TOTAL REPAIR COST LIMIT REPLACEMENT POLICIES: ANALYSES AND COMPARISONS

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A Research Report submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in partial fulfilment of the requirements for the degree of Master of Science.

Johannesburg, 2016

DECLARATION

I declare that this Research Report is my own unaided work. It is being submitted to the degree of Master of Science to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other University.

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(Signature of candidate)

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ABSTRACT

Maintenance of technical systems aims at retaining their reliability and availability, restoring their capability of continuing their operation in case they have failed. A number of reliability and maintenance policies that play a significant role in reducing system failures and in making cost-effective decisions on whether to repair or replace a system will be examined in great detail.

This research report places greater emphasis on the total repair cost (TRC) limit policy and compares it with other polices like the economic lifetime (EL) policy. The maintenance cost rate will be used to obtain the optimal parameters.

When a system fails, it is replaced with an identical unused one if the total repair cost goes beyond a pre-determined repair cost limit by a preventive replacement if in the time period $(0, v)$ no failure forces a replacement of the system. Else, a repair is carried out.

Acknowledgements

Firstly, I would like to thank the School of Statistics and Actuarial Science for giving me the opportunity to further my studies. My sincere gratitude goes to Professor Frank Beichelt for having given me a fascinating topic and for his continued support and guidance.

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Symbols

L	system lifetime, a continuous random variable
$F(v), \bar{F}(v)$	Distribution and Survival functions of L, respectively
	$\bar{F}(v) = 1 - F(v)$
$f(v)$	Probability density of $L : f(v) = F'(v)$
$\lambda(v)$	Failure rate of $L : \lambda(v) = \frac{f(v)}{\bar{F}(v)}$
$\Lambda(v)$	Integrated failure rate $\Lambda(v) = \int_0^v \lambda(d)\, dl$
L_v	Residual lifetime of a system after time v
	provided that it has not failed in interval $[0, v]$
$F_v(l)$	Distribution function of L_v
$\bar{F}_v(l)$	Survival function of $L_v : \bar{F}_v(l) = 1 - F_v(l)$
c_m, c_p, c_e	Average cost of minimal repair, preventive replacement, an emergency
	replacement after a type II failure $c_m < c_p < c_e$
$o(h)$	Landau's order symbol, i.e is any function $o(h)$
	with property $\lim_{h\to\infty} \frac{o(h)}{h} = 0$
$\{N(v), v \geq 0\}$	counting process
$C(v)$	Total maintenance cost in $[0, v]$
K	maintenance cost rate
$H(v)$	Renewal function
Z or $X, G(V)$	replacement cycle length,distribution function of Z or X
$p(v), \bar{p}(v) = 1 - p(v)$	probability of a type II (type I) failure
$c(v)$	repair cost limit at failure time v
$W(v)$	distribution function of one repair cost
c^*	optimal constant repair cost limit
κ^*	optimal constant repair time limit
$\mu(v)$	mean residual lifetime of the system
V	repair time limit
c_R	constant repair cost
C	single repair cost
$F^{(n)}(v)$	n-fold convolution power of $F(V)$
L_n	time to n'th failure
V_0	economic life of the system
v_0	optimal replacement time

n_0	optimal number of failures before replacement
c	constant repair cost limit
μ	average estimated repair cost
B	arbitrary constant of integration

Abbreviations

BM	Brownian Motion
CP	Counting Process
DMRL	Decreasing Mean Residual Lifetime
EL	Economic Lifetime
HPP	Homogneous Poisson Process
IFR	Increasing Failure Rate
MPP	Marked Point Process
MR	Minimal Repair
NHPP	Nonhomogneous Poisson Process
RCL	Repair Cost Limit
RCP	Random Counting Process
RMPP	Random Marked Point Process
RPP	Random Point Process
TRC	Total Repair Cost
TRCR	Total Repair Cost Rate

Chapter 1

INTRODUCTION

Systems, after the fading away of early failures (breakdowns), usually operate to their utmost ability but as time passes with frequent wear and tear, this becomes a challenge and progressively difficult. If thorough and frequent maintenance is managed, then the production volume of, for example, a production company, can be kept at nearly the same level. Maintenance of technical systems aims at retaining their reliability and availability, restoring their capability of continuing their operation in case they have failed.

With the ever increasing complexity of systems or even networks of systems, the role of maintenance in production processes and in the military has been steadily increasing for at least 150 years. Repair cost limit (RCL) replacement policies have gained major credibility, not only in the academic space, but also in the workplace as they play a significant role in making cost-effective decisions of whether to repair (fix) or replace (substitute) a failed system.

Every now and then, news about the need and costliness of maintenance actions is broadcast. Many large organizations use plants that are more than 10 years old. Individuals are faced with challenges that require them to respond to the call of maintenance of systems that they use on an everyday basis.

For example, one pressing problem we hear of a lot in South Africa is the burning down of houses that emerge from the use of worn out electric appliances. Such incidents cause social insecurity and the living environment gets affected in an irreparable way. From that perspective of reliability, we can see the general importance of maintenance and the need for its policies to be perfectly and timely employed.

Maintenance policies are significant because they ensure that systems are always in ready and reliable condition. One other important aspect of maintenance policies is the calibration of systems to provide high-quality results and competitive edge. This aspect eliminates the likelihood of having unforseen and frequent breakdowns and reduces chances of defective products being produced.

Maintenance policies are particularly important in capital-intensive industries because they ensure the absence of serious breakdowns, thus ensuring there are no losses of inventory as that would be costly to replace. More importantly, and in line with the policies stressed in this research report, maintenance policies ensure that costs are always controlled.

Failure to implement effective maintenance policies can lead to costly and life-threatening outcomes. Should organizations be unable to implement functional maintenance policies, full capacity utilization may not be achieved. There could also be a significant increase in costs such as labour which are fixed and can in no way be reduced. Major organizations would see a reduction in the quality of produced products and a rise in wastage. Most importantly and ethically, failure to implement proper maintenance policies could jeopardize the safety of workers and operators.

Very few systems would not require maintenance to operate. As a result, nearly all systems worsen with age and usage, and possess stochastic failures. Normally, systems have to undergo either corrective maintenance or preventive maintenance. The former is carried out when failed systems require repairs or replacements. This type of maintenance restores the functional capabilities of a failed system.

Preventive maintenance on the other hand is when the maintenance engineer or mechanic does not wait for a system to fail, but will carry out inspections of working parts (subsystems), will replace or repair worn-out parts during its operation or will even preventively shut down a system to do major preventive maintenance actions. This makes sense, since, when properly scheduled, preventive maintenance decreases costs, including downtime costs, compared to pure corrective maintenance.

The need for cost-efficient scheduling maintenance by application of suitable maintenance strategies, which are tailored to the particular practical situation, is the bottom line of the theory of preventive maintenance. Preventive maintenance is not only cost-efficient, but also helps in reducing the dangerous effects of sudden failures of system and the risk of disastrous breakdowns.

A system experiences disturbances that fall under the nonhomogenous Poisson process (NHPP). With the occurance of these shocks, the system can experience a type I failure which is amended using minimal repair as these failures are minor. A disastrous failure which requires a system replacement is called the type II failure.

The theory of maintenance is heavily based both on probability theory and stochastic processes because wear and tear and failures of technical systems usually do not follow a strictly deterministic pattern, but are governed by a deterministic process which is superimposed by a stochastic "noise". Hence, this research report will contain a section on those stochastic processes which are crucial for developing efficient maintenance strategies.

It precedes a section introducing basic maintenance strategies based on system age and repair cost limits to prepare for the main subject of this research report, namely the "Total Repair Cost Limit Replacement Policies: Analyses and Comparisons". These replacement strategies deal with a key problem in system maintenance that asks the question "What is the most cost-efficient time point to replace a system with a new one?". The historical knowledge of the system's overall repair cost is used to determine if it needs to be fixed or substituted by an unused one.

Total repair cost limit replacement strategies were proposed for the first time by Beichelt (1981), but their serious analysis only started at the beginning of this millennium. In this research report, some members of the different maintenance policy classes will be compared with regard to their cost efficiency.

The objectives of this research report are as follows:

1. Discussion of basic maintenance strategies (age replacement, block replacement, repair cost limit replacement strategies) for the sake of comparisons and combinations.

2. Up to date detailed survey on TRC limit replacement strategies.

3. Numerical comparisons of TRC limit replacement strategies, in particular with regard to

    (a) Limiting various functionals of the total repair costs, including the total repair cost rate, for the same stochastic process $\{C(v), v \geq 0\}$

    (b) Different stochastic models for $\{C(v), v \geq 0\}$

Chapter 2 lays the theoretical foundation for investigating reliability and maintenance policies. Aims 1 and 2 are covered in Chapters 3 and 4, respectively. In Chapter 5, we outline the optimal TRC limit maintenance policies so as to identify the preferred policy when it comes to cost efficiency. Chapter 6 looks at the numerical evaluation of the maintenance policies outlined in the previous chapter. Lastly, conclusions and recommendations are proposed in Chapter 7.

Chapter 2

THEORETICAL FOUNDATIONS

A number of system reliability measures are of great importance in maintenance theory. One of the many important measures is the system's average number of breakdowns in a particular time interval. Failures usually occur at random time points. The operation of a system over time can be described using stochastic processes and the processes can be broken down into discrete and continuous types.

In reliability engineering, counting processes under the discrete setting are largely used to analyse the arrival of events in time. These events can be number of failures, repairs, etc. The Poisson process is the simplest and commonly studied process. Another well-known counting process which will be comprehensively delt with in the later sections is the renewal process.

A number of books and papers about the principles in this chapter have been published for a number of decades. For the development of this section, we refer to monographs Barlow and Proschan (1975), Beichelt and Tittmann (2012) and Beichelt (2006).

Let L be a system's random lifespan with distribution function $F(v)$. The system's lifetime is the time period from its start-up time point until its failure, where failure is taken as known to be an immediate event. In the engineering context, a system's lifetime does necessarily equate to the end of its useful life.

$F(v)$ is the system's failure probability and $\bar{F}(v) = 1 - F(v)$ is the survival probability with reference to the span $[0, v]$. $F(v)$ and $\bar{F}(v)$ are the system's probabilities of failure and survival in $[0, v]$, respectively.

2.1 Counting Process

We consider a fixable system that starts to operate at time $v = 0$. Like any functioning system, it will breakdown at time v_1, and will need to be fixed to a working condition. The amount of time required to repair is presumed to be negligibly small relative to system lifetime. After the first failure, the next breakdown will occur at v_2 leading to another repair, and so forth. This leads us to an arrangement of breakdown times $v_1, v_2, \ldots$.

Suppose one can quantify Z_i as the time linking breakdowns $(i-1)$ and i with breakdown times V_{i-1} and V_i, respectively for $i = 1, 2, 3, \ldots$, where V_0 is presumed to be 0. Z_i will be termed the inter-event time. We are primarily keen on the random variable $N(v)$, which represents the count of breakdowns in $(0, v]$ and this particular process $\{N(v), v \geq 0\}$ leads us to counting processes.

Counting processes are utilized to represent sequences of events, i.e., the arrival of events. A nonnegative, integer-valued stochastic process $\{N(v), v \geq 0\}$ is a counting process if $N(v)$ denotes the total occurrences in $(0, v]$. As previously mentioned, the random number $N(v)$ of occurances in $(0, v]$ is of great interest compared to the random occurance times, which occur in $(0, v]$:

$$N(v) = \begin{cases} max\,\{n, V_n \leq v\} & \text{if } n = 1, 2, \ldots \\ 0 & \text{if } v < V_1 \end{cases}$$

$\{N(v), v \geq 0\}$ is a random counting process (RCP) associated to the random point process (RPP) $\{V_1, V_2, V_3, \ldots\}$. This stochastic process possesses a parameter space $[0, \infty)$ and state space $\{0, 1, 2, \ldots\}$. See Beichelt and Tittmann (2012) for properties of this process outlined below:

1. $N(0) = 0$
2. $N(u) \leq N(v)$ for $u \leq v$
3. For any u, v with $0 \leq u \leq v$, the increment $N(u, v) = N(v) - N(u)$ is equivalent to the number of occurances in $(u, v]$.

Contrarily, each stochastic process in continuous time that possesses the above mentioned properties is a RCP of exactly one RPP $\{V_1, V_2, V_3, \ldots\}$. Thus, from a statistical perspective, the stochastic processes $\{V_1, V_2, V_3, \ldots\}$, $\{Z_1, Z_2, Z_3, \ldots\}$ and $\{N(v), v \geq 0\}$ are equivalent.

A RPP is said to be stationary if its RCP $\{N(v), v \geq 0\}$ has homogeneous increments. This suggests that for any increment $N(v+u) - N(v)$, the probability distribution relies only on one parameter, namely u. In this case, in any interval of length 1, there occurs on average $E(N(1)) = E(N(v+1) - N(v))$ events. The expected value $E(N(1))$ is termed the intensity of the RCP and symbolized as λ.

A counting process holds independent increments provided the count of occurances that arrive in separate intervals are not dependent. In this case, the count of occurances that arrive till time v are independent of the count of occurances that arrive in interval $(v, v+u)$.

A number of basic concepts related to a counting process that are pivotal in the theory of reliability maintenance are outlined in the following sections without going into further details.

2.2 Point Processes

Failure times of systems or the time points of their (preventive) replacements generate point processes. A point process is a non decreasing sequence of real numbers without a restricted limit. Generally, they are called "event times" or "event time points". Since each repair-failure process will be terminated after a finite time, the obtained point processes should be deemed finite samples from a point process.

The event times are normally random variables. A succession of variables $\{V_1, V_2, V_3, \ldots\}$ with $V_1 < V_2 < \ldots$ is termed a RPP. Consider a repairable system that begins to operate at time $v = 0$. V_1 will represent the failure of the system. After the system breaks down, it is repaired to a performing condition. The amount of time to repair is presumed to be insignificantly small. The next breakdown will arrive at time V_2, and so forth.

By introducing the random inter-event(inter-arrival) times $\{Z_1, Z_2, Z_3, \ldots\}$ with

$$Z_i = V_i - V_{i-1}, i = 1, 2, \ldots, v_0 = 0,$$

a RPP can equally be interpreted as the succession of its random inter-event times $\{Z_1, Z_2, Z_3, \ldots\}$. The sequence of inter-event times will usually not be independent and identically distributed. The only exception is for when the system is substituted upon breakdown to an “as good as new” condition, and the operating conditions stay unchanged over the entire period.

There can be a number of different random point processes. A simple process is when with certain probability, at most one event (failure in our case) can occur at any time v.

A sequence of inter-event times $\{Z_1, Z_2, Z_3, \ldots\}$ can be a succession of independent, identically distributed variables. The subsequent RPP is termed a recurrent RPP. The homogeneous Poisson processes and renewal processes are the most common recurrent point processes which will be covered in detail in the coming sections.

2.2.1 Homogeneous Poisson Process

Stationary RPP with inter-event times that are independent and identically distributed are defined as a homogeneous Poisson process (HPP). These inter-event times have an exponential distribution with the same rate λ. The λ is the rate of the RPP which does not depend on time v.

Definition 2.1. See Beichelt and Tittmann (2012) A counting process is a HPP with rate λ, for $\lambda > 0$, if

1. $N(0) = 0$
2. The process has stationary and independent increments
3. The number of events in any interval of length v is Poisson distributed with mean λv. That is, for all $u, v > 0$,

$$Pr(N(v+u) - N(u) = n) = \frac{(\lambda v)^n}{n!} e^{-\lambda v} \text{ for } n = 0, 1, 2, \ldots$$

Independent increments suggests that the number of breakdowns in an interval are not determined by those in some previous interval. The distribution of the next breakdowns is not influenced by the count of breakdowns encountered in a specific interval.

The property of stationary increments suggests that the distribution of the number of failures in a time interval takes into account only the duration of the interval, and not the interval's distance from the origin.

The HPP can be defined in terms of the Landau order, $o(h)$.

