

Forward and Inverse problems of Hermitian systems in \mathbb{C}^2 .¹

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Abstract

In this thesis, the forward and inverse Spectral Theory for first order Hermitian systems with complex potentials and periodic boundary conditions are studied. The aim of this work is to prove two inverse periodicity Theorems and two uniqueness results for determinants of quasiperiodic boundary value problems.

Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

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at Johannesburg, South Africa.

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1

Introduction

“Some mathematician, I believe, has said that true pleasure lies not in the discovery of truth, but in the search for it.”

– Tolstoy

1.1 Basic Overview

This thesis focuses on forward and inverse spectral problems for the following system of differential equations,

$$\frac{dy_2}{dx} + q_1(x)y_1 + q(x)y_2 = \lambda y_1, \quad (1.1)$$

$$\frac{dy_1}{dx} + q^*(x)y_1 + q_2(x)y_2 = \lambda y_2, \quad (1.2)$$

where q is a complex Lebesgue integrable function, q_1 and q_2 are real Lebesgue integrable functions, $*$ is the conjugate transpose, $\lambda \in \mathbb{C}$ and $x \in [0, \pi]$. Differential equations of the general form above are referred to as Hermitian systems, while more specifically referred to as canonical or Dirac systems. The problem of finding non-trivial solutions of equations (1.1) and (1.2) that satisfy some prescribed condition

on their boundary together with nonzero values of λ , are referred to as the Hermitian system eigenvalue problem.

The primary objective of this thesis is the proof of Borg's periodicity theorems for Hermitian systems, see Theorems 8.4 and 8.5, as well as the uniqueness results contained in Theorems 7.5 and 9.3.

The secondary objective of this work is the indexing results of Chapter 5, in which the indexing of eigenvalues of various boundary value problems are studied.

The ideas in every chapter of this thesis (with the exception of Chapter 2), originate from three original research manuscripts written by the author, with the guidance of his supervisors. Manuscripts [20] and [21] have been submitted for publication in international peer-reviewed journals and are under review and [19] has been accepted for publication.

1.2 Chapter Structure

In Chapter 2, the history of Hermitian systems is discussed and the similarities and differences between the Sturm-Liouville and Hermitian system problems are examined. Applications are also considered wherein the 1-Dimensional Dirac operator is derived from the Dirac equation.

Chapter 3, introduces the main definitions and concepts that will be used in the remainder of the thesis. In particular, the Pauli basis is used to represent the fundamental solutions of the Hermitian system as a linear combination of discriminants of various quasi periodic problems.

In Chapter 4, Floquet theory is studied which is used to define the discriminants of the periodic and antiperiodic eigenvalues problems, these problems are generalised to consider the so called Clifford quasi periodic problems together with the σ_i discriminants. The analytic structures of some of these discriminants are examined.

Chapter 5 uses Prüfer angle arguments to investigate the interlacing of various eigenvalues of Dirichlet and periodic boundary value problems.

Solution asymptotics are obtained for both the absolutely continuous canonical system, as well as the integrable Hermitian system, in Chapter 6. These asymptotics are then used in Chapters 7, 8 and 9 in order to solve various inverse problems. In particular, Chapter 7 contains a uniqueness result for the absolutely continuous canonical system, Chapter 8 deals with some existence results for periodic and anti periodic problems relating to their discriminants of Hermitian systems with integrable potentials, and Chapter 9 is a uniqueness result for a Clifford quasiperiodic problem with an integrable potential.

Finally, Chapter 10 indicates some future possibilities for the direction of research in this thesis.

2

History

“Mathematics, rightly viewed, possesses not only truth, but supreme beauty; a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.”

– Bertrand Russel, *A History of Western Philosophy*

Spectral theory of self adjoint operators has enjoyed a rich history, in part due to its application in the study of numerous physical systems and theories. Spectral theory can be seen as the extension of the finite dimensional theory of systems of linear equations to infinite dimensions. Through his study of integral equations, David Hilbert, together with his doctoral student, Erhard Schmidt, published in 1904-1910, a series of papers, [43, 85, 86, 87], in which they laid down the foundational terminology and theorems of the field. They constructed the framework of the Hilbert space and proved the spectral theorem for infinite dimensional symmetric forms.

The next breakthrough occurred in 1927-1929 with John von Neumann axiomatically describing the abstract concept of a linear operator on a Hilbert space, [73, 74, 75].

This development was, in part, due to the creation of quantum mechanics in 1925-1926 by Werner Heisenberg and Erwin Schrödinger, where observed quantities are represented as self-adjoint operators, [10, 11]. For an extensive survey on the theory for linear operators see, [27, 28, 29]. Specific mention should be made of the spectral theorem, which states that any self-adjoint operator, T , defined on a suitable domain space, has a unique canonical representation in terms of its action on eigenfunctions, which span the space. This relation is concisely represented as

$$T = \int_{\sigma(T)} \lambda dE_\lambda, \quad (2.1)$$

where $\sigma(T)$ is the **spectrum** of T and E_λ is the projection onto the eigenspace of λ , known as the **resolution of the identity**. For an introduction to linear operators and spectral theory see, [53].

Floquet Theory was introduced in 1883 by Gaston Floquet, [33]. Consider the linear differential equation,

$$y' = A(x)y \quad x \in [0, \pi], \quad (2.2)$$

where A is $n \times n$ matrix function. A fundamental matrix solution, \mathbb{Y} , of equation (2.2) is a square matrix whose columns are linearly independent solutions of the differential equation. Floquet showed the existence of a representation called the Floquet normal form. That is, for each matrix Λ satisfying $\mathbb{Y}(0)e^{\Lambda\pi} = \mathbb{Y}(\pi)$, there exists a π -periodic matrix, P , such that

$$\mathbb{Y}(x) = P(x)e^{\Lambda x}. \quad (2.3)$$

This representation is important in the study of dynamical systems, [94], and Hill's equation, in that it is used to investigate the stability of the solutions of equation (2.2).

Since there are many similarities between Sturm-Liouville theory and the study of 2×2 Hermitian systems, it will be useful to compare and contrast them. The Sturm-Liouville operator relating to the Sturm-Liouville eigenvalue problem has been studied extensively with great success. The Sturm-Liouville eigenvalue problem is given by

$$\frac{d}{dx} \left[p \frac{dy}{dx} \right] - qy = \lambda y, \quad (2.4)$$

where p and q are Lebesgue square integrable on compact sets. Classical results have been contributed by J. Sturm in 1836, who proved the Sturm Oscillation theorems, [90, 91]. Furthermore, J. Liouville studied the convergence properties of the eigenfunction expansions, [61, 62], which added to the groundwork for the Spectral Theorem. For a survey of the area see, [18, 54, 72], and the Masters dissertation, [82]. The Sturm-Liouville operator arises from the separation of variables in partial differential equations governing certain physical processes, such as the quantum oscillator, [13], vibrating elastic string/membrane, [77, p. 159], in the latter, angular symmetry of the 2 dimensional vibrating membrane is assumed. For a survey of applications see, [64].

One of the first periodic problems relating to Sturm-Liouville equations was introduced by George William Hill in 1886, [44]. The equation is appropriately called Hill's equation. It is given by equation (2.4) where $p = 1$ and q is π -periodic. This equation was originally used by Hill to study the stability of lunar bodies, but has since then found a wide range of applications. An important special case of Hill's equation is the Mathieu equation given by

$$y'' + (a - 2q \cos 2x)y = 0, \quad (2.5)$$

for $a \in \mathbb{R}$. The Mathieu equation is an interesting special case because it possesses a spectrum with nowhere vanishing instability intervals, [49]. For a comprehensive catalogue of Sturm-Liouville equations see, [31].

Forward and inverse problems for Hermitian systems have been studied extensively in the last century, see [57, 60, 83, 84]. Periodic problems for Hermitian systems with integrable potentials have received consistent attention, [12]. This is especially true recently for the Ambarzumyan and Borg uniqueness-type results, [15, 16, 17, 40, 41, 56, 101]. It should be noted that these results pertain mainly to regular and singular inverse problems with $2n \times 2n$ potentials with matrix valued entries. These classes of problems are not as developed as inverse problems for canonical 2×2 systems, in which many inverse results pertaining to uniqueness have been

investigated, [35, 36, 37, 96]. Never the less, 2×2 Hermitian systems are an active area of study in physics communities in which they are referred to as the Ablowitz-Kaup-Newell-Segur equation, [2, 3, 40, 96], and the Zakarov-Shabat equation, [8, 22, 39]. This alludes to a link between Hermitian systems and completely integrable systems which is being actively investigated, [4, 14, 23, 26].

A particularly important application of Hermitian systems is the 1-Dimensional Dirac system. This comes from the separation of variables of the Dirac equation in quantum electrodynamics under certain symmetry assumptions, see [24, 79, 80, 97].

The results in this thesis were first proved for the Sturm-Liouville eigenvalue problem by Ambarzumyan, [1], Borg, [9], and later Hochstadt, [46, 47]. In particular, in Borg's paper he proved an existence result for periodic potentials which has largely gone unstudied for the Hermitian system. Hermitian systems with absolutely continuous potentials are reducible to Sturm-Liouville equations. This is not possible in general for Hermitian systems with potentials integrable on compact sets. These systems present challenges that make the results of this work a non-trivial extension of the aforementioned works. These challenges are:

- a) Existing asymptotics for Hermitian systems do not allow the generality of potential considered in this thesis. Difficulties in deriving such asymptotics have been discussed in the remark of [89, pp. 1464].
- b) Hermitian systems are spectrally identical to those obtained by certain gauge transformations, thus uniqueness results are not possible in general. These transformations have been investigated in [60] and [17].

Quasiperiodic eigenvalue problems come in two main varieties, the first being where the potential is considered to be quasiperiodic, [5, 6, 92, 30], and the second in which the boundary conditions are considered to be quasiperiodic, [68, 69]. The forward and inverse problems that are considered in this work are of the latter variety. Eigenvalue problems with quasiperiodic boundary conditions were first extensively studied in the book, [55], where boundary conditions of the form $y(\pi) = \omega y(0)$,

$|\omega| = 1$, $\arg(\omega) \neq k\pi$, were considered. A consequence of this is that the periodic and antiperiodic boundary value problems are special cases of the quasiperiodic problem. More recently there has been work carried out by, [20, 21, 71, 101]. Quasiperiodic boundary value problems are also referred to as ω -twisted boundary value problems, as seen in the book, [12, pp. 21].

3

Preliminaries

“A mathematician is a device for turning coffee into theorems”

– Paul Erdős

Let

$$\ell Y := JY' + QY, \tag{3.1}$$

and consider the differential equation

$$\ell Y = \lambda Y \tag{3.2}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} q_1 & q \\ q^* & q_2 \end{pmatrix}, \tag{3.3}$$

q is complex valued, q_1 and q_2 are real and Q is π -periodic and integrable on $[0, \pi)$.

Let $Y_i = \begin{pmatrix} y_{i1}(z) \\ y_{i2}(z) \end{pmatrix}$, $i = 1, 2$, be solutions of (3.2) with initial values given by

$$[Y_1(0) \ Y_2(0)] = \mathbb{I}, \tag{3.4}$$

where \mathbb{I} is the 2×2 identity matrix. Set $\mathbb{Y} = [Y_1 \ Y_2]$. We recall that the Pauli matrices are given by $\sigma_0 = \mathbb{I}$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.5)$$

Here $\sigma_2 = iJ$. The set of Pauli matrices form a basis for the complex general linear group, $GL(2, \mathbb{C})$. Furthermore, $GL(2, \mathbb{C})$ is a vector space with inner and outer products defined by

$$\langle H, F \rangle_{Lin} = Tr\{H^T \overline{F}\}, \quad \text{for } H, F \in GL(2, \mathbb{C}), \quad (3.6)$$

and

$$\sigma_i \sigma_j = i \epsilon_{ijk} \sigma_k + \delta_{ij} \mathbb{I}, \quad \text{for } i, j = 1, 2, 3, k \neq i, j,$$

where ϵ_{ijk} and δ_{ij} are the Levi-Cevita and Kronecker delta symbols, respectively. Note that ϵ_{ijk} is 1 if the number of permutations of (i, j, k) into $(1, 2, 3)$ is even, -1 if the number of permutations of (i, j, k) into $(1, 2, 3)$ is odd, and zero if any of the indices are repeated. For any $H = \sum_{i=0}^3 a_i \sigma_i \in GL(2, \mathbb{C})$, the determinant is given by

$$\det(H) = a_0^2 - a_1^2 - a_2^2 - a_3^2. \quad (3.7)$$

Define the σ_i -symmetric and σ_i -skewsymmetric subspaces $S_+^{\sigma_i}$ and $S_-^{\sigma_i}$ of $GL(2, \mathbb{C})$ as

$$S_+^{\sigma_i} = \{x \in GL(2, \mathbb{C}) : x \sigma_i = \sigma_i x\} \quad \text{and} \quad S_-^{\sigma_i} = \{x \in GL(2, \mathbb{C}) : x \sigma_i = -\sigma_i x\}. \quad (3.8)$$

We have the product space $GL(2, \mathbb{C}) = S_-^{\sigma_i} \oplus S_+^{\sigma_i}$.

The J -decomposition of Q in $GL(2, \mathbb{C}) = S_-^J \oplus S_+^J$ is

$$Q = Q_1 + Q_2, \quad (3.9)$$

where

$$Q_1 = \Re(q) \sigma_1 + \frac{1}{2}(q_1 - q_2) \sigma_3 \quad \text{and} \quad Q_2 = \frac{1}{2}(q_1 + q_2) \sigma_0 + \Im(q) \sigma_2. \quad (3.10)$$

We see that Q_1 and Q_2 are the projections of Q onto S_-^J and S_+^J , respectively. We note for later that $Q_2 J = J Q_2$ and $J Q_2$ and $\int_0^z J Q_2$ commute. A potential Q is said to be in canonical form if $Q_2 = 0$, that is, $Q = Q_1$.

Since \mathbb{Y} is a fundamental system for (3.2), $\mathbb{Y} \in GL(2, \mathbb{C})$. Furthermore setting

$$\begin{aligned}\Delta^{\mathbb{I}} &= y_{11}(\pi) + y_{22}(\pi), & \nabla^{\mathbb{I}} &= y_{11}(\pi) - y_{22}(\pi), \\ \Delta^J &= y_{21}(\pi) - y_{12}(\pi), & \nabla^J &= y_{21}(\pi) + y_{12}(\pi),\end{aligned}\tag{3.11}$$

$\mathbb{Y}(\pi)$ may be represented as

$$\mathbb{Y}(\pi) = \frac{1}{2} \begin{pmatrix} \Delta^{\mathbb{I}} + \nabla^{\mathbb{I}} & \Delta^J + \nabla^J \\ \nabla^J - \Delta^J & \Delta^{\mathbb{I}} - \nabla^{\mathbb{I}} \end{pmatrix}.\tag{3.12}$$

Thus expressed in terms of the Pauli basis for $GL(2, \mathbb{C})$ we have

$$\mathbb{Y}(\pi) = \frac{1}{2}(\Delta^{\mathbb{I}}\mathbb{I} + \Delta^J J + \nabla^{\mathbb{I}}\sigma_3 + \nabla^J\sigma_1).\tag{3.13}$$

A direct computation using (3.7) and (3.13) with $\det(\mathbb{Y}) = 1$ gives

$$(\Delta^{\mathbb{I}})^2 + (\Delta^J)^2 - (\nabla^{\mathbb{I}})^2 - (\nabla^J)^2 = 4.\tag{3.14}$$

Similar relations to (3.11)-(3.14) for $\mathbb{Y}(\frac{\pi}{2})$ and $\mathbb{Y}(-\frac{\pi}{2})$ may be obtained, the symbols contained in these relations are denoted by the subscript $+$ and $-$, respectively. Let $\mathbb{H} = \mathcal{L}_2(0, \pi) \times \mathcal{L}_2(0, \pi)$ be the Hilbert space with inner product

$$\langle Y, Z \rangle = \int_0^\pi Y(t)^T \bar{Z}(t) dt \quad \text{for } Y, Z \in \mathbb{H},$$

and norm $\|Y\|_2^2 := \langle Y, Y \rangle$. The Wronskian of $Y, Z \in \mathbb{H}$ is $\text{Wron}[Y, Z] = Y^T J Z$. We consider the following operator eigenvalue problems

$$L_i Y = \lambda Y, \quad i = 1, \dots, 8,\tag{3.15}$$

where $L_i = \ell|_{\mathcal{D}(L_i)}$ with

$$\mathcal{D}(L_i) = \left\{ Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : y_1, y_2 \in \text{AC}, \ell Y \in \mathbb{H}, Y \text{ obeys } (BC_i) \right\}.\tag{3.16}$$

Here conditions (BC_i) are

$$Y(0) = Y(\pi), \quad (BC_1), \quad (3.17)$$

$$Y(0) = -Y(\pi), \quad (BC_2), \quad (3.18)$$

$$y_1(0) = y_1(\pi) = 0, \quad (BC_3), \quad (3.19)$$

$$y_2(0) = y_2(\pi) = 0, \quad (BC_4), \quad (3.20)$$

$$JY(0) = Y(\pi), \quad (BC_5), \quad (3.21)$$

$$-JY(0) = Y(\pi), \quad (BC_6), \quad (3.22)$$

$$y_1(0) = y_2(\pi) = 0, \quad (BC_7), \quad (3.23)$$

$$y_2(0) = y_1(\pi) = 0, \quad (BC_8). \quad (3.24)$$

4

Characteristic Determinants

“Pure mathematics is, in its way, the poetry of logical ideas.”

– Albert Einstein

4.1 Introduction

In this chapter, the analytic structure of the discriminants, $\Delta^J(\lambda)$ and $\Delta^{\mathbb{I}}(\lambda)$, of the boundary value problems, (3.15), (3.17) and (3.15), (3.21), are studied. Boundary problems of this type will be called *Clifford quasiperiodic* boundary problems. What distinguishes them from the quasiperiodic boundary problems in Chapter 2, is that instead of the parameter ω in $y(\pi) = \omega y(0)$, $|\omega| = 1$, $\arg(\omega) \neq k\pi$, being a complex number of unit length, the parameter ω is a basis element of the Clifford algebra, $GL(2, \mathbb{C})$. That is, they are 2×2 matrices which span $GL(2, \mathbb{C})$.

Classical results similar to the ones in this chapter on the Hill's equation can be found in Magnus and Winkler, [98], and on the Canonical system in Levitan and Sargsjan, [60]. An up to date survey of Floquet theory for differential equations can be found in Brown, Eastham and Schmidt, [12, pages 1-29].

4.2 Floquet Theory

We now show that there is a, possibly multivalued, function $\rho(\lambda)$ so that for each λ there is a nontrivial solution Y of (3.2) on \mathbb{R} with

$$Y(z + \pi, \lambda) = \rho(\lambda)Y(z, \lambda), \quad \text{for all } z \in \mathbb{R}. \quad (4.1)$$

As $\mathbb{Y}(z + \pi, \lambda)$ is a solution matrix of (3.2) and $\mathbb{Y}(z, \lambda)$ is a fundamental matrix of (3.2), $\mathbb{Y}(z + \pi, \lambda)$ can be written as

$$\mathbb{Y}(z + \pi, \lambda) = \mathbb{Y}(z, \lambda)A(\lambda), \quad (4.2)$$

where $A(\lambda)$ is independent of z . Setting $z = 0$ gives $A(\lambda) = \mathbb{Y}(\pi, \lambda)$. Combining this with (4.1) and (4.2) gives that $\rho(\lambda)$ represents the values of ρ for which

$$\mathbb{Y}(z, \lambda)(\rho I - A(\lambda))\underline{c} = 0 \quad (4.3)$$

for some $\underline{c} \neq 0$, i.e. the values of $\rho(\lambda)$ are the eigenvalues of $A(\lambda)$. Thus the values of $\rho(\lambda)$ are the roots, ρ , of the characteristic equation

$$\rho^2 - \rho\Delta(\lambda) + 1 = \det(A(\lambda) - I\rho) = 0. \quad (4.4)$$

Here

$$\Delta(\lambda) := y_{11}(\pi, \lambda) + y_{22}(\pi, \lambda) = \text{trace}(A(\lambda)), \quad (4.5)$$

is called the discriminant of (3.2). In terms of the $\Delta(\lambda)$, from (4.4), $\rho(\lambda)$ is given by

$$\rho(\lambda) = \frac{\Delta(\lambda) \pm \sqrt{\Delta^2(\lambda) - 4}}{2}. \quad (4.6)$$

As $Q(x) = \overline{Q(x)}$, it follows that $\Delta(\lambda)$ is real for $\lambda \in \mathbb{R}$. In this case, if $|\Delta(\lambda)| > 2$ then there are two linearly independent solutions of (3.2) obeying (4.1). Here $\rho(\lambda)$ is real and at least one of these has $|\rho(\lambda)| > 1$, in which case the solution has exponential growth as $z \rightarrow \infty$, so the solutions are unstable for such λ . If λ is real and $|\Delta(\lambda)| \leq 2$ then there are two linearly independent solutions of (3.2) obeying (4.1) both of which

have $|\rho(\lambda)| = 1$, thus making all solutions bounded for $z \in \mathbb{R}$ giving stability of the solution for such λ . The λ -intervals on the real line for which all solutions are bounded will be called the intervals of stability while the intervals for which at least one solutions is unbounded will be called instability intervals. The stability intervals are given by $|\Delta(\lambda)| \leq 2$ while the instability intervals are given by $|\Delta(\lambda)| > 2$. It follows from Corollary 4.2 that the instability intervals are precisely the components of the interior of the set on which $|\Delta(\lambda)| \geq 2, \lambda \in \mathbb{R}$.

