Linear equations: Grade 9 learners' progress and their difficulties over time.

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Abstract

This study investigated Grade 9 learners' progress in solving linear equations over a six month period. The notion of progress is taken to encompass shifts in learners' abilities to solve equations with fluency, and shifts in communicating their reasoning, Data were collected with six learners in a Johannesburg secondary school, by means of four task-based interviews each. The interviews provided periodic glimpses into learners' progress. The interview instrument was a unique task-matrix, which made provision to move flexibly between different linear equations to explore shifts in learners' abilities to solve the equations and in their reasoning. The data analysis focussed on describing learners' progress, the approaches they used in solving linear equations, the difficulties they encountered and resulting errors. Three notable difficulties emerged: the didactic cut (the transition from arithmetic to algebraic equations), difficulties with negativity and difficulties with items of the form $ax = bx$. A key contribution of the study is the closing analysis which accounts for some of the difficulties and successes in the transition from arithmetic to algebraic equations by drawing on the Vygotskian notions of spontaneous and scientific concepts and the zone of proximal development. The findings of the study suggest the need to be aware of the existence and persistence of the didactic cut for some learners, the difficulties which negativity brings and the case of $ax = bx$ which learners treated as a different type of equation.

Keywords

Linear equations, mathematical thinking, didactic cut.

Declaration

I declare that this dissertation is my own work and no part of it has been copied from another source (unless indicated as a quote). All phrases, sentences and paragraphs taken directly from other works have been cited and the reference recorded in full in the reference list. This dissertation is being submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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(Signature of candidate)

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Chapter 1 : Background and Rationale

1.1. Introduction

Mathematics education in South Africa has a reputation for poor achievement from its learners, leading some experts to call it a crisis (Fleisch, 2008) 10 years ago. Research showed that South Africa placed last in the Trends in International Mathematics and Science Study (TIMMS) 2003 study of 50 countries worldwide (Reddy, 2006). While showing the most improvement in the 2015 TIMMS study (Reddy et al., 2016), South Africa remained one of the lowest performing countries of the 39 in the sample. Thus the problems facing South African mathematics education have been around for a long time with marginal improvement being shown.

South Africa is still in need of locally based research to help address this poor achievement. In particular, algebraic skills are a significant part of the problem, with the National Senior Certificate (NSC) Examiners' Report of 2015 noting the following: "The algebraic skills of the candidates were poor. They struggled with Mathematics in Grades 11 and 12 because they could not do the basic mathematics of Grades 8, 9 and 10". (Department of Basic Education (DBE), 2015, p. 151). While the general performance in the NSC exams improved in 2016, the 2016 report noted still "The algebraic skills of the candidates are poor. Most candidates lacked fundamental competencies which should have been acquired in lower grades" (DBE, 2016, p. 152).

It is these basic algebraic skills and knowledge that this research project investigated, particularly the content of linear equations. I present my full reasons for the choice of linear equations later in this chapter.

One of the key aspects in improving algebraic skills is a smooth transition from arithmetic to algebra (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994). Linear equations share a similar transition in moving from arithmetic equations, such as $3 \times \square + 5 = 8$ to algebraic equations such as $3x + 5 = 8x + 2$. And while the study of learning and teaching of linear equations has received significant attention (Kieran, 1992; Herscovics & Linchevski,1994 ; Filloy & Rojano, 1989; Vlassis, 2002), this research presents a unique perspective – that is how learners progress throughout the year. This study looks at the approaches which learners use, and the difficulties that they encounter with linear equations. But moreover, how these two aspects develop over a time period of six months in which four interviews were conducted with each learner.

1.2 Background of the study

This study takes place in the context of the WITS Maths Connect Secondary Project (WMCS), and in particular, the Learning Gains Study (Pournara, Hodgen, Adler, & Pillay, 2015). The primary aim of WMCS is to develop professional development models for secondary schools to enhance teachers' knowledge and skills which culminate in gains in learners' understanding and performance (Pournara et al, 2015). These researchers found that professional development with teachers, focussing on teacher knowledge in what have been classified as "under-performing" or "priority schools" (p. 4), can lead to small, yet statistically significant learning gains in learners. These learning gains were measured with a pre-test and a post-test designed by WMCS.

One of the remaining questions after the learning gains study by Pournara et al. was the nature of the gains made. If learning gains had taken place over an approximate 8 month time period, how and when exactly were these gains made? Was it a lightbulb moment? Was it continual exposure to certain ideas that gradually aligned learners' thinking with correct mathematical approaches?

In my experience as a secondary mathematics teacher, I too have noted learners' gradual improvement. Difficulties which learners may experience during their Grade 9 year are not as apparent in their Grade 10 year. While learners may struggle with linear equations in Grade 9, their difficulties in Grade 10 are often related to quadratic equations in Grade 10. Where and how did the improvement take place?

I chose to investigate these initial questions in a high performing, public, Quintile 5 School in Johannesburg. Because the focus of my study was on *learners'* reasoning, and not on teachers, it was necessary to choose a school where the teaching standard could be considered of good quality. The school achieves consistently noteworthy results by the end of Grade 12, with a 98% to 99% pass rate in Mathematics and an overall Mathematics average of over 60% since the adoption of the South African Curriculum and Assessment Policy Statement (CAPS).

In this study, I tracked the paths that six low-attaining learners followed through a series of 4 task-based interviews over the course of a six-month period. These tasks were carefully designed around the existing literature to gain an understanding of what learners can and cannot do, and how they are reasoning about the content. I chose to investigate this path of learning in the context of equations. Research has shown that a notable challenge facing Grade 9 learners is that of the transition from arithmetical approaches of solving equations (inverse operations) to algebraic approaches (operating on the variable to collect like terms on each side of the equation). This is known as the *didactic cut* (Filloy & Rojano, 1989).

1.3 Terminology used in this report

While there is a more in-depth discussion of these terms in the relevant sections of the literature review and research design, I think it would be helpful to define some common terms that are used in this study.

I distinguish between *arithmetic equations* and *algebraic equations*. Arithmetic equations are those that are similar to elementary school equations where the missing value is replaced with a letter. An *arithmetic equation* such as $3 \times \square + 5 = 11$ can be written with letters as $3x + 5 = 11$ and can be solved by arithmetically 'undoing' the operations. Thus a learner can reverse the original order of operations, undoing 'add 5' by subtracting 5 from 11 to get 6, and then undo '3 times x' with dividing 6 by 3 to get the answer of $x = 2$. An *algebraic equation* requires a fundamental shift in thinking (Kieran, 1992; Filloy & Rojano, 1989) by the need to operate on the letters by using inverse operations (or other models). Thus $3x + 5 = x + 11$ does not allow a learner to 'undo' the operations performed on the unknown since there are two unknowns. A learner is required to operate on the letter in a systematic manner.

I have adopted to the word *progress* to describe the changes in learners' *reasoning* and their *abilities* to solve linear equations over the course of the study. The word *abilities* does not refer to learners' innate talents, but rather what skills they are able to perform. These skills are akin to Kilpatrick, Swafford, and Findell's (2001) notion of procedural fluency as a strand of mathematical proficiency – being able to reliably and efficient solve linear equations. Within *abilities*, there is also Kilpatrick et al.'s (2001) strand of strategic competence – that learners are able to select an appropriate approach for the type of linear equation. These 'types' of linear equations are described in section 3.6.

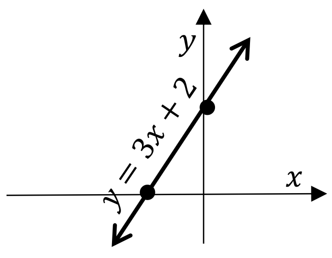
A learner's *reasoning* can only be inferred through the learner's talk. Again, I drew upon Kilpatrick et al.'s (2001) strands of *conceptual understanding* and *adaptive reasoning* in defining learners' reasoning. *Adaptive reasoning* refers whether they are able to explain their approaches in a logical and coherent manner, while *conceptual understanding* refers to their understanding of the concepts of equality (the equals sign) and letter as an unknown. Thus learner *reasoning* refers to the learners' thinking, as expressed through their talk, about the underlying notions in linear equation. Reasoning includes (but is not limited to) how learners justify their choice of approach, how they reconcile different approaches (i.e. informal approaches such as inspection compared to formal approaches), and their reasoning about the nature of their solution (both sides of the equation balance).

The word *progress* implies a forward movement. The learners may *progress* in their abilities by making fewer errors and solving more types of equations, and/or they may *progress* in their reasoning – that they can explain their working or are adopting and implementing an appropriate approach. However, the term *progress* does not imply that all learners necessarily moved forward in their abilities or their reasoning. I will describe cases as of a lack of progress where learners returned to previous errors or where new knowledge, recently acquired, led to new errors.

1.4 Rationale for linear equations and their place in the curriculum

The choice of equations is due to their near ubiquity throughout mathematics. Learners are required to formulate equations to find x intercepts (Figure 1.1), the position (n) of the known term in the number pattern (Figure 1.2), the value of an unknown time period in financial mathematics, and even within Euclidean geometry with certain types of school questions (Figure 1.3).

(1.1) Given $y = 3x + 2$,



Determine the coordinates of the $x -$ intercept.

(1.2)

Given $T_n = 3n + 2$, which term is equal to 17?

(1.3)

Determine the value of x in the following diagram.

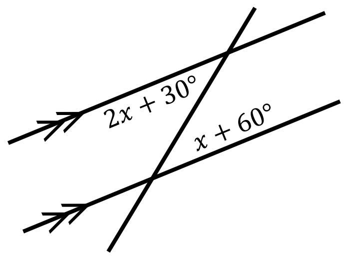


Figure 1.1 ; 1.2 and 1.3: Some other contexts in which linear equations arise in Grade 9

While these other content areas are not part of my research, learners would be repeatedly exposed to solving equations throughout the year, and hence I originally chose linear equations with the hope of being able to notice changes in their reasoning and abilities in solving equations in different contexts. Additionally, learners would be revising work for assessments, and most notably, for the mid-year June exam and final November exams for promotion purposes. These are all possible reasons why learners' abilities in solving linear equations and reasoning may change over the course of the study.

Within the South African mathematics curriculum, linear equations would first have been introduced in Grade 8. In Grade 8, learners are expected to:

"Extend solving equations to include:

-- Using additive and multiplicative inverses" (DBE, 2011, p. 93).

This would mean that by the end of Grade 8, learners are expected to adopt the inverse operational approach and could encounter equations with letters on both sides, such as $2x + 1 = 3x - 5$. The Grade 9 curriculum calls for:

"*Revise* the following done in Grade 8:

-- Set up equations to describe problem situations

-- Analyse and interpret equations that describe a given situation

-- Solve equations by:

♦ inspection

♦ *using additive and multiplicative inverses*

-- Extend solving equations to include:

♦ using factorisation

♦ equations of the form: a product of factors = 0"

(DBE, 2011, p. 131, emphasis mine).

Hence in Grade 9, the shift later in the year is on to simple quadratic equations, e.g. $x^2 + 5x - 6 = 0$ and exponential equations (via inspection) such as $3^{x+1} = 81$. The expectation is that linear equations are therefore acquired and consolidated knowledge by mid-year. My experience as a secondary school teacher is that this is not the case – rather that many learners continue to experience difficulties with linear equations throughout the year.

1.5 Problem statement

Many studies show a snapshot of learner knowledge and abilities – what they can do (or not do) at a specific point in time. And there are many studies which show learning gains at two points in time, i.e. pre and post-test intervention. There are few studies into what happens *over* an extended period of time. The design of this study allows for 4 periodic task-based interviews which periodically 'dip' into learners' abilities and reasoning over the course of six months. Understanding how learner thinking changes over time is important for developing teaching strategies which are informed by this knowledge.

The content of linear equations allows an in-depth study of how learners' reasoning and abilities will progress over time because learners are constantly exposed to equations in other sections in Grade 9 mathematics. It is the nature of this progress which is the problem that this study attempts to investigate.

1.6 Research questions

The aim of this research is to investigate learners' reasoning about, and mathematical abilities in solving linear equations *over time*.

In this study I seek to investigate the following questions:

1. What progress do learners show in solving linear equations over time?
    a. What progress do learners show in their abilities to solve linear equations?
    b. What progress do learners show in their reasoning by explaining their thinking?

2. What approaches do learners use?
    a. Do their approaches change over time?
    b. Do their approaches lead to difficulties?

3. What difficulties do learners experience?
    a. Do their difficulties lead to errors?
    b. Do their difficulties and errors change over time?

1.7 Discussion of research questions

The first research question allows me to describe the extent of learners' progress. Entailed in the notion of progress is the improvement in their abilities in solving equations. Abilities refers to more than their procedural fluency (what questions they got correct). It also refers to their choice of a suitable approach for the type of equation with which they are presented.

This first research question focuses on learner thinking and their explanations of their work, thus investigating their conceptual understanding and adaptive reasoning. It is impossible to gain direct access to learner thinking, but from their written work, I can infer some of their possible thinking. Furthermore, it is through interviews that I can gain more indirect access to their thinking by their verbal descriptions of their work.

The second research question drives at the approaches which learners adopt. It has already been briefly discussed, but *algebraic equations* require a different approach to *arithmetic equations*. In chapter 3, I will discuss the literature on the different approaches to solving linear equations. The learners will have adopted an approach which they use to solve equations. They also may have multiple approaches that are used for different types of equations. In seeking to describe learners' progress, the approaches they use are relevant if they are the source of learner difficulty or success. In adopting an inappropriate approach, learners may encounter difficulties and make errors.

Thus, the third question addresses learner error and difficulties. The notion of learner error is discussed fully in chapter 2 but entailed in this research question is the idea that learners may experience difficulties when solving an equation. The term *difficulties* has two aspects. The first type of difficulty would be that a learner has been exposed to a new approach, i.e. that of operating with letters and performing the same operation on both sides, but is not able to recall this approach when required by a question, i.e. $3x + 2 = 2x - 1$. Thus, while the question may be familiar, the learner does not recall the appropriate approach and thus may operate spontaneously using older, internalised approaches, such as arithmetic undoing.

The second type of difficulty refers to a learner's inability to recognise one equation as a subset of another class, for example that $2x = 3x$ is a special case of $ax + b = cx + d$, and that the same approach is valid. When learners 'see' $2x = 3x$ as a different type of equation, they may see this as a 'new' type of equation and spontaneously operate outside of the standard approach to equations with letters on both sides. In these two cases, learners appear to get 'stuck'. Learners may make attempts to 'get unstuck' which may lead to errors. There are, however, other sources of known errors with linear equations. The literature review in chapter 3 also discusses known errors when solving linear equations which Grade 9 learners are likely to make.

The thread of ‘change over time’ permeates all the research questions. The *time* in question refers to the four periodic interviews over a six-month period. Change over time is the unique aspect this study seeks to contribute to the broader knowledge base. In the context of this study, change over time refers to the change in learners’ abilities and reasoning about linear equations over time. The data analysis will focus on describing the change in learners’ reasoning and their abilities over the course the latter part of their Grade 9 year.

1.8 Conclusion

In this chapter, I have provided the background and rationale for this study. I have defined the important terms *arithmetic* and *algebraic equations,* and *progress, abilities* and *reasoning*. I also clarified the problem I seek to address and described the research questions which will drive the research design and data analysis.

This report contains ten chapters. The following is a brief outline of the content of each chapter.

Chapter 2 defines the conceptual framework which frames this study. Using Ravitch and Riggan’s (2012) description of a conceptual framework as comprising of three primary elements: personal interests, topical research and theoretical frameworks, this chapter focuses on my personal interests (perspective) such as inherent biases, the epistemology and ontology underlying this study and the learning theory (social constructivism) implored as the theoretical lens.

The topical research aspect of the conceptual framework is described in chapter 3, which focusses on reviewing the significant literature on linear equations. This includes the theoretical hierarchy of difficulty for equations, the common approaches taught to solve equations, and known errors and sources of difficulty, such as the didactic cut.

Chapter 4 outlines the research design, sample and data collection process. This entails a discussion of the sampling process in choosing the six learners from the study, and the design of the task for the interview. I also describe the methods adopted in how the data were analysed. Chapter 5 immediately implements this analysis in briefly describing the paths of each of the six learners. Chapters 6, 7 and 8 are rich descriptions of common difficulties experienced by some of learners. Each of these chapters looks at three learners (not the same learners across chapters) and how their thinking changed over time in respect to each one of the difficulties.

Chapter (9) reflects on this data analysis from a Vygotskian perspective, and then the final chapter (10) summarises the findings of the study by answering the research questions using the data analysis from chapters 5 to 8. Finally, in chapter 10, I discuss the limitations of the study, make some recommendations for further research and identify practical implications for the teaching of linear equations.

Chapter 2 : Conceptual Framework

2.1 Introduction

In this chapter, I present the conceptual framework that I have adopted for this study. I begin by defining the conceptual framework of this study as comprising my personal interests, topical research and the theoretical frameworks which underpin my study. I then explain why I chose social constructivism as the learning theory which provides the theoretical perspective on learning for this study. I conclude the chapter with a discussion on learner error from this social constructivist perspective.

There is some debate about the relationship between theoretical and conceptual frameworks, and the respected research design author Maxwell (2005) uses the terms synonymously. I have chosen to adopt Ravitch and Riggan's (2012) view of the conceptual framework comprising of three primary elements: personal interests, topical research and theoretical frameworks.

Personal interests include my curiosities, biases, ideologies and ontological perspective. These will be discussed under the heading of 'personal interests'. Secondly, the 'theoretical framework' will consist of the learning theories and the research design as the lens through which I will seek answers to my research questions. The learning theory chosen for this study, social constructivism is discussed in this chapter. The design of the research is described in chapter 4 under the heading of 'Methodology'.

And finally, topical research refers to the existing research asking similar questions to my own. For this study, this includes learners' difficulties with linear equations (the didactic cut), the concepts which underpin linear equations (the equals sign) and common algebraic errors. These will be addressed in chapter 3 under the heading of the 'literature review'.

2.2 Personal interests (perspective)

According to Ravitch and Riggan (2012), personal interests are curiosities, biases, ideologies and ontological perspective. While some of these are evident from the background to the study discussed in chapter 1, I will briefly provide an account of some pertinent aspects of my perspective. I have been a practicing mathematics teacher for the past 7 years in a South African public school. I have seen the positive effects of good teaching on learner performance, but have also noted that mathematical content can be challenging for the majority of learners, and that there is no ‘perfect’ way to teach mathematics. Part of the beauty of mathematics is its challenging nature, but this is not always appreciated by learners in my experience.

Learning is a complex process, which can occur differently in different contexts, and requires constant reflection on the part of the teacher to maximise the potential of the learner. I now shift to discuss ontology, epistemology and the perspectives I will adopt for the purposes of this study.

I have chosen to adopt the critical realism paradigm, most notably argued for by Roy Bhaskar (1978), for the purposes of this research. Critical realism seeks to traverse carefully the area in between the positivist view and that of interpretivism. Critical realism accepts the ontology that reality can exist, but the apprehension of this knowledge is imperfect due to humanity’s fallible intellectual mechanisms and the complex nature of the phenomena (Guba & Lincoln, 1994).

When we are asked to ‘solve for x’ in $4x - 1 = 5$, the implication is that these symbols represent objects which exist – namely a linear equation, a letter (x) which stands for a specific number and the equals sign signifying an equation. This reflects an ontological realism. However, it is possible to not have a full apprehension of these objects, as one might make an error when solving this equation, such as saying $x = 2$. This answer would not fit with the reality of the object.

This perspective fits my own epistemic stance – that there is a reality but our access to this reality is imperfect. There are two epistemic perspectives at play in this study. The first is the learners’ epistemic access to the mathematical content under investigation – namely their conception of linear equations. And secondly, there is my ability as a researcher to gain access to the learners’ thinking.

The first of these epistemic questions is that of the learners’ access to the content of ‘linear equations’. I can speak of the object of a linear equation as an ontological reality which *can* be accessed by a learner, however their access to this reality could be imperfect. Learners may make errors when solving equations. Thus ‘critical realism’ aligns nicely with the notion of misconception as discussed later in this chapter.

The second epistemic perspective is at the level of a researcher - there is a reality of the learner’s internal cognitive understanding of equation (schema). We can never have perfect access to this knowledge however.

As Scott & Morrison (2005) describe:

> … the observer can only access appearances, but these appearances refer to underlying mechanisms, which are not easily apprehended; finally those deep structures or mechanisms may actually contradict or be in conflict with their appearances. (p. 47)

Through the critical realism paradigm, it is necessary for claims about reality (learner's thinking about solving linear equations) to "be subjected to the widest possible critical examination to facilitate apprehending reality as closely as possible" (Guba & Lincoln, 1994, p. 110). Hence in adopting this epistemic perspective, my study must attempt to validate any claims to reality by examining the phenomena (the learners' abilities and reasoning when solving equations) from different perspectives. These perspectives (discussed in chapter 4) could include analysing the learners' written work as well as their verbal descriptions of their thinking.

2.3 Learning Theory (theoretical framework)

This research is located within the cognitivist paradigm as I am seeking to make inferences about learners' internal cognitive structures when solving linear equations. There is also a social dimension to this research – that the learners are located in a specific context that will affect the development of their internal cognitive structures. But more importantly, I am using interviews to gain access to the learners' thinking. Thus learners are performing tasks in a social setting (with a more knowledgeable other) and this necessitates a discussion of the social dimension of learning. The paradigm of constructivism, and more particularly, social constructivism, is the lens through which I will conduct this research.

2.3.1 Social constructivism

While Vygotsky's theory is better known for explaining the social dimension to learning, his theory still offers a valuable perspective through which to understand the formation of concepts in learners. Vygotsky highlights the link between the cognitive and social dimensions of learning, that

> Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (interpsychological), and then inside the child (intrapsychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. (Vygotsky, 1978, p. 57).

It is this intrapsychological aspect that is of most interest to this study, since the focus of this study is not on the interactions between the learner and teacher, but on how the learner has internalized the concept. This process of internalization occurs through the use of signs and tools, which are: language, systems for counting, algebraic symbols, diagrams, mnemonic techniques, etc. (Daniels, 2009). In a typical Vygotskian manner, there is a fundamental external/internal relationship between signs as tools, referred to as mediation. Tools are externally oriented compared to signs' internal orientation. Berger (2005) argues that "Vygotsky saw action (tasks) mediated by signs as the fundamental mechanism which links the social world to the internal human mental processes" (p. 2).

A tool such as language, which exists on the social level is not immediately internalized. For the mathematics education context, this social level could be the teacher who has access to the tool. But the social can also include the textbook, as Berger (2005) argues that a learner interacting with the textbook is a socially constructed interaction.

As Daniels (2009) eloquently states:

> The social becomes individual not through a process of simple transmission. Individuals construct their own sense from socially available meanings. Inner speech is the result of a constructive process whereby speech from and with others has become speech for the self. (p. 27)

In the context of an algebraic equation, only when the learner has sufficiently engaged with the concept of an equation to the point where the learner's internal 'signs' are aligned with the codified mathematician's view (i.e. the tool), can the learner be said to have internalized the concept of the linear equation (Berger, 2005). In reference to Kilpatrick et al.'s (2001) strands of mathematical proficiency, I argue that full internalisation of a concept would entail demonstrating competence within all the strands of mathematical proficiency – that learners are able to accurately solve all types of linear equations, but that they also share the conceptual understanding of a codified mathematician and can justify the choices they make in executing an approach.

In chapter 9, I dedicate a section to describing how the notion of internalisation is helpful in reflecting and making meaning out of the data analysis conducted in chapters 6 through 8.

2.3.2 Scientific and spontaneous concepts

This notion of internalizing the tools of the community of mathematicians is captured in Vygotky's distinction between scientific and spontaneous concepts. Vygotsky defined scientific concepts as those introduced by a teacher, which are a part of a logical, coherent and generalizable system (Daniels, 2009). For linear equations, a scientific 'concept' refers to operating on the letter using inverse operations and performing the same operation on both sides of the equation (discussed in chapter 2). I place *concept* in quotes because it could be argued that I have described a skill rather than a mathematical concept. Vygotsky appears to use *concept* in a looser sense than we may in mathematics education as he speaks of the *concepts* of 'flower' and 'rose' (Daniels, 2009). Thus, a concept in a Vygotskian sense may include something such as an approach to solving a linear equation.

Spontaneous concepts develop outside of formal instruction. Within childhood development, spontaneous concepts result from children generalising their everyday experiences (Karpov, 2007). For example, a child may generalise the concept of gravity by noticing that everyday objects fall to earth. However, this conception is limited until he encounters helium balloons which rise, or more importantly, is formally instructed in the concept of gravity. I propose this this notion of scientific concepts requiring formal instruction has application within linear equations. Learners may use more informal methods to solve linear equations, such as inspection, or systematic substitution, which are limited in their generalisability. There are more formal scientific approaches (discussed in section 3.2) which are generalisable across all types of linear equations.

Vygotsky held that development in children is so special because adults have the ability to impart their knowledge (from the social) to young children (Karpov, 2003). Children should not be left to discover scientific facts (in a naïve constructivist notion) but rather be brought into the scientific discourse.

Russian Vygotskians argue that the source of misconceptions and error can be due to allowing learners to form spontaneous concepts if allowed to discover the concepts (Karpov, 2003). They also argue that rote learning is problematic – that "rote skills are meaningless and non-transferable" (Karpov, 2003, p. 70) and hence can lead learners to develop their own spontaneous concepts in order to solve problems. These ideas are an important aspect of my research – noting when the learner's procedure is no longer useful and what spontaneous concepts they develop.

It should not be seen as if spontaneous concepts are entirely problematic in their own right. Daniels (2009) argues that Vygotsky saw the two (scientific and spontaneous) concepts as interdependent. A teacher should be able to engage with learners' spontaneous concepts because scientific concepts "are not assimilated in ready-made or pre-packaged forms" (Daniels, 2009, p. 16). However, scientific concepts should be forefronted and, if internalised, should "transform students' everyday life knowledge" (Karpov, 2003, p. 66). Thus the scientific tool is internalised and is able to be used 'spontaneously' by learners to solve mathematical problems.

In the context of this study, learners need to be taught, and to internalise, the scientific approach of 'same operation on both sides' in order to reliably solve equations with letters on both sides. This is a new approach for the learners and is highly unlikely to be spontaneously discovered. Thus the transition from arithmetic approaches to algebraic approaches (discussed fully in chapter 3.2) requires learners to adopt a new scientific concept of operating on the letter across the equals sign.

2.3.3 Zone of proximal development

This notion has become synonymous with Vygotsky, yet was introduced fairly late in his writing and not widely featured (Daniels, 2009). The zone of proximal development (ZPD) describes the space between what a learner is able to do alone, and what they are capable of achieving on a task with the assistance of a more knowledgeable other (Vygotsky, 1978). Vygotsky's initial discussion of the ZPD was regarding assessing the development of two children, which could have the same actual development level (completed development cycles), based on the performance in tests (Vygotsky, 1978). However the two children may perform differently in the company of an adult or a more knowledgeable peer. Thus Vygotsky defines the ZPD as:

> It is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers. (Vygotsky, 1978, p. 86).

Vygotsky asserts the value in assessing the ZPD as not only what has been achieved developmentally, but what is in the course of maturing. This aspect is pertinent to my study in terms of tracking learners' ability to solve equations (development) over time. Through the course of interview based tasks (with a more knowledgeable other), the design of the study should support some form of assessment into what the learners are close to internalising.

Vygotsky (1978) makes the important claim that development processes 'lag' behind the learning process, creating 'zones of proximal development'. This means that, just because a learner has been taught linear equations, does not mean that they have fully internalised this concept. By a concept, Vygotsky defined it as "the bonds between the parts of an idea and between different ideas are logical and the ideas form part of a socially-accepted system of hierarchical knowledge" (Berger, 2005, p. 158). Thus the concept of a linear equation has the ideas of 'letter as a number', 'relational equality' (the equals sign) and 'algebraic syntactical rules'. Learners must be able to operate with all of these ideas to conform to the socially accepted methods of solving linear equations in order to have grasped the concept.

We cannot oversimplify this lagging behind of development – Vygotsky (1978) asserts that the development process is complex and cannot be captured in an "unchanging hypothetical formulation" (p. 91). However, this notion could be particularly helpful the analysis of learner progress in my study, since I am investigating the learner's development (their progress in terms of their abilities and reasoning) over an extended period of time. In chapter 9, I dedicate a section to using the notion of a ZPD that was creating in the interview, for describing some of the types of progress that were noted in the analysis. While my research design does not include interventionist tasks, through the course of task-based interviews conducted by the research, there were occasions in the sessions were learners were prompted with ideas, or were pointed towards inconsistencies in their reasoning. Thus, at times, the learners had opportunities to show their capabilities with some guidance.

2.3.4 Learner error

Thus, if we accept that knowledge cannot be perfectly transmitted between teacher and learner, particularly in the case of scientific concepts, and the learner is constructing his/her own knowledge through internalisation, then we should not be surprised if learners make errors in reproducing the mathematical skills that were taught. As Nesher (1987) states "errors do not occur randomly, but originate in a consistent conceptual framework based on earlier acquired knowledge" (p. 33).

Olivier (1992), coming from a Piagetian perspective, distinguishes between three types of mistake: slips, errors and misconceptions. Slips are mistakes which even experts can make, such that when the mistake is highlighted, the learner can immediately recognize the flaw.

Errors are systematic mistakes that are applied regularly due to the underlying cognitive structure. These deeply-seated underlying cognitive structures are formed because they produce the correct answers at times, but may now produce errors. These underlying structures are called misconceptions. The notion of misconception is beyond the scope of this study and did not feature in the data analysis.

The focus of this study is the difficulties learners encountered and how these led to errors. Systematic errors which learners made were a notable aspect of the data analysis. In chapter 5, I analyse the general progress made by learners and draw on the errors noted from literature as I describe their difficulties. Chapters 6, 7 and 8 feature detailed analyses of the difficulties which learners faced with linear equations and the subsequent errors they made.. The analysis in those chapters drew on the systematic nature of learner error to understand *why* they made the errors they did.

2.4 Conclusion

In this chapter, I have presented two of the three elements of my conceptual framework. My personal interests illustrated my critical realist view of mathematics and mathematical research, that there is a reality I am investigating, but my interpretation of that reality is fallible. I argued for the usefulness in adopting a Vygotskian perspective for my study. Social constructivism provides the notion of internalising a concept – that perfect transmission of externally oriented tools is not possible and that learners construct their own knowledge, in this case, their knowledge of linear equations and the appropriate approaches to use to reliably solve them.

These approaches can be linked to the notion of scientific concepts – that it is not possible for learners to spontaneously develop appropriate approaches, but that they must adopt the codified mathematician approach. Finally the notion of the ZPD is a powerful way of understanding the full extent of learner development (progress). Because the tasks are performed in a task-based interview, I am a "more knowledgeable 'other'". Thus I can probe and prompt learners to describe their potential development – is their potential progress further along than their initial attempts at solving an equation may suggest.

The following chapter presents the relevant linear equations literature as the final part of my conceptual framework. In the literature review, I will fully describe the well-known approaches to solving equations, the known errors specific to linear equations, the notion of the didactic cut. I will also explore the meaning of the equals sign as it relates to equations and draw out a hierarchy of difficulty for the different types of equations.

Chapter 3 : Literature review

3.1 Introduction

The literature reviewed in this chapter was determined by the research questions and the conceptual framework I have employed. While there is a fairly significant body of literature with its focus as equations, the scope of this project required me to focus on the difficulties that learners are known to experience, as well as the standard approaches to solving linear equations. The following chapter will discuss the well-known approaches to solving equations and the associated known difficulties. I discuss two important aspects of linear equations: the didactic cut and the meaning of the equals sign. I propose a hierarchy of expected difficulty of different linear equations based on the literature and other sources of difficulty from algebra and integers. This hierarchy of difficulty is used in chapter 4 to design the research instrument.

3.2 Well known approaches to solve linear equations

Kieran (1992, p. 400) lists the following approaches to solving equations.

(a) Use of number facts
(b) Use of counting techniques
(c) Cover-up
(d) Undoing or working backwards
(e) Trial-and-error substitution
(f) Transposing (change side, change sign)
(g) Performing the same operation on both sides

The first five methods (a to e) are informal arithmetic methods – as could be used for equations with a single letter, i.e. $ax + b = c$, and would be spontaneous concepts learners could use. Kieran (1992) asserts that the last two methods (f and g) are formal and algebraic (structural) in nature and are necessary for solving equations with letters on both sides (the didactic cut), i.e. $ax + b = cx + d$. These are scientific approaches in Vygotskian language.

The use of number facts (a) are methods brought from lower grades from "missing addend sentences" such as $5 + \square = 7$, where the learner is expected to 'know' (fact) the missing number is 2. This 'fact' can still be used in the linear equation $5 + x = 7$. A 'counting technique' (b) is closely related in as much as a learner would count on from 5 to 7 and notice that they counted twice, thus the missing number would be 2.

The cover-up method (c) is described by Bell, O'Brien, and Shiu (1980) for an equation such as $5x + 3 = 8x$, where the learner decomposes $8x$ to $5x$ and $3x$ and then reasons that, since $5x$ plus 3 are the same as $8x$, then $3x$ must be the same as 3, and thus $x = 1$. While this method is more algebraic, it is not normally taught formally. Whitman (1976) found success in its use alongside formal methods such as 'same operation on both sides'.

The 'undoing' approach (d) uses an arithmetic procedure of inverse operations. This method can be seen as reversing the order of operations performed on a letter: This process can be illustrated in a form of flow diagram in Figure 3.1. The equation $3x + 4 = 16$ would mean x is multiplied 3 and then 4 is added to get 16. To reverse (undo) this order, 4 is subtracted from 16 to get 12, which is then divided by 3 to get $x = 4$. This approach can be used for negative values and fractions as the unknown letter, but relies on there being only one letter in the equation. The earlier approaches (a to c) are more difficult when the unknown letter is a fraction or a negative number.

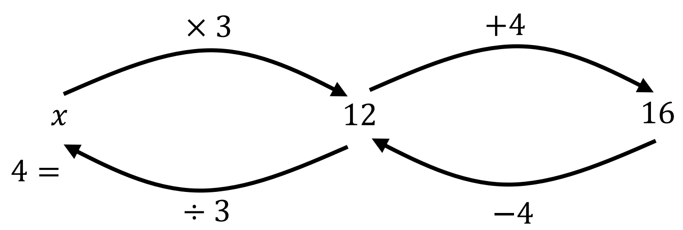


Figure 3.1: Flow diagram of undoing approach to arithmetic linear equations

The trial-and-error approach (e) is when learners try to substitute different values for x until the value on the other side of the equation is reached. This approach may be used on equations of the type (1) $ax + b = c$ and (2) $ax + b = cx + d$. Of course, it is easier to substitute various integer values into (1) and get the 'answer' which is the constant on the right-hand side (RHS). However, this trial-and-error approach becomes more difficult with (2), with the need to substitute the same numbers into the letters on both sides, learners need to have a relational view of the equals sign[1] – that is to see that both sides have the same value, and to keep track of numbers tried. Integer and rational solutions are also less likely to be trialled. Kieran (1992) argues that this method is also tedious and not generalizable. She does note the value in

[1] This term is explained fully in section 3.5 in this chapter on the meaning of the equals sign.

understanding substitution as a way to check solutions to equations even once formal methods have been adopted.

The ‘same operation on both sides approach’ (g) is strongly related to the ‘balance model’ described by Vlassis (2002) in section 3.4 later in this chapter. Filloy and Rojano (1989) call this the Eulerian (referring to Leonhard Euler) model, which is characterised by performing the inverse operation on both sides of the equation. Given the equation $2x - 1 = 4x - 3$, we retain balance (relational equality) if we subtract $2x$ and add 3 to both sides, we end up with the ‘simplified’ equation $2 = 2x$. Thereafter, we divide both sides by 2 (to undo the $2 \times x$) to find the solution to original equation as $x = 1$.

Kieran (1992) states that many view the transposition approach (f) as an abbreviated form of ‘same operation on both sides’ approach. Filloy and Rojano (1989) refer to this as the Viète model (after François Viète). This approach is characterised by the rule: change side, change sign (Kieran, 1992). Thus, given the equation $2x - 1 = 4x + 3$, the first step would be to ‘transpose’ the $4x$ to the left-hand side (LHS) and change it to $-4x$. The -1 on the LHS is transposed to the RHS to become $+1$. Thus the resulting simplified equation is $-4x + 2x = 3 + 1$. There are like terms on either side, which when added, result in $-2x = 4$. From this point, arithmetic inverse operations can be used to find $x = \frac{4}{-2}$ and thus $x = -2$. Kieran (1992) states that there is evidence that learners who are “transposing are not operating on the equation as a mathematical object but rather are blindly applying the Change Side-Change Sign rule” (p. 400). There is one learner, Daniele, who shows some similarities to this approach, which is discussed in section 6.4.

3.3 Specific errors associated with linear equations

In this section, I discuss four specific errors noted in the literature. Kieran (1992) notes two specific errors commonly made by learners when solving linear equations. The first error is called the *redistribution error*, where a learner may add a term to one side but subtract the same term from the other side. For example, given $3x + 1 = 2x - 3$, a learner may wish to undo the $+1$ on the LHS, so they subtract 1 from the LHS. If they commit the redistribution error the would then add 1 to the RHS to result in $3x = 2x - 2$ instead of $3x = 2x - 4$. Hall (2002) explains this may be some form of misappropriation of equilibrium where one compensates by adding, if a weight is removed, to retain equilibrium.

The second specific error Kieran (1992) describes is called *switching addends*. A learner may have committed this error when 'moving' a term across the equals sign if they do not use inverse operations. Thus given $3x + 1 = 2x - 3$, a learner who is guilty of this error would 'move' the $2x$ to the LHS and the $+1$ to the RHS, to result in $3x + 2x = -3 + 1$. Hall (2002) identified a new type of error – that of the *other inverse* error. This error is characterised by using the incorrect inverse operation. For example, given $3x = 1$, a learner may subtract 3 to undo the $3x$, as in $x = 1 - 3$. Thus they have used the other inverse (subtraction) instead of the correct inverse of multiplication (division).