Definition 2.2. See Beichelt and Tittmann (2012) A counting process $\{N(v), v \geq 0\}$ is a HPP with rate λ if:

1. $N(0) = 0$
2. The process $\{N(v), v \geq 0\}$ has stationary and independent increments
3. $Pr\left(N(v+h) - N(v) = 1\right) = \lambda h + o(h)$
4. $Pr\left(N(v+h) - N(v) \geq 2\right) = o(h)$

The main attributes of the HPP infered from the different definitions above are:

1. The HPP is a regular counting process with independent and stationary increments.
2. The rate of the HPP is fixed and independent of time,

$$E(N(v)) = \lambda \text{ for all } v \geq 0$$

3. The number of failures in (u, v) have a Poisson distribution with mean $\lambda(v-u)$,

$$Pr\left(N(u,v) = i\right) = \frac{\lambda(v-u)^i}{i!} e^{-}\lambda(v-u), i = 0, 1, \ldots$$

   Alternatively, we can introduce $\kappa = v - u$ and have

$$Pr\left(N(u, u+\kappa) = i\right) = \frac{(\lambda\kappa)^i}{i!} e^{-\lambda\kappa}, i = 0, 1, \ldots$$

4. The average count of occurances in interval $(u, u + \kappa)$ is

$$E(N(u + \kappa)) - E(N(u)) = \lambda\kappa$$

5. The inter-event times $Z_1, Z_2, \ldots$ are independent and identically distributed random variables. They are exponentially distributed with mean $frac1\lambda$.

6. The time of the n'th failure $V_n = \sum_{i=1}^{n} Z_i$ has an Erlang distribution with parameters (n, λ). Its probability density function is

$$f_{v_0}(v) = \frac{\lambda}{(n-1)!}(\lambda v)^{n-1} e^{-\lambda v} \text{ for } v \geq 0$$

This is a special Gamma distribution. The arrival time v_n has, according to one of the properties above, a gamma distribution with parameters (λ, n).

If $N(v) = n$ only when $V_n \leq v < V_{n+1}$, and inter-event time $Z_{n+1} = V_{n+1} - V_n$, we can apply the total probability law to show that $N(v)$ has a Poisson distribution with mean λv, in agreement with the first definition of the counting process

$$\begin{aligned} Pr(N(v) = n) &= Pr(V_n \leq v < Vn + 1) \\ &= \int_0^v Pr(Z_{n+1} > v - u | V_n = u) f_{U_n}(u)\, du \\ &= \int_0^v e^{-\lambda(v-u)} \frac{\lambda}{(n-1)!}(\lambda u)^{n-1} e^{-\lambda u}\, du \\ &= \frac{(\lambda v)^n}{n!} e^{-\lambda v} \end{aligned}$$

Another important aspect of this counting process is the conditional distribution of the breakdown time. Assuming only a single event of a HPP with rate λ has occured in interval $(0, v_0]$. The time v_1 at which the failure occured is given by the distribution:

$$\begin{aligned} Pr(V_1 \leq v | N(v_0) = 1) &= \frac{Pr(V_1 \leq v \cap N(v_0) = 1)}{Pr(N(v_0) = 1)} \\ &= \frac{Pr(1 \text{ failure in } (0, v] \cap 0 \text{ failures in } (v, v_0)}{Pr(N(v_0) = 1)} \\ &= \frac{Pr(N(v) = 1)Pr(N(v_0) - N(v) = 0)}{Pr(N(v_0) = 1)} \\ &= \frac{\lambda v e^{-\lambda(v_0 - t)}}{\lambda v_0 e^{-\lambda v_0}} \\ &= \frac{v}{v_0} \text{ for } 0 < v \leq v_0 \end{aligned}$$

If only one failure is recorded in the interval $(0, v_0]$, then the failure time is uniformly distributed in $(0, v_0]$. Thus, each interval of same length in $(0, v_0]$ has an equal chance of containing the failure. The average time at which the breakdown arrives is

$$E(v_1 | N(v_0) = 1) = \frac{v_0}{2}$$

2.2.2 Nonhomogeneous Poisson Process

The NHPP is a generalization of the HPP in that the rate of arrival of failures varies with time rather than being fixed. The inter-event times of the NHPP are independent, but not indentically distributed. This process is mostly used to model repairable systems that are contingent on a minimal repair strategy, with insignificant repair times.

A broken down system is reinstated to its operating condition and this is termed a "minimal repair" (MR). Succeeding a MR, a system carries on to operate as if it had not experienced a breakdown. It is presumed that the failure distribution of system failure is equivalent immediately before and after a failure. This repair strategy reinstates a system to an "as bad as old" condition.

Definition 2.3. See Beichelt and Tittmann (2012) The counting process $\{N(v), v \geq 0\}$ is said to be a NHPP with rate function $\lambda(v), v \geq 0$, if

1. $N(0) = 0$
2. The process $\{N(v), v \geq 0\}$ has independent increments
3. $Pr\left(N(v + h) - N(v) = 1\right) = \lambda(v)h + o(h)$

4. $Pr\left(N(v+h)-N(v)\geq 2\right)=o(h)$, which suggests that a system will experience at most one failure at a point in time.

The assumption of stationary increments is no longer valid with the NHPP. This means that events (failures) have a higher chance of occuring at particular times than others, and the inter-event times are usually not identically distributed. Therefore, statistical techniques that use the assumption that the random vaviables are independent and identically distributed cannot be applied to a NHPP.

Under such variations, the rate λ depends on time. This function leads to the trend function of the RCP with the following expected increments:

$$m(v)=E\left(N(v)\right)=\int_0^v \lambda(l)\,dl$$

and

$$m(u,v)=m(v)-m(u)=\int_u^v \lambda(l)\,dl, 0\leq u\leq v$$

The parameter $\lambda(v)$ is the arrival rate of failures of system which is repairable. If we let the integrated failure rate be

$$\Lambda(v)=\int_0^v \lambda(u)\,du \tag{2.1}$$

then we can conclude that for a NHPP for all s, $v\geq 0$ and

$$Pr\left(N(v)-N(u)=n\right)=e^{-(\Lambda(v)-\Lambda(u))}\frac{(\Lambda(v)-\Lambda(u))^n}{n!}\text{ for } n=0,1,\ldots$$

The important aspects of the NHPP are:

1. The probability distribution of increments $N(u,v)$.

   Let $p_n(u,v)=Pr(N(u,v)=n); 0\leq u\leq v, n=0,1,\ldots.$

With $\{N(v), v \geq 0\}$ having independent increments for $h \geq 0$, then

$$\begin{aligned} p_0(u, v+h) &= Pr(N(u, v+h) = 0) \\ &= Pr(0 \text{ events in } (u, v) \text{ and } 0 \text{ events in } (v, v+h)) \\ &= Pr(N(u, v) = 0, N(v, v+h) = 0) \\ &= Pr(N(u, v) = 0 \cdot N(v, v+h) = 0) \\ &= p_0(u, v)[1 - \lambda(v)h + o(h)] \end{aligned}$$

This further means

$$\frac{p_0(u, v+h) - p_0(u, v)}{h} = -\lambda(v)p_0(u, v) + \frac{o(h)}{h}$$

If we let $h \to 0$ then

$$p_0^{'}(u, v) = \lambda(v)p_0(u, v)$$

which is a first order partial differential equation. By means of the initial condition $N(0) = 0$ or $p_0(0, 0) = 1$,

$$\ln p_0(u, v) = -\int_u^v \lambda(u)\, du$$

Using the integrated failure rate, we get

$$p_0(u, v) = e^{-(\Lambda(v) - \Lambda(u))} \tag{2.2}$$

$N(u, v)$ has a Poisson distribution with mean $\Lambda(v) - \Lambda(u)$, implying that

$$\begin{aligned} E(N(u, v)) &= \Lambda(v) - \Lambda(u) \\ &= \int_u^v \lambda(u)\, du \end{aligned}$$

By induction, for $n \geq 1$ it can be shown that

$$p_0(u, v) = \frac{[\Lambda(v) - \Lambda(u)]^n e^{-(\Lambda(v) - \Lambda(u))}}{n!}; n \geq 1 \tag{2.3}$$

Proof of equation 2.3: For $n > 0$

$$\begin{aligned} p_n(u, v+h) &= Pr(N(u, v+h) = n) \\ &= Pr\{(N(u, v) = n), Pr(N(v, v+h) = 0\} + \\ &\quad Pr\{(N(u, v) = n-1), Pr(N(v, v+h) = 1\} + \\ &\quad \sum_{k=2}^{n} (Pr(N(u, v) = n-k), N(v, v+h) = k) \end{aligned} \tag{2.4}$$

Using condition (iii) of a NHPP definition,

$$\sum_{k=2}^{n} (Pr(N(u, v) = n-k), N(v, v+h) = k) = o(h)$$

By the assumption of independent increments,

$$\begin{aligned} p_n(u, v+h) &= Pr(N(u, v) = n)Pr(N(v, v+h) = 0) + \\ &\quad Pr(N(u, v) = n-1)Pr(N(v, v+h) = 1) + o(h) \\ &= p_n(u, v)p_0(v, v+h) + p_{n-1}(u, v)p_1(v, v+h) + o(h) \\ &= p_n(u, v)[1 - \lambda(v)h - o(h) - o(h)] + \\ &\quad p_{n-1}(u, v)[\lambda(v)h + o(h)] + o(h) \\ &= p_n(u, v) - p_n(u, v)\lambda(v)h + p_{n-1}(u, v)\lambda(v)h \end{aligned}$$

Therefore,

$$\frac{p_n(u, v+h) - p_n(u, v)}{h} = p_n(u, v)\lambda(v) + p_{n-1}(u, v)\lambda(v) \tag{2.5}$$

Taking the limit in equation 2.5 as h tends to 0 gives

$$p_n'(u, v) = -p_n(u, v)\lambda(v) + p_{n-1}(u, v)\lambda(v) \tag{2.6}$$

Multipying both sides of equation 2.6 by $e^{(\Lambda(v) - \Lambda(u))}$ we get

$$e^{(\Lambda(v)-\Lambda(u))}[p_n'(u, v) + p_n(u, v)\lambda(v)] = e^{(\Lambda(v)-\Lambda(u))}[p_{n-1}(u, v)\lambda(v)]$$

where the left hand side is basically $\frac{d}{dv}[e^{(\Lambda(v)-\Lambda(u))}p_{n-1}(u, v)\lambda(v)]$.

For $n - 1$

$$\begin{aligned}\frac{d}{dv}[e^{(\Lambda(v)-\Lambda(u))}p_1(u,v)] &= e^{(\Lambda(v)-\Lambda(u))}p_0(u,v)\lambda(v)\\ &= e^{(\Lambda(v)-\Lambda(u))}e^{-(\Lambda(v)-\Lambda(u))}\lambda(v)\\ &= \lambda(v)\end{aligned}$$

which is a differential equation. Therefore

$$e^{(\Lambda(v)-\Lambda(u))}p_1(u,v) = \int_u^v \lambda(u)\,du + B$$

Seeing that $p_0(0,0) = 0$,

$$\begin{aligned}p_1(u,v) &= \int_u^v \lambda(u)\,du e^{(\Lambda(v)-\Lambda(u))}\\ &= [\Lambda(v) - \Lambda(u)]e^{(\Lambda(v)-\Lambda(u))}\end{aligned}$$

To prove that

$$p_n(u,v) = \frac{[\Lambda(v) - \Lambda(u)]^n e^{(\Lambda(v)-\Lambda(u))}}{n!}; n \geq 1 \qquad (2.7)$$

we suppose that it holds for $n = 1$. Then

$$\begin{aligned}\frac{d}{dv}[e^{(\Lambda(v)-\Lambda(u))}p_n(u,v)] &= e^{(\Lambda(v)-\Lambda(u))}e^{-(\Lambda(v)-\Lambda(u))}\frac{[\Lambda(v) - \Lambda(u)]^{n-1}}{(n-1)!}\lambda(v)\\ &= \frac{[\Lambda(v) - \Lambda(u)]^{n-1}}{(n-1)!}\lambda(v)\end{aligned}$$

Using the method of substitution (letting $u = \Lambda(v) - \Lambda(u)$) and the fundamental theorem of calculus to solve the above differential equation, we get

$$e^{(\Lambda(v)-\Lambda(u))}p_n(u,v) = \frac{[\Lambda(v) - \Lambda(u)]^n}{(n)!} + B$$

Using the condition that $p_n(0,0) = 0$ leads to

$$p_n(u,v) = \frac{[\Lambda(v) - \Lambda(u)]^n e^{(\Lambda(v)-\Lambda(u))}}{n!}; n \geq 1$$

2. The density function of the time of occurrence V_n of the n'th Poisson event.

Suppose V_1 is the time of occurrence of the first Poisson event with distribution function $F_{V_1}(v) = Pr(V_1 \leq v)$ and density function $f_{V_1}(v)$. Then

$$\begin{aligned} p_0(0, v) &= Pr(V_1 > v) \\ &= 1 - F_{V_1}(v) \end{aligned}$$

From equation 2.2

$$p_0(0, v) = e^{-\Lambda(v)} = e^{-\int_0^v \lambda(u)\, du}$$

Then $F_{V_1}(v) = 1 - e^{-\int_0^v \lambda(u)\, du}$ and $f_{V_1}(v) = \lambda(v)e^{-\int_0^v \lambda(u)\, du}, v \geq 0$.

The distribution function of V_n is

$$\begin{aligned} F_{V_n}(v) &= Pr(V_n \leq v) \\ &= Pr(N \leq v) \\ &= \sum_{i=n}^{\infty} \frac{[\Lambda(v)]^i}{i!} e^{-\Lambda(v)}, n = 1, 2, 3, \ldots \end{aligned}$$

The probability density function is given by

$$f_{V_n}(v) = \frac{[\Lambda(v)]^{n-1}}{(n-1)!} \lambda(v) e^{-\Lambda(v)}; v \geq 0, n = 1, 2, \ldots$$

3. Joint probability density of $(V_1, V_2, \ldots, V_n)$:

Conditioning on the first Poisson event at the $V_1 = v_1$, the distribution function of V_2 is

$$F_{V_1}(v_2|v_1) = Pr(V_2 \leq v_2|v_1 = v_1)$$

The above is the probability that one or more Poisson events occurs in the interval $(v_1, v_2]$. Using equation 2.7, we get

$$\begin{aligned} F_{V_2}(v_2|v_1) &= 1 - p_0(v_1, v_2) \\ &= 1 - e^{-[\Lambda(v_2) - \Lambda(v_1)]} \end{aligned}$$

The corresponding density function is

$$f_{V_2}(v_2|v_1) = \lambda(v_2)e^{-[\Lambda(v_2)-\Lambda(v_1)]}; 0 \leq v_1 \leq v_2$$

It is known that the joint probability density $f_{X,Y}(x, y)$ of any two random variables X and Y is given by

$$f_{X,Y}(x, y) = f_Y(y|x)f_X(x)$$

Therefore, the joint probability density of (V_1, V_2) is

$$f(v_1, v_2) = \begin{cases} \lambda(v_1)f_{V_1}(v_2) & \text{for } v_1 \leq v_2 \\ 0 & \text{Otherwise} \end{cases}$$

By induction, the joint density of $(V_1, V_2, \ldots, V_n)$ is given by

$$f(v_1, v_2, \ldots, v_n) = \begin{cases} \lambda(v_1)\lambda(v_2)\ldots\lambda(v_n)f_{v_1}(v_n) & \text{for } 0 \leq v_1 \leq v_2 \ldots < v_n \\ 0 & \text{Otherwise} \end{cases}$$

2.3 Residual Lifetime

The lifetime of a system is its lifespan from its starting point up to its failure, where the failure is taken to be an immediate event. The residual lifetime L_v of a system which has operated for t time units with no failure has the distribution function $F_v(l)$:

$$\begin{aligned} F_v(l) &= Pr(L_v \leq l) \\ &= Pr(L - v \leq l|L > v) \\ &= \frac{Pr(L - v \leq l \cap L > v)}{Pr(L > v)} \\ &= \frac{Pr(v < L \leq v + l)}{Pr(L > v)} \\ &= \frac{F(v + l)}{\bar{F}(v)} l \geq 0, v \geq 0 \end{aligned}$$

The equivalent conditional survival probability $\bar{F}_v(l) = 1 - F_v(l)$ is

$$\bar{F}_v(l) = \frac{\bar{F}(v+l)}{\bar{F}(v)} l \geq 0, v \geq 0$$

The mean residual lifetime is

$$m_l(v) = E(L - v|L > v) = \int_0^\infty \bar{F}_v(l)\, dl$$

Thus $F(l)$ has a decreasing mean residual lifetime (DMRL) if $m_l(v)$ is decreasing in v or if

$$m_l(v_1) \leq m_l(v_2) \text{ for } 0 \leq v_1 \leq v_2$$

Based on the residual lifetime, one can characterize the aging behavior of a system.

2.4 Renewal Theory

2.4.1 Foundations

A renewal process is a generalization of the Poisson process with inter-event times that are independent and identically distributed, but not necessarily exponentially distributed. After a breakdown, the component of a system is replaced or returned to an "as good as new condition".