4.3 The σ_i -Discriminant

Consider the problems

$$Y(\pi) = \sigma_i \rho_i(\lambda) Y(0), \quad i = 1, \dots, 3, \quad (4.7)$$

where Y is a non-trivial solution of (3.2), and $\rho_i(\lambda) \in \mathbb{C}$. Here $\rho_i(\lambda)$ is multivalued and Y can be represented as $Y = \mathbb{Y}(x)\underline{v}$, for some $\underline{v} \in \mathbb{R}^2 \setminus \{0\}$, considering this with (4.7) yields

$$(\mathbb{Y}(\pi) - \rho_i \sigma_i)\underline{v} = 0. \quad (4.8)$$

A necessary and sufficient condition for the existence of nontrivial solutions of (4.8) is $\det(\mathbb{Y}(\pi) - \rho_i \sigma_i) = 0$. This may be expressed, via (3.13), as

$$\det(\Delta^{\mathbb{I}}\mathbb{I} + \Delta^J J + \nabla^{\mathbb{I}}\sigma_3 + \nabla^J\sigma_1 - 2\rho_i\sigma_i) = 0. \quad (4.9)$$

Using (3.7) and (3.14) to simplify (4.9), we obtain

$$\rho_i^2 - \rho_i \langle \mathbb{Y}(\pi), \sigma_i \rangle_{Lin} + 1 = 0. \quad \text{for } i = 0, \quad (4.10)$$

and

$$\rho_i^2 - \rho_i \langle \mathbb{Y}(\pi), \sigma_i \rangle_{Lin} - 1 = 0. \quad \text{for } i = 1, 2, 3. \quad (4.11)$$

The quantities $\langle \mathbb{Y}(\pi), \sigma_i \rangle_{Lin}$ will be called the σ_i -discriminant of the problem (3.2) on $[0, \pi)$, and the solutions

$$\rho_i = \frac{\langle \mathbb{Y}(\pi), \sigma_i \rangle_{Lin} \pm \sqrt{\langle \mathbb{Y}(\pi), \sigma_i \rangle_{Lin}^2 - 4}}{2} \quad \text{for } i = 0, \quad (4.12)$$

and

$$\rho_i = \frac{\langle \mathbb{Y}(\pi), \sigma_i \rangle_{Lin} \pm \sqrt{\langle \mathbb{Y}(\pi), \sigma_i \rangle_{Lin}^2 + 4}}{2} \quad \text{for } i = 1, 2, 3, \quad (4.13)$$

of (4.10) and (4.11), respectively, are called *Floquet multipliers*. A direct calculation shows that the σ_i -discriminants, $\langle \mathbb{Y}(\pi), \sigma_i \rangle_{Lin}$, $i = 0, \dots, 3$, are given by $\Delta^{\mathbb{I}}$, ∇^I , $i\Delta^J$, and ∇^J respectively. A discriminant featured in this work is the J -discriminant, Δ^J , with characteristic equation

$$\rho^2 - \rho\Delta^J + 1 = 0, \quad (4.14)$$

and Floquet multiplier, $\rho = \frac{\Delta^J \pm \sqrt{(\Delta^J)^2 - 4}}{2}$.

Define the sets

$$S^{\mathbb{I}} := \{\lambda \in \mathbb{R} : |\Delta^I| \geq 2\} \quad (4.15)$$

$$S^J := \{\lambda \in \mathbb{R} : |\Delta^J| \geq 2\}. \quad (4.16)$$

For each $\lambda \in \mathbb{R} \setminus S^{\mathbb{I}}$, there exists two linearly independent solutions of (3.2) and (4.8) with $i = 0$, both of which have $|\rho| < 1$. Hence for such $\lambda \in \mathbb{R} \setminus S^{\mathbb{I}}$, every solution is bounded on \mathbb{R} . From the discussion at the end of Section 4.2, the maximally connected subsets of $\mathbb{R} \setminus S^{\mathbb{I}}$ are referred to as *intervals of stability*. Furthermore, for $\lambda \in S^{\mathbb{I}}$, there is at least one solution of (4.21) and (4.8) with $i = 0$ and $|\rho| > 1$, thus there is at least one unbounded solution on \mathbb{R} . Thus maximally connected subsets of $S^{\mathbb{I}}$ are referred to as *intervals of instability*.

Define the transformation

$$R(z) := e^{J \int_0^z (Q_2 - \frac{1}{\pi} \int_0^\pi Q_2 d\tau) dt}, \quad (4.17)$$

thus $Y = R\tilde{Y}$ transforms $\ell Y = \lambda Y$ into

$$J\tilde{Y}' + \tilde{Q}\tilde{Y} = \lambda\tilde{Y}, \quad (4.18)$$

where $\tilde{Q} = \tilde{Q}_1 + \tilde{Q}_2$ so that

$$\tilde{Q}_1(z) = R^{-1}(z)Q_1(z)R(z), \quad (4.19)$$

$$\tilde{Q}_2(z) = \frac{1}{\pi} \int_0^\pi Q_2 dt. \quad (4.20)$$

We have that $R(0) = \mathbb{I} = R(\pi)$, so that the above transformation preserves boundary conditions. Consider the equation

$$J\tilde{Y}'_a + \tilde{Q}_1\tilde{Y}_a = \left(\lambda - \frac{1}{2\pi} \int_0^\pi (q_1 + q_2) dt \right) \tilde{Y}_a, \quad (4.21)$$

then

$$\mathbb{Y}(\lambda, z) = R(z)e^{-i \int_0^z \Im q dt} \tilde{\mathbb{Y}}_a \left(\lambda - \frac{1}{2\pi} \int_0^z (q_1 + q_2) dt, z \right). \quad (4.22)$$

Setting $z = \pi$ in equation (4.22) and calculating $\langle \mathbb{Y}(\pi), \sigma_i \rangle_{Lin}$, $i = 0, 2$, we have

$$\Delta^{\mathbb{I}}(\lambda) = e^{-i \int_0^\pi \Im q dt} \tilde{\Delta}_a^{\mathbb{I}} \left(\lambda - \frac{1}{2\pi} \int_0^\pi (q_1 + q_2) dt \right), \quad (4.23)$$

$$\Delta^J(\lambda) = e^{-i \int_0^\pi \Im q dt} \tilde{\Delta}_a^J \left(\lambda - \frac{1}{2\pi} \int_0^\pi (q_1 + q_2) dt \right). \quad (4.24)$$

Equations (4.23) and (4.24) show that $\Delta^{\mathbb{I}}$ and Δ^J map $\lambda \in \mathbb{R}$ into the line $\eta e^{-i \int_0^\pi \Im q dt} \subset \mathbb{C}$, $\eta \in \mathbb{R}$. Thus the important information about the characteristic determinants of equation (3.2) are encoded in the discriminant of equation (4.21).

4.4 Analytic structure of $\Delta^{\mathbb{I}}$

What follows is a characterisation of the $\Delta^{\mathbb{I}}$ discriminant in terms of the eigenparameter λ . For brevity we refer to $\Delta^{\mathbb{I}}$ as Δ . This is necessary for the eigenvalue characterisation of Lemma 5.1 and inverse problem of Theorem 7.5.

Lemma 4.1 (a) *The λ -derivative of Δ is given by*

$$\frac{d\Delta}{d\lambda} = y_{21}(\pi) \int_0^\pi Y_1^T Y_1 dt + (y_{22}(\pi) - y_{11}(\pi)) \int_0^\pi Y_1^T Y_2 dt - y_{12}(\pi) \int_0^\pi Y_2^T Y_2 dt. \quad (4.25)$$

which can also be expressed as

$$\frac{d\Delta}{d\lambda} = y_{12}(\pi) \left\{ \frac{\Delta^2 - 4}{4y_{12}^2(\pi)} \|Y_1\|_2^2 - \left\| Y_2 - \frac{y_{22}(\pi) - y_{11}(\pi)}{2y_{12}(\pi)} Y_1 \right\|_2^2 \right\}, \quad y_{12}(\pi) \neq 0, \quad (4.26)$$

$$\frac{d\Delta}{d\lambda} = y_{21}(\pi) \left\{ \left\| Y_1 + \frac{y_{22}(\pi) - y_{11}(\pi)}{2y_{21}(\pi)} Y_2 \right\|_2^2 - \frac{\Delta^2 - 4}{4y_{21}^2(\pi)} \|Y_2\|_2^2 \right\}, \quad y_{21}(\pi) \neq 0. \quad (4.27)$$

(b) If $\Delta(\lambda) = \pm 2$ and $\frac{d\Delta}{d\lambda}(\lambda) = 0$ then $y_{12}(\pi) = 0 = y_{21}(\pi)$ and $\mp \frac{d^2\Delta}{d\lambda^2}(\lambda) > 0$.

(c) If $|\Delta| \leq 2$ then

$$\frac{1}{y_{12}(\pi)} \frac{d\Delta}{d\lambda} < 0, \quad \text{for } y_{12}(\pi) \neq 0, \quad (4.28)$$

$$\frac{1}{y_{21}(\pi)} \frac{d\Delta}{d\lambda} > 0, \quad \text{for } y_{21}(\pi) \neq 0. \quad (4.29)$$

(d) If $y_{12}(\pi) = 0$ or $y_{21}(\pi) = 0$, then $\Delta \cdot \operatorname{sgn} y_{11}(\pi) \geq 2$.

Proof: (a) Taking the λ -derivative of \mathbb{Y} in (3.15) and (3.4), we obtain that \mathbb{Y}_λ obeys the non-homogeneous initial value problem

$$J\mathbb{Y}'_\lambda + (Q - \lambda I)\mathbb{Y}_\lambda = \mathbb{Y},$$

with the initial condition $\mathbb{Y}_\lambda(0) = \mathbf{0}$. The homogeneous equation $JY'_\lambda + (Q - \lambda I)Y_\lambda = 0$ has the fundamental matrix solution \mathbb{Y} . Using the method of variation of parameters, see [18, pp. 74], we obtain

$$\frac{\partial \mathbb{Y}(x)}{\partial \lambda} = - \int_0^x \mathbb{Y}(x)\mathbb{Y}^{-1}(t)J\mathbb{Y}(t) dt. \quad (4.30)$$

Using (4.30), the λ -derivative of the discriminant (4.5) can be rewritten as

$$\frac{d\Delta}{d\lambda} = y_{21}(\pi) \int_0^\pi Y_1^T Y_1 dt + (y_{22}(\pi) - y_{11}(\pi)) \int_0^\pi Y_1^T Y_2 dt - y_{12}(\pi) \int_0^\pi Y_2^T Y_2 dt. \quad (4.31)$$

Completing the square in the (4.31) and using that the Wronskian $\det \mathbb{Y} = 1$ with the definition of Δ we obtain the remaining forms for the λ -derivative of Δ .

(b) If $\Delta = \pm 2$ and $\frac{d\Delta}{d\lambda} = 0$ then as Y_1 and Y_2 are linearly independent in $L^2(0, \pi)$, (4.26) leads to a contradiction if $y_{12}(\pi) \neq 0$ and (4.27) leads to a contradiction if $y_{21}(\pi) \neq 0$. Thus $y_{12}(\pi) = 0 = y_{21}(\pi)$.

As $[y_{11}y_{22} - y_{12}y_{21}](\pi) = 1$, it now follows that $\Delta = y_{11}(\pi) + \frac{1}{y_{11}(\pi)}$. The function $f(t) = t + (1/t)$ on $\mathbb{R} \setminus \{0\}$ attains the value 2 only at $t = 1$ and the value -2 only at $t = -1$. Thus $y_{11}(\pi) = y_{22}(\pi) = \pm 1$ and $\mathbb{Y}(\pi) = \pm I$.

Taking the λ -derivative of \mathbb{Y}_λ in (3.15) gives

$$J\mathbb{Y}'_{\lambda\lambda} + (Q - \lambda I)\mathbb{Y}_{\lambda\lambda} = 2\mathbb{Y}_\lambda, \quad (4.32)$$

and we obtain that $\mathbb{Y}_{\lambda\lambda}$ obeys the initial condition $\mathbb{Y}_{\lambda\lambda}(0) = 0$. Using the method of variation of parameters as in (4.30) gives

$$\frac{1}{2} \frac{\partial^2 \mathbb{Y}}{\partial \lambda^2}(x) = \int_0^x \int_0^t \mathbb{Y}(x)\mathbb{Y}^{-1}(t)J\mathbb{Y}(t)\mathbb{Y}^{-1}(\tau)J\mathbb{Y}(\tau) d\tau dt, \quad (4.33)$$

which with $x = \pi$ and $\mathbb{Y}(\pi) = \pm I$ yields

$$\frac{1}{2} \frac{\partial^2 \mathbb{Y}}{\partial \lambda^2}(\pi) = \pm \int_0^\pi \mathbb{Y}^{-1}(t)J\mathbb{Y}(t) \int_0^t \mathbb{Y}^{-1}(\tau)J\mathbb{Y}(\tau) d\tau dt. \quad (4.34)$$

Here

$$\mathbb{Y}^{-1}J\mathbb{Y} = \begin{bmatrix} Y_2^T Y_1 & Y_2^T Y_2 \\ -Y_1^T Y_1 & -Y_1^T Y_2 \end{bmatrix}$$

giving

$$\begin{aligned} \frac{\pm 1}{2} \frac{d^2 \Delta}{d\lambda^2}(\lambda) &= \frac{\pm 1}{2} \text{trace} \left(\frac{\partial^2 \mathbb{Y}}{\partial \lambda^2}(\pi) \right) \\ &= - \int_0^\pi Y_2^T Y_2 \int_0^x Y_1^T Y_1 dt dx + 2 \int_0^\pi Y_2^T Y_1 \int_0^x Y_2^T Y_1 dt dx \\ &\quad - \int_0^\pi Y_1^T Y_1 \int_0^x Y_2^T Y_2 dt dx. \end{aligned}$$

As Y_1, Y_2 have real entries for $\lambda \in \mathbb{R}$, by Fubini's Theorem applied to the above double integrals we obtain

$$\begin{aligned} \frac{\pm 1}{2} \frac{d^2 \Delta}{d\lambda^2}(\lambda) &= - \int_0^\pi Y_2^T Y_2 dt \int_0^\pi Y_1^T Y_1 dt + \left(\int_0^\pi Y_2^T Y_1 dt \right)^2 \\ &= -\|Y_1\|_2^2 \|Y_2\|_2^2 + \langle Y_1, Y_2 \rangle^2 < 0, \end{aligned}$$

for $\lambda \in \mathbb{R}$. Now Hölder's inequality gives that Y_1 and Y_2 are linearly independent.

(c) If $|\Delta| \leq 2$ then $\Delta^2 - 4 \leq 0$ so (4.26) and (4.27) respectively yield (4.28) and (4.29).

(d) If $y_{12}(\pi) = 0$ or $y_{21}(\pi) = 0$ then as $\det \mathbb{Y}(\pi) = 1$, it follows that $y_{11}(\pi)y_{22}(\pi) = 1$ giving $\Delta = y_{11}(\pi) + \frac{1}{y_{11}(\pi)}$ so $\Delta \geq 2$ if $y_{11}(\pi) > 0$ and $\Delta \leq -2$ if $y_{11}(\pi) < 0$. ■

Corollary 4.2 *For $\lambda \in \mathbb{R}$, the function $|\Delta(\lambda)|$ attains the value 2 only on the boundary of the set $\Gamma = \{\lambda \in \mathbb{R} \mid |\Delta(\lambda)| \geq 2\}$.*

Proof: Suppose that $\lambda \in \text{int}(\Gamma)$ and $\Delta(\lambda) = \pm 2$. As $\lambda \in \text{int}(\Gamma)$ there is $\delta > 0$ so that $I := (\lambda - \delta, \lambda + \delta) \subset \Gamma$. The continuity of Δ and connectedness of I give that $\pm\Delta \geq 2$ on I . Hence $\pm\Delta$ attains a local minimum at λ . Thus $\Delta'(\lambda) = 0$. Lemma 4.1(b) can now be applied to give $\pm\Delta''(\lambda) > 0$. From the analyticity of Δ , Δ'' is continuous, making $\pm\Delta'' < 0$ on a neighbourhood, say N , of λ . Hence $\pm\Delta < 2$ on $N \setminus \{\lambda\}$, which contradicts $\pm\Delta \geq 2$ on I . ■

4.5 Analytic structure of Δ^J

The following lemma proceeds in a similar manner to that of Lemma 4.1. It is used for the indexing results of Theorem 5.4.

Lemma 4.3 (a) *The λ -derivative of Δ^J is given by*

$$\frac{d\Delta^J}{d\lambda} = -y_{22}(\pi) \int_0^\pi Y_1^T Y_1 dt + (y_{21}(\pi) + y_{12}(\pi)) \int_0^\pi Y_1^T Y_2 dt - y_{11}(\pi) \int_0^\pi Y_2^T Y_2 dt, \quad (4.35)$$

which can be expressed as

$$\frac{d\Delta^J}{d\lambda} = -y_{11}(\pi) \left\{ \left\| Y_2 - \frac{y_{21} + y_{12}}{2y_{11}} Y_1 \right\|_2^2 + \frac{4 - (\Delta^J)^2}{4y_{11}^2} \|Y_1\|_2^2 \right\}, \quad y_{11}(\pi) \neq 0, \quad (4.36)$$

$$\frac{d\Delta^J}{d\lambda} = -y_{22}(\pi) \left\{ \left\| Y_1 - \frac{y_{21} + y_{12}}{2y_{22}} Y_2 \right\|_2^2 + \frac{4 - (\Delta^J)^2}{4y_{22}^2} \|Y_2\|_2^2 \right\}, \quad y_{22}(\pi) \neq 0. \quad (4.37)$$

(b) *If $\Delta^J(\lambda) = \pm 2$ and $\frac{d\Delta^J}{d\lambda}(\lambda) = 0$ then $y_{11}(\pi) = 0 = y_{22}(\pi)$ and $\pm \frac{d^2\Delta^J}{d^2\lambda}(\lambda) > 0$.*

(c) *If $|\Delta^J| \leq 2$ then*

$$\frac{1}{y_{11}(\pi)} \frac{d\Delta^J}{d\lambda} < 0, \quad \text{for } y_{11}(\pi) \neq 0, \quad (4.38)$$

$$\frac{1}{y_{22}(\pi)} \frac{d\Delta^J}{d\lambda} < 0, \quad \text{for } y_{22}(\pi) \neq 0. \quad (4.39)$$

(d) If $y_{11}(\pi) = 0$ or $y_{22}(\pi) = 0$, then $\Delta^J \cdot \operatorname{sgny}_{21}(\pi) \geq 2$.

Proof: (a) Following the reasoning of [19], we take the λ -derivative of \mathbb{Y} in (3.15) and (3.4), to see that \mathbb{Y}_λ obeys the non-homogeneous initial value problem

$$J\mathbb{Y}'_\lambda + (Q - \lambda I)\mathbb{Y}_\lambda = \mathbb{Y}, \quad (4.40)$$

with the initial condition $\mathbb{Y}_\lambda(0) = \mathbf{0}$. The homogeneous equation related to (4.40), $JY'_\lambda + (Q - \lambda I)Y_\lambda = 0$, has the fundamental matrix solution \mathbb{Y} . Thus using the method of variation of parameters, see [18, page 74], we have

$$\frac{\partial \mathbb{Y}(x)}{\partial \lambda} = - \int_0^x \mathbb{Y}(x)\mathbb{Y}^{-1}(t)J\mathbb{Y}(t) dt. \quad (4.41)$$

Multiplying equation (4.41) by J and taking the trace of both sides, we see that the λ -derivative of the J -discriminant can be rewritten as

$$\frac{d\Delta^J}{d\lambda} = -y_{22}(\pi) \int_0^\pi Y_1^T Y_1 dt + (y_{21}(\pi) + y_{12}(\pi)) \int_0^\pi Y_1^T Y_2 dt - y_{11}(\pi) \int_0^\pi Y_2^T Y_2 dt. \quad (4.42)$$

Completing the square in (4.42) and using $\det \mathbb{Y} = 1$ with the definition of Δ^J we obtain the remaining forms for the λ -derivative of Δ^J .

(b) Assuming that $\Delta^J = \pm 2$ and $\frac{d\Delta^J}{d\lambda} = 0$ we see that if $y_{11}(\pi) \neq 0$ or $y_{22}(\pi) \neq 0$ then (4.36) and (4.37) respectively, shows that Y_1 and Y_2 are not linearly independent, which is false. Thus $y_{11}(\pi) = 0 = y_{22}(\pi)$.

However $[y_{11}y_{22} - y_{21}y_{12}](\pi) = 1$ thus $\Delta^J = y_{21}(\pi) + \frac{1}{y_{21}(\pi)}$. Furthermore, the function $f(t) = t + (1/t)$ on $\mathbb{R} \setminus \{0\}$ attains the value 2 only at $t = 1$ and the value -2 only at $t = -1$. Thus $y_{21}(\pi) = \pm 1$ and $y_{12}(\pi) = \mp 1$, so that $\mathbb{Y}(\pi) = \pm J$.

Taking the λ -derivative of \mathbb{Y}_λ in (3.15) gives

$$J\mathbb{Y}'_{\lambda\lambda} + (Q - \lambda I)\mathbb{Y}_{\lambda\lambda} = 2\mathbb{Y}_\lambda, \quad (4.43)$$

and we obtain that $\mathbb{Y}_{\lambda\lambda}$ obeys the initial condition $\mathbb{Y}_{\lambda\lambda}(0) = 0$. Using the method of variation of parameters as in (4.41) we have

$$\frac{1}{2} \frac{\partial^2 \mathbb{Y}}{\partial \lambda^2}(x) = \int_0^x \int_0^t \mathbb{Y}(x)\mathbb{Y}^{-1}(t)J\mathbb{Y}(t)\mathbb{Y}^{-1}(\tau)J\mathbb{Y}(\tau) d\tau dt, \quad (4.44)$$

Let $x = \pi$ in (4.44). As $\mathbb{Y}(\pi) = \pm J$ from (4.14) we have

$$\frac{1}{2} \frac{\partial^2 \mathbb{Y}}{\partial \lambda^2}(\pi) = \pm \int_0^\pi J \mathbb{Y}^{-1}(t) J \mathbb{Y}(t) \int_0^t \mathbb{Y}^{-1}(\tau) J \mathbb{Y}(\tau) d\tau dt. \quad (4.45)$$

Here

$$\mathbb{Y}^{-1} J \mathbb{Y} = \begin{bmatrix} Y_2^T Y_1 & Y_2^T Y_2 \\ -Y_1^T Y_1 & -Y_1^T Y_2 \end{bmatrix}$$

giving

$$\begin{aligned} \mp \frac{1}{2} \frac{d^2 \Delta^J}{d\lambda^2}(\lambda) &= \mp \frac{1}{2} \text{trace} \left(J \frac{\partial^2 \mathbb{Y}}{\partial \lambda^2}(\pi) \right) \\ &= - \int_0^\pi Y_2^T Y_2 \int_0^x Y_1^T Y_1 dt dx + 2 \int_0^\pi Y_2^T Y_1 \int_0^x Y_2^T Y_1 dt dx \\ &\quad - \int_0^\pi Y_1^T Y_1 \int_0^x Y_2^T Y_2 dt dx. \end{aligned}$$

Since Y_1, Y_2 have real entries for $\lambda \in \mathbb{R}$, Fubini's Theorem applied to the above double integrals gives

$$\begin{aligned} \mp \frac{1}{2} \frac{d^2 \Delta^J}{d\lambda^2}(\lambda) &= - \int_0^\pi Y_2^T Y_2 dt \int_0^\pi Y_1^T Y_1 dt + \left(\int_0^\pi Y_2^T Y_1 dt \right)^2 \\ &= - \|Y_1\|_2^2 \|Y_2\|_2^2 + \langle Y_1, Y_2 \rangle^2 < 0, \end{aligned}$$

for $\lambda \in \mathbb{R}$. Now Hölder's inequality gives that Y_1 and Y_2 are linearly independent.