Another error particular to equations had been noted by Sanders (2017), which she has termed *familiar structure*. Sanders describes this error as arising after learners had made a conjoining error with the form $ax + b = cx + d$ so that the resulting form of the equation is $ax = bx$ (the unlike terms are conjoined on both sides). With a 'simplified' equation such as $4x = 8x$, Sanders (2017) describes a learner would first divide by 4 to get $x = 2x$. This 'unfamiliar' equation didn't look right, and they know that the answer is $x = \dots$, so they ignored the x on the RHS to get an answer of $x = 2$. She also notes some learners who divide both sides by x to arrive at the same answer. She argues that learners are aware of familiar structures, which some authors have referred to as *canonical* equations (McNeil et al. (2006)) and that the number on the right is the 'answer'.

The following sections outline essential concepts and the possible difficulties that face learners when solving linear equations.

3.4 The didactic cut

The term didactic cut was coined by Filloy and Rojano (1989) to capture the change in thinking where learners are required to transition from solving equations arithmetically to operating on the unknown. An equation such as $2x + 3 = 15$, can be solved arithmetically, in a manner similar to how younger learners would 'work out' $2 \times \square + 3 = 15$. Here the learner can arithmetically undo the operations in the way described in section 3.2 earlier.

However, this arithmetic thinking of 'undoing' cannot be applied to an equation with variables on both sides, i.e. $2x + 3 = 5x - 2$. Other models have to be introduced, such as the balance model (Vlassis, 2002), a concrete object progressing to a metaphor where the equals sign is used as the fulcrum for a balance scale in which identical operations must be performed on both sides of the equation to maintain balance. This concrete model is formalised by what Kieran (1992, p. 400) calls the “performing the same operation on both sides”.

Kieran (1992) argues that there is a cognitive shift that needs to take place from arithmetic, which she calls ‘procedural’, to the algebraic (structural) thinking required when there are letters on both sides of the equal sign. She explains that learners are required to go from letters which stand for a number that can be found using purely numerical methods, i.e. that $3x + 5 = 14$ can be seen to a learner in an arithmetic manner of $3 \times \square + 5 = 14$. However, Kieran asserts that when letters are present on both sides of the equals sign, learners are required to not operate on numbers, but on algebraic objects (terms including a letter). Furthermore, the result of these operations is not a number, but an algebraic expression. Given the equation $3x + 5 = 2x + 14$, a learner is required to operate on either $3x$ or $2x$ and the result (if $2x$ is subtracted from both sides) is another algebraic object (x on the LHS).

Sfard and Linchevski (1994) argue that learners struggle to transition from the operational view of algebra to the structural, and the meaning of the equals sign being key in moving from a process to structure[2]. Mathematical notions can be either viewed as operational, that is as a process, or as a structure, which is an abstract object. Sfard and Linchevski (1994) record an interview in which the child was trying to make sense of an equation with variables on both sides, which “looked to him like ‘two equations’, two processes waiting to be performed while the relation between them was not clear at all” (p. 210). Hence the child cannot see the structural aspect of the equation, only as two separate processes.

The notion of the didactic cut has come in for some criticism. Herscovics and Linchevski (1994) found that learners were able to solve equations with variables on both sides, even without instruction. They argue that the demarcation between arithmetic and structural algebra as rather a ‘cognitive gap’, which is characterized by the students inability to operate with, or on, the unknown.

In reading Herscovics and Linchevski (1994), I am not convinced they have sufficiently shown that the didactic cut is absent. In their study, they had three equations with variables on both sides, $n + 15 = 4n$; $4n + 9 = 7n$; $5n + 12 = 3n + 24$ (p. 73). While there was a high success rate (91%), the majority of the learners used ‘systematic substitution’ (SS) to arrive at

[2] Sfard’s use of the term ‘structure’ is not exactly the same as Mason, Stephens and Watson (2009), but is shared by Kieran (1992). Mason et al (2009) define structure as the generalisation of particular generalisations discerned through variation.

their correct answers (with the aid of calculators). SS does require an equivalence view of the equals sign (that both the LHS and RHS have the same value), but this is hardly a structural understanding (Kieran, 1992), nor efficient method for solving these equations.

Lima and Healey (2010) also did not find the didactic cut emerging in their study when transitioning from what they term 'evaluation' to 'manipulation' equations. In their study, learners treated all equations as 'manipulation', which in the terms of my study would be 'same operation on both sides' [3]. Thus for these learners, the new method for solving equations supplanted the previous arithmetic understanding. This should not be entirely surprising - that learners adopt a new method that works for both types of equations. The method of 'perform the same operation on both sides' can still be used for an equation with a letter on one side only, such as $3x - 1 = 5$. However, again it does not discount that learners may have difficulties in understanding or adopting the new procedure if they are attached to an arithmetic view of linear equations.

Thus it cannot be argued that the didactic cut (or cognitive gap) is a barrier which all learners will encounter. It would appear that the didactic cut arise for learners who use an arithmetic 'undoing' model for solving equations. When learners are taught more generalizable approaches such as 'perform same operation on both sides', they are expected to replace their earlier approaches with these new approaches. However, many fail to do so.

3.5 The meaning of the equals sign

Early research into the meaning that learners attach to the equals sign (Baroody & Ginsburg, 1981; Behr, Erlwanger, & Nichols, 1980; Kieran, 1981) found that learners often interpret the symbol as a "do something signal" (Kieran, 1981, p. 324), which is defined as an 'operational' view. This is contrasted with a 'relational' view of equivalence between the two sides. Kieran (1981) argues that this operational conception of the equals sign can remain all the way into college, where students use the symbol in advanced calculus in finding derivatives where every line is equal to every line (even after taking the limit), i.e.

$$f'(x) = \lim_{h \to 0} 2x + h$$
$$f'(x) = \lim_{h \to 0} 2x + 0$$
$$f'(x) = 2x$$

[3] It should be noted that Lima and Healey's sample was Grade 10 and 11 Brazilian students, who could be expected to have progressed to the manipulation model.

When the learner has substituted the h with 0 in the second line, they should have dropped the $\lim_{h \to 0}$ as they have 'evaluated the limit'. However, the learner who sees the equals sign in operational manner, sees the answer at the end as being 2.

This early operational conception of the equals sign as producing the 'answer', is not necessarily problematic with equations such as $2x + 3 = 15$, where the 15 is seen as the 'answer' to 2 multiplied by a number and then 3 is added. However, questions such as $7 + 2 = \square + 5$ can be very problematic for learners (Falkner, Levi, & Carpenter, 1999; Essien & Setati, 2006), where an answer of 9 in the box would show an operational as opposed to relational view of the equals sign. The implications of an operational view of the equals sign can be seen when Grade 9 learners encounter equations with unknowns on both sides, i.e. $2x + 3 = 5x - 2$, where the equals sign cannot be seen as a 'do something' symbol because it is now a relation between two expressions, where equivalence must be maintained by performing the same operation on both sides of the equation. Arithmetical thinking leads them to a "sum equals a sum" notion (Sfard & Linchevski, 1994).

Knuth, Stephens, McNeil, and Alibali (2006) found that at multiple grade levels, a greater proportion of students who had shown a relational understanding of the equals sign, were able to accurately solve the equations. In a South African context, Essien & Setati (2006) found many Grade 8 and 9 learners who had difficulties with arithmetic problems such as $14 \times 3 = \square - 3$ or $24 + \square = 27 + 31$, and then with equations such as $2x + 3 = 5x - 2$, thus underlying the importance of a relational view of the equals sign in solving equations. In particular, they noted that Grade 9 leaners could not "explain how changing the sign when a quantity is taken to the other side, or adding the same quantity to both sides of the equation, was necessary to keep the quantities on both sides equal. They only knew the 'rules without reason' (Essien & Setati, 2006, p. 55).

The following sections present a hierarchy of difficulty within linear equations, drawn from the literature.

3.6 A proposed theoretical hierarchy of difficulty

For the design of the study it is necessary to distinguish between easier and more difficult equations. In this section, I draw on the literature that describes the different levels of difficulty which linear equations can present to learners.

Various authors have made some attempt to establish a sequence of increasing difficulty. Pirie and Martin (1997, p. 161) describe the following trajectory:

(1) $x + b = d$
(2) $ax = d$
(3) $ax + b = d$
(4) $x - b = d$
(5) $ax - b = d$
(6) $ax + b = cx + d$.

The authors did not justify the transition from one form to another, but do assert that this is the order in which learners would *encounter* equations, meaning the order in which they would normally be taught. Despite them not explaining the features which distinguish each subsequent level, there are a number of aspects that can be *inferred* as to what could possibly make this progression a representation of increasing difficulty for learners.

The first 5 types of equations can seen as *arithmetic* since they have only one letter (x) as an unknown. Both forms (1) and (2) involve a single arithmetic operation to be 'undone'. Pirie and Martin appear to treat multiplication as more difficult than addition with subtraction (4) as possibly more difficult still. This may depend upon the learner's grasp of integers. Forms (3) and (5) involve two operations to be undone. Pirie and Martin (1997) do not discuss how the values of a and b, etc. can impact upon the difficulty. An example in level 2 of $2x = 8$ may not have the same difficulty as $3x = 8$, where the answer would be a fraction. Indeed, Tossavainen (2009) looks at the cognitive load (difficulty) experienced by $2x = 1$ and how different approaches may affect the difficulty, i.e. one could use a routine procedure, which may actually be easier than solving by inspection and certainly easier than solving the equation graphically.

Ngu and Phan (2016) adopted Pirie and Martin's (1997) basic progression. However they disagreed that $x - b = d$ is more difficult than $ax + b = d$ because it involved only a single operation to be undone. I concur with Ngu and Phan, and thus I argue that $ax + b = d$ is more difficult than $x - b = d$ since it involves two operations (subtract b and divide by a). Linsell (2009) supports this notion with his empirical research with 621 learners, ranging from 13 to 16 year olds.

Linsell (2009) states that:

> In general one-step equations were easier to solve than two-step, which in turn were easier than equations with unknowns on both sides. However it should also be noted that equations involving division were harder than similar equations with other operators. Nevertheless one-step equations involving division were easier than two-step equations involving division. (p. 334).

Thus, I propose that in combining Ngu and Phan (2016) and Linsell (2009) with Pirie and Martin (1997), there would appear to be three distinct levels of difficulty. In Table 3.1 is the general form of each level with an example. In all cases, a, b, c and d are integers, which allows for the subtractive forms from Pirie and Martin.

Level	**Form**	**Example**	**Distinguishing feature**
1	$x + b = c$	$x - 5 = 8$	single operation to be undone
2	$ax + b = c$	$3x - 5 = 10$	two operations to be undone
3	$ax + b = cx + d$	$2x + 5 = x + 1$	letters on both sides

Table 3.1: The three levels of difficulty for linear equations

Linsell mentions division in the quote above and offers examples such as $\frac{n}{20} = 5$ and $\frac{n+12}{4} = 18$. In the Grade 9 curriculum in South Africa, the equations that involve fractions are normally of the form $\frac{2x}{3} + \frac{5}{2} = 3x$, where the standard procedure is to multiply through by the lowest common denominator (LCD). Although Linsell's equations are arithmetic, with Grade 9 learners new knowledge of equations with fractions, these items could appear more complex for them and invoke another approach (to multiply by the LCD). Hence all items involving equations with fractions were beyond the scope of this study.

The literature reviewed does not distinguish between having larger coefficients on the right side of the equation, entailing either collecting the unknowns on the right hand side or dividing by a negative coefficient. For example, given the equation $3x + 2 = 5x - 6$, a learner could avoid dividing by negative coefficients (which would increase the cognitive load) if he/she grouped the x's on the RHS and the constants on the LHS, i.e. $2 + 6 = 5x - 3x$. In my experience as a teacher, mathematically stronger learners are more comfortable with this idea.

The inclusion of brackets is a known source of algebraic error. Rittle-Johnson and Star (2007; 2009) deem the inclusion of brackets (parentheses) to constitute higher cognitive demand. Pournara, Hodgen, Sanders, and Adler, (2016) noted in the context of algebra, that brackets often signalled learners to multiply, often applying the distributive law inappropriately.

As discussed earlier in section 3.4, Lima and Healey (2010) showed that once learners had acquired the same operation on both sides approach (they term the manipulation model), that this model was pervasive even on arithmetic (single unknown) equations. Hence a learner with a strong balance model approach, might solve $1 - x = 7$ by adding x to both sides, and then subtracting 7, to end up with $-6 = x$. It may be less difficult than the learner who procedurally subtracts 1 from both sides, and is then required to divide by -1. Based on Linsell's argument, this resulting equation of $-x = 6$ may be a more difficult equation, although the learner may arrive at the correct solution but be operating on symbols with very little understanding of what they were doing (Lima and Healey, 2010).

The notable landmark identified above was the transition from the arithmetic conception of equation to the need to operate on the variable (level 3 in Table 3.1). This transition is known as the didactic cut.

3.7 Integer operations and algebraic simplification errors

In this section, I discuss errors which learners make due to integer operations and algebraic simplification. I will limit my discussion of these types of errors since they are not errors which stem directly from the concept of equation itself. These errors are due to the ambiguous minus symbol and the algebraic expressions on either side of the equals sign. Both of these aspects are almost always present in equations, and may be the source of learner error.

3.7.1 Errors associated with the minus symbol

The minus symbol can be viewed as an *operator* (subtraction) which Gallardo & Rojano (1994) refer to as its *binary* function, or what they term its *unary* function, as being *negative* (sign). Hence when learners encounter an equation such as $2 - 7x = 6 - 5x$ the -5 can be seen as subtracting $5x$ or as negative $5x$. This duality of the minus symbol can pose significant difficulty for learners. In chapter 7, I discuss learners' difficulties with negativity, particularly in this regard. I also use the terms 'negative letter' (negative 5 x), 'subtracting x' (6 subtract $5x$) and 'negative coefficient to x' (the coefficient of $-5x$ is negative 5) in different sections depending on which aspect I wish to highlight.

Vlassis (2004) extends the work of Gallardo and Rojano (1994) to identify some of the errors associated with the minus symbol. She describes 'right to left' reasoning where learners will simplify $5x - 7x$ as $2x$ since it is more comfortable. She further describes 'brackets' reasoning as in the insertion of invisible brackets, where $5 - 5x - 3x$ would simplify to $5 - 2x$, with invisible brackets around $5x - 3x$. Over-simplified rules drawn from other contexts can also pose as source of error. A rule such as 'a minus and a minus gives a plus' can result in $-5x - 2x$ being simplified to $+7x$. There are other types of errors mentioned by Vlassis (2004) and Gallardo and Rojano (1994), such as the insertion of 'taking the following sign'. However the errors I have discussed are the most noteworthy for my study.

3.7.2 Errors associated with algebraic expressions and variable

Küchemann (1981) describes the different conceptions of variable – and when they can be problematic. Of relevance to this study, are the notions of 'letter not used' and 'letter as a specific unknown' (p. 104). With 'letter not used', Küchemann describes this understanding of letters as ignoring, or barely acknowledging, the letter when solving problems. I would assert that this is fairly closely aligned to the discussion earlier of learners solving equations with an arithmetic understanding – that is $2x + 3 = 15$ can be thought of as $2 \times \square + 3 = 15$. This may seem like learners are required to use the 'letter as a specific unknown'. Küchemann however describes this interpretation as seeing the letter as representing a number, but that the learner can *operate* upon the letter directly (emphasis mine). This would apparently link with the 'didactic cut' - that learners may have to have a different understanding of letter in order to be able to manipulate the equation successfully.

Booth (1988), MacGregor and Stacey (1997) and in South Africa, Pournara et al. (2016) and Mashazi (2014) all note the prevalent use of 'conjoining', whereby two unlike terms, $5x + 3$, are combined into a single term, $8x$. Another specifically algebraic error is term 'deletion' by Carry, Lewis and Bernard (1980), where a learner would simplify $2x - 2$ as just x (because $2 - 2 = 0$ and you are left with x).

If learners are making these sort of algebraic simplification errors, even with simple equations such as $2x + 1 = 3$, they will not arrive at the correct solutions to equations even if their approaches (procedures) are correct.

3.8 Conclusion

In this chapter, I first discussed the well-known approaches to solving equations found in the local and international literature. The arithmetic approach of *undoing* (using inverse operations) was contrasted with the algebraic approach of *same operation to both sides* which was a necessary progression learners would need to make when faced with equations with letters on both sides. The other approaches which were discussed and have particular relevance to this study were *systematic substitution* and *transposition* of terms across the equals sign, both of which are seen later in the data.

The specific errors to equations that learners are known to make were the *redistribution error, switching addends, other inverse* and *familiar structure*. These will be used to inform part of the data analysis process. The well-known difficulty in transitioning from arithmetic equations to algebraic, termed the *didactic cut*, was also discussed. This known source of difficulty formed a significant part of the design of the research instrument, which is discussed in the following chapter.

The meaning which learners attach to the equals sign points towards their conception of equality – necessary especially when equations have letters on both sides. The equals sign can no longer be seen as an *operator* which gives an answer (on the RHS) but rather needs to be seen as a *relational* symbol of equality between two expressions, i.e. $2x + 5 = x - 1$.

With these underlying concepts of equations in place, I discussed the theoretical hierarchy of difficulty. In section 3.6, I defined the three types of equations in Table 3.1 as representing three levels of difficulty, single operation, two operations and letters on both sides.

These three levels will form the basis for the different levels of task items in the design of the research instrument (*vertical dimension*) in the following chapter. However, it was also noted that algebraic errors, and errors due to the minus sign (negativity) can also affect the difficulty of linear equations. Thus these elements also need to be incorporated into the design of the research instrument. These algebraic and negativity features form the basis of the *horizontal dimension* of the research instrument. The following chapter will specify the design of the research, and how the literature was used in designing the research instrument and informing the data analysis process.

Chapter 4 : Methodology

4.1 Introduction

In this chapter, I discuss the design of the research. The primary aim of the research was to describe the progress made by learners in solving linear equations, the approaches they used and the difficulties they encountered. In this chapter, I firstly define research design as part of conceptual framework identified in chapter 2. I then outline the research methodology and sampling procedures that I employed, before describing the data collection process and how the data was analysed. Finally, I discuss some of the inherent limitations of the study as determined by its design and scope.

4.2 Research design

The research questions must drive the design of the research in order that they can be answered effectively. This study is focussed on describing learners' progress over a six month time period. Did their abilities in solving equations increase and did their reasoning progress? What difficulties did they encounter? Did these difficulties lead to errors? While the notion of progress could appear to be quantitative – measuring their ability, this study sought to describe the *nature* of the progress. Thus, the nature of learner progress, the types of difficulties encountered and errors made are fundamentally qualitative questions. Van Mannen (1979) defines qualitative research as an:

> … array of interpretive techniques which seek to describe, decode, translate, and otherwise come to terms with the meaning, not the frequency, of certain more or less naturally occurring phenomena in the social world. (p. 520).

This study describes the phenomena of learners' progress in linear equations over time. The phenomena include their abilities in solving linear equations and their reasoning, and these phenomena can only be qualitatively *described* as they appear to me, since it is impossible to gain direct access to their thinking. Their thinking is made evident through their written work and their verbal explanations (in interviews), but these qualitative elements only point towards their actual thinking. Mirriam (2002) asserts that the "product of qualitative inquiry is *richly descriptive*" (p.16, emphasis in original) of complex phenomena which are supported by a combination of data sources.

This research design also aligns with the critical realist perspective that was discussed in chapter 2. Critical realism accepts the ontology that reality can exist, but the apprehension of this knowledge is imperfect due to humanity's fallible intellectual mechanisms and the complex nature of the phenomena (Guba & Lincoln, 1994). Thus, the design of this research necessitates a rich description to approach a fuller apprehension of the reality of learners' thinking.

Mirriam (2002) also situates the logic associated with qualitative research as primarily inductive in nature. A qualitative researcher builds concepts from existing data by extracting general themes and categories in order to build theory. This is contrasted with a positivist perspective, in which a hypothesis is tested deductively.

4.3 Research methodology

At the outset of this research project, the methodology was initially conceived of as a multiple case study. This was due to having 6 participants and the aim was to track all 6 learners in depth. Mirriam (2002) describes a case study as a holistic description and explanation of a phenomenon, or common phenomenon to a number of cases. Thus for a multiple case study, each case must be treated "as a comprehensive case in and of itself" (Mirriam, 2002, p. 204).

However, based on the enactment of the data collection and analysis, the study does not fit neatly within this definition of a multiple case study because the scope and gaps in the data did not permit a holistic and in depth analysis of all six learners. Chapter 5 provides a brief analysis of all six learners, but this analysis is not as holistic as case study design warrants because the analysis focusses only the progress in their abilities. I do not delve into their reasoning, nor do I examine other holistic elements such as their work ethic or revision of work in-between periods.

I did however draw inspiration from multiple case study design. Mirriam (2002) goes further to say that after the individual cases have been described, a cross-case analysis can be performed to identify themes and categories present in more than one case. I found this process helpful in beginning my data analysis, which led to the extraction of common themes across cases. These common themes formed the basis of my data analysis: chapter 6 analyses the common difficulty of negotiating the didactic cut, chapter 7 analyses three learners' difficulties with negativity and chapter 8 analyses the difficulties learners experienced with the special case of $ax = bx$.

Thus a more fitting label for the methodology that I have employed in this study would be what Mirriam (2002) refers to as a *basic, interpretive* study. In such a study, a researcher is "interested in understanding the meaning a phenomenon has for those involved" (p. 22). In my study, this can framed as follows: I, as a researcher, am interested in understanding the meaning that linear equations have for the participants. This *meaning* in the context of this study refers to the abilities that learners demonstrate, particularly their procedural fluency and their strategic competence, and their reasoning – how they explain their thinking.

4.4 Sampling process

The initial sample was not chosen by me. The school which was chosen for this study by the WMCS project is a Quintile 5 school in northern Johannesburg. The initial WMCS test was given to approximately 190 learners. The test was provided with instructions such that the teachers of the school could administer the test.

From this group that wrote the initial test, I invited six learners to participate in the in-depth, task-based interviews. I chose to use purposeful sampling. Patton (2002) describes the benefits of purposeful sampling as such: "the logic and power of purposeful sampling lies in selecting *information-rich* cases for study in depth. Information - rich cases are those from which one can learn a great deal about matters of central importance to the purpose of the inquiry, thus the term *purposeful* sampling " (p. 230, emphasis in original).

I invited the six learners to participate in my study. For the scope of this study, and with my full-time teaching commitments, I had to balance what would be feasible to study in the given time, with the need to have a reasonable sample to support trustworthy conclusions. I sampled the six learners on three factors. The first factor, was that they got only one of the 5 equations of the original WMCS test correct. This was the first arithmetic equation, similar to $3x - 5 = 12$. None of the six learners got the remaining questions correct. The second question on the WMCS test had letters on both sides, similar to $3x - 5 = x + 2$. This indicated they had experienced significant difficulties with solving equations, but that they were also somewhat capable of solving basic arithmetic linear equations. But all six learners *could* be experiencing the didactic cut, since none of them solved the equation with letters on both sides correctly.

Their responses to the remaining equations questions were either incorrect or missing. In South African schools, 40% is considered a pass mark for Grade 9 learners. It was important to have learners who had *some* ability to solve equations in order to be able to engage with their reasoning, but who were also experiencing difficulties.

The second factor was the types of errors made and the working of their answers. In particular, I looked for learners who appeared to be struggling operating with the letters when letters were on both sides of the equals sign (possible evidence of a didactic cut) and those who wrote down answers only (possibly solving by inspection). One learner (Daniele) gave different values for x for the item which had letters on both sides, indicating a relational view of the equals sign but difficulties with the nature of *letter as an unknown number*.

I limited my sampling to those learners from two teachers. These teachers were available for further enquiry as to what methods the learners had been exposed to and to keep them up to date with their learners' progress for ethical reasons (the concern for the well-being of the learners).

The third factor was the learners' general performance in mathematics and algebra. The teachers of the learners provided me with the learners' results from a summative class assessment (focused on algebra) and a summative general exam. The six learners results are provided in Table 4.1. The names provided are pseudonyms. The performances of the chosen six learners varied. On the class-test, the lowest score was 13%, (Yusuf) up to maximum of 70% (Edith). On the June exam, their performances ranged from 24% for the weakest learner (again Yusuf), up to 72% for the strongest (Guban).

Learner (Pseudonym)	***Algebra Class test (%)***	***June Exam (%)***
Sarah	43%	64%
Yusuf	13%	24%
Guban	53%	72%
Daniele	50%	42%
Edith	70%	46%
Tebogo	57%	57%
	$\bar{x} = 48\% \quad \sigma = 18\%\ pp$	$\bar{x} = 51\% \quad \sigma = 16\%\ pp$

Table 4.1: Sampled learners' performance on school assessments

The mean average for the six learners on the class-test was 48%, with $\sigma = 18\ pp$ and the class-test mean was 51%, with $\sigma = 16\ pp$. The standard deviation indicates the notable spread of the learners' performance and the mean substantiates the claim that they may be seen as neither low-attaining nor high performing learners.

4.5 Research Instruments

The first research question focussed on the progress learners made in their abilities in solving equations. The second research question focussed on the learners approaches, in particular, did their approaches change for different equations? The third research question looked at the difficulties they encountered. Thus the research instrument needed to be able to be used in a task-based interview setting to ascertain which types of equations learners could solve, and conversely which types of equations they had difficulties with. Furthermore, this research investigated these two aspects over time: did they show progress in which equations they could solve and did they experience fewer difficulties as time progressed? Thus the instrument needed to be consistent over time, but still allow for progress to be noted.

Thus there was a need to have a selection of linear equations that was adaptable in an interview task setting which could challenge learners, but also have ‘easier’ equations to hand. The task interview matrix was designed in such a way as to allow me to be able to move flexibly between providing learners with easier and/or difficult equations, but also to map to the theoretical hierarchy of difficulty that was derived from the literature in chapter 3.

4.5.1 The task interview matrix

The design on this instrument (grid of tasks) was motivated by two interests:

1. To assess the full range of linear equations identified in the literature, i.e. the hierarchy of difficulty.

2. To be adaptable in a condensed interview situation (limited to 20 minute sessions), that it can challenge a learner who is correctly answering questions, but that it can also help identify what weaker learners *can* do (not just what they *cannot*).

The grid of tasks in Figure in 4.1 is also attached in its full scale as Appendix A. The design of the instrument has two dimensions: a *vertical* and a *horizontal*. The vertical dimension (moving from row A to B to C, etc.) is based on the increasing cognitive demands of equations as described in the literature. Rows A to C map onto the three levels identified in Table 3.1 namely:

(1) $x + b = c$ Row A
(2) $ax + b = c$ Row B
(3) $ax + b = cx + d$ Row C

The horizontal dimension (from column 3 to 4 to 5) varies the features of the equation within the same type of question (one operation, or two operations, etc.) to see whether features such as the order of terms, the inclusion of negatives, or the side on which the higher coefficient of the unknown is, have an impact on learners' abilities to solve the equation.

Equations in 2 Dimensions

	0	1	2	3	4	5	6
A	$\square + 5 = 8$	$5 + \square = 8$	$x + 5 = 8$	$x - 5 = 8$	$5 + x = 8$	$5 - x = 8$	$8 = x + 3$
B	$5 + 2 = \square + 3$	$3x = 6$	$3x = 8 - 5$	$3x - 5 = 10$	$3x + 5 = 6 + 5$	$5 - 3x = -10$	$-(5 - 3x) = 10$
C	$\square + \square = \square + 3$	$2x = x + 5$	$2x + 5 = x$	$2x + 5 = x + 1$	$4x - 5 = x + 1$	$3x = 4x$	$4 - 3x = 5x - 4$
D		$5 + 2 = 2x - x$	$x + 5 = 2x - 2$	$2x + 5 = 3x - 2$	$5 - 2x = 3x - 10$	$2(5 - 2x) = 10 - 3x$	$2x + 1 = 3x - x$

Figure 4.1 : The grid of tasks to be used in the interviews

It is important to note that all the questions in the grid would be considered standard 'Level 1' (knowledge) and 'Level 2' (routine procedures) questions according to the South African CAPS curriculum (Department of Education, 2011). These first two levels are 'low cognitive demand' as they are seen to be routine – items familiar to learners from textbook exercises.

The factors which determine the levels of difficulty within both dimensions have a basis in literature and were discussed in detail earlier. What follows will be a brief description based on the factors identified earlier.

The vertical dimension is the clearest in terms of the increasing cognitive demand from literature reviewed in chapter 3. In Table 4.2, the first column shows Pirie and Martin's (1997) progression of difficulty in equations. Ngu and Phan (2016) argued that subtraction did not necessarily increase the cognitive demand. I agreed with Ngu and Phan, and I determined that negativity was a component of the horizontal dimension. Linsell (2009) argued that distinction was rather between how many operations on the letter are required to be undone. One operation items (1) and (2) and (4) of Pirie and Martin became the first level, and thus row A. The third column shows the core progression (column 3) from the task-matrix (Figure 4.1.)

Pirie and Martin	*Ngu & Phan, and Linsell (adapted)*	*Vertical Dimension (Rows)*
(1) $x + b = d$ (2) $ax = d$ (3) $ax + b = d$ (4) $x - b = d$ (5) $ax - b = d$ (6) $ax + b = cx + d.$	(A) $x + b = c$ (B) $ax + b = c$) (C) $ax + b = cx + d$	(A) $x - 5 = 8$ (B) $3x - 5 = 10$ (C) $2x + 5 = x + 1$

Table 4.2: The vertical dimension from the task-matrix from literature

The most significant transition is from row B to C, whereby the unknown (x) appears on both sides of the equation. This transition was Filloy and Rojano's (1989) classification of a 'didactic cut', discussed in detail in section 3.4. This requires moving from arithmetic views of equation as 'undoing' to an operational (algebraic) view, that is operating on the variable.

The literature reviewed did not distinguish between having higher coefficients on the right side of the equation, entailing either collecting the unknowns on the right hand side or dividing by a negative coefficient. In my experience as a Grade 9 teacher over 5 years, this does appear to cause difficulties for some learners. Thus I designed row D to investigate these anecdotal observations more systematically.

(D) $2x + 5 = 3x - 2$

Finally, the inclusion of brackets is a known source of algebraic error and an increase in cognitive demand (Rittle-Johnson and Star, 2007, 2009). However, I have included these in the horizontal component relating to algebraic features (item D5 in Figure 4.1). Again, these errors would be classified as algebraic features rather than relating specifically to equations, but nonetheless could still be of note.

In general, the variation in questions along a row was to see whether the specific features and layout of each question caused difficulties for learners. This variation has some basis in variation theory (Runesson, 2005), where variation of aspects is employed so that the object of learning is in focus. Example, by varying $3x - 2 = 7$ and $2 - 3x = 7$, the only dimension of variation is the order of terms $3x$ and 2, while fixing the constant on the RHS (7) and the operation (subtraction).

For the design of the horizontal dimension, there was variation in the features of the equations. These features were based on the known algebraic and integer based errors discussed in chapter 3. Figure 4.2 shows the horizontal dimension of variation in row B.

Column	0	1	2	3	4	5	6
Equation	$5+2=\square+3$	$3x=6$	$3x+5=8$	$3x-5=10$	$3x+5=6+5$	$5-3x=-10$	$-5-3x=10$

Figure 4.2: The horizontal variation in row B.

Comparing items B3 and B4, the $+5$ is introduced on the RHS of (B4) to introduce like terms on the RHS. In B5, the order of the terms was changed so that x had a negative coefficient (-3). Item B6 then makes the first term negative as well. Towards the left of the row, item B2 removes the minus symbol, intended to check whether a learner getting B3 incorrect could cope with a similar item without negatives. Item B1 was intended to investigate whether they are able to divide by the coefficient. B4 is similar to an item that Stephens et al. (2013) had used. Stephens el al. noted that learners who have a strong relational understanding of the equals sign, could see this equation simply as $3x=6$ (that the $+5$ on either side is negated).

In Figure 4.1, items A0; A1; B0 and C0 have replaced the unknown letter (x) with a box. This is aligned with other authors (Stephens et al, 2013, Essien & Setati, 2006) uses of boxes (☐) or question marks (?) by Warren and Cooper (2005) or underscores (__) by Stephens et al. (2013). Watson (2013) has described the many meanings of 'letter' when used in algebra and the problems that can arise with the use of letters. Thus, the inclusion of these items was designed to ascertain whether learners could solve the equation if letters were not present.

4.5.2 How the task-matrix was adapted for the different sessions.

The features of task items were varied slightly for each of the four sessions. I was concerned that learners would become overly familiar with the answers if the solutions to the equations and the equations were the same in each session. Thus, the variation in constants and coefficients was to prevent rote recall and to ensure learners were faced with familiar but slightly different equations each time.

The variation in tasks between sessions also allowed me to incorporate brief analysis and reflection after each session. The adapted task matrix for session 2 is included as Appendix B. This shows that the basic design of the items remained in place. An example of a change that was made to the original matrix was to include an item with two letters, but on the *same* side after realising that I did not have an 'easier' item with two letters in row C. In Appendix B, this was item (C1) $4x - 2x = 10$. The use of this adapted item is discussed in chapter 8, particularly section 8.3.

Because of learners' difficulties with items such as $3x = 4x$ (C5 in session 1) I realised that I needed an equation that resulted in a solution of $x = 0$, but was in a more familiar form. Thus, I designed the item (B6) $20 = 2(x + 10)$ for session 2 (in Appendix B). These difficulties with items like $3x = 4x$ are discussed in chapter 8.

4.5.3 How the task-matrix was used

The interview-task would begin with me asking the learner to solve equation A3. Every interview would begin with item A3. Column three is designated as the 'core' progression – that is the progression aligned to the major changes in cognitive level. This is the middle column and would allow me to move to the right in the row to more difficult items in that row, or to the left for easier items.

The movement in the task matrix is captured in Figure 4.3 where the symbol (✓) represents the learner correctly solving the equation, while (✗) represents an incorrect solution. The order of tasks given is represented by the numbers inside the ovals.

Equations in 2 Dimensions

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	0	1	2	3	4	5	6
A	$\square + 5 = 8$	$5 + \square = 8$	$x + 5 = 8$	✓ $x - 5 = 8$ 1	✓ $5 + x = 8$ 2	✗ $5 - x = 8$ 3	$8 = x + 3$
B	✓ $5 + 2 = \square + 3$ 7	✗ $3x = 6$ 6	✗ $3x = 8 - 5$ 5	✗ $3x - 5 = 10$ 4	$3x + 5 = 6 + 5$	$5 - 3x = -10$	$-(5 - 3x) = 10$
C	$\square + \square = \square + 3$	$2x = x + 5$	$2x + 5 = x$	$2x + 5 = x + 1$ 8	$4x - 5 = x + 1$	$3x = 4x$	$4 - 3x = 5x - 4$
D		$5 + 2 = 2x - x$	$x + 5 = 2x - 2$	$2x + 5 = 3x - 2$	$5 - 2x = 3x - 10$	$2(5 - 2x) = 10 - 3x$	$2x + 1 = 3x - x$

Figure 4.3: A possible path of questions used in the interview

If a learner is successful in solving A3, I would give them A4, then A5 and then A6. When they fail at any level higher than A3, or progress all the way to A6, I would proceed to B3. If they fail to answer B3 correctly, I would then recede to B2, and even B1 or B0. I would then proceed to C3 and again move right or left horizontally according to their success, all the while following the interview protocol until the allotted 20 minutes period was up.

4.5.4 Interview protocol

Kvale (2003) traces the value of interview-based research back to the great psychologists, through psychoanalysis. But in the educational field, Piaget made extensive use of interviews for his initial work. Piaget argued that the way children explained their errors would reveal their underlying mental structures (Kvale, 2003). The key aspect inherited from the psychoanalytic interview method was the use of interview enquiries directed towards the interpretation of meaning. I am following a semi-structured format, allowing learners to answer the questions and then explain their thinking in order for me to make meaning of their responses. Importantly, Kvale argues that it is goal of the interview to allow your ‘subject’ to talk freely, that is to minimize your questions and maximise their responses.

Kvale (1996, p. 133 - 135) offers 9 types of interview questions, listed below:

> Introducing questions ; follow-up questions ; probing questions ; specifying questions; direct questions ; indirect questions ; structuring questions ; silence ; interpreting questions.

The first relevant type of question I used was an “introductory question”, such as, “how have you been doing recently in maths”? Since my interview would be centred on a grid of tasks, I began by writing down the equation and ask the learner “Can you try and work this out?”. I would then ask the learner to explain their working out, which could be categorized as a “follow-up question”. If the learner’s explanation was vague, I would ask the learner to explain more through probing questions such as “can you say more?”. If the learner had used a noteworthy word, such as ‘inverse operation’, I would then follow up with a “specifying question” such as “can you explain what you mean by ‘inverse operation’?”.

It was sometimes necessary to ask interpreting questions such as “do you mean that there is no number that can take the place of x?” to clarify what a learner may mean by “no solution”. However, these types of questions could have lead to problematic “leading questions” (Kvale, 2003) in which learners feel the need to agree with a teacher in an unequal power relationship. For this reason, I tried to minimize the need for such questions. To contrast this, Kvale (2006) notes “silence” as a type of question, whereby I remained silent and allow the learner to speak as much as possible.

Kvale (2003) argues that the very nature of the term inter-view means that total objectivity is not possible. I was aware that the interpretations and meaning that I make from interviews are to be based on making the raw evidence explicit, and the reasoning based on that evidence, to be as clear as possible for the reader to evaluate.