In maintenance, a renewal process explains the situation that a broken down system is substituted with a similar one in negligibly small time. The probability that there are exactly n failures occurring by time v can be written as

$$Pr\{N(v) = n\} = Pr\{N(v) \geq n\} - Pr\{N(v) > n\}$$

A sequence of nonnegative, independent and identically distributed random variables

$\{Z_1, Z_2, \dots Z_n\}$ are an ordinary renewal process where Z_n is the time between the (n-1)'th and n'th renewal and $L_0 = 0$. The time of the n'th renewal is

$$V_n = \sum_{i=1}^{n} Z_i$$

As an application to replacement models, consider a repairable system which, upon failure, is either repaired or replaced. We assume that maintenance takes an insignificant amount of time, and the repair times are the independent, nonnegative, and identically distributed random variables $Z_1, Z_2, \dots, Z_n$.

When the system starts to operate at time $v = 0$, Z_1 is the residual lifetime of the new system. And $Z_2, Z_3, \dots, Z_n$ is considered the residual lifetime of the system after renewal. Hence the distribution of Z_1 will be different from the assumed common distribution of $Z_2, Z_3, \dots$.

A delayed renewal process $\{Z_1, Z_2, \dots\}$ is a sequence of independent, nonnegative random variables such that L_1 is distributed as

$$F_1(v) = Pr(Z_1 \leq v)$$

and $L_2, L_3, \dots$ are identically distributed with a distribution function

$$F(v) = Pr(Z \leq v)$$

The count of events until time v is represented as $N(v)$, which is the counting process and is called the renewal process. More specifically, $N(v)$ is the maximum n for which $V_n \leq v$ provided $N(v) = 0$ if $Z_1 \geq v$. To get the distribution of $N(v)$, we note that $N(v) \geq n$ iff $V_n \leq v$, hence

$$F_{V_n}(v) = Pr(V_n \leq v) = Pr(N(v) \geq n) \tag{2.8}$$

Therefore, by independence of the Z_i's and using the convolution operator:

$$F_{V_n}(v) = F_1 \times F^{*(n-1)}(v) \text{ for } n = 1, 2, \dots \tag{2.9}$$

From equations 2.8 and 2.9 we see that

$$\begin{aligned} Pr(N(v) = n) &= Pr(Z_1 + Z_2 + \ldots + Z_n \leq v) \text{and } Z_1 + Z_2 + \ldots + Z_n > v \\ &= F_{V_n}(v) - F_{V_{n+1}}(v) \text{ for } n = 0, 1, 2, \ldots \end{aligned}$$

The renewal function is the "expected number of renewals in $[0, v]$". We denote the renewal function of an ordinary renewal process as by $H(v)$, and that of the delayed renewal process by $H_1(v)$. Using equations 2.8 and 2.9 above,

$$\begin{aligned} H(v) &= E(N(v)) \\ &= \sum_{n=1}^{\infty} Pr(N(v) \geq n) \\ &= \sum_{n=1}^{\infty} F^{*(n)}(v) \end{aligned} \tag{2.10}$$

And similarly,

$$H_1(v) = \sum_{n=1}^{\infty} F_1 \cdot F^{*(n-1)}(v) \tag{2.11}$$

The integral equations of the renewal functions is obtaining by noting that $F^{(n+1)}(v) = \int_0^v F^n(v - l)\, dF(l)$ and $H_1(v) = \sum_{n=0}^{\infty} F_1 \cdot F^{*(n)}(v)$

Hence

$$\begin{aligned} H_1(v) &= F_1(v) + \sum_{n=1}^{\infty} \int_0^v F_1 \cdot F^{*(n-1)}(v - l)\, dF(l) \\ &= F_1(v) + \int_0^v \sum_{n=1}^{\infty} F_1 \cdot F^{*(n-1)}(v - l)\, dF(l) \end{aligned}$$

From equation 2.11

$$H_1(v) = F_1(v) + \int_0^v H_1(v - l)\, dF(l) \tag{2.12}$$

Similar arguments lead to

$$H(v) = F(v) + \int_0^v H_1(v - l)\, dF(l) \tag{2.13}$$

Equations 2.12 and 2.13 are known as the fundamental renewal equation of the delayed and ordinary renewal process respectively. Upon differentiating $H_1(v)$ and $H(v)$, we obtain the equivalent renewal densities $h_1(v)$ and $h(v)$ with provision that the densities of Z_1 and Z exist

$$\begin{aligned} h_1(v) &= \frac{d}{dv} H_1(v) \\ &= f_1(v) + \int_0^v h_1(v-l) f(l)\, dl \end{aligned} \tag{2.14}$$

and

$$H(v) = F(v) + \int_0^v h(v-l) f(l)\, dl \tag{2.15}$$

Alternatively, we can differentiate equation 2.10,

$$H(v) = \frac{d}{dv} H(v) = \sum_{n=1}^{\infty} f^{*(n)}(v)$$

where $H(v)dv$ is, for small dv, approximately the probability of a renewal occurring in $[v, v + dv]$.

One easy way to solve for the above integral equations is to obtain their corresponding Laplace-Stieljies transform

$$\hat{H}(u) = \int_0^{\infty} e^{-ul}\, dH(l)$$

Hence, taking the transform in equations 2.12 and 2.13 we obtain

$$\hat{H}_1(u) = \hat{F}_1(u) + \hat{H}_1(u)\hat{F}_1(u) \text{ and } \hat{H}(u) = \hat{F}(u) + \hat{H}(u)\hat{F}(u)$$

so that

$$\hat{H}_1(u) = \frac{\hat{F}_1(u)}{1 - \hat{F}_1(u)} \text{ and } \hat{H}(u) = \tfrac{\hat{F}(u)}{1 - \hat{F}(u)}$$

or equivalently

$$\hat{F}_1(u) = \frac{\hat{H}_1(u)}{1 - \hat{H}_1(u)} \text{ and } \hat{F}(u) = \tfrac{\hat{H}(u)}{1-\hat{H}(u)}$$

Therefore, we obtain $H(v)$ or $H_1(v)$ by knowing either $F(v)$ or $F_1(v)$ respectively and vice-versa. We can note that the entire number of renewals as $t \to \infty$ is ∞, i.e.

$$\lim_{v\to\infty} N(v) = \infty$$

Also of importance is the asymptotic behavior of $H(v)$ as $v \to \infty$, and proofs are given in a number of probability theory textbooks. Letting $\mu = E(X) < \infty$, we can deduce the following:

$$\lim_{v\to\infty} \frac{H(v)}{v} = \frac{1}{\mu}$$

And the above is known as the Elementary renewal theorem. It states that the average renewal rate approaches $\frac{1}{\mu}$ with certainty. If $F(v)$ is non-arithmetic and $g(v)$ is an integrable function on $[0, \infty)$, then we have a Key renewal theorem given by

$$\lim_{v\to\infty} \int_0^v g(v-l)\, dH_1(l) = \frac{1}{\mu} \int_0^v g(l)\, dl \text{ where } \mu = \textstyle\int_0^\infty \bar{F}(v)\, dv$$

Blackwell's theorem states that if $F(v)$ is non-arithmetic, then the expected number of renewals in an interval of length h is given by

$$\lim_{v\to\infty} [H(v+h) - H(v)] = \frac{h}{\mu} \text{ for any } h > 0$$

2.4.2 Renewal Reward Theorem

A replacement cycle is described as time linking neighboring substitutions. Suppose M is the total random maintenance cost during a replacement cycle and V be the random duration of a replacement cycle. Then the expected maintenance cost per unit time in the long-run are

$$K = \frac{E(M)}{E(V)}$$

This formula is called the renewal reward theorem. This theorem means the rate at which rewards are earned is equivalent to the expected reward over the cycle divided by an expected cycle length.

Theorem 2.4. *Renewal reward theorem*

$$\lim_{v\to\infty} \frac{M(v)}{v} = \frac{E(M_1)}{E(Z_1)}$$

The proof for Theorem 2.4 stems from Lemma 2.5.

Lemma 2.5. For all $v \geq 0$, suppose $N(v)$ counts the cycles accomplished until time v. Thus

$$\lim_{v\to\infty} \frac{N(v)}{v} = \frac{E(N_1)}{E(Z_1)}$$

Proof By definition of a counting process, we have

$$Z_1 + Z_2 + \ldots + Z_{N(v)} \leq v < Z_1 + Z_2 + \ldots + Z_{N(v)+1}$$

Since $P\{Z_1 + Z_2 + \ldots + Z_n < \infty\} = 1$ for all $n \geq 1$, one can validate that

$$\lim_{v\to\infty} N(v) = \infty \text{ with certain probability}$$

This inequality leads to

$$\frac{Z_1 + Z_2 + \ldots + Z_{N(v)}}{N(v)} \leq \frac{v}{N(v)} < \frac{Z_1 + Z_2 + \ldots + Z_{N(v)+1}}{N(v)+1} . \frac{N(v)+1}{N(v)}$$

For a succession of independent and identically distributed random variables, we can infer from the strong law of large numbers that

$$\lim_{v\to\infty} \frac{Z_1 + Z_2 + \ldots + Z_n}{n} = E(Z_1) \text{ with probability 1}$$

Therefore, the desired result follows by letting $v \to \infty$ in the above inequality.

Proof of Theorem 2.4 Suppose that the maintenance costs are non-negative. Then, for any $v > 0$,

$$\sum_{i=1}^{N(v)} M_i \leq M(v) \leq \sum_{i=1}^{N(v)+1} M_i$$

This gives

$$\frac{\sum_{i=1}^{N(v)} M_i}{N(v)} . \frac{N(v)}{v} \leq \frac{M(v)}{v} \leq \frac{\sum_{i=1}^{N(v)+1} M_i}{N(v)+1} . \frac{N(v)+1}{v}$$

By the strong law of large numbers we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{N(v)} M_i = E(M_i) \text{ with probability 1}$$

As clearly outlined in the proof of Lemma 2.5, $N(v) \to \infty$ with probability 1 as $v \to \infty$. Letting $v \to \infty$ in the above inequality and utilizing Lemma 2.5, the desired outcome follows for the case that the rewards are positive. If the rewards can assume both positive and negative values, then the theorem is proved by treating the positive and negative parts of the rewards separately.

In a normal way, Theorem 2.4 "relates the characteristics of the renewal-reward process over time to the characteristics of the process over a single renewal cycle. It is worth noting that the result of the long-run average actual reward per time unit can be estimated with certainty. If we are going to run the process over a long period of time, then we can say beforehand that in the long run the average actual reward per time unit will be the same as the constant $\frac{E(M_1)}{E(Z_1)}$ with probability 1. This is a much stronger and more useful statement than the statement that the long-run expected average reward per time unit equals $\frac{E(M_1)}{E(Z_1)}$. It indeed holds that

$$\lim_{n\to\infty} \frac{M(v)}{v} = \frac{E(M_1)}{E(Z_1)}$$

which is the expected-value version of the renewal-reward theorem".

2.5 Marked Point Processes

2.5.1 Compound Stochastic Process

Usually, events arrive with another piece of information over and above their arrival times. In maintenance, a number of situations such as the following are common: If V_i is the system's i'th falure time point, then the time (or cost) necessary for eliminating the failure may be assigned to V_i. Or if at time V_i a repaired system resumes its operation, then the random profit arising from using the system to its next failure, may be assigned to V_i. Generally, a decision of any kind may be assigned to V_i (decision processes). This leads us to the notion of a marked point process (MPP).

Let $\{V_1, V_2, \ldots\}$ be a random point process with random marks M_i out of a marked space M assigned to V_i. The succession $\{(V_1, M_1), (V_2, M_2), \ldots\}$ is referred to as a random MPP.

Suppose $\{(V_1, M_1), (V_2, M_2), \ldots\}$ is a random MPP with $\{N(v), v \geq 0\}$ being the RCP with rate λ belonging to the RPP $\{V_1, V_2, \ldots\}$, the stochastic process $\{C(v), v \geq 0\}$ defined by

$$C(v) = \sum_{i=0}^{N(v)} M_i \text{ where } M_0 = 0$$

is referred to as a compound stochastic process.

For example, if V_i is the time point at which the i'th failure of a system occurs and M_i the corresponding repair cost, then $C(v)$ is the total repair cost in (0,v]. M_i is a family of independent and identically distributed random variables with a common distribution.

The CP $\{N(v), v \geq 0\}$ and the sequence $\{M_i, i = 1, 2, \ldots\}$ are assumed to be independent. This means the value of M_i's is independent of the arrival of the events which $N(v)$ is counting. Compound stochastic processes play a key role in TRC limit replacement strategies.

Suppose M has distribution function $G(l)$, then $C(v)$ has the conditional distribution function

$$Pr(C(v) \leq l | N(v) = n) = G^{*(n)} \tag{2.16}$$

where $G^{*(n)}(l)$ is the n-fold convolution of G with itself, $G^{*(0)}(l) \equiv 1$ and $G^{*(1)}(l) = G(l)$. By applying the probability rule,

$$\begin{aligned} F_{C(v)}(l) &= Pr(C(v) \leq l) \\ &= \sum_{n=1}^{\infty} G^{*(n)}(l) Pr(N(v) = n) \end{aligned} \tag{2.17}$$

The above can be proved as follows:

$$\begin{aligned} Pr\{C(v) \leq c\} &= Pr\{\sum_{j=0}^{N(v)} M_i \leq c\} \\ &= \sum_{k=0}^{\infty} Pr\{\sum_{j=0}^{k} |N(v) = k\} Pr\{N(v) = k\} \\ &= \sum_{k=0}^{\infty} Pr\{W_0 + W_1 + \ldots + W_k \leq c\} \frac{(\lambda v)^k}{k!} e^{-\lambda v} \\ &= \sum_{k=0}^{\infty} G^{(k)}(l) \frac{(\lambda v)^k}{k!} e^{-\lambda v} \end{aligned}$$

Properties of the distribution of $C(v)$ are

1. If $\{N(v), v \geq 0\}$ is a HPP with intensity λ, then

$$F_{C(v)}(l) = e^{-\lambda v} \sum_{n=1}^{\infty} G^{*(n)}(l) \frac{(\lambda v)^n}{n!} \text{for } l > 0, v > 0$$

2. If $M = N(\mu, \sigma^2)$ with $\mu > 3\sigma^2$, then

$$F_{C(v)}(l) = e^{-\lambda v}[1 + \sum_{n=1}^{\infty} \varphi(\frac{l-\mu}{\sigma\sqrt{n}} \frac{(\lambda v)^n}{n!})] \text{for } l > 0, v > 0$$

More often, tractable formulas for $F_{C(v)}$ are not obtained from equation 2.16. Constructing bounds and asymptotic expansions for $F_{C(v)}$ is particularly of importance. For v sufficiently large,

$$C(v) \approx N\Big(\frac{E(M)}{E(X)} v, [E(X)]^{-3} \gamma^2 v\Big)$$

where

$$\gamma^2 = Var(E(X)M - E(M)X) > 0$$

Suppose at time $X(l)$ the compound stochastic process $\{C(v), v \geq 0\}$ goes beyond the level l for the first time, $X(l) = \inf_v\{v, C(v) \geq l\}$ for the first time. The subsequent connection between the distribution of $C(v)$ and $X(l)$ can be inferred since the process $\{C(v), v \geq 0\}$ is assumed to have nondecreasing sample paths

$$Pr(X(l) \leq v) = Pr(C(v) \geq l)$$

or

$$F_{X(l)}(v) = 1 - F_{C(v)}(l)$$

If $\{N(v), v \geq 0\}$ is the HPP, then, using equation 2.17

$$F_{X(l)}(v) = 1 - e^{-\lambda v} \sum_{n=0}^{\infty} G^{*(n)}(l) \frac{(\lambda v)^n}{n!}, v > 0, l > 0$$

Chapter 3

BASIC MAINTENANCE POLICIES

Failure of a system can lead to two possible actions, renewal or repair of the failed system. The former is an extreme form of maintenance action as it requires the failed system to be replaced by an unused one, which, regarding the lifespan distribution is similar to that of used system. Another major, but less costly action is the minimal repair.

NHPP are used to model the stochastic failures under minimal repairs. The other two process, HPP and renewal process, are both used under perfect repairs. Extensive discussions in this section will provide formulas which are needed for examining policies in the impending sections.