(c) Assuming that $|\Delta^J| \leq 2$, (4.36) and (4.37) gives (4.38) and (4.39), respectively.

(d) If $y_{11}(\pi) = 0$ or $y_{22}(\pi) = 0$, then the Wronskian being 1 gives, $y_{12}(\pi)y_{21}(\pi) = 1$, so that $\Delta^J = y_{21}(\pi) + \frac{1}{y_{21}(\pi)}$. Thus if $y_{21}(\pi) > 0$ then $\Delta^J > 2$ and if $y_{21}(\pi) < 0$ then $\Delta^J < 2$. ■

5

Eigenvalues

“ The art of doing mathematics consists in finding that special case which contains all the germs of generality.”

– David Hilbert

5.1 Introduction

In this chapter, important indexing results for the eigenvalues of Clifford quasiperiodic and Dirichlet boundary problems are proved. It is shown that the instability intervals and zeros of $\Delta^{\mathbb{I}}$ and Δ^J interlace, furthermore interlacing results for the \mathbb{I}, J -periodic, \mathbb{I}, J -antiperiodic and \mathbb{I}, J -Dirichlet eigenvalues are derived. These zeros are important because they give a complete description of the zeros of the monodromy matrix, $\mathbb{Y}(\pi)$, as well as the zeros of $\Delta^{\mathbb{I}} \pm 2$ and $\Delta^J \pm 2$. For an excellent introduction to results of this type see the monograph, [60].

5.2 The Prüfer Angle

Let $\Psi(z) = \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \end{pmatrix}$ be the non-trivial solution of (3.2) satisfying the initial condition $\begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} = \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix}$ where $\gamma \in [0, \pi)$. Define $R(z, \lambda, \gamma)$ and $\theta(z, \lambda, \gamma)$ by

$$\Psi(z) = \begin{pmatrix} R(z, \lambda, \gamma) \cos \theta(z, \lambda, \gamma) \\ R(z, \lambda, \gamma) \sin \theta(z, \lambda, \gamma) \end{pmatrix}, \quad (5.1)$$

where $R(z, \lambda, \gamma) > 0$ and $\theta(z, \lambda, \gamma)$ is a continuous function of z with $\theta(0, \lambda, \gamma) = \gamma$. From now on θ will be referred to as the angular part of Ψ . The function $R(z, \lambda, \gamma)$ is differentiable in z, λ, γ , and $\theta(z, \lambda, \gamma)$ is analytic in λ and γ for fixed z , and differentiable in z for fixed λ and γ . Here $\theta(z, \lambda, \gamma)$ is the solution to a first order initial value problem

$$\theta' = \lambda - q \sin 2\theta - q_1 \cos^2 \theta - q_2 \sin^2 \theta, \quad (5.2)$$

$$\theta(0) = \gamma. \quad (5.3)$$

This initial value problem obeys the conditions of [67, Section 69.1], from which it follows that $\theta(z, \lambda, \gamma)$ is jointly continuous in (z, λ, γ) . Moreover, for fixed $z > 0$ and γ , $\theta(z, \lambda, \gamma)$ is strictly increasing in λ , $\lambda \in \mathbb{R}$, see Weidmann [97, p. 242], with $\theta(z, \lambda, \gamma) \rightarrow \pm\infty$ as $\lambda \rightarrow \pm\infty$, see [7]. Thus the eigenvalues, ν_n, μ_n, β_n and ζ_n , $n \in \mathbb{Z}$, of L_3, L_4, L_7 and L_8 , respectively, are simple and determined uniquely by the equations

$$\theta(\pi, \nu_n, \pi/2) = n\pi + \frac{\pi}{2}, \quad n \in \mathbb{Z}, \quad (5.4)$$

$$\theta(\pi, \mu_n, 0) = n\pi, \quad n \in \mathbb{Z}. \quad (5.5)$$

$$\theta(\pi, \beta_n, \pi/2) = (n+1)\pi, \quad n \in \mathbb{Z}, \quad (5.6)$$

$$\theta(\pi, \zeta_n, 0) = n\pi + \frac{\pi}{2}, \quad n \in \mathbb{Z}. \quad (5.7)$$

As a consequence of the above observation it follows that $\mu_n, \nu_n, \beta_n, \zeta_n \rightarrow \pm\infty$ as $n \rightarrow \pm\infty$.

5.3 Periodic and Antiperiodic Interlacing

The following lemma characterises the eigenvalues μ_n and ν_n in terms of the instability intervals of the discriminant $\Delta^{\mathbb{I}}$.

Lemma 5.1 (a) For each $n \in \mathbb{Z}$,

$$\max\{\mu_n, \nu_n\} < \min\{\mu_{n+1}, \nu_{n+1}\}. \quad (5.8)$$

(b) If $\lambda \in (\min\{\nu_n, \mu_n\}, \max\{\nu_{n+1}, \mu_{n+1}\})$ and $|\Delta(\lambda)| \leq 2$ then

$$(-1)^n \Delta'(\lambda) < 0.$$

(c) The set $|\Delta(\lambda)| \geq 2$ consists of a countable union of disjoint closed finite intervals, each of which contains precisely one of the sets $\{\nu_n, \mu_n\}$, $n \in \mathbb{Z}$. The end points of these intervals are the only points at which $|\Delta(\lambda)| = 2$.

Proof: (a) For fixed λ , $\theta(\pi, \lambda, \gamma)$ is monotonic increasing in γ (this follows from the fact that θ is a solution to a first order differential equation which has a unique solution for each initial value - giving that if a solution θ_1 begins below θ_2 then it remains below θ_2 for all values of the independent variable). Thus

$$\theta(\pi, \mu_n, \pi/2) < \theta(\pi, \mu_n, \pi) = (n+1)\pi < (n+1)\pi + \frac{\pi}{2} = \theta(\pi, \nu_{n+1}, \pi/2),$$

which, as $\theta(\pi, \lambda, \pi/2)$ is increasing in λ , gives $\mu_n < \nu_{n+1}$. As $\theta(\pi, \lambda, 0)$ is increasing in λ , $\nu_n < \nu_{n+1}$. Combining these inequalities gives $\max\{\mu_n, \nu_n\} < \nu_{n+1}$. Similarly

$$\theta(\pi, \nu_n, 0) < \theta(\pi, \nu_n, \pi/2) = n\pi + \frac{\pi}{2} < (n+1)\pi = \theta(\pi, \mu_{n+1}, 0)$$

giving $\nu_n < \mu_{n+1}$ and $\mu_n < \mu_{n+1}$ giving $\max\{\mu_n, \nu_n\} < \mu_{n+1}$. Hence (5.8) follows.

(b) From the monotonicity of $\theta(\pi, \lambda, \pi/2)$ in λ , for $\lambda \in (\nu_n, \nu_{n+1})$,

$$n\pi + \frac{\pi}{2} = \theta(\pi, \nu_n, \pi/2) < \theta(\pi, \lambda, \pi/2) < \theta(\pi, \nu_{n+1}, \pi/2) = (n+1)\pi + \frac{\pi}{2},$$

giving

$$(-1)^n y_{21}(\pi, \lambda) = (-1)^n R(\pi, \lambda, \pi/2) \cos \theta(\pi, \lambda, \pi/2) < 0. \quad (5.9)$$

Similarly, for $\lambda \in (\mu_n, \mu_{n+1})$,

$$n\pi = \theta(\pi, \mu_n, 0) < \theta(\pi, \lambda, 0) < \theta(\pi, \mu_{n+1}, 0) = (n+1)\pi,$$

giving

$$(-1)^n y_{12}(\pi, \lambda) = (-1)^n R(\pi, \lambda, 0) \sin \theta(\pi, \lambda, 0) > 0. \quad (5.10)$$

From (5.8) we have that $(\nu_n, \nu_{n+1}) \cap (\mu_n, \mu_{n+1}) \neq \emptyset$ and thus

$$(\nu_n, \nu_{n+1}) \cup (\mu_n, \mu_{n+1}) = (\min\{\nu_n, \mu_n\}, \max\{\nu_{n+1}, \mu_{n+1}\}).$$

Now by Lemma 4.1(c) along with (5.9) and (5.10), if

$$\lambda \in (\min\{\nu_n, \mu_n\}, \max\{\nu_{n+1}, \mu_{n+1}\}) \quad \text{and} \quad |\Delta(\lambda)| \leq 2$$

then $(-1)^n \Delta'(\lambda) < 0$.

(c) Since $|\Delta(\lambda)|$ is continuous, the set of $\lambda \in \mathbb{R}$ for which $|\Delta(\lambda)| \geq 2$ consists of a countable union of disjoint closed finite intervals. From the definition of ν_n , we have $y_{21}(\pi, \nu_n) = 0$ and $y_{22}(\pi, \nu_n) = (-1)^n R(\pi, \nu_n, \pi/2)$. Hence $y_{11}(\pi, \nu_n) = (-1)^n / R(\pi, \nu_n, \pi/2)$ and $(-1)^n \Delta(\nu_n) \geq 2$. Similarly $y_{11}(\pi, \mu_n) = (-1)^n R(\pi, \mu_n, 0)$ and $y_{12}(\pi, \mu_n) = 0$. Hence $y_{22}(\pi, \mu_n) = (-1)^n / R(\pi, \mu_n, 0)$ and $(-1)^n \Delta(\mu_n) \geq 2$. Hence, for each $n \in \mathbb{Z}$,

$$\min\{(-1)^n \Delta(\min\{\nu_n, \mu_n\}), (-1)^n \Delta(\max\{\nu_n, \mu_n\})\} \geq 2. \quad (5.11)$$

Let

$$S := \{\lambda | (-1)^n \Delta(\lambda) < 2\} \cap (\min\{\nu_n, \mu_n\}, \max\{\nu_n, \mu_n\}).$$

If $S \neq \emptyset$ then there is $\lambda^* \in S$. Here $K := (-1)^n \Delta(\lambda^*) < 2$ and by (5.11), $(-1)^n \Delta(\max\{\nu_n, \mu_n\}) \geq 2$. So from the intermediate value theorem there is λ with

$\lambda^* \leq \lambda \leq \max\{\nu_n, \mu_n\}$ having $(-1)^n \Delta(\lambda) = (2 + K)/2$. The set of such λ is compact and thus has a least element, say λ^\dagger . By part (b) of this lemma $(-1)^n \Delta'(\lambda) < 0$ for all $\lambda^* \leq \lambda \leq \lambda^\dagger$ giving the contradiction

$$K = (-1)^n \Delta(\lambda^*) \geq (-1)^n \Delta(\lambda^\dagger) = (2 + K)/2.$$

Thus $S = \emptyset$ and for each $n \in \mathbb{Z}$ both μ_n and ν_n lie in the same component of $\{\lambda \mid |\Delta(\lambda)| \geq 2\}$. Due to the sign alternation in (5.11) as n changes, each component of $\{\lambda \in \mathbb{R} \mid |\Delta(\lambda)| \geq 2\}$ contains at most one pair $\{\mu_n, \nu_n\}, n \in \mathbb{Z}$.

It remains to show that every component of $\{\lambda \in \mathbb{R} \mid |\Delta(\lambda)| \geq 2\}$ contains μ_n for some $n \in \mathbb{Z}$. If not then there is a component, say T , of $\{\lambda \in \mathbb{R} \mid |\Delta(\lambda)| \geq 2\}$ and $n \in \mathbb{Z}$ so that $T \subset (\mu_n, \mu_{n+1})$. Let $[\tilde{\lambda}_{-1}, \tilde{\lambda}_0]$ and $[\tilde{\lambda}_3, \tilde{\lambda}_4]$ denote the components of $\{\lambda \in \mathbb{R} \mid |\Delta(\lambda)| \geq 2\}$ containing μ_n and μ_{n+1} respectively. The set $T := [\tilde{\lambda}_1, \tilde{\lambda}_2]$ is compact and we may, without loss of generality, assume that $\tilde{\lambda}_1$ is the least $\lambda > \tilde{\lambda}_0$ with $|\Delta(\lambda)| \geq 2$. Here $\tilde{\lambda}_0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 < \tilde{\lambda}_3$. From (5.5) we have $(-1)^n \Delta(\mu_n) \geq 2$, however, from part (b) of this lemma, $(-1)^n \Delta'(\lambda) < 0$ for $\lambda \in (\tilde{\lambda}_0, \tilde{\lambda}_1)$. Thus $\Delta(\lambda) \leq -2$ for $\lambda \in T$. Again, as $(-1)^n \Delta'(\lambda) < 0$ for $\lambda \in (\mu_n, \mu_{n+1}) \setminus T$, $\Delta(\lambda) \leq -2$ for all $\lambda \in [\tilde{\lambda}_1, \mu_{n+1}]$. Hence T contains μ_{n+1} , contradicting the definition of T , and giving that no such T exists.

The last part of the claim follows directly from Corollary 4.2. \blacksquare

We denote the components (maximal connected subsets) of the set $\{\lambda \in \mathbb{R} \mid \Delta(\lambda) \geq 2\}$ by $[\lambda_{2k-1}, \lambda_{2k}]$ indexed so that $\{\mu_{2k}, \nu_{2k}\} \subset [\lambda_{2k-1}, \lambda_{2k}]$ (this indexing is possible and uniquely defined by the previous lemma). Similarly we denote the components of the set $\Delta(\lambda) \leq -2$ by $[\lambda'_{2k-1}, \lambda'_{2k}]$, labeled so that $\{\mu_{2k-1}, \nu_{2k-1}\} \subset [\lambda'_{2k-1}, \lambda'_{2k}]$. With this indexing

$$\lambda'_{2k-1} \leq \{\mu_{2k-1}, \nu_{2k-1}\} \leq \lambda'_{2k} < \lambda_{2k-1} \leq \{\mu_{2k}, \nu_{2k}\} \leq \lambda_{2k} < \lambda'_{2k+1}. \quad (5.12)$$

Here by $\lambda'_{2k-1} \leq \{\mu_{2k-1}, \nu_{2k-1}\} \leq \lambda'_{2k}$ we mean that both μ_{2k-1} and ν_{2k-1} are greater than or equal to λ'_{2k-1} and less than or equal to λ'_{2k} with analogous interpretation

for $\lambda_{2k-1} \leq \{\mu_{2k}, \nu_{2k}\} \leq \lambda_{2k}$. The instability intervals are thus $I_{2k} := (\lambda_{2k-1}, \lambda_{2k})$ and $I_{2k-1} = (\lambda'_{2k-1}, \lambda'_{2k})$, $k \in \mathbb{Z}$, which might be the an empty interval. From (4.6) the solutions of $\Delta(\lambda) = 2$ and $\Delta(\lambda) = -2$ are the eigenvalues of the periodic and anti-periodic problems respectively, as these are respectively where $\rho(\lambda) = 1$ and $\rho(\lambda) = -1$. Hence the eigenvalues of L_1 and L_2 are (λ_j) and (λ'_j) respectively. This can be visualized in Figure 5.1.

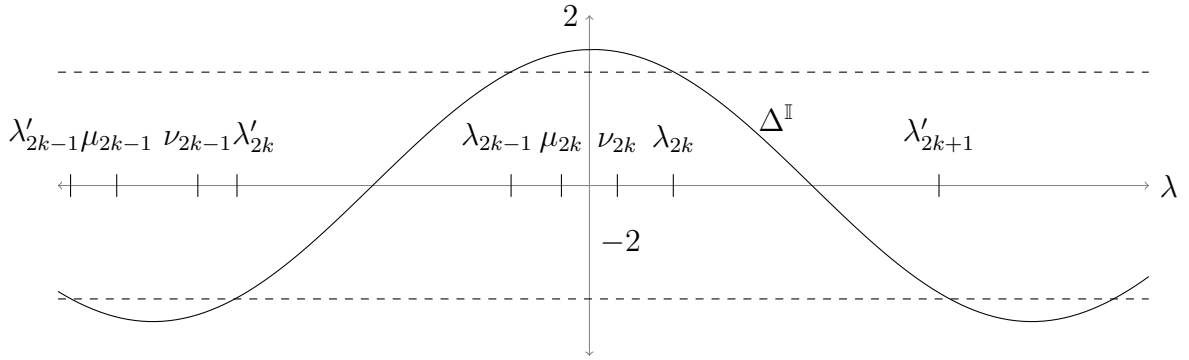


Figure 5.1: Plot of Δ^{II} showing the interlacing of various dirchlet eigenvalues.

Corollary 5.2 *The eigenvalue λ_{2k} (resp. λ'_{2k}) is a double eigenvalue if and only if the interval $[\lambda_{2k-1}, \lambda_{2k}]$ (resp. $[\lambda'_{2k-1}, \lambda'_{2k}]$) is reduced to a single point.*

Proof: If the interval $[\lambda_{2k-1}, \lambda_{2k}]$ reduces to a single point then $\lambda_{2k-1} = \mu_{2k} = \nu_{2k} = \lambda_{2k}$ giving $y_{12}(\pi) = 0 = y_{21}(\pi)$. Thus, $\mathbb{Y}(\pi, \lambda_{2k})$ is diagonal with trace $2 = \Delta(\lambda_{2k}) = y_{11}(\pi) + y_{22}(\pi)$ and determinant $1 = [y_{11}y_{22} - y_{12}y_{21}](\pi) = y_{11}(\pi)y_{22}(\pi)$. Hence $\mathbb{Y}(\pi, \lambda_{2k}) = I$. Thus Y_1 and Y_2 are both periodic eigenfunctions and the eigenspace attains its maximal dimension of 2.

Conversely if λ_{2k} is a double eigenvalue then all solutions are π -periodic as the solution space is only 2-dimensional. In particular Y_1 and Y_2 are eigenfunctions. Thus $y_{11}(\pi) = 1 = y_{22}(\pi)$ and $y_{12}(\pi) = 0 = y_{21}(\pi)$ giving $\Delta(\lambda_{2k}) = 2$. Now by Lemma 4.1(a) $\Delta'(\lambda_{2k}) = 0$ but by Lemma 4.1(b) $\Delta''(\lambda_{2k}) < 0$ so the interval $[\lambda_{2k-1}, \lambda_{2k}]$ reduces to a single point.

Similar reasoning can be applied to the case of λ'_{2k} . ■

We now turn our attention back to the translated equation (7.1).

Theorem 5.3 *Let $\mu_i(\tau)$ denote the eigenvalue μ_i but for the differential equation in which $Q(z)$ has been replaced by the shifted potential $Q(z + \tau)$. In terms of the above eigenvalues, for $k \in \mathbb{Z}$, we obtain*

$$\begin{aligned} \lambda_{2k-1} &= \min_{\tau} \mu_{2k}(\tau), \quad k \neq 0, & \lambda_{2k} &= \max_{\tau} \mu_{2k}(\tau), \quad k \neq 0, \\ \lambda'_{2k-1} &= \min_{\tau} \mu_{2k-1}(\tau) & \lambda'_{2k} &= \max_{\tau} \mu_{2k-1}(\tau). \end{aligned}$$

Proof: From Lemma 7.1, the eigenvalues $\lambda_i, \lambda'_i, i \in \mathbb{Z}$, are independent of τ . Let $\Phi_{\tau}(z, \xi, \lambda, \gamma)$ be the solution of the equation (5.2) with initial condition $\Phi_{\tau}(\xi, \xi, \lambda, \gamma) = \gamma$ and $Q(z)$ replaced by $Q(z + \tau)$. Here $\Phi_{\tau}(z, \xi, \lambda, \gamma)$ is continuous in $(z, \xi, \lambda, \gamma)$ by [67, Section 69.1]. In addition, as

$$\Phi_{\tau}(z, \xi, \lambda, \gamma) = \Phi_0(z + \tau, \xi + \tau, \lambda, \gamma),$$

it follows that Φ_{τ} is continuous in τ , and

$$\Phi_{\tau}(\pi, 0, \mu_n(\tau), 0) = n\pi, \quad n \in \mathbb{Z},$$

defines $\mu_n(\tau)$.

As for $\theta(z, \lambda, \gamma)$, the derivative of $\Phi_{\tau}(z, \xi, \lambda, \gamma)$ with respect to λ is positive. Thus the inverse function theorem applied to $\Phi_{\tau}(z, \xi, \lambda, \gamma)$ gives that $\mu_n(\tau)$ is continuous in τ . Now from Lemma 7.1 the sets $\{\lambda_i | i \in \mathbb{Z}\}$ and $\{\lambda'_i | i \in \mathbb{Z}\}$ do not depend on τ , while, from the continuity of $\mu_n(\tau)$, the indexing of the eigenvalues λ_i, λ'_i does not depend of τ . Hence $\mu_{2k}(\tau) \in [\lambda_{2k-1}, \lambda_{2k}]$, for all τ , giving

$$\lambda_{2k-1} \leq \inf_{\tau} \mu_{2k}(\tau) \leq \sup_{\tau} \mu_{2k}(\tau) \leq \lambda_{2k}$$

and

$$\lambda'_{2k-1} \leq \inf_{\tau} \mu_{2k-1}(\tau) \leq \sup_{\tau} \mu_{2k-1}(\tau) \leq \lambda'_{2k}.$$

If Y is an eigenfunction to the periodic eigenvalue λ_{2k-1} then Y has angular part $\theta(x, \lambda_{2k-1}, \gamma)$ where without loss of generality $\gamma \in [0, \pi)$. Now $\mu_{2k-1} \leq \lambda'_{2k} < \lambda_{2k-1} \leq \mu_{2k}$. For $k \geq 1$, as $\theta(x, \lambda, \gamma)$ is increasing in γ , we have

$$\theta(0, \lambda_{2k-1}, \gamma) = \gamma < \pi \leq (2k - 1)\pi = \theta(\pi, \mu_{2k-1}, 0) < \theta(\pi, \lambda_{2k-1}, 0) \leq \theta(\pi, \lambda_{2k-1}, \gamma)$$

so by the intermediate value theorem there exists $\tau \in (0, \pi]$ with $\theta(\tau, \lambda_{2k-1}, \gamma) = \pi$. As Y is π -periodic, so is $Y(x + \tau)$. Thus $\lambda_{2k-1} = \mu_n(\tau)$ for some n , but the only n for which $\mu_n(\tau)$ is in $[\lambda_{2k-1}, \lambda_{2k}]$ is $n = 2k$. Hence $\lambda_{2k-1} = \min_{\tau} \mu_{2k}(\tau)$. In the case of $k \leq -1$ we have

$$\theta(\pi, \lambda_{2k-1}, \gamma) < \theta(\pi, \lambda_{2k-1}, \pi) = \pi + \theta(\pi, \lambda_{2k-1}, 0) \leq \pi + \theta(\pi, \mu_{2k}, 0) = (2k+1)\pi \leq -\pi.$$

But $0 \leq \gamma = \theta(0, \lambda_{2k-1}, \gamma)$ so there exists $\tau \in [0, \pi)$ such that $\theta(\tau, \lambda_{2k-1}, \gamma) = 0$. Proceeding as in the previous case, $\lambda_{2k-1} = \min_{\tau} \mu_{2k}(\tau)$.

