a and you cancelled it; that is not allowed, you can't do that."*

This shows a very low level of attending since the lecturer simply described the strategy as incorrect or invalid. Furthermore, rather than interpreting the reasoning behind the cancelling misconception, he said *"you know you can't do that*." The lecturer missed a good opportunity to probe the student and find out why he wanted to cancel.

In response to the cancelling error, Lecturer A explained that:

"Only if it's multiplication, you know that you can cross cancel the terms, isn't? But if it's a minus or a plus, you can only cancel on that fraction only if they are exactly the same, the numerator and the denominator, or if you remove a common factor in the numerator and in the denominator, then you can cancel. For example, if you have $\frac{x^2+x}{2x^2-x}$ *and you decide to remove a common factor of* $\frac{x(x+1)}{x(2x-1)}$ *then you can cancel."*

Similarly, when explaining the same question, Lecturer B simply describes the students' strategy as correct:

> *"Yes, that's right madam, continue. Right. What would be the next step?"*

The student than indicates that she is unsure of the next step, and Lecturer B mentions: *"*

> *We don't want you to be sure, we just want you to tell us about your thinking regarding that."*

What becomes clear is that despite the Lecturer B indicating that he wanted to understand the students' thinking, he fails to use the opportunity to find out what the student is uncertain about

or pay attention to why the student is reluctant to proceed. Hence, the lecturer merely concludes that the student does not understand and therefore, responds by asking the class to assist her.

> "*Yes, help her to factorise. You [talking to the student at the board] write as she is leading you."*
> *"Now we add like terms, isn't. This is how you approach it." (Lecturer B)*

Also, in question one, when a student cancelled incorrectly, a powerful learning opportunity was lost when Lecturer B simply stated, *"This is not a mathematical principle."* Although he interpreted the students' reasoning as flawed and asked the student *"why did you take that approach (of incorrectly cancelling)"*, the lecturer placed no emphasis on exposing the misconception and dealing with it. Rather, provided a general explanation *"the first thing we have to do is to take out a common factor, before we can do the cancellation."*

Additionally, when Lecturer A discussed the concept of the LCD, I observed the following:

The lecturer took it for granted that the student understood how to add or subtract fractions using an LCD, however this was not the case.

> *"So, your problem was just here in the beginning, in this step. You did not understand why you needed an LCD to make it one fraction? Haai, man" (Lecturer A)*

To respond to this student, the lecturer used an example *"let's say we have* $\frac{1}{2} + \frac{3}{4}$*, what would you do?"* At first a student said cross cancel, and the lecturer responded:

> *"Always if it's a plus or minus, turn it into a single fraction by LCD. But if its multiplication or division that's when you can cancel common factors, and multiply and all that. But if its plus/minus, you can't cross cancel, across fractions." (Lecturer A)*

Here, the lecturer missed the opportunity to understand why the student wanted to cross cancel, or what it means to cross cancel. Furthermore, after accepting that he needs an LCD, the student suggested that the LCD will be 2, and the lecturers' response was:

> *"No, you are looking for a number that is common to both sides. So which number are you looking for? It is 4, because 2 goes into 4 and 4 goes into 4." (Lecturer A)*

In this instance the lecturer responds by telling the student the correct answer and pays no attention to discovering the source of the misconception, consequently overlooking a significant mathematical opportunity.

4.3 Conclusion

In keeping with the main and specific objectives of the study, in this chapter, the findings from the data collected were presented. This presentation was organised according to the research questions and other themes I identified from the data coding and analysis processes. The three strands of professional noticing as well as the levels of sophistication framework (see Ergene & Mine, 2021; Jacobs et al., 2010) served as an analytical tool for analysing the lesson observations and written scripts. The key findings that I identified from the data suggest both lecturers were more attentive to students' algebraic reasoning in the written scripts as compared to in the moment teaching. However, it is expedient to discuss the findings of the study in relation to extant literature and through the lens of the theoretical framework which informs the study. Having noted the significance of discussing the findings, in the next chapter, I provide a detailed discussion of the results in relation to extant literature and the study's theoretical framework.

CHAPTER FIVE: DISCUSSION OF THE FINDINGS

5.1 Introduction

The aim of this study was to explore the effect of a professional noticing intervention program on the pedagogical practices of NCV L3 lecturers in relation to the simplification algebraic fractions. In line with this objective, the study focused on how NCV L3 lecturers *attend* to students' mathematical reasoning; *interpret* students mathematical reasoning; and *respond* to students' mathematical reasoning in relation to algebraic fractions. Based on the findings, in this chapter, I discuss the findings in relation to extant literature and the theoretical framework used for the purpose of this study.

5.2 Discussion of findings

The study aimed to address the ways in which NCV L3 lecturers made sense of students' mathematical reasoning (Research Question 1); interpretated the mathematical reasoning as reflected in students' strategies (Research Question 2); and used what they learned about students' understanding in deciding how to respond (Research Question 3). In the sub-sections that follow (sections 5.2.1; 5.2.2; 5.2.3; 5.2.4; and 5.2.5), the key findings of the study are discussed according to the three research questions. In addressing the research questions, the professional noticing expertise of the participants are characterised by means of a range of sample responses. Responses are shared for each component (attending, interpreting, and responding) and discussed according to the levels of sophistication framework and in relation to the study's theoretical framework. Thereafter, the results in terms of noticing in action (lesson observation) versus noticing in reflection (written scripts) are examined.

5.2.1 How do NCV L3 lecturers *attend* to students mathematical reasoning in relation to simplifying algebraic fractions?

Professional noticing of students' mathematical reasoning requires lecturers to accurately attend to nuanced aspects of students' strategies; details that are likely to advise or influence lecturers' instruction (Jacobs et al., 2010). In addressing the first research question, I share sample responses that present evidence of medium (1) and low (0) proficiency of attendance to students' strategies.

Evidence of Medium (1) attention to students' strategies

Responses coded as medium (1) included a limited or vague description of mathematically significant aspects of students' reasoning. The response was correct, but key information, such as not explicitly stating the incorrect conjoining of terms, or common errors associated with attempting to get rid of the fraction, were omitted. Furthermore, correct mathematical terminology was used; and although participants were aware of the incorrect use of mathematical laws and principles, they did not indicate explicitly why law was applied incorrectly. According to Baidoo et al. (2020), the misconceptions that students commit in relation to incorrect conjoining of terms or getting rid of fractions, are conceptual errors. These type of errors shows no appreciation of the sub-constructs of the fraction. Hill et al. (2005) states that overlooking or not explicitly attending to the conceptual nature errors, may be indicative of either a lack of knowledge in teaching algebraic fractions, or too much knowledge, which causes the lecturer to fail to recognise the difficulties experienced by their students. For example, consider the following description of Student 8's strategy (see Table 4, figure 17-Q4):

> *In term 1, the student combined the two terms of the denominator, then divided the 3 of numerator with the 2 in the denominator; in term 2, the student divided the numerator with 2 in the denominator; in term 3, the student divided the 3 in the numerator with the 4 of the denominators, ignoring the x in the numerator. By doing this the student got rid of the fraction. In line 3, the student added the co-efficient on the first and second term and added the variables incorrectly using incorrect exponential laws. In the last step, the student subtracted the co-efficient of the first two terms and subtracted the exponents of the variables incorrectly.*

In this response, lecturer B did not capture the mathematical essence of the strategy in its entirety. Specifically, he did not make explicit the incorrect application of exponential rules, by stating that the student confused the multiplication and division rules of exponents with addition and subtraction. Also, there was no explanation of how the student combined the terms and whether it was correct or incorrect. While lecturer B may have correctly tracked the strategy, it lacked an in-depth account of the mathematical significant details embedded in the strategy. In line with the mathematical knowledge for teaching (MKT) framework employed in the study, a deficient account of students mathematical reasoning may be attributed to a lack of specialised content knowledge (SCK) (Ball et al., 2005). SCK is a distinctive form of knowledge that refers to the ability of the lecturer to go beyond sizing up the correctness of a

particular strategy and stimulates dynamic thinking and offers an intuitive attentiveness to detail (Ball et al., 2005), thus supporting a deeper appreciation of the students' strategy.

Evidence of Low (0) attention to students' strategies

Responses coded as low (0), provided a general or vague description of how the algebraic fraction was solved. For example, consider the following imprecise explanation (see Table 4; Student 6, Figure 13-Q2):

> *The student wrote down the LCD but then did not work with a fraction. He/she multiplied the numerator and denominator of the first term together and then multiplied the* $6a^2$ *term by the denominator of the second term and then subtracted* $2x^2$ *(not sure why).*

This is evidence of a vague and generalised description, devoid of any reference to what the LCD was, whether it was correct or incorrect, or what the student did after recognising the LCD. The response demonstrates confusion and uncertainty about the students' approach which suggests that the lecturer did not fully understand the strategy. Drawing on the study's framework, Ball et al. (2005) assert that misunderstanding or not understanding students thinking has detrimental consequences for teaching and learning. A lack of MKT, specifically a lack of content and students, makes it difficult to anticipate the misconceptions that students will commit as well as the difficulties associated with the topic. A deficient MKT, impacts the decision-making of the lecturers, making it difficult to accurately identify the students ZPD and gear instruction appropriately to resource the misconceptions and construct "new" concepts. Supporting students in using their misconceptions as opportunities to stimulate mathematical inquiry (Borasi, 1994) is pivotal in advancing conceptual understanding and mathematical proficiency.

The participants obscurity in relation to students reasoning was expressed when they explicitly stated, "*not sure why*", or "*To be honest when I first looked at this response I was confused"*. This form of tentative language suggests confusion and uncertainty in understanding students' mathematical reasoning as reflected in the strategies. Similarly, Simpson and Haltiwanger (2017) in their study showed that when attending to students' approach, some participants did not always provide evidence of recognising what the student did or did not do, resulting in vague details. For example, while one of the lecturer participants stated that the student "continued the pattern all the way down" he did not explain "the way in which the student continued the pattern" (p. 346). Furthermore, uncertainty and confusion were also expressed

by another participant "I was a little bit confused at first by the way they did it." (p. 346). By implication and in relation to the study's theoretical framework, what becomes evident is the fundamental role of MKT in fully capturing the mathematical essence of students' strategies. Thomas et al. (2017) claim that the ability of lecturers to *attend* to students reasoning proficiently occurs at the intersection of knowledge (MKT) and practice (noticing). Thus, specialised mathematical knowledge (SCK) is integral to making sense of students' strategies; lecturers who lack SCK and knowledge of their students and knowledge of the content (KCS) find it challenging and almost impossible, at times, to accurately attend to and make sense of students thinking (Ball et al., 2005). Drawing on Cockcroft (2009), since explicit instruction is necessary to assist students transitioning from amateur concepts to building scientific concepts, it is fundamental for lecturers to understand students' strategies. If lecturers are unsure of students thinking, they are unable to offer guided instruction within the ZPD (Vygotsky, 1978), which may result in the fortification of misconceptions.