4.6 Data collection

The initial data collection consisted of the class teacher administering the WMCS pre-test in March 2017 for a 60 minute period. Instructions were given verbally to learners and calculators were not allowed. Learners were informed that these tasks would not be 'marked', and would not count towards their school marks, but the research team would look at their answers. The original WMCS test was coded by postgraduate students from the WMCS project. The items were coded as correct, incorrect or missing. This process was repeated in September 2017.

After I sampled the six learners I wished to interview, I met with them to invite them to participate and hand out ethics letters. I arranged to do 2 rounds of interviews in the second term, and two in the third. As it turned out, the last session had to be moved into the fourth term.

Figure 4.4 represents the key data collection points, as well as other key events in the year for the sampled learners. The four interview sessions are represented by S1 to S4. The start of 2017 is represented by Term 0, with the end of each terms as Term 1 ; Term 2, etc.

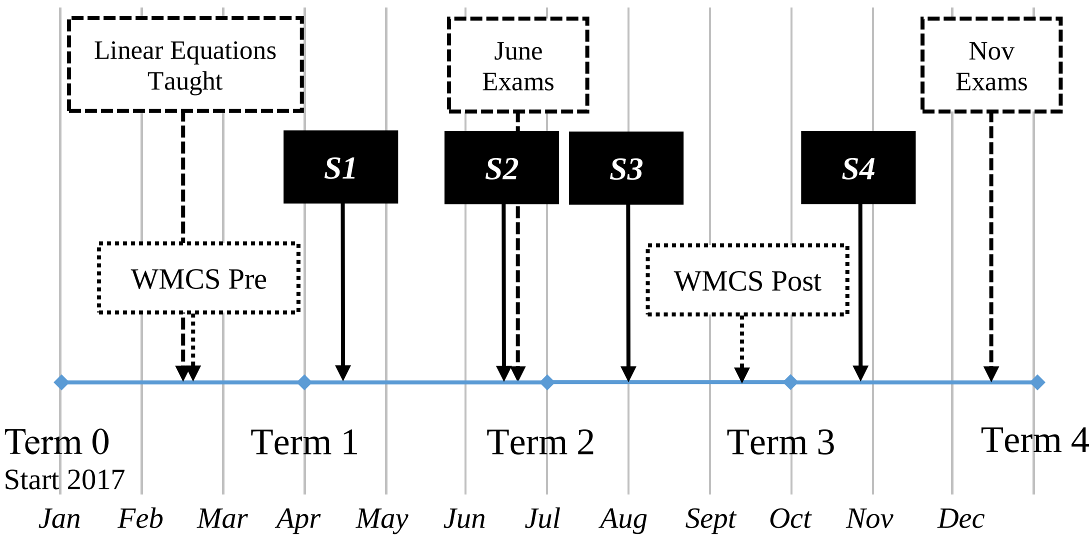


Figure 4.4: Timeline of key events in data collection.

All of the interviews were completed by the end of October 2017. Figure 4.4 shows that linear equations were being taught very close to the time that the WMCS pre-test was administered. Also included in Figure 4.4 are the exams that the learners would have written. These are points in time where learners should have been revising linear equations for assessment purposes and thus their abilities *may* have changed as a result of the revision.

There were two primary sources of data: the learners' written responses to the interview task, and the video recordings of the learners' working. The video recording of the learners' work was deemed necessary in order to capture the indexical nature of their working and explanations. It was valuable to note precisely when learners paused in their working and for how long. When learners were explaining their working, indexical terms such as 'this' and 'that' can be matched to their gestures and pointing. The video also included a clear audio recording which enabled precise transcriptions.

4.7 Data analysis process

4.7.1 Preparing the data for analysis.

The data collected consisted of the learners' written work and the video recording of their working which was collected and scanned with the learner's unique code for ease of retrieval. Through the mini-reflections in-between sessions, I realised that Edith, Tebogo and Daniele were experiencing the most difficulties with equations and all three had tried to explain their thinking through talk. Thus, Edith, Tebogo and Daniele's videos were selected for analysis. These videos were transcribed by myself and another professional transcription service. I then checked the transcription for accuracy. Only the relevant sections of the other learners' interviews were transcribed. For example, Guban only experienced difficulty with items of the type $ax = bx$. Thus only the sections of the sessions where he was working on this type of item were transcribed by myself.

4.7.2 The data analysis approach

The data for this research comes from the interviews conducted using the task matrix and the learners' written responses. The data analysis contained in this chapter serves to answer the questions:

1. What progress do learners show in solving linear equations over time?
    a. What progress do learners show in their abilities to solve linear equations?
    b. What progress do learners show in their reasoning by explaining their thinking?

2. What approaches do learners use?
    a. Do their approaches change over time?
    b. Do their approaches lead to difficulties?

3. What difficulties do learners experience?
    a. Do their difficulties lead to errors?

The approach the data analysis drew heavily upon Merriam's (2002) description of the process to be followed when analysing a multiple case study. Again although this study is not a true case study, Merriam's ideas were helpful to me in beginning to look at the data. Merriam (2002) asserts that there are two stages of analysis: the *within-case* and the *cross-case* analysis.

The within-case analysis tells the story of the individual, while the cross-case looks at the common themes across the cases (learners). For the individual story, my data analysis needed to tell the stories of the learners – their successes and difficulties. But more than that, I needed to tell the story as these aspects changed over time. I realised that coding of errors with such a small sample was not sufficient. I needed to track their errors over time. I used two representations to represent the data in order to analyse it: the *interview progress table* and the *progress diagram*.

4.7.3 The interview progress table

The *interview progress table* represents a learner's progress through the items in each session. Table 4.3 describes the coding of the items for the interview progress table.

(C)	Correct: The final value for x was the correct value.
(C – MA)	Correct with multiple attempts: Because of the nature of the task taking place within an interview (with learners checking their answers and justifying their thinking), learners would realise an answer was incorrect, and re-attempt the item, sometimes correctly.
(W)	Incorrectly answered or not completed.

Table 4.3: The coding of items in the interview progress table

In Table 4.4, an example of an interview progress table is provided. There are 4 large columns for each of the sessions, and within each column is a comparison of the item with its corresponding code. The order of the items presented from top to bottom mirrors the order of the items that the learner would have encountered.

Session 1		*Session 2*		*Session 3*		*Session 4*	
Item	**Code**	**Item**	**Code**	**Item**	**Code**	**Item**	**Code**
A3	C	A3	C	A3	C	A3	C
A5	W	A4	C	B3	C	A5	C-MA
B3	C	B3	C	B4	C-MA	B3	C
B5	C-MA	B4	C	A6	C-MA	B4	C
C0	C-MA	C3	C	C3	C	C3	C
C3	W	C4	C	C5	W	C4	C
D3	W	C5	W	D3	C	C5	C-MA
		D3	C			B5	C

Table 4.4: An example of an interview progress table for a learner

To appreciate fully the meaning of the item codes, it is necessary to remind the reader of the design of the interview task. The letters for the item correspond to the rows in the task matrix. These rows were the distinctive cognitive levels derived from literature. The meaning of the first 3 rows (which were the most significant for this study) is given in Table 4.5.

ROW	**General form**	**Features**
A	$x + b = c$	A single arithmetic operation
B	$ax + b =$	Two arithmetic operations
C	$ax + b = cx + d$	Letters on both sides of the equals sign.

Table 4.5: The three cognitive levels

The number attached to each letter, i.e. 3 in A3, represented its location in the horizontal dimension of the matrix. The higher the number, the more difficult for the learner (in terms of the features). The variation in the horizontal dimension focussed on the algebraic and integer based features of the question that may cause difficulty. In the interview progress Table 4.4 (provided earlier), this learner was able to correctly solve (A3) $x - 5 = 8$ but was unable to solve (A5) $5 - x = 8$. This suggest that the order of the terms made A5 more difficult for her.

The common algebraic, integer and equations errors that were noted in the literature will be referred to for the specific cases of (W) if applicable. Due to the small sample, it was not appropriate to be coding common errors. However, the design of the study allows for a detailed analysis of learner error and explanation. These detailed discussions of learner error are provided in chapters 6, 7 and 8.

The code of (W) also includes learners who stop at some point in their working or leave work incomplete. For example, Daniele left an answer as $-x = 8$ (shown in Figure 7.15 in chapter 7). This was coded as wrong, even though her working to this point was not wrong, just incomplete. Similarly, due to the nature of interviews with learners explaining their work, learners would realise they had made an error and make another attempt. Such an example is provided in Figure 6.4, which was coded as C-MA, because her final attempt is correct.

The interview progress table provides a detailed account of a learner's specific progress through each of the four sessions but can be overwhelming in the details. Thus I offer a second illustration of the individual learners' progress – that of a *progress diagram*.

4.7.4 The progress diagram

After reflecting between sessions, it became clear that there were four levels of equations that learners were progressing through. These levels maps closely, but not exactly onto the distinction between rows (A – C) discussed above.

The levels of equations are:

(1) $x + b = c$ (row A)

(2) $ax + b = c$ (row B)

(3) $ax + b = cx + d$ (row C)

(4) $ax = bx$ (this type of item was in row C, but it became apparent that learners were treating this item differently)

This fourth level requires some justification as to why it deserves its own level. This fourth level was discussed in section 3.3 in regards to Sanders's (2017) findings. Originally this item was in the same horizontal dimension as $ax + b = cx + d$, i.e. a more difficult form *within* row C. However, after reflecting on the learners' responses after the first couple of sessions, it became apparent that learners were treating this item as a different type to other equations with letters on both sides. It is the consistent difficulties which learners had with items of this type that lead me to classify it as its own level. These difficulties are analysed in chapter 8.

An example of a progress diagram is provided in Figure 4.5. This diagram offers a level of abstraction into how a learner's *general* progress throughout the six months of the study. It presents the big picture of the types of equations which learners had difficulties with. It condenses the specifics from the *interview progress table* by collecting the general performance of learners within the broad types of equations. The progress diagram for Edith shows her progress through four levels and through four sessions.

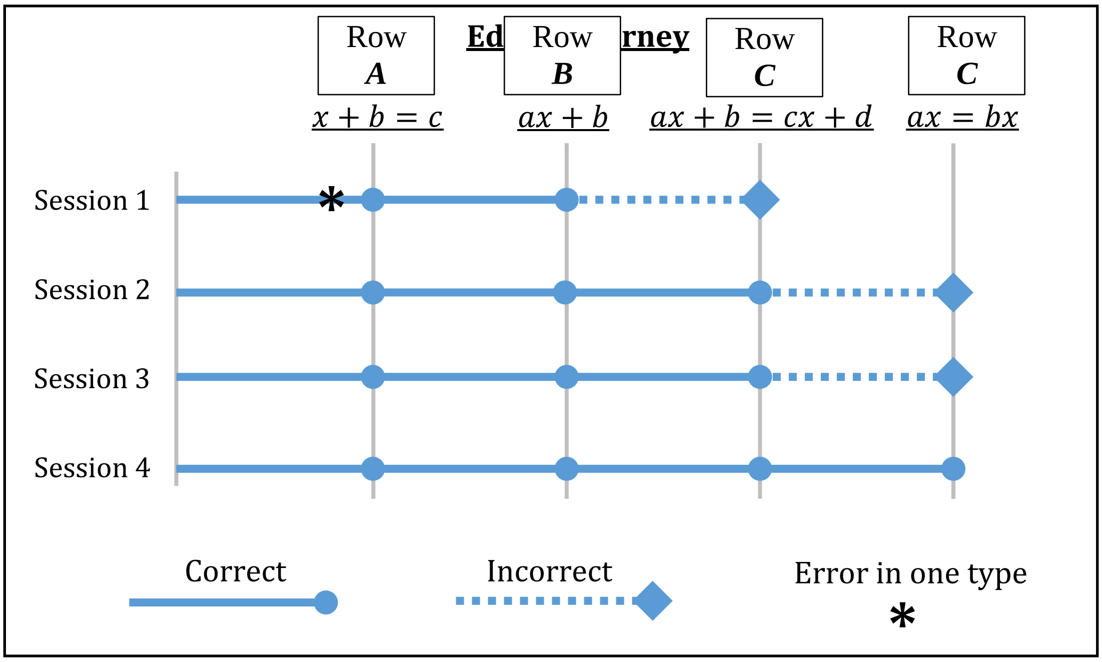


Figure 4.5: An example of a progress diagram for a learner

This diagram illustrates that a learner could be generally procedurally fluent with equations of one type, i.e. $x + b = c$, but that there are still specific difficulties that are experienced *within* this type (by the * symbol in the diagram). One of these difficulties included the three learners who had difficulties with negativity, particularly with an item from row A like $5 - x = 8$ where x had a coefficient of -1. These difficulties are analysed in chapter 7.

These "errors within type" represented by the *, are linked to the code of (W) in the interview progress table. If learners got the correct answer after multiple attempts (C- MA), this would not reflect as a * in their progress diagrams as they got the answer correct. However, if learners could not do any of the items within the row, i.e. Row C in Figure 4.5, this was represent as the dashed line as "Incorrect".

Other difficulties lay in the specific type of item, for example items which had letters on both sides (the didactic cut). The progress diagrams revealed three learners who encountered this type of difficulty. The progress diagrams also revealed that another three learners encountered difficulties with items of the type $ax = bx$. Mirriam (2002) asserts that the findings in a multiple case study should take the form of "*organized descriptive accounts*, themes, or *categories* that cut across the data" (p. 176, emphasis mine). Drawing on this framework, the progress diagrams and common errors that were noted allowed for the extraction of general categories to be discussed further.

These categories were: "the didactic cut", "difficulties with negativity" and the special case of "$ax = bx$". Each of these categories have a detailed discursive chapter dedicated to describing the learners' difficulties and how these changed over the four sessions. Furthermore, thanks to detailed transcripts, the relationship between what the learners wrote and what they were thinking can be analysed in detail.

4.8 Difficulties with analysis

With a small group of participants, it was not appropriate to look at every type of error as a code. Drawing on more inspiration from case study research design, I chose to rather provide detailed and rich accounts of the common difficulties and errors *across* learners, rather than telling the story of each individual learner.

The most significant difficulty I faced was whether or not to include the item $ax = bx$ as a separate category from $ax + b = cx + d$ in the progress diagrams. It became clear in my in my initial analysis of the interviews that learners were experiencing qualitatively different types of difficulty with this special case ($ax = bx$) than they were with the general category of letters on both sides ($ax + b = cx + d$). However, this $ax = bx$ form is an extreme case in linear equations – it is not normally included in textbook exercises and assessments that learners would have encountered. Sanders (2017) notes however that this form of item can appear if learners were to add unlike terms on either side of the equation, that is $7x + 3 = 5x - 2$, which would simplify to $8x = 3x$ and thus is of the form $ax = bx$. Hence learners may end up with this type of equation even if it is not commonly given to them.

However, I then chose to not include the difficulty with negativity as its own category in the progress diagrams despite dedicating a chapter to its analysis (chapter 7). The difficulties with negativity were not limited to a specific form of item, although most of the difficulties learners experienced were with a negative coefficient to the letter, and in particular, a coefficient of -1, i.e. $5 - x = 8$. These difficulties spanned two of the existing categories, $x + b = c$ and $ax + b = c$, and thus I could not extract these difficulties as a unique 'type' within the progress diagrams.

4.9 Rigour

Merriam (1998) notes that the qualitative paradigm, and in particular, post-positivist perspectives (in my case critical realism) have moved away from the measures of validity and reliability to the notion of the 'trustworthiness' of the research. Lincoln and Guba (1985) suggest that the concepts of: credibility, transferability, dependability and confirmability should replace the terms: internal validity, external validity, reliability and objectivity respectfully. In the following section I shall discuss these issues of trustworthiness in relation to my study.

The first aspect of trustworthiness is credibility, which is similar to the notion of internal validity, and describes how likely the findings are to be valid representations of the phenomena. Lincoln and Guba (1985) and Merriam (1998) assert the need for prolonged engagement and observation of the subjects. Drawing on the design of a qualitative case study, it is necessary to examine the situations deeply and over a significant period of time. My study has a sufficient amount of time allocated to it, and I am examining the aspect of equations in significant depth. I have sampled six learners, four of whom were interviewed four times at regular intervals and the other two had three interviews. These interviews covered a time span of six months which lends credibility to the findings due to prolonged engagement.

Another significant facet of credibility noted by both authors is that of triangulation. This means using multiple sources to evaluate judgments. My design focusses on the relationship between the learners' written work and their vocalisation of their thinking through interviews. This triangulation can also include asking other researchers to examine my data and come to their own conclusions (which Merriam (1998) refers to as peer examination). My supervisor has worked closely with me throughout the data analysis periods and challenged my thinking in order to refine the analysis.

Lincoln and Guba (1985) ascribe the term transferability to the idea of external validity, but argue that the notion of general applicability or of replicable results in other context do not fit with context bound case studies. Merriam argues that in

> … qualitative research, a single case or small non-random sample is selected precisely because the researcher wishes to understand the particular in depth, not to find out what is generally true of many. (1998, p. 208).

However, the notion of transferability can have some relevance if rich, 'thick' descriptions are used. If the contexts are sufficiently similar, then some transfer may be possible. The term *thick descriptions* is explained by Geertz (1973), particularly pertaining to ethnographic research of complex social phenomena. Thus thick descriptions must take into account the complex nature of the phenomena, and not diminish these complex situations into overly simplified ones. Phenomena must be examined from multiple points of view.

In my context, thick descriptions entail detailed descriptions of both the learners' work and the learners' talk about their work. Kvale (2003) argues that the researcher must present the interpretations of the interviews in such a manner as to ensure that the reader can follow, and critically evaluate the claims. This entails that the data is presented in as raw a form as possible and linked to the conceptual framework in clear and open ways.

Reliability is problematic in the social sciences because of the complex and non-static nature of human behaviour and thought (Merriam, 1998). However, the notion of dependability, along with confirmability can be used to evaluate the reliability of the data collection and analysis. Lincoln and Guba (1985), and Merriam (1998) both put forward the notion of an inquiry audit. This is based on a financial audit, whereby the records and decisions made by a business are confirmed to follow acceptable norms. In the same way, the research must explain in detail the data collection process, the formulation and use of categories used and the basis for all decisions made. In this way the research process is transparent and could be considered dependable and confirmable if this audit process is followed. Other research can also be involved in this process, much in the same way as the previously mentioned concept of triangulation.

4.10 Ethics

Ethics clearance was obtained from the University of the Witwatersrand and from the Gauteng Department of Education in April 2017. These clearance letters are attached as Appendices C and D respectively.

The study consists of six participants. The identities of the learners were protected at all times, with pseudonyms used throughout the study. The school's name has also been protected, as was the district in which it is situated.

Ethical consent was obtained from the parents of all the learners in the original group that wrote the WMCS test. Further ethical consent was obtained from the parents of the sample of six learners. The ethics letters for the individual learners and their parents are attached as Appendix E. The ethics letters mentioned the need to video record the work of the learners but that their faces would not appear in the footage. Each learner was informed that I would video record the interview, but that only I and my supervisor would watch the recordings and that their faces would not appear in the footage. I informed the learners that the purpose of the recording was to make sure I was clear on exactly what they had said. I explained the interview questions I would ask, and that I was interested in understanding their thinking.

The interests of the child were paramount. As was noted in the interview discussion, I am in a privileged position as a more knowledgeable adult figure. Although the study involved interviews, the design of the study was not for me to teach them equations, but to observe their progress. And while during the interview, I probed their thinking, prompted them with counter-examples and pushed the boundaries of their knowledge, I purposefully would not tell them when they were correct or not – either in their solutions to equations, nor their explanations. This may leave them anxious and distressed at 'not knowing'. I told the learners however that I would keep their teachers informed of their progress and leave it up to the teacher to intervene should they deem fit. After the final session, I did assist some of learners with questions that they had.

The raw data, including the learners' written work, audio and video recordings, was stored on a password protected computer and their physical work in a lockable cupboard.

4.11 Limitations of the study from its design and the scope of the analysis

In the description of the sampling process, I described purposeful sampling for learners who showed evidence of struggling with the didactic cut. Thus, it cannot be surprising for the data to show that learners were *still* struggling with the didactic cut during the period of the study. However, the strength of this research design lies in the periodic interviews, which provide regular access to the learners' progress in terms of the didactic cut, in both their abilities and their reasoning.

The scope of the data analysis was also limited by the design of the interview task-matrix. If learners made an error with C3 (the middle column), the design of task-matrix necessitated moving to the left (i.e. C2) to an 'easier' question. Thus I cannot actually claim that this particular learner could not do items C4, C5 etc. if they never encountered them. My experience as a secondary teacher and the literature on known algebraic and integer errors informed this horizontal dimension. But it was never trialled to see whether learners *could* answer questions to the right *if* they had made an error.

Finally, the scope of the project meant that some items included in the matrix, and encountered by learners, were never analysed. These items featured to the right of row D and focussed on the nature of solution. An example from session 1 was (D6) $2x + 1 = 3x - x$, which would have 'no solution'. From session 2, the item (D6) changed to $12(x - 1) = 6(2x - 2)$, which would be true for all real values of x. These items do appear in the *interview progress* tables of some learners (notably Sarah) but were not analysed further.

4.12 Conclusion

In this chapter, I have discussed the design of my research. While drawing on inspiration from multiple case study design, the study is a basic qualitative *interpretive* study. I described purposefully sampling six learners who showed ability to solve basic arithmetic equations but did not solve equations with letters on both sides in the WMCS test. These learners were interviewed 4 times each over a period of six months using a task-matrix. The task matrix was designed in line with the 3 levels of difficulty for equations derived from literature. Row A had a single operation ($x + b = c$), row B had two operations ($ax + b = c$) and row C had letters on both sides. Row D was informed by experience as a teacher, that learners can find larger coefficients on the RHS more difficult.

I also presented the process I followed in analysing the data. I developed two representations of the data to aid my analysis. The first was termed the *interview progress table,* and provided detail on what the learners could, and could not do, with the items from the task-matrix.

The other representation of the data was the *progress diagram*. This diagram provides of broader summary of learner progress in regards to 4 types of equation items. These types of item mapped onto the first 3 rows (A to C) of the task matrix, thus describing learners' general progress through the matrix. However, a fourth type of item was introduced into these diagrams, the type $ax = bx$, The learners' difficulties with this item became apparent in the small reflections I performed between interview sessions. In the following chapters, it will become apparent how learners treated this item differently from $ax + b = cx + d$.

The next chapter will present the *interview progress table* and *progress diagram* for each of the six learners. These diagrams will be used to provide a general analysis of each learner's progress throughout the six month time period. From these general analyses, the key, common difficulties which learners experienced are drawn out. These are *negotiating the didactic cut, difficulties with negativity* and *the special type $ax = bx$*. Each of these difficulties has a chapter's worth of discussion devoted to understanding why these learners experienced the difficulties they did, and how (or if) they overcame these difficulties.

Chapter 5 : The progress of learners over time

5.1 Introduction

In this chapter, I present the within-case data analysis for all six learners. Each learner was interviewed based on their responses to items provided from the task matrix described in chapter 4. A detailed description of how the data analysis was performed was provided in the previous chapter. There are two illustrations of each learner's data provided: the progress diagram and interview progress table. The first gives a broad picture of the learner's progress throughout the four sessions. The latter provides a detailed analysis of what the learner could and could not do in each session, as well as the order of items that the learner faced.

The order of the learners discussed here generally mirrors the order in which they are discussed in the following chapters which analyse the common difficulties in significantly more detail. The last two learners, Sarah and Yusuf are not discussed in other chapters and only a brief account will be given of their journey, although the discussion will make clear why these learners did not feature more in the data analysis.

5.2 Edith

Edith's *progress diagram* is provided in Figure 5.1. The diagram shows her overall progress through four sessions, though her difficulties with $ax = bx$ remained through session 2. The progress diagram shows her initial difficulties in session 1 with equations with letters on both sides (level 3) before overcoming these in session 2. The (*) symbol indicates she had difficulty with one item of the type $x + b = c$. This difficulty will be highlighted in the interview progress table which follows this discussion.

In sessions 2 and 3, she was unable to solve the item type $ax = bx$. However, in session 4, she overcame these difficulties with this type of item. Again, a full discussion of her difficulties with this type of item is provided in chapter 8.

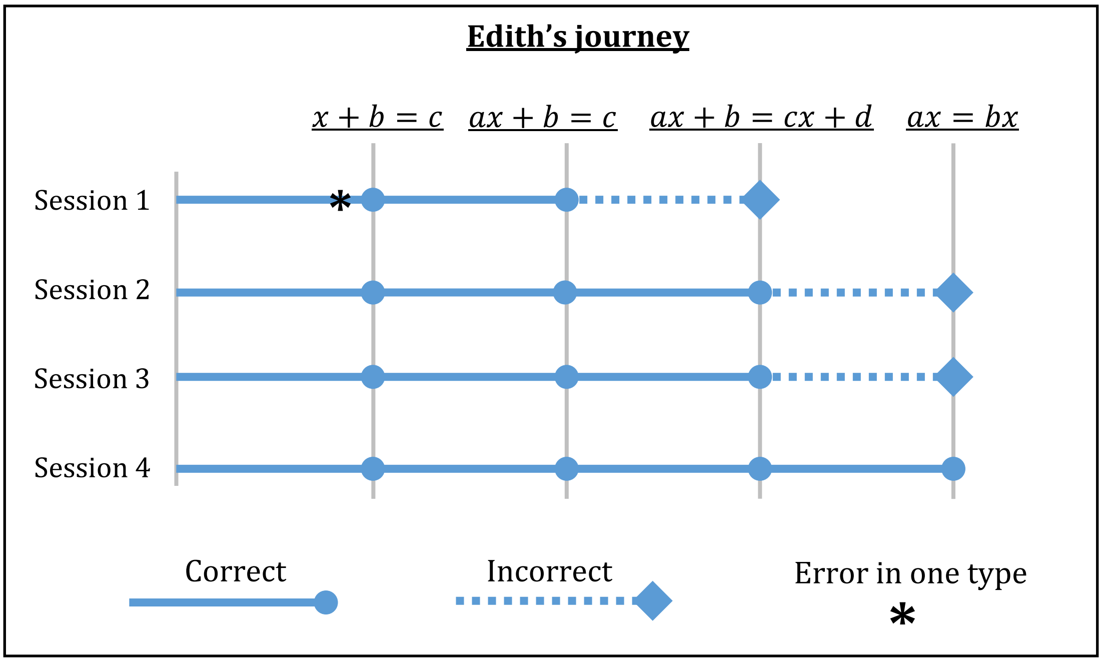


Figure 5.1: Edith's progress diagram

The following *interview progress table* illustrates Edith's journey through items in each interview. In section 4.7.3, the design of this table was explained. But in summary, each row (A to C) corresponds to a different cognitive level of question. Row D had equations with letters on both sides, but where the coefficient of the letter on the RHS was a larger number than the coefficient of x on the LHS. The numbers attached to the letter correspond with the variation within a row, i.e. C3 is the middle item in row C. In general, when moving to a new row, I would start in column 3. The codes were described in Table 4.3, but at a reminder, C is correct, $C - MA$ is correct after multiple attempts, and W is incorrect.

Session 1		*Session 2*		*Session 3*		*Session 4*	
Item	**Code**	**Item**	**Code**	**Item**	**Code**	**Item**	**Code**
A3	C	A3	C	A3	C	A3	C
A5	W	A4	C	B3	C	A5	C-MA
B3	C	B3	C	B4	C-MA	B3	C
B5	C-MA	B4	C	A6	C-MA	B4	C
C0	C-MA	C3	C	C3	C	C3	C
C3	W	C4	C	C5	W	C4	C
D3	W	C5	W	D3	C	C5	C-MA
		D3	C			B5	C

Table 5.1: Edith's interview progress table

A detailed analysis the major difficulties that Edith experienced is provided in the dedicated following chapters (6, 7 and 8). However, I will provide a brief indication of the errors and difficulties Edith faced when moving between the sessions. The (*) in session 1 from Edith's progress diagram (Figure 5.1) was her incorrect response to (A5) $5 - x = 8$ where she left her answer as $-x = 3$. In session 1 she made multiple attempts at (B5) $5 - 3x = -10$ and (C0) $\square + \square = \square + 3$ before arriving at the correct answers for both. For (B5), her initial error was taking all terms the LHS and not being left with 'anything' on the RHS. She then made an arithmetic integer error on her second attempt, which she corrected when using her calculator.

Item (C0) $\square + \square = \square + 3$ was a non-standard item and given to her earlier (before C3) to ascertain her understanding of the meaning of solution as she was having difficulty checking her solution to equations (she understood checking to mean re-doing the item instead of substituting the value of x back into the original equation). Her initial error was to substitute different numbers into the boxes, i.e. $\mathbf{3} + \mathbf{5} = \mathbf{5} + 3$. When I told her that the numbers should be the same, she gave the correct answer of $\mathbf{3} + \mathbf{3} = \mathbf{3} + 3$**.**

In session 2, she progressed smoothly through the task matrix until she encountered (C5) $3x = 4x$, which was coded (W). Again, there is a dedicated chapter to this type of item (chapter 8), however her difficulties with this item remained in session 3. In session 3, she needed multiple attempts (C-MA) for (B4) $12 - 5x = -18$ and (A6) $30 = 4 - x$. In both cases her initial error was using the incorrect inverse operation to move the constant from one side to the other. In (B4) she added 12 to -18 to get -6 on the RHS. This error is what Kieran (1992) refers to as 'switching addends'. However, in both cases, she picked up her own error when checking her working. I thus think that both errors could be classified as slips due to her picking up on her own error.

In session 4 she needed a second attempt (C-MA) at (A5) $17 - x = 28$ where she only divided the LHS by -1. Again a full discussion of her difficulties with this type of item are discussed in chapter 7. But she did correct her error in her second attempt. She finally solved (C5) $5x = 2x$ on her second attempt, but this was through a lengthy engagement with me which is covered in chapter 8.

5.3 Tebogo

Tebogo's progress diagram is provided in Figure 5.2. The progress diagram shows that Tebogo did show progress, and a similar progression through levels as Edith. In session 1, Tebogo was unable to solve equations with letters on both sides (level 3). In session 2, she was able to solve this third level, although still making an error which will be addressed later in this section. She did not encounter the fourth level ($ax = bx$) in session 2 due to time constraints, but when she did face this type in session 3, she was unable to solve it. Finally in session 4 she was able to solve this type. The 5 (*) symbols indicates she made a number of errors within a type throughout the four sessions. These particular errors will be addressed in the following section when discussing her interview progress table.

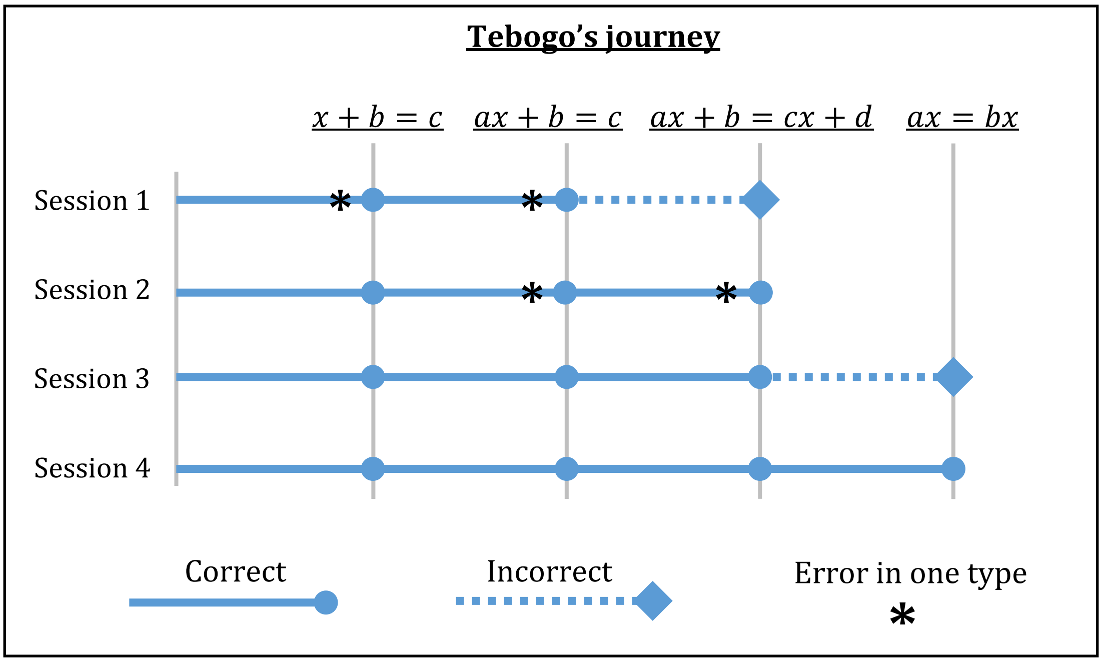


Figure 5.2: Tebogo's progress diagram

The following interview progress table illustrates Tebogo's journey through her four interviews.

Session 1		*Session 2*		*Session 3*		*Session 4*	
Item	**Code**	**Item**	**Code**	**Item**	**Code**	**Item**	**Code**
A3	C	A3	C	A3	C	A3	C
A4	C	A4	C	A5	C	A4	C
A5	W	B3	C	B3	C	A5	C
B3	C	B4	C	B4	C	B3	C
B4	C	B5	C*	C3	C	B5	C
B5	W	C3	W	C4	C	C3	C-MA
C3	W	C1	C-MA	D3	C	C1	C
				C5	W	C5	C
				C6	W		

Table 5.2: Tebogo's interview progress table

In session 1, the (*) in her progress diagram (Figure 5.2) for $x + b = c$ refers to her error with (A5) $5 - x = 8$ which she could not finish. She was attempting to solve the equation by inspection and with her calculator, and after trying $x = -13$ to get -18 she said she was stuck. The second (*) in session 1 was for item (B5) $5 - 3x = -10$, which she got the incorrect answer of -5 because of not dividing the negative 3 coefficient by negative 3. Her difficulties with negativity are discussed in chapter 7. However she could answer other items on these two levels (rows A and B) so long as the letter did not have a negative coefficient. In session 1, she was unable to correctly solve the third level - equations with letters on both sides.

In session 2, she left her answer to (B3) $20 - 2x = -15$ as $= \frac{-35}{-2}$, which is not mathematically wrong, but is not simplified and thus had C* in the table. She did manage to solve equations with letters on both sides (on her second attempt) with (C1) $4x - 2x = 10$. This item was at the low end on row C and designed to see whether she could operate with the variable. In her working, she initially added $2x$ to both sides before subtracting it again (in Figure 5.3), showing that she could operate with letters and thus was classified as getting it correct. I did not have time to return to C3 (W), but based on her working to C1 (Figure 5.3), I determined that she was able to solve equations with letters on both sides in session 2, and thus of her progress diagram (Figure 5.2) indicates this. A full discussion of Tebogo's difficulties with the didactic cut is in chapter 6.

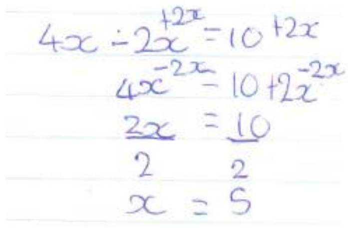


Figure 5.3: Tebogo's working for item C1 in session 2

In session 3, her only notable difficulty (in terms of levels) was with the item (C5) $5x = 2x$, which again is addressed in its own chapter. Her other minor difficulty was with item (C6) $-3(x - 2) = 6 - 3x$. She did not finish this item. This item was originally included to investigate the learners understanding of "infinite solutions" to equations, but this aspect of linear equations was too far from the scope of the original research design (in section 4.11) and thus won't be discussed further.

Finally, in session 4, Tebogo again experienced difficulties, needing multiple attempts (C-MA) to solve equations with letters on both sides (C3). After successfully completing (C1), she returned to (C3) and was able to correctly solve it.

5.4 Daniele

Daniele's progress diagram in Figure 5.4 actually shows a notable lack of progress. The story throughout the four sessions is that she can generally solve equations with a single letter but did not overcome these difficulties in this study to solve equations with letters on both sides, i.e. the didactic cut.

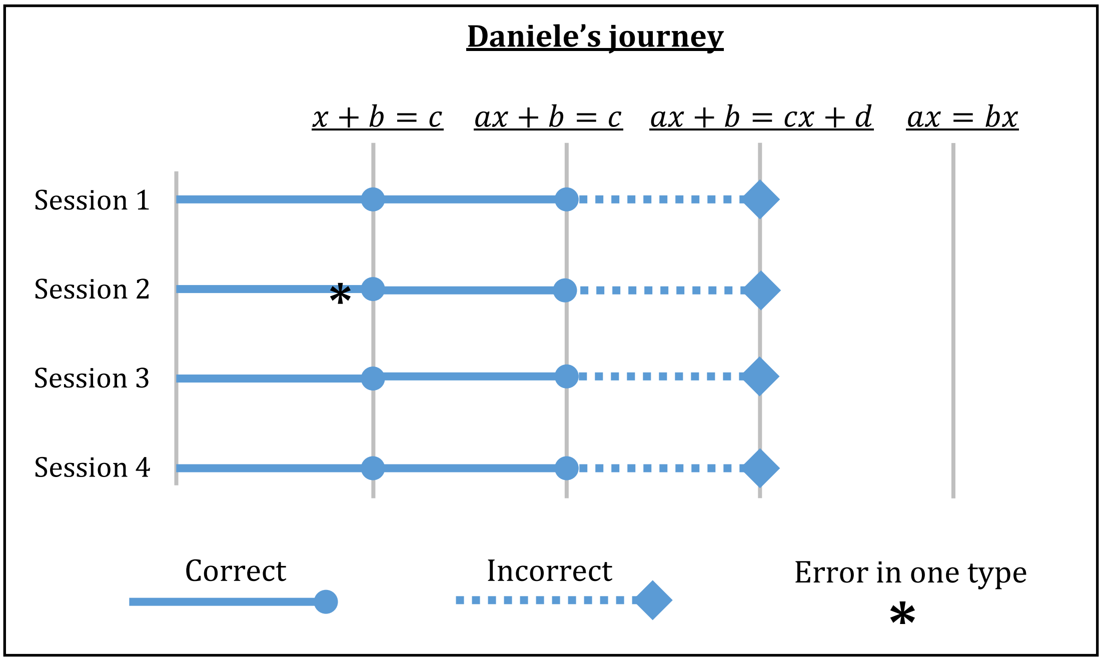


Figure 5.4: Daniele's progress diagram

Daniele's interview task progress table below tells the same story of general success in rows A and B but unable to solve any equations in row C.