Consider a system to be maintained that has a lifetime L characterised by the failure distribution function $F(v)$ which simplifies to $F(v) = Pr(L \leq v) = 1 - e^{-\Lambda(v)}$. Other noteworthy elements are the survival probability function $\bar{F}(v)$, the failure rate of L given by $\lambda(v) = \frac{f(v)}{F(v)}$, and integrated failure rate $\Lambda(v) = \int_0^v \lambda(v)\, dl$, the probability density function $f(v)$, c_m the minimal repair, c_e the replacement cost post breakdown (emergency replacement), c_p the preventive replacement cost, where $c_e > c_p > c_m$, K the maintenance cost rate.

Policy: Each breakdown is fixed by minimal repair. This policy means that a system's breakdown rate immediately prior to breakdown is equivalent to the breakdown rate immediately post its repair.

The main assumptions underlying the basic maintenance policies are as follows:

1. All repair and replacement times are negligibly small, unless stated otherwise.

2. The lifetime distributions of unused and used systems are equivalent, meaning that replaced systems are “as good as new”.

3. All failures are sudden failures, i.e. they occur instantaneously.

4. The maintenance process is carried to infinity (a sufficiently long time).

5. The systems under preventive maintenance are aging, i.e. they have an increasing failure rate.

The MR process $\{N(v)\}$, represents the random frequency of minimal repairs in $(0, v]$. The increments $N(u, v) = N(v) - N(u), u < v$ follow a Poisson distribution with mean value

$$\begin{aligned} m(u, v) &= \int_u^v \lambda(l)\, dl \\ &= \Lambda(v) - \Lambda(u), n = 0, 1, 2, \ldots \end{aligned}$$

and probability function

$$p_n(u, v) = Pr(N(u, v) = n) = \frac{(\Lambda(v) - \Lambda(u))^n}{n!} e^{-(\Lambda(v) - \Lambda(u))} \tag{3.1}$$

Specifically, the MR process has state probabilities

$$p_n(v) = Pr(N(v) = n) = \frac{(\Lambda(v))^n}{n!} e^{-(\Lambda(v))} n = 0, 1, 2, \ldots$$

This leads us to the trend function

$$m(v) = \Lambda(v), v \geq 0$$

The above further show that the number of repairs $\{N(v), v \geq 0\}$, under the condition that every failure is fixed by minimal repairs, fall under a NHPP characterised by the intensity $\Lambda(v)$.

The time to the n’th disturbance distribution function, V_n, is

$$F_{V_n}(v) = e^{-\Lambda(v)} \sum_{i=n}^{\infty} \frac{(\Lambda(v))^n}{n!}, v \geq 0, n = 1, 2, \ldots \tag{3.2}$$

If we differentiate equation 3.2 with respect to v, we get the following density function

$$f_{V_n}(v) = \frac{(\Lambda(v))^{n-1}}{(n-1!)} f_{V_1}(v), v \geq 0, n = 1, 2, \ldots$$

Since $E(V) = \int_0^\infty v \cdot f(v)\, dv$, then

$$E(V_n) = \frac{1}{(n-1)!} \int_0^\infty t(\Lambda(v))^{(n-1)} f(v)\, dv \tag{3.3}$$

The inter-arrival time of failures, $Z_n = V_n - V_{n-1}$, are defined as the time linking the (n-1)'th and the n'th breakdowns. A system would start to operate at $v = 0$. If failure happens at $V = v$, then a minimal repair would be conducted. By definition, the failure rate is $\lambda(v)$. The expected time linking the inter-arrival time of breakdowns is given by

$$E(Z_n) = \frac{1}{(n-1)!} \int_0^\infty (\Lambda(v))^i e^{-\Lambda(v)}\, dv, n = 1, 2, 3, \ldots$$

Another important property is the joint density of the minimal repair times $(V_1, V_2, \ldots, V_n)$. The likelihood that at the very least a minimal repair takes place in $(v_1, v_2], v_1 < v_2$ is given by

$$F_{V_2}(v_2|v_1) = Pr(V_2 \leq v_2|V_1 = v_1)$$

By the distribution of increments in equation 3.1, the conditional probability becomes

$$\begin{aligned} F_{V_2}(v_2|v_1) &= 1 - Pr(N(v_1, v_2) = 0) \\ &= 1 - e^{-[\Lambda(v_2) - \Lambda(v_1)]} \end{aligned}$$

By differentiating with regard to v_2, the corresponding density then becomes

$$f_{V_2}(v_2|v_1) = \lambda(v_2) e^{-[\Lambda(v_2) - \Lambda(v_1)]}, 0 \leq v_1 < v_2$$

Since by the law of probabilities

$$f_{(V_1,V_2)}(v_1, v_2) = f_{V_2}(v_2|v_1) f_{V_1}(v_1)$$

we have

$$f_V(v) = \begin{cases} \lambda(v_1)\lambda(v_2)\ldots\lambda(v_n)e^{-\Lambda(v_n)}, & v_1 < v_2 < \ldots < v_n \\ 0, & \text{elsewhere} \end{cases}$$

3.1 Minimal Repair Process with Embedded Renewals

Many times it is impossible to reinstate the capability of a broken-down system to continue to operate only by minimal repair. Wear and tear, and other causes eventually force a system to a state of failure which in turn requires major maintenance. To model maintenance for such complexities, we make the assumption that a failure at time v will be eliminated by minimal repair and removed by a renewal with probabilities $1 - p(v)$ and $p(v)$, respectively, where $0 < p(v) < 1$.

Suppose that there exists two failure types that a system can experience. Minor failures (Type I) do not require major maintenance. They can be simply taken care of by minimal repairs. Failures that could lead to system breakdowns and requiring replacement are termed Type II failures.

Beichelt (1976) first recommended and applied this type of failure to maintenance modelling. A number of developments on this model were brought forth and applied by a number of specialists in this field. Types I and II failures are presumed to be independent of each other. Times required to repair and replace systems are taken to be insignificantly small. These systems resume operation immediately, and the process continues indefinitely.

Policy: A system's breakdown occurs with some probability. The type I failures with likelihood $1-p(v)$ and type II failures with probability $p(v)$ are eliminated using minimal repair and substitutions respectively.

The time intervals linking two adjacent renewals, referred to as "renewal cycles", generate a RPP. The renewal cycles at different time points are assumed to be statistically equivalent. Suppose Z is the random time interval between replacements, and $G(v)$ denotes its distribution function, where

$$G(v) = Pr(Z \leq v)$$

If a failure occurs at time v, the Type I and Type II intensity functions would be $\lambda Z(v) = p(v)\lambda(v)$, $\lambda Z(v) = (1 - p(v))\lambda(v)$, respectively. The survival function corresponding to $\lambda(v)$ is given by

$$G(v) = e^{\int_0^v \lambda(l)p(l)\,dl}$$

and

$$G(v) = 1 - e^{-p(v)\Lambda(v)}$$

The above is generally applicable to all choices of the underlying lifetime distributions. The survival function, $G(v)$, of a system is only applicable if $p(v) \equiv 1$, which further implies

$$G(v) = e^{-\Lambda(v)} = F(v)$$

Suppose the number of minimal repairs between two adjacent renewal time intervals is N. Then

$$Pr(N = 0) = \int_0^\infty p(v)f(v)\,dv$$

where

$$f(v) = f_{V_1}(v) = \lambda(v)e^{-\int_0^v \lambda(l)p(l)\,dl}, v \geq 0$$

For $n \geq 1$

$$Pr(N = n) = \int_0^\infty \int_0^{v_{n+1}} \dots \int_0^{v_2} \prod [\bar{p}(v_i)\lambda(v_i)\,dv_i] p(v_{n+1})f(v_{n+1})\,dv_{n+1}$$

The probability distribution of $Pr(N = n)$ becomes

$$Pr(N = n) = \frac{1}{n!}\int_0^\infty \left(\int_0^v \bar{(p)}(l)\lambda(l) \right)^n p(v)f(v)\,dv$$

and

$$E(N) = \sum_{n=1}^{\infty} n \cdot Pr(N = n) = \int_0^{\infty} \Lambda(v)G(v) - 1$$

For a special case when $p(v) = p, p > 0$

$$\bar{G}(v) = 1 - G(v) = [\bar{F}(v)]^p \text{ and } E(N) = {}^{1-p}/_{p}$$

If $p(v) > 0$, then N is characterized by a geometric distribution with parameter p such that

$$E(N) = \frac{1-p}{p}, \bar{G}(v) = [\bar{F}_Z(v)]^p, v \geq 0$$

Suppose N_v is the random frequency of minimal repairs in the region $(0, min\{N, v\}$ and

$$r_v(n) = Pr(N_v = n | Z = v), n = 0, 1, 2, \ldots$$

Then

$$r_v(n) = \lim_{\Lambda(v) \to 0} \frac{Pr(N_v = n \cap v \leq Z \leq v + \Lambda(v))}{Pr(v \leq Z \leq v + \Lambda(v))} \tag{3.4}$$

Dividing equation 3.4 by $\Lambda(v)$ and letting $\Lambda(v) \to 0$ gives

$$r_v(n) = \frac{1}{n!}\Big(\int_0^z \bar{p}(l)\lambda(l)\, dl\Big)^n e^{-\int_0^v \bar{p}(l)\lambda(l)\, dl}$$

Under a special condition $Z = v$, the random variable N_v is seen to be Poisson distributed with expected value

$$E(N_v | Z = v) = \int_0^v \bar{p}(l)\lambda(l)\, dl$$

3.2 Time-Dependent Replacement Strategies

Systems wear out due to constant usage, and at a certain point they have to be maintained. Preventive maintenance is appropriate if the system is aging. We generally assume that $F(v)$ is increasing failure rate (IFR) in $[0, \infty)$. In the following sections,we will investigate a number of basic maintenance policies that are designed for simple systems but can be applicable to the cost economical maintenance of large systems. Under these policies, the assumptions that all times required to repair and replace are insignificantly small and replaced systems are "as good as new" are still applicable.

3.2.1 Age Replacement Policy

Policy: A system that breaks down undergoes an emergency replacement. If it has survived κ time units after the former substitution, it will be preventively replaced.

In a renewal cycle, the random maintenance cost C and the random length Z of a renewal cycle are

$$C = \begin{cases} c_e & \text{with probability } F(\kappa) \\ c_p & \text{with probability } \bar{F}(\kappa) \end{cases}$$

and

$$Z = \begin{cases} L & \text{if } L \leq \kappa) \\ \kappa & \text{if } L > \kappa \end{cases}$$

The expected maintenance cost $E(M)$ within a replacement cycle is

$$E(M) = c_e F(\kappa) + c_p \bar{F}(\kappa)$$

and the expected replacement cycle length is

$$E(V) = \int_0^\kappa \bar{F}(v)\, dv$$

Using the renewal reward theorem leads to the analogous maintenance cost rate

$$K(\kappa) = \frac{c_e F(\kappa) + c_p \bar{F}(\kappa)}{\int_0^\kappa \bar{F}(v)\, dv}$$

An optimum preventive replacement interval satisfies the first order differentiation of the maintenance cost rate above at $\kappa = 0$:

$$\lambda(\kappa) \int_0^\kappa \bar{F}(v)\,dv - F(\kappa) = \frac{c_p}{c_e - c_p} \tag{3.5}$$

An adequate condition for the existance of an optimal solution κ^* of equation 3.5 is that the rate, $\lambda(v)$, is surely increasing and $\lambda(\infty) > \frac{c_e}{\mu(c_e - c_p)}$. This is fulfilled if $\lambda(v) \to \infty$ for $v \to \infty$. The analogous minimal maintenance cost rate is

$$K(\kappa^*) = (c_e - c_p)\lambda(\kappa^*) \tag{3.6}$$

It has been found that equation 3.5 does not only apply to an optimal preventive maintenance interval in conjunction with the age replacement policy, but also with the optimal maintenance of a two-unit-system regarding its expected lifetime.

Consider a system consisting of two similar elements. At first, one of them is not operating. If the working unit breaks down, it undergoes maintenance and the standby element is put into operation. The difference under this case compared to the other policies is that the times needed for maintenance are no longer assumed to be insignificantly small. The working unit is fixed using the age replacement policy with interval of length κ. To eliminate system breakdowns altogether, a modification of the age replacement policy was proposed.

Policy: If the substitute unit is still under maintenance when the active unit goes through maintenance, then the preventive replacement of the active unit is postponed until the replacement of the other unit is completed. If the active unit fails during the maintenance of the substitute unit, then the system fails.

Let $F(v)$ be the cumulative density function of the lifespan L of a working unit and $F_e(v)$ and $F_p(v)$ be the distribution functions of the random times L_e and L_p to perform emergency and preventive replacements, respectively, with non-negaive and finite average values $\mu_e = E(L_e)$, $\mu_p = E(L_p)$ and $\mu = E(L)$.

Suppose $\mu_p \ll \mu$, $\mu_e \ll \mu$ and $\mu \lessapprox \kappa$, then $F_p(\kappa) \lessapprox 1$ and $F_e(p) \lessapprox 1$ so that

$$\alpha \approx \int_0^\infty \bar{F}_e(v)\,dF(v), \beta \approx \int_0^\infty \bar{F}_p(v)\,dF(v)$$

The approximative formula to determine the expected lifespan is then given by

$$E(L_U) \approx \tilde{E}(L_U) = (1+\alpha)\frac{\int_0^\kappa \bar{F}(v)\,dv}{\alpha F(\kappa) + \beta\bar{F}(\kappa)} \tag{3.7}$$

which suggests that a replacement interval of length $\kappa = \kappa^*$ being optimal satisfies equation 3.7 with $\alpha = c_e$ and $\beta = c_p$.

3.2.2 Block Replacement

Policy: A system gets preventively substituted with an unused one at set times $\kappa, 2\kappa, \ldots$. This translates to an emergency replacement.

The analogous maintenance cost rate using the renewal reward theorem is

$$K(\kappa) = \frac{c_p + c_e H(\kappa)}{\kappa}$$

where $H(\kappa)$ is the renewal function affiliated with the lifespan distribution function $F(v)$ and the optinum replacement interval meets the equation

$$\kappa \cdot h(\kappa) - H(\kappa) = \frac{c_p}{c_e}$$

From the point of view of its cost rate structure, the block replacement with MR strategy is a special case of the following general replacement strategy.

3.2.3 Economic Lifetime

At time points $\kappa, 2\kappa, \ldots$, a system is preventively replaced. A deterministic loss $\nu(v)$ comes from the time of replacement $n\kappa$ to $n\kappa + v, v < \kappa, n = 0, 1, 2, \ldots, \nu$. $\nu(v)$ stems from maintenance actions. After a number of minimal repairs, repairs are planned preventively or after breakdowns.

The analogous cost rate is

$$K(\kappa) = \frac{c_p + \nu(\kappa)}{\kappa}$$

The economic lifetime of the system corresponds to the replacement interval κ that minimizes $K(\kappa)$

$$\kappa\nu'(\kappa) - \nu(\kappa) = c_p \tag{3.8}$$

Removing every failure by a minimal repair is sometimes not feasible. Assuming that failure comes with some information about next feasible step, minimal repair or replacement, we can distinguish two failure types as already outlined in section 3.1.

3.2.4 Block Replacement with Minimal Repair

Policy: Systems get preventively substituted with unused ones at fixed time points $\kappa, 2\kappa, \ldots$. In between, minimal repairs are performed. The cost rate, where $\Lambda(\kappa)$ is the expected number of repairs in the renewal cycle of length κ, becomes

$$K(\kappa) = \frac{c_p + c_m\Lambda(\kappa)}{\kappa} \tag{3.9}$$

and the ideal replacement interval satisfies the solution

$$\kappa\lambda(\kappa) - \Lambda(\kappa) = \frac{c_p}{c_m} \tag{3.10}$$

Assuming that $\lambda(v)$ is only increasing in v and combining equations 3.9 and 3.10 above, the analogous cost rate is

$$K(\kappa^*) = c_m\lambda(\kappa)$$

A number of policies building up from the block replacement policy with minimal repair have been considered. A special case of the policy above allows for a system replacement at the initial failure which only happens beyond a fixed time κ, with minimal repairs removing failures occurring between replacements. Suppose that a system that has yet to fail in the region $[0, \kappa]$ has the average residual lifetime X_κ given by

$$\mu(\kappa) = E(X_\kappa) = e^{\Lambda(\kappa)} \int_\kappa^\infty e^{-\Lambda(\kappa)}\, dv$$

The corresponding cost rate under this emergency replacement is

$$K(\kappa) = \frac{c_e + c_m \Lambda(\kappa)}{\kappa + \mu(\kappa)}$$

And the optimal solution satisfies the equation

$$[\Lambda(\kappa) + \frac{c_e}{c_m} - 1]\mu(\kappa) = \kappa$$

The above policy considers the system lifetime. It may seem better or preferred than the above policy but tends to lead to higher replacement costs.