A Low (0) evidence of attending was also evident during the lesson observation of lecturer A,

> *"You said that this a is the same as this a and you cancelled it; that is not allowed, you can't do that."*

Similarly, Lecturer B when observed mentioned:

> *"Yes, that's right madam, continue. Right. What would be the next step?"*

Evidently, while emphasis is placed on correct or incorrect responses, there is a lack of *uncovering* the mathematical aspects of the approach. Coskun et al. (2023) suggest that teacher knowledge may impact what [lecturers] attend to and how they use the details of what they attended to, to interpret. However, Jacobs et al. (2010) advise that

> demonstrating expertise does not require that individuals always recall and understand everything; even [lecturers] who have acquired expertise in attending to [students'] strategies can lose focus and miss a strategy or not fully understand a particular aspect of a strategy" (p. 179).

By implication, while Jacobs et al. (2010) consider it sufficient for participants to accurately attend to at least two out of three of students' strategies, the authors do not suggest possible reasons for not completely understanding certain strategies. Thus, it can be argued that MKT alone is insufficient in providing a comprehensive understanding of students' reasoning. This

brings to the fore a dire need to consider the critical role that social context plays in making sense of students reasoning.

In line with the study's theoretical framework, from a socio-cultural perspective, it is important to consider the influence of values and beliefs in shaping the noticing skills of the lecturers (van Es et al., 2022). Noticing students reasoning effectively is not only related to a strong MKT, but it is also deeply rooted within an amalgamation of socio-cultural aspects. Thus, overlooking certain aspects of students reasoning cannot solely be attributed to MKT or a lack thereof, or due to negligence; instead, a deeper theoretical explanation describes noticing as a "socially situated activity" (Goodwin, 1994, p. 606, as cited in König et al., 2022, p. 3). Consequently, in the context of this study, it can be argued that participants may have failed to consider the social realities of their students whilst noticing, and/or the lecturers' own social realities may have led to a biased account of students' reasoning.

5.2.2 How do NCV L3 lecturers *interpret* students mathematical reasoning in relation to simplifying algebraic fractions?

Once lecturers have *attended* to students' thinking, it is important that these lecturers *interpret* students' mathematical reasoning reflected in the strategies used. According to Jacobs et al. (2010),emphasis should be placed on the "extent to which participants' reasoning was consistent with the details of the specific [students] strategies and the research on [students] mathematical development" (p. 185), rather than looking for the best interpretation. In addressing the second research question, I share sample responses that present evidence of medium (1) and low (0) proficiency of interpretation of students' strategies.

Medium evidence of interpretation of students' reasoning

Responses coded as medium (1) provided a fair interpretation of students reasoning and connected it to research on students algebraic reasoning in relation to simplification of fractions, however at times overgeneralised it. For example, consider the following (see Table 4, Student 3, Figure 1- Q1):

> *This student fell into the trap that most students fall into. As soon as they see something that looks the same in the numerator and denominator, they want to cancel it. The student did not understand that there were two terms in the numerator and that there was a common bracket or common factor that he/she could remove and only then cancel. This student did not know the*

difference between cancelling factors and cancelling terms. Also, by cancelling the denominator, the student no longer had a fraction, making it easier for them to work with. This is another common error that students make, they find it easier to work with a linear expression. This student does not know when to cancel or how to factorise. The student knew how to distribute the negative into the bracket but did not recognise how the sign would change when he/she put the bracket back.

Drawing on research on students reasoning in relation to the simplification of algebraic fractions, the participants uncovered common misconceptions, such as incorrect cancelling and the desire to get rid of fractions (Figueras et al., 2008; Mhakure et al., 2014). Understanding students' misconceptions and working with them are an important aspect of noticing. In terms of the study's MKT framework, the participates narrative "*…the trap most students fall into"* as well as *"...they find it easier to work with*" indicates the participant's knowledge of the content and students, which is a dimension of PCK, and also points to SCK which allows participants to uncover common misconceptions.

Low (0) evidence of interpretation of students' reasoning

Responses that were coded as low (0) did not display any evidence of interpretation of students' reasoning. Instead, these responses provided a general account of what students did not understand, with no identification of errors and misconceptions. For instance, consider the following (see Table 4; Student 4; Fig 6-Q1):

Shows no consideration for DOTS and does not understand the difference between addition and multiplication of fractions. Does not understand when to invert and multiply. No concept of LCD.

Often times, lecturers lack interpretation of students reasoning because they do not have a rich mathematical knowledge to assist in making sense of the strategies; "to interpret [students' understanding, one must not only attend to [students] strategies but also have sufficient understanding of the mathematical landscape to connect how those strategies reflect understanding of mathematical concepts" (Jacobs et al., 2010, p. 195). A lack of evidence of interpretation was also evident in the study by Simpson and Haltiwanger (2017); due to the positive correlation between interpreting and responding, low interpretation often overlooks misconceptions and errors and influences the level of sophistication when lecturers respond to students' thinking.

5.2.3 What course of action do NCV L3 lecturers decide upon based on students mathematical reasoning in relation to algebraic fractions?

Efficient responses are rooted in the expertise of using what was learned about students' understanding of algebraic fractions and deciding accordingly. Responses that lacked detail but were specific were coded as medium (1); responses that provided general suggestions were coded as low (0). For example, the response below suggests medium evidence (see table 4, Student 3).

> *This student seemed to make similar mistakes throughout the test. To help this student I would take him/her back to basic factorisation to help identify common factors and then common brackets. I would show the difference between* $\frac{(x+1)(x+2)}{(x+1)}$ *and* $\frac{(x+1)(x+2)-a(x+1)}{(x+1)}$ *, when you can cancel and when you can't cancel. While the student knows how signs change when you remove a bracket, he/she did not consider the change of signs when putting the brackets back. This may be a careless mistake or not, I will have to talk to the student first to find out. The student needs to be taken back to basic addition and subtraction of arithmetic fractions, to help with the concept of an LCD. Then move to algebraic fractions. I would also show the difference between an expression and an equation and when it is possible to solve for a variable and when you need to simplify only and can't solve for a variable in an expression. So, basically, I would give the student examples of algebraic expressions and equations and show when he/she can solve for a certain variable and when it can't be done. He/she also needs to see when he/she can cross multiply and when can't. This student needs assistance with seeing the fraction as a whole and not as separate parts which can be manipulated separately. I will also give questions on division and multiplication of algebraic fractions; this can assist with also showing when to cancel and when not to.*

Most of the responses provided evidence of what the students *did not* understand, with very little emphasis placed on the existing understanding or what the student *did* understand.

Low (0) evidence of responding based on students' understanding

Responses that made no reference to advancing students understanding or anticipated strategies were coded as low (0). Consider the example below (researcher observation notes, 2023).

> *"Always if it's a plus or minus, turn it into a single fraction by LCD. But if its multiplication or division that's when you can cancel common factors, and multiply and all that. But if its plus/minus, you can't cross cancel, across fractions." (Lecturer A)*

Here, Lecturer A decided to respond by offering an explanation and telling the student what not to do, with no reference made to build on the students' understanding.

Results indicate that participants responses were mostly focused on correcting students thinking through telling, re-explaining, or highlighting students' misunderstandings with little or no consideration of existing understanding and the need to build on these understandings. A constant focus on students' misconceptions and deficits perpetuates messages regarding their incompetence as students of mathematics and impedes the development of mathematical identities (Jilk, 2016). According to Jilk (2016), teachers experience great difficulty in focusing on students' strengths rather than deficiencies because:

> [we] are often immersed in a culture that *attends* (emphasis added) to students' shortcomings. We are unknowingly trained to identify [students'] mistakes and misunderstandings. We analyse what students do not know and cannot do, and then we try and close the gap with what they need to understand. (p. 189)

This habitual desire to *fix* students misconceptions and "close" learning gaps prevents lecturers from appreciating errors and misconceptions as evidence of learning. From a social constructivist approach to learning, upon which the study is grounded, it hinders the processes involved in gearing instruction to support students in transitioning from amateur concepts to scientific concepts (Cockcroft, 2009; Karpov, 1995). According to social constructivism, it becomes obvious that the lecturers neglected the social aspects, such as the students' background or foundational knowledge when deciding how to respond. The disconnect between students' background and the lecturer's ability to notice oftens leads to misguided instruction.

5.2.4 Discussing a convergence of the study's results on the three research questions: *Attending, interpreting*, and *responding*

A holistic examination of attending, interpreting, and responding, illuminates the nested relationship or interdependence of the three strands. In emphasising the interrelatedness Jacobs et al. (2010) postulate that efficient decision making in response to students' understandings are founded on detailed or substantial interpretations of students reasoning, which in turn stem from an accurate, nuanced attention of students' strategies. The findings of the study supported the positive correlation of the three strands in that participants ability to attend was linked to their ability to interpret and respond accordingly. Most times, a lack of attention to students'

strategies led to a poor interpretation of the students reasoning which resulted in poor decision making. In response to the research questions, by converging the participants level of attending, interpreting, and responding to students' mathematical reasoning in relation to algebraic fractions, results indicate that participants displayed low (0) or a medium (1) level of professional noticing skills when analysing students' artefacts. Worth noting is that lecturers were not always perceptive of students' ideas and therefore showed no understanding thereof. Also, it can be argued that at times participants may have responded by posing problems or examples that they deemed appropriate for the problem situation rather than based on the students' current understanding. These findings are supported by Simpson and Haltiwanger (2017) analysis of noticing, which reveals that at times participants displayed no evidence of students thinking and often expressed their uncertainty and confusion by simply stating "I don't know" or "I guess" (p. 345). Additionally, the study by Simpson and Haltiwanger (2017) shows that participants often displayed contradicting ways of attending, interpreting, and responding to students written work because their personal experiences often influenced what they noticed. Similarly, Tyminski et al. (2021) suggest that due to the complex nature of students' representations, it is often difficult for teachers to offer precise inferences about students reasoning. Making sense of or drawing accurate inferences of students' often "messy", incomplete strategies is a strenuous task and often requires lecturers to be mathematically in tune. Thomas et al. (2017) advise that proficient professional noticing depends on a robust mathematical knowledge for teaching (MKT), in particular horizon content knowledge allows teachers to appreciate the mathematical landscape and undercover nuances in students' articulations (verbal and written). Simpson and Haltiwanger (2017) resonate with the importance of MKT and surmise that participants deficiencies in noticing can be attributed to their lack of content knowledge and insufficient knowledge of their students' (Ball et al., 2008). Furthermore, as mentioned earlier, although MKT is a fundamental component of professional noticing, it does not suffice in explaining the "weaknesses" associated with the lecturers' noticing skills. Drawing from social constructivism, it is expedient to recognise the influence of the social realities of both lecturers and students (see Van Es et al., 2022).