For $k \in \mathbb{Z}$, we have that $\mu_{2k} \leq \lambda_{2k} < \mu_{2k+1}$. If Y is an eigenfunction to the periodic eigenvalue λ_{2k} then Y has angular part $\theta(x, \lambda_{2k}, \gamma)$ where without loss of generality $\gamma \in [0, \pi)$. For $k \geq 1$,

$$\theta(0, \lambda_{2k}, \gamma) = \gamma < \pi < 2k\pi = \theta(\pi, \mu_{2k}, 0) \leq \theta(\pi, \lambda_{2k}, 0) \leq \theta(\pi, \lambda_{2k}, \gamma)$$

so there exists $\tau \in (0, \pi]$ for which $\theta(\tau, \lambda_{2k-1}, \gamma) = \pi$ and $\lambda_{2k} = \mu_{2k}(\tau)$. In the case of $k \leq -1$ we have

$$\theta(\pi, \lambda_{2k}, \gamma) < \theta(\pi, \lambda_{2k}, \pi) = \pi + \theta(\pi, \lambda_{2k}, 0) < \pi + \theta(\pi, \mu_{2k+1}, 0) = (2k+1)\pi \leq -\pi.$$

Now $-\pi < 0 \leq \gamma = \theta(0, \lambda_{2k}, \gamma)$ so there exists $\tau \in [0, \pi)$ with $\theta(\tau, \lambda_{2k}, \gamma) = 0$ giving $\lambda_{2k} = \mu_{2k}(\tau)$. Thus for $k \in \mathbb{Z} \setminus \{0\}$, $\lambda_{2k} = \max_{\tau} \mu_{2k}(\tau)$.

For an eigenfunction of the Y of the anti-periodic problem at eigenvalue λ'_j , where $j = 2k - 1$ or $2k$, we have $Y(0) = -Y(\pi)$ giving that the angular part $\theta(x, \lambda'_j, \gamma)$ of Y necessarily changes by an odd multiple of π over the interval $[0, \pi]$. In particular this ensures that there is some $\tau \in [0, \pi]$ for which $\theta(\tau, \lambda'_j, \gamma) = \pm\pi$. Setting $Z(x) = Y(x)$ for $x \in [0, \pi]$ and $Z(x) = -Y(x - \pi)$ for $x \in (\pi, 2\pi]$ we have that Z is a solution of the periodically extended equation on $[0, 2\pi]$ for $\lambda = \lambda'_j$ and that $Z(x + \tau)$ is an eigenfunction to the eigenvalue $\mu_{2k-1}(\tau)$. Thus showing that $\mu_{2k-1}(\tau)$ attains both λ'_{2k-1} and λ'_{2k} . ■

Remark In the above theorem we have that $\mu_0(\tau) \in [\lambda_{-1}, \lambda_0]$, but in general λ_{-1} is not the minimum of $\mu_0(\tau)$ nor is λ_0 the maximum of $\mu_0(\tau)$. To see this consider the

following example.

Example Consider the case of $Q(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then $\mu_0(\tau) = 0 = \nu_0(\tau)$ for all τ , but $\Delta(0) = 2 \cosh(\pi) > 2$ so $\lambda = 0$ is not an eigenvalue of the periodic problem. Thus here we have

$$\lambda_{-1} < \inf_{\tau} \mu_0(\tau) = 0 = \sup_{\tau} \mu_0(\tau) < \lambda_0.$$

Remark If $Q(x)$ is constant then $\mu_n(\tau)$ and $\nu_n(\tau)$ are independent of τ and from the above all the instability intervals vanish except possibly $I_0 = (\lambda_{-1}, \lambda_0)$. The $\Delta^{\mathbb{I}}$ inverse Theorem 7.5, gives a partial converse to this.

5.4 J-Periodic and J-Antiperiodic interlacing

Consider $S^{\mathbb{I}}$ and S^J as defined in equations (4.15) and (4.16). It has been shown, [19], that $S^{\mathbb{I}}$ has the representation

$$S^{\mathbb{I}} = \bigcup_{k=-\infty}^{\infty} [\lambda_{2k-1}, \lambda_{2k}] \cup [\lambda'_{2k-1}, \lambda'_{2k}], \quad (5.13)$$

where, λ_n and λ'_n are the eigenvalues of the periodic and anti-periodic eigenvalue problems, respectively. The following results will be used to prove the interlacing of various eigenvalues and zeros of $\Delta^{\mathbb{I}}$ and Δ^J .

Theorem 5.4 (a) For each $n \in \mathbb{Z}$,

$$\max\{\beta_n, \zeta_n\} < \{\nu_{n+1}, \mu_{n+1}\} < \min\{\beta_{n+1}, \zeta_{n+1}\}, \quad (5.14)$$

$$\max\{\mu_n, \nu_n\} < \{\beta_n, \zeta_n\} < \min\{\mu_{n+1}, \nu_{n+1}\}. \quad (5.15)$$

(b) If $\lambda \in (\min\{\beta_n, \zeta_n\}, \max\{\beta_{n+1}, \zeta_{n+1}\})$ and $|\Delta^J(\lambda)| \leq 2$ then

$$(-1)^n \Delta^J_\lambda(\lambda) > 0, \quad (5.16)$$

- (c) Each interval $[\min\{\beta_n, \zeta_n\}, \max\{\beta_n, \zeta_n\}]$ contains precisely one of the zeros of $\Delta^{\mathbb{I}}$ and every zero of $\Delta^{\mathbb{I}}$ is contained in such an interval. Similarly, each interval of $[\min\{\mu_n, \nu_n\}, \max\{\mu_n, \nu_n\}]$ contains precisely one of the zeros of $\Delta^{\mathbb{J}}$ and every zero of $\Delta^{\mathbb{J}}$ is contained in such an interval.
- (d) The set $S^{\mathbb{J}}$ ($S^{\mathbb{I}}$ respectively) consists of the countable union of closed finite intervals, each of which contains precisely one zero of $\Delta^{\mathbb{I}}$. ($\Delta^{\mathbb{J}}$ respectively)
- (e) The zeros of $\Delta^{\mathbb{I}}$ are contained within $S^{\mathbb{J}}$, with each maximally connected subset of $S^{\mathbb{J}}$ containing exactly one zero of $\Delta^{\mathbb{I}}$ and exactly one of the sets $\{\beta_n, \zeta_n\}$, $n \in \mathbb{Z}$.

Proof: (a) Since $\theta(x, \lambda, \gamma)$ is monotonic in λ we have

$$\begin{aligned} (n+1)\pi &= \theta(\pi, \beta_n, \frac{\pi}{2}) < (n+1)\pi + \frac{\pi}{2} \\ &= \theta(\pi, \nu_{n+1}, \frac{\pi}{2}) < \theta(\pi, \beta_{n+1}, \frac{\pi}{2}) = (n+2)\pi, \end{aligned} \quad (5.17)$$

giving

$$\beta_n < \nu_{n+1} < \beta_{n+1}, \quad (5.18)$$

furthermore

$$\begin{aligned} n\pi + \frac{\pi}{2} &= \theta(\pi, \zeta_n, 0) < (n+1)\pi \\ &= \theta(\pi, \mu_{n+1}, 0) < \theta(\pi, \zeta_{n+1}, 0) = (n+1)\pi + \frac{\pi}{2}, \end{aligned} \quad (5.19)$$

showing that

$$\zeta_n < \mu_{n+1} < \zeta_{n+1}. \quad (5.20)$$

Suppose $\beta_n \geq \mu_{n+1}$ or $\mu_{n+1} \geq \beta_{n+1}$. By assumption and the monotonicity of θ and the eigenparameter, we have

$$(n+1)\pi = \theta(\pi, \mu_{n+1}, 0) \leq \theta(\pi, \beta_n, 0) < \theta(\pi, \beta_n, \frac{\pi}{2}) = (n+1)\pi, \quad (5.21)$$

for $\beta_n \geq \mu_{n+1}$ and

$$(n+2)\pi = \theta(\pi, \beta_{n+1}, \frac{\pi}{2}) \leq \theta(\pi, \mu_{n+1}, \frac{\pi}{2}) < \theta(\pi, \mu_{n+1}, \pi) = (n+2)\pi, \quad (5.22)$$

for $\mu_{n+1} \geq \beta_{n+1}$. Thus equations (5.21) and (5.22) show by contradiction that $\beta_n < \mu_{n+1}$ and $\mu_{n+1} < \beta_{n+1}$, respectively. Thus

$$\beta_n < \mu_{n+1} < \beta_{n+1}. \quad (5.23)$$

Similarly, assume that $\zeta_n \geq \nu_{n+1}$ and $\nu_{n+1} \geq \zeta_{n+1}$ thus again from the monotonicity of θ and the eigenparameter, we have

$$(n+1)\pi + \frac{\pi}{2} = \theta(\pi, \nu_{n+1}, \frac{\pi}{2}) \leq \theta(\pi, \zeta_n, \frac{\pi}{2}) < \theta(\pi, \zeta_n, \pi) = (n+1)\pi + \frac{\pi}{2} \quad (5.24)$$

for $\zeta_n \geq \nu_{n+1}$, and

$$(n+1)\pi + \frac{\pi}{2} = \theta(\pi, \zeta_{n+1}, 0) \leq \theta(\pi, \nu_{n+1}, 0) < \theta(\pi, \zeta_{n+1}, \frac{\pi}{2}) = (n+1)\pi + \frac{\pi}{2}. \quad (5.25)$$

for $\nu_{n+1} \geq \zeta_{n+1}$. Hence equations (5.24) and (5.25) show by contradiction that $\zeta_n < \nu_{n+1}$ and $\nu_{n+1} < \zeta_{n+1}$, respectively. Thus

$$\zeta_n < \nu_{n+1} < \zeta_{n+1}. \quad (5.26)$$

Combining inequalities, (5.18), (5.20), (5.23) and (5.26) gives (5.14), thus also (5.15) holds. **(a)** is proved.

(b). From the monotonicity of $\theta(\pi, \lambda, \frac{\pi}{2})$ in λ , we have for $\lambda \in (\beta_n, \beta_{n+1})$ that

$$(n+1)\pi = \theta(\pi, \beta_n, \frac{\pi}{2}) < \theta(\pi, \lambda, \frac{\pi}{2}) < \theta(\pi, \beta_{n+1}, \frac{\pi}{2}) = (n+2)\pi, \quad (5.27)$$

thus

$$(-1)^n y_{22}(\pi, \lambda) = (-1)^n R(\pi, \lambda, 0) \sin \theta(\pi, \lambda, 0) < 0. \quad (5.28)$$

Similarly for $\lambda \in (\zeta_n, \zeta_{n+1})$ we have

$$n\pi + \frac{\pi}{2} = \theta(\pi, \zeta_n, 0) < \theta(\pi, \lambda, 0) < \theta(\pi, \zeta_{n+1}, 0) = (n+1)\pi + \frac{\pi}{2}, \quad (5.29)$$

thus

$$(-1)^n y_{11}(\pi, \lambda) = (-1)^n R(\pi, \lambda, \frac{\pi}{2}) \cos \theta(\pi, \lambda, \frac{\pi}{2}) < 0. \quad (5.30)$$

From (5.14) we have that $(\beta_n, \beta_{n+1}) \cap (\zeta_n, \zeta_{n+1})$ is not empty, thus for

$$\lambda \in (\beta_n, \beta_{n+1}) \cup (\zeta_n, \zeta_{n+1}) = (\min\{\beta_n, \zeta_n\}, \max\{\beta_{n+1}, \zeta_{n+1}\}), \quad (5.31)$$

with $|\Delta^J(\lambda)| \leq 2$, we have using (5.30), (5.28) and Lemma 4.3 (c) that $(-1)^n \Delta_\lambda^J(\lambda) > 0$.

(c) From monotonicity of $\theta(\pi, \lambda, 0)$ in λ we have for $\lambda \in (\mu_n, \mu_{n+1})$ that

$$n\pi = \theta(\pi, \mu_n, 0) < \theta(\pi, \lambda, 0) < \theta(\pi, \mu_{n+1}, 0) = (n+1)\pi, \quad (5.32)$$

thus

$$(-1)^n y_{12}(\pi, \lambda) = (-1)^n R(\pi, \lambda, 0) \sin \theta(\pi, \lambda, 0) > 0. \quad (5.33)$$

Similarly for $\lambda \in (\nu_n, \nu_{n+1})$ we have

$$n\pi + \frac{\pi}{2} = \theta(\pi, \nu_n, \frac{\pi}{2}) < \theta(\pi, \lambda, \frac{\pi}{2}) < \theta(\pi, \nu_{n+1}, \frac{\pi}{2}) = (n+1)\pi + \frac{\pi}{2}, \quad (5.34)$$

thus

$$(-1)^n y_{21}(\pi, \lambda) = (-1)^n R(\pi, \lambda, \frac{\pi}{2}) \cos \theta(\pi, \lambda, \frac{\pi}{2}) < 0. \quad (5.35)$$

Combining inequalities (5.30), (5.28) and (5.15) we have for

$$\lambda \in (\beta_n, \beta_{n+1}) \cap (\zeta_n, \zeta_{n+1}) = (\max\{\beta_n, \zeta_n\}, \min\{\beta_{n+1}, \zeta_{n+1}\}), \quad (5.36)$$

that $(-1)^n \Delta^{\mathbb{I}} > 0$. Since $\Delta^{\mathbb{I}}$ is continuous, there must be at least one zero of $\Delta^{\mathbb{I}}$ in $[\max\{\beta_n, \zeta_n\}, \min\{\beta_{n+1}, \zeta_{n+1}\}]$. Combining (a) with Lemma 5.1 (b) shows that there can be at most one zero of $\Delta^{\mathbb{I}}$ in each $[\max\{\beta_n, \zeta_n\}, \min\{\beta_{n+1}, \zeta_{n+1}\}]$. Similarly, combining (5.33) and (5.35) we have for

$$\lambda \in (\mu_n, \nu_{n+1}) \cap (\mu_n, \nu_n) = (\max\{\mu_n, \nu_n\}, \min\{\mu_{n+1}, \nu_{n+1}\}), \quad (5.37)$$

that $(-1)^n \Delta^J < 0$. Thus there must be at least one zero of Δ^J in $[\min\{\mu_n, \nu_n\}, \max\{\mu_n, \nu_n\}]$. Combining (a) with (b) shows that there is no more than one zero of Δ^J in $[\min\{\mu_n, \nu_n\}, \max\{\mu_n, \nu_n\}]$.

(d) Since Δ^J is absolutely continuous, S^J consists of closed intervals. Let S_0^J be the set of all maximally connected subsets of S^J containing at least one zero of $\Delta^{\mathbb{I}}$ and let S_1^J be the set of all maximally connected subsets of S^J containing no zeros of $\Delta^{\mathbb{I}}$. Thus we have

$$S^J = \bigcup S_0^J \cup \bigcup S_1^J. \quad (5.38)$$

We will show that $S_1^J = \emptyset$. Assume the contrary, then there exists some $I \neq \emptyset$ with $I \in S_1^J$. From Theorem 5.4 (c) it follows that since I contains no zeros of $\Delta^{\mathbb{I}}$, it must be strictly contained in $A_n := [\min\{\beta_n, \zeta_n\}, \max\{\beta_{n+1}, \zeta_{n+1}\}]$, for some fixed $n \in \mathbb{Z}$. Hence $I = [a, b]$ for some $a \leq b$, $a, b \in \mathbb{R}$. Furthermore, $I \subset A_n$ is a maximally connected component of S^J , thus either $\Delta^J(a) = 2 = \Delta^J(b)$ or $\Delta^J(a) = -2 = \Delta^J(b)$. However, this contradicts (5.16). Hence no such I exists.

What has been shown is that every maximally connected subset of S^J contains at least one zero of $\Delta^{\mathbb{I}}$. Furthermore, every maximally connected subset of S^J contains at most one zero of $\Delta^{\mathbb{I}}$, otherwise it would contradict (a) and (c), which show that the zeros of $\Delta^{\mathbb{I}}$ and Δ^J alternate. Lemma 5.1 with (c) show that the maximally connected subsets of $S^{\mathbb{I}}$ contain precisely one zero of Δ^J .

(e) This follows directly the fact that $\beta_n, \zeta_n \in S^J$ with Theorem 5.4 (c) and (d). ■

The *winding number* of a solution $Y(x)$ of equation (3.2) on $[0, \pi]$, counts the number of rotations of Y around the origin from the point $Y(0)$ to $Y(\pi)$. The next theorem shows that the \mathbb{I} and J instability intervals interlace, with the exception of the I -instability intervals containing the periodic eigenvalues of winding number 0.

Theorem 5.5 *The \mathbb{I} -instability and J -instability intervals, $S^{\mathbb{I}}$ and S^J , respectively, satisfy the following inclusion,*

$$S^J \subset (\mathbb{R} \setminus S^{\mathbb{I}}) \cup [\lambda_{-1}, \lambda_0], \quad (5.39)$$

where λ_{-1} and λ_0 are the periodic eigenvalues of solutions with winding number 0 about the origin.

Proof: Assume the contrary. Since S^J is the countable union of closed intervals, we may assume that there exists a $\lambda^* \in S^J$ such that $\lambda^* \in S^{\mathbb{I}} \setminus [\lambda_{-1}, \lambda_0]$ and $\mathbb{Y}(\lambda^*, \pi) = \pm J$. Since $S^{\mathbb{I}}$ is the union of disjoint closed intervals, we have that $\lambda^* \in [\lambda_{k^*-1}, \lambda_{k^*}]$ for some $k^* \in 2\mathbb{Z}$, $k^* \neq 0$. Consider the equation

$$JU'(z) + Q(z + \tau)U(z) = \lambda U(z), \quad (5.40)$$

where $U(\lambda, z) = \mathbb{Y}(\lambda, z + \tau)$. Since \mathbb{Y} is jointly continuous in its arguments, so is y_{12} thus $\mu_{k^*}(\tau)$ and $\lambda^*(\tau)$ are τ -continuous. Furthermore, the \mathbb{I} -discriminant of equation (5.40) is independent of τ , see [19], hence Theorem 5.3 implies that there exists $\tau_1, \tau_2 \in \mathbb{R}$, $\tau_1 < \tau_2$, such that $\mu_{k^*}(\tau_1) = \lambda_{k^*-1}$ and $\mu_{k^*}(\tau_2) = \lambda_{k^*}$. Without loss of generality we may assume that $\tau_1 < \tau_2$. Thus the mapping

$$\mu_{k^*} : [\tau_1, \tau_2] \rightarrow [\lambda_{k^*-1}, \lambda_{k^*}], \quad (5.41)$$

is surjective and $\lambda^*([\tau_1, \tau_2]) \subset [\lambda_{k^*-1}, \lambda_{k^*}]$. By the Intermediate value theorem, there exists a $\tau^* \in [\tau_1, \tau_2]$ such that $\lambda^*(\tau^*) = \mu_{k^*}(\tau^*)$. Thus $y_{12}(\pi + \tau^*) = 0$ and $y_{12}(\pi + \tau^*) = \pm 1$. Since $k^* \neq 0$ was chosen arbitrarily, no such λ^* exists in $S^{\mathbb{I}} \setminus [\lambda_{-1}, \lambda_0]$. ■

The next example shows the necessity of the previous theorem, that is, $S^J \not\subset (\mathbb{R} \setminus S^{\mathbb{I}})$.

Example 5.6 Consider the equation

$$JY' + m\sigma_3 Y = \lambda Y, \quad \text{where } m \in \mathbb{R}. \quad (5.42)$$

The above equation is $Y' = AY$ where $A := (-\lambda J - m\sigma_1)$. We may now decouple this equation by considering the eigenvalue decomposition, $A = P\Lambda P^{-1}$, where

$$P = \begin{pmatrix} \frac{\sqrt{m^2 - \lambda^2}}{m - \lambda} & -\frac{\sqrt{m^2 - \lambda^2}}{m - \lambda} \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} -\sqrt{m^2 - \lambda^2} & 0 \\ 0 & \sqrt{m^2 - \lambda^2} \end{pmatrix}. \quad (5.43)$$

Setting $Z = P^{-1}Y$ we have $Z' = \Lambda Z$, thus $Z = e^{\Lambda z}$, so that $Y = Pe^{\Lambda z}$. The superposition principle gives $\mathbb{Y} = Pe^{\Lambda z} P^{-1}$. Thus from a direct calculation we have

$$\mathbb{Y}(z, \lambda) = \begin{pmatrix} \cos \sqrt{\lambda^2 - m^2} z & \frac{\sqrt{\lambda^2 - m^2}}{\lambda - m} \sin \sqrt{\lambda^2 - m^2} z \\ \frac{m - \lambda}{\sqrt{\lambda^2 - m^2}} \sin \sqrt{\lambda^2 - m^2} z & \cos \sqrt{\lambda^2 - m^2} z \end{pmatrix}, \quad (5.44)$$

thus

$$\Delta^{\mathbb{I}}(\lambda) = 2 \cos \sqrt{\lambda^2 - m^2} \pi \quad \text{and} \quad \Delta^J(\lambda) = \frac{2\lambda}{\sqrt{\lambda^2 - m^2}} \sin \sqrt{\lambda^2 - m^2} \pi. \quad (5.45)$$

At $\lambda = m$ we have $\Delta^{\mathbb{I}} = 2$, and L'Hôpital's rule gives $\Delta^J = 2m\sqrt{\pi}$, so for $m = 1$ we have $S^J \cap [\lambda_{-1}, \lambda_0] \neq \emptyset$, hence $S^J \not\subset (\mathbb{R} \setminus S^{\mathbb{I}})$. For $m = 1$, we have the following graph in Figure 5.2.