Another policy in the block replacement family replaces a system at the initial breakdown which arrives post a fixed time κ_1. Should there be an absence of breakdown in $[\kappa_1, \kappa_2], 0 < \kappa_1 < \kappa_2$, then a preventive replacement is performed at time κ_2. Breakdowns in $(0, \kappa_1)$ are eliminated by MR. The replacement cycle is

$$L = \kappa_1 + \min\{X_{\kappa_1}, \kappa_2 - \kappa_1\}$$

The expected cycle then becomes

$$E(L) = \kappa_1 + \mu(\kappa_1, \kappa_2) \text{ where } \mu(\kappa_1, \kappa_2) = \int_0^{\kappa_2 - \kappa_1} \bar{F}_{\kappa_1}(v)\, dv$$

The replacement cost rate is

$$K(\kappa_1, \kappa_2) = \frac{c_m \Lambda(\kappa_1) + c_e F_{\kappa_1}(\kappa_2 - \kappa_1) + c_p \overline{F}_{\kappa_1}(\kappa_2 - \kappa_1)}{\kappa_1 + \mu(\kappa_1, \kappa_2)}$$

The optimal vector (κ_1^*, κ_2^*) satisfies the first order partial differentiation at $\kappa_i, i = 1, 2$

$$\lambda(\kappa_2)\mu(\kappa_1, \kappa_2) + \overline{F}_{\kappa_1}(\kappa_2 - \kappa_1) - \frac{c_m}{(c_e - c_p)} = 0, (c_e - c_p)\kappa_1 \lambda(\kappa_2) + c_m \Lambda \kappa_1 = c_e - c_m$$

Another policy suggests that you remove the initial $n-1$ failures using minimal repairs, and an emergency replacement is applied for the n'th failure. With the recycle length V_n, the cost rate becomes

$$K(n) = \frac{c_e + (n-1)c_m}{E(V_n)}$$

where $E(V_n)$ is given by equation 3.3. The ideal smallest integer n satifies the following equation if we analyse the behavioral difference of $K(n) - K(n-1)$:

$$E(V_n) - [n - 1 + \frac{c_e}{c_m}]E(X_{n+1}) \geq 0$$

where $E(X_{n+1})$ is the average time linking the n'th and $(n+1)$'th minimal repairs.

3.2.5 Generalized Age Replacement

Minor and disastrous breakdowns are eliminated by MR and renewals respectively. κ time units post the former substitution, preventive replacements will be carried out. Suppose Z represents the random time joining adjacent disastrous breakdowns. The duration of a replacement cycle is

$$L(\kappa) = min(Z, \kappa) \text{ with } E(L(\kappa)) = \int_0^\kappa \bar{G}(v)\, dv$$

where

$$\bar{G}(v) = e^{-\int_0^v p(l)\lambda(l)\, dl}, v \geq 0$$

The corresponding cost rate is

$$K(\kappa) = \frac{c_m[\int_0^\kappa \bar{G}(v)\lambda(v)\, dv - G(\kappa)] + c_e G(\kappa) + c_p \bar{G}(\kappa)}{\int_0^\kappa \bar{G}(v)\, dv}$$

with the optimal replacement interval satisfying

$$(p(\kappa) + c)\lambda(\kappa) \int_0^\kappa \bar{G}(v)\, dv - c \int_0^\kappa \bar{G}(v)\lambda(v)\, dv - G(\kappa) = {}^{c_p}/_{c_m},$$

with $c = c_m(c_e - c_p - c_m)$.

Policy: The system is preventively substituted at set times $\kappa, 2\kappa, \ldots$. In between the points, the system is maintained analogous to breakdown type.

Suppose $H_m(v)$ and $H_e(v)$ are the expected number of minimal repairs and replacements respectively, that arrive in $(0, v]$. The corresponding cost rate is

$$K(\kappa) = \frac{1}{\kappa}[c_m H_m(\kappa) + c_e H_e(\kappa) + c_p]$$

The above policy is a generalization of the block replacement policy and block replacement policy with MR.

Another extension in this family of policies using breakdown type is that of performing preventive replacement after (n-1) successive minimal repairs at the point of the n'th breakdown.

Policy: The system is maintaned analogous to the failure type. A preventive replacement is carried out at a time point of the n'th failure after $(n-1)$ consecutive minimal repairs.

Suppose $Z(Y)$ denotes, as before, the time linking neighbouring disastrous breakdowns. The random cycle length is $L(n) = min\{Z, V_n\}$, and the cost rate is considered to be

$$K(n) = \frac{[c_m E(Y|Z < V_n) + c_e]P(Z < V_n) + [(n-1)c_m + c_p]P(Z \geq V_n)}{E(L(n))}$$

3.3 System Replacement Based on Cost Limits

Cost limit strategies are widely accepted because they are easy to use and have proved to be economical in formulating the maintenance of technical systems. Not similar to the most purely time-dependent maintenance policies, RCL policies acknowledge that repair costs are random variables.

3.3.1 Repair Cost Limit Replacement Strategy

Policy: After each and every system breakdown, the essential repair cost is determined. Thereafter, a broken-down system is substituted with a similar unused one provided the repair cost goes above a pre-determined threshold c. Else a MR is performed.

The theoretical foundation of this policy and modifications is the 2-failure type model developed under Section 3.1, titled "minimal repair process with embedded renewals".

Suppose a system that fails at time v has a random repair cost C_v. A breakdown is minor or disastrous if

$$C_v \leq c(v) \text{ or } C_v > c(v),$$

respectively.

If $R_v(l)$ is the total distribution function of C_v given by

$$R_v(l) = Pr(C_v \leq l),$$

then the corresponding probabilities of minor and disastrous breakdowns are

$$R_v(c(v)) = 1 - p(v) \text{ and } \bar{R}_v(c(v)) = p(v),$$

respectively.

The expected cost of a repair is

$$c_m = E(C|C \leq c) = \frac{1}{R(c)}[\int_0^c \bar{R(l)}\, dl - c\bar{R}(c)].$$

The analogous maintenance cost rate is

$$K(c) = \frac{\frac{1}{\bar{R}(c)} \int_0^c \bar{R}(l)\, dl + c_e - c}{\int_0^\infty [\bar{F}(v)]^{\bar{R}(c)}\, dv}$$

The optimum limit c has to be determined from the equation

$$\frac{dK(c)}{dc} = 0.$$

Explicit solutions exist in special cases.

3.3.2 Combined Age-Repair Cost Limit Replacement Strategy

The RCL strategy can be combined with the age replacement strategy.

Policy: A repair cost limit strategy is used to maintain a system. However, a preventive replacement is performed after an interval of length κ should there be no substitution within κ time units post the preceding substitution.

This is a special case of the generalized age replacement policy, and the corresponding cost rate is

$$K(c,\kappa) = \frac{\left\{\frac{1}{\bar{R}(c)}\int_0^c \bar{R}(l)\,dl + c_e - c\right\}G(\kappa) + c_p\bar{G}(\kappa)}{\int_0^\kappa \bar{G}(v)\,dv}$$

where $\bar{G}(v) = [\bar{F}(v)^{\bar{R}(c)}]$. A solution, which is a function of c is

$$\bar{R}(c)\lambda(\kappa)\int_0^\kappa \bar{G}(v)\,dv - G(\kappa) = \frac{\bar{R}(c)c_p}{(c_e - c_p - c)\bar{R}(c) + \int_0^c \bar{R}(l)\,dl}$$

Deciding to repair or completely substitute a broken-down system by an unused one solely depends on a single repair cost. It is known that repairs tend to be frequent as the system ages leading the repair cost rate, as well as the TRC to rise uncontrollably fast, even when overall costs do not surpass the RCL.

Thus, making replacement decisions by using the TRC $C(v)$ occurring in $[0, v]$ appears to be cost-efficient. This introduces us to the following replacement strategy.

3.3.3 Total Repair Cost Limit Replacement Strategy

Policy: The system is susbtituted with a similar unused one immediately after the TRC $C(v)$ goes beyond a preset level c.

Assuming that the stochastic process $\{C(v), v \geq 0\}$ has continuous sample paths, the respective maintenance cost rate is

$$K(c) = \frac{c_e + c}{E(X_C(c))}$$

where $X_C(c)$ is the first passage time of the stochastic process $\{C(v), v \geq 0\}$ with regard to threshold c.

The advantages of employing a TRC limit strategy over the RCL strategy are as follows:

1. Information about the system's underlying lifetime distribution is not necessary. This enables us to completely disregard the assumption of minimal repairs and only focus on the cost implications of failures.

2. In addition to the repair costs, the policy is able to handle costs due to monitoring, servicing, stock keeping, etc.

3. It is easy to implement this strategy because of its simple structure and the inputs it requires (the maintenance cost development in time). Due to the policy's simplicity and flexibility, it serves as a standard policy for setting up cost efficient repair/renewal cycles.

4. The running maintenance cost data are used to make decisions under the practical setting.

Instead of restricting the TRC $C(v)$, limiting functions of $C(v)$, for instance the total repair cost rate $\left\{\frac{C(v)}{v}, v \geq 0\right\}$, may be the groundwork for replacement decisions.

Policy: A system is substituted with a similar unused one immediately after the repair cost rate $\left\{R(v) = \frac{C(v)}{v}\right\}$ surpasses a given level r.

The analogous cost rate is

$$K(c) = r + \frac{c_p}{E(X_R(v))}$$

where $X_R(v)$ is the first passage time of the process $\{R(v), v > 0\}$ with regard to level r.

If the processes $\{C(v), v \geq 0\}$ and $\{R(v), v \geq 0\}$ have non-decreasing sample paths, then

$$C(v) \leq c \Leftrightarrow X_C(c) \geq v, R(v) \leq r \Leftrightarrow L_R(r) \geq v$$

The property of non-decreasing sample paths stated in equation 3.11 implies that if the total repair cost limit $C(v)$ observed at time v has not reached the pre-determined optimal repair cost, c, then it goes without proof that the first passage time of the TRC with respect to level c will be beyond time v.

Hence

$$F_{C(v)}(l) = Pr(C(v) \leq l) = Pr(X_C(l) \geq v) = 1 - F_{X_{C(l)}}(v)$$
$$F_{R(v)}(z) = Pr(R(v) \leq z) = Pr(X_R(z) \geq v) = 1 - F_{X_{R(z)}}(v)$$

so that

$$E(X_C(c)) = \int_0^\infty F_{C(v)}(c)\,dv$$
$$E(X_R(r)) = \int_0^\infty F_{R(v)}(r)\,dv,$$

3.3.4 Combined Age-Total Repair Cost Limit Replacement Strategy

Using the TRC limit policy to schedule replacements has its limitations as the policy does not explicitly consider the reliability requirements imposed on the system. These reliability aspects can be incorporated into the policy in a number of ways.

Limiting the replacement cycle length by a constant κ is one of the simplest ways of taking into account the reliability aspect of a system. By choosing a suitable κ, a number of severe breakdowns would be prevented with a given probability. This idea then introduces us to the next replacement policy.

Policy: The system is substituted with an unused one immediately when the TRC surpasses a preset threshold c or after κ units, whichever arrives first.

The cycle length $Y = min\{X_C(c), \kappa\}$ has a probability distribution function

$$Pr(Z \leq v) = \begin{cases} 1 - F_{C(v)}(c) & \text{for } 0 \leq v \leq \kappa \\ 1 & \kappa < v \end{cases}$$

and the expected cycle duration is

$$E(Z) = \int_0^\kappa F_{C(v)}(c)\,dv$$

The cost rate is given by

$$K(c,\kappa) = \frac{\int_0^c \bar{F}_{C(\kappa)}(l)\,dl - (c_e - c_p)F_{C(\kappa)}(c) + c_e}{\int_0^\kappa F_{C(v)}\,dv}$$

where κ is a fixed parameter since it arises from meeting reliability requirements. Numerical methods are usually applied to optimize $K(c,\kappa)$ with respect to c.

Chapter 4

LITERATURE REVIEW

For most equipment, maintenance costs rise with age and usage. The increasing significance of maintenance has resulted in increasing use of maintenance policies that refine system reliability, eliminating incidents of breakdowns and further decreasing costs of worsening systems.

An equipment's value gets used up over time and we refer to that as depreciation, which is the root where reliability maintenance stems from. Clapham (1957) identified that there is an optimal time for which the equipment should be kept before being removed from service. This will occur when the entirety of the capital depreciation payments and the maintenance costs have reached a minimum, and he referred to this as the 'economic life' of a system.

Clapham (1957) is one of the founding fathers of cost-based replacement policies for technical systems by deriving the economic life and its properties in a general case. He further suggested that if the unit is kept for less than this time, larger expenses will be incurred due to heavier depreciation charges; if it is kept for longer, the rising maintenance charges will cause extra loss costs.

Clapham (1957) found that there is a point at which it is on average cheaper to replace a system than to continue with repairs. He derived a replacement time that will minimize the total life cycle cost, while taking into account repair cost and the replacement cost. The results were applied to the underground diesel locomotives.

In this experiment, Clapham (1957) realised that it may, in practise, be worthwhile scrapping out the machine before the economic life, since the improved features of the newer model may well be worth the slight increase in cost. However, the fact that cost does not vary much according to how long the equipment is kept does not mean that the determination of the economic life is irrelevant. It is necessary to know the economic life fairly exactly to predict the replacement costs of any operating system.

Replacing a system that does not wear out does not seem cost-effective because the replaced system would have the same or a higher probability of failing compared to a system that was not replaced.

It has been many decades since the application of the repair limits, their effectiveness steming from the fundamentals that costs incured to fix an item broken down should not go beyond the actual value of the item, using the assumption that the system under operation breaks down, and the breakdown can be quantified using probability laws.

Whenever a system breaks down, a decision is undertaken to determine if it is cost-effective to substitute the system by an unused one; else the broken down system is repaired and returned to operation.

If the parts of a system break down and the repair/replacement methods applied reinstate the system to its operating status but the system's failure rate stays unaltered, then this leads us to the important concept called minimal repair.

Suppose the tyre of a car flats out and is substituted, the car's entire failure rate is in essence unaltered. As a system's failure rate rises with age and usage, it becomes rather too costly to sustain proper functioning by minimal repairs.

It is expected that some systems experience major failures which prompt replacement of the system. When that happens, one is faced with the following question: "when is it optimal to replace the entire system instead of performing minimal repairs?"

4.1 Age Preventive Replacement Policies

Possibly, the most general and widely held maintenance policy is the age-dependent preventive policy, and its studies are traced back to the work done by Senju (1957).

Under the policy formulated by Senju (1957), a system is on every occassion replaced at some pre-set age V, or repaired when it fails, until it receives perfect maintenance.

When systems reach a certain age observed from the previous replacement, they get replaced.

While along the introduction of age replacement policy, the concepts of minimal repairs emerged. The original work on a maintenance policy which takes into account replacements, as well as minimal repairs, was first introduced and discussed by Barlow and Hunter (1960).

As the MR concept became more established, another "periodic replacement under minimal repair" at failures policy was established. Under this policy, a unit is substituted by an unused one at preset times kV and breakdowns are removed by MR. This is referred to as Policy II.

Under the policy proposed by Barlow and Hunter (1960), at predetermined fixed time intervals $kV(k = 1, 2, 3, \ldots)$ a unit is substituted by unused one, and minimal repairs are carried out to remove failures. If the failure of a system only requires minimal repair, then this policy is equivalent to the "periodic replacement with minimal repairs" at failure.

The main fundamental of the minimal repair model was to determine the replacement age which yields the least "long-run expected cost per unit time of replacement and minimal repairs". Breakdowns that arrive prior to v_0 are removed by minimal repairs, and replacement with a similar unnused system at time v_0. The long run expected cost per unit time after employing the replacement age v_0 for the simple model is

$$K(v) = \frac{c_m E[N(v)] + c_p}{v}$$

where $E[N(v)]$ represents the average number of failures(minimal repairs) in the interval $(0, v]$.

For Weibull system lifetime, the minimum cost per unit time is

$$K(v_0) = c_m \lambda \beta \Big[\frac{c_p}{(\beta - 1)c_m}\Big]^{\frac{(\beta-1)}{\beta}}$$

Tahara and Nishida (1975) made extensions to the age replacement policy. Their suggestion was to substitute a system by a similar one when the initial failure occurs beyond

v_0 or when the overall working time gets to $V(0 \leq v_0 \leq V)$, whichever takes place first. Minimal repairs are carried out for failures that occur in $[0, v_0]$.