Session 1		*Session 2*		*Session 3*		*Session 4*	
Item	**Code**	**Item**	**Code**	**Item**	**Code**	**Item**	**Code**
A3	C	A3	C	A3	C	A3	C
A4	C	A4	W	A4	C	A5	C-MA
A5	C	B3	C	B3	C-MA	B3	C
A6	C	B5	C	B4	C	B4	C
B3	C	C3	W	A5	C-MA	C3	W
B5	C	C1	C	C3	W		
C3	W	C2	W				
C2	W						
C1	W						

Table 5.3: Daniele's interview progress table

In session 1, Daniele appeared to have adopted an inspection approach for all the items in rows A and B as she wrote down answers only (i.e. just $x = \ldots$ with no working). This appeared to be successful for her, and with the help of her calculator in verifying her solutions, she got all these items correct. She did not solve either of the equations with letters on both sides (C3 and C2). The details of these difficulties will be discussed in the didactic cut chapter (chapter 6).

In session 2, she had adopted an algebraic approach to solving equations and this served her well apart from (A4) $10 - x = 18$ where she gave an answer of $x = 8$ and thus was coded

(W). In session 2, she was still experiencing difficulties with letters on both sides (row C). It is notable that she got (C1) $4x - 2x = 10$ correct, but as was discussed earlier with Tebogo, this was a special item with both letters on the same side, designed to see if she could operate with letters (i.e. add like terms). Despite being able to do this, she did not manage to solve item (C2) $4x = 3x + 4$ directly after.

In sessions 3 and 4, she experienced the same difficulties (as she had previously) with negativity (A5) $17 - x = 28$ and the didactic cut (C3). Item A5 was coded as C-MA (meaning correct after multiple attempts) as we returned to this item later in the session and she was able to correctly solve it then. Again, the full description of this process is given in chapter 7 on 'negativity'.

5.5 Guban

Guban is the last learner to feature in the following in-depth analysis chapters. His progress diagram in Figure 5.5 shows a generally strong performance, with only items in the form $ax = bx$ causing him difficulty. It is also notable that he was unable to attend session 3, and hence no data is provided for that session.

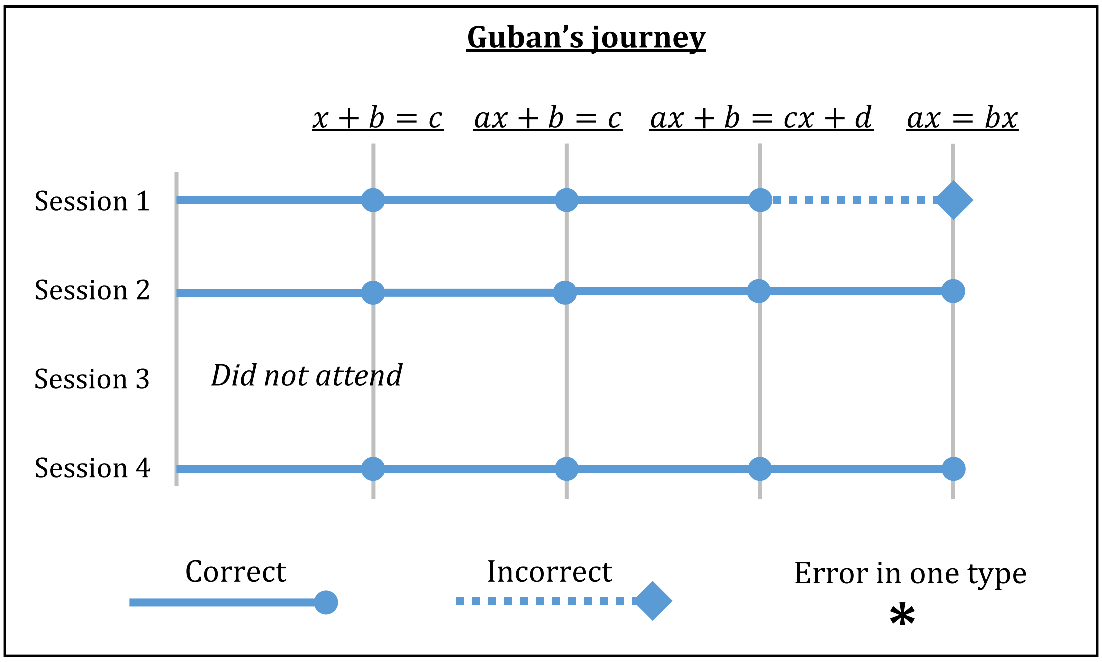


Figure 5.5: Guban's progress diagram

Guban's interview progress table does provide some additional detail on his other difficulties.

Session 1		Session 2		Session 3	Session 4	
Item	**Code**	**Item**	**Code**		**Item**	**Code**
A3	C	A3	C	Absent from interview	A3	C
A5	C	A4	C		A4	C
A6	C	B3	C		A5	C
B3	C	B4	C		B3	C
B6	C	B6	C		B5	C
C3	C	C3	C		C3	C
C4	C	C4	C		C5	C-MA
C6	C	C5	C*		C6	C
C5	W	C6	W		D3	C
D5	W				D5	C
					D6	W

Table 5.4: Guban's interview progress table

Item C5 (of the type $ax = bx$) throughout all 3 sessions caused Guban some difficulty. This item is discussed in detail in chapter 8. The C* in session 2 for C5 indicates an error in his working but he still got the correct answer. In session 1, he also made an error with item (D5) $2(5 - 2x) = 10 - 3x$ where he subtracted 2 from both sides first. This is what Hall (2002) referred to as using the 'other inverse', i.e. subtraction instead of division. This was an error in one equation and the other equations with letters on both sides had been solved correctly, so I am inclined to think of it as a slip rather than a persistent error. However the study of the effects of brackets (parentheses) was outside of the scope of this analysis.

In session 2, item (C6) $2(x + 1) = 2x - 2$ has 'no solution' and again, I later decided that this aspect of linear equations was outside the scope of this research. He was troubled by the item (long pauses, over 10 seconds) and eventually stopped working on it. Another item outside the scope of the research was item (C6) $-3(x - 2) = 6 - 3x$ where x can be any real number. His eventual answer was $x = x$, and he explained that "x could be any number", showing some understanding of an infinite set of solutions.

5.6 Sarah

Sarah did not experience difficulties as the other learners discussed thus far. Her progress diagram in Figure 5.6 shows that she did not have difficulties with negativity, i.e. $5 - x = 10$, nor the didactic cut, nor the special case of $ax = bx$. The items with (*) will be addressed in regards to her interview progress table.

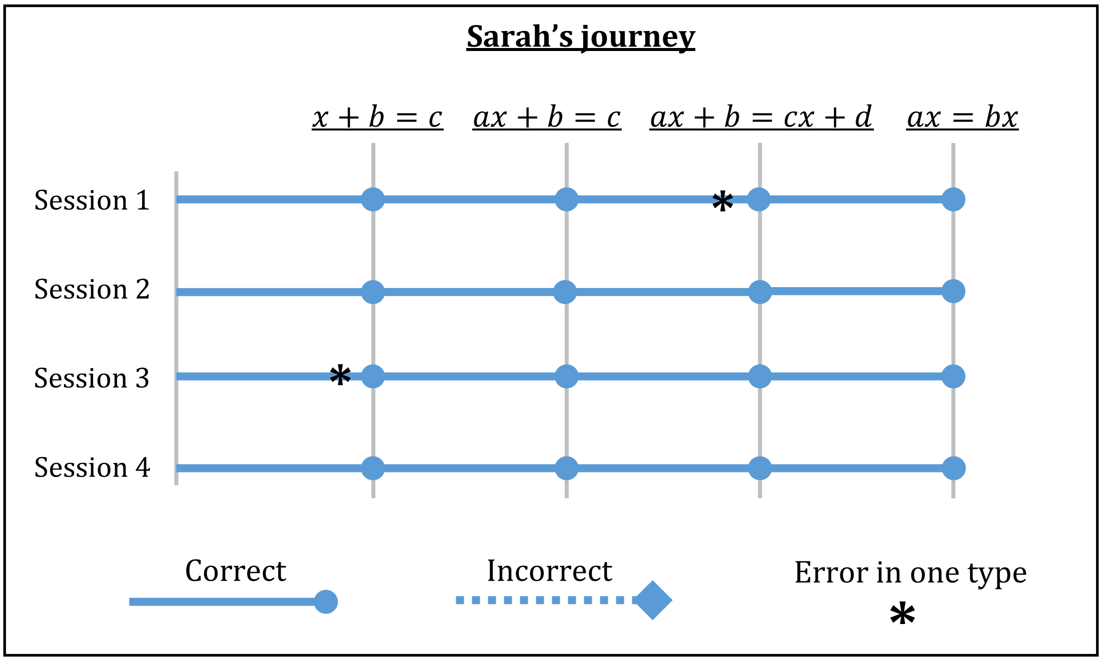


Figure 5.6: Sarah's progress diagram

Sarah's interview progress table shows her strong performance across rows A to D.

Session 1		*Session 2*		*Session 3*		*Session 4*	
Item	**Code**	**Item**	**Code**	**Item**	**Code**	**Item**	**Code**
A3	C	A3	C	A3	C	B3	C
A5	C	A4	C	A4	W	C3	C
B3	C	B3	C	B3	C	C5	C
B5	C	B4	C	B5	C-MA	B5	C
B6	C	B6	C	C4	C	C6	C
C3	W	C3	C	C5	C	D6	C
C2	C	C5	C	C6	W		
C4	C	C6	W	D3	C		
C6	C	D3	C	D6	W		
C5	C	D6	W	E3	W		
D5	W			E4	W		
E3	C			E5	C		
				E6	C		

Table 5.5: Sarah's interview progress table

Although there are errors that Sarah made throughout the four sessions, these errors were not persistent. In session 1, with (C3) $2x + 5 = x + 1$, she wrote $-x$ on the RHS but then added x to the LHS to get $3x + 5 = 1$. This error is what Kieran (1992) refers to as 'redistribution'. Her incorrect answer for this item led me to give her item (C2) $2x + x = 6$, which she solved correctly. She then did not make that error again, hence I think that it as a slip. With item (D5)

$2(5 - 2x) = 10 - 3x$, she wrote $+3x$ on the RHS, but then subtracted $3x$ on the LHS to get $10 - 7x = 10$, again showing the redistribution error. This may suggest the error is persistent but then she didn't make this error with the other items with letters on both sides.

In session 2, she had difficulty with (C6) $2(x + 1) = 2x - 2$, which has no solution. As has been discussed, this item is outside the scope of this research. However, it can be noted that in session 3, she still could not fully explain why (D3) $12(x - 1) = 6(2x + 5)$ had no solution. Her last line in this item was $0 = 42$ but couldn't explain verbally what this meant. However in session 4 with (D6) $12(x - 1) = 6(2x + 5)$, she was able to write "no solution" at the end and explain that there 'was no answer for x'.

In session 3, she needed multiple attempts to get item (B5) $\frac{x+5}{2} = -3$ correct. This item introduced fractions, which again I later determined to be outside the scope of this research. Her first error was to multiply the numerator on the LHS also by 2, i.e. $2x + 10 = -6$. When she checked her solution of -8, she realised her error and correctly only multiplied the RHS by 2.

Items (E3) $x(x + 1) = x^2 + 3$ and (E4) $x(x - 1) = 0$ were also later determined to be outside the scope of this study. While item (E3) is linear, Sarah made an error in distribution on the LHS so that she got $2x + x = x^2 + 3$ which left her with a quadratic equation, which she was unable to solve. And for item (E4), she had not yet been exposed to the zero product rule for quadratics, and thus for both of these items she did not have the mathematical tool available to her.

5.7 Yusuf

Like Daniele earlier, Yusuf also did not show progress throughout the 4 sessions. The progress diagram in Figure 5.7 shows this lack of progress, in particularly not successfully solving equations with letters on both sides during this study. Thus the didactic cut was present for him throughout the time of the study. Like Guban, he also was absent for the third session.

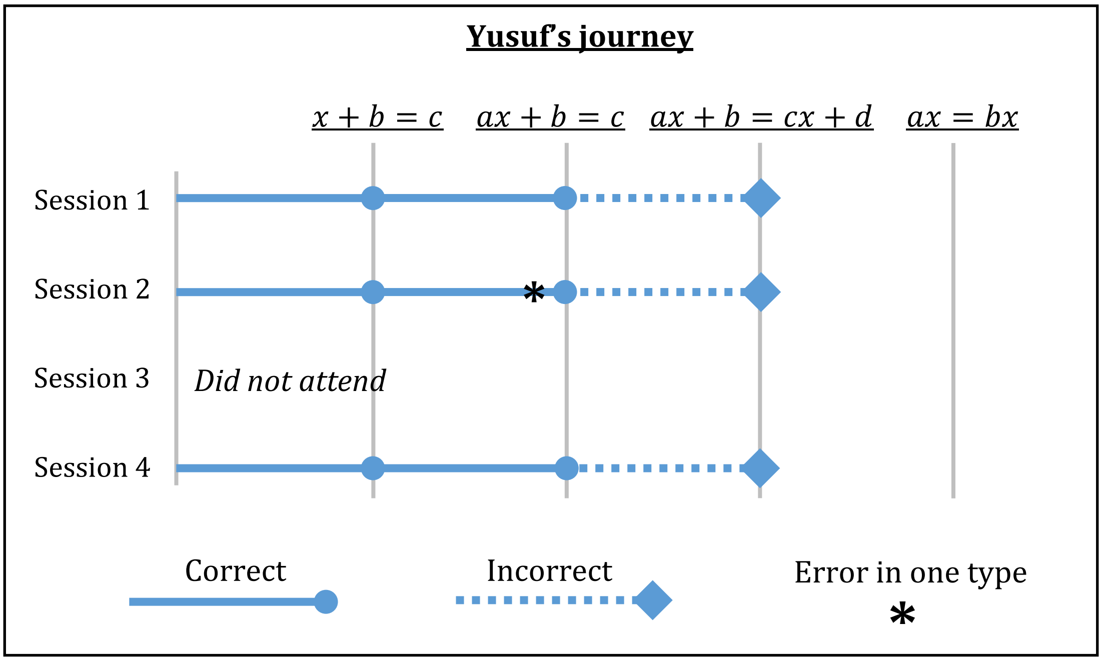


Figure 5.7: Yusuf's progress diagram

Yusuf's interview task progression table shows a remarkable consistency in what he could and could not do across the three sessions we had together – generally proficient with rows A and B, but struggling in row C.

Session 1		*Session 2*		*Session 3*	*Session 4*	
Item	**Code**	**Item**	**Code**		**Item**	**Code**
A3	C	A3	C	Absent from interview	A3	C
A5	W	A5	C		A4	C
B3	C	B3	C		A5	C
B4	C	B4	C		B3	C
B5	C	B5	W		B5	W
C3	W	C3	W		C3	W
C1	C	C2	W		C1	C
		C1	C			

Table 5.6: Yusuf's interview progress table

In session 1, Yusuf appeared to be using arithmetic inspection to solve equations. However, I had difficulty with Yusuf in the interviews as he was not forthcoming regarding explaining his thinking. He would often say "I can't explain, I just see it like this". For this reason, he does not feature in the subsequent chapters which provide detailed analyses of the common difficulties.

But I think I can gain some insight into his thinking from his working. In Figure 5.8, his working shows a strong arithmetic notion of ‘undoing’. He also appears to see the equals sign from an operational viewpoint – simply using it as “which gives me”.

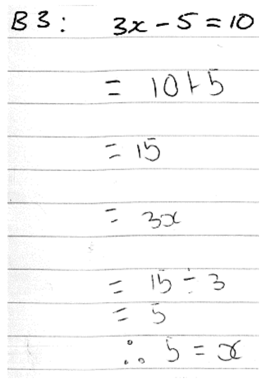


Figure 5.8: Yusuf’s working for item B3 in session 1

In Figure 5.8, he uses the inverse operation of -5 by adding 5 to 10 to get 15. Then he uses the inverse of multiplication (division) to undo $3x$ by dividing 15 by 3.

His first error in session 1 was for item (A5) $5 - x = 8$, where he gave the answer of $x = -13$ showing $5 - -13$ and saying “two negatives are a positive” and thus getting positive 8. However, he did not repeat this error in session 2 with (A5) $37 - x = -12$, correctly writing $x = 49$.

In session 1, Yusuf also struggled with (C3) $2x + 5 = x + 1$, not writing down an answer. He said he was “blank” (he couldn’t recall the method) and didn’t “know what to try”. However he was able to solve (C1) $7x - 4x = -3$ (with two letters, but both on the same side) first by subtracting the like terms and solving $3x = -3$. But this didn’t help him to see what to do with letters on both sides (C3).

In session 2, he was able to solve all ‘arithmetic’ items in rows A and B correctly, but again unable to solve (C3) $3x - 15 = x + 1$, and (C2) $4x = 3x + 4$, with letters on both sides. Again he was able to solve (C1) $4x - 2x = 10$, with two letters on the same side. This pattern continued in session 4 with no response to item (C3) $7x + 4 = 2x - 1$, but correctly answering (C1) $7x - 2x = 10$.

In session 4, he did make an error with (B5) $20 = 2(x + 10)$. He initially correctly wrote $x = 0$, but changed his mind and wrote $x = 5$, after I asked him to check his answer. He explained why he changed his answer: “if you minus 10 from 20, you get 10 and therefore

$x = 5$". He thus appears to be ignoring the brackets and seeing the item as $20 = 2x + 10$, but still showing a strong notion of arithmetically undoing by inverse operation. Thus Yusuf has held strongly onto his method of solving equations and not engaging with any algebraic methods to solve equations with letters on both sides.

5.8 The overall progression of the six learners

I have outlined the journeys of all six learners through the interview task matrix over the course of 4 sessions in this chapter. In Figure 5.9, all the learners have been combined onto a single progress diagram. The combined progression diagrams show that, for those who did progress, they appeared to follow a similar path. Firstly, 4 of the learners experienced difficulty with the didactic cut (level 3). After overcoming this difficulty, learners then experienced difficulty with the special case of $ax = b$.

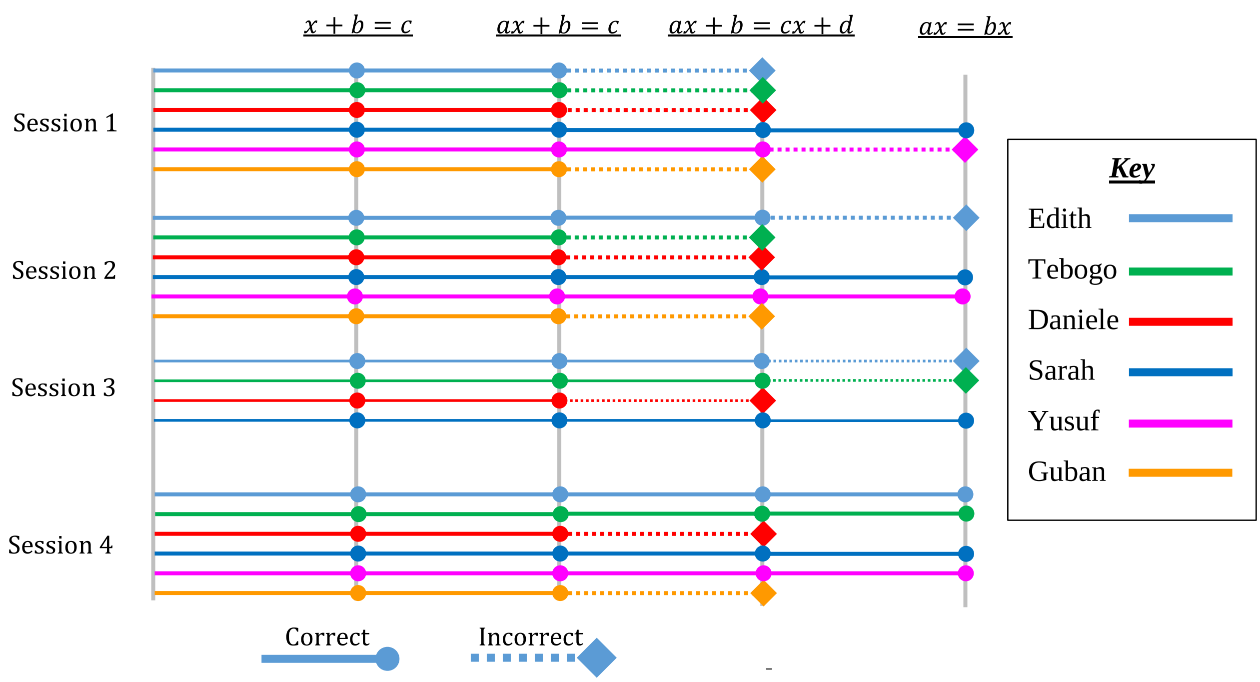


Figure 5.9: Overall progression of six learners over four sessions

These common difficulties form the basis on the subsequent in-depth data analysis. Of the four learners who experienced difficulties with the didactic cut, Edith, Tebogo and Daniele, are discussed in chapter 6. Yusuf is not discussed for the reasons mentioned in section 5.7 earlier. For each of the three learners, I analyse their explanations of what made equations with letters on both sides difficult and describe how Edith and Tebogo overcame their difficulties, whilst Daniele's persisted.

The learners who experienced difficulties with the special case of $ax = bx$ were Edith, Tebogo and Guban. Chapter 8 analyses how these learners saw this item differently to $ax + b = cx + d$ and how they overcame their difficulties. All three learners required prompting with another item from the matrix, (B6) $20 = 2(x + 10)$ to compare to $5x = 2x$ in overcoming their difficulties.

Although not shown on this diagram due to space constraints, a number of learners experienced difficulty with negativity, in particular with a negative letter (i.e. $-3x$ or $-x$). These difficulties were indicated by (*) on the learners' original progress diagrams. Tebogo, Edith and Daniele all experienced these difficulties. Tebogo's difficulties were more confined to having a negative coefficient to x, i.e. $5 - 3x = 11$, whereas Edith and Daniele's difficulties were confined to x not having an explicit coefficient, such as $5 - x = 12$. All three learners overcame these difficulties, i.e. made progress. Chapter 7 analyses how the learners made this progress and how they found these items more difficult.

Chapter 6 : Negotiating the didactic cut

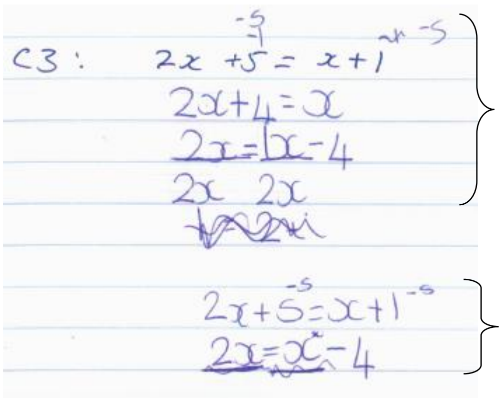
6.1 Introduction

A key focus of this research was describing the progress of learners and the difficulties they encountered. One of the foremost difficulties associated with linear equations was the transition from solving arithmetic equations to algebraic equations – the didactic cut. Indeed, the learners in this study were sampled on their difficulties with letters on both sides in the WMCS test. In this chapter, I discuss three learners in respect to the didactic cut. Firstly Edith, who showed initial difficulties, but overcame those difficulties by the second session. Tebogo's journey was more fraught, though she too overcame the cut to successfully solve equations with letters on both sides. And then finally Daniele, who did not successfully solve equations with letters on both sides during this study, but did show a change in her approach over time.

Vlassis (2002) and Lima and Healey (2010) have argued the approach the learner uses can make the cut less prominent, and thus for each learner, I describe their approaches, and how these changed over time in respect to the transition to letters on both sides of the equation. I also describe the errors learners made and how these changed over time.

6.2 Edith

In Edith's first session, she was able to solve the equations with a single letter (in rows A and B of the interview task matrix) but unable to correctly solve (C3) $2x + 5 = x + 1$ nor (D3) $2x + 5 = 3x - 2$. Figure 6.1 is her working for item C3.



1^{st} *attempt*

2^{nd} *attempt*

Figure 6.1: Edith's working for item C3 in session 1

In her first attempt at C1, she begun by subtracting 1 from both sides, leaving her with constants on the LHS and letters on both sides. At this point she paused. In the interaction below, she explains that it is not the x on the RHS that concerns her, rather the $+4$ on the LHS in line 2. She then explains how she would deal with the two letters.

UTTR	SPKR	TRANSCRIPT
103	AH	(after pause) okay, what are you thinking now?
104	EM	**I'm thinking of this 4 here…** $(2x + 4 = x)$**… I know what I'd do with these two** (the $2x$ and x)**, but not sure here** $(+4)$**.**
105	AH	Okay tell me what you'd do with these two here…
106	EM	***Cos* this is like multiplication because the 2 stands here** $(2x)$**, so I could <u>divide</u> both these two, and I'd get 2 equals 4, but we're trying to find the value of x.**
107	AH	mmm…
108	EM	That's what running through my mind, but then I want to try… another sum... If that feels right to me
109	AH	Do you want to carry on then?
110	AH	Why are you shaking your head? What are you thinking?
111	EM	I'm thinking this $\left(\frac{2x}{2x}\right)$ would work out but this $\left(\frac{x}{2x}\right)$ wouldn't work out
113	AH	Okay, so what would this $\left(\frac{2x}{2x}\right)$ become?
114	EM	That would be, that wouldn't be… (pause) … that would be 1… can I carry on…

She says she knows what to do with the $2x$ and x in utterance (104), explaining in (106) that the 2 means multiplication, and that she would divide by 2. This would point to an arithmetic notion of undoing (inverse operations). However, she has not divided all terms by 2 (the -4 on the RHS is left alone), which is problematic in terms of a relational view of the equals sign. Secondly, she also divides by x, which could point towards the notion of 'cancelling' (from exponents and fractions). In (111) she appears to think that she has 'undone' $2x$, but that $\frac{x}{2x}$ is still incorrect. In 114 she was about to explain herself, but then asked to reattempt the question. The second attempt is also shown in Figure 6.1, the only difference being her first step to subtract 5 from both sides leaving her with the same state of $2x = x - 4$. At this point she gets stuck saying "it doesn't represent x", i.e. her answer is not $x = \ldots$ Thus in both attempts she first deals with the constant, comfortably moving it to either side. However she is unable to operate on the letter in the same way (of adding or subtracting).

After this question, I presented her with another equation with letters on both sides (D3) $2x + 5 = 3x - 2$. Below is her working out for this equation, following a very similar approach as before – subtracting the constant on the LHS but then 'undoing' the $3x$ by dividing by $3x$.

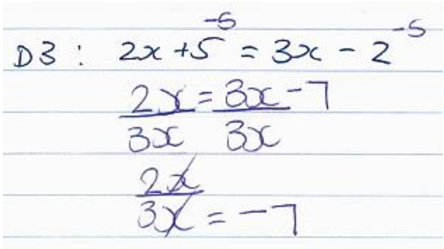


Figure 6.2: Edith's working for item D3 in session 1

Again she paused at this point. In the transcript below she expands on her thinking of inverse operations, particularly in utterance (138) below where she explicitly invokes the notion of inverse operation and 'crosses out' to indicate cancelling an x.

UTTR	SPKR	TRANSCRIPT
133	AH	Can you tell me what you're thinking now?
134	EM	I understand what's in front of the… I know it's meant to be one (pointing at x in the above question)
135	AH	So what would you do now?
136	EM	I always do the minus, so this (-2) will be plus 2, will give me 7 there (on the RHS). **Okay so I'm trying to get that** $(3x)$ **on that side** (LHS) **so divide by $3x$.** And then, it'll be $2x$ over $3x$ and negative 7
137	AH	Okay… (pause) what are you thinking?
138	EM	I'm thinking here $\left(\frac{2x}{3x}\right)$, I know we're (*inaudible*) that because **the inverse of times is divide, and whatever you do here** (LHS)**, you do that side** (RHS)**, and then that one** (x) **and that one** (x) **crosses out**.
139	AH	But you seem concerned?
140	EM	I am
141	AH	Explain why
142	EM	I know your answer has to be **like x equals something**

She also clarifies what her general approach is in (136). She states that she is attempting to collect variables on one side when she says "Okay so I'm trying to get that $(3x)$ on that side

(LHS) so divide by $3x$". Thus it appears as if she has an idea of the standard approach for collecting the variables on one side and the constants on the other. However, she also has a strong notion of inverse operation and thus sees the means to 'undo' the $3x$ on the RHS is by using its inverse operation, namely division. This however does not lead to her to what she expects the answer to look like, namely "I know your answer has to be like x equals something" (142).

Edith had a very strong notion of inverse operation which appears to have been causing her difficulties with items with letters on both sides. This inverse operations approach had served her well in solving equations of the form $ax + b = c$. The notable exception to this was the equation $5 - x = 8$, but these difficulties are addressed in chapter 7, regarding 'negativity'. Her working out (in Figure 6.3 below) is explicit in showing operations being performed on both sides of the equation. Indeed, in the sessions, she used the phrase "What you do to the right, you do the left" repeatedly. Figure 6.3 below show her correct working for item (B3) $3x - 5 = 10$. Her working shows her adding 5 in line 1, the inverse of subtracting 5. In her second line, she divides both sides by 3, utilizing the inverse of multiplication.

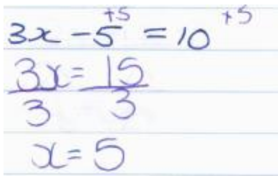
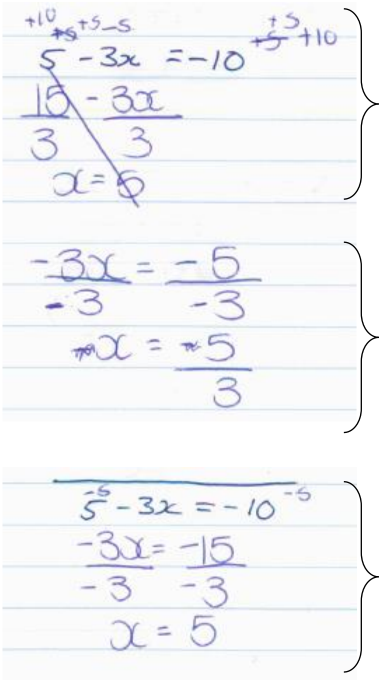


Figure 6.3: Edith's working for item B3 in session 1

She initially struggled with item (B5) $5 - 3x = -10$. She ended up using three attempts shown in Figure 6.4 below. Initially she added 10 to both sides but then stopped and explained "This would be wrong because they're ($15 - 3x$) both on the same side (LHS)". She asked to reattempt the equation. In this next attempt, she appeared to add 5 to both sides. This would not follow correct inverse operations as she should have subtracted 5 from both sides. In asking her where the $+5$ (in line 1) came from, she realised her error and correctly solved the equation at the third attempt by subtracting 5 from both sides and then dividing by negative 3.



1st attempt

2nd attempt

3rd attempt

Figure 6.4: Edith's working for item B5 in session 1

The following transcript shows her realisation of her error, and she then explains that the 5 is positive and therefore it should be negative 5 on both sides.

UTTR	SPKR	TRANSCRIPT
55	AH	Can you explain where the +5 came from?
56	EM	The **plus 5** came from… because this is… oh no…
57	AH	What are you thinking?
58	EM	**Because this** (5 on LHS) **is a positive, this** (the two mini -5 on both sides) **should be a negative.**

The terms 'positive' and 'negative' are not associated with inverse operations, as we may expect the terms 'adding' and 'subtracting', which are indicative of operations. They are more adjectival – describing the sign of the algebraic term, i.e. $-3x$ is *negative* $3x$. However, it does not necessarily mean that Edith used 'positive' and 'negative' in this way.

I discussed earlier in this chapter that Edith does show an understanding of the inverse of multiplication being division, where she points to the 2 in $2x$ and says "*Cos* this is like multiplication because the 2 stands here, so I could divide both these two". In items of the form $ax + b = c$, she is required to operate on the letter with addition or subtraction and the then divide by the coefficient of x (i.e. a). This method serves her generally well with these items, but she is then unable to adapt her method when letters are on both sides, still dividing by the coefficient of the letter instead of adding or subtracting the term with a letter.

As was discussed in the literature review, numerous authors (Knuth et al, 2006; Essien & Setati, 2006) noted the importance of a relational view of the equals sign in solving equations with letters on both sides. While Edith was not exposed to the diagnostic questions relating to her view of the equals sign, there are two reasons to suggest she has a relational view of the equals sign.

Her working out explicitly shows performing the same operation on both sides of the equals sign. The second reason is her language – that she used the phrase "what you do to the left, you do to the right". These aspects indicate that she sees the equals sign in a relational manner compared to an operational one. However, she was still unable to successfully solve either of the equations with letters on both sides. Thus I can conclude that in respect to Edith, a relational view of the equals sign was a necessary, but not sufficient condition for her ability to solve equations with letters on both sides.

In session 2, Edith was able to correctly solve equations with letters on both sides. Figure 6.5 below shows her correct solution to item (C3) $3x - 15 = x + 1$. Her working out is similar in respect to showing operating on both sides of the equals sign, but she is now able to correctly subtract x from both sides, leaving her with letters on one side ($2x$) and constants (16) on the other.

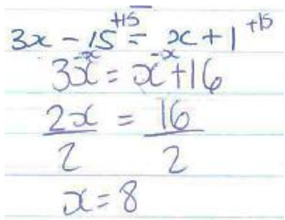


Figure 6.5: Edith's working for item C3 in session 2

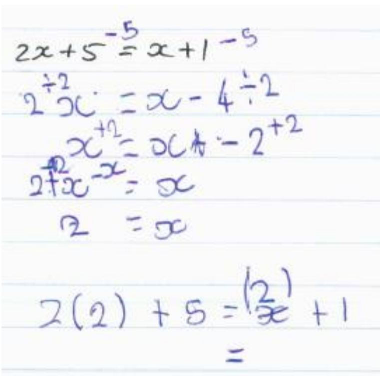
She was clearly feeling more confident with these types of equations, by correctly solving items C3 (above) and (C4) $4x = x + 1$. And so I asked her what had changed since our last interview. She explained that her Mom had been helping her and she had been practicing. She said "I'm good at, like doing equations. When it came to doing this last time, I forgot some stuff". In sessions 3 and 4, she did not experience difficulties with equations of the type $ax + b = cx + d$, apart from the specific case with no constants, i.e. $ax = bx$. These difficulties with $ax = bx$ will be discussed in chapter 8. Thus, her progress happened in-between sessions, and I cannot pinpoint exactly how she negotiated the cut. Her lack of 'recall' points towards the external location of this *tool,* which is discussed in chapter 9, as looking through a Vygotskian lens at this data. It is evident however, that Edith was experiencing an obstacle in the first session, and her inverse operations approach was fore-fronted.

6.3 Tebogo

Tebogo's progress diagram (earlier in section 5.3) showed that she successfully solved equations with letters on both sides in sessions 2, 3 and 4. However, this diagram belies the process she went through in the four sessions we had together. Tebogo is noteworthy for eventually overcoming the didactic cut, but only after exposure to examples with two letters on the same side. It is only after this 'reminder' of the notion of adding algebraic like terms, that she is able to succeed in solving these.

In the first session, Tebogo made very similar errors to Edith, inappropriately using inverse operations to try and remove a letter from one side. Due to a lack of time in the interview (this was the last item), she did not explain verbally what she was thinking in each step. But from her working, I provide the following analysis.

Figure 6.6 shows her working for (C3) $2x + 5 = x + 1$. She clearly shows subtracting 5 from both sides (line 1) but then attempts to divide both sides by 2 (line 2), to undo the multiplication within $2x$. Yet, she does not divide ALL terms by 2 (x is left alone) indicating the first signal that either she might not have a full relational view of the equals sign or that she doesn't see x and -4 as separate terms.



(1)
(2)
(3)
(4)
(5)
(6)

Figure 6.6: Tebogo's working for item C3 in session 1

In line 3, she applies her known procedure for dealing with constants by adding 2 to both sides. In line 4, she shows that she is able to operate with the variable in the sense of subtracting x, but does this to the LHS only, again indicating a problem with maintaining the equality of the two sides. This, however, helps her in getting a familiar answer of $x = \ldots$

Line 6 shows her attempt to check her working out, but she said she didn't think this answer was correct, as she struggled to check it on her calculator. She had been checking her answers to the previous items (which had only single letters) by substituting her value for x back into the equations. For this item she said "Ja, like usually I am used to checking, like the previous ones like, they were easy to check, but then these ones are like a little bit different." This shows that she may have been working with an operational view of the equals sign in the first session as she was used to pushing equals on her calculator and matching the 'answer' to the RHS of an equation.

In session 2, she continued to make similar errors involving inverse operations. In Figure 6.7, we can again see her attempting to undo $3x$ by dividing through by 3. At least by this second session, she is dividing all terms by 3, indicating that her thinking about the equals sign and/or individual terms has progressed from her error in the first session.

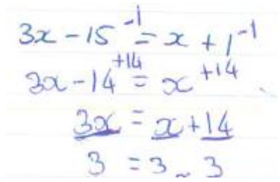


Figure 6.7: Tebogo's working for item C3 in session 2

In the transcript below, she explains that she wants to separate the 3 from x (76). But it also becomes clear in (80) that she isn't confident about this method and then by (86) realises that it's not productive.