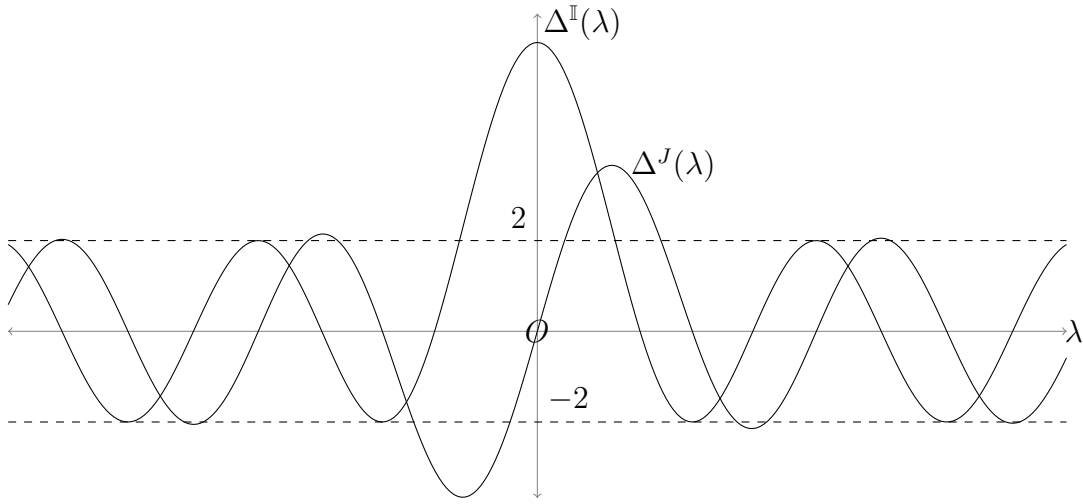


Figure 5.2: Plot of $\Delta^{\mathbb{I}}$ and Δ^J as a function of λ

We are now in the position to state the indexing result for the eigenvalues of the J periodic and J anti-periodic problems associated with L_7 and L_8 , respectively, with the Dirichlet problems associated with L_7 and L_8 . Let $S^J = S_{L_5}^J \cup S_{L_6}^J$ where

$$S_{L_5}^J = \{\lambda \in \mathbb{R} \mid \Delta^J \geq 2\} \quad \text{and} \quad S_{L_6}^J = \{\lambda \in \mathbb{R} \mid \Delta^J \leq -2\}. \quad (5.46)$$

We have the representations

$$S_{L_5}^J = \bigcup_k [\lambda_{2k-1}^*, \lambda_{2k}^*] \quad \text{and} \quad S_{L_6}^J = \bigcup_k [\lambda_{2k-1}^\dagger, \lambda_{2k}^\dagger], \quad (5.47)$$

where $[\lambda_{2k-1}^*, \lambda_{2k}^*]$ and $[\lambda_{2k-1}^\dagger, \lambda_{2k}^\dagger]$ are maximally connected subsets of S^J with boundaries being the eigenvalues of L_5 and L_6 , respectively. Theorems 5.4 (e) and 5.5 give the indexing

$$\lambda_{2k-1}^* < \{\beta_{2n-1}, \zeta_{2n-1}\} \leq \lambda_{2k-1}^* < \lambda_{2k-1}^\dagger \leq \{\beta_{2n}, \zeta_{2n}\} \leq \lambda_{2k}^\dagger < \lambda_{2k+1}^*. \quad (5.48)$$

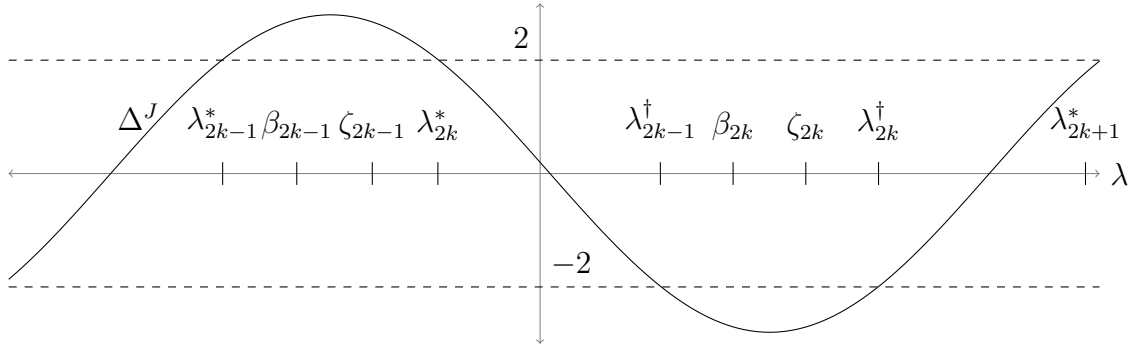


Figure 5.3: Plot of Δ^J showing the interlacing of various Dirichlet eigenvalues.

Also

$$\lambda'_{2k-1} \leq \lambda'_{2k} < \lambda^{\dagger}_{2k-1} \leq \lambda^{\dagger}_{2k} < \lambda_{2k-1} \leq \lambda_{2k} < \lambda^*_{2k-1} \leq \lambda^*_{2k} < \lambda'_{2k+1} \quad \text{for } k \neq 0, \tag{5.49}$$

this is expressed visually in Figure 5.4.

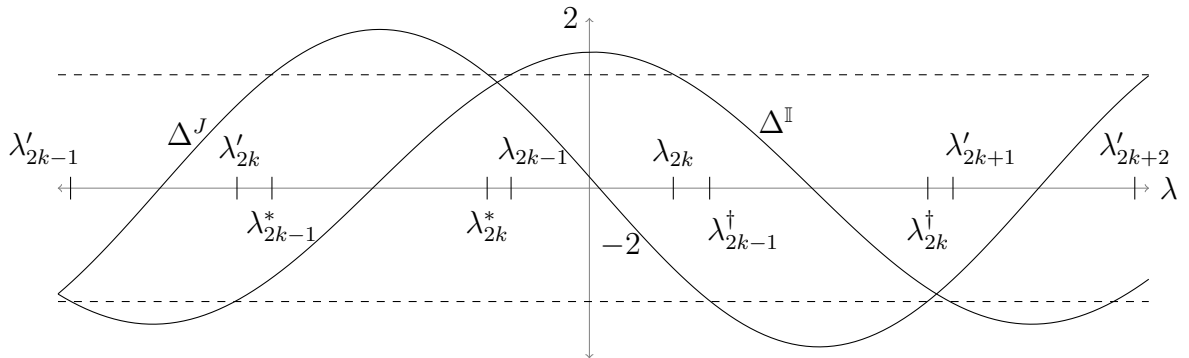


Figure 5.4: Plot of Δ^J showing the interlacing of J -periodic eigenvalues

6

Solution Asymptotics

“The essence of mathematics lies in its freedom.”

– Georg Cantor

6.1 Introduction

In this Chapter, resolution of the first term in the solution asymptotics for all values of the eigenparameter in \mathbb{C} are established for Hermitian Q integrable on compact sets, furthermore, the first and second terms are obtained if the potential is considered to be absolutely continuous. The literature contains various results for absolutely continuous potentials, [60], [89, pp. 1462], however, a resolution of the second term is absent. Similarly, there exist solution asymptotics on open sectors in \mathbb{C} for canonical systems with potentials integrable on compact sets, see [60, pp. 191], [17, pp. 3492], however asymptotics on all \mathbb{C} are absent from the literature.

6.2 Absolutely continuous canonical Q

We say that the potential Q is in canonical form if

$$Q(z) = \begin{pmatrix} q_1(z) & q_2(z) \\ q_2(z) & -q_1(z) \end{pmatrix}, \quad (6.1)$$

where q_1 and q_2 are real valued measurable functions. A direct computation gives that if Q is in canonical form then $JQ = -QJ$. Through out the remainder the norm of a matrix denotes the operator matrix norm

$$||[c_{ij}]|| = \max_j \sqrt{\sum_i |c_{ij}|^2}.$$

Solution asymptotics will be given only for the case of (3.2) with potential in canonical form as these are all that are required for the study of the inverse problem.

Theorem 6.1 *Let Q be in canonical form with entries absolutely continuous and $|Q'|$ integrable on $[0, \pi]$. The matrix solution \mathbb{Y} of (3.2) with initial condition (3.4) is of order 1 and for $|\lambda|$ large takes the asymptotic form*

$$\mathbb{Y}(z) = e^{-\lambda Jz} \left(I - \frac{Q(0)}{2\lambda} \right) + \frac{Q(z)e^{-\lambda Jz}}{2\lambda} + \int_0^z \frac{e^{\lambda J(t-z)}}{2\lambda} (JQ^2 - Q') e^{-\lambda Jt} dt + O\left(\frac{e^{|\Im \lambda|z}}{\lambda^2}\right).$$

Proof: Using variation of parameters we can represent equation (3.2) as the integral equation

$$\mathbb{Y}(z) = e^{-\lambda Jz} + \int_0^z e^{-\lambda J(z-t)} JQ\mathbb{Y} dt. \quad (6.2)$$

In the above equation take $\mathbb{Y}(z) = e^{|\Im \lambda|z} \mathbb{V}(z)$ giving

$$\mathbb{V}(z) = e^{-|\Im \lambda|z} e^{-\lambda Jz} + \int_0^z e^{-|\Im \lambda|(z-t)} e^{-\lambda J(z-t)} JQ\mathbb{V}(t) dt. \quad (6.3)$$

From (6.3) we have

$$|\mathbb{V}(z)| \leq 1 + \int_0^z |Q| |\mathbb{V}(t)| dt. \quad (6.4)$$

Applying Gronwall's Lemma [51, Lemma 6.3.6] to (6.4) gives

$$|\mathbb{V}(z)| \leq \exp\left(\int_0^\pi |Q| dt\right).$$

Hence $\mathbb{V}(z) = O(1)$ and thus $\mathbb{Y}(z) = O(e^{|\Im\lambda|z})$.

Let $\mathbb{Y}(z) = e^{-\lambda Jz}\mathbb{W}(z)$. In terms of \mathbb{W} (3.2) becomes

$$\mathbb{W}' = J e^{2\lambda Jz} Q \mathbb{W}. \quad (6.5)$$

and thus

$$Q \mathbb{W}' = -J e^{-2\lambda Jz} Q^2 \mathbb{W}. \quad (6.6)$$

Integrating (6.5) from 0 to z gives

$$\mathbb{W}(z) = \mathbb{W}(0) + \frac{1}{2\lambda} \int_0^z \frac{d(e^{2\lambda Jt})}{dt} Q(t) \mathbb{W}(t) dt, \quad (6.7)$$

which, when integrated by parts, yields

$$\mathbb{W}(z) = \mathbb{W}(0) + \frac{1}{2\lambda} [e^{2\lambda Jt} Q \mathbb{W}]_0^z - \frac{1}{2\lambda} \int_0^z e^{2\lambda Jt} (Q' \mathbb{W} + Q \mathbb{W}') dt. \quad (6.8)$$

Combining (6.6) and (6.8) gives

$$\mathbb{W}(z) = \mathbb{W}(0) + \frac{1}{2\lambda} [e^{2\lambda Jt} Q \mathbb{W}]_0^z - \frac{1}{2\lambda} \int_0^z e^{2\lambda Jt} Q' \mathbb{W} dt + \frac{J}{2\lambda} \int_0^z Q^2 \mathbb{W} dt. \quad (6.9)$$

Thus

$$\mathbb{Y}(z) = e^{-\lambda Jz} \left(I - \frac{Q(0)}{2\lambda} \right) + \frac{Q(z)\mathbb{Y}(z)}{2\lambda} + \int_0^z \frac{e^{\lambda J(t-z)}}{2\lambda} (JQ^2 - Q') \mathbb{Y} dt. \quad (6.10)$$

Here $|\mathbb{Y}(t)| = O(e^{|\Im\lambda|t})$ and $|e^{\lambda J(t-z)}| = O(e^{|\Im\lambda|(z-t)})$ for $0 \leq t \leq z$, so from (6.10)

$$\mathbb{Y}(z) = e^{-\lambda Jz} + O\left(\frac{e^{|\Im\lambda|z}}{\lambda}\right). \quad (6.11)$$

Substituting (6.11) into (6.10) gives

$$\mathbb{Y}(z) = e^{-\lambda Jz} \left(I - \frac{Q(0)}{2\lambda} \right) + \frac{Q(z)e^{-\lambda Jz}}{2\lambda} + \int_0^z \frac{e^{\lambda J(t-z)}}{2\lambda} (JQ^2 - Q') e^{-\lambda Jt} dt + O\left(\frac{e^{|\Im\lambda|z}}{\lambda^2}\right)$$

proving the theorem. \blacksquare

Applying the Riemann-Lebesgue Lemma [67] to Theorem 6.1 gives the courser but simpler asymptotic approximation

$$\mathbb{Y}(z) = e^{-\lambda Jz} \left[I - \frac{Q(0)}{2\lambda} + \frac{J}{2\lambda} \int_0^z Q^2 dt \right] + e^{\lambda Jz} \frac{Q(z)}{2\lambda} + o\left(\frac{e^{|\Im\lambda|z}}{\lambda}\right). \quad (6.12)$$

6.3 Integrable Hermitian Q

We now give an asymptotic approximation for \mathbb{Y} in the case of $|\lambda|$ large. To insure that the integral of a matrix is bounded, we will make use of the following operator matrix norm

$$|[c_{ij}]| = \max_j \sum_i |c_{ij}|.$$

Lemma 6.2 *Let $Q = Q_1 + Q_2$ (as in (3.10)) be complex valued and integrable on $[0, \pi)$. The matrix solutions \mathbb{Y} and \mathbb{U} of $\ell Y = \lambda Y$ satisfying the conditions, $\mathbb{Y}(0) = \mathbb{I} = \mathbb{U}(\pi)$, are of order 1. For $z \in \mathbb{R}$ and $\lambda = re^{i\theta}$ with $r \rightarrow \infty$, we have uniformly in θ and z , that*

$$\mathbb{Y}(z) = e^{-J\lambda z} e^{J \int_0^z Q_2 dt} + o(e^{|\Im \lambda z|}), \quad (6.13)$$

$$\mathbb{U}(z) = e^{-J\lambda(\pi-z)} e^{J \int_0^{(\pi-z)} Q_2 dt} + o(e^{|\Im \lambda(\pi-z)|}). \quad (6.14)$$

Proof: Consider the transformation $\mathbb{Y}(z) = e^{J \int_0^z Q_2 dt} \tilde{\mathbb{Y}}(z)$ for $z \geq 0$. Substituting this transformation into (3.1) gives

$$J\tilde{\mathbb{Y}}' + \tilde{Q}\tilde{\mathbb{Y}} = \lambda\tilde{\mathbb{Y}} \quad (6.15)$$

where

$$\tilde{Q}(z) = e^{-J \int_0^z Q_2 dt} Q_1(z) e^{J \int_0^z Q_2 dt}. \quad (6.16)$$

Notice that \tilde{Q} is a real canonical matrix. Let $\tau := \Im \lambda$ and $\rho := \Re \lambda$. Using variation of parameters, [18, pp. 74], $\tilde{\mathbb{Y}}$ obeys the integral equation

$$\tilde{\mathbb{Y}}(z) = e^{-\lambda J z} + \int_0^z e^{-\lambda J(z-t)} J\tilde{Q}(t)\tilde{\mathbb{Y}}(t) dt. \quad (6.17)$$

Setting $\tilde{\mathbb{Y}}(z) = e^{\tau z} \mathbb{V}(z)$ we have

$$\mathbb{V}(z) = e^{-(\lambda J + \mathbb{I}\tau)z} + \int_0^z e^{-(\lambda J + \mathbb{I}\tau)(z-t)} J\tilde{Q}(t)\mathbb{V}(t) dt, \quad (6.18)$$

giving

$$|\mathbb{V}(z)| \leq 1 + \int_0^z |\tilde{Q}||\mathbb{V}|dt. \quad (6.19)$$

Using Gronwall's inequality, [51, Lemma 6.3.6], we have the estimate $\mathbb{V} = O(1)$, thus $\tilde{\mathbb{Y}} = O(e^{\tau z})$. Set $W(z) := e^{-(J\lambda + \mathbb{I}\tau)z}$. Substituting (6.18) back into itself gives

$$\mathbb{V}(z) = W(z) + \int_0^z JW(z-t)\tilde{Q}(t)W(t)dt + \int_0^z \int_0^t W(z-t)\tilde{Q}(t)W(t-s)\tilde{Q}(s)\mathbb{V}(s)dsdt, \quad (6.20)$$

since $\tilde{Q}J = -J\tilde{Q}$. For $x, y \in \mathbb{R}$, $z \geq 0$, we have

$$W(x)\tilde{Q}(z)W(y) = e^{-(\lambda J + \mathbb{I}|\tau|x)e^{(\lambda J - \mathbb{I}|\tau|)y}}\tilde{Q}(z), \quad (6.21)$$

$$= e^{-\rho J(x-y)}e^{-|\tau|(x+y)}e^{-i\tau J(x-y)}\tilde{Q}(z), \quad (6.22)$$

Furthermore, setting $f(x, y) := e^{-|\tau|(x+y)}e^{-i\tau J(x-y)}$ we have

$$f(x, y) = \frac{1}{2}\mathbb{I}(e^{-2|\tau|x} + e^{-2|\tau|y}) + \frac{\text{sgn}\tau}{2i}J(e^{-2|\tau|x} - e^{-2|\tau|y}), \quad (6.23)$$

thus combining (6.22) and (6.23) gives

$$W(x)\tilde{Q}W(y) = O(|\tilde{Q}|e^{-2|\tau|\min\{x,y\}}). \quad (6.24)$$

From (6.24) we have the following bound

$$|W(z-t)\tilde{Q}(t)W(t)| \leq k|\tilde{Q}(t)|e^{-2\min\{z-t,t\}}, \quad (6.25)$$

for some $k > 0$, independent of λ , x and y . Using (6.25), the Lebesgue dominated convergence theorem shows that

$$\int_0^z W(z-t)\tilde{Q}(t)W(t)dt = O\left(\int_0^z |\tilde{Q}(t)|e^{-2|\tau|\min\{z-t,t\}}dt\right), \quad (6.26)$$

tends to zero as $|\tau|$ tends to infinity. While for $|\tau| = c < c'$, using (6.22), we have that the second term on the right hand side of (6.20) is equal to

$$\int_0^z (\mathbb{I} \cos \rho(z-2t) - J \sin \rho(z-2t))f(z-t, t)\tilde{Q}(t)dt, \quad (6.27)$$

where $f(z-t, t)\tilde{Q}(t)$ is integrable on $[0, \pi]$. Thus by the Riemann-Lebesgue Lemma, (6.27) tends to zero as $|\rho|$ tends to infinity. Hence the second term on the right hand

side of (6.20) tends to zero uniformly in $\arg(\lambda)$ as $|\lambda|$ tends to infinity. The uniformity here follows from the uniformity of this limit as $|\tau|$ tends to infinity, thus this limit holds as $|\rho|$ tends to infinity for fixed c .

By changing the order of integration, the double integral in (6.20) is equal to

$$\int_0^z \left(\int_\tau^z W(z-t)\tilde{Q}(t)W(t-\tau)dt \right) \tilde{Q}(\tau)\mathbb{V}(\tau)d\tau. \quad (6.28)$$

From the reasoning above, the inner integral in (6.28) tends to zero as $|\lambda|$ tends to infinity, thus, as \mathbb{V} is bounded, so does the double integral. So from (6.20) for large $|\lambda|$,

$$\mathbb{V}(z) = e^{-(\lambda J + \mathbb{I}|\tau|)z} + o(1). \quad (6.29)$$

Substituting (6.29) back into the expression for $\tilde{\mathbb{Y}}$ gives

$$\mathbb{Y}(z) = e^{-J\lambda z} e^{J \int_0^z Q_2 dt} + o(e^{|\tau|z}) \quad \text{for } z \geq 0. \quad (6.30)$$

Assuming that $z \leq 0$, we may apply the transformation $\hat{z} = -z$, $\hat{Y}(\hat{z}) = Y(z)$, $\hat{Q}(\hat{z}) = -Q(z)$ and $\hat{\lambda} = -\lambda$ to transform $\ell Y = \lambda Y$ into

$$J\hat{Y}'(\hat{z}) + \hat{Q}(\hat{z})\hat{Y}(\hat{z}) = \lambda\hat{Y}(\hat{z}). \quad (6.31)$$

From the above work $\hat{\mathbb{Y}}(\hat{z})$ is given by

$$\hat{\mathbb{Y}}(\hat{z}) = e^{-J\hat{\lambda}\hat{z}} e^{J \int_0^{\hat{z}} \hat{Q}_2 dt} + o(e^{|\tau|\hat{z}}). \quad (6.32)$$

Thus substituting the transformations above we have

$$\mathbb{Y}(z) = e^{-J\lambda z} e^{J \int_0^z Q_2 dt} + o(e^{-|\tau|z}) \quad \text{for } z \leq 0. \quad (6.33)$$

Combining (6.30) and (6.33) gives (6.13). To obtain (6.14), set $\check{\mathbb{U}}(\check{x}) := \mathbb{U}(\pi - x)$ where $\check{x} = \pi - x$. Thus $\check{\mathbb{U}}$ with $\check{\mathbb{U}}(0) = \mathbb{I}$ is a solution to $\ell Y = \lambda Y$ with potential $\check{Q}(\check{x}) := -Q(\pi - x)$. Finally we can apply (6.13) to obtain (6.14). ■

7

Δ^{II} inverse problem

“Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.”

– David Hilbert

7.1 Introduction

In this chapter, a shifting lemma is proved which is the Hermitian systems analogue to [46]. The main theorem of this chapter is Theorem 7.5 in which we consider a canonical system in \mathbb{R}^2 with real symmetric absolutely continuous π -periodic matrix potential. We prove that if all instability intervals are empty, then the matrix potential is diagonal with the two diagonal entries equal, analogous results for Hill’s equation can be found in [12, pages 94-111] and [46].