5.2.5 Discussing 'In the moment noticing' versus 'noticing in reflection'.

To engage in "highly individualised and responsive instructional tactics" (Thomas et al., 2017, p.16) requires lecturers to notice students reasoning during instruction. Noticing in action

enables lecturers to adapt instructional practices based on students' reasoning, thus promoting meaningful learning (Olanoff, 2021). An important finding from this study was the misalignment or inconsistency between noticing during instruction and noticing in reflection (written artefacts of students). Results of this study suggest that lecturers paid more attention to students' thinking when analysing the written scripts than they did during instruction. An overall analysis of in-the-moment noticing shows that lecturers failed to attend to students reasoning, which led to a poor understanding of students thinking, causing participants to neglect significant mathematical opportunities that may have supported students thinking from amateur concepts to scientific concepts. This finding was also evident in the study conducted by Kilic and Dogan (2022), which examined how participants notice mathematical opportunities in action and how they reflect on these opportunities orally and in writing. Results indicate that two of the study participants overlooked more occurrences in action but were able to consider all instances in their written explanations. In line with the negative consequences attached to overlooking students reasoning, Kilic and Dogan (2022) warn that if lecturers are unable to recognise the reasoning behind students' misconceptions than they are more likely to ignore students' work and thus fail to elicit further thinking. Furthermore, during instruction, it was observed that participants often responded by re-explaining or telling students what to do. These responses may be due to the need for staying the course of the pre-planned learning trajectory, without deviating from the goal, due to time constraints or other purposes. An important observation of the participants revealed that participates may lack crucial pedagogical skills in terms of using professional noticing as a tool to mediate the learning process (McLeod, 2018). Fundamental to the idea of noticing students mathematical reasoning in relation to algebraic fractions, in accordance with the social constructivist framework of the study, is the ability to work with the student in the ZPD and use various techniques, such as probing, questioning, or modelling to facilitate the transition from amateur to scientific concepts (McLeod, 2018). Vygotsky (1978) notes that the ZPD consists of (algebraic) concepts that are in the process of being developed but need guided instruction. Additionally, Vygotsky (1978) suggests that instructional practices be geared above the students' current cognitive capabilities to mediate learning. However, in the specific case of this current study, since lecturers mostly responded by telling the student what to do or why cancelling cannot be done or by assuming that students were being careless, the construction of new knowledge may have been impeded.

An important aspect to consider when understanding the mismatch between in action noticing and noticing in reflection, is that lecturers analysed students written scripts in a quiet and peaceful environment as compared to during instruction where lecturers had to make sense of students' contributions in a loud, buzzing classroom space infiltrated by high volumes of sensory inputs. As a result, while analysing students' contributions, participants had more time and were also able to revisit and reflect on students reasoning several times before providing a final analysis. Whereas in the mathematics classroom, teachers are expected to respond almost instantly, enacting all three strands of professional noticing simultaneously. Olanoff (2021) claims that the gap between noticing in reflection and noticing in action can be narrowed if teachers can change their beliefs regarding student learning and work towards a more student-centred approach when teaching. In terms of social constructivism, knowledge is not transmitted, but is constructed through social negotiation; thus, student-centred instructional practices are fundamental for the construction of knowledge.

5.3 Conclusion

Reiterating that the main objective of the study was to explore the effect of a professional noticing intervention program on the pedagogical practices of NCV L3 lecturers in relation to the simplification algebraic fractions, in this chapter, I discussed the study's findings in accordance with the three research questions. Beyond the ordering of the discussion in line with the research questions, I provided a detailed discussion of the findings in relation to extant literature and the theoretical framework used for this study. One main point that stands out is the realisation that there is indeed a dire need to actively engage in the practice of professional noticing in the subject of mathematics, specifically for algebraic fractions in the TVET sector. Along this line, in the next chapter, I present an integrated conclusion based on the research questions this study sought to answer and the discussed findings. Thereafter, a reflection on the theoretical and analytical framework of the study, the limitations of the study, and recommendations for future research are also considered.

CHAPTER 6: CONCLUSION

6.1 Introduction

Being the final chapter of the study, I draw final conclusions based on the research questions stated in chapter 1 as well as the findings. Thereafter, a summary of the key findings is illustrated in table 8 below followed by a reflection on the framework of professional noticing. The limitations of the study as well as recommendations for TVET Colleges, NCV mathematics lecturers and researchers are discussed.

6.2 An integrated conclusion

The main objective of this qualitative study was to explore the professional noticing practices of NCV L3 lecturers in relation to the simplification of algebraic fractions. This was done by means of a professional noticing developmental program or workshop where lecturers were introduced to the phenomenon by means of relevant literature as well as through discussion sessions. Lecturers were requested to analyse the written scripts of students and were observed teaching the topic of algebraic fractions. Data were analysed according to the three strands of professional noticing as well as by using levels of sophistication framework. The study sought to answer the three research questions:

1) How do NCV L3 lecturers attend to students mathematical reasoning in relation to simplifying algebraic fractions?
2) How do NCV L3 lecturers interpret students mathematical reasoning in relation to simplifying algebraic fractions?
3) What course of action do NCV L3 lecturers take based on students mathematical reasoning in relation to algebraic fractions?

Results of the findings indicate that participants displayed mostly a low (0) or medium (1) level of noticing when analysing the students written work. In terms of noticing in action, participants were observed exhibiting mainly a low (0) level of noticing. Table 8 below illustrates the summary of the findings.

Table 8: Summary of the study's main findings in relation to the specific research objectives

Specific research objectives	Summary of the main findings in relation to the research objectives
Attending	
How NCV L3 lecturers *attend* to students' mathematical reasoning in relation to the simplification of algebraic fractions	The lecturers displayed mainly low (0) and medium (1) levels of attending to students' reasoning. In terms of low (0) levels of attending, lecturers mainly described students' strategies as incorrect or correct with a level of uncertainty and vagueness. In terms of medium (1) levels of attending, lecturers provided general descriptions of the students' strategy.
Interpreting	
How NCV L3 lecturers *interpret* students' mathematical reasoning in relation to the simplification of algebraic fractions	Lecturers displayed mainly low (0) and medium (1) levels of interpreting students' mathematical reasoning. In terms of low (0) levels of interpreting, lecturers provided mainly general interpretations of students' understanding. In terms of medium (1) levels of interpreting, lecturers provided primarily limited interpretations of students' understanding, often in broad terms.
Responding	
The course of action taken by NCV L3 lecturers based on students' mathematical reasoning in relation to the simplification of algebraic fractions.	The lecturers displayed mainly low (0) or medium (1) levels of responding to students' reasoning. In terms of low (0) levels of responding, lecturers provided mainly general suggestions. In terms of medium (1) levels of responding, lecturers provided primarily specific undetailed suggestions.

While the study's framework offers a nuanced insight into the intricacies of mathematics teaching, it also has certain constraints. The findings from the study show that it was difficult for lecturers to implement a robust level of noticing. One of the reasons for this may be attributed to the limitations associated with the framework, in terms of its failure to account for

the different social realities of the classroom context. As a result, there is a need to reflect on the professional noticing framework and its realistic use during teaching. In line with the above stated, the section that follows considers the limitations in terms of the framework.

6.3 Reflection on the theoretical and analytical framework of the study

Ineffective professional noticing presents numerous challenges for both lecturers and students. In relation to lecturers, poor noticing skills have detrimental consequences within the classroom space as it impedes the ability to identify errors and misconceptions, making it difficult to offer specific support and advance conceptual understanding. Failure to notice students thinking obstructs their ability to articulate their mathematical reasoning, preventing opportunities to engage in meaningful mathematical discourse. Although the framework serves as a powerful lens for lecturers to hone into their skills of attending, interpreting, and responding, the framework does not consider the social realities and realistic contextual circumstances. Thus, to assist in understanding and mitigating these challenges associated poor noticing skills and the consequences thereof, it is expedient to reflect on the framework in terms of the limitations.

The way lecturers attend, interpret, and respond to students is largely informed by their beliefs, values, epistemological, and ontological assumptions of the discourse of mathematics (van Es et al., 2022). Given the subjective nature of professional noticing, it is important to consider how these socio-cultural aspects influence the level of noticing displayed by the participants. Since noticing is not objective, it is imperative to recognise that the inconsistencies displayed when participants attend, interpret, and respond to students reasoning do not only indicate poor noticing or a lack of MKT, rather the levels of noticing need to be considered within a particular socio-cultural context. Furthermore, appreciating the socio-cultural realities of the students as well as the realistic setting of the classroom are integral components of effective noticing. Factors such as students background knowledge, classroom culture, and resources may affect the efficiency of noticing. Reflecting on the noticing framework, suggests that adaptations be made to acknowledge and accommodate the diversity and contextual circumstances of our South African classrooms to cultivate a more inclusive approach to noticing.

6.4 Limitations to the study

The main limitation of the study centred around the realistic classroom situation. Factors such as the high degree of absenteeism experienced in the NCV program and poor background

knowledge of students, and time constraints made it difficult in terms of both the teaching of algebraic fractions and in terms of students writing the pre-test. However, to overcome this limitation, we had to work with the sample available at the time of data collection and within the time frame that lecturers allocated to teaching the topic.

Additionally, as the researcher, I acknowledge that the study was limited to a single case study and that the findings cannot be generalised to other mathematics classrooms. Moreover, the limitations in terms of the study being conducted at only one TVET College as well as the sample size is recognised. However, Lewis and Ritchie (2003) note that the aim of qualitative research is not to generalise findings, but rather to produce an in-depth understanding of a phenomenon and determine *transferability* to other contexts. In this regard, in chapter 3, I provided detailed, thick descriptions of the research site, sampling, and data collection methods to facilitate the possibility of transferring the findings to other *similar* contexts.

Furthermore, acting as the primary instrument of data collection, I recognise that my own beliefs, values, and philosophy may have caused bias in reporting the findings. However, periodic meetings with my supervisor during data collection and analysis assisted in overcoming this limitation. Moreover, triangulating the findings from both observations and written scripts were useful in overcoming these challenges.