UTTR	**SPKR**	**TRANSCRIPT**
76	TB	Well it's like, basically I wanted to, you know, to **separate the 3 from the what do you call it, $3x$**.
77	AH	Yes.
78	TB	**So I just, just, I divide by 3, so then like, I thought like that we need to divide everything by 3, so it's well ja.**
79	AH	Do you know why, why did you want to do that, because I know that you didn't divide by this (x) by 3 and you came back and did it. Why, why were you doing that?

80	TB	Well like I thought like maybe, maybe it will help the answer maybe a little bit. So ja.
81	AH	But now?
82	TB	Uhm, I am just ***trying to choose which one to cancel out.***
83	AH	Mmmm.
84	TB	Ja. I think I am ...[intervened]
85	AH	...[laughing]
86	TB	**Ja. It's going wrong somewhere.**

Using the design of the research instrument task matrix, I used an equation in the same row (C), meaning it had two letters, but they were on the same side of the equation: $4x - 2x = 10 + 2x$. Initially she struggled with the equation, which can be seen in her first attempt on the left of Figure 6.8. She didn't see the like terms but did operate on the letter by adding $2x$. But she did this only to the LHS. She then again reverted to trying to undo the $2x$ on the RHS by dividing through by 2. In the last line she appeared to try to undo the x by subtracting x but again from the RHS only. When I asked her to explain her thinking she said "Ja you see, I want to get rid of the x but I know you also have to do, like whatever you do on this side (LHS) you must do on that (RHS)." Despite saying that she needs to perform the same operation on both sides of the equation, she doesn't see the contradiction that she was actually operating one side only.

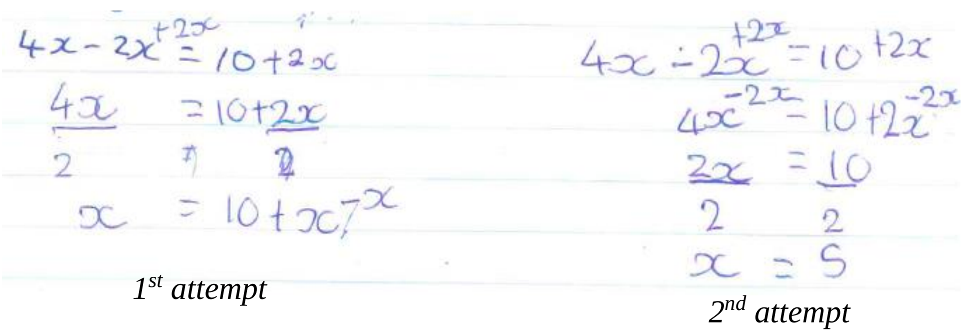


Figure 6.8: Tebogo's two attempts for item C1 in session 2

I asked her if she could 'simplify' anything on the LHS, actually writing "Simplify $4x - 2x$" for her. Her response was "Well since they are both like terms I think you can." She then exclaimed "Okay! Wait, Okay!". This prompted her second attempt on the right of Figure 6.8.

She didn't actually add the like terms on the LHS, but my 'simplification question' appeared to prompt her into recalling that she could operate on the letters (by adding like terms) – by adding $2x$ to both sides in line 1, and then subtracting $2x$ in the second line. This incident is discussed in chapter 9 in regards to a ZPD being created during the interview in which she was able to show progress in negotiating the didactic cut. This item was given at the end of our second session, so she didn't have time to return to try item C3 again.

In session 3, she showed no difficulties with the two equations which had variables on both sides. She successfully solved both items (C3) $5x + 4 = 2x + 1$; (C4) $5x = 4 - 3x$ and (D3) $5x + 20 = 12 - 8$. Her working for the latter is shown below in Figure 6.9. Here she is again explicit in the operations she is performing on both sides. She also shows she is comfortable with collecting the letters on the RHS and having an answer of $4 = x$. Tebogo was also showing more maturity in her ability to check her solution (underneath). She shows both sides have a value of 40 when $x = 4$. The collection of letters ($7x$) on the RHS and checking each side independently both strongly point to a relational view of the equals sign.

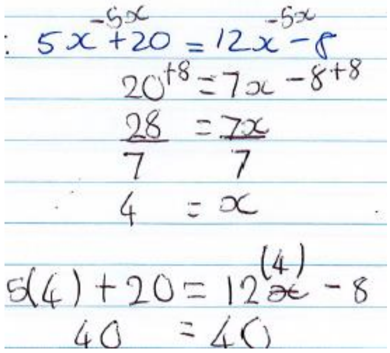


Figure 6.9: Tebogo's working for item C4 in session 3

However, in session 4, she reverted back to a similar inverse operation approach as described in session 1 and 2, with the equation $7x + 4 = 2x - 1$ by dividing through by 2 to undo the $2x$ on the RHS. This is shown in Figure 6.10.

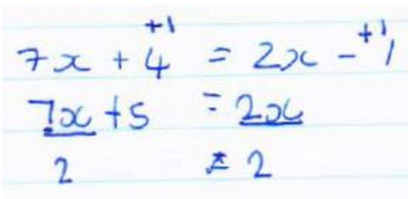


Figure 6.10: Tebogo's working for item C3 in session 4

Again, prompted by an equation with two letters on the same side ($7x - 2x = 10$), appeared to remind her of the need to operate on the variable. Indeed, she explains this regression in the transcript below:

UTTR	SPKR	TRANSCRIPT
102	TB	I don't know, I dunno, it's most likely, something like I kind of forgot how to work out that. But ja I like I always ***forget that both like, like these are both like terms***.

Tebogo in an interesting case in terms of 'progression'. The design of the study was to investigate exactly this possibility – that learning is not always linear. Learning is not necessarily a switch that is flipped and suddenly a learner can perform consistently. By session 3, Tebogo appeared to be very confident in solving equations with letters on both sides – thus overcoming the didactic cut. But in session 4 she appeared to fall back into difficulties with solving equations with letters on both sides. Thus Tebogo did not appear to have fully internalised the method to correctly solve equations with letters on both sides, but could successfully solve them with prompting (assistance of a more knowledgeable other). A full discussion of this Vygotskian perspective is provided in chapter 9.

6.4 Daniele

Daniele's progress diagram (Figure 5.4) shows that she was unable to correctly solve an equation with letters on both sides during this study. Thus she provides the strongest evidence from this research of the existence, and persistence, of the didactic cut. What her progress diagram did not illustrate were her changing approaches and the different errors she made over time.

In session 1 she had been using inspection and systematic substitution to solve equations with a single letter (discussed further in section 7.4). When she encountered item (C3) $2x + 5 = x + 1$, she said that "both values are missing from both sides." The plural value*s* might imply that there could be different values for each letter. After she paused before beginning, I asked her if she remembered what to do. She replied that she didn't but she would try.

Figure 6.11 shows the errors she made. The first attempt was scratched out and then she reattempted alongside.

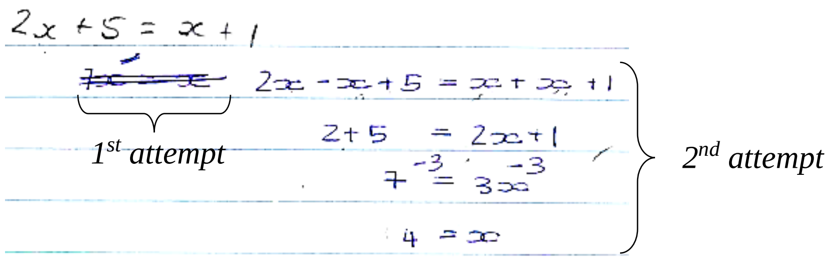


Figure 6.11: Daniele's working for item C3 in session 1

In her first attempt (scratched out), she attempted to add unlike terms (by conjoining) on both sides of the equation, resulting in the $7x = x$. I asked her whether she was finished. The following transcript shows her explanation of why it was a problem and how she began her reattempt.

UTTR	SPKR	TRANSCRIPT
83	AH	Well, do you feel like is that the answer? Are you finished or do you think you need to go further?
84	DR	I think I have to go further **because x isn't solved**.
85	AH	What does your answer look like normally?
86	DR	**x equals something**.
87	AH	Okay.
88	DR	I remember, uhm, **I think something gets subtracted from both sides**, I think, uhm, no …

She then reattempted the question (2nd attempt in Figure 6.11), and correctly subtracted x from the LHS but then added x to the RHS, meaning that instead of no letters ($0x$) left on the RHS, she now had $2x$. Kieran (1992) calls this error the redistribution error. On the LHS she simplified $2x - x$ as 2, another algebraic error, similar to the deletion error (Carry, Lewis & Bernard, 1980). She explain this step as "so that we can get 2 and 5 on their own", meaning that there would only be constants on the LHS.

It is noteworthy that in this first session, although the transcript reveals her uncertainty, she was 'operating' on the letters (subtracting x from LHS and adding x to RHS) but making algebraic errors (adding unlike terms, conjoining, redistribution, and deletion). These non-symmetric operations (redistribution) to the LHS and RHS are problematic in terms of a relational view of the equals sign. However, she had not made any of these errors when solving equations with only a single letter. In session 1, she had solved equations of the type

$ax + b = c$ with either inspection, or systematic substitution. Thus equations with letters on both sides were the first time that she was required to use a more formal approach to find the solution to the equation.

Even with her difficulties with letters on both sides, she was able to understand that both sides needed to have the same value for the same value of x. In the item (C3) $2x + 5 = x + 1$ (shown in Figure 6.11 earlier), she finally got to an answer $x = 4$. Daniele was accustomed to checking her answers with her calculator. On the LHS, by substituting $x = 4$, she got a value of 13. She explains her thinking in the transcript below.

UTTR	SPKR	TRANSCRIPT
107	AH	So this (LHS) is 2 times 4 plus 5 is 13?
108	DR	And … if this (RHS) was **12 plus 1** it would be 13.
109	AH	Okay but then what does this ($4 = x$), so here you said, 4 equals x?
110	DR	It's supposed to be the **other way around**.
111	AH	It's the other way around?
112	DR	Ja.
113	AH	Okay so this, you put 4 here (in place of x on LHS), but you were saying that this (x) could be 12 (on the RHS)?
114	DR	**Ja, it doesn't sound right?**
115	AH	Why?
116	DR	Because I think **x should be the same value**, ja.

In utterance (108), she is thinking that if the x on the RHS was 12, then both sides would have a value of 13. This points towards a relational view of the equals sign – that both sides balance for a combined value of 13. But by (116) she realises that x must be the same value on both sides. She does make a curious statement in (111) about the answer $4 = x$ being the wrong way around, indicating that it should have been $x = 4$. This could indicate that her understanding of equals has both operational and relational meanings to her.

I did provide her with another equation with letters on both sides (C2) $2x + 5 = x$. Her initial response to this item was that "x could be anything". When I probed her to explain, she stated that "Because the *answer* is the unknown value so this x (on the LHS) could be anything. I think." This statement is more characteristic of an operational view of the equals sign (as an answer). This inconsistent view of the equals sign could possibly be part of the reason she struggled so much with equations with letters on both sides.

In session 2, she immediately reacted to the equation $3x + 15 = x + 1$ by stating "Oh I hate this so much" because "x is on both sides. And it doesn't make sense to me because we are trying to find out what x is. But I don't know. It's hard". But she proceeded to try and solve the equation, where her working out is shown in Figure 6.12. This time, we see a correct start to add 15 to both sides. But she then makes a similar algebraic error (to session 1) in line 2, conjoining $x + 16$ as $16x$.

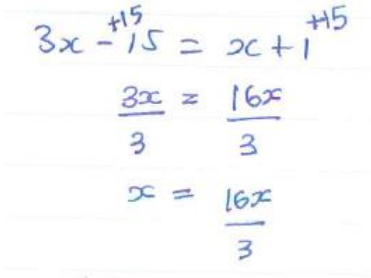


Figure 6.12: Daniele's working for item C3 in session 2

Being prepared for her difficulties with this of equation, I followed up with the equation (C1) $4x - 2x = 10$, which she correctly solved (shown on the left in Figure 6.13). I then followed this up with questioning designed to prompt her to see that (in C3 above) it was possible to collect the like terms on each side (on the right of Figure 6.13).

Figure 6.13: Daniele's working for items C1 and C2 in session 2

The transcript of this interaction is below. The transcript below shows that she sees item C1 entirely differently to how she sees C3. I have also included her working for (C1) $4x - 2x = 10$, where letters are on the same side, and (C2) $4x = 3x + 4$ where there is only one constant on the RHS, and the only operation would be to transpose the $3x$ to the LHS.

UTTR	SPKR	TRANSCRIPT
164	AH	So you have managed to work it out when both of the x's were on the same side, you said.
165	DR	Ja.
166	AH	But you said the issue was here you don't like it when the x' s are on different sides?
167	DR	Ja
168	AH	So you don't know how to go from here (C3) to that (C1)?
169	DR	I don't know how ...[intervened]
170	AH	Like if I gave you this question (C3), you could not get x's on both sides?
171	DR	**I don't, like x, on this side only?**
172	AH	Ja to make it look like this, what happens if I gave you this one (C2: $4x = 3x + 4$)? Now the x's are on both sides?
173	DR	Are they?
174	AH	What do you read it as?
175	DR	$4x$ equals $3x$ plus 4.
176	AH	Because you said you don't like it like this right? What didn't you like about this equation?
177	DR	**That there was a sum equals and then there is a sum.**
178	AH	Mmmm is that, does that, is that not the case here? (C1)
179	DR	No.
180	AH	Okay what do you see this one (C1) as?
181	DR	It's a sum and an answer.

In (169) she states she doesn't know how to 'get' both x's on the same side. The item I refer to in line 172 was (C2) $4x = 3x + 4$, i.e. leaving the constant already on the RHS, and only requiring operating on the $3x$. However, she still sees it as a 'sum equals a sum' (177), implying that x's on both sides are a sum. This is reminiscent of Sfard and Linchevski's (1994) 'two equations' – that is two separate processes waiting to be performed. And unlike Tebogo earlier, provoking her with two letters on the same side did not help her to see a way to operate on the letters on both sides of the equals sign. This comparison is discussed in the ZPD section (9.2.3) of chapter 9.

There was a fairly dramatic shift in session 3, with her approach to the first item with letters on both sides shown below. She appears to have some notion of the need to have letters on one side and constants on the other. Figure 6.14 reveals that she uses arrows to indicate the movement of terms from one side of the equation to the other.

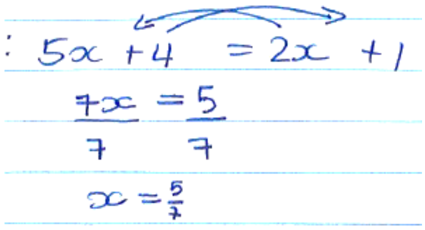


Figure 6.14: Daniele's working for item C3 in session 3

The arrows do strongly suggest that she is employing some form of the transposition model, 'moving' letters (shown by the arrows) to one side and constants to another. However, she has not executed the approach correctly because she maintained the sign $(+)$ of $2x$ and 4 to get $7x$ and 5 respectively. Thus there are no indications of inverse operation being employed her, rather some form of 'transposition'. This type of error was noted by Kieran (1992), which she termed 'switching addends'.

Her arrow transpositional method is also problematic in terms of the equals sign, as she is not performing the same operation on both sides of the equals sign in order to maintain balance. Simply 'moving' objects across the equals sign without changing the sign of the term is an error in terms of the transpositional model. However, she does not see any problem with this operation, even as it contradicts previous methods where she does perform the same operation (inverse) on both sides.

In session 3, she had made an initial error when solving (B3) $5x + 12 = 7$. Her error in shown on the left side below in Figure 6.15, where she appears to have made a similar mistake to that in Figure 6.14, of adding 12 and 7 to get 19 on the RHS. Figure 6.16 shows her reattempt after checking $x = \frac{19}{5}$ and realising it is incorrect. The final line in Figure 6.16 shows only -1 and not $x = -1$. I think that this is a slip on her part as she then checked her solution on her calculator by typing $5 \times -1 + 12$ (indicating that $x = -1$) and appeared satisfied when her calculator gave her an answer of 7.

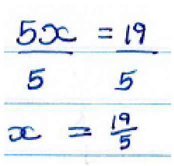


Figure 6.15: Daniele's working for item B3 in session 3

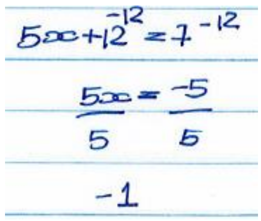


Figure 6.16: Daniele's reattempt at item B3 in session 3

In Figure 6.16, she is also explicit in indicating the operations she performs on both sides, showing subtracting 12 in the first line and then dividing by 5 in the second. This shows some conception of inverse operation and a relational view of the equals sign – that the same operation is performed on both sides. Her error in line 1 (of Figure 6.15) of adding 7 to 12 instead of subtracting 7 from 12 is similar to the error she makes when using her arrows in Figure 6.14.

Thus in session 3, I confronted her with this contradiction in the following interaction. I am comparing her approach in Figure 6.16 with her error she made in Figure 6.14.

UTTR	**SPKR**	**TRANSCRIPT**
135	AH	Yes, okay so, so explain to me how do you get $7x$ (on LHS in Figure 6.14)?
136	DR	**You say $5x$ plus $2x$. Which is $7x$.**
137	AH	Okay and the 5 (on RHS in Figure 6.14)?
138	DR	**4 plus 1**.
139	AH	Okay but then you didn't, didn't you do something like that to get the 19 (on RHS in Figure 6.15)? Where you said 12 plus 7 was 19?
140	DR	Ja.
141	AH	And then that was wrong and then what, you changed it to this one (her reattempt in Figure 6.15), where you did that (subtracted 12 to both sides – Figure 6.16). What is different between that and that (comparing 6.15 and 6.16)? Instead of saying 12 plus 7, what did you do here (in Figure 6.16)?
142	DR	**I just got rid of** ...[intervened]
143	AH	12?
144	DR	Ja. **The whole number**.

145	AH	Okay is that not what you are doing here (Figure 6.14) though?
146	DR	**I don't think I can do that**.
147	AH	Ja carry on. Well wait, what are you saying… you said you don't think you could do that. Why?
148	DR	Mmmm, it **wouldn't make sense to do that.**
149	AH	Okay.
150	DR	I can try.
151	AH	What would you do?
152	DR	**Because the last time I did this it was wrong**.
153	AH	Okay ...[laughing]
154	DR	Ja.
155	AH	Right so, you are talking about with me (in a previous interview)?
156	DR	No, by myself.
157	AH	Oh, okay.
158	DR	(re-attempts the question – Figure 6.17) **No, that's wrong**.
159	AH	Why?
160	DR	**Because you can't get anything out of that because these two** (on RHS in Figure 6.17 below) **aren't like terms.**
161	AH	Alright.
162	DR	So, it's wrong. Completely.

Her responses in (146) and (148) were noteworthy. She appears to be saying that she cannot use the same successful method from Figure 6.16 in an equation with letters on both sides (Figure 6.14). In-between utterances 156 and 157, she reattempted the question. Her reattempt is shown below in Figure 6.17. In her reattempt, she does use inverse operations to subtract 4 from both sides but then leaves the $2x$ on the RHS. This is when she says "that's wrong" (158) and says that she continue because those two $(2x + (-3)$ in line 3) are not like terms.

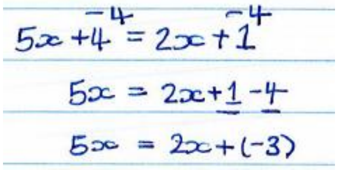


Figure 6.17: Daniele's reattempt for item C3 in session 3

This shows progression from earlier sessions where she was conjoining, but also shows that she sees inverse operations as being limited to equations with a single letter. She doesn't appear to see the possibility of subtracting $2x$ from both sides in the same way as she can subtract 4.

In session 4, she appears to be experiencing the same difficulties from session 3. She was still utilizing her arrows without inverse operations for letters on both sides, but using inverse operations correctly for single letter equations. I tried to again drill down to this contradiction in methods. Below in Figure 6.18 and 6.19, are her two approaches to items A3 (using inverse operations) and C3 (arrow method) respectively.

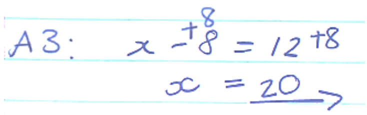


Figure 6.18: Daniele's working for item A3 in session 4

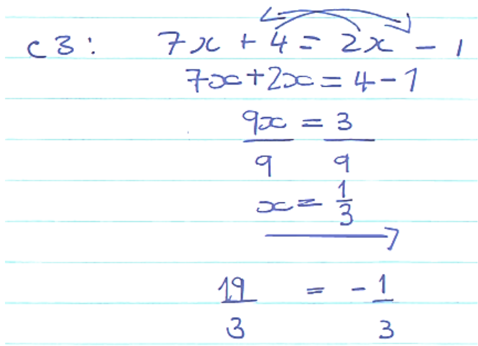


Figure 6.19: Daniele's working for item C3 in session 4

In the following transcript, I again try to probe her into seeing the contradiction between methods she is using in (151). She appears to see the contradiction (152 & 156), but then reaffirms that this is a different type of question in (166).

UTTR	**SPKR**	**TRANSCRIPT**
143	AH	So does the arrow thing work like what anything that you did here, or are these two different methods?
144	DR	I just don't know if it is ...[intervened]

145	AH	So you said you want to get the variables on the one side and the numbers on the other side?
146	DR	Ja.
147	AH	Ja?
148	DR	I just do that with **these types of equations**.
149	AH	Okay but it seems like it could, we have a problem here.
150	DR	Ja.
151	AH	**So, if I ask you here the $2x$ you have moved it from that side onto that side? And the 4 you have moved from that side here onto that side. Is that different to say moving the eight from this side onto that side? Should I take minus 8 and just put minus 8 on that side?**
152	DR	**You could because it would, no, wait, no you couldn't.**
153	AH	Ja?
154	DR	You couldn't.
155	AH	Why not?
156	DR	**Because it won't equal 20** and ...[intervened]
157	AH	So what did you do with the minus 8 here? To get it onto the other side?
158	DR	I added it.
159	AH	Ja, and the 17?
160	DR	Subtracted.
161	AH	So what do you think you could do here to get 4 onto that side?
162	DR	I want to say you could minus 4 but ...[intervened]
163	AH	Ja?
164	DR	This ...[intervened]
165	AH	So you want to try? So what are you thinking?
166	DR	**Because it is difficult for me to read it that way. So I, all I do, well usually do, I flip it the other way, so I imagine that this side is this side and that side. So like that, the thing is you can't have two outstanding variables on both sides. It doesn't work.**

Her response in (166) is interesting, but not entirely clear. When she says “difficult for me to read it that way”, I think that she means that she doesn’t see equations with letters on both sides in the same way she sees an equation with only a single letter. She then explains her method to get the ‘variables’ on the same sides by ‘flipping’ it.

She then tries to use a similar method to reattempt (C3) $7x + 4 = 2x - 1$ (possibly feeding off my probing as telling her that her method was wrong) in Figure 6.20 below. It is striking to note the use of inverse operations in the first line, but the $2x$ is moved across and added to $7x$ to get $9x$.

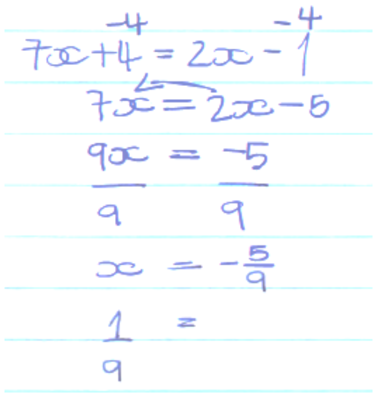


Figure 6.20: Daniele's 2nd attempt at item C3 in session 4

This difference in operating with letters compared to constants indicates that she possibly sees $2x$ as a fundamentally different entity to 4 – that we can subtract positive 4 from both sides in line 1, but not able to subtract $2x$ from both sides.

6.5 Conclusion

In this chapter, I have described three learners' very different journeys in relation to the didactic cut. Edith experienced the cut in the first session only, before successfully overcoming her difficulties and correctly solving most[4] of the equations with letters on both sides. Tebogo was able to solve equations with letters on both sides after being prompted by an equation with letters on the same side, namely (C1) $4x - 2x = 10$. However, she continued to struggle, as after a successful session 3, in session 4 she regressed into making the same mistakes of dividing through by the coefficient of x, as she did in session 2. But again, by prompting her with $7x - 2x = 10$, she recalled being able to add like terms, and successfully returned to correctly solve (C3) $7x + 4 = 2x - 1$. And finally Daniele who was not able to overcome the didactic cut during this study. Daniele adopted her new 'arrow' method (Figure 6.14), which conflict with her approach with items that had only one letter, since she did not use inverse

[4] The only type of equation with letters on both sides that she experienced difficulties with was of the type $Ax = Bx$, which has its own chapter devoted to it.

operations with her arrow method. Prompting Daniele with the same type of example which had helped Tebogo, i.e. $4x - 2x = 10$, did not help Daniele in overcoming the didactic cut.

In the following chapter, I analyse another common difficulty experienced by learners – that of *negativity*.

Chapter 7 : Difficulties with negativity

7.1 Introduction

The literature reviews discussed the known errors in research regarding the minus symbol. Based upon known difficulties with the minus symbol, I expected that learners may make errors due to operating with negative numbers and terms in equations. The progress diagrams in chapter 5 for Edith, Tebogo and Danielle all show errors 'within a type' of equation. Most of these errors had something to do with negativity, i.e. the minus symbol appeared to be causing learners difficulty.

In this chapter, I discuss three learners, with two separate sources of difficulty with negativity. Firstly Tebogo, whose difficulties stem from the coefficient of x being a negative number since she had difficulties with both $5 - x = 8$ and $5 - 3x = -10$. But then I discus Edith and Danielle, whose difficulties were confined to the coefficient of x being -1, i.e. $-x$. Thus, the equations lacked an explicit negative coefficient (implied negative 1), such as $5 - x = 8$. In this chapter I will expand upon their initial difficulties and how their thinking changed over time as they attempted to overcome these difficulties.

7.2 Tebogo

Tebogo had been discussed already as showing progress in her ability to solve linear equations over the course of four sessions. She had experienced difficulties with equations with letters on both sides but had mostly overcome these by sessions 3 and 4. She also experienced difficulties with the item $ax = bx$ which be discussed in the following chapter.

In session 1, Tebogo successfully solved the items (A3) $x - 5 = 8$, and (A4) $5 + x = 8$. She appeared to be using a mixture of inverse operations and inspection, as she asked whether she had to show working. This could imply that she could 'see' the answer (not required to show working) or that she was doing the working (inverse operations) in her head. Indeed, her working shown in Figure 7.1 does indicate her use of inverse operations to arrive at the correct answers for both A3 and A4.

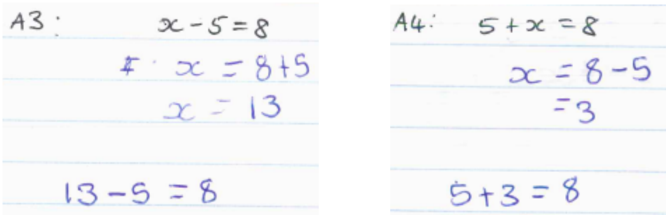


Figure 7.1: Tebogo's working for items A3 and A4 in session 1

hen she encountered item (A5) $5 - x = 8$, she appeared to be stuck – not writing any working and staring at the item. She tried systematic substitution on her calculator, trying $5 - (-13)$ but got an answer of 18 instead of 8. Then again she appeared to be stuck. After giving her time, I probed her as to what she was thinking. Her explanation below indicates that she was trying to use inspection with a solution of 13.

UTTR	SPKR	TRANSCRIPT
31	AH	Okay explain to me what you are thinking.
32	TB	Like I am thinking **it (x) might be like thirteen**, right?
33	AH	Mmmm…
34	TB	But then, uhm, like, I am saying but it probably could be only to the number if I am not, you know, careful. So ***I don't want the 8 to become negative***. So I am trying to like Figure out.
…		
40	TB	**Mmmm. Ja this one is very hard ... [laughing]**
41	AH	...[laughing] okay. Okay, uhm do you remember any methods your teacher taught you on how to solve equations?
42	TB	The, like, **I think there might have been a method but then I think I must have like have forgot it** or something.
43	AH	Okay do you want to try, try and do a method here? Like if this was a test, what would you do next? Don't stress, just, if this was a test right, what do you think your best guess would be?
44	TB	Mmmm. Uhm.
45	AH	What are you thinking to do there?
46	TB	**I am thinking of changing, uhm, like I am trying to Figure out a way to change the negative x to a positive, I mean negative 8 to a positive 8?**
47	AH	Uhuh?

48 TB But then I am trying to figure out how.

If we look at the item more closely, she may be seeing the minus symbol as a difference – that is that the difference between 5 and another number (x) is 8. That number would be 13, i.e. $|5 - 13| = 8$ (the absolute value on the LHS meaning the positive difference). However, she is also apparently aware that if $x = 13$, the 'value' on the RHS would be negative 8, as in $5 - 13 = -8$. In (46) she want to change negative 8 to positive eight. Of course, in this example, $x = -3$, i.e. a negative number. Thus she may not have the mathematical notion of 'subtracting a negative making bigger' at hand.

She stated in (42) that she couldn't remember a 'method' (approach). Nonetheless, she then proceeded to apply a 'method' to the item which followed A5, namely (B3) $3x - 5 = 10$. In Figure 7.2 below, her working to item B3 is shown on the left. For this item, she appears to be showing only one inverse operation performed on one side, i.e. adding 5 to the RHS in the second line and dividing only the RHS by 3 in the third line. However, in item (B4) (included on the right in Figure 7.2), she clearly indicates her use of inverse operations as she shows the opposite of $+5$ as -5 on both sides and then dividing both sides by 3 (the opposite of $3 \times x$).

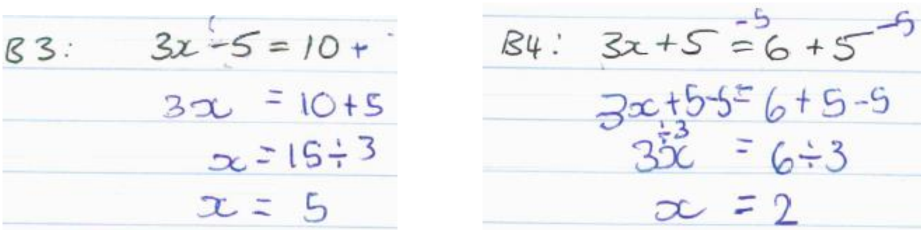


Figure 7.2: Tebogo 's working for items B3 and B4 in session 1

But when faced with (B5) $5 - 3x = -10$ after (B4) , the minus symbol ($-3x$) on the LHS again introduced an error. She made two attempts at this question, show in Figure 7.3. Her initial attempt shows adding 5 to the RHS instead of subtracting and then she subtracted 3 in the second line instead of dividing by 3. She realised her mistake in the first line only however. Her second attempt shows her subtracting 5 from both sides but then added 3 to both sides. Earlier she had said she did the opposite sign of the number, so here she did the opposite of minus 3 (in $-3x$) After checking her solution of -12, she realised she had made an error. She said her "head was somewhere else" and she was meant to divide by 3 in line 2. However, when she did this, she only divided both sides by positive 3, possibly 'ignoring' the minus symbol attached to $-3x$. This item was 18 minutes into the session and it was possible she was tired by this stage.

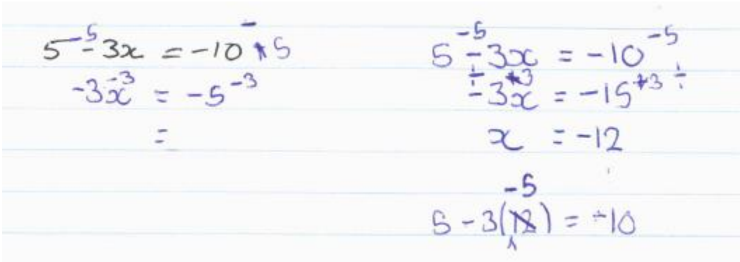


Figure 7.3: Tebogo's two attempts at item B5 in session 1

Although she leaves her answer as $x = -12$, she shows she really considers value of x to be -5 (from $-15 \div (+3)$) in her last line of working. After checking this line $(5 - 3(-5))$, she got an answer of 20 on her calculator, and realised she had made another error. For the sake of time, I didn't pursue this item further, but it appeared that Tebogo's initial difficulties were more due to the minus symbol as both $5 - x = 8$ and $5 - 3x = -10$ resulted in similar difficulties, although her approach was different for each item.

Session 2 helped me to hone in on this issue. She correctly solved item A4, $10 - x = 18$ as shown below. However, she struggled when I asked her to check her working (attempt 1 in Figure 7.4), and she felt her answer was wrong. She then reattempted the question in Figure 7.5 and obtained the same answer.

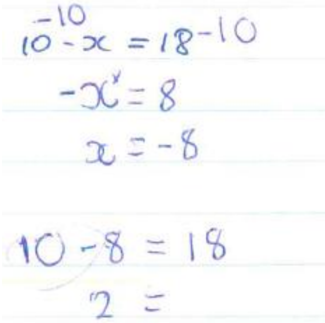


Figure 7.4: Tebogo's first attempt at item A4 in session 2

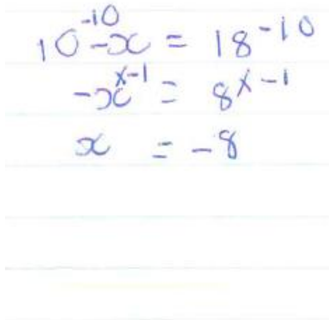


Figure 7.5: Tebogo's second attempt at item A4 in session 2

Thus, upon a first look at her work, she had apparently overcome her difficulties with the minus symbol. However, when she checked her work, she failed to include the minus attached to 8, i.e. the $10 - 8$ instead of $10 - (-8)$ at the bottom of Figure 7.4. When I asked her to explain her work, she expressed very similar concerns regarding 'difference' as in session 1 in the following transcript.

UTTR	SPKR	TRANSCRIPT
17	AH	Okay what was your thinking there (the $\times\ -1$ on the 2nd line of her second attempt in Figure 7.5)? Can you explain what you were thinking?
18	TB	Since I was thinking like why don't you like, I ***think like times it by the negative or something.***
19	AH	Mmmm. Okay well do you feel like that is the right answer? Could you check that answer?
20	TB	Let me see? Uhh ***it doesn't seem right***.
21	AH	Okay explain.
22	TB	Because this will ***eventually equal to 2 or something since this like 10 minus 8 will equal to 2***.

In (22), it could be asserted that she still sees the subtraction as a difference, even though her answer is correct. Her substitution indicates she may be struggling with the two minus symbols, that the minus symbol in $10 - x$ would indicate subtraction, while her x value of -8 indicates a negative. Hence she should be subtracting a negative number to result in an increase of 10 to 18. After her second attempt in (24) I did not pursue this item anymore in session 2 due to time constraints and her tone of frustration.

Finally in session 3, it appears as she has a better grasp over the method she is applying. In her work shown in Figure 7.6, she is explicitly showing division by negative 1 (as the coefficient).

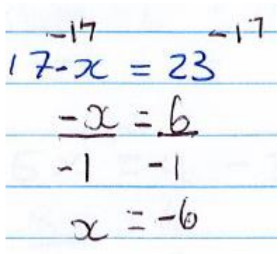


Figure 7.6: Tebogo's working for item A5 in session 3

However, her difficulties with subtracting a negative number when checking her solution still remain. In the transcript below, I challenged her as to why she had substituted 6 when checking her work, i.e. $17 - 6$ instead of $17 - (-6)$ in (15). She explains in (18) the similar confusion between the minus in $17 - x$ and the negative attached to the value of x. She then checked her solution on her calculator as $17 - (-6)$ (between utterances (17) and (18)) to get the RHS of 23. Although her explanation (20) is still unclear as to why she is dividing by negative 1, it does show a better understanding that the letter must be 'positive'. I understand the 'it' in (20) refers to x, and the need for it to be positive.

UTTR	SPKR	TRANSCRIPT
18	TB	Well you see, **I assume that since the 6 is negative right you didn't have to put another, uhm, negative thing**, so I assume that, just put the 6 there, then up until you pointed it out then I realised that ...[intervened]
19	AH	But does this make sense to you, why does this give you 23?
20	TB	Well because, mmmm, I am not sure how to explain it but like since **it *(solution) is a negative 6 right with a…***, I like to kind of ***turn it (x) into a positive*** in a way.

Finally in session 4, she successfully solved all the items with subtracting a letter. For item (A4) $17 - x = 8$, her working in Figure 7.7 below shows her division by negative 1 to remove the minus symbol.

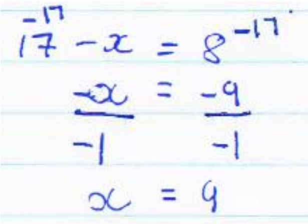


Figure 7.7: Tebogo's working for item A4 in session 4

She explains the need to divide by negative 1 as "…since I got the negative, uhm x, so we can't really delete the uhm the negative, so I kind of like used 1 to like divide the x then take away the negative and then I am sitting with a 9 and then I got $x =$ 9". She sees the need to remove the minus attached to x when she wishes to "delete" the negative. She doesn't justify as to why she uses division, but she was confident that her method was producing the correct answer as she was checked her solution on her calculator to get the RHS of 8.

Furthermore, she has come to terms with subtracting a negative 'making bigger'. For the item (A5) $17 - x = 28$ as she explained her check of $x = -11$, that "So it is like ***adding*** 11 to 17, then you get 28". In her working shown below in Figure 7.8, she explicitly shows the addition indicating an understanding of subtracting a negative being equivalent to addition of a positive.