Nakagawa (1984) introduced the extension of the "age replacement policy that replaces a system at age V or at N, where N is the number of failures, whichever takes place first. Between replacements, only minimal repair is carried out". The policy has two decision variables, namely V and N. The policy has been found to be effective for analysing systems that deteriorate with age and use. Optimal time V^* and number N^* were determined under a numerical example with failure following a Weibull distribution.

Beichelt (1976) introduced a policy where the "lifetime of a system is presumed to be a random variable with an increasing failure rate distribution function". He identified that a system fails at time v according to one of the two types: Type I failures of this kind occur with probability $p(v)$ and are removed using minimal repair, while Type II failures have a $q(v) = 1 - p(v)$ likelihood of happening and are rectified by replacement of a failed system.

Block et al. (1993) established a "generalized age replacement policy and termed it the repair replacement policy". This policy preventively maintains a system after an amount of time has passed from the previous repair. If a complete repair was done in the previous failure, then this policy is no different to the age replacement policy, with only exception under the minimal repair.

The above policy seems very appropriate since at repair a plan to maintain the unit is already known and put in place. Moreover, it seems worthwhile to have some replacement policy rather than to do nothing for systems that are aging and minimal repairs have already been applied.

Sheu et al. (1995) made further extension to the age replacement policy by incorporating findings by Beichelt (1976). They assumed that "a unit of a system has two types of failures at age v, and is replaced either at the n'th type I failure with probability $p(v)$ or at the first type II failure with probability $q(v) = 1 - p(v)$, or at age V, whichever occurs first".

Wang and Pham (1999) extended the age replacement policy by introducing the "mixed age preventive maintenance policy". Their policy introduces two types of failures after the n'th imperfect repair. Type I failure refers to the absolute breakdowns which occur with probability $p(v)$ and dependent upon perfect repairs, while a type II refers to breakdowns that are minor and can be easily fixed with likelihood $q(v) = 1 - p(v)$ and subject to minimal repairs.

4.2 Periodic Preventive Maintenance Policies

The periodic replacement policy was introduced in earlier studies conducted, and the policy means that at fixed time intervals kV a unit is preventively maintained. This maintenance is not governed by a unit's past failures. The unit is repaired as failures occur where V is a constant.

Some researchers have come up with many remarkable and substantial results for variations of the age replacement model. Most of these variations incorporate the generalization and modifications of the basic MR model developed by Barlow and Hunter (1960) to fit more convincing situations.

Tango (1978) suggested that some used items can be of effective use without being thrown away to reduce wastage. Under this policy, he proposed that used items can be reused. "Failed units are replaced by new ones at times $kV(k = 1, 2, \ldots)$". In between, failures are fixed with used items.

Functioning systems are not always substituted by unused ones at fixed times. This allows for fairly new systems to continue in operation. Not different to v_0 in Berg and Epstein (1978), a time limit r is preset in the policy.

The modification of this policy then states that should a broken-down system's age be below the preset time limit r, it is substituted with an unused one; else substituted with a used one.

The policy by Tango (1978) is not the same as that from Berg and Epstein (1978) because it adjusts the ordinary periodic replacement by recognizing rules on the broken down systems rather than on operating ones.

Nakagawa (1981) initially presented three alterations to the "periodic replacement with MR at failure policy". The three modifications introduce a reference time V_0 and periodic time V^*.

"If failures occur prior to V_0, then minimal repair takes place. If the system is working at time V^*, then it is replaced at that time". If failures occur between V_0 and V^*, then the system is not substituted by a new one and will stay failed until V^*; the failed system is substituted by an unused system irrespective of how much it is needed until V^*; the broken down system is substituted by an unused one.

4.3 Repair Time Limit Models

Nakagawa and Osaki (1974) suggested the "repair time limit policy" which repairs a system at failure. Should the repair not be finished within the allocated time V, then the system is substituted with an unused one; else the repaired system continues to work, where V is the repair time limit.

There is an assumption that a system failure is instantly identified and the failed unit is immediately repaired or replaced upon failure. Also, the unit is presumed "as good as new" upon repair or replacement.

The total expected maintenance cost rate until time t is

$$K(T) = \frac{c_R \bar{R}(V) + \int_0^V \bar{R}(v)\, dc_R(v)}{\lambda + \int_0^V \bar{R}(v)\, dv} \text{ for } \lambda > 0$$

where λ is the mean failure time and c_R is the expected repair cost during $(0, v]$, which takes into account all costs experienced due to repair and system down cost during $(0, v]$.

Nguyen and Murthy (1980) studied the "repair time limit replacement policy with imperfect repair (the repaired system is not as good as new)" in which they identified two repair types; the perfect repair where the failed system is repaired on site and imperfect repair which may be time consuming. They got the ideal solution with the use of semi-markov processes theory.

Dohi et al. (1997) considered a "nonparametric approach to determining the optimal repair time limit". They studied the repair time limit model by taking into account the imperfect repair and lead time conditional to time constraints.

Koshimae et al. (1996) thought-out another "repair time limit policy". Under this policy, the repair of the failed system is initiated immediately. Should the repair be finished within the preset time limit v_0, the installation of the repaired system will be underway as soon as the repair is done.

Under their policy, should the repair time go beyond the preset time limit V^*, the broken down system is no longer used and the order of a spare unit is placed straightaway.

Makabe and Morimura (1963) recommended that a system be preventively maintained at the n'th breakdown. For the $(n-1)$ breakdowns, only minimal repair is carried out. This is the repair number counting policy with decision variable n. Assuming the lifetime distribution of a system $F(v)$ with finite mean μ and density $f(v)$, then $\bar{F}(v) = e^{[-\Lambda(v)]}$, where $\bar{F}(v) = 1 - F(v)$.

They assumed that the "failure rate $\lambda(v)$ is continuous, monotone increasing and remains unchanged by minimal repairs". Then, the expected cost rate is

$$K(n) = \frac{(n-1)c_m + c_p}{E(L_n)}$$

where $E(L_n)$ refers to equation 3.3.

Well along, Morimura (1970) built on this policy by incorporating another variable V which is defined as "the critical reference time and is a non-negative number". Before the n'th breakdown, minimal repairs are carried out. Should the n'th failure occur before an total working time V, it is rectified by MR with the next breakdown calling for a substitution.

However, if the n'th breakdown arrives post V, it requires substitution of the system by a new one. This policy brings together the concepts of determining the number of fixes and reporting the time passed.

Another alternative to the periodic replacement policy with minimal repair was proposed by Nakagawa (1986). He suggested that replacements be carried out at periodic times kV and minimal repairs be performed for other failures. If the sum of failures is at least the preset number N, then a replacement is carried out at the upcoming preset time; else there would be no maintenance performed.

Similar to the work originally done by Morimura (1970), Park (1979) investigated a policy along the idea of the repair number counting policy in which at the n'th failure a system is substituted with an unused one and for the first $(n-1)$ failures minimal repairs are carried out.

Park (1979) presented a "closed-form solution of the model for the case when the failure distribution of the system is Weibull". Park (1979) also compared the findings obtained from this solution with the optimal age replacement policy as stated by Barlow and Hunter (1960). His policy leads to lesser "long-run expected cost per unit time" than a fixed time replacement policy.

The analogous cost function is

$$K(n) = \frac{(n-1)c_m + c_p}{Ecycle}$$

where the value for $Ecycle$ equivalent to $E(L_n)$ relies on the failure distribution of the system and on the number of breakdowns n prior to substitution.

Phelps (1981) showed that the count policy by Park (1979) leads to less number of failures prior to replacement than in the fixed time policy. That said, all the results are shown numerically for the Weibull distribution without any mathematical proof.

In a later work, Park (1983) considered an optimal "minimal repair cost limit policy". It minimally repairs a broken down system (or substitutes with an unused one) should the estimated repair cost be below (above) the constant cost limit c. At this point, the structure of the model by Park (1983) is the same as that presented by Beichelt (1976). A Weibull distribution of the time to failure and a negative exponential distribution of estimated repair cost were assumed for analytical evaluation.

Later on, Park (1985) considered a variant of his earlier repair cost limit replacement policy by assuming an "exponentially declining repair cost limit, negative exponentially distributed repair cost and a Weibull time to failure distribution".

He presented a mathematical model to assess a "cost limit replacement policy for a system that follows a general time to failure distribution". If a system breaks down and has to be repaired, examination will take place and thereafter a repair cost is determined. Should the approximated cost be below the exponentially declining RCL, then a minimal repair will be carried out.

Yeh (1988) and Stadje and Zuckerman (1991) later investigated the "repair number counting policy" assuming the intervals do not increase while the lengths of the repair increase in a number of ways.

Muth (1977) examined the replacement policy identical to the "reference time plan" of the extended policy by Morimura (1970). This policy minimally repaires a system up to time V and substitutes with an unused one at initial failure after time V.

He assumed that a system's mean residual life function is surely non-increasing after some age V. Under this condition, which is termed "positive aging", the system worsens and finally gets to a point where it's not cost-effective and it is practical to simply fix the system after failure. He solved for V using a simple calculus approach.

The optimal cost function is

$$K(v) = \frac{c_m E[N(v)] + c_p}{v + \mu(v)}$$

where $\mu(v)$ is the "mean residual life of the system" at time t and $\mu(v) = E[\kappa - v|\kappa > v]$.

In general, repair number counting policy is practical when there is no recording of the system's operating time or it prolongs time and it becomes costly and tedious to substitute a system in operation with a new one.

In his work, Phelps (1981) compared the "periodic replacement with minimal repair at failure policy" by Barlow and Hunter (1960) to those policies of the reference time policy by Muth (1977) and the "repair number counting policy" by Park (1979) given an IFR distribution.

Phelps (1981) showed that "using the long-run expected cost per unit time, the reference time policy by Muth (1977) which replaces after the failure that happens after reference time V is the optimal of the three policies, while the number counting policy is more cost-efficient compared to the periodic replacement with minimal repair at failure policy proposed by Barlow and Hunter (1960)".

Park (1979) suggested that since the "optimal age for the model by Muth (1977) is not easy to obtain, an estimate generated from his model can be a good replacement and still qualify as a good replacement policy".

4.4 Repair Cost Limit Policies

The failure limit rule is considered the optimal replacement rule in terms of "average long run maintenance cost rate". This implies that it is "optimal to replace with a new system either at failure or when the state variable has surpassed some preset value, whichever occurs first". There exists two kinds of repair limit strategies in existing literature, namely the RCL policy and repair time limit policy.

When a system breaks down the repair cost is calculated and repair is carried out if the approximated cost falls below the pre-set limit; else, the system is substituted by a new one. In literature, this is the RCL policy which was first introduced by Garnet and Nonat (1963), and Drinkwater and Hastings (1967).

Drinkwater and Hastings (1967) derived the repair limit value, which they defined as the "average future cost per vehicle-year" when the broken-down vehicle is fixed. They found the repair limit value to be no different to "the cost at which the failed vehicle is replaced and a new one is installed".

The repair limit can be explained in monetary terms as the limit on the value of money you are willing to spend to repair a system. The repair limits values are heavily based on the type, as well as the age of the system. Therefore, repair costs were assumed to depende on the age of the vehicle.

In their paper, Drinkwater and Hastings (1967) recommended three methods for optimizing the repair-limit policy. Optimization can be done by simulation, or by using particular frequency distribution functions which are found to represent the repair cost data, and finally by using dynamic programming methods.

In his later work, Hastings (1969) further assumed "year by year failure rates and costs formulating the problem as a Markov decision process". He also formulated and discussed cases of finite and infinite planning horizons on the replacement model and the issue of discounted and undiscounted costs.

Yun and Bai (1987) generalized the policy by Drinkwater and Hastings (1967) and proposed a policy in which "when a unit fails, the repair cost is determined and repair is done if the predicted cost falls below the preset limit c". Else, the unit is substituted by a new one. The cost function for the model by Yun and Bai (1987) is

$$K(c) = \frac{c_r \bar{R}(c) + c_m R(c)}{\bar{R}(c) \int_0^\infty e^{-\Lambda(v)\bar{R}(c)\,dv}}$$

Kapur et al. (1989) proposed three extended models that generalizes model due to Park (1983) and Park (1987). Under all models the MR cost is approximated by inspection. "The system begins to operate at time zero ($v = 0$) and is replaced by a new one at the n'th failure or when the estimated repair cost goes beyond the preset repair limit".

In Model a and Model b, "the failed system is replaced at the n'th type I failure; type II failure or when the repair cost due to type I failure exceeds the limit. Type I failures are minimal failures and occur with constant probability and Model c, the failed system is replaced at n'th type I failure, type II failure occurring before n'th type I failure or when the repair cost due to type I failure exceeds the cost limit".

The costs involved in the models presented in Kapur et al. (1989), take into account, other than the standard MR cost at breakdown c_m, and cost of expected replacement c_p, and the cost due to inspection c_s.

Kapur et al. (1989) formulated the "long-run expected cost per unit time for each of the models a, and b", respectively, as

$$K_a(n, c) = \frac{c_p + (c_s + c_m R(c)) \sum_{n=0}^{k-2} R(c)^n}{\sum_{n=0}^{k-1} R(c)^n E(Z_n)} \text{ for k = 2,3,...}$$

$$K_b(n, c) = \frac{c_p + \bar{p}(c_s + c_m R(c)) \sum_{n=0}^{k-2} \bar{p} R(c)^n}{\sum_{n=0}^{k-1} \int_0^\infty \frac{\bar{p}\Lambda(v)^n}{n!} \bar{F}(v)\,dt} \text{ for k = 2,3,...}$$

where $R(c)$ is the cumulative distribution function of the MR cost and $\bar{p}$ is the probability of type I failure.

Beichelt and Krober (1992) considered maintenance policy with time-dependent repair cost limits. Aging systems usually see a rise in their average breakdown numbers, as well as their average repair costs. They assumed that for a given $c_1 > c_2$, the repair cost limit, $c(v)$, be given by

$$c(v) = \begin{cases} c_1 & \text{for } 0 \leq v \leq V \\ c_2 & v > V \end{cases}$$

They also assumed that the cost of minimal repairs depends on the repair cost limit. The probability to carry out a replacement at failure time v, is

$$Pr(C > c(v)) = p(v) = \begin{cases} \bar{R}(c_1) & \text{for } 0 \leq v \leq V \\ \bar{R}(c_2) & v > V \end{cases}$$

The cycle length Z has survival function

$$\bar{G}(v) = e^{-\int_0^v p(l)\lambda(l)\,dl} = \begin{cases} \bar{F}(v)^{\bar{R}(c_1)} & \text{for } 0 \leq v \leq V \\ \frac{\bar{F}(v)^{\bar{R}(c_1)}}{\bar{F}(v)^{\bar{R}(c_2)}}\bar{F}(v)^{\bar{R}(c_2)} & v < V \end{cases}$$

The maintenance cost rate is

$$K(c_1, c_2, V) = \frac{G(V)(\frac{\int_0^{c_1}\bar{R}(l)\,dl}{\bar{R}(c_1)} - c_1) + \bar{G}(V)(\frac{\int_0^{c_2}\bar{R}(l)\,dl}{\bar{R}(c_2)}) + c_r}{\int_0^{\infty}\bar{G}(v)\,dv}$$

Mamabolo and Beichelt (2004) further investigated the impact of the "continuous age-dependent repair cost limits on the maintenance cost rate". Contrary to the model by Beichelt and Krober (1992) above, they assumed that the repair cost limit does not have an influence or effect on the cost of minimal repair.

They considered the hyperbolic and linearly decreasing repair cost limits as follows: for hyperbolical repair cost limit with $0 \leq c \leq c_r$ they assumed

$$c(v) = \begin{cases} c_r & \text{for } 0 \leq v \leq \frac{d}{c_r - c} \\ c + \frac{d}{v} & \frac{d}{c_r - c} \leq v < \infty \end{cases}$$

4.4.1 Total Repair Cost Limit Models

A disadvantage to the repair cost limit is that desicion making solely depends on the cost of one repair. Situtations characterised by numerous repairs that have costs that do not exceed the preset limit do not have a direct influence on the replacement time, although a replacement might be justified when using the repair cost rate.

Financial savings have been obsereved when the replacement decision takes into account the past information of the repair process. Due to this limitation, Beichelt (1982) looked at the RCL policy and used the "repair cost rate (repair cost per unit time) as a benchmark of replacement or repair". A system is substituted by an unused one immediately when the repair cost rate goes beyond a preset level; else it is fixed. Under the policy by Beichelt (1982), "the replacement intervals are assumed to be independently and identically distributed random variables".