6.5 Recommendations

In line with the findings of the study, the following are recommendations for TVET colleges; NCV mathematics lecturers; and researchers on professional noticing of students thinking in relation to the simplification of algebraic fractions.

For TVET colleges

Since professional noticing can be learned (Jacobs et al., 2010), to support lecturers in professionally noticing students' mathematical reasoning, researchers (Jacobs et al., 2010; Jilk, 2016; LaRochelle et al., 2019; Sherin et al., 2011) advise that professional noticing of students thinking be made explicit in teaching courses, and training and development programs to ensure that noticing skills are constantly and continuously developed. In line with the above mentioned, TVET colleges should provide training programs and workshops that explicitly focus on learning and implementation of professional noticing. Colleges need to cultivate a culture where student thinking is valued and appreciated. This will create a productive,

meaningful teaching and learning environment where students, lecturers, management, and society can appreciate a positive image of mathematics.

For NCV mathematics lecturers

It is recommended that lecturers be available and willing to learn the skills of noticing by taking part in training and development programs and implementing professional noticing in their pedagogical practices. Lecturers can practice noticing by recording their own teaching, observing other lecturers, or by analysing the written work of students. By researching their own practice, they can develop their noticing skills through reflecting on their teaching, and work towards being proficient in noticing in action and in reflection. It is also important for lecturers to have a robust MKT and be willing identify their weaknesses (if any), in the content that they teach and seek support accordingly. Moreover, lecturers should be encouraged to share their strengths in both content and noticing; to ensure the intersection of both knowledge (MKT) and practice (noticing) which can assist in empowering each other and their students. Also, it is fundamental for lecturers to appreciate the social realities of their classrooms, having knowledge of their students in terms of their socio-cultural background, mathematical background as well as their identities as students of mathematics. Furthermore, an in-depth knowledge of both students and the mathematical content can assist lecturers in advancing their noticing skills. By acknowledging each student as a learner of mathematics, in his/her own right, may assist lecturers in moving away from considering students as a group to considering individual students, in terms of how each student makes sense of the mathematics. This may assist lecturers in shifting from offering general feedback to individualising and tailoring responses to meet the specific needs of the individual student.

For researchers

Firstly, since the study was limited to a single case study with a small sample size, it is recommended that the research be extended to multiple TVET Colleges with an increase in sample size to gain a better understanding of the effect of a professional noticing intervention program.

Secondly, it is suggested that the study be conducted over a greater time frame to provide lecturers with a more intensive professional noticing intervention program or incorporating a learning to notice training program as part of the research.

Thirdly, since professional noticing is influenced by teacher knowledge, beliefs, values, and culture, it is expedient to consider the role of social realities in noticing students reasoning.

Finally, it is suggested that researchers consider the use of pre and post interviews and pre and post-tests to investigate the challenges that lecturers experienced while noticing and to discover any shifts in the pedagogical practices of lecturers after the intervention.

6.6 Conclusion

Being the final chapter of the study, this chapter presented an integrated conclusion in terms of the research questions, followed by a summary of the key findings in relation to the specific objectives of the study. This is illustrated in table 8 above. A reflection on the theoretical and analytical framework employed in the study was described. Additionally, the limitations of the study were acknowledged and recommendations for TVET colleges, NCV mathematics lecturers, as well as for researchers were outlined in this chapter.

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APPENICES

Appendix A: DHET ethical clearance form

higher education & training
Department:
Higher Education and Training
REPUBLIC OF SOUTH AFRICA

APPLICATION FORM FOR STUDENTS TO CONDUCT RESEARCH IN THE DEPARTMENT OF HIGHER EDUCATION AND TRAINING

1. APPLICANT INFORMATION

1.1.	Title (Dr /Mr /Mrs /Ms)	Mrs	
1.2	Name and surname	Salome Nadason	
1.3	Postal address	33 Umgeni Street Farrarmere Benoni 1501	
1.4	Contact details	Tel	
		Cell	0724083430
		Fax	
		Email	0201103P@students.wits.ac.za
1.5	Name of institution where enrolled	University Of Witwatersrand	
1.6	Field of study	Mathematics Education	
1.7	Qualification registered for	*Please tick relevant option:*	
		Doctoral Degree (PhD)	
		Master's Degree	√
		Other (please specify) Honours Degree	

2. DETAILS OF THE STUDY

2.1	Title of the study
The Effect of Professional Noticing Intervention in the Context of Algebraic Fractions: A Case Study of NCV L3 Lecturers in a TVET College	

2.2	Purpose of the study
The purpose of the study is to determine the effect of a professional noticing intervention program on the pedagogical practices of NCV L3 lecturers in relation to the simplification of algebraic fractions.	

3. PARTCIPANTS AND TYPE/S OF ACTIVITIES TO BE UNDERTAKEN IN THE DEPARTMENT

Please indicate the types of research activities you are planning to undertake in Department as well as the categories of persons who are expected to participate in your study (for example, DDGs, Chief Directors, Directors, Deputy Directors etc.), including the number of participants for each activity

3.1	Complete questionnaires	Expected participants	Number of participants
		a)	
		b)	
		c)	
		d)	
		e)	
3.2	Participate in individual interviews	Expected participants	Number of participants
		a) Lecturers	2
		b)Students	40
		c)	
		d)	
		e)	
3.3	Participate in focus group discussions/ workshops	Expected participants	Number of participants
		a)	
		b)	
		c)	
		d)	
		e)	
3.4	Complete standardised tests (e.g. Psychometric Tests)	Expected participants	Number of participants
		a)	
		b)	
		c)	
		d)	
		e)	
3.5	Undertake observations *Please specify*		

2

3.6	**Other** *Please specify*	Test scripts of students for document analysis

4. SUPPORT NEEDED FROM THE DEPARTMENT

***Please indicate** the type **of support required** from the Department (Please tick relevant option/s)*			
Type of support		*Yes*	*No*
4.1	The Department will be required to identify participants and provide their contact details to the researcher.	√	
4.2	The Department will be required to distribute questionnaires/instruments to participants on behalf of the researcher.		
4.3	The Department will be required to provide official documents. ***Please specify the documents required below***		
4.4	The Department will be required to provide data. ***Please specify the data fields required below***		
4.5	***Other, please specify below***	√	
	The Department will be required to approve document analysis of students test scripts.		

5. DOCUMENTS TO BE ATTACHED TO THE APPLICATION

***The following 2 (two) documents must be attached as a prerequisite for approval to undertake research in the** Department*	
5.1	Ethics Clearance Certificate issued by a University Ethics Committee
5.2	Research proposal approved by a University

6. DECLARATION BY THE APPLICANT

I undertake to use the information that I acquire through my research, in a balanced and a responsible manner. I furthermore take note of, and agree to adhere to the following conditions:

a) I will schedule my research activities in consultation with the Department and research participants;

b) I agree that involvement by participants in my research study is voluntary, and that participants have a right to decline to participate in my research study;

c) I will obtain signed consent forms from participants prior to any engagement with them;

d) I will inform participants about the use of recording devices such as tape-recorders and cameras, and participants will be free to reject them if they wish;

e) I will honour the right of participants to privacy, anonymity, confidentiality and respect for human dignity at all times. Participants will not be identifiable in any way from the results of my research, unless written consent is obtained otherwise;

f) I will not include the names of research participants in my research report, without the written consent of each of the said individual/s;

g) I will send the draft research report to research participants before finalisation, in order to validate the accuracy of the information in the report;

h) I will not use the resources of the Department when I am conducting research (such as stationery, photocopies, faxes, and telephones), for my research study;

i) I will include a disclaimer in any report, publication or presentation arising from my research, that the findings and recommendations of the study do not represent the views of the Department of Higher Education and Training; and

j) I will provide a summary of my research report to the Department for information purposes.

I declare that all statements made in this application are true and accurate. I accept the conditions associated with the granting of approval to conduct research and undertake to abide by them.

SIGNATURE	Wadase
DATE	03/03/2023

FOR OFFICIAL USE

DECISION BY DIRECTOR-GENERAL

Please tick relevant decision and provide conditions/reasons where applicable		
Decision		***Please tick relevant option***
1	Application approved	✓
2	Application approved subject to certain conditions. ***Specify conditions below***	
	To contact the Campus Manager, Mr Mkosi to make arrangements.	
3	Application not approved. ***Provide reasons for non-approval below***	
SIGNATURE		
DATE	07 · 03 · 2023	

Appendix B: Wits University ethical clearance certificate

UNIVERSITY OF THE WITWATERSRAND JOHANNESBURG | WSoE Wits School of Education

SCHOOL OF EDUCATION ETHICS COMMITTEE

CONSTITUTED UNDER THE UNIVERSITY HUMAN RESEARCH ETHICS COMMITTEE (NON-MEDICAL)

CLEARANCE CERTIFICATE	PROTOCOL NUMBER: 2023ECE013
PROJECT TITLE	The Effect of Professional Noticing Intervention in the Context of Algebraic Fractions: A Case Study of NCV L3 Lecturers in a TVET College
INVESTIGATOR	NADASON SALOME
SCHOOL/DEPARTMENT OF INVESTIGATOR	Wits School of Education
DATE CONSIDERED	06-Mar-23
DECISION OF THE COMMITTEE	Approved unconditionally.
RISK LEVEL	Minimal risk
EXPIRY DATE	Date of submission of the Research Report
ISSUE DATE OF CERTIFICATE	20-Mar-23
CHAIRPERSON	Dr. Batseba Mofolo-Mbokane

cc: Prof Judah Makonye

DECLARATION OF INVESTIGATOR

To be completed in duplicate and **ONE COPY** returned to the Chairperson of the School/Department ethics committee.

I fully understand the conditions under which I am are authorized to carry out the abovementioned research and I guarantee to ensure compliance with these conditions. Should any departure to be contemplated from the research procedure as approved I/we undertake to resubmit the protocol to the Committee.