7.2 Inverse Problem

The following lemma shows that $\Delta(\lambda)$ is independent of replacement of $Q(z)$ by $Q(z + \tau)$, that is $\Delta(\lambda)$ is independent of shifts of the independent variable in the potential. We first prove a necessary lemma for the inverse problem.

Lemma 7.1 *Let $\Delta(\lambda, \tau)$ denote the discriminant of*

$$JU'(z) + [Q(z + \tau) - \lambda I]U(z) = 0, \quad (7.1)$$

for $\tau \in \mathbb{R}$, then $\Delta(\lambda, \tau)$ is independent of τ .

Proof: Let $U_1(z, \tau) = \begin{pmatrix} u_{11}(z, \tau) \\ u_{12}(z, \tau) \end{pmatrix}$ and $U_2(z, \tau) = \begin{pmatrix} u_{21}(z, \tau) \\ u_{22}(z, \tau) \end{pmatrix}$ be the solutions of (7.1) which satisfy the initial conditions

$$[U_1 \ U_2](0, \tau) = I. \quad (7.2)$$

Let $\mathbb{U} = [U_1 \ U_2]$. Since $Y_1(z + \tau)$ and $Y_2(z + \tau)$ are solutions of (7.1) and a basis for the solution set of (7.1), we may represent $U_1(z, \tau)$ and $U_2(z, \tau)$ as a linear combination of $Y_1(z + \tau)$ and $Y_2(z + \tau)$ giving

$$\mathbb{U}(z, \tau) = \mathbb{Y}(z + \tau)B(\tau), \quad (7.3)$$

where $B(\tau)$ is an invertible matrix. Inverting $B(\tau)$ and setting $z = 0$ we obtain $B^{-1}(\tau) = \mathbb{Y}(\tau)$.

Thus

$$\mathbb{U}(\pi, \tau) = \mathbb{Y}(\pi + \tau)\mathbb{Y}^{-1}(\tau) = \mathbb{Y}(\tau)\mathbb{Y}(\pi)\mathbb{Y}^{-1}(\tau). \quad (7.4)$$

From above we have

$$\Delta_\tau = \text{Trace}(\mathbb{U}(\pi, \tau)) = \text{Trace}(\mathbb{Y}(\pi)) = \Delta \quad (7.5)$$

■

We are now in a position to characterise the class of real symmetric matrices, Q , with absolutely continuous entries for which the instability intervals of (3.2) vanish, see Theorem 7.5.

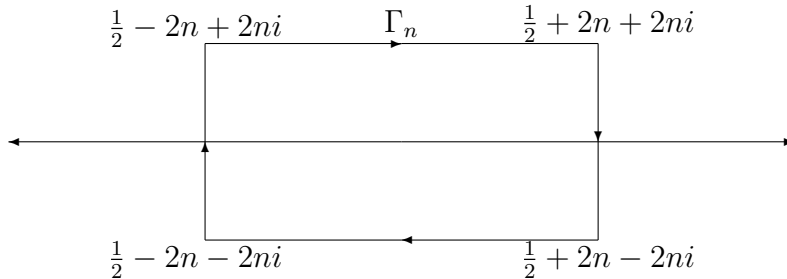
Lemma 7.2 *Let Q be in canonical form and have absolutely continuous entries which are π -periodic on \mathbb{R} . All instability intervals of (3.2) vanish if and only if $Q = 0$.*

Proof: If $Q(z) = 0$, then $\mathbb{Y}(z) = e^{-\lambda J z}$ giving $\Delta(\lambda) = 2 \cos(\lambda z)$. So for all real λ , $|\Delta| \leq 2$. Thus all instability intervals vanishes.

From the converse, suppose that all instability intervals of (3.2) vanish. Now $\lambda'_{2k-1} = \lambda'_{2k}$ and $\lambda_{2k-1} = \lambda_{2k}$, $k \in \mathbb{Z}$. So from (5.12) $\lambda'_{2k-1} = \mu_{2k-1}(\tau) = \nu_{2k-1}(\tau) = \lambda'_{2k}$ and $\lambda_{2k-1} = \mu_{2k}(\tau) = \nu_{2k}(\tau) = \lambda_{2k}$ for all $\tau \in \mathbb{R}$. In the notation of Lemma 7.1, as a consequence of the above equality, the λ -zeros of the entire functions $u_{ij}(\pi, \tau)$, $i \neq j$, are $\{\lambda_{2k} | k \in \mathbb{Z}\} \cup \{\lambda'_{2k} | k \in \mathbb{Z}\}$, for each $\tau \in \mathbb{R}$. In addition the zeros of $u_{ij}(\pi, \tau)$, $i \neq j$, are simple for each $\tau \in \mathbb{R}$. Here $[u_{ij}]_{(j,i)} = \mathbb{U}$. Thus $u_{ij}(\pi, \tau)/u_{ij}(\pi, 0)$, for each $\tau \in \mathbb{R}$ and $i \neq j$, is an entire function of λ . However, from Theorem 6.1,

$$u_{ij}(\pi, \tau) = (-1)^j \sin \lambda \pi + O\left(\frac{e^{|\Im \lambda| \pi}}{\lambda}\right), \quad i \neq j. \quad (7.6)$$

Let Γ_n , $n \in \mathbb{N}$, denote the closed paths in \mathbb{C} consisting of the squares with corners at $2n(1 \pm i) + \frac{1}{2}$ and $-2n(1 \mp i) + \frac{1}{2}$.



On the edges of Γ_n parametrised by $\lambda = \pm(2n + it) + \frac{1}{2}$, $t \in [-2n, 2n]$, for $i \neq j$, we have

$$u_{ij}(\pi, \tau) = (-1)^j \cosh \pi t + O\left(\frac{e^{\pi|t|}}{n}\right) = (-1)^j \frac{e^{\pi|t|}}{2} \left(1 + O\left(\frac{1}{n}\right)\right),$$

giving

$$\frac{u_{ij}(\pi, \tau)}{u_{ij}(\pi, 0)} = 1 + O\left(\frac{1}{n}\right).$$

On the edges of Γ_n parametrized by $\lambda = \pm(2ni - t) + \frac{1}{2}, t \in [-2n, 2n]$, we have

$$u_{ij}(\pi, \tau) = (-1)^j \cos \pi(t - 2ni) + O\left(\frac{e^{\pi|t|}}{n}\right) = (-1)^j \frac{e^{2\pi n}}{2} \left(e^{\pi it} + O\left(\frac{1}{n}\right) \right),$$

giving

$$\frac{u_{ij}(\pi, \tau)}{u_{ij}(\pi, 0)} = 1 + O\left(\frac{1}{n}\right).$$

Thus by the maximum modulus principal, for $i \neq j$,

$$\left| \frac{u_{ij}(\pi, \tau)}{u_{ij}(\pi, 0)} - 1 \right| = O\left(\frac{1}{n}\right).$$

on the region enclosed by Γ_n for each $n \in \mathbb{N}$. Taking $n \rightarrow \infty$ gives

$$\frac{u_{ij}(\pi, \tau)}{u_{ij}(\pi, 0)} = 1, \quad i \neq j,$$

on \mathbb{C} , and $u_{ij}(\pi, \tau) = u_{ij}(\pi, 0)$, for all $\tau \in \mathbb{R}$, $i \neq j$, on \mathbb{C} . By Lemma 7.1, $\Delta(\lambda, \tau) = \Delta(\lambda)$ for $\tau \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. Thus, as functions of λ , we have

$$u_{ij}(\pi, \tau) = y_{ij}(\pi) \quad \text{for } i \neq j, \quad (7.7)$$

$$u_{11}(\pi, \tau) + u_{22}(\pi, \tau) = y_{11}(\pi) + y_{22}(\pi). \quad (7.8)$$

Setting $\gamma(\tau, \lambda) := u_{11}(\pi, \tau) - y_{11}(\pi)$ it follows that $u_{22}(\pi, \tau) = y_{22}(\pi) - \gamma(\tau, \lambda)$ and

$$\mathbb{U}(\pi, \tau) = \mathbb{Y}(\pi) + \gamma(\tau, \lambda)\sigma_3, \quad (7.9)$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Combining (6.12) and (7.9) gives

$$\gamma(\tau, \lambda)\sigma_3 = \mathbb{U}(\pi, \tau) - \mathbb{Y}(\pi) = \frac{\sin \lambda \pi}{\lambda} J [Q(\tau) - Q(0)] + o\left(\frac{e^{|\Im \lambda| \pi}}{\lambda}\right). \quad (7.10)$$

Considering equation (7.7) the off diagonal components in (7.10) are equal to zero, thus in the notation of (6.1) we have

$$(q_1(\tau) - q_1(0)) \sin(\lambda \pi) = o\left(e^{|\Im \lambda| \pi}\right). \quad (7.11)$$

Now setting $\lambda = 2n + \frac{1}{2}$ for $n \in \mathbb{N}$ in (7.11) gives $q_1(\tau) - q_1(0) = o(1)$, from which it follows that $q_1(\tau) = q_1(0)$ for all $\tau \in \mathbb{R}$. Hence q_1 is constant.

Let $\tilde{Y}(z) = e^{J\omega}Y(z)$, $\omega \in \mathbb{R}$. This unitary transformation transforms (3.2) to

$$J\tilde{Y}' + \tilde{Q}\tilde{Y} = \lambda\tilde{Y}, \quad (7.12)$$

where $\tilde{Q} = e^{2J\omega}Q$. Such unitary transformations are isospectral, thus the periodic eigenvalues of problem (3.2) and (7.12) are the same, and similarly for the antiperiodic eigenvalues, see [60, Ch. 7.1]. Setting $\omega = \pi/4$, then, in the notation of (6.1),

$$\tilde{Q}(z) = \begin{pmatrix} q_2(z) & -q_1(z) \\ -q_1(z) & -q_2(z) \end{pmatrix}, \quad (7.13)$$

which is in canonical form. The first part of the proof can now be applied to (7.12) to give q_2 constant.

Having established that q_1 and q_2 , and thus Q , are constant we set $\omega = \frac{1}{2} \arctan\left(\frac{q_2}{q_1}\right)$ in the above transformation, to give

$$\tilde{Q} = m\sigma_3 \quad \text{where} \quad m = \sqrt{q_1^2 + q_2^2}. \quad (7.14)$$

Equation (7.12) with \tilde{Q} as in (7.14), is the free particle Dirac system studied in [101, Appendix]. Using the fundamental matrix obtained in [101, Appendix], or by direct computation, we have that

$$\Delta(\lambda) = 2 \cos \sqrt{(\lambda^2 - m^2)\pi}.$$

Since, by assumption, all instability intervals vanish $|\Delta(\lambda)| \leq 2$ for all $\lambda \in \mathbb{R}$. In particular $|\Delta(0)| \leq 2$, giving $\cosh(m\pi) \leq 1$ and so $m = 0$. Thus $Q = 0$. \blacksquare

Lemma 7.3 *Let Q be a symmetric matrix with real valued absolutely continuous π -periodic entries. If all instability intervals of (3.2) vanish then $Q = pI$ where $p := \frac{\text{trace}(Q)}{2}$. In this case $\lambda_{2k-1} = \lambda_{2k} = 2k + \frac{1}{\pi} \int_0^\pi p dt = 1 + \lambda'_{2k-1} = 1 + \lambda'_{2k}$, for $k \in \mathbb{Z}$.*

Proof: Let

$$h(z) = \frac{\pi - z}{\pi} \int_0^z p dt - \frac{z}{\pi} \int_z^\pi p dt,$$

then $h(0) = 0$ and $h(\pi) = 0$, so h can be extended to a π -periodic function on \mathbb{R} . Let $Y(z) = e^{Jh(z)}X(z)$ then $Y(0) = X(0)$ and $Y(\pi) = X(\pi)$, so the transformation preserves boundary conditions. Here $X(z)$ obeys the equation

$$JX' + \tilde{Q}X = \tilde{\lambda}X \quad (7.15)$$

where

$$\tilde{Q} = e^{-Jh(z)} (Q(z) - pI) e^{Jh(z)}, \quad (7.16)$$

$$\tilde{\lambda} = \lambda - \frac{1}{\pi} \int_0^\pi p dt, \quad (7.17)$$

and \tilde{Q} is a real symmetric matrix valued function with π -periodic absolutely continuous entries. As $\text{trace}(Q(z) - pI) = 0$ we have $\text{trace}(\tilde{Q}) = 0$ and \tilde{Q} is in canonical form. In addition the $\tilde{\lambda}$ -eigenvalues of (7.15) with periodic and anti-periodic boundary conditions are precisely the λ -eigenvalues of (3.2) with respectively periodic and anti-periodic boundary conditions, but shifted by $-\frac{1}{\pi} \int_0^\pi p dt$. If all instability intervals of (3.2) vanish, so do those of (7.15). Lemma 7.2 can now be applied to (7.15) to give $\tilde{Q} = 0$. Hence $Q(z) = pI$, from which the first claim of the lemma follows. In this case direct computation gives

$$\tilde{\Delta}(\tilde{\lambda}) = 2 \cos \tilde{\lambda}\pi, \quad (7.18)$$

where $\tilde{\Delta}$ is the discriminant of (7.15). From Section 4, (7.18) and direct computation we see that for $\tilde{Q} = 0$, $\tilde{\lambda}_{2k-1} = \tilde{\lambda}_{2k} = 2k$ and by (5.12) and (7.18), $\tilde{\lambda}'_{2k-1} = \tilde{\lambda}'_{2k} = 2k - 1$, $k \in \mathbb{Z}$, from which along with (7.17) the remaining claims of the lemma follow. ■

Lemma 7.4 *If p is a real (scalar) valued π -periodic function which is integrable on compact sets then all instability intervals vanish for the equation*

$$JY' + pY = \lambda Y. \quad (7.19)$$

Proof: A direct computation yields that for (7.19) we have

$$\mathbb{Y}(z) = e^{J(\int_0^z p dt - \lambda z)}.$$

Taking the trace of $\mathbb{Y}(\pi)$ gives

$$\Delta(\lambda) = 2 \cos \left(\lambda \pi - \int_0^\pi p dt \right)$$

from which it follows that all instability intervals vanish. ■

Combining Lemma 7.3 and Lemma 7.4 we obtain our main theorem.

Theorem 7.5 *Let Q be a real symmetric matrix valued function with absolutely continuous π -periodic entries. All instability intervals of (3.2) vanish if and only if $Q = pI$ for some absolutely continuous real (scalar) valued π -periodic function p .*

8

Borg's Periodicity Theorems

“If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.”

– John von Neumann

8.1 Introduction

In this chapter, Lemmas 8.2 and 8.3 establish an important relation between the \mathbb{L} -discriminant of a Hermitian system and the behaviour of the fundamental solution at π and $\frac{\pi}{2}$ (these are also referred to as monodromy matrices). These Lemmas are essential for studying the inverse problem. The main results of this chapter, Theorems 8.4 and 8.5 are the Hermitian system analogues to the Sturm-Liouville results obtained in [9, 47] and [48], respectively. Corollary 8.6 shows that uniqueness is possible only in the case when Q is in canonical form. Finally, it is shown as a pleasant consequence, that Borg's uniqueness result for canonical systems is derivable from the Borg Periodicity Theorems. Furthermore, the extent to which this uniqueness result fails for Hermitian systems is characterised. This work uses ideas presented in [47],

however, as far as the author is aware, the results presented here are new.

8.2 Lemmas

The following lemmas are necessary for the inverse problem. The first such lemma follows the method in [34, p. 30], for Sturm-Liouville problems.

Lemma 8.1 *Let \mathbb{Y} and $\tilde{\mathbb{Y}}$ be solutions of $\ell Y = \lambda Y$ satisfying the initial conditions (3.4) with canonical potentials $Q = Q_1$ and $\tilde{Q} = \tilde{Q}_1$, respectively. If $\tilde{\mathbb{Y}}(\pi, \lambda) = \mathbb{Y}(\pi, \lambda)$, for all $\lambda \in \mathbb{C}$, then $\tilde{\mathbb{Y}}(z, \lambda) = \mathbb{Y}(z, \lambda)$, for all $z \in \mathbb{R}$, $\lambda \in \mathbb{C}$.*

Proof: Define the linear boundary operators, $U(Y) := y_2(0)$ and $V(Y) := y_2(\pi)$. Let $\Phi(z, \lambda)$ be defined by

$$\Phi := Y_2 + MY_1, \quad (8.1)$$

where M is chosen so that $V(\Phi) = 0$. Note $U(\Phi) = 1$. Thus $M = -\frac{V(Y_2)}{V(Y_1)} = -\frac{y_{22}(\pi)}{y_{12}(\pi)}$. Setting $Y_3 := y_{12}(\pi)Y_2 - y_{22}(\pi)Y_1$, we have

$$Y_3 = \Delta_0 \Phi \quad \text{where} \quad \Delta_0 := \text{Wron}[Y_1, Y_3] = y_{12}(\pi). \quad (8.2)$$

Let $P(z, \lambda)$ be given by

$$P(z, \lambda) \begin{pmatrix} \tilde{y}_{11} & \tilde{\Phi}_1 \\ \tilde{y}_{12} & \tilde{\Phi}_2 \end{pmatrix} = \begin{pmatrix} y_{11} & \Phi_1 \\ y_{12} & \Phi_2 \end{pmatrix}. \quad (8.3)$$

Since $\text{Wron}[\tilde{Y}_1, \tilde{\Phi}] = 1$, a direct calculation gives

$$P(z, \lambda) = \begin{pmatrix} y_{11}\tilde{\Phi}_2 - \tilde{y}_{12}\Phi_1 & \tilde{y}_{11}\Phi_1 - y_{11}\tilde{\Phi}_1 \\ y_{12}\tilde{\Phi}_2 - \tilde{y}_{12}\Phi_2 & \tilde{y}_{11}\Phi_2 - y_{12}\tilde{\Phi}_1 \end{pmatrix}. \quad (8.4)$$

Substituting (8.1) into the above equation gives

$$P(z, \lambda) = \begin{pmatrix} y_{11}\tilde{y}_{22} - y_{21}\tilde{y}_{12} & y_{21}\tilde{y}_{11} - y_{11}\tilde{y}_{21} \\ y_{12}\tilde{y}_{22} - y_{22}\tilde{y}_{12} & y_{22}\tilde{y}_{11} - y_{12}\tilde{y}_{21} \end{pmatrix} + (\tilde{M} - M) \begin{pmatrix} y_{11}\tilde{y}_{12} & -y_{11}\tilde{y}_{11} \\ y_{12}\tilde{y}_{12} & -y_{12}\tilde{y}_{11} \end{pmatrix}. \quad (8.5)$$

Since $\mathbb{Y}(\pi) = \tilde{\mathbb{Y}}(\pi)$, we have that $M(\lambda) = \tilde{M}(\lambda)$ for every λ , thus $P(z, \lambda)$ is entire for each $z \in \mathbb{R}$, as is $P(z, \lambda) - \mathbb{I}$. Combining (8.1) and (8.5) with the identity $\Delta_0 \mathbb{I} = \langle Y_1, Y_3 \rangle \mathbb{I}$ gives

$$\Delta_0(P(z, \lambda) - \mathbb{I}) = \begin{pmatrix} y_{11}(\tilde{y}_{32} - y_{32}) - y_{31}(\tilde{y}_{12} - y_{12}) & y_{31}\tilde{y}_{11} - y_{11}\tilde{y}_{31} \\ y_{12}\tilde{y}_{32} - y_{32}\tilde{y}_{12} & y_{32}(\tilde{y}_{11} - y_{11}) - y_{12}(\tilde{y}_{31} - y_{31}) \end{pmatrix}. \quad (8.6)$$

Substituting the asymptotic expressions from Lemma 6.2 for Y_1 and Y_3 into the right hand side of the above equation gives

$$\Delta_0 P(z, \lambda) = \Delta_0 \mathbb{I} + o(e^{|\Im \lambda| \pi}) \quad \text{for } \lambda \in \mathbb{C}. \quad (8.7)$$

We also note from Lemma 6.2, the asymptotic estimate

$$\Delta_0 = -\sin \lambda \pi + o(e^{|\Im \lambda| \pi}). \quad (8.8)$$

Define the sets $\tilde{D}_\epsilon^k := \{\lambda : |\sin \lambda \pi| < \epsilon, |\lambda - k| < \frac{1}{2}, k \in \mathbb{Z}\}$. Let $\tilde{D}_\epsilon = \bigcup_k \tilde{D}_\epsilon^k$ and notice that for large $|\lambda|$ and some $\epsilon > 0$, \tilde{D}_ϵ^k contains exactly one zero of Δ_0 . Furthermore for large $|\lambda|$,

$$|\Delta_0| e^{-|\Im \lambda| \pi} \geq \epsilon + o(1) \quad \text{for every } \lambda \in \mathbb{C} \setminus \tilde{D}_\epsilon. \quad (8.9)$$

This shows that for some $\epsilon > 0$ there is a $C^* \in \mathbb{R}$ such that for large $|\lambda|$ we have

$$|\Delta_0| \geq C^* e^{|\Im \lambda| \pi} \quad \text{for every } \lambda \in \mathbb{C} \setminus \tilde{D}_\epsilon. \quad (8.10)$$

Thus

$$\frac{1}{\Delta_0} = O(e^{-|\Im \lambda| \pi}) \quad \text{for } \lambda \in \mathbb{C} \setminus \tilde{D}_\epsilon. \quad (8.11)$$

Combining equations (8.7), (8.8) and (8.11) gives that

$$P(z, \lambda) = \mathbb{I} + o(1) \quad \text{for } \lambda \in \mathbb{C} \setminus \tilde{D}_\epsilon. \quad (8.12)$$

The maximum modulus principle shows that relation (8.12) holds on \mathbb{C} . Thus P is bounded on \mathbb{C} , hence by Liouville's Theorem, $P = \mathbb{I}$ on \mathbb{C} . Finally, equation (8.3) completes the Lemma. \blacksquare

We now prove some important lemmas for the Borg inverse theorems.