Some improved policies that take into account the overall repair history have been suggested. Beichelt (2001a) proposed a TRC limit policy where a system is substituted with an unused one immediately after the TRC goes beyond a preset limit.

Beichelt et al. (2006) considered two replacement policies for systems which, during their useful life, are subject to deterioration. The first policy they considered stated that "after a failure, the repair cost is estimated. If the repair cost exceeds a given limit, the system is replaced by a new one."

In their paper, Beichelt et al. (2006) investigated the effect of applying "time-dependent repair cost limits on the long run maintenance cost rate". In the examples they used, they restricted their work by assuming that a system has a Weibull lifetime distribution and power distributed repair costs. They found that using the "optimal-age dependent decreasing repair cost limit function is more economical than using the constant repair cost limits", and reduction of the maintenance cost between 5% and 10% can be expected.

The second policy they considered is the policy which replaces a system immediately when the total repair cost limit policy goes beyond a given limit. This considered policy, with a Frechet distribution function, was compared to the economic lifetime policy developed by Clapham (1957) which is based on the average repair cost development. They found that the TRC limit is better and can lead to maintenance cost savings of up to 20%.

Probably the first comparison of the TRC limit policy to the EL policy was done by Beichelt (2001a). He investigated a cost function that follows a Weibull distribution and found that taking into account the past information of the repair process leads to beneficial cost savings.

Later that year, Beichelt (2001b) further investigated the long-run total maintenance cost rate and found that it is preferable to the famous economic lifetime approach. He provided examples that assumed that a system has the following lifetime distributions, namely power, Raleigh and Maxwell. His analysis was in favor of organizing replacements based on the TRC limit policy over the EL policy.

Continuing on his earlier work, Beichelt (2008) further tested the superiority of using the TRC limit policy over the EL policy by using examples with cost functions that had the power and Frechét distributions. His findings were that the TRC limit policy produces cost savings that average between 4% and 30%.

Using data of all repair cost to decide whether to replace or repair seems to be very important. Chang et al. (2010) established the optimal number of minimal repairs prior to a substitution that minimizes the TRC limit policy. A system with a Weibull lifetime distribution was considered for numerical purposes.

Chang et al. (2010) found that "the minimum expected cost per unit time will be decreased when the probability of minimal repair is increased; the optimal number of minimal repairs before a substitution will be lengthy when the probability of minimal repair is raised; and if the optimal number n^* of minimal repairs before replacement and the probability of minimal repair are increased, then the minimun expected cost per unit time $C(n^*)$ will be reduced".

Chien et al. (2009) looked at the 'age-replacement model with minimal repair based on a cumulative repair cost limit and random lead time for replacement delivery". The model showed the "cost of storing a spare as well as the cost of system downtime. The minimum-cost policy time was derived and illustrated with a numerical example". They verified that "if the optimal replacement age V^* is longer, then the minimum expected cost per unit time $C(V^*)$ will be reduced".

Later on, Chien et al. (2010) presented the "age-replacement model with minimal repair, which is based on a cumulative repair cost limit including the random lead time for replacement". As they ouline in their paper, "the long-run expected cost per unit time for operating the system were developed incorporating costs due to holding a spare unit, shortage, minimal repairs, and different kinds of replacement state".

Their findings were that "the minimum expected cost per unit time will be reduced, when the probability of minimal repair (type I failure) is greater; the minimum expected cost per unit time will be reduced, when the probability of age-dependent minimal repair is greater; and that if the optimal replacement age V^* is longer, then the minimum expected cost per unit time $C(V^*)$ will be reduced".

Sheu et al. (2010) considered "a periodic replacement model with minimal repair based on a cumulative repair-cost limit. Under this policy, the system is expectedly substituted at the n'th type I failure, or at the k'th type I failure ($k < n$) at which the total repair cost goes beyond the preset limit, or at any type II failure, whichever takes place first". In their analysis, they concluded that the optimal TRC limit will be higher when the probability of MR is raised.

Lai (2007) also investigated the "periodical replacement model based on cumulative repair-cost limit". He found that the "long-run expected cost per unit time is formulated and the optimal period V^* minimizing that cost is also verified to be finite and unique under some specific conditions". Using the Weibull distributed cost function, he also came to the same conclusion seen by Chien et al. (2009) that "if the optimal V^* is longer under the fixed optimal cost, the long-run expected cost per unit time will be smaller".

Continuing on his previous work in Lai (2007), Lai (2012) later published a paper on "a periodical replacement model combining the concept of cumulative repair cost limit for a two-unit system with failure rate interaction is presented". In this model, "on any occasion unit 1 fails, unit 2 experiences a certain amount of damage that leads to an increase in the failure rate of unit 2 of a certain degree".

He found that with age, the two units' failure rates also increase when the two units do not have any failure rate interaction. "When unit 1 fails, the repair cost is estimated and is included to the total repair cost. If the total repair cost is below the predetermined limit, unit 1 is fixed by minimal repair. Else, the system is preventively replaced by a new one".

Lai (2012) studied the "periodical replacement policy and cumulative repair cost limit and derived the long-run expected cost per unit time by incorporating relative costs as a criterion of optimality". He further discussed the ideal time V^*.

Going forward, we will only focus on the TRC limit policies and our optimality criterion will be minimizing the maintenance cost rate under a boundless time horizon.

Chapter 5

TOTAL REPAIR COST LIMIT REPLACEMENT POLICIES

The RCL policy states that the system is substituted by a similar unused one if cost of repair goes beyond the pre-determined threshold of $C(v)$; else a MR is performed. As previously mentioned, this is a major short-coming of the RCL policy as it bases the judgement to fix or substitute on a single repair cost.

Aging systems tend to be characterized by frequent repairs. These repairs cause the repair cost rate, as well as the total repair costs to increase rapidly regardless of the cost of total repairs falling below the repair cost limit. As such, replacement decisions should incorporate the complete history of the repair process of the machine.

The limitations of the RCL policy call for the use of better optimal repair cost limit policies. Making replacement decisions based on the TRC $C(v)$ in $[0, v]$, where v is the time of the previous replacement has been found to be cost-efficient.

Comparing the TRC limit to the widely used RCL policy, it has been found to be advantageous as information about the system's lifetime distribution is not necessary. With that said, one can also disregard the assumption of minimal repairs and focus on the cost implications of failures as these play a crucial role.

Beichelt (2001b) found that the lifetime distribution concept is "not applicable when compiling a basic strategy for organizing cost-optimal replacement cycles of large systems like trucks, etc". Minor costs that contribute to the existence and operation of the system like servicing, etc., can be incorporated into the actual repair costs, thus placing favour on the use of the TRC limit policy.

The assumptions underlying the TRC limit policy, see Beichelt (2001b), are as follows:

- The planning horizon is boundless.
- The times of replacement are considered to be insignificantly small.
- The durations of the replacement cycles are independent, identically distributed random variables with finite mean.
- $C(v)$ does not incorporate costs of replacement.

Amongst many other flexibilities of the TRC limit policy, one can introduce different functionals of $C(v)$ like the repair cost rate $R(v) = C(v)/v, v > 0$ and make replacement decisions on the basis of these functionals.

The use of a functional like the repair cost rate changes the TRC limit policy, which replaces a system with a similar one immediately after the total cost $C(v)$ surpasses a predetermined level c, to a better and widely used policy. This policy states that a system is substituted with a similar unused one immediately after the repair cost rate $R(v) = \frac{C(v)}{v}$ goes beyond a preset level r.

Assuming that $X_C(l)$ and $X_R(y)$ are the first passage times of the processes $\{C(v), v \geq 0\}$ and $\{R(v), v > 0\}$ regarding levels l and y respectively, the TRC limit policy consists of finding the ideal cost $c = c^*$ with respect to the cost rate below

$$K_{15}(c) = \frac{c + c_e}{E(X_C(c))}$$

The total repair cost rate (TRCR) policy will be compared to the EL policy. The EL policy seeks to determine the ideal replacement interval of length κ corresponding to the cost rate

$$K_5(c) = \frac{c + c_e}{E(X_C(c))}$$

Furthermore, the TRCR policy is given by

$$K_{16}(c) = r + \frac{c_e}{E(X_R(r))}$$

where c_e is the cost of a replacement. It should be noted that the length of the replacement cycles are independent random variables.

If the processes $\{C(v), v \geq 0\}$ and $\{R(v), v > 0\}$ have non decreasing paths, then it can be deduced that

$$C(v) \leq c \Leftrightarrow X_C(c) \geq v, R(v) \leq r \Leftrightarrow X_R(r) \geq v,$$

respectively.

The relation above simply means that if the total repair cost $C(v)$ at time v is still below the ideal RCL c, then the first passage time to level c is beyond time v.

Different stochastic models for $C(v)$ can be used in reliability maintenance to analyse the TRC of a system, and this leads us to the following section about the special case of the Brownian motion (BM).

5.1 Special Case of a Brownian Motion

A continuous-time stochastic process $\{A(v), v \geq 0\}$ is termed a BM process if it consists of the following properties, see Beichelt (2006):

1. $A(0) = 0$.
2. $\{A(v), v \geq 0\}$ has homogeneous and independent increments.
3. $A(v)$ has a normal distribution with

$$E(A(v)) = 0, Var(A(v)) = \sigma^2 v, v > 0$$

BM functionals can be applied to maintenance to analyse the TRC arising over a specified period. These functionals can also be used to characterize the TRCR policy. The analysis under these policies is not solely based on sample paths of the process but on the trend functions and mean initial passage times of the policies.

A stochastic process $\{Q(v), v \geq 0\}$ is a BM with drift with properties, see Beichelt (2006):

1. $Q(0) = 0$.
2. $\{Q(v), v \geq 0\}$ has homogeneous and independent increments.
3. Every increment $Q(v) - Q(u)$ has a normal distribution with mean $(v - u)\mu$ and variance $|v - u|\,\sigma^2$

This process with drift has structure $Q(v) = \mu v + A(v)$, where $\{A(v), v \geq 0\}$ is the BM with constant μ.

Consider a total repair cost $C(v)$ that has a structure $C(v) = c_0\left[e^{Q(v)-1}\right]$, where $\{Q(v), v \geq 0\}$ is the BM process. The parameter μ is positive drift and σ^2 volatility. It can be stated that $A(v)$ is a BM process with $Var(A(1)) = \sigma^2, 0 < c_0 \leq c_e$, and $X_Q(d)$ is the first passage time of $\{Q(v), v \geq 0\}$ regarding threshold $d : X_Q(d) = \min v, Q(v) = d$.

It should be noted that $c_0\left[e^{Q(v)-1}\right] = c \Longleftrightarrow Q(v) = ln\left(\frac{c+c_0}{c_0}\right)$. This then leads to the TRC limit policy being

$$K_{15}(c) = \frac{c_e + c}{ln\left(\frac{c+c_0}{c_0}\right)}\mu$$

The ideal level c^* satisfies the equation

$$ln\left(\frac{c + c_0}{c_0}\right) = \frac{c_e + c}{c_0 + c}$$

and under the TRC limit policy, the optimal minimal cost is $K_{15}(c^*) = (c^* + c_0)\mu$.

The EL policy has the minimal cost rate $K_5(\kappa) = \frac{c_0\left[e^{(\mu+\sigma^2/2)\kappa}-1\right]+c_e}{\kappa}$ and $\kappa = \kappa^*$ satisfies the equation

$$\frac{c_0}{c_e}\left[(\mu + \sigma^2/2)\kappa - 1\right]e^{(\mu+\sigma^2/2)\kappa} = \frac{c_e - c_0}{c}$$

Chapter 6

COMPUTATIONAL ANALYSIS

In this chapter, we contrast in a number of illustrations the cost rates stemming from TRC limit policy and TRCR policy to the cost rates that come from functions assumed to be in continuous-time under the EL approach.

The stochastic process $\{C(v), v \geq 0\}$ with different models that can take any probability distribution will be considered. The optimization code used to calculate the optimal replacement costs and intervals which minimize the expected cost rates for the different policies is presented in the Appendix.

Consider the cost function $C(v) = A_2v^2 + A_3v^3$, where A_2 and A_3 are exponentialy distributed independent random variables with parameters λ_2 and λ_3, respectively. The distribution of the cost function is given by convolution:

$$F_{C(v)}(v) = \int_0^x \left(1 - e^{-\dfrac{\lambda_2(x-u)}{v^2}}\right) \frac{\lambda_3}{v^3} e^{\frac{\lambda_3 u}{v^3}}\, du$$

which, when simplified, gives

$$F_{C(v)}(v) = 1 - e^{\frac{\lambda_3 u}{v^3}} - \frac{\lambda_3}{\lambda_3 - \lambda_2 v}\left(e^{-\dfrac{\lambda_2 x}{v^2}} - e^{-\dfrac{\lambda_3 x}{v^3}}\right)$$

If $c_e = 50$, $\lambda_2 = 2$ and $\lambda_3 = 3$, then the corresponding rate for the total repair cost policy is

$$K_{15}(c) = \frac{c + 50}{\displaystyle\int_0^\infty \left[1 - e^{\frac{-\lambda_3 c}{v^3}} - \frac{3}{3 - 2v}\left(e^{\frac{-\lambda_2 c}{v^2}} - e^{\frac{-\lambda_3 c}{v^3}}\right)\right] dv}$$

Minimizing the above equation using a mathematical software, MATLAB, produces the following results: $c^* = 31$ and $K_{15}(c^*) = 16.5172$.

The TRC limit policy will be compared to the EL policy given by the following mean cost

$$E(C(v)) = E(A_2)v + E(A_3)v^2 = \frac{v^2}{\lambda_2} + \frac{v^3}{\lambda_3} = \frac{v^2}{2} + \frac{v^3}{3}$$

and the average cost rate with replacement interval of duration τ given by

$$K_5(\tau) = \frac{20 + \frac{\tau^2}{2} + \frac{\tau^3}{3}}{\tau}$$

The EL policy replaces a failed system by an unused one when the maintenance cost rate is the least possible. The optimal replacement interval for the EL policy is $\tau^* = 4$ and $K_5(\tau^*) = 19.8333$.

When we compare the EL and TRC limit policies, we see that the latter policy reduces the cost rate by 16.72%. The EL policy is inefficient because it does not factor in the stand-alone variations of maintenance cost rates of each system. The advantages of the TRC limit policy are also evident in the efficiency of the policy when compared to the EL policy, as seen in Beichelt (2001b).

TRC limit and TRCR policies seem similar and obvious, but the results show a different case. The repair cost rate for the cost function $C(v)$ is given by $R(v) = A_2v + A_3v^2$.

Assuming the same distribution structure as above, the distribution function then becomes

$$R_{R(v)}(y) = r + \frac{50}{\int_0^\infty \left[1 - e^{\frac{-\lambda_3 r}{v^2}} - \frac{3}{3-2v}\left(e^{\frac{-\lambda_2 r}{v}} - e^{\frac{-\lambda_3 r}{v^2}}\right)\right] dv}$$

The TRCR policy has the optimal values $r^* = 6$ and $K_{16}(r^*) = 15.7469$.

To overcome the shortcomings of the EL policy, Beichelt (1982) proposed the use of the maintenance cost rate policy. Applying this policy further decreases the cost rate by 4.96% compared to the TRC limit policy.

One would then be interested in how the policies do when the parameters are kept unchanged while the emergency cost is increased by a constant factor; when the parameters of the exponentially distributed cost function are varied by a constant factor while keeping the emergency cost constant, and finally investigating a special case of a BM.

6.1 Varying Costs by Constant Factor

Under the TRC limit policy, we increase the emergency cost by a constant factor $(2, 4, 6, 8)$ during each iteration to find the optimal costs with their corresponding cost rates. The optimal cost rate gradually increases as the cost is increased, and starts to spike after a factor of 8 is applied. This shows that systems with higher emergency costs generally have very high optimal costs with corresponding very high cost rates.

When the increase applied in the TRC limit policy is also applied in the EL policy, the cost rate increases drastically after a factor of 8. The increase observed is much more than that seen in the TRCR policy. This still confirms the savings that the TRC limit policy has when compared to the EL policy.

Under the TRCR policy, the cost rate also increases but the increase is not as significant as in the EL policy and the TRC limit policy. Significant savings are still offered by the TRCR policy even when the costs are increased by a costant factor.