Signature

Date 31/ 03 /2023

Appendix C: Faculty of Humanities' approval letter for research proposal

UNIVERSITY OF THE JOHANNESBURG | FACULTY OF HUMANITIES

05 April 2023

Prof Judah Makonye
Wits University
Wits School of Education
Parktown
2193

Dear Prof Makonye

Student Name: Salome Nadason
Student Number: 0201103P

I am enclosing Salome's reader's report for her MEd proposal. The Graduate Studies Committee of the Faculty of Humanities, acting on behalf of the Senate recommended that the research proposal can pass subject to minor revisions to the satisfaction of the supervisors. The EGSC further commented the following:

Based on the reviewer's comments, the recommendation would be for the research proposal to pass subject to minor revisions to the satisfaction of the supervisor. The proposal demonstrates a high ability to work independently, and the research procedures and techniques are in order. The articulation of the problem is well done, and the literature review shows the candidate's understanding and mastery of the field of study at a master's level. However, there are some areas that need improvement, such as revising the literature review to include related empirical studies, discussing all the listening strategies identified by Davis, justifying the choice of interpretive and case study design, citing the relevant researchers, and making minor revisions to the ethical and referencing issues.

Please liaise with the student regarding the content of the reader's report.

Kind regards
Faith Herbert
Faith Herbert
Senior Faculty Officer
Faculty of Humanities Education
Tel: +27 11 717 3018
Email: Faith.Herbert@wits.ac.za

Appendix D: Pre-test simplification of algebraic fractions

Simplify the following fractions and explain your reasoning:

1. $\frac{(x+1)(X+2)-a\,(x+1)}{x+1}$
2. $\frac{x^2-1}{2(1-x)^2}$
3. $\frac{a^2-b^2}{a-b}+\frac{a+b}{2}$
4. $\frac{ab}{2c}+\frac{(a-b)}{b}$
5. $\frac{3}{2x+y}+\frac{5}{2x-y}-\frac{3x}{4x^2-y^2}$

Appendix E: Lecturer participants' analysis of students' responses to the pre-test questions (attending, interpreting, and responding)

Lecturer A's analysis of test scripts of Student 3, 4 and 7

Student 3's Script:

Q1: This student was drawn to the fact that the term $(x + 1)$ *appeared in both the numerator and the denominator and therefore proceeded to cancel them instinctively. The incorrect cancelling also assisted the student to get rid of the fraction, forming a linear expression that most students find easier to work with. The student then mistakenly changed* $(x + 1)$ *to* $(x - 1)$ *and then applied the distributive law, correctly, to remove the brackets. The student then added brackets again, however this was done incorrectly. The student realised that there were no like terms, hence did not proceed further.*

Q2: This student cross multiplied the denominator of each term to get rid of the denominator, then the student applied the distributive law to remove the brackets (this was done correctly). The student then turned the expression into an equation, collected like terms (incorrectly) and then equated the equation to zero. Thereafter, the student took out a common factor and solved for a*. To the student it seemed as if he/she solved for both* a *and* x*. The fact that there were two unknowns did not deter him/her from working towards a final answer.*

Q3: The student took out a common factor in the numerator (correctly) and in the denominator (incorrectly) of the third term. Then cancelled one of the $p's$ *in the numerator with the one in the denominator. Also, in line 4, in the third term, the term* $(p^2 - 4)$ *just disappeared. In line 5, in the first term, the student multiplied the numerator by the same denominator to cancel both the numerator and denominator; in term 2 the student saw that both the numerator and denominator contained a* p *and so cancelled them (incorrectly), so he/she was left with a one in the numerator and a one plus two in the denominator. In term 3, the two in the denominator with the term* $(p - 2)$ *in the numerator (this was done incorrectly in terms of the signs) and completely ignored the other* p *in the numerator. The student then tried to add and subtract like terms.*

Q4: The student formed an equation by equating the first two terms to the third term. Then cross multiplied each denominator and then correctly identified the denominator of the third term as DOTS (a difference of two squares) and factorised it. He/she then saw the one of the squares $(2x + y)$ *also existed in term 2, on the left-hand side of the equation. This stimulated the student to cancel the* $(2x + y)$ *in both terms. In line 5, he/she multiplied each term on the left side by* $\frac{1}{(2x+y)}$ *and on the right-hand side by* $\frac{(2x+y}{1}$ *to cancel the denominators and form a linear*

equation. Then he/she made the equation equal to 0 and was left with three unlike terms and could not proceed.

Q5: The student correctly factorised the numerator as a DOTS, however swopped the two terms within the bracket of the denominator so that he/she could cancel the $(x-1)$. *Then the student used distributive law correctly in the denominator and couldn't proceed.*

Q6: In term one, the student incorrectly factorised the numerator, this could have been a careless mistake (instead of one positive and one negative bracket, he put a negative in both brackets, then cancelled one of the terms in the numerator and denominator. Then multiplied both terms by the LCD to get rid of the two and form a linear equation. Then distributed the two in the first term and instead of adding or subtracting like terms, he/she completely ignored the plus sign and bracketed the first two terms together and the last two terms together to form one term.

Student 4's Script:

Q1: The student overlooked the common factor and multiplied the brackets out, correctly with the first term but made an error with one of the signs in the second term. Then incorrectly collected like terms in the numerator and cancelled the x *in the denominator with the second term of the numerator and divided the 2 in the numerator with the 1 in the denominator, this then got rid of the fraction. Then added like terms again until he/she couldn't proceed.*

Q2: In both terms the student cancelled both variables in the numerator with the same variables in the denominator and was left with a numeric fraction that he/she calculated using a calculator.

Q3: In the first term, the student added the two unlike terms in the numerator to form a single term, he/she did the same thing in the denominator of the second term. In the term, there was no recognition of a common factor, rather he/she used the laws of exponents, incorrectly, to form a single term in the numerator and incorrectly combined the two terms in the denominator to form a single term also. In line 2, he/she cancelled $2p$ *in the numerator of the first term with* $2p$ *in the denominator of second term to form a single term. Then the law of exponents was incorrectly applied, in the second term, when dividing the* p^5 *and* p^2 *the student got* p^3 *which is correct, but still had another* p *in the denominator, which could've been an error. He/she then ended up with two terms.*

Q4: The student changed the addition to multiplication and inverted all three terms. He/she then went back to original question and multiplied the numerators and denominators (incorrectly) of the first two terms, then cross multiplied the second and third terms (incorrectly). Then distributed $5(4x^2-y^2)$ *incorrectly and did not go further.*

Q5: The student did not see the numerator as DOTS. Rather, he/she wrote the bracket out twice and incorrectly multiplied the denominator out and instead cancelled the like terms in the denominator leaving him/her with the numerator as a final answer.

Q6: The student changed the addition to multiplication and inverted the second term, then multiplied the numerator out and the denominator and saw that there was no like terms and ended there.

Student 7's Script:

Q1: The student separated the fraction into terms and cancelled the common factor correctly within the first term. In the second term, however, he/she multiplied the bracket out instead of cancelling the common factor, then multiplied by the conjugate of the denominator, then multiplied the denominators of the second term only to factorise it out again and then finally cancel the common factor. The student took a long route but arrived at the correct answer.

Q2: This student cross multiplied to get rid of the fraction, then applied distributive law but made a mistake with the last term (forget the co-efficient of a^2 of the first bracket). Then he/she collected like terms to arrive with a single term as a final answer.

Q3: The student wrote down the numerators and denominators of all three terms as a single fraction, then took out p as a common factor in the numerator (although it was not common). He/she ignored the denominator, then just gave up.

Q4: The student used the same approach as the previous question, combined the numerators and denominators to form a single fraction, then collected like terms. The fact that he/she noticed that there were squares in the denominator stimulated him/her to factorise as if it were DOTS, despite that there was a + instead of a -. Also, the co-efficient of 4 was ignored.

Q5: This student correctly factorised the numerators as DOTS but incorrectly distributed the 2 inside the square bracket of the denominator then multiplied the brackets out and collected like terms and ended there.

Q6: This student ignored the + sign and cross multiplied to get rid of the fraction and then multiplied brackets out and collected like terms to reach a final answer consisting of two unlike terms. He/she didn't continue further by considering 3 as a common factor or recognise the squares as DOTS.

Lecturer B's analysis of test scripts of Student 5, 6 and 8:

Student 5's Script:

Q1: The student was drawn to the same term in the numerator and denominator and rushed to cancel instead of factorising first. Then changed the x to a; in the second term he/she distributed the $-a$ into the bracket must made a mistake with one of the signs and then combined like terms. Then took out a as a common factor of the first two terms and then just put brackets around the $a + 3$ so that the result consisted of two brackets.

Q2: To be honest when I first looked at this response I was confused, then I realised that the student took out $(2x - a)$ as a common factor in the numerator and $(x - a)$ in the denominator despite there being two separate terms, by doing this he/she was left with the co-efficient of each of the variables in the square brackets, that he/she removed as common factor. Then added/subtracted the numerical values without considering the – signs, then the student distributed the 2 into the numerator and then took it out as a common factor in the last step. Then cancelled the $(2x - a) with\ (x - a)$ even though the terms are not the same.

Q3: In the third term, the student correctly factorised the numerator and denominator and recognised DOTS and correctly cancelled the $(p - 2)$ but then mistakenly wrote p^2 instead of p and when he/she cancelled the common bracket added the 1 instead of multiplying it which led to two terms in the numerator of the third term. He/she correctly identified an LCD and combined the three terms into one fraction, but then was drawn to the common bracket and the common co-efficient of 2 in the numerator and denominator and decided to cancel them. Then added the like terms and ended the solution.

Q4: The student correctly factorised the denominator of the third term using DOTS and found an LCD and correctly formed a single fraction. Then correctly distributed the co-efficient to remove brackets and combined like terms but made a calculation error with the x term, everything else was correct.

Q5: The student applied DOTS in the numerator correctly, but incorrectly squared each term with the bracket of the numerator and tried to make it look like the numerator so that he/she can factorise it using DOTS. Then the student cancelled and got an answer of two, here the numerator of 1 was ignored.

Q6: The student factorised the numerator of term one using DOTS, then cancelled the common bracket, then found an LCD of 2 but then cancelled the LCD with the 2 in the numerator to get rid of the fraction and formed a linear expression making it easy to add like terms.

Student 6's Script:

Q1: This student gave two responses. In the first one, he/she multiplied the numerator out and correctly, added the like terms but saw that there weren't any more like terms and left it.

In the second attempt, cancelled the same bracket in the numerator and denominator to get rid of the fraction and then cancelled the x and ax terms as though combining them would give a 0, and remained with two terms.

Q2: The student wrote down the LCD but then did not work with a fraction. He/she multiplied the numerator and denominator of the first term together and then multiplied the $6a^2$ term by the denominator of the second term and then subtracted $2x^2$ (not sure why).

Q3: In term 3, the student was correct in factorising the numerator and the denominator but did not recognise DOTS in the denominator. Then he/she cancelled the numerator of term 1 with the co-efficient of term 3's denominator and formed a single term by subtracting term 2 and term 3's denominator. In line 3, term two was factorised by taking out p as common, then he/she cancelled the two brackets of the numerator and tried to factorise the denominator as a trinomial.