Lemma 8.2 *Suppose Q in $\ell Y = \lambda Y$ is a 2×2 π -periodic matrix function integrable on $[0, \pi)$, of the form $Q = Q_1$ then $\Delta + 2$ has only double zeros if and only if $\mathbb{Y}(\pi) = \mathbb{Y}(\frac{\pi}{2})^2$.*

Proof: Assume $\mathbb{Y}(\pi) = \mathbb{Y}(\frac{\pi}{2})^2$. A direct calculation gives

$$\mathbb{Y}^2(\frac{\pi}{2}) = \frac{1}{4}(\Delta_+^{\mathbb{I}}\mathbb{I} + \Delta_+^J J + \nabla_+^{\mathbb{I}}\sigma_3 + \nabla_+^J\sigma_1)^2, \quad (8.13)$$

$$= \frac{1}{4}((\Delta_+^{\mathbb{I}})^2 + (\nabla_+^J)^2 + (\nabla_+^{\mathbb{I}})^2 - (\Delta_+^J)^2)\mathbb{I} + \sum_{i=1}^3 d_i\sigma_i, \quad (8.14)$$

where d_i are analytic functions of order 1. Using equation (3.14) we have

$$\mathbb{Y}^2(\frac{\pi}{2}) = \frac{1}{4}(2(\Delta_+^{\mathbb{I}})^2 - 4)\mathbb{I} + \sum_{i=1}^3 d_i\sigma_i. \quad (8.15)$$

However, by assumption $\langle \mathbb{Y}(\frac{\pi}{2})^2 - \mathbb{Y}(\pi), \mathbb{I} \rangle_{Lin} = 0$, thus using (8.15) and the fact that the Pauli matrices form an orthonormal basis, gives

$$(\Delta_+^{\mathbb{I}})^2 = \Delta + 2. \quad (8.16)$$

The above relation shows that the zeros of $\Delta + 2$ are at least of order 2, but the maximal dimension of every eigenspace of L_2 is 2. Thus $\Delta + 2$ has only double zeros.

Conversely, assume $\Delta + 2$ has only double zeros. $\mathbb{Y}(\pi)$ is an entire matrix valued function of order 1, thus $\Delta + 2$ is an entire function of order 1.

At every double zero, $\lambda = \tilde{\lambda}$, of $\Delta + 2$, the corresponding instability interval vanishes, furthermore the eigenspace of L_2 is of dimension 2, thus every solution is π anti-periodic, giving

$$F(z, \lambda) := \mathbb{Y}(z + \pi) + \mathbb{Y}(z) = 0. \quad (8.17)$$

This condition is also necessary for an anti-periodic eigenvalue to be double. Since $\Delta + 2$ is an entire function of order 1 with all zeros being double, it follows from the Hadamard expansion of $\Delta + 2$ as an infinite product that $\sqrt{\Delta + 2}$ is an entire function of order $\frac{1}{2}$ with all zeros simple. Now $F(z, \lambda)$ is an entire function of order 1, and the zeros of $\sqrt{\Delta + 2}$ and $F(z, \lambda)$ coincide. Thus $\frac{F(z, \lambda)}{\sqrt{\Delta + 2}}$ is an entire function.

Lemma 6.2 and (3.11) give

$$\Delta + 2 = 2 \cos \lambda \pi + 2 + o(e^{|\Im \lambda| \pi}), \quad (8.18)$$

thus

$$|\Delta + 2|e^{-|\Im \lambda| \pi} = |(2 \cos \lambda \pi + 2)e^{-|\Im \lambda| \pi} + o(1)|. \quad (8.19)$$

Define the sets

$$D_\epsilon^k := \{\lambda : |2 \cos \lambda \pi + 2| < \epsilon, |\lambda - (2k + 1)| < \frac{1}{2}\}, \quad (8.20)$$

for each $k \in \mathbb{Z}$ and a fixed $\epsilon > 0$ so small so that every D_ϵ^k is a single simply connected set. For brevity we write $D_\epsilon = \cup_k D_\epsilon^k$ and note that for large $|\lambda|$ each D_ϵ^k contains exactly one zero of $\Delta + 2$. For $\lambda \in \mathbb{C} \setminus D_\epsilon$, large $|\Im \lambda|$, we have $|2 \cos \lambda \pi + 2|e^{-|\Im \lambda| \pi} \geq \frac{1}{2}$. For $c > 0$ and $\lambda \in \mathbb{C} \setminus D_\epsilon$ with $|\Im \lambda| \leq c$ and for large $|\Re \lambda|$ we have $|2 \cos \lambda \pi + 2|e^{-|\Im \lambda| \pi} \geq \epsilon e^{-c\pi}$. Thus there exists a $k > 0$ so large that

$$|\Delta + 2|e^{-|\Im \lambda| \pi} \geq \min\{\frac{1}{2}, \epsilon e^{-c\pi}\} + o(1), \quad \text{for all } |\lambda| \geq k, \quad (8.21)$$

for $\lambda \in \mathbb{C} \setminus D_\epsilon$. Hence

$$\frac{1}{\sqrt{\Delta + 2}} = O(e^{-|\Im \lambda| \frac{\pi}{2}}) \quad \text{for } \lambda \in \mathbb{C} \setminus D_\epsilon. \quad (8.22)$$

Lemma 6.2 and $e^{-\lambda J \frac{\pi}{2}} + e^{\lambda J \frac{\pi}{2}} = 2\mathbb{I} \cos(\lambda \frac{\pi}{2})$ yield

$$F(-\frac{\pi}{2}, \lambda) = e^{-\lambda J \frac{\pi}{2}} + e^{\lambda J \frac{\pi}{2}} + o(e^{|\Im \lambda| \frac{\pi}{2}}), \quad (8.23)$$

$$= 2\mathbb{I} \cos(\lambda \frac{\pi}{2}) + o(e^{|\Im \lambda| \frac{\pi}{2}}). \quad (8.24)$$

Combining (8.22) and (8.24) yields

$$\tilde{F} := \frac{F(-\frac{\pi}{2}, \lambda)}{\sqrt{\Delta + 2}} = O(1), \quad (8.25)$$

for $\lambda \in \mathbb{C} \setminus D_\epsilon$. However \tilde{F} is entire in \mathbb{C} , so the maximum modulus principle gives that $\tilde{F} = O(1)$ in \mathbb{C} , thus is constant in \mathbb{C} by Liouville's Theorem. So there exists $a, b, c, d \in \mathbb{C}$ such that

$$\frac{F(-\frac{\pi}{2}, \lambda)}{\sqrt{\Delta + 2}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{for all } \lambda \in \mathbb{C}. \quad (8.26)$$

For $\lambda = i\zeta$, $\zeta \rightarrow \infty$, equation (8.18) gives $\frac{1}{\Delta+2} = e^{-\pi\zeta}(1+o(1))$, thus $\frac{1}{\sqrt{\Delta+2}} = e^{-\frac{\pi}{2}\zeta}(1+o(1))$. Furthermore, (8.24) gives

$$F\left(-\frac{\pi}{2}, i\zeta\right) = e^{\frac{\pi\zeta}{2}}(\mathbb{I} + o(1)), \quad (8.27)$$

hence $a = 1 = d$ and $c = 0 = b$, thus

$$\mathbb{Y}\left(\frac{\pi}{2}\right) + \mathbb{Y}\left(-\frac{\pi}{2}\right) = \mathbb{I}\sqrt{\Delta+2}. \quad (8.28)$$

So the analogues of (3.13) at $\mathbb{Y}\left(\frac{\pi}{2}\right)$ and $\mathbb{Y}\left(-\frac{\pi}{2}\right)$ combined with (8.28) give

$$(\Delta_+^{\mathbb{I}} + \Delta_-^{\mathbb{I}})\mathbb{I} + (\Delta_+^J + \Delta_-^J)J + (\nabla_+^{\mathbb{I}} + \nabla_-^{\mathbb{I}})\sigma_3 + (\nabla_+^J + \nabla_-^J)\sigma_1 = 2\mathbb{I}\sqrt{\Delta+2}. \quad (8.29)$$

Applying the inner product $\langle \cdot, \sigma_i \rangle_{Lin}$, $i = 0, \dots, 3$, to both sides of the above equation gives

$$\Delta_+^{\mathbb{I}} + \Delta_-^{\mathbb{I}} = 2\sqrt{\Delta+2}, \quad (8.30)$$

$$\Delta_+^J = -\Delta_-^J, \quad (8.31)$$

$$\nabla_+^{\mathbb{I}} = -\nabla_-^{\mathbb{I}}, \quad (8.32)$$

$$\nabla_+^J = -\nabla_-^J. \quad (8.33)$$

Furthermore, equations (3.14), (8.31), (8.32) and (8.33) gives $\Delta_+^{\mathbb{I}} = \pm\Delta_-^{\mathbb{I}}$, but if $\Delta_+^{\mathbb{I}} = -\Delta_-^{\mathbb{I}}$, equation (8.30) shows that $\sigma(L_2) = \mathbb{C}$, which is not possible as L_2 is a self-adjoint operator and thus has $\sigma(L_2) \subset \mathbb{R}$. Hence

$$\Delta_+^{\mathbb{I}} = \Delta_-^{\mathbb{I}}. \quad (8.34)$$

A direct calculation shows that

$$\mathbb{Y}^{-1}\left(-\frac{\pi}{2}\right) = \frac{1}{2}(\Delta_+^{\mathbb{I}}\mathbb{I} - \Delta_-^J J - \nabla_-^{\mathbb{I}}\sigma_3 - \nabla_-^J\sigma_1). \quad (8.35)$$

Applying (8.31)-(8.34) to (8.35) shows that $\mathbb{Y}^{-1}\left(-\frac{\pi}{2}\right) = \mathbb{Y}\left(\frac{\pi}{2}\right)$. Since Q is π -periodic, the fundamental matrix solutions $\mathbb{Y}(z)$ and $\mathbb{Y}(z + \pi)$ of (3.2) are related by

$$\mathbb{Y}(z + \pi) = \mathbb{Y}(z)\mathbb{Y}(\pi). \quad (8.36)$$

Setting $z = -\frac{\pi}{2}$ in (8.36) gives

$$\mathbb{Y}\left(\frac{\pi}{2}\right) = \mathbb{Y}\left(-\frac{\pi}{2}\right)\mathbb{Y}(\pi), \quad (8.37)$$

thus

$$\mathbb{Y}(\pi) = \mathbb{Y}\left(\frac{\pi}{2}\right)^2. \quad (8.38)$$

■

This lemma is used in the antiperiodic Borg inverse Theorem 8.4.

Lemma 8.3 *Suppose Q in $\ell Y = \lambda Y$ is a 2×2 π -periodic matrix function integrable on $[0, \pi)$, of the form $Q = Q_1$ then $\Delta - 2$ has only double zeros if and only if $\mathbb{Y}(\pi) = (\sigma_2 \mathbb{Y}(\frac{\pi}{2}))^2$.*

Proof: Let us assume that $\mathbb{Y}(\pi) = (\sigma_2 \mathbb{Y}(\frac{\pi}{2}))^2$. We have

$$\sigma_2 \mathbb{Y}\left(\frac{\pi}{2}\right) = \frac{1}{2}(\Delta_+^{\mathbb{I}} \sigma_2 - i \Delta_+^J \mathbb{I} - i \nabla_+^{\mathbb{I}} \sigma_1 + i \nabla_+^J \sigma_3), \quad (8.39)$$

thus using (3.14) a direct computation gives

$$(\sigma_2 \mathbb{Y}\left(\frac{\pi}{2}\right))^2 = \frac{1}{4}(4 - 2(\Delta_+^J)^2) \mathbb{I} + \sum_{i=1}^3 d_i \sigma_i, \quad (8.40)$$

where d_i are analytic functions of order 1. Considering that the Pauli matrices form an orthonormal set, using $\mathbb{Y}(\pi) - (\sigma_2 \mathbb{Y}(\frac{\pi}{2}))^2 = 0$, we calculate $\langle \mathbb{Y}(\pi) - (\sigma_2 \mathbb{Y}(\frac{\pi}{2}))^2, \mathbb{I} \rangle_{Lin} = 0$, to find

$$(\Delta_+^J)^2 = 2 - \Delta^{\mathbb{I}}. \quad (8.41)$$

The zeros of $(\Delta_+^J)^2$ are at least of order 2, however the maximal dimension of every eigenspace of L_1 is 2. Thus all the zeros of $\Delta - 2$ are double.

For sufficiency, assume that all the zeros of $\Delta - 2$ are double. Define

$$H(x, \lambda) := \mathbb{Y}(x + \pi) - \mathbb{Y}(x). \quad (8.42)$$

Using similar reasoning to Lemma 8.2, we have that $\frac{H(z, \lambda)}{\sqrt{2 - \Delta}}$ is an entire function, thus we have

$$H\left(-\frac{\pi}{2}, \lambda\right) = 2J \sin\left(\lambda \frac{\pi}{2}\right) + o\left(e^{|\Im \lambda| \frac{\pi}{2}}\right). \quad (8.43)$$

Lemma 6.2 and (3.11) give

$$2 - \Delta = 2 - 2 \cos \lambda \pi + o(e^{|\Im \lambda| \pi}), \quad (8.44)$$

thus

$$|2 - \Delta| e^{-|\Im \lambda| \pi} = |(2 - 2 \cos \lambda \pi) e^{-|\Im \lambda| \pi} + o(1)|. \quad (8.45)$$

Define the sets

$$\hat{D}_\epsilon^k := \{\lambda : |2 \cos \lambda \pi - 2| < \epsilon, |\lambda - 2k| < \frac{1}{2}\}, \quad (8.46)$$

for each $k \in \mathbb{Z}$ and a fixed $\epsilon > 0$ so small so that every \hat{D}_ϵ^k is a single simply connected set. For brevity we write $\hat{D}_\epsilon = \cup_k \hat{D}_\epsilon^k$ and note that for large $|\lambda|$ each \hat{D}_ϵ^k contains exactly one zero of $\Delta - 2$. Following reasoning as in Lemma 8.2 we have

$$\frac{1}{\sqrt{2 - \Delta}} = O(e^{-|\Im \lambda| \frac{\pi}{2}}) \quad \text{for } \lambda \in \mathbb{C} \setminus \hat{D}_\epsilon. \quad (8.47)$$

Thus

$$\tilde{H} := \frac{H(-\frac{\pi}{2}, \lambda)}{\sqrt{2 - \Delta}} = O(1), \quad (8.48)$$

for $\lambda \in \mathbb{C} \setminus \hat{D}_\epsilon$. Since \tilde{H} is entire on \mathbb{C} , the maximum-modulus theorem shows that it is bounded on \mathbb{C} . Hence by Liouville's theorem \tilde{H} is constant on \mathbb{C} . For $\lambda = i\zeta$, $\zeta \rightarrow \infty$, equation (8.44) gives $\frac{1}{\sqrt{2 - \Delta}} = e^{-\zeta \frac{\pi}{2}}(-i + o(1))$, also equation (8.43) gives

$$H(-\frac{\pi}{2}, \lambda) = e^{\zeta \frac{\pi}{2}}(iJ + o(1)), \quad (8.49)$$

thus

$$\frac{H(-\frac{\pi}{2}, \lambda)}{\sqrt{2 - \Delta}} = J + o(1). \quad (8.50)$$

Giving

$$\mathbb{Y}(\frac{\pi}{2}) - \mathbb{Y}(-\frac{\pi}{2}) = J\sqrt{2 - \Delta}. \quad (8.51)$$

Similarly to Lemma 8.2, the analogues of (3.13) at $\mathbb{Y}(\frac{\pi}{2})$ and $\mathbb{Y}(-\frac{\pi}{2})$ combined with (8.51), give the expansion

$$(\Delta_-^{\mathbb{I}} - \Delta_+^{\mathbb{I}})\mathbb{I} + (\Delta_-^{\mathbb{J}} - \Delta_+^{\mathbb{J}})J + (\nabla_-^{\mathbb{I}} - \nabla_+^{\mathbb{I}})\sigma_3 + (\nabla_-^{\mathbb{J}} + \nabla_+^{\mathbb{J}})\sigma_1 = 2J\sqrt{2 - \Delta}. \quad (8.52)$$

Applying the inner product $\langle \cdot, \sigma_i \rangle_{Lin}$ to the above equation yields

$$\Delta_-^J - \Delta_+^J = 2\sqrt{2 - \Delta}, \quad (8.53)$$

$$\Delta_-^{\mathbb{I}} = \Delta_+^{\mathbb{I}}, \quad (8.54)$$

$$\nabla_-^{\mathbb{I}} = \nabla_+^{\mathbb{I}}, \quad (8.55)$$

$$\nabla_-^J = \nabla_+^J. \quad (8.56)$$

The identity (3.14) together with (8.54)-(8.56) gives $\Delta_-^J = -\Delta_+^J$, otherwise (8.53) yields $\sigma(L_1) = \mathbb{C}$, which is not possible since L_1 is self-adjoint. A direct calculation shows that

$$\sigma_2 \mathbb{Y}(\frac{\pi}{2}) \sigma_2 = \frac{1}{2}(\Delta_+^{\mathbb{I}} \mathbb{I} + \Delta_+^J J - \nabla_+^{\mathbb{I}} \sigma_3 - \nabla_+^J \sigma_1). \quad (8.57)$$

Comparing the above equation with (8.35) and (8.54)-(8.56) shows that $\sigma_2 \mathbb{Y}(-\frac{\pi}{2}) \sigma_2 = \mathbb{Y}(\frac{\pi}{2})^{-1}$. Thus using (8.37) we have

$$\mathbb{Y}(\pi) = (\sigma_2 \mathbb{Y}(\frac{\pi}{2}))^2. \quad (8.58)$$

■

8.3 Main Results

We are now in a position to prove the main theorems of this chapter. For brevity we recall the transformation defined in section 4.2. Let

$$R(z) := e^{J \int_0^z (Q_2 - \frac{1}{\pi} \int_0^\pi Q_2 d\tau) dt}, \quad (8.59)$$

thus $Y = R\tilde{Y}$ transforms $\ell Y = \lambda Y$ into

$$J\tilde{Y}' + \tilde{Q}\tilde{Y} = \lambda\tilde{Y}, \quad (8.60)$$

where $\tilde{Q} = \tilde{Q}_1 + \tilde{Q}_2$ in which

$$\tilde{Q}_1(z) = R^{-1}(z)Q_1(z)R(z), \quad (8.61)$$

$$\tilde{Q}_2(z) = \frac{1}{\pi} \int_0^\pi Q_2 dt, \quad (8.62)$$

with $\tilde{Q}_1 \in S_-^J$ and $\tilde{Q}_2 \in S_+^J$. Notice that $R(0) = \mathbb{I} = R(\pi)$, thus the above transformation preserves boundary conditions. If we consider the equation

$$J\tilde{Y}'_a + \tilde{Q}_1\tilde{Y}_a = \left(\lambda - \frac{1}{2\pi} \int_0^\pi (q_1 + q_2) dt \right) \tilde{Y}_a, \quad (8.63)$$

then

$$\mathbb{Y}(\lambda, z) = R(z)e^{-i \int_0^z \Im q dt} \tilde{Y}_a \left(\lambda - \frac{1}{2\pi} \int_0^z (q_1 + q_2) dt, z \right). \quad (8.64)$$

Setting $z = \pi$ in equation (8.64) and taking the trace we have

$$\Delta(\lambda) = e^{-i \int_0^\pi \Im q dt} \tilde{\Delta}_a \left(\lambda - \frac{1}{2\pi} \int_0^\pi (q_1 + q_2) dt \right). \quad (8.65)$$

Equation (8.65) shows that Δ maps $\lambda \in \mathbb{R}$ into the line $\{\eta e^{-i \int_0^\pi \Im q dt} : \eta \in \mathbb{R}\}$.

We say that Q is $\frac{\pi}{2}$ - σ_2 -**similar** if $Q(x + \frac{\pi}{2}) = \sigma_2 Q(x) \sigma_2$, this is equivalent to Q_1 being $\frac{\pi}{2}$ -anti-periodic and Q_2 being $\frac{\pi}{2}$ -periodic.

Theorem 8.4 *Suppose Q in $\ell Y = \lambda Y$ is a Hermitian 2×2 complex π -periodic matrix function integrable on $[0, \pi)$, then the following hold:*

- (a) *If Q is a.e. $\frac{\pi}{2}$ -periodic then $\Delta + 2e^{-i \int_0^\pi \Im q dt}$ has only double zeros.*
- (b) *If $\Delta + 2e^{-i \int_0^\pi \Im q dt}$ has only double zeros then \tilde{Q} is a.e. $\frac{\pi}{2}$ -periodic, where \tilde{Q} is as given in (8.60)-(8.62).*

Proof: To prove **(a)**, assume Q is $\frac{\pi}{2}$ -periodic. The fundamental solutions $\mathbb{Y}(z + \pi)$ and $\mathbb{Y}(z + \frac{\pi}{2})$ are both solutions of $\ell Y = \lambda Y$ thus

$$\mathbb{Y}(z + \pi) = \mathbb{Y}(z + \frac{\pi}{2})B, \quad (8.66)$$

for some invertible matrix B , which may depend on λ . Setting $z = -\frac{\pi}{2}$ in the above equation gives $B = \mathbb{Y}(\frac{\pi}{2})$, thus

$$\mathbb{Y}(z + \frac{\pi}{2}) = \mathbb{Y}(z)\mathbb{Y}(\frac{\pi}{2}) \quad \text{and} \quad \mathbb{Y}(z + \pi) = \mathbb{Y}(z + \frac{\pi}{2})\mathbb{Y}(\frac{\pi}{2}). \quad (8.67)$$

Setting $z = 0$ in the second equation of (8.67) gives

$$\mathbb{Y}(\pi) = \mathbb{Y}(\frac{\pi}{2})^2. \quad (8.68)$$

Since Q_2 is $\frac{\pi}{2}$ -periodic, a direct calculation shows that $R(\pi) = \mathbb{I} = R(\frac{\pi}{2})$, for $R(z)$ defined by (8.59). Thus (8.64) and (8.68) give

$$\tilde{\mathbb{Y}}_a(\pi) = \tilde{\mathbb{Y}}_a(\frac{\pi}{2})^2. \quad (8.69)$$

Furthermore, following the method used in (8.13)-(8.16) we obtain

$$(\tilde{\Delta}_{a+}^{\mathbb{I}})^2 = \tilde{\Delta}_a + 2. \quad (8.70)$$

Equation (8.70) shows that $\tilde{\Delta}_a + 2$ has only zeros of order $2n$, $n \in \mathbb{N}$, but the maximal dimension of the eigenspace of $\sigma(L_2)$ is 2, thus $\tilde{\Delta}_a + 2$ has only double zeros. Hence equation (8.65) shows that $e^{i \int_0^\pi \Im q dt} \Delta + 2$ has only double zeros.