The following table shows the optimal cost rates with regard to an increase in emergency cost by a constant factor:

Increase Cost by	**EL Policy**	**TRC Limit Policy**	**TRCR Policy**
2	30.8333	25.4403	24.1978
4	75.3333	60.9374	57.6697
6	242.5000	191.5730	180.1818
8	955.0000	737.2085	689.3385

Table 6.1: Increasing Costs by a Constant Factor

6.2 Varying the Input Parameters

When the parameters of the cost function, λ_2 and λ_3, are increased by a constant factor, the cost rate decreases under all the investigated policies. Under the total repair cost limit policy, as the parameters are increased, the difference in cost rates observed is not that significant. As a result, the optimal cost converges to some optimal c^* since the difference becomes minimal as the parameters increase.

Under the EL policy, even though the cost rate decreases as the parameters increase, the optimal length does not eventually converge as in the TRC limit policy. Instead, it appears to be constant after every second optimal time interval. This leads us to an observation that the optimal time interval seems to have a slowly increasing trend.

The TRCR policy has similar findings to the TRC limit policy. The cost rate decreases and the decrease is not significant as the parameters increase. The TRCR policy also reaches an optimal cost of 2 as the parameters increase.

The following table shows the optimal cost rates with regard to an increase in input parameters by a constant factor:

Increase Parameters by	**EL Policy**	**TRC Limit Policy**	**TRCR Policy**
2	15.4167	12.7202	12.1284
4	12.0833	9.8285	9.3800
6	10.4484	8.4641	8.0182
8	9.4167	7.6172	7.2398

TABLE 6.2: Increasing Parameters by a Constant Factor

Savings under this special case across the different policies are not as high as when you increase costs by a constant factor.

6.3 Special Case of a Brownian Motion

Suppose $\mu = 0.10$ and $\sigma^2 = 0.02$. Then $K_{15}(c^*) = 5.7708$ and $K_5(c^*) = 6.1328$.

As seen above, using the TRC limit policy rather than the EL policy results in a savings of 5.90%. When the volatility gets increased by a constant factor while keeping the same assumptions, the following results were obtained:

Increase Parameters by	**EL Policy**	**TRC Limit Policy**	**Savings (%)**
2	6.6897	5.7708	13.74
3	7.2458	5.7708	20.36
4	7.8046	5.7708	26.06
5	8.3700	5.7708	31.05

TABLE 6.3: Increasing Variance by a Constant Factor

From Table 6.3 above, it can be seen that as the volatility of the cumulative repair cost process $\{C(v), v \geq 0\}$ increases, the cost-effectiveness of using TRC limit policy rather than the EL policy is improved.

Lower maintenance cost rate are observed when using TRC limit policy over the EL policy. The relative gain for using the TRC limit policy over the EL policy as the variance increases is represented graphically.

Figure 6.1 below indicates that the ideal cost is easily affected by the change in the variance of the distribution. As the variance increases, the EL policy becomes worse off compared to the TRC limit policy, both under the case of the BM. For the variance greater or equal to 0.04, the gain in using the TRC limit policy over the EL policy is more than 10%.

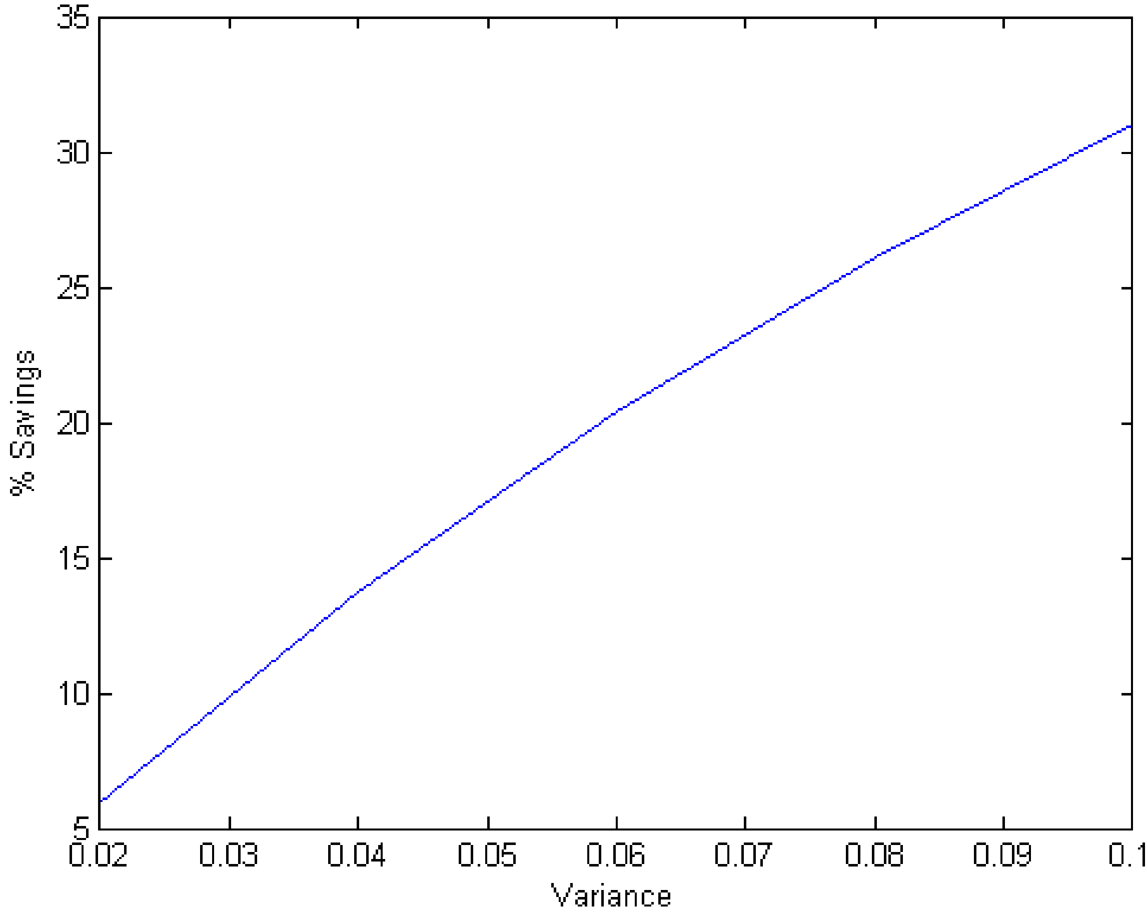

FIGURE 6.1: Relative Gain when the Variance Increases

Overall, the special cases (increasing the costs by a constant parameter, increasing the parameters by a constant factor and the special case of the BM) considered and explained above have demonstrated that the TRC limit policy is superior than the use of the EL policy to schedule repairs or replacements. The above examples show that using all past information of the repair process leads to greater financial savings. The TRCR policy provides further financial savings across all the cases investigated.

Chapter 7

CONCLUSION

Different types of policies under age, block and cost categories were studied and analysed, with emphasis on the repair cost policies. Repair cost limit replacement policies are feasible and cost efficient maintenance policies for complicated systems.

The replacement policy we looked at solely limits the total repair cost within a replacement cycle. This policy does not require knowledge of any underlying lifetime distribution. The required cost data are usually available. These properties facilitate the practical use for planning substitutions of large systems, as well as technical systems subjected to wear and tear.

In this research report, the TRC limit policy is analysed and compared to the EL policy, which uses the same input data as the TRC limit policy. Examples analysed show strong reasoning to the advantage of scheduling replacements using the TRC limit policy as opposed to the EL policy. Nonetheless, there is still room for more theoretical and practical work to be done to further explore the relationship between the TRC limit policy and the EL policy.

Additional studies are carried out to understand the advantages of using the TRCR policy over the TRC limit policy. Our analysis shows that the TRCR policy is more advantageous and leads to additional cost savings of approximately 5%.

The research done leans towards using two or more policy decision variables in a policy. A number of numerical special cases are investigated for all three policies. These cases and their findings are:

1. Varying emergency costs by a constant factor - leads to increased optimal costs. Savings are still observed as in the findings above.

2. Varying input parameters of the cost function - cost rate increases under all policies.

3. Special case of a BM - TRC limit policy leads to savings compared to EL policy. When volatility is increased by a constant factor, the savings of using the TRC limit policy are very high. After a cetrain level of variation, the gain is more than 10%.

The combination of these established maintenance policies, as well as their numerical applications, are regarded as the greatest contribution of this research report. Lastly, some further research on the combined age-total repair cost limit policy would be of great value.

Appendix A

EL Source Code

Source code used for investigating the economic lifetime policy is given below:

```
clc;
clear all
format long g;

ce = 50; %initial cost
l2 = 2;
l3 = 3;
BCost = [];
BParam = [];

for i=1:1:10000
    K(i,1) = i;
    K(i,2) = (ce + i^2/l2 + (1/l3)*i^3)/i;
%     if i>2 && K(i-1,2)>K(i,2)
%         break
%     end
end

K5_50 = min(K(:,2))
BCost= K;
BParam = K;
```

```
%% Varying Costs by Constant Factor %%
for a=2:2:8
    ce = a*ce;
    for i=1:1:10000
        K(i,1) = i;
        K(i,2) = (ce + i^2/l2 + (1/l3)*i^3)/i;
%         if i>2 && K(i-1,2)>K(i,2)
%             break
%         end
    end
    BCost = [BCost K];
end

K5_C2 = min(BCost(:,4))
K5_C4 = min(BCost(:,6))
K5_C6 = min(BCost(:,8))
K5_C8 = min(BCost(:,10))
%K5_C10 = min(BCost(:,12))

%% Varying Parameters by constant factor %%
for a=2:2:20
    ce = 50;
    l2 = 2*a;
    l3 = 3*a;
    for i=1:1:10000
        K(i,1) = i;
        K(i,2) = (ce + i^2/l2 + (1/l3)*i^3)/i;
        %         if i>2 && K(i-1,2)>K(i,2)
        %             break
        %         end
    end
    BParam = [BParam K];
end
```

```
K5_P1 = min(BParam(:,2))
K5_P2 = min(BParam(:,4))
K5_P4 = min(BParam(:,6))
K5_P6 = min(BParam(:,8))
K5_P8 = min(BParam(:,10))
```

Appendix B

TRC Limit Source Code

Source code used for investigating the TRC limit policy is given below:

```
clc;
clear all
format long g;

ce = 50; %initial cost
l2 = 2;
l3 = 3;
BCost = [];
BParam = [];

for c=1:1:15000
    fun = @(t,c) 1-exp(-l3*c./(t.^3))-(l3./(l3-l2*t).*(exp(-l2*c./(t.^2))))...
      + (l3./(l3-l2*t).*(exp(-l3*c./(t.^3))));
    q = integral(@(t)fun(t,c),0,Inf);
    K(c,1) = c;
    K(c,2) = (c+ce)./q;
%     if c>2 && K(c-1,2)>K(c,2)
%         break
%     end
end

K15_50 = min(K(:,2))
BCost= K;
```

```
BParam = K;

%% Varying Costs by Constant Factor %%
for a=2:2:8
    ce = a*ce;
for c=1:1:15000
    fun = @(t,c) 1-exp(-l3*c./(t.^3))-(l3./(l3-l2*t).*(exp(-l2*c./(t.^2))))...
      + (l3./(l3-l2*t).*(exp(-l3*c./(t.^3))));
    q = integral(@(t)fun(t,c),0,Inf);
    K(c,1) = c;
    K(c,2) = (c+ce)./q;
%     if c>12000 && K(c-1,2)>K(c,2)
%         break
%     end
end
 BCost = [BCost K];
end

K15_C2 = min(BCost(:,4))
K15_C4 = min(BCost(:,6))
K15_C6 = min(BCost(:,8))
K15_C8 = min(BCost(:,10))
%K15_C10 = min(BCost(:,12))

% % x=[31 60 231 1326 10246];
% % y = [16.5172 25 60 191 737];
% % figure
% % plot(x,y)

%% Varying Parameters by constant factor %%
for a=2:2:8
    ce = 50;
    l2 = 2*a;
    l3 = 3*a;
```

```
for c=1:1:15000
    fun = @(t,c) 1-exp(-l3*c./(t.^3))-(l3./(l3-l2*t).*(exp(-l2*c./(t.^2))))...
     + (l3./(l3-l2*t).*(exp(-l3*c./(t.^3))));
    q = integral(@(t)fun(t,c),0,Inf);
    K(c,1) = c;
    K(c,2) = (c+ce)./q;
%     if c>2 && K(c-1,2)>K(c,2)
%         break
%     end
end
 BParam = [BParam K];
end
K15_P1 = min(BParam(:,2))
K15_P2 = min(BParam(:,4))
K15_P4 = min(BParam(:,6))
K15_P6 = min(BParam(:,8))
K15_P8 = min(BParam(:,10))
```

Appendix C

TRCR Source Code

Source code used for investigating the TRCR policy is given below:

```
clc;
clear all
format long g;

ce = 50; %initial cost
l2 = 2;
l3 = 3;
BCost = [];
BParam = [];

for r=1:1:5000
    fun = @(t,r) 1-exp(-l3*r./(t.^2))-(l3./(l3-l2*t).*(exp(-l2*r./(t)))) ...
      + (l3./(l3-l2*t).*(exp(-l3*r./(t.^2))));
    q = integral(@(t)fun(t,r),0,Inf);
    K(r,1) = r;
    K(r,2) = r + (ce./q);
%     if c>2 && K(c-1,2)>K(c,2)
%         break
%     end
end

K16_50 = min(K(:,2))
```

```
BCost= K;
BParam = K;

%% Varying Costs by Constant Factor %%
for a=2:2:8
    ce = a*ce;
for r=1:1:5000
    fun = @(t,r) 1-exp(-l3*r./(t.^2))-(l3./(l3-l2*t).*(exp(-l2*r./(t)))) ...
      + (l3./(l3-l2*t).*(exp(-l3*r./(t.^2))));
    q = integral(@(t)fun(t,r),0,Inf);
    K(r,1) = r;
    K(r,2) = r + (ce./q);
end
 BCost = [BCost K];
end

K16_C2 = min(BCost(:,4))
K16_C4 = min(BCost(:,6))
K16_C6 = min(BCost(:,8))
K16_C8 = min(BCost(:,10))
%K16_C10 = min(BCost(:,12));

%% Varying Parameters by constant factor %%
for a=2:2:20
    ce = 50;
    l2 = 2*a;
    l3 = 3*a;
for r=1:1:5000
    fun = @(t,r) 1-exp(-l3*r./(t.^2))-(l3./(l3-l2*t).*(exp(-l2*r./(t))))...
      + (l3./(l3-l2*t).*(exp(-l3*r./(t.^2))));
    q = integral(@(t)fun(t,r),0,Inf);
    K(r,1) = r;
    K(r,2) = r + (ce./q);
%     if c>2 && K(c-1,2)>K(c,2)
```

```
%          break
%      end
end
 BParam = [BParam K];
end

K16_P1 = min(BParam(:,2))
K16_P2 = min(BParam(:,4))
K16_P4 = min(BParam(:,6))
K16_P6 = min(BParam(:,8))
K16_P8 = min(BParam(:,10))
```

Appendix D

BM Source Code

Source code used for investigating the special case of a BM is given below:

```
clear all;
clc;
format long g;

l2 = 2;
l3 = 3;
mu = 0.10;
sigma = 0.02;
ce = 50;
c0 = 10;

BCost = [];

for c=1:1:30
    K(c,1) = c;
    K(c,2) = mu*(c+ce)./log((c+c0)/c0);
end
K15 = min(K(:,2))
BCost= K;

clc;
clear all;
```

```
format long g;


l2 = 2;
l3 = 3;
mu = 0.10;
sigma = 0.02; %this is the variance
ce = 50;
c0 = 10;
BParam = [];

%% Investigate increasing variance


for tau=1:1:25
    K(tau,1) = tau;
    K(tau,2) = (c0*(exp(tau.*(mu+(sigma./2))) - 1) + ce) ./tau;
end
K5 = min(K(:,2))
BParam = K;


for a=2:1:5
    sig = 0.02;
sigma = a*sig;
for tau=1:1:25
    K(tau,1) = tau;
    K(tau,2) = (c0*(exp(tau.*(mu+(sigma./2))) - 1) + ce) ./tau;
end
    BParam = [BParam K];
end


K5_P2 = min(BParam(:,4))
K5_P3 = min(BParam(:,6))
K5_P4 = min(BParam(:,8))
K5_P5 = min(BParam(:,10))
```

```
x = [0.02 0.04 0.06 0.08 0.10];
y = [5.90 13.74 20.36 26.06 31.05];
figure
plot(x,y)
xlabel('Variance')
ylabel('% Savings')
```

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