Q4: The student factorised the denominator of the third term by DOTS and found the LCD and multiplied each term correctly to form one fraction and added/subtracted like terms correctly in the numerator but then ignored the denominator in the final step.

Q5: The student left the numerator as is and saw the denominator as DOTS and tried to multiply the brackets of the denominator and combine like terms to get $(1 - x^2)$. Then cancelled the numerator with the bracket in the denominator and got an answer of two instead of $\frac{1}{2}$.

Q6: The student factorised term 1 numerator as DOTS and used an LCD of 2 and $(a - b)$. He/she saw that there was a common bracket of $(a - b)$ but only cancelled the bracket in the numerator not in the denominator which then led to using the LCD of $(a - b)2$. The student then formed a single fraction, carrying the error from previous step and tried to collect like terms which he/she tried to factorise as a trinomial and then cancelled one of the brackets, $(a - b)$, in the numerator with the $(a - b)$ in the denominator.

Student 8's Script:

Q1: The student immediately cancelled the same bracket in the numerator of the first term with the denominator. Then multiplied the $-a$ into $(x + 1)$ incorrectly, then said $x + 2 = 2x$ and $2x + x = 2x^2$ to get an answer of $2x^2 - a$.

Q2: The student cross multiplied the denominator of one term with the numerator of the other term but got a binomial rather than a trinomial (did not use foil correctly). Then he/she disregarded the co-efficient of 2 in both terms of the denominator so that he/she was left with the same numerator and denominator that could be cancelled to get an answer of 1.

Q3: In term 1and 2, the student said $p + 2 = 2p$, then combined the two terms of term 3 with incorrect exponential laws (subtracted the exponents). In line 2, he/she subtracted the terms in the denominator to get a single term. In line 3, term 1, cancelled the 2 in the numerator and denominator; in term 2, cancelled one of the $p's$ and got a co-efficient of $-\frac{1}{2}$ (not sure where to place the p (numerator or denominator); in term 3, the student divided the numerator and denominator but incorrectly applied exponential laws. In line 5, the student combined the two terms, again using incorrect laws of exponents.

Q4: In term 1, the student combined the two terms of the denominator, then divided the 3 of numerator with the 2 in the denominator; in term 2, the student divided the numerator with 2 in the denominator; in term 3, the student divided the 3 in the numerator with the 4 of the denominators, ignoring the x in the numerator. By doing this the student got rid of the fraction. In line 3, the student added the co-efficient on the first and second term and added the variables incorrectly using incorrect exponential laws. In the last step, the student subtracted the co-efficient of the first two terms and subtracted the exponents of the variables incorrectly.

Q5: The student did not factorise the numerator; in the denominator the student just squared each term in the bracket and then noticed that the numerator and denominator had similar terms and so cancelled them, leaving him/her with an answer of 2 instead of a fraction.

Q6: The student did not see the numerator as DOTS but saw a^2 as $(a + a)$ and the b^2 as $(b + b)$, tried foiling the two brackets despite the negative sign between them. Then he/she added the numerator of term 1 to get two terms, and then got rid of the squares and the denominator of term 1. Then the student combined the two fractions into 1 term using 2 as the denominator and adding the $a's$ to get a^2 and the $b's$ to get b^2, ignoring the negative sign.

Lecturer A's interpretation of Student 3, 4 and 7 based on the written responses:

Student 3's Script:

***Q1:** This student fell into the trap that most students fall into. As soon as they see something that looks the same in the numerator and denominator, they want to cancel it. The student did not understand that there were two terms in the numerator and that there was a common bracket or common factor that he/she could remove and only then cancel. This student did not know the difference between cancelling factors and cancelling terms. Also, by cancelling the denominator, the student no longer had a fraction, making it easier for them to work with. This is another common error that students make, they find it easier to work with a linear expression. This student does not know when to cancel or how to factorise. The student knew how to distribute the negative*

into the bracket but did not recognise how the sign would change when he/she put the bracket back.

Q2: *This student does not understand how to add or subtract fractions, shows no concept of an LCD. He/she confuses an expression with an equation and does not understand when cross multiplication is possible. He/she knows how to FOIL correctly and combine like terms and move terms correctly to equate to 0 and solve for a correctly. Even though he/she knows this, the thinking in dealing with the fraction is completely wrong. This student finds it easy to work with an equation instead of a fraction so created the equation.*

Q3: *Again, no concept of LCD. However, in term 3, he/she knows how to take put a common factor but did not notice DOTS. In line 5, term 1 and 2, the student incorrectly tried to take out a common factor in each numerator in order to cancel the denominator. This shows the misunderstanding of factorisation and the desire to get rid of fractions. The student does not know when to cancel and when not to cancel or how to add/subtract fractions. The student just manipulates the expression to make it easier to work with, with no understanding of what he/she is doing or what needs to be done.*

Q4: *The student does not understand the difference between an algebraic expression and equation. He/she knows how to do DOTS. No understanding of when cross multiplication is applicable and seems to prefer working with equations, hence the removal of the denominators. No understanding of addition/subtraction of fractions.*

Q5: *Displays an understanding of DOTS. Swopped the terms in the bracket of the denominator so that it can cancel with the numerator. No understanding of the effect of the negative sign or how to change the sign so that it becomes possible to cancel.*

Q6: *Correctly identified DOTS but it could be a careless error of the sign in the one bracket. No concept of addition/subtraction of fractions or LCD. Again, doesn't understand fractions so gets rid of the denominator. No understanding of how an expression changes when bracketing the terms.*

Student 4's Script:

Q1: *This student did not understand that the expression needs to be factorised, instead he/she correctly removed the brackets of the first term but has no understanding of distributing a negative sign into a bracket. There is no understanding of when to cancel. This student knows that by dividing* $\frac{3x}{x}$ *would get 3 but does not know when it is appropriate to do so. By dividing the*

numerator with the denominator, the student could get rid of the fraction, which students find easier to work with.

Q2: *No understanding of subtraction of fraction, or concept of LCD. The student was just drawn the same variables in the numerator and denominator. Like in Q1, the student does not know when to cancel. By getting rid of the variables the student was left with an arithmetic fraction which was solved using a calculator, which shows that he/she does not know how to subtract basic arithmetic fractions.*

Q3: *Shows no understanding of how to add/subtract fractions, also no concept of an LCD. Shows no understanding between variables and constants and adding/subtracting of like or unlike terms. Lacks knowledge of basic algebra and arithmetic. Wrong application of exponential rules but knows that when dividing like bases exponents are subtracted.*

Q4: *No understanding of when to invert the fraction, or the context in which this operation is applicable. Does not know how to add/subtract fractions. No recognition of DOTS or the idea of an LCD. Also shows no understanding of cross multiplication when it can or can't be used.*

Q5: *No recognition of DOTS. Does not know how to multiply brackets out correctly in the denominator and shows no consideration of how cancelling the terms in the denominator can affect the entire fraction. The student does not see the fraction as a whole but looks at it separately.*

Q6: *Shows no consideration for DOTS and does not understand the difference between addition and multiplication of fractions. Does not understand when to invert and multiply. No concept of LCD.*

Student 7's Script:

Q1: *The student understands how to separate one term into two fractions again. I find it strange that the student didn't recognise the common bracket in the second term, this could be because it was not bracketed. He/she went about a roundabout before cancelling the common factor. This showed an understanding of DOTS, multiplying by conjugate of denominator and correct factorisation eventually.*

Q2: *No understanding of subtraction of fraction, or the concept of an LCD. Student does not understand when to cross multiply the denominator, so decided to get rid of denominators by cross multiplying so that he/she does not have to work with a fraction. Understands how to multiply the brackets out, except for a careless mistake in the one term. Student also knows how to add and subtract like terms.*

Q3: *Again, no concept of LCD or addition/subtraction of fractions. Does not know how to factorise correctly. This student did not know what to do so just combined the numerator and denominator and incorrectly cancelled.*

Q4: *As previously, no concept of LCD or how to add/subtract fractions. Just combined numerator and denominator and added/subtracted like terms in the denominator. No consideration for the fraction as a whole but works on separate parts.*

Q5: *The student understands that the numerator is DOTS and correctly factorises it. But does not understand how to introduce a negative sign into the numerator to enable cancelling of common brackets. The student showed not consideration of BODMAS by deciding to distribute the 2 into the bracket of the denominator. The student however did not do what most students do (just square each term in the bracket), rather he/she understood that* $(1-x)^2$ *means that the bracket is repeated.*

Q6: *Shows no understanding of addition of fractions, or LCD. Does not know when not to cross multiply, show no consideration of the fact that there are 2 terms. Does not consider the numerator of term 1 as DOTS, which is strange as he/she identified it in the previous question. The student wanted to get rid of the fraction because he/she finds it easier to work without a fraction. The student knows how to add/subtract like terms but did not see that 3 was a common factor in his/her final answer and therefore also did not notice that if he/she removed the 3 it would leave DOTS again.*

Lecturer B's interpretation of Student 5, 6 and 8 based on written responses:

Student 5's Script:

Q1: *Student does not understand how to factorise, does not see the common factor but because there is something similar in the numerator and denominator, he/she rushes to just cancel. This is a common mistake that they make. There is no understanding of the fact that there is more than one term in the numerator, so cancelling is wrong. Another thing that he/she does not understand is how the negative sign influences the terms in the brackets when you remove the brackets. The student has no understanding of accurate grouping, so just adds, and removes brackets as they feel.*

Q2: *Lacks knowledge of basic factorisation, of how to identify a common factor. No understanding of addition or subtraction of fractions, which means there is no understanding of the idea or even importance of an LCD. Looking at how he/she added the integers, shows that*

he/she doesn't even know basic primary school adding or subtracting of integers. Which is very worrying. The student also does not know when to cancel.