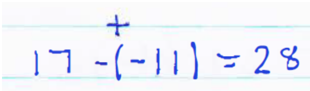


Figure 7.8: Tebogo's check for item A5 in session 4

In session 1, she struggled with both items which involved a negative letter; both (A5) $5 - x = 8$ and (B5) $5 - 3x = -10$. However, in session 2 she was able to solve both types of items but could not verify her solution of $x = -8$ to $10 - x = 18$. In session 3 she shows a clearer understanding of the coefficient of $-x$ being negative 1, but her explanation as to why she needs to divide by this negative 1 is still unclear in terms of the inverse operation of $-1 \times$ as needing to divide by negative 1. In session 4, is still not explicit about the inverse operation but she illustrates the progress she has made in understanding the difference between the minus symbol as subtraction and negative by checking her solution – showing that subtracting a negative will result in adding a positive.

7.3 Edith

As discussed in chapter 6, Edith had used a strong notion of inverse operations to solve equations. She had experienced difficulties in solving equations with letters on both sides, but had overcome these by session 2. Edith's difficulties with negativity are confined to items with a coefficient of -1 to x, i.e. $5 - x = 8$.

In session 1, Edith had correctly solved (A3) $x - 5 = 8$. When faced with item (A5) $5 - x = 8$, she initially wrote down $x = 3$. I asked her to check her solution, and she added in a minus symbol in front of x. Her working is shown below in Figure 7.9.

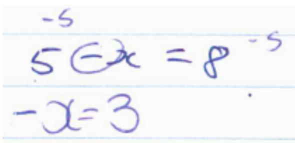


Figure 7.9: Edith's working for item A5 in session 1

When I probed as to why she had made the change, she explained that she had forgotten about the minus symbol. And while she appeared to understand the minus symbol as representing negative, she also appeared to be "happy" that this was the final answer.

UTTR	SPKR	TRANSCRIPT
19	AH	So why did you change that (x) to a minus?

20	EM	I saw the minus (above)
21	AH	So what does this ($-x$) mean to you now? Can you read this ($-x = 3$) to me now?
22	EM	Umm, negative x equals 3
23	AH	Okay, so you're finished, you're happy with this now?
24	EM	Ja I'm happy

Thus it would appear as the negative coefficient may be at the root of her difficulties. However, in the same session, with item (B5) $5 - 3x = -10$, she divided by -3 in her second last line to get the correct answer of $x = 5$, shown in Figure 7.10 below.

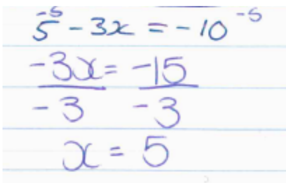


Figure 7.10: Edith's working for item B5 in session 1

When asked to explain her reasoning, and in particular regarding leaving her earlier answer as $-x = 3$, she explained the LHS as negative 3 dividing by negative 3 to result in a positive. I thus confronted her with the similarity in the question to A5 earlier. Her explanation indicates that she sees $-x$ differently to how she sees $-3x$.

UTTR	SPKR	TRANSCRIPT
44	EM	**Because negative 3 divided negative 3 would be a positive,** so this side would be negative 5 divided by (a) negative which should be positive.
45	AH	But here ($-x = 3$) **you left it as negative x equals 3**. **You stopped there.** But then here ($-3x = -5$) you can divide by negative 3.
46	EM	Because this is all it's x, it's all like umm, it's meant to be because the **x stands for 1, because that would be like this variable replaces the 1, so I can't just remove the, if I didn't do anything.**
47	AH	So, if there **was a number between the negative and the x**, would you have done something differently?
48	EM	What do you mean like?
49	AH	So here you said there was number (3) here, but here there is no number in front of the x.

50 EM **Ja, there's no number in front.** There's a 5 which I moved to the RHS. That negative x is equal to the 3.

Edith initially shows that she wanted positive x at the end with this item (44), but explains her difficulty with $-x$ in (46) when she says that she can't remove the minus if she doesn't do anything. Thus she apparently cannot see to 'remove' the minus symbol in the same way she undid $-3x$ earlier.

She uses the phrase "x stands for 1". I don't think she actually means that the value of x is 1 (as I will show later in this section) but rather she has some idea that there is an 'invisible' 1 as a coefficient. However, she clearly doesn't see $-x$ as $-1x$ in the same way she sees $-3x$. Perhaps she didn't quite understand what I was saying in (49), as her response in (50) is thinking the 'number' refers to the constant. This might also explain her language of "x stands for 1", as underlying the notion that she doesn't see a coefficient in $-x$.

It also should be noted her use of the term "negative". This would be the unary purpose of the minus symbol – that of the 'sign' of the number. However, she does not appear to attach the sign to a letter in the same was as she can to a number, as in $-3x$. Thus we might deduce that Edith's difficulties in session 1 are more to do with the 'invisible' coefficient than the minus symbol.

In session 2, I followed up with the same type of question ($a - x = b$) with item (A4) $10 - x = 18$. This time, she was explicit regarding the coefficient of $-x$, as she inserted -1 as a coefficient and correctly divided by -1 as shown below in Figure 7.11. She introduced a new 'error' – the x in the denominator on the LHS. At first I thought this error may have been a slip but she did the same thing again in the following session, and again only when dividing by negative 1.

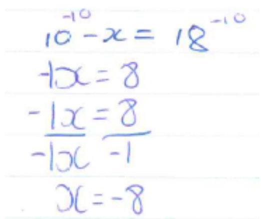


Figure 7.11: Edith's working for item A4 in session 2

In the transcript below, she explains that she is using the ‘a negative and a negative make a positive’ to make the x positive. But she also reiterates her written work of “negative x” and “negative x” to make a positive.

UTTR	SPKR	TRANSCRIPT
11	AH	Can you explain to me what you are thinking?
12	EM	Umm, here I did uh, that umm, because um, **you never let a variable ever be a negative, because I think you can't get a variable with a negative. So what I learnt in class is that you always try and put it over 1 and 1 and that will cancel out everything. And a negative and a negative make a positive. So, if I make x, negative x and negative x, it will equal a positive (LHS),** then it's a positive and a negative will make it negative and 8 divide negative will be negative.

I had asked her about what she had learnt in class (that she mentions in (12)) and did it help her. She also admitted that in the previous session she had “forgot” how the “negative stuff” worked and explained that after her difficulties in the last session, she went back to her notes to revise.

I did not pursue the division by x in session 2 as I was more concerned with her understanding of coefficient and our time was limited. I also did not code it as an error in her interview progress table (Table 5.1) since her final answer was correct, but it was a concern for me.

In session 3, she repeated the same division by $-x$ for item (A6) $30 = 4 - x$ as shown in Figure 7.12. She had not divided by x when correctly solving (B4) $12 - 5x = -18$ just previously.

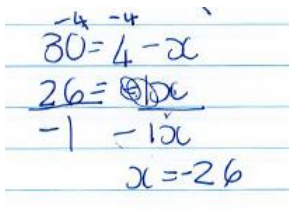


Figure 7.12: Edith’s working for item A6 in session 3

This allowed me to pursue this line of thinking. In the transcript below, she reiterates the need for her final answer to have a positive x by performing the operation on both sides.

UTTR	SPKR	TRANSCRIPT
46	EM	Okay, so here (line 1) I do the inverse. I do minus, minus, that'll leave me with x, **but we know that when you're doing that you have to make sure x goes for positive. So the value of x is equal to $1x$, x is the value of 1. So it's a standby**. You don't always have to put the 1 there, just to make your, I dunno, **just to make it like equal, divide it on both sides... negative 1...**
47	AH	Yah but you also had an x at the bottom here $\left(\frac{}{-1x}\right)$. So here (LHS) you've done divided by negative 1, and this side here (RHS) you have…
48	EM	**Wouldn't it be the same thing?**
49	AH	Okay
50	EM	I think it would be the same thing.

In (46), she again explains that x needs to be positive. She uses the phrase "value of x is equal to $1x$". By value, I do not think she means the actual solution, but rather is the coefficient (which she explains as a 'standby'). Thus, if she sees $1x$ as having an value of 1, this would be in contradiction to a final answer of $x = -26$. She had shown (with other questions) that she understood the final answer of $x = \ldots$ to represent the value of the letter which 'balances' both sides. Thus, I argue that she meant "value of x is equal to $1x$" as the coefficient of x was 1 (or indeed negative 1), but did not have the vocabulary to express this.

She repeated this division by $-1x$ again in session 4, with item (A5) $17 - x = 28$, again only where the coefficient of x was negative 1. Her explanation was very similar, again showing that she thought that $-x$ was the same as -1.

UTTR	SPKR	TRANSCRIPT
34	EM	So you have to divide it by 1, *cos* x … **means 1… times 1 something**… and what you do on the LHS, you do on the RHS. So I was dividing here (LHS) and dividing that (RHS)…
35	AH	So you're dividing both sides by negative…
36	EM	**One**
37	AH	Negative one (**pointing at x in denominator**)
38	EM	**You can say negative x**

In (36), and (38) she again conflates dividing by negative 1 as the same as dividing by negative x. However, she doesn't appear to hold that the value of x is negative 1, as she checks her 'solution' on the calculator. In (37), my pointing at x may have elicited her response in (38) as she did pause (less than 2 seconds) before responding.

In summary, after Edith's initial difficulties with $5 - x = 8$, she was able to correctly solve equations with a coefficient of x as negative 1. However, she retained the method of dividing one side of the equation (that with the letter) by x as well, showing a lack of progress in this regard. Thus, I would argue that the negative x was causing her to change her working out compared to when there was a visible coefficient, such as $-3x$.

7.4 Daniele

In session 1, Daniele appeared to be utilizing the inspection approach. For the first six items below, Daniele wrote down only answers in the form $x =$ ... without any 'working out'.

(A3)	(A4)	(A5)	(A6)	(B3)	(B5)
$x - 5$	$5 + x = 8$	$5 - x = 8$	$8 = x + 3$	$3x - 5 = 10$	$5 - 3x = -10$

The item (A5) $5 - x = 8$ had caused some difficulties for Tebogo's inspection approach (in section 7.2), but Daniele had correctly identified the solution as $x = -3$. She verified the solution on her calculator shown in the screenshot in Figure 7.13.

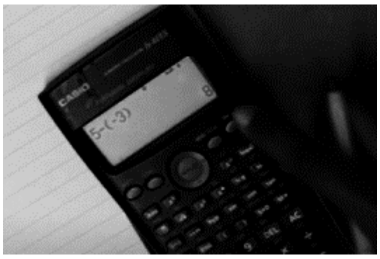


Figure 7.13: Daniele's working for item A5 in session 1

I then asked Daniele why she chose negative 3 and why she had put brackets around the minus 3 on her calculator (Figure 7.13). In the following transcript she explains that she also used the difference (as Tebogo had done earlier), but focussed on the difference between 5 and 8, rather than Tebogo, who had focussed on the difference between 5 and a number, as being 8.

UTTR	SPKR	TRANSCRIPT
36	DR	Uhm, because these **two signs** ($5-(-3)$ on the calculator) **next** to each other might confuse somebody so I just decided to put it in **brackets** to **differentiate the subtraction sign and the negative number**.
37	AH	Okay, okay, and how did you come up with negative 3? How did you know it was negative 3 to put in?
38	DR	Because the **difference of these two numbers is 3** and if it's **not a positive** number then it **should be a negative number**.

She explains that she uses the difference between the two numbers, but does not seem to have the idea that subtraction of negative makes bigger, rather implying it must be negative because subtracting positive 3 wouldn't give the correct 'answer' (as in the RHS value of 8).

I thought that an item such as $5-3x=-10$, would challenge her inspection abilities. But she used her calculator to substitute various values for x, till she found one that gave her the 'answer' negative 10. She had started with $x=1$, then $x=2$, etcetera, till she reached $x=5$, which she explains below.

UTTR	SPKR	TRANSCRIPT
58	DR	I worked out, uhm, what 5 times 3 would be and it's 15 and then I said 5 minus 15, which is negative 10.
59	AH	Okay and how did you chose this five? To start off with?
60	DR	I just, I worked like I worked down so I said 3 times 1 which is 3 and then I tried to work it out which was a ... and then the answer was a negative 10, then I did 2 times 3, then 3 times 3, and 4 times 3, 5 times 3.

Thus in session 1, she had not experienced difficulty with either of the items that involved subtracting a letter (i.e. a negative coefficient to the letter). Her systematic substitution may have been challenged with an item where the value of x was a fraction or a negative number itself.

In session 2, she also began solving the linear equations from the first row using inspection, but encountered some difficulty with (A4) $10-x=18$. In this question, she reasoned that x would be negative, but instead of finding the difference, she added 10 and 18 to get $x=-28$, which she said was wrong. This appears to be similar to Tebogo earlier who focussed on the difference between 10 and x as being 18.

She then shifted into a difference of 8 between 10 and 18, but when checking on her calculator, tried only $10 - 8$ to get 2. She said this was wrong, because it didn't equal 18. Thus, she appears to have forgotten (from session 1) her use of brackets on her calculator when substituting in place of x. Daniele was making the same mistake as Tebogo had been with difference and not distinguishing subtraction from the negative value of x. And like Tebogo, she couldn't put these two ideas together – that the difference was 8 but that it needed to be negative 8 in order to go to a larger number (18).

In the same session, she began using a more algorithmic balance approach, so I asked her if that approach could apply to item A4. She explained she hadn't seen the need because "x was already on its own", but was willing to try with her other method. This indicates that she wasn't initially engaging with the negative 1 coefficient – that x did not have anything to be undone. She only had to worry about the constant (10).

In Figure 7.14 her initial reattempt at solving the equation algebraically is shown. Her initial attempt did not have the negative in front of x. She left her answer as $x = 8$, which after checking, she realised was incorrect. Earlier she had correctly solved item (B5) $20 - 2x = -15$. I've also included that working out on the right of Figure 7.14 to show that she can deal with negative coefficients when the coefficient is visible.

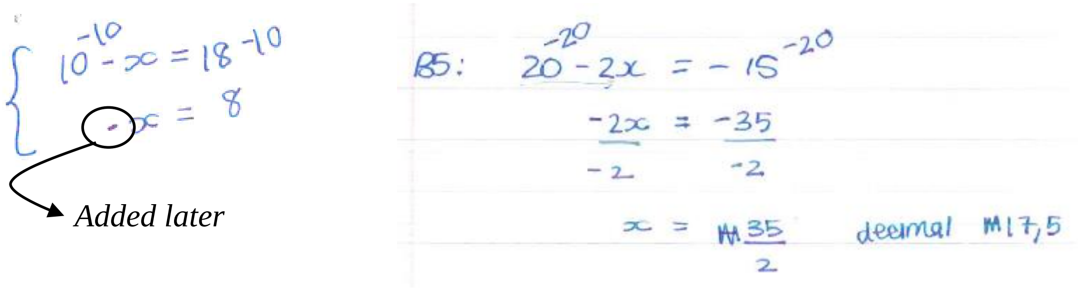


Figure 7.14: Daniele's 2nd attempt at item A4 compared to B5 in session 2

I confronted her with her why she divided by -2 in B5, but had ignored the minus symbol in A4.

UTTR	**SPKR**	**TRANSCRIPT**
117	DR	A little bit. I think I, I made a mistake, I don't know. x looks like it is supposed to be a negative x... I don't know, I am not sure (looking at minus in $10 - x = 18$).
118	AH	Why are you saying that?

119	DR	**Because it's** (x) **got this sign** (minus)**.**
120	AH	Mmmm okay. Does that, do you think that that makes a difference?
121	DR	I think so because it made a **difference here** (in B5)**,** so it, it could possibly.

She then tried $10 - -8$ on her calculator (not including brackets around -8), which gave her the answer on the RHS of 18. Although she appeared to see the need to deal with a coefficient of negative 1, she was never explicit in verbalising this and may have been using some form of trial and error.

In session 3, she again experienced difficulties with the item (A4) $17 - x = -3$ leaving her 'answer' as $-x = -20$. This is shown in Figure 7.15.

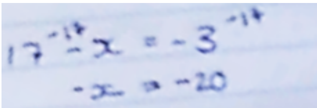


Figure 7.15: Daniele's working for item A4 in session 3

She paused (for a few seconds) and said "that doesn't look right". She then checked her solution on her calculator as $17 - (-20)$. This immediately implies she has read her original solution as $x = -20$, effectively ignoring the negative in front of the letter. She realised this solution was not valid since it gave her an answer of 37 instead of -3. On her calculator, she then tried $17 - 20$, which gave her the 'answer' of the RHS value of -3.

The following interaction shows her conflict between her inspection (or trial and error) answer of $x = 20$ from her calculator and her algebra which she understood to be $x = -20$.

UTTR	**SPKR**	**TRANSCRIPT**
11	AH	Okay so what have you done there now (on her calculator)?
12	DR	I just **said 17 minus 20**.
13	AH	So what would that imply that x is?
14	DR	**20. And not negative.**
15	AH	But what does this (minus symbol in $-x = -20$) mean to you?
16	DR	It's a **negative sign**.
17	AH	Ja.

18	DR	...[laughing]
19	AH	So you read this (her final line) to me as?
20	DR	**Negative x equals negative 20**.
21	AH	Okay and can you simplify that any further?
22	DR	**Uhm, no. I don't think so.**

In (14) she recognises $x = 20$ should be her final answer. But when I asked whether she could go any further (21), she replied that she didn't think so (22). Thus her algebraic method couldn't produce the solution that she knew she should get. This could be similar to her session 2 notion of x 'already being on its own'.

However, again in the same session she was able to correctly solve (B4) $12 - 5x = -18$, which has a a visible numerical coefficient, by dividing by -5 at the end.

A very similar interaction to session 2's took place, where I again asked her to compare her working for the two items. In the transcript below, I picked up from her idea of "not going further" in (A4).

UTTR	**SPKR**	**TRANSCRIPT**
109	AH	Okay so do you have an idea of what you could have done here (item A4: $-x = -20$) now? To finish off?
110	DR	No.
111	AH	Because do you remember that you, when you put the number 20 you said 17 minus 20 and you got that answer (-3). Which kind of meant that x equals 20 hey? Do you remember that? 17 minus 20. Do you remember that, try it (on her calculator). You did that and then you said to me, maybe x equals, you said, maybe x equals 20.
112	DR	Ja.
113	AH	But how do you get to x equals 20 from there ($-x = -20$)? *(DN tries* $-20 \div -1$ *in her calculator)* Okay so what are you thinking?
114	DR	**So, there could be an imaginary 1 so that it is visible and then divide by negative 1 which gives us positive 20**

This time she referred to the imaginary 1 (114), but still appears uncertain with the language of "could be". Thus I would argue that $-x$ having a coefficient of -1 is still an issue for her.

Immediately following this, I gave her item (A5) $17 - x = 23$. Her working below (Figure 7.16) shows her explicitly adding in the coefficient of negative 1 to x and showing both sides dividing by -1.

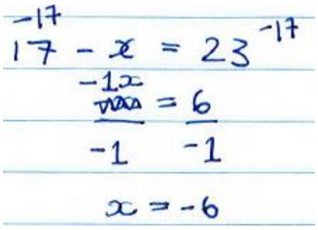


Figure 7.16: Daniele's working for item A5 in session 3

However, despite her apparent realisation with her reattempt at item A4 above, she was still concerned that this answer of $x = -6$ was not correct. However, after checking her solution on her calculator, she stated that is was correct. Her voice still sounded unsure. After probing this concern, she explained "Because I, I think to me, x can't equal any kind of negative number but it can. I just had to check." This response was curious since in the very first session, she had got the answer of $x = -3$ to $5 - x = 8$ by inspection. She also apparently still does not trust her algebra, even though her method of checking her answers is now supporting her answers.

Remarkably, in session 4, she again left her answer to (A5) $17 - x = 28$ as $-x = 11$, again not dealing with the negative 1 coefficient to x. She checked this on her calculator as $17 - 11$, implying that $x = 11$. She then tried $17 - (-11)$ to get the answer of 28. Only then did she change the RHS to be negative 11. Both of these steps are in line with her previous sessions in regards to treating $(-x)$ as if the letter is already isolated. Thus, her final line was $-x = -11$, shown in Figure 7.17.

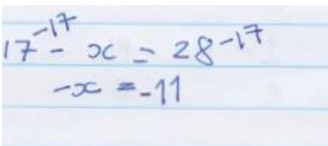


Figure 7.17: Daniele's working for item A5 in session 4

I confronted her with the fact that the RHS $(28 - 17)$ doesn't give her -11, which she agreed with. I also repeated her original working that she written $-x = 11$ but then she had used $x = -11$ in her calculator. She didn't respond to this directly but focussed on the LHS of the original equation. "I am **concerned about that sign ($-x$)**, okay, because I am thinking **they**

are like together (the minus symbol and the letter x)". Thus, she apparently couldn't see the connection between $-x = 11$ and $x = -11$, but was aware of the sign attached to x. Her language of "together" suggests that she cannot see a way to separate the minus symbol from x.

Based on our experiences with negative coefficients in session 3, I gave her (B4) $16 - 7x = 9$. Her working for this item is included in Figure 7.18. She immediately divided the LHS by negative 7 and typed $\left(\frac{-7}{-7}\right)$ into her calculator and paused at the answer of 1, but left the LHS as $-x = 1$.

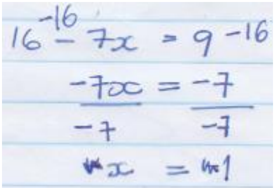


Figure 7.18: Daniele's working for item B4 in session 4

This answer of 1 on her calculator prompted a discussion on the coefficient of $-x$. She appears be focussed on the LHS simplifying to $-x$.

UTTR	SPKR	TRANSCRIPT
57	AH	So tell me why you thought to do this (divide by -7 on LHS) immediately? But then what stopped you from putting it ($\div -7$) here (on RHS)?
58	DR	**I should not have put that** (negative attached to 7) **there** (in the denominator on LHS) because, because I am trying to get rid of ...(paused)
59	AH	Yes? Trying to get rid of?
60	DR	The 7, but it doesn't**. Isn't negative $1x$ and negative x the same thing**?
61	AH	Okay so you are saying to me, negative $1x$ and x are the same thing?
62	DR	And negative x.
63	AH	Okay.
64	DR	**Because I know there is always an invisible 1 there**.

While saying (66) Daniele wrote down $-x = -1$. But then she checked the RHS on her calculator and changed the RHS to positive 1. Even though her final line was $-x = 1$, she checked this answer of on her calculator as $16 - 7(1)$ and got the 'answer' she wanted of the RHS as 9. This appeared to be the same mistake she was making earlier with $-x = -11$.

I then probed her to compare the LHS and the RHS. She had simplified $\frac{-7}{-7}$ on RHS as 1, but had left the LHS as $-x$, even though the line above was $\frac{-7x}{-7}$. She then recalled that "a negative and a negative make a positive". I asked if it was different when we had an x, to which she replied it wasn't and crossed out the negative in front of x.

After she had brought up the coefficient of negative 1 when faced with $-x$, I returned to the earlier item (A5) $17 - x = 28$. I pointed out the $-x$ and asked her about the 'invisible 1' she had recently referred to.

UTTR	SPKR	TRANSCRIPT
103	AH	Isn't this $(-x)$, is this… so you said to me, you know when the x is by itself, there should be an invisible 1?
104	DR	Ja.
105	AH	So here $(-x)$, what's the, what goes in front of the x? **We have a negative 1 don't we**?
106	DR	Ja.
107	AH	Is that different to having negative 7?
108	DR	No.
109	AH	You know like if I compare those two lines there ($-7x = -7$ and $-x = 11$), so you have here a negative $7x$ equals negative 7 and you very quickly said, okay, I need to divide by negative 7?
110	DR	Mmmm.
111	AH	So here if we have negative $1x$ equals 11, **how would we undo negative 1**?
112	DR	You divide it by negative 1.

Possibly in this interaction I led her too much in (105) and (111). However, based on our lengthy discussion, I felt that she had some of the concepts – inverse operations, a coefficient of negative 1, and integer rules but was struggling to connect them all within this procedure. And she had shown in session 3 that she was capable of making the connections. However, it is striking to note the similarity to the previous two sessions and how this notion of dividing by a negative 1 coefficient is evidently quite fragile in her mind.

7.5 Conclusion

In this chapter I have discussed three learners and their difficulties with equations where the letter has a negative coefficient, i.e. $5 - 3x = 8$. My initial suspicion was that errors in this type may be due to the minus symbol. This appeared to be the case with Tebogo, as she made the same errors when x had a coefficient other than 1. However, both Edith and Daniele's difficulties were confined to equations without an explicit coefficient, i.e. $5 - x = 8$. At times, they both showed they understood that there was an 'invisible' 1 in front of x, but that this entailed an extra layer of difficulty for them.

Edith was aware of the 'invisible' -1, and was able to operate with inverse operations by dividing both sides of the equation. However, she also introduced the curious aspect of dividing by x on the LHS, and that this was only present when faced with $-x = \dots$ as she felt that x was already isolated. Tebogo showed the most progress in solving these types of equations, but struggled to make meaning of her answers until the final session where she explained that subtracting a negative results in a larger number. Daniele displayed the most difficulty with equations of this type. Her difficulties were due to trying to match her 'algebraic working' with what she knew the solution to be (either due to inspection or systematic substitution).

In the following chapter, I analyse the final type of equation which caused learners difficulty, namely equations of the type $ax = bx$. This type of item does not feature constants on either side of the equals sign. It is apparent that Edith, Tebogo and Guban treat this type of item differently to items of the type $ax + b = cx + d$.

Chapter 8 : Difficulties with equations of the type $ax = bx$

8.1 Introduction

As was discussed in chapter 4, an item of the type $ax = bx$ was originally included in row C of the interview task matrix. Thus it originally fell under the broader set of equations $ax + b = cx + d$, but where c and d were zero. This item was included to see what learner would do without explicit constants. In addition, Sanders (2017) had found that learners could generate this form of an equation after making conjoining errors, e.g. given the equation $3x - 1 = 7x - 2$, a learner could make a conjoining error which results in $2x = 5x$. And while this item is not typical within South African Grade 9 textbooks, this type of 'extreme case' has proved to be illuminating in revealing a learner's understanding of equations and the resiliency of their approaches.

As was shown in the progress diagrams in chapter 5, $ax = bx$ became an obstacle for three of the four learners who encountered it. It became apparent that learners who were not experiencing difficulties with other equations with letters on both sides were experiencing difficulties with this item.

Two of the learners (Daniele and Yusuf) did not successfully solve the first item in row C (namely C3) and hence did not encounter this from (C5) along row C during this study. However three of the other four learners experienced difficulty to varying degrees. One learner (Sarah) experienced minor concern not worthy of discussion. In this chapter, I discuss the difficulties that Edith, Tebogo and Guban experienced with items of the form $ax = bx$.

8.2 Edith

Edith only encountered (C5) $3x = 4x$ in her second session because she was experiencing difficulties with 'simpler' equations of the type $ax + b = cx + d$ in session 1. When she encountered (C5), she said that she recognised the question from class and her textbook, but wasn't sure what to do with it and asked for help. She said she wasn't sure what to do in this item, but that she would try. In Figure 8.1, her working out shows that she tried substituting 1 in place of x, but then realised that this results in $3 = 4$, which is not true, and thus crossed out. Below that, is my writing $3 \times \square = 4 \times \square$, which was an attempt to prompt her into inspection by suggesting she read the equation as "3 times a number is the same as 4 times the number". This prompted her to substitute 4 into the block on the LHS, and 3 into the block on the RHS.

While this shows a relational view of the equals sign, it is problematic in the box representing a single unknown number.

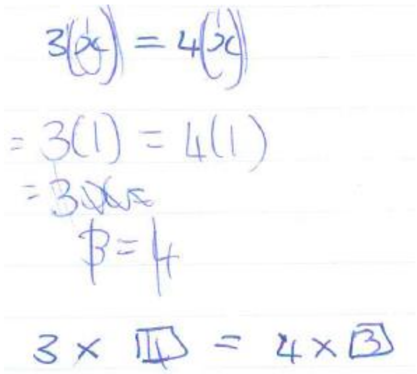


Figure 8.1: Edith's working for item C5 in session 2

In the transcript below, she explains her uncertainty of her approach and why she stopped at $3 = 4$. I also prompt her (from 78 to 82) with the item $3 \times \square = 4 \times \square$ (as shown in Figure 8.1), which she sees the relation to $3x = 4x$ but cannot use it to use inspection to think of the number 0 which would satisfy the boxes.

UTTR	SPKR	TRANSCRIPT
76	EM	I'm thinking... **Maybe if I substitute 1 here** (LHS)**…. I'm not sure, might be very, very wrong. I'm thinking if you do this.. What you do to the one side you do to the other side…**
77	AH	Explain to me what you're doing… why you stopped there.
78	EM	Because I wouldn't… **now I don't know the value of x. Because if I'd had to write it like that** ($3x =$)**, it would be way easier… like this… I'm clueless (laughing)**
79	AH	I'm just want to try something new here, what if I wrote it like this? If **I read it to you as, 3 times a number is the same as 4 times a number? Does that make you think any differently? Is there such a number?**
80	EM	**Three times… because that** (coefficient)**, this here, means times, you just put another block... It** (block) **means the same thing** (as a letter)**.**
81	AH	So do you think there's any number when you times it by 3, it's the same as timsing the number by 4
82	EM	**I would look for the LCD maybe… or the multiples… ja…** (silence)

Her uncertainty with her approach is evident in (76) and she explains her initial approach would be to substitute $x = 1$. However, during (79) she has stopped at $3 = 4$ because (in 78) there is no x left. She continues to try to explain what would make the equation easier (78) presumable that $3x = \textit{number}$ rather than another letter, but admits that she is "clueless". From (78) to (82) I try prompting her with $3 \times \square = 4 \times \square$ (as shown in Figure 8.1). She understands (80) that the boxes are the same as the letters and that the operation for $3x$ is multiplication. But she appears to be grasping for ideas (82). This grasping at ideas is discussed in section 9.2.2 as an example of learners operating with spontaneous concepts when they don't have the scientific concept fully internalised. After (82), she wrote down $3 \times 4 = 4 \times 3$ (shown in Figure 8.1), but due to time constraints, I did not pursue this problematic substitution of different numbers in each box.

It is also notable that she has changed her successful approach to solving equations with letters on both sides when faced with $ax = bx$. By session 2, Edith was operating on the letter, using the inverse operation of the letter on the RHS in order to collect letters on the LHS. She does not use this 'same operation on both sides' approach, rather starting to substitute values for x to see if it 'balances'. I gave her the more 'difficult' (in the following row) item (D3) $7x + 23 = 12x - 12$, which she correctly solved. This contradiction in expected difficulty prompted me to think that the unfamiliar nature of this item (without constants) may be seen as an entirely different sort of equation for learners.

In session 3 it became clearer why she wasn't comfortable using the same approach as she had in questions of the type $ax + b = cx + d$. Figure 8.2 below shows her response to item (C5) $5x = 2x$ and her response to (C3) $5x + 4 = 2x + 1$. In the discussion which followed, I was comparing her responses to these two items.

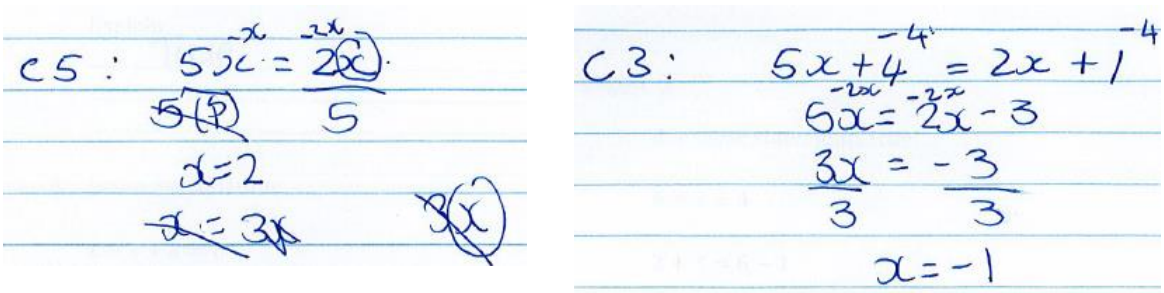


Figure 8.2: Edith's working for items C5 and C3 in session 3

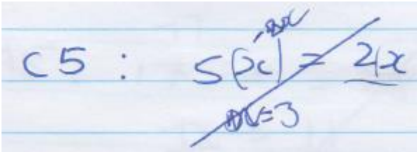
We spent almost 3 minutes discussing her response to (C5) $5x = 2x$. She said that she recalled this question from the previous session, but could not remember what to do. She reverted to her 'inverse operations' method of dividing both sides by the coefficient of x (with letters on both sides of the equation in session 1 – Figure 6.1), but said she wasn't sure what to do with $\frac{2x}{5}$. I asked her if this was different to previous question she had answered successfully, namely C3 (on the right of Figure 8.2). She said that it was similar, but also different. I asked her what line in the previous question it was similar to, and in the conversation which followed it become clear that her concern was what would happen on the RHS if she subtracted $2x$ from both sides.

UTTR	**SPKR**	**TRANSCRIPT**
92	EM	Line 2 - I just minussed from both sides here *(in C3, she minused $2x$ from both sides)*
93	AH	And you don't think you can do that here *(C5, line 1)*?
94	EM	Probably could**… it would be equal to $3x$** (LHS). But then**… I need to get the value of x**… the thingy….
95	AH	So how did you get this (the small minus x on LHS), what were you thinking then?
96	EM	Nah, I was thinking that maybe I could get like… minus from this side, minus from this side… but that would give me $3x$. And then I wouldn't be determining that (x)…
97	AH	What's left on the other side? (RHS)
98	EM	**Nothing.. Like.. This is why I didn't't…**

Utterance (98) is notably revealing. She appears to think that by subtracting the $2x$ from the RHS, because there is no explicit number as a coefficient, she would be left with 'nothing'. Furthermore, this value of zero on the RHS may be more difficult to see as zero, since the zero results from the subtraction of letters. It is not $2 - 2$ which results in 0, but rather $2x - 2x$ which equals 0. Thus it would appear she doesn't see that it is possible for zero to be the constant on the RHS.

I originally had included the item (B6) $20 = 2(x + 10)$ in session 2's interview task. This item had a solution of $x = 0$ and resulted in a constant value of zero (on either side) that came from the 20 of the LHS, and the 20 on the RHS, so long as learners correctly distributed 2 to $+10$. This item was not originally designed with the intent of comparison to C5. However after Guban had encountered it in session 2 (discussed later in section 8.4), I realised that it was valuable to compare learners' response to $20 = 2(x + 10)$ to their difficulties with $5x = 2x$ as it became clear they may be slightly more comfortable with a zero which resulted from constants, rather than a zero from $2x - 2x$. Edith had not yet encountered this item and thus in session 4, I gave her $20 = 2(x + 10)$ as item B5. Her response to B5 is included on the right in Figure 8.3.

In session 4, Edith had already tried and got stuck with item (C5) $5x = 2x$ again. Her working out on the left in the Figure 8.3 shows similar difficulties (attempting inspection with $x = 3$, and a tentative attempt to subtract $2x$ from both sides) involving zero. The working on the right shows her eventual solution of $x = 0$ to item B5, though not without difficulties along the way.



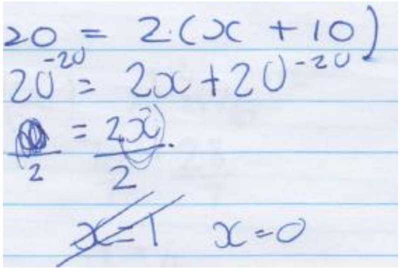


Figure 8.3: Edith's working for items C5 and B5 in session 4

In her initial attempt, she crosses out the zero, because she says "you can't divide zero by…" (but then trailed off). I asked her what she was dividing by, and she said 2. She again replied that it wasn't possible, but then tried to do it on her calculator and got an answer of zero. I asked her if that was possible, for x to equal 0? She seemed a little confused still, but managed to check her solution by substitution and replied that it was true – that both sides had a value of 20.

I asked her if she wanted to try C5, i.e. $5x = 2x$ again now that she had seen that zero can be the value on one side of the equation. In the following transcript, I am asking her to compare her method for B5 while working through C5 again.

UTTR	SPKR	TRANSCRIPT
164	EM	C5? Okay… Should I do it the exact same way I did here (pointing at working for B5)
165	AH	Well… what would you… **what's different between these questions is that we've got no constants here (C5), that's what tricky right?**
166	EM	Ja.

167	AH	But what were you doing when you had two variables and one constant?
168	EM	**I made them (variables) go to the same side.**
169	AH	So could you do that here (C5)?
170	EM	**I could minus $2x$ from this side and minus $2x$ from this side… gives me $3x$…**
171	AH	On which side?
172	EM	**This side** (LHS)**.**
173	AH	And what's on the other side then?
174	EM	**Would it not be zero?**
175	AH	Do you want to try? See what happens… That's what happened here (B5), 20 minus 20 left us with zero…

Thus it's clear that she now realises that it is possible to have a zero on one side, which is not the same as the 'nothing' she referred to in session 3. This zero also results from the subtraction of letters ($2x - 2x = 0$) with which she apparently is coming to terms. I asked her if the equation balances (using her words) if $x = 0$? She replied that both sides would be zero, and thus it would balance.

In conclusion, when Edith again gets stuck with an item that is either unfamiliar to her, or where she can't remember what to do, she resorts to trying to do 'something that may help'. I argue that Edith initially appeared to have a notion of zero as 'nothing' which was problematic when it came to solving $ax = bx$ in session 3. However, in session 4, she first encountered an equation where the subtraction of constants ($20 - 20$) resulted in zero. This appeared to prompt her to use zero as a number. She then needed confirmation that zero divided by a constant is zero. Thus, she came to see that the solution to an equation could be $x = 0$ and hence she was able to solve the equation $5x = 2x$.

8.3 Tebogo

Tebogo did not encounter this type of question in sessions 1 and 2, as she was experiencing difficulties with equations with letters on both sides. In session 2, she did eventually overcome these difficulties but there was not sufficient time in the session to give her an item in the form $ax = bx$.