For **(b)**, suppose $e^{i \int_0^\pi \Im q dt} \Delta + 2$ has only double zeros, thus $\tilde{\Delta}_a + 2$ has only double zeros. From Lemma 8.2 we have

$$\tilde{\mathbb{Y}}_a(\pi) = \tilde{\mathbb{Y}}_a(\frac{\pi}{2})^2. \quad (8.71)$$

Consider the problem

$$J\tilde{Y}_b' + \tilde{Q}_b\tilde{Y}_b = \left(\lambda - \frac{1}{2\pi} \int_0^\pi (q_1 + q_2) dt \right) \tilde{Y}_b, \quad (8.72)$$

where $\tilde{Q}_b(x) := \tilde{Q}_1(x \bmod \frac{\pi}{2})$ a.e., where $x \bmod \frac{\pi}{2} \in [0, \frac{\pi}{2})$ for all $x \in \mathbb{R}$. It follows that \tilde{Q}_b is a.e. $\frac{\pi}{2}$ -periodic, then proceeding as in (8.66)-(8.69) we have

$$\tilde{\mathbb{Y}}_b(\pi) = \tilde{\mathbb{Y}}_b(\frac{\pi}{2})^2. \quad (8.73)$$

However, by construction $\tilde{Y}_b(\frac{\pi}{2}) = \tilde{Y}_a(\frac{\pi}{2})$, thus (8.71) and (8.73) show that $\tilde{Y}_b(\pi) = \tilde{Y}_a(\pi)$. Using Lemma 8.1 we have that $\tilde{Y}_b(\lambda, x) = \tilde{Y}_a(\lambda, x)$, for $\lambda \in \mathbb{C}$, $x \in \mathbb{R}$. Thus as

$$\tilde{Q}_b - \tilde{Q}_1 = J(\tilde{Y}'_a \tilde{Y}_a^{-1} - \tilde{Y}'_b \tilde{Y}_b^{-1}) = 0, \quad (8.74)$$

we have $\tilde{Q}_b = \tilde{Q}_1$, and \tilde{Q}_1 is a.e. $\frac{\pi}{2}$ -periodic. Since \tilde{Q}_2 is constant, we have that \tilde{Q} is a.e. $\frac{\pi}{2}$ -periodic. ■

Theorem 8.5 *Suppose Q in $\ell Y = \lambda Y$ is a Hermitian 2×2 complex π -periodic matrix function integrable on $[0, \pi)$, then the following hold:*

- (a) *If Q_1 is a.e. $\frac{\pi}{2}$ -anti-periodic and Q_2 is a.e. $\frac{\pi}{2}$ -periodic then $\Delta - 2e^{-i \int_0^\pi \Im q dt}$ has only double zeros.*
- (b) *If $\Delta - 2e^{-i \int_0^\pi \Im q dt}$ has only double zeros then \tilde{Q}_1 is a.e. $\frac{\pi}{2}$ -anti-periodic and \tilde{Q}_2 is a.e. $\frac{\pi}{2}$ -periodic, where \tilde{Q}_1 and \tilde{Q}_2 are as given in (8.60)-(8.62)*

Proof: To prove (a), assume that Q_1 is a.e. $\frac{\pi}{2}$ -anti-periodic and Q_2 is a.e. $\frac{\pi}{2}$ -periodic, then $\mathbb{Y}(x)$ and $\sigma_2 \mathbb{Y}(x + \frac{\pi}{2})$ are both solutions of $\ell Y = \lambda Y$, thus they are related by a transformation matrix B as

$$\sigma_2 \mathbb{Y}(z + \frac{\pi}{2}) = \mathbb{Y}(z)B. \quad (8.75)$$

Setting $z = 0$ in the above equation gives $B = \sigma_2 \mathbb{Y}(\frac{\pi}{2})$, thus

$$\mathbb{Y}(z + \frac{\pi}{2}) = \sigma_2 \mathbb{Y}(z) \sigma_2 \mathbb{Y}(\frac{\pi}{2}). \quad (8.76)$$

At $z = \frac{\pi}{2}$ we have

$$\mathbb{Y}(\pi) = (\sigma_2 \mathbb{Y}(\frac{\pi}{2}))^2. \quad (8.77)$$

Since Q_2 is $\frac{\pi}{2}$ -periodic we have that $R(\pi) = \mathbb{I} = R(\frac{\pi}{2})$. Thus (8.64) and (8.68) gives

$$\tilde{Y}_a(\pi) = \tilde{Y}_a(\frac{\pi}{2})^2. \quad (8.78)$$

Following the method used in (8.39)-(8.41), we have

$$(\tilde{\Delta}_{+a}^J)^2 = 2 - \tilde{\Delta}_a^{\mathbb{I}}. \quad (8.79)$$

The above equation shows that $\tilde{\Delta}_a - 2$ has only zeros of order $2n$, $n \in \mathbb{N}$, but the maximal dimension of the eigenspace of $\sigma(L_1)$ is 2, thus $\tilde{\Delta}_a - 2$ has only double zeros. Combining this with (8.65) proves **(a)**.

For **(b)**, suppose $e^{i \int_0^\pi \Im q dt} \Delta - 2$ has only double zeros, thus $\tilde{\Delta}_a - 2$ has only double zeros. From Lemma 8.3 we have

$$\tilde{Y}_a(\pi) = (\sigma_2 \tilde{Y}_a(\frac{\pi}{2}))^2. \quad (8.80)$$

Consider the problem

$$J\tilde{Y}_b' + \tilde{Q}_b \tilde{Y}_b = \left(\lambda - \frac{1}{2\pi} \int_0^\pi (q_1 + q_2) dt \right) \tilde{Y}_b, \quad (8.81)$$

where $\tilde{Q}_b(x \bmod \pi) := \tilde{Q}_1(x)$ a.e. on $[0, \frac{\pi}{2})$ and $\tilde{Q}_b(x)$ extended to $[0, \pi)$ to be $\frac{\pi}{2}$ -anti-periodic, then following (8.75)-(8.78) in **(a)** of this proof we have

$$\tilde{Y}_b(\pi) = (\sigma_2 \tilde{Y}_b(\frac{\pi}{2}))^2. \quad (8.82)$$

However, by construction $\tilde{Y}_b(\frac{\pi}{2}) = \tilde{Y}_a(\frac{\pi}{2})$, thus (8.80) and (8.82) show that $\tilde{Y}_b(\pi) = \tilde{Y}_a(\pi)$. Using Lemma 8.1 we have that $\tilde{Y}_b(\lambda, x) = \tilde{Y}_a(\lambda, x)$, for $\lambda \in \mathbb{C}$, $x \in \mathbb{R}$. Thus by the uniqueness of solutions to a differential equation, $\tilde{Q}_b = \tilde{Q}_1$, and \tilde{Q}_1 is a.e. $\frac{\pi}{2}$ -anti-periodic. Since \tilde{Q}_2 is constant, we have that \tilde{Q}_2 is a.e. $\frac{\pi}{2}$ -periodic. ■

If Q is in a canonical form then $R = \mathbb{I}$ so that $Q = Q_1 = \tilde{Q}_1$. This leads to the following Corollary to Theorems 8.4 and 8.5.

Corollary 8.6 *If Q in $\ell Y = \lambda Y$ is a 2×2 canonical π -periodic matrix function integrable on $[0, \pi)$ then the following hold:*

- (a) Q is a.e. $\frac{\pi}{2}$ -periodic if and only if $\Delta + 2$ has only double zeros.
- (b) Q is a.e. $\frac{\pi}{2}$ -anti-periodic if and only if $\Delta - 2$ has only double zeros.

Corollary 8.7 (Ambarzumyan) *If Q in $\ell Y = \lambda Y$ is a Hermitian 2×2 complex π -periodic matrix function integrable on $[0, \pi)$, then every instability interval vanishes if and only if $Q = r\sigma_0 + q\sigma_2$ a.e., where r and q are real and integrable on $[0, \pi)$.*

Proof: Assuming that $Q = r\sigma_0 + q\sigma_2$ a.e., we may rewrite equation (3.1) as $Y' = (pJ - iq\mathbb{I} - \lambda J)Y$, thus $\mathbb{Y}(x) = e^{J \int_0^z p dt - i\mathbb{I} \int_0^z q dt - J\lambda z}$, so that

$$\Delta = 2 \cos \left(\lambda\pi - \int_0^\pi p dt \right) e^{-i\mathbb{I} \int_0^\pi q dt}. \quad (8.83)$$

The above equation shows that $|\Delta| \leq 2$, thus every instability interval vanishes.

For necessity, assume that every instability interval vanishes, thus for any fixed $e^{i \int_0^\pi \Im q dt}$ all zeros of $e^{i \int_0^\pi \Im q dt} \Delta + 2$ and $e^{i \int_0^\pi \Im q dt} \Delta - 2$ are double zeros. Thus every zero of $\tilde{\Delta}_a + 2$ and $\tilde{\Delta}_a - 2$ is a double zero. Applying Theorems 8.4 and 8.5 we have that \tilde{Q}_1 is both a.e. $\frac{\pi}{2}$ -periodic and a.e. $\frac{\pi}{2}$ -anti-periodic, thus $\tilde{Q}_1 = 0$ a.e.. So that $\tilde{Q} = \tilde{Q}_2$ a.e.. Thus equation (8.61) shows that $Q_1 = 0$ a.e. and $Q = r\sigma_0 + q\sigma_2$ a.e.. ■

The following example shows that the converse of **(a)** in Theorem 8.4 is not possible in general.

Example 8.8 Suppose $Q = Q_1$ is a.e. $\frac{\pi}{2}$ -periodic and consider the transformation $Y = \hat{R}\hat{Y}$, where

$$\hat{R}(z) = e^{J \int_0^z (-2t + \pi)(\mathbb{I} + \sigma_2) dt}. \quad (8.84)$$

From Theorem 8.4 **(a)** we have that the zeros of $\Delta + 2$ are all double. Furthermore $-2x + \pi$ has mean value zero on $[0, \pi]$, thus $\hat{R}(0) = \hat{R}(\pi) = \mathbb{I}$, so that \hat{R} preserves the boundary conditions. The transformation $Y = \hat{R}\hat{Y}$ gives

$$J\hat{Y}' + (\hat{Q}_1 + \hat{Q}_2)\hat{Y} = \lambda\hat{Y}, \quad (8.85)$$

where

$$\hat{Q}_1(z) = e^{2J(z^2 - \pi z)} Q_1(z), \quad (8.86)$$

$$\hat{Q}_2(z) = (-2z + \pi)(\mathbb{I} + \sigma_2). \quad (8.87)$$

Notice that $\hat{Q}_2(\frac{\pi}{4}) = \frac{\pi}{2}(\mathbb{I} + \sigma_2)$ while $\hat{Q}_2(\frac{3\pi}{4}) = -\frac{\pi}{2}(\mathbb{I} + \sigma_2)$, thus \hat{Q}_2 is not $\frac{\pi}{2}$ -periodic even though zeros of $\hat{\Delta} + 2$ are all double.

9

Δ^J Inverse problem

“It has become almost a cliché to remark that nobody boasts of ignorance of literature, but it is socially acceptable to boast ignorance of science and proudly claim incompetence in mathematics.”

– Richard Dawkins

9.1 Introduction

The main result of this chapter is the uniqueness result of Theorem 9.3 which is a Clifford quasiperiodic inverse problem. This result depends on the analytic structure of $\Delta^{\mathbb{I}}$ and Δ^J from Chapter 4, and the interlacing of the zeros of $\Delta^J + 2$ and $\Delta^J - 2$ with the zeros of $\Delta^{\mathbb{I}}$ from Section 5.4. What makes this problem interesting is that Lemma 7.1, used in the proof of Theorem 7.5, cannot be used as $\Delta^J(\tau, \lambda)$ is dependant on translations of the independent variable, τ , of equation (7.1).

The discriminant Δ^J appears in the Pauli matrix decomposition of the monodromy matrix, $\mathbb{Y}(\pi)$, see (3.13). This representation is important because it was used essentially in the proof of Borg’s periodicity theorem proved in Chapter 8. This inverse

problem is a quasiperiodic variation of a result obtained by Borg, [9], [19] and [47]. however, as far as the author is aware, these results are new.

9.2 Main result

Firstly we state an important theorem.

Theorem 9.1 [19] (*Ambarzumyan*) *If Q in $\ell Y = \lambda Y$ is a canonical 2×2 real π -periodic matrix function integrable on $[0, \pi)$, then every instability interval vanishes if and only if $Q = 0$ a.e..*

We are now in the position to prove the main theorems of this chapter.

Theorem 9.2 *If Q in $\ell Y = \lambda Y$ is a canonical 2×2 π -periodic matrix function integrable on $[0, \pi)$, all J -instability intervals vanish if and only if $Q = 0$ a.e..*

Proof:

Assume every J -instability interval vanishes. From Theorem 5.4 (e), we have that the λ -zeros of $\Delta^{\mathbb{I}}$ are contained within the instability intervals of Δ^J . Thus the zeros of $\Delta^{\mathbb{I}}$ correspond to the double zeros of $\Delta^J + 2$ and $\Delta^J - 2$, hence $\frac{(\Delta^J + 2)(\Delta^J - 2)}{(\Delta^{\mathbb{I}})^2}$ is an analytic function in \mathbb{C} . The asymptotic estimates in Lemma 6.2 show that for large $|\lambda|$ with $\lambda \in \mathbb{C}$, we have

$$\Delta^{\mathbb{I}} = 2 \cos \lambda \pi + o(e^{|\Im \lambda| \pi}) \quad \text{and} \quad \Delta^J = -2 \sin \lambda \pi + o(e^{|\Im \lambda| \pi}). \quad (9.1)$$

The above equations yield

$$(\Delta^{\mathbb{I}})^2 = 4 \cos^2(\lambda \pi) + o(e^{|\Im \lambda| 2\pi}) \quad \text{and} \quad (\Delta^J)^2 - 4 = -4 \cos^2(\lambda \pi) + o(e^{|\Im \lambda| 2\pi}). \quad (9.2)$$

Equations (9.2) give for large $|\lambda|$ that

$$G(\lambda) := \frac{(\Delta^J + 2)(\Delta^J - 2)}{(\Delta^{\mathbb{I}})^2} = \frac{-4 \cos^2(\lambda \pi) + o(e^{|\Im \lambda| 2\pi})}{4 \cos^2(\lambda \pi) + o(e^{|\Im \lambda| 2\pi})}. \quad (9.3)$$

Using (9.2) we have

$$|(\Delta^{\mathbb{I}})^2|e^{-2|\Im\lambda|\pi} = |4e^{-2|\Im\lambda|\pi} \cos^2 \lambda\pi + o(1)| \quad (9.4)$$

Define the sets

$$E_\epsilon^k := \{\lambda : |\cos \lambda\pi| < \epsilon\}, \quad (9.5)$$

and choose $\epsilon > 0$ so small such that for every $k \in \mathbb{Z}$, E_ϵ^k is a single simply connected set. Let $E_\epsilon := \bigcup_k E_\epsilon^k$ and note that for large $|\lambda|$, each E_ϵ^k contains exactly one zero of Δ^J . Thus for large $|\Im\lambda|$ we have $|4e^{-2|\Im\lambda|\pi} \cos^2 \lambda\pi| \geq \frac{1}{4}$, furthermore for large $|\Re\lambda|$ and $|\Im\lambda| \leq c$ gives $|4e^{-2|\Im\lambda|\pi} \cos^2 \lambda\pi| \geq 4\epsilon^2 e^{-2c\pi}$. Thus there exists a $\lambda^* \in \mathbb{C}$ so large such that for $\lambda \in \mathbb{C} \setminus E_\epsilon$ we have

$$|(\Delta^{\mathbb{I}})^2|e^{-2|\Im\lambda|\pi} = \min\left\{\frac{1}{4}, 4\epsilon^2 e^{-2c\pi}\right\} + o(1), \quad \text{for } |\lambda| \geq |\lambda^*|. \quad (9.6)$$

Hence

$$|(\Delta^{\mathbb{I}})^2|e^{-2|\Im\lambda|\pi} = O(1), \quad \text{for } \lambda \in \mathbb{C} \setminus E_\epsilon, \quad (9.7)$$

so that

$$\frac{1}{(\Delta^{\mathbb{I}})^2} = O(e^{-2|\Im\lambda|\pi}), \quad \text{for } \lambda \in \mathbb{C} \setminus E_\epsilon. \quad (9.8)$$

Combining (9.2) and (9.8) we have

$$\frac{(\Delta^J + 2)(\Delta^J - 2)}{(\Delta^{\mathbb{I}})^2} = O(1), \quad \text{for } \lambda \in \mathbb{C} \setminus E_\epsilon. \quad (9.9)$$

Applying the maximum-modulus theorem to the above equation shows that $G(\lambda)$ is bounded on \mathbb{C} , furthermore, applying Liouville's theorem shows that $G(\lambda)$ is constant. Considering the sequence $\lambda_n = n\pi$, $n \in \mathbb{Z}$, we have from equation (9.3) that $G(\lambda_n) \rightarrow -1$ as $n \rightarrow \infty$, thus

$$(\Delta^{\mathbb{I}})^2 + (\Delta^J)^2 = 4. \quad (9.10)$$

Combining (9.10) with (3.14) gives

$$(\nabla^{\mathbb{I}})^2 + (\nabla^J)^2 = 0. \quad (9.11)$$

By factorising the above equation, we have

$$\nabla^{\mathbb{I}} = i\nabla^J \quad \text{or} \quad \nabla^{\mathbb{I}} = -i\nabla^J. \quad (9.12)$$

However, $\nabla^{\mathbb{I}}$ and ∇^J are pure real functions since they are composed of components of solutions of a first order canonical system with a real matrix potential on \mathbb{R} with real initial conditions. Thus $\nabla^{\mathbb{I}} = 0$ and $\nabla^J = 0$, so (3.13) becomes

$$\mathbb{Y}(\pi) = \frac{1}{2}(\Delta^{\mathbb{I}}\mathbb{I} + \Delta^J J). \quad (9.13)$$

At every λ -zero of Δ^J we have from (9.13) that $\mathbb{Y}(\pi)$ is in diagonal form so that the Floquet multipliers, ρ_1 and ρ_2 , of the \mathbb{I} -discriminant are equal to $\rho_1 = \frac{1}{2}\Delta^{\mathbb{I}} = \rho_2$. But since $\det \mathbb{Y}(\pi) = 1$, we have $\rho_1 \rho_2 = 1$, so that $\Delta^{\mathbb{I}} = \pm 2$. Thus at every λ -zero of Δ^J , the $\Delta^{\mathbb{I}}$ instability interval containing this zero, vanishes. Using Theorem 5.4 (d) we can conclude that every $\Delta^{\mathbb{I}}$ instability interval vanishes. Thus by Theorem 8.7 we have $Q = 0$ a.e..

For necessity, assume that $Q = 0$ a.e.. Then $Y' = -\lambda JY$, so that $\mathbb{Y}(x) = e^{-J\lambda x}$, hence

$$\Delta^J = -2 \sin \lambda \pi. \quad (9.14)$$

Thus $|\Delta^J| < 2$ for every $\lambda \in \mathbb{R}$, so that every J -instability interval vanishes. ■

Theorem 9.3 *If Q in $\ell Y = \lambda Y$ is a Hermitian 2×2 complex π -periodic matrix function integrable on $[0, \pi)$, all J -instability intervals vanish if and only if $Q = p\sigma_0 + q\sigma_2$ a.e., where p and q are real and integrable on $[0, \pi)$.*

Assume that every J -instability interval vanishes. Apply the transformation (8.59) to obtain (8.60)-(8.62). It follows that every J -instability interval of $\tilde{\Delta}_a^J$ vanishes. By Theorem 9.2, we have $\tilde{Q}_1 = 0$, so that $Q_1 = 0$. Finally giving $Q = p\sigma_0 + q\sigma_2$ a.e..

Assuming that $Q = p\sigma_0 + q\sigma_2$ a.e., we have that $Y' = (pJ - iq\mathbb{I} - \lambda J)Y$, so that $\mathbb{Y}(x) = e^{J \int_0^x p dt - i\mathbb{I} \int_0^x q dt - J\lambda x}$, hence

$$\Delta^J = -2 \sin \left(\lambda \pi - \int_0^\pi p dt \right) e^{-i\mathbb{I} \int_0^\pi q dt}. \quad (9.15)$$

Thus $|\Delta^J| < 2$ for every $\lambda \in \mathbb{R}$, so that every J -instability interval vanishes. ■

10

Conclusion

“Give me a place to stand, and I will move the earth.”

– Archimedes

10.1 Summary

What has past has been a window into the developments of the field of inverse periodic and Clifford quasiperiodic 2×2 Hermitian eigenvalue problems. Similarities between this and the inverse theory of the extensively studied Hill’s equation have been discussed and complications extending this theory to Hermitian systems have been overcome. It is the hope of the author that this thesis serves as a solid introduction and thinking tool for future research into $n \times n$ Hermitian boundary value problems.

10.2 Future Directions

The work in this thesis can and should be expanded and generalised. This section gives some possible future research paths on the topic of this thesis.

The Pauli representation of the monodromy matrix $\mathbb{Y}(\pi)$, played a central role in the definition of the Clifford quasiperiodic problems and the resolution of the first term of solution asymptotics for integrable potentials for Hermitian systems, see equations (6.23) and (6.27). A possible future direction would be to consider the so called generalised Dirac matrices, which are a basis for $GL(2n, \mathbb{C})$, for investigating asymptotics for $n \times n$ Hermitian systems.

In Chapter 7, the structure of Q was determined for when 0 and 1 instability interval fail to vanish, it is an interesting open problem to investigate the structure of Q when n instability intervals fail to vanish. This was investigated in the case of the Sturm-Liouville problem with great success, see [42].

The periodicity result in Chapter 8 could be generalised in three ways. Firstly, Q could be extended to be an $n \times n$ matrix. Secondly, we could ask the question, if every instability interval which is not a multiple of n vanishes, does there exist a potential Q which is $\frac{\pi}{n}$ periodic? Such a result exists in the Sturm-Liouville case, see [50]. Finally, it might be possible to extend the results in Chapter 8 to quasiperiodic problems.

An alternative direction of study would be to consider a wider range of Clifford quasiperiodic uniqueness problems as in Chapter 9. It might be possible to generalise the indexing and inverse results of Chapters 5 and 9 to continuous families of quasiperiodic problems and their associated Dirichlet boundary value problems.

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