Q3: *To my surprise, he/she understands how to factorise, in term 3, he understands how to do DOTS. He also understood what the LCD is here. Except for the mistake he/she made when cancelling in term 3, the student understood how to combine the three terms as one using the LCD. But the student does not know that he/she could not just cancel but had to remove brackets and collect like terms.*

Q4: *Like in the previous question, the student understands how to do DOTS and how to find the LCD. Also understands how to add and subtract the fractions using an LCD and correctly distributes the co-efficient into the brackets. The student made a careless mistake when adding the* x *terms however did the* y *terms correctly.*

Q5: *Understands that the numerator is DOTS but does not know how to change the numerator to make it easy to cancel the denominator. Does not understand how the square of the bracket in the denominator effects the entire brackets and can't just square each term in the bracket. Because of the need to cancel, he/she manipulates the expression to allow cancelling. The student also does not understand that when cancelling the numerator, it leaves him/her with a 1 in the numerator meaning the answer will be* $\frac{1}{2}$ not 2 *. The student does not understand the fraction in its entirety and therefore can't acknowledge the 1 in the numerator.*

Q6: *Sees the DOTS in term 1 and cancels correctly, and interestingly acknowledges that he/she is left with 1 in the denominator when cancelling, unlike in Q5. He/she also knows how to select an LCD but then does not understand that he/she can't cancel. After incorrect cancelling, understands how to add like terms.*

Student 6's Script:

Q1: *Based on the two solutions the student provided, he/she does not understand the idea of factorisation or looking for a common factor and removing it. The student does not understand why he/she can't cancel, showing no acknowledgement of the fraction as a whole. The student understands how to distribute or FOIL brackets but looking at response 2, he/she does not fully understand when to add/subtract like terms.*

Q2: *The student does not understand how to subtract fractions but does know what the LCD is but does not know what to do with the LCD. Lack of understanding of the concept of a fraction can cause the mistake of him/her multiplying the numerator and denominator of the first term*

then trying to multiply that by denominator of term 2. This student has no appreciation of fractions and tries to rather get rid of the denominators.

Q3: *The student displays no understanding of fractions or idea of an LCD. In term 3, has a partial understanding of factorisation when he/she removes* p *and not* p^2*, also does not recognise DOTS in the denominator. No understanding of how to identify a quadratic to be factorised.*

Q4: *Interestingly, unlike Q3, he/she understands DOTS in term 3 and also understands the LCD and how to use it appropriately to form one fraction. Understands how to correctly distribute the co-efficient to remove brackets and add/subtract like terms. But then just ignores the denominator in the final step, which indicates that there is no appreciation of the fraction in its entirety.*

Q5: *Just like student 5, does not think about how to match the numerator and denominator to make it possible to cancel. Makes the same mistake of ignoring the 1 in the numerator so gives an answer of 2, instead of* $\frac{1}{2}$*. No understanding of the concept of a fraction so wants to get rid of the fraction.*

Q6: *Correctly interprets DOTS but does not realise that he/she is left with 1 in the denominator after cancelling the common bracket. Thus, this effects the LCD which effects the entire solution. Understands how to add/subtract like terms. Again, shows no conceptual understanding of the whole fraction.*

Student 8's Script:

Q1: *No understanding of a common factor, or factorisation. Does not know how to distribute the* $-a$ *correctly. No understanding of add/subtracting like terms. The student also does not know the difference between addition or multiplication of terms, so he/she said* $x + 2 = 2x$ *and* $2x + x = 2x^2$*.*

Q2: *No understanding of subtraction of fractions, or the idea of an LCD. Does not understand when not to cross multiply or how to Foil two brackets, he/she forgot about inner terms. The student has no idea when to cancel or not. Due to a lack of knowledge he/she manipulated the fractions to get an answer of 1.*

Q3: *Does not understand how to factorise or identify a common factor. No idea of what an LCD is or its use. Shows no respect for rules of adding/subtracting like terms. He/she does not know how to work with exponents (no adherence to basic exponential rules).*

Q4: *Student shows no understanding of addition/subtraction of fractions or the value of an LCD. No concept of how to do basic operations of algebra. There is no comprehension of basic algebra*

and no knowledge of fractions itself, so when you put algebra and fractions together it's a disaster because he/she does not get basic arithmetic fractions.

Q5: *Does not know how to interpret the square in the denominator or acknowledge 2 as a co-efficient. Does not recognise DOTS in the numerator or that it can be manipulated for cancellation purposes. No understanding of the fraction in its entirety, which makes him/her forget about the 1 in the numerator to get an answer of 2 instead of* $\frac{1}{2}$.

Q6: *No recognition of DOTS. No basic understanding of algebra or exponents so the student thinks that* a^2 *is* $(a + a)$*and not* $a \times a$*, same with* b*. Shows no understanding of how to add/subtract variables or how to work with fractions or that he/she need an LCD. The student does not know how or when to cancel and there is no understanding of basic fractional rules. No appreciation of the fraction in its entirety or any of the relationships within.*

Lecturer A's choice of response to Student 3, 4 and 7 based on written responses:

Student 3's Script:

This student seemed to make similar mistakes throughout the test. To help this student I would take him/her back to basic factorisation to help identify common factors and then common brackets. I would show the difference between $\frac{(x+1)(x+2)}{(x+1)}$ *and* $\frac{(x+1)(x+2)-a(x+1)}{(x+1)}$*, when you can cancel and when you can't cancel. While the student knows how signs change when you remove a bracket, he/she did not consider the change of signs when putting the brackets back. This may be a careless mistake or not, I will have to talk to the student first to find out. The student needs to be taken back to basic addition and subtraction of arithmetic fractions, to help with the concept of an LCD. Then move to algebraic fractions. I would also show the difference between an expression and an equation and when it is possible to solve for a variable and when you need to simplify only and can't solve for a variable in an expression. So, basically, I would give the student examples of algebraic expressions and equations and show when he/she can solve for a certain variable and when it can't be done. He/she also needs to see when he/she can cross multiply and when can't. This student needs assistance with seeing the fraction as a whole and not as separate parts which can be manipulated separately. I will also give questions on division and multiplication of algebraic fractions; this can assist with also showing when to cancel and when not to.*

Student 4's Script:

To help this student, I would go back to basic arithmetic fractions involving adding, subtracting, multiplying, and dividing. This student needs help with the correction application of exponential

rules and the context in which these rules are relevant. Introduce the student to basic factorisation, DOTS and provide questions where cancellation is appropriate and not appropriate. Once the student understands the basic algebra and arithmetic fractions then I can slowly introduce him/her to algebraic fractions. The student may also benefit from working with addition and subtraction and multiplication and division of both arithmetic and eventually algebraic fractions.

Student 7's Script:

To help this student I would take him/her back to addition/subtraction of arithmetic fractions. Maybe start with 2 terms than move on to more terms and introduce the concept of an LCD. I would also start with simple problems on factorisation and then move on to factorisation of a common bracket and show how to manipulate the negative or positive sign inside the bracket so that it becomes a common bracket. The student also needs help to identify DOTS. Like the other students, the student needs help to understand the algebraic fraction as a whole, knowing that you can't just add/subtract all the denominators or numerators.

All these students need to be taken back to basic arithmetic fractions, so I'm I can see myself having a class devoted to finding an LCD. Which is unbelievable because these students are expected to know this especially at this level.

Lecturer B's choice of response to Student 5, 6 and 8 based on written responses:

Student 5's Script:

In assisting the student, I will start off with basic factorisation examples of how to take out a common factor. We will show the difference between when cancelling can be done, using examples of multiplication and division of algebraic fractions then moving to addition and subtraction of fractions. I will use examples to explain how an algebraic fraction is made up, it component parts and that one needs to view it in its entirety and not separately. We will have to look at part-whole relationships so that the student can build conceptual understanding rather than just cancel without thinking.

Student 6's Script:

To assist, the student needs to be introduced to basic arithmetic fractions and the concept of a fraction as a whole. There needs to be an awareness of the components parts and how they come together to form a fraction. As with student 5, we will look into part-whole relationships so assist

in foundational fraction knowledge. The concept of an LCD and how to use it needs to be worked on. We will work on examples involving correct/incorrect cancelling as well as examples whereby getting rid of fractions seem to be attractive.

Student 8's Script:

I honestly do not know how this student managed to get to L3. This student needs help with the basic of the basics. We will start by showing what an algebraic expression is, what is the difference between variables, constants, co-efficient, what are like/unlike terms. How to perform different operations in algebra. Show the student how a fraction is composed, how to work with arithmetic fractions first. Need to show him/her rules of exponents, how to apply the rules. Basically, this student needs to be taken through all of algebra again, build concepts of fractions only then can we move to algebraic fractions. There should be serious intervention. I honestly do not why this was never noticed before.

Appendix F1: Information sheet - The principal

Wits School of Education
27 St Andrews Rd
Parktown

Johannesburg

2193

South Africa

Date:

Dear Madam Principal of Ekurhuleni East TVET College

My name is Salome Nadason and I am a Masters student in Mathematics Education at the University of the Witwatersrand, Johannesburg. As part of my studies, I must undertake a research project, and I am investigating the effect of professional noticing intervention in the context of algebraic fractions, under the supervision of Prof Judah Makonye. The aim of this research project is to find out the effect of professional noticing intervention on the pedagogical practices on NCV L3 lecturers in relation to the simplification of algebraic fractions.

I hereby request permission to carry out my research at your college; the study will be conducted after lecturing hours, so that teaching and learning will not be disrupted. As part of this project, I would like to invite two of your NCV L3 Mathematics lecturers and their students to participate in the study. The lecturers will be invited to participate in a professional noticing intervention program, this may require 4 to 5 sessions. Each session will be approximately an hour long and will be conducted in the campus boardroom, making it convenient for the lecturers as no travelling is required of them. Thereafter, lecturers will give students a test on algebraic fractions and will analyse their responses. Lecturers will also be observed and recorded teaching two sessions of algebraic fractions. This recording will be stored in a password protected computer that only I will have access to. It will be deleted after 3-5 years.

There will be no personal costs to you, or any of the participants, who choose to participate in this project. Your college will not receive any direct benefits from participation but there are no disadvantages or penalties if you do not choose to participate or if you withdraw from the study. You may withdraw at any time. The interview and results from the analysis of the students test scripts will be completely confidential and anonymous as I will not be asking for any names or any identifying information, and the information given to me will be held securely in a password protected computer and will not be disclosed to anyone else. I will be using a pseudonym (false name) to represent your participation in my final research report.

If you have any questions during or afterwards about this research, feel free to contact me on the details listed below. The data collected from this research project will be stored in a password protected computer and will be kept for 3-5 years. If you have any concerns or complaints regarding the ethical procedures of this study, you are welcome to contact the University Human Research Ethics Committee (Non-Medical), telephone +27(0) 11 717 1408, email hrecnon-medical@wits.ac.za

Yours sincerely,
Salome Nadason

Researcher:
Salome Nadason 0201103P@students.wits.ac.za

Supervisor:
Prof Judah Makonye Judah.Makonye@wits.ac.za

Appendix F2: Information sheet - The campus manager

Wits School of Education
27 St Andrews Rd
Parktown

Johannesburg

2193

South Africa

Date:

Dear Campus Manager of Ekurhuleni East TVET College,

My name is Salome Nadason and I am a Masters student in Mathematics Education at the University of the Witwatersrand, Johannesburg. As part of my studies, I must undertake a research project, and I am investigating the effect of professional noticing intervention in the context of algebraic fractions, under the supervision of Prof Judah Makonye. The aim of this research project is to find out the effect of professional noticing intervention on the pedagogical practices on NCV L3 lecturers in relation to the simplification of algebraic fractions.