However, in session 3, she was more confident with equations with letters on both sides (section 6.3) and thus I gave her the item $5x = 2x$. She studied the question for 42 seconds before writing anything down. She then made her first attempt but stopped at $2{,}5x = x$, shown in Figure 8.4. In this first attempt she has divided both sides by 2. She then reattempted the equation on the right, this time subtracting $2x$ from both sides, but again stopped on the second line at $3x = \ldots$ (not writing anything on the RHS).

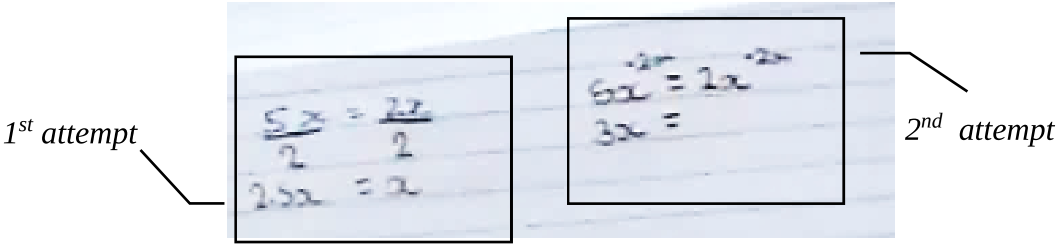


Figure 8.4: Tebogo's two attempts at item C5 in session 3

She then expressed her uncertainty at what to do with this equation and the following transcript reveals that, like Edith in section 82, it appears that the lack of constants (numbers) is at the root of her concern.

UTTR	SPKR	TRANSCRIPT
72	TB	Ja, I am not sure about this one here.
73	AH	What are you thinking there, what can do you?
74	TB	Because like it is not like you can ***depend on another number*** like in this case *(referring to the LHS of C5, implying the lack of a constant)* .
75	AH	Uhuh, uhuh.
76	TB	It's not like you can, like, ***there is a plus one over here*** *(pointing to the RHS, implying a missing constant)*, so you can like you know, so basically it's kind of like a tricky thing so say for example you have to like minus 2 from 5 over here, ***then when we have a number here (on the RHS)*** or say if you had to like, I am not sure about this one.
77	AH	So, so what are you saying, there is no number left here (on the RHS)?
78	TB	Well ***maybe there is but, I am considering putting 1 (writes 1 on empty RHS)*** but like I am not exactly sure.

In her first attempt (on the left of Figure 8.4) she reverts to an approach she had earlier employed (in session 2, Figure 6.7) with equations with letters on both sides – that of using division of both sides by the coefficient of the letter in order to 'undo' that side (Figure 6.7). She had not been using this 'divide both sides' strategy in this session with equations of the type $ax + b = cx + d$ in session 3.

In her second attempt, she is using the approach which she had previously used in session 3 (with equations that had letters on both sides – Figure 6.9) by subtracting $2x$ from both sides. However, she had left the RHS empty. In (76) above, the 2 and 5 are referring to $5x - 2x$, which she knows simplifies to $3x$. However, she in then clearly concerned at what the ‘answer’ to $2x - 2x$ is in (78). In (74), (76) and (78), she is expressing her need to have a constant on either side of the equation. Thus, like Edith earlier, Tebogo appears to be uncomfortable with the notion that $2x - 2x = 0$, which would result in 0 on the RHS. She does not explicitly mention ‘nothing’ as Edith had, but choosing a 1 may imply the concern that ‘nothing’ cannot be left over. Indeed, the need for the constant appears to result in her putting a 1 on the RHS (78). But she could not continue from there.

Thus she appeared to be experiencing similar difficulties related to zero as had Edith. Hence in session 4, I gave her the item (B5) $20 = 2(x + 10)$ which resulted in both a constant of zero on one side (resulting from $20 - 20$) and thus a final solution of $x = 0$.

In session 4, Tebogo appeared uncomfortable in solving this item. In her working on the left of Figure 8.5, she had paused before writing 0 on the LHS and then stopped at the line $0 = 2x$ (indicated by dashed line). She then reattempted the item on the right of that page, changing her strategy to dividing first by 2 (although she did not divide 20 on the RHS by 2. This gave her the constant of 10 on the LHS, which she continued through to get a solution of $-10 = x$.

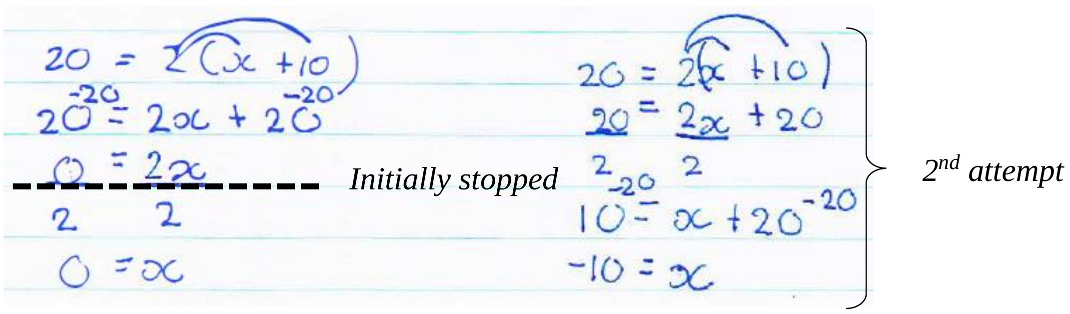


Figure 8.5: Tebogo’s two attempts at item B5 in session 4

She then checked this solution of $x = -10$, typing $2(-10 + 10)$, which gave her an answer of 0, not 20 (as on the RHS). I asked her why she had stopped at $0 = 2x$ in her first attempt. She then added in $\frac{0}{2} = \frac{2x}{2}$ to her first attempt, but explained that she was afraid that $\frac{0}{2}$ was ‘undefined’. I asked her if she could put $\frac{0}{2}$ into her calculator, which she saw that $\frac{0}{2} = 0$. She then carried on to her solution of $x = 0$. I asked her if that was 'possible'? She said it 'actually is' – and performed the 'check' (in Figure 8.6) where she substitutes x with 0.

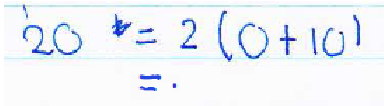


Figure 8.6: Tebogo's checking item B5 in session 4

Although she didn't write anything on the line below, she checked the RHS on her calculator to get an answer of 20. Thus she felt confident that $x = 0$ was correct.

In this session, she again struggled with the equation (C5) $5x = 2x$. I asked her if she couldn't use the same approach which she had used to correctly solve (C3) $7x + 4 = 2x - 1$? Her working and checking of her solution are shown in Figure 8.7.

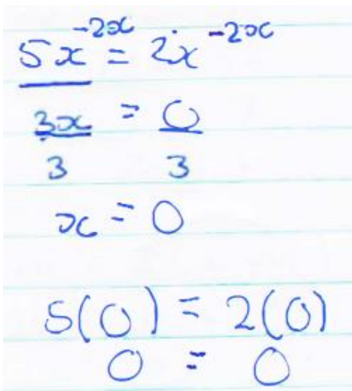


Figure 8.7: Tebogo's working for item C5 in session 4

Her working shows her correct solution and ability to check that solution, however the following transcript shows it was a rather gradual realisation that she could have zero (nothing) left over on one side.

UTTR	SPKR	TRANSCRIPT
111	AH	Could you use, I mean here (C3) we also have the, the x' s on both sides, so could you use what you did here (C3), there (C5)?
112	TB	I could but then it means that like ***one side would not have a variable***, I mean, ja.
113	AH	There would not be anything there (on the RHS)? But hasn't that happened before? *(referring to item B5 discussed above)*
114	TB	Mmmm, ja.

115	AH	Didn't that happen over here (B5)? Where is it here now, with this one here (C5)?
116	TB	Mmmm.
117	AH	That (zero on LHS) happened right?
118	TB	Ja. Maybe it could work (continues working).
119	AH	Do you think about that (her final solution of $x = 0$) in the original question?
120	TB	It is not ...
121	AH	Could that work?
122	TB	Ja, it could work.
123	AH	Explain?
124	TB	Because, I am ***not sure how to explain this but it could work***. There is a possibility that it could work.
125	AH	Like, I mean how would you, could you check it like this (her checking by substitution in the previous equation)?
126	TB	Ja, I think so.
127	AH	What does that (her check in Figure 8.7) mean? Is that balanced?
128	TB	Ja.
129	AH	Why?
130	TB	***Because, like both the, like both numbers still equal the same thing.***
131	AH	***You mean the zero?***
132	TB	Ja.

Tebogo and Edith had both experienced difficulties in accepting that zero can be the constant on one side of an equation, and that $x = 0$ is a valid solution to an equation. The item $20 = 2(x + 10)$ resulted in constants of 20 on either side but was more familiar to Tebogo and Edith, and appeared to support them in making sense these two issues - that of zero is the resultant constant on one side, and a final solution of $x = 0$.

8.4 Guban

Guban had not experienced any of the difficulties with letters on both sides, nor $-x$, that Edith, Tebogo and Daniele had experienced. However, in the first session, he paused when faced with $3x = 4x$, seemingly unsure of how to continue. His working is shown in Figure 8.8. I asked him if this item felt more difficult, to which he replied "Well, you're going to have to subtract

($3x$) and that's going to give you a zero over here (on the LHS). And then dividing it ($1x$ on RHS) is going to be… is going to be more difficult.” He does not seem to share the notion that ‘nothing’ will remain on one side, but rather his concern appeared to be dividing with (or by) zero.

I encouraged him to try the item. He made two similar attempts, shown below in Figure 8.8. His method for solving these equations was different to that which he used on an item such as (C6) $4 - 3x = 5x - 4$.

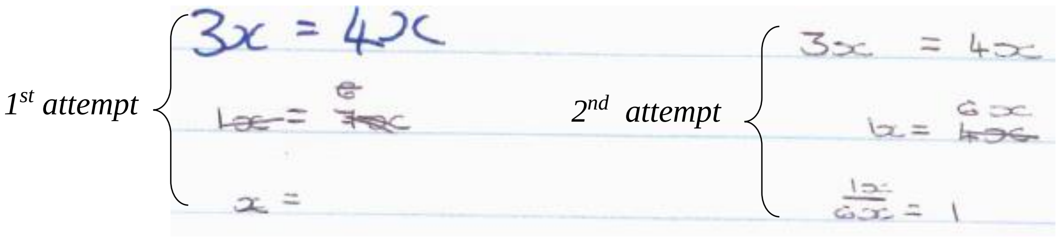


Figure 8.8: Guban’s two attempts at item C5 in session 1

In both of his attempts, he appears to be operating with the letters by adding and subtracting. And again, in both attempts, his second line is $1x = 6x$, which would appear to be some idea of subtracting $2x$ from the left which is then added to the right – like Kieran’s (1992) redistribution error. He did not make this error in other equations with letters on both sides. His initial $7x$ on the RHS of his first attempt (scratched out) could be from adding $3x$ to the $4x$, but then realising he would be left with ‘nothing’ on the LHS. When I asked to him to explain his working, he didn’t explain this compensation (redistribution) but did give some insight into why he left $1x$ on the LHS.

UTTR	**SPKR**	**TRANSCRIPT**
20	GZ	Well, I divided both sides by $6x$ to get rid of the x on this (LHS) side.
21	AH	Sorry. I meant to get from this line (line 1) to this line (line 2)?
22	GZ	Oh from this line to this line? **I subtracted 2. So that there was still an amount left on this side, sir.**
23	AH	Okay. But then you got to here and you **divided both sides by $6x$.**
24	GZ	Yes sir
25	AH	And do you recognise what this side **would simplify to?**
26	GZ	Umm, no sir.

27	AH	Okay. Thank you!

In (22) he states that he subtracted from both sides, even though the $6x$ would result from adding $2x$ to $4x$. However, he also indicates that he only subtracted $2x$ in order that there would still be "something left" on the LHS. As mentioned earlier, he appeared to be concerned about being left with zero, and then needing to divide, possibly concerned about division by zero being undefined.

Unfortunately, this item was at the end of the first session and I ran out of time to pursue his thinking further. In session 2 however, it was clear that he was still experiencing difficulties with equations of the type $ax = bx$, and we could spend more time on it.

Again in session 2, Guban had not experienced difficulties with equations with letters on both sides. That is until I gave him the equation (B6) $20 = 2(x + 10)$. As discussed earlier in section 8.2, this item was helpful as a comparison as it provoked zero as a constant (from $20 - 20$) and a final solution of zero. Guban had used his standard approach of collecting letters on one side and constants on the other. He wrote down his final answer of $x = 0$ but then paused. I then probed his thinking as to what his concern was.

UTTR	SPKR	TRANSCRIPT
1	AH	Okay, you paused for a little second there (at writing $x = 0$). What were you thinking?
2	GZ	Oh, I was thinking that x can't be zero. But then it is.
3	AH	Do you think it can be zero?
4	GZ	Yes, it can.
5	AH	Can you explain why it can?
6	GZ	Because **zero is a number**, just like any other number. And x is the variable that represents that number.

It is clear from (6) that Guban does not share the 'nothing' meaning to zero which Edith and Tebogo initially had. Although he may have been initially concerned, he appears to have reasoned that it is possible for $x = 0$. Utterance (6) appears to imply that he thinks that his answer it correct, but the conversation was still theoretical as to whether x *could* be zero. Utterance (2) suggest that Guban thinks it is zero. However, I still asked Guban to 'check' his solution, where he was satisfied that zero was the correct solution to the equation.

However, he still appeared uncomfortable while working out (C5) $3x = 4x$, pausing at times, and crossing out the 1 (on the LHS of Figure 8.9 below) before continuing, as his working shows.

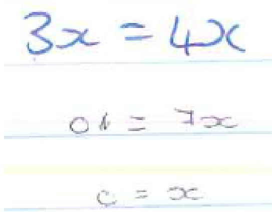


Figure 8.9: Guban's working for item C5 in session 2

On the LHS, he initially wrote 1, but then crossed it out. He also made an error in adding $3x$ to $4x$. Again he had not made this error in other equations with letters on both sides, and is similar to the 'compensation' error he made in session 1 where he subtracted from the one side and added to the other. Indeed, in the transcript of his explanation below, he again says that he subtracted, even though he would have added to the RHS. Although his working out is actually correct, the transcript below shows that he thinks that his answer to be incorrect, due to dividing by zero. He also explains where the one had initially come from.

UTTR	**SPKR**	**TRANSCRIPT**
20	AH	Okay, can you explain what you've done? And why you crossed out, and what you were thinking doing this one?
21	GZ	Oh no, because I subtracted these ones (the LHS). I was thinking of dividing. **And if you divide $3x$ by $3x$, it will give you one.** But I was **subtracting and then it gave me zero**. So then I divided **<u>$7x$ by zero</u>** and then… oh no. **That's undefined**.
22	AH	Why do you say that? Can you explain what you're thinking?
23	GZ	Oh, **zero divided by any number is undefined**.
24	AH	Okay. Do you feel like this ($x = 0$) is the correct answer?
25	GZ	No.
26	AH	Explain why.
27	GZ	Because **you can't divide anything by zero** and still get zero. And **I divided by 7**.
28	AH	**Okay, so I'm hearing you say you divided by 7. But then you're also saying to me that you divided by 0?**

29	GZ	Yah… Oh no, **I'm saying I divided by 7,** but **I still got 0**, which is… you can't **divide a number by 0** and still get zero.
30	AH	Okay, so do you think that this ($x = 0$) is wrong here (imply sub into original question)?
31	GZ	Yah, I think it's wrong (paused to think).
32	AH	Do you want to think about it? What are you thinking?
33	GZ	Well the **equation could still work if the variable was zero.**
34	AH	Okay, explain why.
35	GZ	So **3 times 0, which would be 0 is equal to 4 times 0, which is zero.**
36	AH	Okay
37	GZ	Yah, so… what are you thinking?
38	AH	I think my **answer is correct but I don't think the method I used is correct.**

Guban firstly explains in (21) where the 1 had come from – that dividing $3x$ by $3x$ is a 1. He appears to be using a similar idea as in session 1 (with $ax = bx$) of dividing both sides by the same thing in order to isolate one letter. Although Guban ended up with the correct answer, he appeared concerned at the end of his working. Guban's concern appears to emanate from division by zero from his utterances in (27) and (29). However, his language fluctuates between saying he divided by 7 and that he divided by 0. Thus he shares Tebogo's uncertainty as to whether $\frac{0}{7}$ or $\frac{7}{0}$ is undefined and order of what he dividing by.

Guban is however able to interpret $x = 0$ as the correct solution, as he explains in (35) that the two sides are balanced when $x = 0$, that is that both sides have a value of 0. He doesn't appear satisfied with his method, even though it produced what he understands to be the correct method (38).

Unfortunately, Guban was not able to make the third session. So the next opportunity I had to pursue his understanding of $ax = bx$ was 4 months after session 2 (Figure 4.4). In session 4, he had again solved (B5) $20 = 2(x + 10)$ correctly. However, when faced with (C5), $5x = 2x$, he again experienced difficulty. His working in Figure 8.10 below shows two attempts made, each with different approaches.

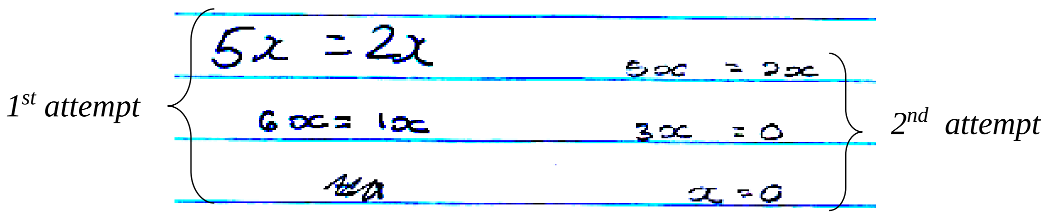


Figure 8.10: Guban's two attempts at item C5 in session 4

In his first attempt, he appears to have reverted to his approach that he used in session 2 – that of not leaving a zero, so he 'compensates' by subtracting $1x$ on the RHS and adding $1x$ to the RHS, again making the redistribution error.

Then he stated that he was trying to use division again to remove the x on the RHS (the 'one' from session 2), but stops short because it "didn't work" and thus crossed out in the final line. As he had solved (B5) $20 = 2x(x + 10)$ earlier, that also contained zero on the RHS and division with zero $\left(\frac{0}{20}\right)$, I asked him to compare items B5 and C5.

UTTR	SPKR	TRANSCRIPT
9	AH	Okay, how are these questions (B5 & C5) different to you?
10	GZ	Okay, this **one (B5) has constants in it**, and **values** which you can **use to work out x**. And this one (C5) does not.
11	AH	Okay, so this one (B5) has x's on both sides… so how did you … you subtracted $2x$ from both sides? So what would happen if you did that here (in C5)? Can you do that here?
12	GZ	I think so.
13	AH	Can you do that (pointing at $5x$ on second line of working in B5) here (in C5)?
14	GZ	Ummm
15	AH	Do you want to try?

Again, like Tebogo and Edith before him, Guban appears to share the concern of the lack of constants (10). This is in spite of him not appearing to have this concern in either session 1 or 2. Because I knew that Guban had previously used subtraction of the letters from both sides, I was direct in prompting him with this idea in (11) and (13).

This prompted to him make his second effort in Figure 8.10. This time he has actually subtracted $2x$ from sides (not his redistribution method from session 1 and 2), which has allowed him to get to an answer. Again, I asked him to check his answer and explain whether or not he thinks $x = 0$ is correct.

UTTR	SPKR	TRANSCRIPT
17	AH	Okay, so you get an answer now. Do you think that that's correct?
18	GZ	I think so.
19	AH	Why?
20	GZ	**So, if you put the variables on the one side, you can get $3x$. And then $3x$ divided by zero is zero.**
21	AH	Okay, but does this work back in the original equation?
22	GZ	Yes, yes…
23	AH	Why?
24	GZ	**Because if you substitute the x with a zero, 5 times zero is zero, which is equal to 2 times zero, which is also zero**

Although Guban finally got to the correct solution in session 2 and 4, it is clear that items of the form $ax = bx$ still caused him concern in session 4. His initial thought in session 4 was still to leave a term on the RHS (Figure 8.10). However, Guban then also appears to have progressed in his reasoning about his main difficulty – dividing by zero (utterance (20) above). He also appears more confident (24 above) in justifying that his solution of $x = 0$ is valid.

8.5 Conclusion

The extent of the difficulties experienced by learners with items in the form $ax = bx$ were surprising to me. I originally treated $ax = bx$ as a special case of $ax + b = cx + d$, and I did not expect it to present difficulty for as many as three of the four learners who encountered it. Sanders (2017) had noted learners attempts to make the item ($ax = bx$) more canonical by ignoring the x on the RHS. However, it became clear that for three of the four learners who encountered it (excepting Sarah), it brought up some new, and noteworthy challenges. At the root of these challenges appeared to be the lack of constants. Thus Edith and Tebogo were both concerned with what was left on the one side after subtracting the term with letters. As was noted in the introduction to this chapter, items of the type $ax = bx$ are not common within textbooks. Equations almost always have a constant on one side of the equation.

The item $20 = 2(x + 10)$ proved useful as a comparison for learners. The prompting which came along with this item, created a ZPD in which learners could show progress. This perspective of a ZPD allowing progress to be shown for Edith, Tebogo and Guban is discussed in section 9.2.3 in the following chapter. In this item, they had numbers (20 on each side after distribution), but they ended up with a zero (normally on the LHS) after subtracting 20 from both sides. While Tebogo initially had some difficulties, this item appeared to be more canonical (McNeil et al. (2006)) and thus learners arrived at the solution with fewer difficulties than $ax = bx$. This solution of $x = 0$ is also not common within linear equations, but was slightly easier for learners to check by substitution - that both sides were equal to 20. Thus they could accept that an acceptable solution for an equation could be zero.

Another challenge that this type of equation elicited was that of division by zero, or rather zero divided by another number. Both Tebogo and Guban were concerned that they were dividing by zero which is undefined. Tebogo overcame this through checking $\frac{0}{2}$ on her calculator. Guban didn't use his calculator and was apprehensive of his method despite understanding that $x = 0$ satisfied the original equation.

In the following chapter, I present a Vygotskian perspective on the previous three chapters' data analysis. While conducting the data analysis, I was aware of Vygotskian ideas that were helpful in making sense of some of the incidents which occurred. I have deliberately not discussed these Vygotskian ideas in the separate chapters. It seemed more appropriate to address these common ideas together in a new chapter.

Chapter 9 : Theoretical perspective on the data analysis

9.1 Introduction

In this chapter, I present a Vygotskian perspective on the data presented in the previous 4 chapters. Throughout those chapters, I made references to certain incidents which I argued a Vygotskian perspective would be illuminating. This brief chapter summarises those incidents in line with the Vygotskian notions of *internalisation*, *scientific concepts* and the *zone of proximal development*.

9.2 A Vygotskian perspective on the data analysis

9.2.1 Internalisation

To refer back to the discussion of Vygotskian theory in chapter 2, the process of internalisation refers to the learner's construction of their own knowledge through tasks mediated by signs. The externally located tool needs to become an internally located sign which is fully aligned with the codified mathematician's view. I would assert that this perspective can provide us with some understanding of the difficulties that learners in this study faced in overcoming the didactic cut and what enabled some of them to do so.

The didactic cut refers to the difficult transition from arithmetic equations to algebraic equation – with letters on both sides, i.e. $3x + 2 = 2x - 1$. Their existing sign (approach) for linear equations with one letter on one side is no longer helpful in these new items. They need a new approach. This new approach will be externally located for the learner (and with the teacher). Through semiotic mediation, this new approach needs to be internalised. However, as Daniels (2009) points out, this is not a simple mode of transmission – meaning is negotiated.

Thus there are two elements at play: the external sign of the new approach being internalised, and the nature of the internalisation. Both Tebogo and Edith are eventually successful in solving linear equations with letters on both sides. However, both admit to not remembering the correct method initially. Edith had overcome her difficulties after she had done some personal revision after the second session. But Tebogo needed prompting during the sessions with items in which the letters were on the same side of the equation, i.e. $7x - 2x = 10$. This suggests that this approach of 'perform the same operation on both sides' is still externally located for Tebogo. Some of the pieces are in place, she is able to perform the same operation on both sides with *constants*, however she requires external reminders (from me) of the notion of like terms – that she could add and subtract terms with letters.

As she said in the fourth session "I don't know, I dunno, it's most likely, something like I kind of forgot how to work out that. But ja, I like I always ***forget that both like, like these are both like terms***." For someone like Daniele, who did not overcome the didactic cut during this study, this required approach of 'same operation on both sides' appears to be located almost entirely externally, as even with prompting with the same example ($7x - 2x = 10$), she is still unable to use this approach. More will be said about this aspect in the discussion about the ZPD later in this chapter.

Berger (2005) also asserts that in order for a sign to be fully internalised, the sign needs to be aligned with the codified mathematician's understanding. I consider the sign as 'using the same operation on both sides to operate on the letter' approach. While it is not expected for Grade 9 learners to share the same full meaning of linear equations as fully-fledged mathematicians, I would assert that the full notion of internalisation does help us understand the role that the special case $ax = bx$ plays within the set of equations of the form $ax + b = cx + d$.

For a mathematician, the special case of $ax = bx$ is simply a subset of the larger group of equations with letters on both sides. The same approach is still used and $x = 0$ is a valid solution. However for the three learners discussed in chapter 8, this does not appear to be the case. All three learners had difficulties with this item that they did not experience with the more general items with letters on both sides. Hence I would argue that they still have not fully internalised the sign of 'same operation on both sides' approach for *all* cases of $ax + b = cx + d$, even when $b = 0$ and $d = 0$.

9.2.2 The shift from spontaneous to scientific concepts

In chapter 2, I discussed Russian Vygotskians' views that spontaneous concepts (approaches) can be the source of misconceptions and that rote skills are meaningless and non-transferrable (Karpov, 2007). Thus I expect that the errors noted in the data analysis could be due to spontaneous approaches that learners have developed, or to rote skills executed with little meaning, as opposed to scientific approaches that they have internalised.

When dealing with equations with letters on both sides, a new algebraic approach is required for learners to operate with letters on both sides of the equation. This is a scientific concept (approach). Spontaneous concepts would include informal approaches such as inspection and systematic substitution. However, I argue that learners reverting to arithmetic approaches with equations with letters on both sides can be seen as operating spontaneously – not adopting the generalised approach of operating on the letter.

It was noted with both Edith and Tebogo that, in their uncertainty with equations with letters on both sides, they could not recall the approach and therefore that they were 'trying stuff'. Initially both used inverse operations in trying to divide by the coefficient of the letter on the RHS of the equation (Figures 6.1 and 6.6). This is characteristic of a learner error due to an approach which is successful sometimes, i.e. when the equation has only one letter, but then leads to incorrect results when used in inappropriate contexts (letters on both sides). They had not internalised the scientific concept (approach) of 'perform the same operation on both sides'.

The item $ax = bx$ also appeared to draw on spontaneous reasoning from learners. Because this item appeared to be unfamiliar, they resorted to 'trying things'. Their language often conveyed this grasping at helpful ideas (approaches). Edith expressed the approaches at which she was grasping: "Maybe if I substitute 1 here (LHS)…. I'm not sure, might be very, very wrong" and later "I would look for the LCD maybe… or the multiples… yah". Tebogo's initial thought was to substitute a value of 1, also reverting to basic substitution. Daniele's unique 'arrow' method (Figure 6.14) could certainly be an example of a spontaneous concept as well. It appears to have the rote characteristics (procedure without understanding) of 'moving terms' over the equals sign (misapplication of the transposition model) which is not aligned with the codified mathematical view of equality.

Both Edith and Tebogo, as well as Guban, continued to revert back to inverse operations when experiencing difficulties with the item $ax = bx$. This is in spite of all of them being able to solve items of the form $ax + b = cx + d$. Because $ax = bx$ did not appear to fit within this same type of equation, they appeared to revert to arithmetical inverse operations rather than the scientific (algebraic) concept of 'perform the same operation on both sides'.

9.2.3 Establishing a zone of proximal development in the interviews

In this section, I examine two cases from the interviews, and examine those incidents in terms of a ZPD being created which allowed learners to show progress.

A Vygotskian perspective was argued for primarily as a result of the data being collected in task-based interviews. Thus learners were doing mathematics with a more knowledgeable other (myself) present. Although my intention was not to assist the learners, but rather to investigate the progress of the learners in their abilities and reasoning. Vygotsky (1978) introduced the notion of the ZPD in the context of determining the developmental level of children. Hence it is this notion of the ZPD that I would assert is helpful in understanding the progress of the learners in this study.

In the data analysis, there were a number of instances where learners displayed their potential progress within a ZPD in the interview setting. The first is the case of Tebogo's difficulties with the didactic cut. In session 1, while working on item (C3) $2x + 5 = x + 1$, she subtracted 5 from both sides to get $2x = x - 4$, but then was using inverse operations to 'undo' the $2x$ by dividing all terms by 2. She continued to show these difficulties in session 2, but later in this session I provided her with the item $4x - 2x = 10$ as an item which had two letters, but both were on the same side. This appeared to trigger her recall of 'adding like terms' and allowed her to go onto successfully solve other items in the form $ax + b = cx + d$.

While in the third session she did not have difficulties with the didactic cut, her difficulties reappeared in session 4. Again it was the prompting of an item with letters on the same side which appeared to help her recollection of adding like terms ($4x - 2x$ in Figure 6.8) and enabled her to negotiate the didactic cut again. Thus she appeared to be situated in a ZPD in terms of being able to operate with letters over the equals sign if prompted with an example with letters on the same side.

Tebogo's case can be contrasted with that of Daniele. Daniele consistently had difficulties with letters on both sides through all four sessions, but had been successful with instances of only one letter (arithmetic equations). When I gave her the item with two letters on the same side ($4x - 2x = 10$ in session 2, Figure 6.13), she was able to solve it successfully. However she did not transfer this knowledge to equations with letters on both sides. Hence I would argue that she was not in the ZPD in terms of getting over the didactic cut.

I would argue that for all three learners discussed in chapter 8, a ZPD was established between me and the learner relating to equations of the form $ax = bx$ in each interview. For each of Edith, Tebogo and Guban, the difficulties which they were experiencing appeared to involve 'nothing' left on one side, i.e. that 0 was the constant, and possible division with (not by) zero (Tebogo and Guban were concerned that $\frac{0}{7}$ was undefined).

The item $20 = 2(x + 10)$ was designed to bring up these same issues but in a more familiar context (where there were constants present). Through the interview process, I provoked their thinking about these two items, comparing $20 = 2(x + 10)$ with $5x = 2x$ and what the differences were. The item $20 = 2(x + 10)$ was more familiar (canonical) for the learners, and drew out an answer of 0 on the LHS (normally) from $20 - 20$ (Figures 8.3 and 8.5). This zero came from numbers subtracted $(20 - 20)$ as opposed to letters in $5x = 2x$ (where $2x - 2x$

would equal 0). They also had to confront difficulties with dividing 0 by 2, which Tebogo and Guban were concerned was undefined, but these difficulties were overcome by checking these calculations on their calculator.

My prompting their comparisons between $20 = 2(x + 10)$ and $5x = 2x$ appeared to enable them to get the correct answers for $ax = bx$, even if they still were not confident in their reasoning. However this provocation does point towards them being in some developmental state ready to be able to solve this type of item independently in the near future.

These instances provide evidence of learning in and through the interview. As the more knowledgeable other, I was involved in co-constructing knowledge with them in the interview. The interviews were not only access their 'stored' knowledge, but I was helping them (by prompting) to construct new knowledge in the interview. While learning is the process of internalising (interpersonal) these ideas (operating with letters on both sides of the equals, special cases of general forms), this learning can take place in an intrapersonal space, namely the interview. Thus we have a greater understanding of learner progress, not just what they are capable of independently, but what their potential progress is.

9.3 Conclusion

A Vygotskian perspective was an appropriate lens through which to look at the data due to the social nature of the interviews. Since Vygotsky was concerned with the development of learners' thinking, I was able to describe the nature of learners' progress from the data analysis. The Vygotskian notion of internalisation was useful in understanding the external location of the 'same operation on both sides' approach for Edith, Tebogo and Daniele. For Edith and Tebogo, their initial difficulties with the didactic cut were due to not 'recalling' (externally located) the appropriate method. Daniele did not successfully solve equations with letters on both sides during this study, and thus this approach did not appear to have ever been internalised.

The shift from arithmetic approaches for algebraic equations requiring a new approach was likened to the difference between scientific concepts (approaches) and spontaneous concepts. Learners who struggled with the didactic cut had not adopted the appropriate scientific approach. Daniele showed the most notable instance of a spontaneous approach with her 'arrow method'.

Finally, the task-based interviews allowed for a ZPD to be created in which learners could show their potential development (progress). There were two instances of ZPDs being established which were discussed. Tebogo was prompted with the item $7x - 2x = 10$, which had two letters on the same side. This helped her in overcoming her difficulties with the didactic cut by reminding her that she can add like terms (operate on the letters). The second instance of a ZPD being established was with learners' difficulties with the equation type $5x = 2x$. The item $20 = 2(x + 10)$ allowed me to prompt learners with a more familiar item that produced a value of 0 on the LHS and thus allowed them to later accept that 0 could be 'left over' after subtracting $2x$ from both sides in $5x = 2x$.

The following chapter will conclude this study by presented the findings of this study in relation to the original research questions. The research questions are answered by providing evidence from the data analysis from the chapters 5 through 8 and in the light of this Vygotskian perspective. Finally, I critically reflect on this study and provide some recommendations for the classroom and for further research.

Chapter 10 : Findings, reflection and recommendations

10.1 Introduction

In this final chapter, I present the findings of the study by answering the research questions based on the data analysis covered in chapters 5 to 8. After I discuss these findings, I present a critical reflection of the study and make some recommendations for teaching and for further research.

10.2 Findings

The findings of this study are presented under 5 headings which answer the original research questions using the data analysis from chapters 5 to 8. As a reminder, the research questions were:

1. What progress do learners show in solving linear equations over time?
    a. What progress do learners show in their abilities to solve linear equations?
    b. What progress do learners show in their reasoning by explaining their thinking?

2. What approaches do learners use?
    a. Do their approaches change over time?
    b. Do their approaches lead to difficulties?

3. What difficulties do learners experience?
    a. Do their difficulties lead to errors?
    b. Do their difficulties and errors change over time?

Thus, the first section discusses the progress that learners showed over time. The aspect of time is embedded within all three questions, and thus has its own section dedicated to it. The second section addresses the second research question which focussed on learners' approaches and how these led to difficulties. The final research question enquired into learners' difficulties and the errors they made. Thus, the last 3 sections address the 3 major difficulties noted in the data analysis: the didactic cut, negativity and $ax = bx$.

10.2.1 Learner progress over time

A unique aspect of this research was investigating change in learners' abilities and reasoning over time. In chapter 1, I argued for the use of the term *progress* to indicate the changes made over time even if there was an apparent lack of progress. The findings of this study must represent the complex nature of this progress. Some learners showed definite progress in their abilities and reasoning. Other learners showed some progress in their reasoning, even if their ability did not necessarily follow. And other learners did not show significant progress in either their abilities or reasoning.

Edith appeared to be the most notable in overcoming her difficulties quickly, and not regressing. Her difficulties with the didactic cut and negativity were confined to session 1, even if she continued to struggle to *explain* the need to divide by $-1x$ with items like $5 - x = 8$. Her difficulties with $ax = bx$ were more persistent however, only overcoming them in session 4 with and in-depth discussion after successfully solving $20 = 10(x + 2)$. While it is beyond the scope to look at *why* Edith improved so quickly, she had explained that after her difficulties in the first session, she had gone back over her work. This shows her productive disposition strand (Kilpatrick et al., 2001) which apparently aided her abilities in the following sessions.

For other learners, the progress they showed was gradual and not entirely incremental. The progress which they showed in one session may have regressed in some aspects in following sessions.

Tebogo was the best example of this type of progress. Her difficulties with the didactic cut were overcome with exposure to an item with letters on the same side in session 2. In session 3 she continued to show progress in not needing this provocation. However in session 4, she appeared to regress slightly with the didactic cut, again needed prompting with an item with letters on the same side. In chapter 9, I discussed this regression as possibly as a result of her not fully internalizing the approach to solving equations with letters on both sides. While she had the notion of 'same operation on both sides', she needed reminding of the idea to 'add like terms' in order to combine the letters down to a single term.

Tebogo also showed progression in her reasoning. While she overcame her difficulties with negativity in session 2, she still struggled to reconcile her inspection and algebraic approaches. Initially she was seeing the minus symbol in $10 - x = 18$ as difference (between 10 and 18) but could not see that x needed to be negative. However by the fourth session she explained that subtracting a negative is "like adding", showing that her reasoning had progressed in terms of the duality of the minus symbol (that it can be both sign (negative) and operator (subtraction)).

Guban also illustrated the complex nature of progress. Guban did not show the difficulties from the WMCS test which led him to being sampled, which may suggest rapid progress. But he experienced prolonged difficulties with $ax = bx$, despite achieving the correct answers. His difficulties lay in his uncertainty in his algebraic approach, which remained until session 4. He doubted his approach since he was concerned that he needed to divide by 0 (when faced with $3x = 0$). But he could reason by substitution that $x = 0$ was the correct solution to $5x = 2x$. Thus, his procedural fluency was fairly strong (achieving correct answers) in solving items of the type $ax = bx$. But his conceptual understanding was problematic in not seeing that he was dividing 0 by a number (which is not undefined) and thus he struggled to reconcile his approach with his conception of what he was doing.

The lack of progress was best illustrated in chapter 5 by Yusuf and Daniele. Both of their progress diagrams (chapter 5) show that they were not able to successfully solve equations with letters on both sides during the study. They were able to solve equations with only a single letter, or when the letters were on the same side. This supports Kieran (1992) and Filloy and Rojano's (1989) argument of the difficulty in transitioning from arithmetic to algebraic (structural) approaches. Yusuf showed the strongest evidence of maintaining an arithmetic approach. Daniele tried to shift approaches, moving to an arrow approach of transposing terms over the equals sign but not changing their signs.