I hereby request permission to carry out my research at your college; the study will be conducted after lecturing hours, so that teaching and learning will not be disrupted. As part of this project, I would like to invite two of your NCV L3 Mathematics lecturers and their students to participate in the study. The lecturers will be invited to participate in a professional noticing intervention program, this may require 4 to 5 sessions. Each session will be approximately an hour long and will be conducted in the campus boardroom, making it convenient for the lecturers as no travelling is required of them Thereafter, lecturers will give students a test on algebraic fractions and will analyse their responses. Lecturers will also be observed and recorded teaching two sessions of algebraic fractions. This recording will be stored in a password protected computer that only I will have access to. It will be deleted after 3-5 years.

There will be no personal costs to you, or any of the participants, who choose to participate in this project. Your college will not receive any direct benefits from participation but there are no disadvantages or penalties if you do not choose to participate or if you withdraw from the study. You may withdraw at any time. The interview and results from the analysis of the students test scripts will be completely confidential and anonymous as I will not be asking for any names or any identifying information, and the information given to me will be held securely in a password protected computer and will not be disclosed to anyone else. I will be using a pseudonym (false name) to represent your participation in my final research report.

If you have any questions during or afterwards about this research, feel free to contact me on the details listed below. The data collected from this research project will be stored in a password protected computer and will be kept for 3-5 years. If you have any concerns or complaints regarding the ethical procedures of this study, you are welcome to contact the University Human Research Ethics Committee (Non-Medical), telephone +27(0) 11 717 1408, email hrecnon-medical@wits.ac.za

Yours sincerely,
Salome Nadason

Researcher:
Salome Nadason 0201103P@students.wits.ac.za

Supervisor:
Prof Judah Makonye Judah.Makonye@wits.ac.za

Appendix F3: Information sheet - The lecturer

Wits School of Education
27 St Andrews Rd
Parktown

Johannesburg

2193

South Africa

Date:

Dear Lecturer of Ekurhuleni East TVET College,

My name is Salome Nadason and I am a Masters student in Mathematics Education at the University of the Witwatersrand, Johannesburg. As part of my studies, I must undertake a research project, and I am investigating the effect of professional noticing intervention in the context of algebraic fractions, under the supervision of Prof Judah Makonye. The aim of this research project is to find out the effect of professional noticing intervention on the pedagogical practices on NCV L3 lecturers in relation to the simplification of algebraic fractions.

As part of this project, I would like to invite you to participate in a professional noticing intervention program, this will involve 4 to 5 sessions where we will discuss the construct of professional noticing and its explicit use in the teaching of algebraic fractions. Each session will be approximately an hour long and will be conducted in the campus boardroom, making it convenient in that no travelling is required. These discussions will occur after school hours so that your teaching time will not be disrupted. Thereafter, students will be given a test on algebraic fractions, and you will be asked to analyse their responses. With your permission, I would like to observe two of your lecturers on algebraic fractions. With your permission, I would also like to audio record the lesson using a digital device. This recording will be stored in a password protected computer that only I will have access to. It will be deleted after 3-5 years.

There will be no personal costs to you, or any of the participants, who choose to participate in this project. Your college will not receive any direct benefits from participation but there are no disadvantages or penalties if you do not choose to participate or if you withdraw from the study. You may withdraw at any time. The interview and results from the analysis of the students test scripts will be completely confidential and anonymous as I will not be asking for any names or any identifying information, and the information given to me will be held securely in a password protected computer and will not be disclosed to anyone else. I will be using a pseudonym (false name) to represent your participation in my final research report.

If you have any questions during or afterwards about this research, feel free to contact me on the details listed below. The data collected from this research project will be stored in a password protected computer and will be kept for 3-5 years. If you have any concerns or complaints regarding the ethical procedures of this study, you are welcome to contact the University Human Research Ethics Committee (Non-Medical), telephone +27(0) 11 717 1408, email hrecnon-medical@wits.ac.za

Yours sincerely,
Salome Nadason

Researcher:
Salome Nadason 0201103P@students.wits.ac.za

Supervisor:
Prof Judah Makonye Judah.Makonye@wits.ac.za

Appendix F4: Information sheet - The student

Wits School of Education
27 St Andrews Rd
Parktown

Johannesburg

2193

South Africa

Date:

Dear Student of Ekurhuleni East TVET College,

My name is Salome Nadason and I am a Masters student in Mathematics Education at the University of the Witwatersrand, Johannesburg. As part of my studies, I must undertake a research project, and I am investigating the effect of professional noticing intervention in the context of algebraic fractions, under the supervision of Prof Judah Makonye. The aim of this research project is to find out the effect of professional noticing intervention on the pedagogical practices on NCV L3 lecturers in relation to the simplification of algebraic fractions.

I hereby request permission for you to complete a written test on the simplification of algebraic fractions. The test will be written under formal test conditions and will be approximately 30 minutes long. Furthermore, I request permission to analyse your responses of the test. The test will not affect your academic performance or count towards your final mark. The results from this test will be used for research purposes and will remain confidential.

There will be no personal costs to you, or any of the participants, who choose to participate in this project. Your college will not receive any direct benefits from participation but there are no disadvantages or penalties if you do not choose to participate or if you withdraw from the study. You may withdraw at any time. The results from the analysis of the test scripts will be completely confidential and anonymous as I will not be asking for any names or any identifying information, and the information given to me will be held securely in a password protected computer and will not be disclosed to anyone else. I will be using a pseudonym (false name) to represent your participation in my final research report.

If you have any questions during or afterwards about this research, feel free to contact me on the details listed below. The data collected from this research project will be stored in a password protected computer and will be kept for 3-5 years. If you have any concerns or complaints regarding the ethical procedures of this study, you are welcome to contact the University Human Research Ethics Committee (Non-Medical), telephone +27(0) 11 717 1408, email hrecnon-medical@wits.ac.za

Yours sincerely,
Salome Nadason

Researcher:
Salome Nadason 0201103P@students.wits.ac.za

Supervisor:
Prof Judah Makonye Judah.Makonye@wits.ac.za

Appendix G1: Principal consent form

The Effect of Professional Noticing Intervention in the Context of Algebraic Fractions: A Case Study of NCV L3 Lecturers
Researcher name: Salome Nadason

I, ……., agree to participate in this research project.

I agree to the following:

(Please circle the relevant options below)

The research study was explained to me. I understand what this study is about.	YES	NO
I understand that my lecturers (staff) can volunteer to take part in the study	YES	NO
I agree that direct quotations from the observations, and document analysis may be used by the researcher in their research report	YES	NO
I agree that the participation of my lecturers (staff) will remain anonymous (their names will not be used by the researcher in their research report/manuscript/book chapter)	YES	NO
I agree that other researchers may use the information provided by my lecturers (staff) in the observations, and document analysis (depending on their own ethics clearance being obtained) but their names and any personal information will not be used or passed on	YES	NO
I agree that the researcher may analyse the test scripts of my students for her research report	YES	NO

…… (signature)
…… (college principal)
…… (date)

…… (signature)
…… (name of researcher/person seeking consent)
…… (date)

Appendix G2: Campus manager consent form

The Effect of Professional Noticing Intervention in the Context of Algebraic Fractions: A Case Study of NCV L3 Lecturers
Researcher name: Salome Nadason

I, ……., agree to participate in this research project.

I agree to the following:

(Please circle the relevant options below)

The research study was explained to me. I understand what this study is about.	YES	NO
I understand that my lecturers (staff) can volunteer to take part in the study	YES	NO
I agree that direct quotations from the observations, and document analysis may be used by the researcher in their research report	YES	NO
I agree that the participation of my lecturers (staff) will remain anonymous (their names will not be used by the researcher in their research report/manuscript/book chapter)	YES	NO
I agree that other researchers may use the information provided by my lecturers (staff) observations, and document analysis (depending on their own ethics clearance being obtained) but their names and any personal information will not be used or passed on	YES	NO
I agree that the researcher may analyse the test scripts of my students for her research report	YES	NO

…… (signature)
…… (campus manager)
…… (date)

…… (signature)
…… (name of researcher/person seeking consent)
…… (date)

Appendix G3: Lecturer consent form

The Effect of Professional Noticing Intervention in the Context of Algebraic Fractions: A Case Study of NCV L3 Lecturers
Researcher name: Salome Nadason

I, ……., agree to participate in this research project.

I agree to the following:

(Please circle the relevant options below)

The research study was explained to me. I understand what this study is about.	YES	NO
I understand that I can volunteer to take part in the study	YES	NO
I agree to participate in 4-5 sessions (approximately one hour per session) of the planned intervention program	YES	NO
I agree to be observed teaching two sessions of algebraic fractions	YES	NO
I agree that direct quotations from the observations, and document analysis may be used by the researcher in their research report	YES	NO
I agree that my participation will remain anonymous (my name will not be used by the researcher in their research report/manuscript/book chapter)	YES	NO
I agree that other researchers may use the information I provide in my observations, and document analysis (depending on their own ethics clearance being obtained) but my name and any personal information will not be used or passed on	YES	NO
I agree that the researcher may analyse the test scripts of my students for her research report	YES	NO

…… (signature)
…… (name of participant)
…… (date)

…… (signature)
…… (name of researcher/person seeking consent)
…… (date)

Appendix G4: Student consent form

The Effect of Professional Noticing Intervention in the Context of Algebraic Fractions: A Case Study of NCV L3 Lecturers
Researcher name: Salome Nadason

I, …….., agree to participate in this research project.

I agree to the following:

(Please circle the relevant options below)

The research study was explained to me. I understand what this study is about.	YES	NO
I understand that I can volunteer to take part in the study	YES	NO
I agree that my participation will remain anonymous (my name will not be used by the researcher in their research report/manuscript/book chapter)	YES	NO
I agree that other researchers may use the information I provide in my test (depending on their own ethics clearance being obtained) but my name and any personal information will not be used or passed on	YES	NO
I agree that the researcher may analyse my written test for her research report	YES	NO

…… (signature)
…… (name of participant)
…… (date)

…… (signature)
…… (name of researcher/person seeking consent)
…… (date)