Daniele did show progress in overcoming her difficulties with negativity (after a prolonged engagement) by comparing her approaches to items which had a coefficient as an explicit number (i.e. $-7x$) to items where the coefficient was not visible, i.e. $-x$ (which has a coefficient of -1). This engagement was discussed from a Vygotskian perspective in the previous chapter where it was show that she was in the ZPD of this item by being able to show progress in collaboration with me.

To summarize, the nature of learner progress is complex. Some learners do not show progress. But those who do, tend to do so in gradual steps and possibly regress. The relationship between progress in terms of abilities and reasoning is also complex. Some learners showed progress in their reasoning (Daniele) even if their difficulties remained. And others were procedurally fluent, but their reasoning lagged behind (Tebogo and Guban).

10.2.2. The role of the approach when solving equations

The second research questions was:

2. What approaches do learners use?
    a. Do their approaches change over time?
    b. Do their approaches lead to difficulties?

This question was addressed throughout the data analysis where there was relevance. The finding of this study is that some learners had a single approach they used throughout (Sarah, Guban and Yusuf) while others used an approach as long as it was useful (Daniele, Edith and Tebogo). If they encountered difficulties as a result of their approach, they would try and adapt their approach, some successfully and others not.

The learners Sarah and Guban appeared to have one approach and use it consistently throughout. In Kieren's (1992) language, this would be described as 'performing the same operation on both sides'. Guban doubted his approach with the $ax = bx$ items despite thinking that the answer was correct. Yusuf appeared to also use one approach: 'undoing or working backwards' but could also be said to be using 'use of number facts' (Kieren, 1992).

Daniele was a relatively unique case in this study in that she developed a totally separate approach for letters on both sides – her arrow method. This appeared to be related to the 'transposition' (Kieren, 1992) approach, although she didn't change sign of the term when moving it over the equals sign. Daniele would get the correct answers for arithmetic equations using her 'undoing approach'. But her change in approach with algebraic equations led her to difficulties which she couldn't resolve. She also saw this arrow approach as incommensurate with her 'undoing' approach. For her, different types of equations require different approaches. This is perhaps understandable since quadratic equations do require another approach. But she was not getting the correct answers, which I thought may have prompted her to reflect on her approach and my probing.

Edith and Tebogo also shifted approaches back and forth for different items. Initially, they shifted from inspection (number facts) to and undoing (inverse operation) approach. Inspection led to difficulties with the items with subtraction making bigger, i.e. $5 - x = 8$. But the undoing approach also presented difficulties in seeing $-x$ as $-1x$. The undoing approach was generally successful for them when the items had only one letter. Then they ran into difficulties with this undoing approach with equations with letters on both sides.

Edith and Tebogo continued to try to use inverse operations to operate on the letter in order to undo it. They both realised this approach was unsuccessful. Edith determined this through revision between sessions but Tebogo required prompting through an item with two letters on the same side, which reminded her of adding like terms. For Edith and Tebogo, they appeared to adopt an approach as long as it was useful. However, when it was no longer useful they were able eventually to adopt new approaches. This may be a significant experience to go through – to realise in mathematics some ideas may have limited merit – they work sometimes. However other approaches are more generalizable and can be used throughout a section.

The following three sections address the research question:

3. What difficulties do learners experience?
    a. Do the difficulties lead to errors?
    b. Do their difficulties and errors change over time?

There were three chapters devoted to the analysis of the difficulties that learners encountered. The findings which follow are mapped onto the analysis from chapters 6 through 8 respectively.

10.2.3 Overcoming the didactic cut

The findings of this study support the presence of the didactic cut as described by Filloy and Rojano (1989). For two of the learners, Daniele and Yusuf, the didactic cut was persistent and not successfully overcome. Another two learners, Edith and Tebogo, were able to eventually solve equations with letters on both sides.

The didactic cut requires learners to shift from arithmetic thinking to algebraic approaches (Filloy & Rojano. 1989). As discussed in the literature review, Kieran argues that the root of the difficulties lie in the need for learners to operate not on numbers, but on algebraic objects (terms including a letter). The data from Edith, Tebogo, and to some extent Yusuf, support the difficulty resulting from this shift from arithmetic thinking towards algebraic. As was discussed in chapter 6, both Edith and Tebogo could solve equations of the type $ax + b = c$ using inverse operations (undoing). But when faced with letters on both sides, i.e. $2x + 5 = 3x + 2$, both tried to undo the $3x$ on the RHS by dividing by 3 or $3x$. This may be seen as *premature* inverse operations as in the standard approach of 'same operations on both sides', one would divide by the coefficient of the letter, but only once the like terms have been reduced to a single term. Yusuf too had shown strong arithmetic thinking, but was unable to even begin to solve items with letters on both sides. Thus, it would appear that these learners were trying to use arithmetic approaches (or at least premature inverse operations) and were unable to operate algebraically on the letter.

Daniele's initial difficulties lay in her algebraic errors, such as conjoining, where she combined the unlike terms $x + 16$ as $16x$ (Figure 6.12). She had not made similar mistakes with arithmetic equations. In session 3, she shifted to her arrow approach, simply moving terms to the other side of the equals sign, what Kieran (1992) calls shifting addends. Daniele also appeared to have both relational and operational views of the equals sign although this wasn't investigated extensively. Furthermore she could not see the contradiction between her arrow approach, which she used with letters on both sides, and her inverse operational approach when she had only a single letter – that the sign of the term changed using inverse operations but did not change when she 'moved' the term over the equals sign. This again shows the distinct 'cut' between the two types of equations for her – they are incompatible with each other's approaches.

The item $7x - 2x = 10$ was helpful to Tebogo in overcoming her difficulties and reminding her that she could add or subtract like terms, and thus operate with the letters over the equals sign. Both Daniele and Yusuf were successful with this item but still could not use this notion of adding like terms (operating on the letter) to overcome the didactic cut.

Despite the small sample size (discussed later under limitations), the findings of this study do support the notion of a distinct difficulty and shift in thinking required for learners when letters are on both sides of the equals sign, i.e. the presence and persistence of the didactic cut.

10.2.4 Difficulties with negativity

While errors with the minus symbol were expected, the difficulty which learners experienced with a negative coefficients was a surprising finding of this study. Tebogo's difficulties included items which had a visible negative coefficient of x, e.g. $5 - 3x = -10$, but Edith and Daniele only experienced difficulties with items involving $-x$ such as $10 - x = 18$.

The first finding with regards to negativity was that items such as $5 - x = 8$ required learners to understand that 'subtracting a negative' will result in a larger number, *if* the learners were using inspection. Both Tebogo and Daniele originally used an inspection approach. Tebogo was focussed on the minus symbol as representing a difference of 8 between 5 and a number. This pointed her towards 13, but neither $x = 13$ nor $x = -13$ satisfied the equation. Daniele also focussed on minus as difference, but saw x as representing the difference between 5 and

8, and it needed to be -3 because $+3$ wouldn't work. This did satisfy the equation. However, in session 2, Daniele reverted to the difference between 10 and 28 as being 18 in $10 - x = 18$. Thus the inspection approach relies on understanding the minus symbol as both a sign and an operator, that the operation in these items is subtraction and the value of x needs to be negative in order to have a larger number as the result (on the RHS).

The second finding was that learners experience difficulties with this type of item when using arithmetic approaches because these items did not have an explicit numerical coefficient. This notion of non-explicit numerical coefficients was first seen when Edith left her answer as $-x = 3$ and was "happy" that this was her answer. Edith saw $-3x$ and $-x$ differently; "there's no number in front" when referring to $-x$. In session 2, she appeared to have some notion of the invisible -1 coefficient but her language was problematic in explaining this by saying "So if I make x, negative x and negative x, it will equal a positive (LHS)", thus reinforcing division by $-x$ and not just -1. However, she was able to arrive at the correct solution to type of item despite dividing by $-x$ instead of -1 (Figure 7.12). Again, she only divided by x in these particular items with no visible coefficient.

Daniele also ignored the minus attached to x in her answer of $-x = -20$. When checking her answer to item $17 - x = -3$, she typed $17 - (-20)$ into her calculator, implying that $x = -20$ rather than $x = 20$. When Daniele used inspection to verify that $x = 20$, she could not see how algebra could get her there (i.e. to divide by negative 1). She did have some idea of an 'invisible' coefficient when she said "There could be an imaginary 1". Eventually in session 4, after a lengthy discussion with me in comparing her responses to $16 - 7x = 9$ and $17 - x = 9$, she saw the similarities between the two items (that -7 was the coefficient to x and thus $-x$ had a coefficient of -1). She saw that she was able to divide by -7 in the first, and thus she could divide by -1 in $-x = -8$.

10.2.5 Difficulties with a equations of the type $ax = bx$

The final finding of this research is the difficulties that learners experienced with the special case of $ax = bx$. This difficulties with this item were surprising – that learners treated this item differently to more *canonical* equations of the form $ax + b = cx + d$. Difficulties with this type of item were not noted in the international research, and it is not a common item in textbooks. However, difficulties with this item were noted by Sanders (2017) where learners arrived at this form after making conjoining errors, e.g. $2x + 1 = 7x - 3$ became $3x = 4x$.

All the learners who experienced difficulty with this type of item had been successful in solving other items with letters on both sides. However, learners saw this item as a different type of equation because they changed their approach. In session 2, Edith reverted to substitution of values for x. Then in session 3, she again reverted to using inverse operations (division) like she had for $ax + b = cx + d$ in session 1. Tebogo also reverted to inverse operations, dividing both sides by 2 in $5x = 2x$ but then getting stuck at $2{,}5x = x$ and reattempted the item. In her second attempt, she did subtract $2x$ from both sides but the left the RHS blank (both attempts are shown in Figure 8.4).

Guban had not experienced difficulties at any stage with other equations with letters on both sides, but adapted his approaches for this item. This indicates this this type of item presents challenges to 'stronger' learners. The sources of his difficulties were his concern at having 'nothing' left on one side (after subtracting the term on the RHS) and then division by 0 (even though it was dividing 0 by a number).

Thus I found that there appeared to be two main sources of the difficulties which learners were experiencing with this type of item. The first concern for learners appeared to be what would be left on the RHS (normally) if the letters were subtracted, i.e. if we subtract $2x$ from both sides in $5x = 2x$, then there is 'nothing' left over on the RHS. Tebogo said she was considering "putting 1" and Guban only subtracted $2x$ (in the equation $3x = 4x$) so that there would be something ($1x$) left over on the LHS.

The item $20 = 2(x + 10)$ appeared to be helpful to all three learners in determining that it was not 'nothing' remaining but rather the number zero. When distributing the 2 into $(x + 10)$, they get to $20 = 2x + 20$. When collecting the like terms by subtracting 20 from both sides, they get a 0 on one side (probably the LHS in the example), thus being left with $0 = 2x$. This zero appeared to be more acceptable for learners since the learners experienced fewer difficulties with this item.

I also conjectured that the resultant zero from subtracting $2x$ from both sides in $5x = 2x$ could be more difficult for learners to accept, since it is the simplification to the subtraction of letters, i.e. $2x - 2x = 0$. Learners are taught to treat constants, and terms with letters, differently – the rule being to only add or subtract 'like terms'. So, this is a unique case where by subtracting terms with letters ($2x$ and $2x$), your simplification is a constant, namely zero.

The second difficulty was expressed by Guban and Tebogo. Given $2x = 5x$, after subtracting $2x$ from both sides you will be left with $0 = 3x$. Both learners were concerned at this point that they would be dividing by zero. For Tebogo, this actually arrived in the item $20 = 2(x + 10)$, and was only convinced that $\frac{0}{2}$ was defined after I suggested putting this into her calculator. Guban appeared to be confused about the order of the division, with his language alternating between dividing by 7 and dividing by 0 when trying to proceed from $0 = 7x$.His concern was that he was dividing 7 by 0, which would be undefined, rather than 0 by 7.

Thus, the findings of this study present the unique case of $ax = bx$ as a special case of letters on both sides. While this item is not commonly found in textbooks, this study found that learners treat it differently. Furthermore, it leverages the important notions of *zero* and *undefined*. The next type of equations which these learners would encounter would be quadratic, where the factors equal to 0 is the standard approach for integer solutions.

10.3 Critical reflections

Some of the inherent limitations in the design of the research were discussed in chapter 4. In this section, I will focus on the limitations of the findings of this study.

10.3.1 Design of the study

The design of the study included sampling learners based on their results in the WMCS test and school assessments. As part of investigating known difficulties from literature, learners were sampled on the basis of appearing to experience difficulties with the didactic cut on the WMCS test. A finding of this study is the presence and persistence of the didactic cut. Hence it could be argued that it is not surprising that a finding of this study was the presence of the didactic cut, i.e. a self-fulfilling prophecy. The study found what it was intended to find.

While this criticism has merit, I would argue that the findings of this study refer more to the nature of the difficulties inherent in the didactic cut, i.e. confirming that the transition from arithmetic to algebraic thinking is at the root of the cut. And this study presented the stories of the difficult *process* of overcoming the cut. Tebogo and Danielle both experienced difficulties, but Tebogo overcame this difficulties with exposure to specific provoking examples.

10.3.2 Sampling

After acknowledging that I sampled learners who displayed evidence of difficulty with the didactic cut, there was another issue with my sample. As was shown in chapter 5, both Sarah and Guban did not experience as many difficulties as I expected based on their WMCS test.

Both had been unable to correctly solve the item with letters on both sides in the WMCS test and had only solved the arithmetic equation correctly. However, by their first session, they had overcome these difficulties.

Thus it would appear that they were too strong to be selected in the original sample according to the criteria I had originally described. While Guban did experience difficulties with the item $ax = bx$, and both he and Sarah experienced difficulties with items which had no solution or infinite solutions, they did appear to stand out as being 'stronger' learners than the rest of the sample.

10.3.3 Natural maturity over time

Because this study spanned six months of the learners' lives, we may expect some natural maturity to take place without intervention. It is my anecdotal experience as a teacher that learners can show improvement over time in mathematics without explicit teaching of a particular concept or skill. Content such as linear equations is dealt with (even if not explicitly) within other topics learners were exposed to during the time, such as 'determining the $x-$ intercept" of a linear function, and "finding n" in a pattern such as $T_n = 3n + 1$ where $T_n = 13$. These other sections were highlighted in section 1.4. Thus, the learner progress which was described in the findings may partly have had to do with natural maturation and exposure to the content. These two aspects were not dealt with in the analysis due to the scope of the project but *may* have had an impact on learner progress.

10.3.4 Research instrument

One of the unique aspects to this research was the design of the research instrument. I had hoped that such an instrument could be helpful to other researchers, and possibly teachers, in helping to understand what aspects of linear equations learners could find difficult. As I reflect on the matrix and how I used it, there are some limitations.

In the design of the task-matrix, rows A, B and C were designed based on the distinct cognitive levels derived from the literature. Row D was not informed by the literature but rather based upon my own experiences as a secondary teacher as to what learners could find more difficult. In Figure 10.1, the distinct feature in Row 4 is that the coefficient of x on the RHS is a larger value.

C	$\square + \square = \square + 3$	$2x = x + 5$	$2x + 5 = x$	$2x + 5 = x + 1$	$4x - 5 = x + 1$	$3x = 4x$	$4 - 3x = 5x - 4$
D		$5 + 2 = 2x - x$	$x + 5 = 2x - 2$	$2x + 5 = 3x - 2$	$5 - 2x = 3x - 10$	$2(5 - 2x) = 10 - 3x$	$2x + 1 = 3x - x$

Figure 10.1: Rows C and D of the task-matrix

I had thought that learners who were flexible in their approaches may have collected the letters on the RHS and thus avoided ending up with a negative coefficient on the LHS. However, it became apparent relatively quickly in my mini reflections that Row 4 not serving a significant purpose in this research as learners unless they experience difficulties with negativity. And these difficulties with negativity normally preceded row C, thus should be overcome by row D.

Row C was also initially (session 1) limited in having 'easier' equations for letters on both sides. Edith did not understand what the item with boxes with (C0) $\square + \square = \square + 3$ entailed in session 1, where she put different numbers in boxes ($3 + 5 = 5 + 3$). After reflection from the first session, I introduced the item $2x + x = 6$ as the 'easiest' equation in this row. This item had two letters, but both letters were on the same side. This was the type of item that had helped Tebogo overcome her difficulties. However, it could be argued that this item is not of the same 'type' as the rest of row C, as both letters are already on the same side.

The other limitation of the research instrument is aspect of difficulty. Originally, the rows were theoretically informed distinct levels of cognitive 'difficulty', i.e. row A had a single operation to be undone, row B two operations and row C had letters on both sides. However, I have also referred to giving learners an 'easier' equation from row B. The variation within a row was determined by the *features* of a question, i.e. order of terms, brackets, location of minus sign. This horizontal dimension was informed by variation learning theory (Runesson, 2005) as to what the learner is focussed on – the object of learning. This is not the same type of difficulty as shifting from row B to C.

10.3.5 Analysis

The progress diagrams categorised the item $ax = bx$ as a distinct level of progress. The other levels were aligned with the progression in cognitive difficulty in rows A, B and C. I did argue for the distinct difficulty that this item presented to learners. However it could be argued that this difficulty is not of the same type as the first three levels of difficulty. Learners' difficulties with negativity were not treated as a distinct level in the progress diagram. Edith and Daniele's difficulties were confined to items with a coefficient of -1 only. This would have been an item near the end of row A, and may have presented a distinct level in the way $ax = bx$ did. However, Tebogo's difficulties included difficulties with $5 - 3x = -10$. This item crossed into row B, and hence I decided that it should not be included as a distinct level in the progress diagram.

The difficulties that learners experienced with negativity were also influenced by them being allowed to use calculators. Thus the findings of learners' difficulties with negativity are to be considered in the light of them having access to calculators. They may have experienced many more difficulties if they had not been allowed calculators. However, the focus of this research was not particularly on their difficulties with integers. Additionally, these learners would always be allowed calculators in their assessments, and would be used to having access to calculators.

10.3.6 Reflections on methodology and analysis

My previous experiences with educational research had focussed on larger groups and coding errors. Thus it was a new experience for me to shift into providing rich descriptions of learners' thinking. It was always a challenge to code learners' work from only their written responses. Learners may have written work down that did not necessarily have as much intention and reason behind as I, the researcher, may be reading into it. Hence it was gratifying to perform in-depth task based interviews where I could extensively probe learners' written work by discussing their thinking with them. Even still, the time limitations on each interview were a frustration at times because I still felt that more could be discussed with the interviewees.

The interviews were still not an open window into the learners' mind. I have discussed the power imbalance between myself and the learners which appeared to cause of Edith's nerves in session 1 and 2, and Tebogo's nerves in session 1. Their thinking was also not always explicit in the way they expressed themselves in their vocalisations. Some examples of this are Tebogo trying to explain why she divided by negative 1 (chapter 7), Daniele explaining her method for letters on both sides (chapter 6) and Edith trying to explain how she sees the coefficient to $-x$ as negative 1 (chapter 7).

10.3.7 Learning not teaching

In adopting a Vygotskian perspective, I am acknowledging that learning is a social process. For the learners in this study, they would have been taught how to solve linear equations through negotiated meaning with their teacher. Hence one may expect a study that adopts this perspective to involve the teacher as part of the process. The teachers were not included due to the scope of the study since the focus was on the learners' abilities and reasoning. This should not undermine the influence of the teacher. But the effectiveness of the teaching is not within the scope of this study. Later in this chapter, I will make some recommendations in respect to teaching.

10.4 Recommendations

I have split my recommendations into two sections. As a practicing teacher, I have naturally reflected on the implications of my findings for teaching and learning. However, as a researcher, I also have remaining questions that I would hope other researchers can pick up and pursue.

10.4.1 Teaching

I wish to make 4 recommendations for teaching. The task interview matrix can be useful. The findings of this study include some interventions which appeared to yield positive results. The use of 'extreme' cases such as $ax = bx$ can help engagement with big mathematical ideas. And finally, the need for teachers to engage with the 'invisible' coefficient of -1 in $-x$.

The task interview matrix, or something similar, may be useful for teachers looking to do revision with learners. The ability to adapt to easier or more challenging items dynamically, may be a useful diagnostic tool for teachers to see which aspects of linear equations their learners are still finding difficult. The structure of the matrix has also value. Similar task-matrices could be designed for other content areas with the same underpinnings of distinct levels of difficulty (rows) and variation within each row. Obviously algebraic content lends itself well to this design. And different types of equations, could also be designed into a matrix. Quadratic equations in Grades 10 and 11 would also suit distinct rows – those that could factorise with a highest common factor, simple trinomials and complex trinomials with irrational roots.

I also think that this research has provided some idea of what positive interventions could take place to help learners overcome difficulties. Although it only worked for Tebogo, providing learners with equations with letters on the same side may help them to be able to operate on letters and overcome the didactic cut. Similarly, the item $20 = 2(x + 10)$ was more canonical to learners than $3x = 4x$, but still raised the issues of 'nothing' left on one side of the equation, and that an acceptable solution to an equation can be $x = 0$.

Although the item $ax = bx$ is not common within textbooks and can be seen as an 'extreme' example, it does raise important mathematical ideas: those of zero as a number and $x = 0$ as a solution. The next significant algebraic equation which learners would need to engage with would be quadratic equations. For quadratic equations, learners may often have a solution of $x = 0$ from $x(x - 1) = 0$. And learners will need to get used to having a zero on one side of the equation, either to factorise the other side, or to use the quadratic formula.

The final recommendation for teachers is to ensure to write in the explicit coefficient of -1 for $-x$ in the initial stages of teaching linear equations. Although the learners in this study (Edith and Daniele) appeared to be aware of this invisible -1, it wasn't internalised in the same way that $-3x$ was.Once learners have internalised this notion, it is no longer required to be explicitly stated. But teachers should be aware of the possible difficulties faced by learners.

10.4.2 Research

I would like to make four recommendations for further research: what are helpful interventions in overcoming these difficulties, the further usefulness of the task-matrix, difficulties with zero and other forms of solution.

The first recommendation for further research is to pursue helpful interventions in helping learners overcome some of their difficulties. This research found positive results for Tebogo in overcoming the didactic cut by including items which had two letters on the same side, i.e. $2x + x = 6$. This item did not help Daniele nor Yusuf in overcoming their difficulties. Thus additional research is needed into which interventions are useful and reliable.

The task-matrix could be further developed for research. While it lends itself to the dynamic nature of interviews, it could also hold value for further research on larger scale studies. The extreme case of $ax = bx$ could feature more prominently in the matrix. Larger samples would provide a better idea of how common these issues are across larger population from different backgrounds.

The difficulties with zero that the item $ax = bx$ elicited, deserve further research. The notion of zero is important for quadratic equations. It is my experience that learners may divide by x given the quadratic equation $2x^2 = 10x$ which bears a visual similarity to $2x = 10x$. Dividing by x is problematic in quadratic equations if $x = 0$, which in the above equation, it does.

It was outside the scope of this research for items which had different *types* of solution – namely *no solution* or $x \epsilon \mathbb{R}$. But these types of items are also extreme examples (like $ax = bx$) which would draw out further notions of solution. Learners are expected to engage with two solutions for quadratic equations in Grade 9, where an equation such as $x^2 + 9 = 0$ does not have real solutions. In Grade 10, they will encounter linear inequalities where the 'solution' is a range of solutions, i.e. $x < 5$ in $2x < 10$.

10.5 Final reflection and conclusion

My previous experiences as a secondary teacher had convinced me of the existence of the didactic cut as described by Filloy and Rojano (1989). After reading Herscovics and Linchevski's (1994) criticisms of the term and Lima and Healey's (2010) research with older learners where the algebraic approach had replaced the arithmetic, I was concerned as to whether my findings would include these Grade 9 learners still experiencing difficulties. Thus I was relieved that they did experience difficulties – certainly not for their sake, but it allowed me to gain some insight into why their difficulties were so persistent. Indeed, one of the more surprising findings was Tebogo's regression in session 4 after appearing to have overcome her difficulties in session 3. Thus it was surprising to me just how persistent the cut could be. Although I was also relieved to see progress in Edith and Tebogo to maintain hope that the cut can be overcome.

I was also surprised at the difficulties which learners experienced with negativity. My previous research had focussed on learner error with integers when operating without a calculator. Hence I had expected fewer errors when allowed to use a calculator. However, the persistent difficulties with negativity reminded me of the importance of strong integer foundations for success in algebra. Although I had noticed errors before in my teaching with $-x$, the unique difficulties which Edith and Daniele experienced were also surprising.

The findings regarding item $ax = bx$ were also unexpected. Since this is not a common item within school textbooks and exams, I did not expect to see the difficulties which I found. Although Sanders (2017) had noted the error, her learners were conjoining terms and probably weaker than my sample. Thus I was surprised when three of the six learners experienced difficulties. In fact, Daniele and Yusuf did not encountered it because of their persistent difficulties with $ax + b = cx + d$, thus three of the four learners who encountered it, experienced difficulties.

And finally, I am aware of the inability to generalise my findings extensively. I would like to someday to be part of larger scale research which could generalise. However, in reading Mirriam (2002), I have been convinced that qualitative research (such as mine) can add value to the education community in understanding the difficulties which learners face. I may not be able to make strong claims on how to remediate learners, but I do think that the rich descriptions of their difficulties, and how they change over time, provide some valuable insight.

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Appendices

Appendix A – The interview task matrix

Equations in 2 Dimensions

	0	1	2	3	4	5	6
A	$\square + 5 = 8$	$5 + \square = 8$	$x + 5 = 8$	$x - 5 = 8$	$5 + x = 8$	$5 - x = 8$	$8 = x + 3$
B	$5 + 2 = \square + 3$	$3x = 6$	$3x = 8 - 5$	$3x - 5 = 10$	$3x + 5 = 6 + 5$	$5 - 3x = -10$	$-(5 - 3x) = 10$
C	$\square + \square = \square + 3$	$2x = x + 5$	$2x + 5 = x$	$2x + 5 = x + 1$	$4x - 5 = x + 1$	$3x = 4x$	$4 - 3x = 5x - 4$
D		$5 + 2 = 2x - x$	$x + 5 = 2x - 2$	$2x + 5 = 3x - 2$	$5 - 2x = 3x - 10$	$2(5 - 2x) = 10 - 3x$	$2x + 1 = 3x - x$

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Appendix B - The interview task matrix for session 2

Equations in 2 Dimensions – Session 2

	0	1	2	3	4	5	6
A	$\square + 8 = 10$	$8 - \square = 10$	$x + 10 = 18$	$x - 10 = 18$	$10 - x = 18$	$37 - x = -12$	$100 - x = -100$
B	$2 \times \square = 5$	$2x = 1$	$2x = 9$	$2x - 5 = 25$	$2x + 5 = 20$	$20 - 2x = -15$	$20 = 2(x + 10)$
C	$2x + x = 6$	$4x - 2x = 10$	$4x = 3x + 4$	$3x - 15 = x + 1$	$4x = x + 1$	$3x = 4x$	$2(x + 1) = 2x - 2$
D		$5 + 2 = 2x - x$	$x + 5 = 2x - 2$	$7x + 23 = 12x - 12$	$5 - 2x = 10 - 3x$	$-2(12 - x) = 9 - x$	$12(x - 1) = 6(2x - 2)$

Appendix C – Ethics clearance from WITS

Wits School of Education

WITS UNIVERSITY

27 St Andrews Road, Parktown, Johannesburg, 2193 Private Bag 3, Wits 2050, South Africa. Tel: +27 11 717-3064 Fax: +27 11 717-3100 E-mail: enquiries@educ.wits.ac.za Website: www.wits.ac.za

24 April 2017

Student Number: 0614020A

Protocol Number: 2017ECE013M

Dear Andrew Halley

Application for Ethics Clearance: Master of Education

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate, has considered your application for ethics clearance for your proposal entitled:

Tracking Grade 9 Learners' thinking when solving linear equations over time

The committee recently met and I am pleased to inform you that **clearance was granted**.

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

All the best with your research project.

Yours sincerely,

MMabeta

Wits School of Education

011 717-3416

cc Supervisor - Dr Craig Pournara

Appendix D – Ethics clearance from Gauteng Department of Education (GDE)

GAUTENG PROVINCE
Department: Education
REPUBLIC OF SOUTH AFRICA

8/4/4/1/2

GDE RESEARCH APPROVAL LETTER

Date:	**19 April 2017**
Validity of Research Approval:	**06 February 2017 – 29 September 2017** **2017/80**
Name of Researcher:	**Halley A.G**
Address of Researcher:	**Flat 20 Tugela Terrace**
	Jacaranda Street, Linden Ext
	Randburg, 2194
Telephone Number:	**084 503 3787**
Email address:	**aghalley@gmail.com**
Research Topic:	**Tracking Grade 9 Learner errors over time in the context of solving equations**
Number and type of schools:	**One Secondary School**
District/s/HO	**Johannesburg North**

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved to conduct the research. A separate copy of this letter must be presented to both the School (both Principal and SGB) and the District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted.

19/04/2017

The following conditions apply to GDE research. The researcher may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:

1

Making education a societal priority

Office of the Director: Education Research and Knowledge Management
7th Floor, 17 Simmonds Street, Johannesburg, 2001
Tel: (011) 355 0488
Email: Faith.Tshabalala@gauteng.gov.za
Website: www.education.gpg.gov.za

1. *The District/Head Office Senior Manager/s concerned must be presented with a copy of this letter that would indicate that the said researcher/s has/have been granted permission from the Gauteng Department of Education to conduct the research study.*
2. *The District/Head Office Senior Manager/s must be approached separately, and in writing, for permission to involve District/Head Office Officials in the project.*
3. *A copy of this letter must be forwarded to the school principal and the chairperson of the School Governing Body (SGB) that would indicate that the researcher/s have been granted permission from the Gauteng Department of Education to conduct the research study.*
4. *A letter / document that outlines the purpose of the research and the anticipated outcomes of such research must be made available to the principals, SGBs and District/Head Office Senior Managers of the schools and districts/offices concerned, respectively.*
5. *The Researcher will make every effort obtain the goodwill and co-operation of all the GDE officials, principals, and chairpersons of the SGBs, teachers and learners involved. Persons who offer their co-operation will not receive additional remuneration from the Department while those that opt not to participate will not be penalised in any way.*
6. *Research may only be conducted after school hours so that the normal school programme is not interrupted. The Principal (if at a school) and/or Director (if at a district/head office) must be consulted about an appropriate time when the researcher/s may carry out their research at the sites that they manage.*
7. *Research may only commence from the second week of February and must be concluded before the beginning of the last quarter of the academic year. If incomplete, an amended Research Approval letter may be requested to conduct research in the following year.*
8. *Items 6 and 7 will not apply to any research effort being undertaken on behalf of the GDE. Such research will have been commissioned and be paid for by the Gauteng Department of Education.*
9. *It is the researcher's responsibility to obtain written parental consent of all learners that are expected to participate in the study.*
10. *The researcher is responsible for supplying and utilising his/her own research resources, such as stationery, photocopies, transport, faxes and telephones and should not depend on the goodwill of the institutions and/or the offices visited for supplying such resources.*
11. *The names of the GDE officials, schools, principals, parents, teachers and learners that participate in the study may not appear in the research report without the written consent of each of these individuals and/or organisations.*
12. *On completion of the study the researcher/s must supply the Director: Knowledge Management & Research with one Hard Cover bound and an electronic copy of the research.*
13. *The researcher may be expected to provide short presentations on the purpose, findings and recommendations of his/her research to both GDE officials and the schools concerned.*
14. *Should the researcher have been involved with research at a school and/or a district/head office level, the Director concerned must also be supplied with a brief summary of the purpose, findings and recommendations of the research study.*

The Gauteng Department of Education wishes you well in this important undertaking and looks forward to examining the findings of your research study.

Kind regards

Ms Faith Tshabalala
CES: Education Research and Knowledge Management

DATE: 19/04/2017

2

Making education a societal priority

Office of the Director: Education Research and Knowledge Management
7th Floor, 17 Simmonds Street, Johannesburg, 2001
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Appendix E – Ethical consent from learners and parent

GRADE 9 EQUATIONS RESEARCH PROJECT

INFORMATION SHEET LEARNERS

25 April 2017

Dear Learner

My name is Andrew Halley and I am a Masters in Education student at the University of the Witwatersrand. I am doing research on Tracking Grade 9 Learners' thinking when solving equations over time.

Earlier this year, you agreed to help with the research into Learning Gains in Mathematics. You wrote a test in February, and will write another in September. I am interesting in interviewing you 4 times between the two tests. I am interesting in understanding how your thinking changes over the time, particularly with equations.

You are one of only 6 learners chosen to be interviewed. I'm afraid I won't be helping you to solve the equations, only interviewing you about how you go about solving them. I was hoping that you would agree to help with this research. You will be helping us to learn more about how grade 9 learners understand mathematics and overcome difficulties.

The interview questions are not for marks and you are not expected to study for them. Your participation is voluntary, which means that you don't have to do it. Also, if you participate and then decide that you don't want to continue, this is completely your choice and will not affect you negatively in any way.

I will not be using your own name so no-one can identify you. All information about you will be kept confidential in all my writing about the research. Also, all collected information will be stored safely and destroyed 5 years after I have completed the project.

Your parents/guardians have also been given an information sheet and consent form, but at the end of the day it is your decision to join us in the research. I look forward to working with you!

Please feel free to contact me if you have any questions.

Andrew Halley
University of the Witwatersrand
aghalley@gmail.com, 084 503 3787

LEARNER ASSENT FORM: Grade 9 Equations Research Project

Please fill in the reply slip below and return it to your teacher as soon as possible.

First name: ______________________ Surname: ______________

School: ____________________ Grade: ______ Class: ________

	Circle one
Permission to be audiotaped	
I agree to be audiotaped during the interview	YES/NO
I know that the audiotapes will be used for this project only	YES/NO
Permission to be interviewed	
I would like to be interviewed for this study.	YES/NO
I know that I can stop the interview at any time and don't have to answer all the questions asked.	YES/NO
Permission to be videotaped	
I agree to be videotaped in the interview.	YES/NO
I know that the videotapes will be used for this project only.	YES/NO

Informed Consent
I understand that:

- my name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the research at any time.
- all the data collected during this research study will be destroyed 5 years after completion of the project.

Sign____________________ Date__________________

GRADE 9 EQUATIONS RESEARCH PROJECT

INFORMATION SHEET PARENTS AND GUARDIANS

25 April 2017

Dear Parent

My name is Andrew Halley and I am a Masters in Education student at the University of the Witwatersrand and a mathematics teacher at Northcliff High School. I am doing research on Tracking Grade 9 Learners' thinking when solving equations over time.

Earlier this year, you gave permission for to participate in the Learning Gains in Mathematics project. is one of six learners who have been selected to participate in 4 one-on-one follow-up interviews which are seeking to track each child's progress in-between the two tests. We will be focusing particularly on the important content of equations.

These interviews will provide valuable insight into the way in which Grade 9 learners are reasoning about equations. They will help us to understand better what difficulties learners are having and ultimately how their teachers might help them overcome these difficulties.

I will conduct the interviews directly after school. Each interview will last a maximum of 20 minutes. Three learners will be interviewed per day, which means may have to wait at most 40 minutes before the interview. You will be informed of the schedule at least a week in advance. We would like to conduct 4 interviews, one month apart, starting in May and ending in September. Light snacks and a drink will be provided and Savern will be encouraged to do homework while waiting.

It is important to clarify that the interviews are not extra lessons. I will observe what your child is doing and ask probing questions but will not "teach" your child during the interview. However, your child's math's teacher will be kept informed as to your child's progress and will be able to offer the necessary support, if required. I will offer a personalized extra lesson to each participating child after the second test in September to address any persistent errors we have noticed.

Would you mind giving your consent for to participate? s participation is voluntary, which means that doesn't have to do it. Also, if your child decides halfway through that she would like to withdraw from the research, this is completely her choice and will not affect her negatively in any way.

We will not be using your child's own name so no-one can identify your child. For the purposes of accuracy, we wish to video record the interview. However, your child's face will not appear on the video. The camera will focus on your child's work and audio will be recorded. All information about your child and the school will be kept confidential in all our writing about the study. Also, all collected information will be stored safely and destroyed 5 years after we have completed the project.

Your child has also been given an information sheet and consent form. At the end of the day it is your child's decision to participate in the research.

The first interview would be either the 8th or 9th May.

Please feel free to contact me if you have any questions.

Thank you

Andrew Halley
University of the Witwatersrand
aghalley@gmail.com, 084 503 3787

My supervisor's details are the following:

Dr Craig Pournara
Project Manager
Wits Maths Connect Secondary Project
Marang Block West, Wits Education Campus
craig.pournara@wits.ac.za, 082 696 8381

PARENT CONSENT FORM: Grade 9 Equations Research Project

Please fill in the reply slip below and return it to you teacher as soon as possible.

Child's First name: ______________________________

Surname: ______________________________

School: ____________________________ Grade: _______ Class: ____________

Parent's Name: ____

Preferred contact method (please provide email address or cell number):

____

Permission to be audiotaped	**Circle one**
I agree that my child may be audiotaped during interview or observations.	YES/NO
I know that the audiotapes will be used for this project only	YES/NO
Permission to be interviewed	
I agree that my child may be interviewed for this study.	YES/NO
I know that he/she can stop the interview at any time and doesn't have to answer all the questions asked.	YES/NO
Permission to be videotaped	
I agree my child may be videotaped in class.	YES/NO
I know that the videotapes will be used for this project only.	YES/NO

Informed Consent
I understand that:

- my child's information will be kept confidential and safe and that his/her name and the name of the school will not be revealed.
- my child does not have to answer every question and can withdraw from the research at any time.
- all the data collected during this research study will be destroyed 5 years after completion of the project.

Sign Date