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Smoothness Conditions and Symmetries of Partial Differential Equations

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A research report submitted to the Faculty of Science, University of the Witwatersrand, in fulfilment of the requirements for the degree of Master of Science.

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DECLARATION

I declare that the contents of this thesis are original except where due references have been made. It has not been submitted before for any degree to any other institution.

S Mamba

Abstract

We obtain a solution of the Black-Scholes equation with a non-smooth boundary condition using symmetry methods. The Black-Scholes equation along with its boundary condition are first transformed into the one dimensional heat equation and an initial condition respectively. We then find an appropriate general symmetry generator of the heat equation using symmetries of the heat equation and the fundamental solution of the heat equation. The method we use to find the symmetry generator is such that the boundary condition is left invariant and yet the symmetry can still be used to solve the heat equation. We then use the help of Mathematica to find the solution to the heat equation. Then the solution is then transformed backwards to a solution of the Black-Scholes equation using the same change of variables that were used for the forward transformations. The solution is then finally checked if it satisfies the boundary condition of the Black-Scholes equation.

Contents

1 Introduction **1**

2 Notation and Preliminaries **4**

2.1 Prolongations and Computing Lie Symmetries 6

2.2 Application of Symmetries to Differential Equations 11

3 Symmetries and Boundary Value Problems **15**

3.1 Deriving the Fundamental Solution 16

3.2 Non-invariant Boundary Conditions 23

4 Solving the Black-Scholes equation **27**

Discussion and Conclusions **36**

Chapter 1

Introduction

Our main objective in this research report is to obtain a detailed solution to a boundary valued problem using symmetry techniques. There are many well-known methods for doing this but we will investigate the use of symmetry methods for solving a Partial Differential Equation (PDE) with a non-smooth boundary condition. In particular, the problem we are going to tackle is the linear parabolic PDE version of the Black-Scholes equation which is used for pricing European call options. It can be defined as follows:

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1.1}$$

with the terminal condition

$$V(S,T) = \begin{cases} S-K, & \text{if } S > K \\ 0, & \text{if } S < K \end{cases}$$

One may be interested in the derivation of (1.1), which can be found in [1], where a financial concrete interpretation of (1.1) can also be found. Alternative methods have been used for obtaining solutions of (1.1). For example, Fourier transforms, the variation iteration method and numeric finite methods, see [2]. One may be tempted to believe that the symmetry method is easier to understand, more effective and is a more direct approach in computing exact solutions of linear and non-linear differential equations as compared to other methods. Lie Symmetry methods have been used in the study of linear and non-linear ODEs and PDEs and they have been used to obtain their solutions and solutions of many other PDEs like Burgers' equation and The Boussinesq equation, see [7]. This research report will consists of a section defining the notation used, will consists of a section about the literature review which will give a theoretical base of the research report, a section where the Black-Scholes equation is transformed into the one dimensional heat equation using change of variables along with it's terminal condition transformed into an initial condition using the same change of variables, a section identifying symmetries of the heat equation and using those symmetries to construct the most general symmetry of the heat equation, using the

fundamental solution of the heat equation to obtain a solution that can be converted back into a solution of Black-Scholes equation, showing that the solution is indeed a solution to Black-Scholes equation and checking whether the solution satisfies the terminal condition and a section on discussions and conclusions.

Chapter 2

Notation and Preliminaries

We begin this chapter with a definition explaining what a symmetry is; details can be found in [1].

> **Definition:** A transformation $\Gamma : G \to G$ acting on the space $G \simeq X \times U$ of dependent and independent variables is called a *symmetry* of a certain partial derivative if, whenever $u = f(x)$ is a solution, and the transformation $\hat{f} = \Gamma(f)$ $\hat{f} = g \cdot f$ is well defined, then $\hat{u} = f(\hat{x})$ is also a solution to the differential equation.

Lie Symmetries were introduced by Sophus Lie about 120 years ago in order to solve and study a wide range of ordinary differential equations and to reduce *nth* order partial differential equations. The classical symmetry method is the most popular technique used for finding symmetry reductions for PDEs in one dependent variable u and k independent variables i.e $x = (x_1, x_2...x_k)$.

Consider a PDE of the form

$$\triangle(\mathbf{x}, u, u_{(1)}, ..., u_{(n)}) = 0, \tag{2.1}$$

where $u_{(k)}$ represents the collection of all $kth - order$ derivatives of u. One could find a one-parameter Lie group of transformations in infinitesimal form, in (X,U)-space given by

$$\begin{aligned} X^* &= X + \epsilon\xi(X, U) + O(\epsilon^2) \\ U^* &= U + \epsilon\eta(X, U) + O(\epsilon^2) \end{aligned} \tag{2.2}$$

that leave (2.1) invariant. The coefficients $\xi = (\xi_1, \xi_2, ..., \xi_p)$ and $\eta = (\eta_1, \eta_2, ..., \eta_q)$ of the infinitesimal symmetry (2.2) are normally referred to as the infinitesimals of the group transformations and ϵ is the group parameter. The invariance requirement is determined by

$$\Gamma^{(n)}\Delta \mid_{\Delta=0} = 0,$$

where

$$\Gamma = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}$$

are vector fields spanning the associated Lie algebra. These are called the infinitesimal generators of the transformations (2.2). Symbolically $\Gamma^{(n)}$ represents the nth extension of Γ extended to the nth jet space. Since Γ acts on functions, it also acts on their derivatives.

2.1 Prolongations and Computing Lie Symmetries

We begin this section with a theorem, which can be found in [4]:

Theorem 1: Suppose

$$\triangle_v(x, u^{(n)}) = 0, v = 1, ..., l,$$

is a system of differential equations of maximal rank where $u^{(n)}$ represents the collection of all $kth-order$ derivatives of u defined over $M \subset X \times U$. If G is a local group of transformations acting

on M, and

$$\Gamma^{(n)} \triangle_v (x, u^{(n)}) = 0, v = 1, ..., l,$$

whenever

$$\triangle(x, u^{(n)}) = 0,$$

for every infinitesimal generator Γ of G, then G is a symmetry group of the system.

We will use Theorem 1 at a later stage but for now we will refer to it as the 'infinitesimal criterion'.

In this section, we briefly discuss the concept of prolongations as they are essential in the computation of symmetries. More details can be obtained from P.J.Olver's book, in the prolongation section, see [4]. We will see how the prolongation of the vector field $\Gamma^{(n)}$ plays a role in the computation of Lie symmetries. The prolongation formula is given by

$$Pr\Gamma = \Gamma + \sum_{\alpha,J} \phi_J^\alpha(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}$$

where the coefficients ϕ of the prolonged vector field are given by the formula

$$\phi_J^\alpha = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha$$

and

$$Q^{\alpha}(x, u^{(1)}) = \phi^{\alpha} - \sum_{i=1}^{p} \xi^{i} \frac{\partial u^{\alpha}}{\partial x^{i}}.$$

This may be illustrated more clearer with the aid of an example.

Example 1: We compute the Lie Symmetries of the following ODE:

$$u_{xx} = 0$$

with infinitesimal symmetry generator:

$$\Gamma = \xi(x, u)\frac{\partial}{\partial x} + \phi(x, u)\frac{\partial}{\partial u}.$$

We note that the equation above is of order 2 (i.e. $n = 2$), so we use the second prolongation $\Gamma^{(2)}$. The second prolongation $\Gamma^{(2)}$ is given by the following:

$$\Gamma^{(2)} = \xi(x, u)\frac{\partial}{\partial x} + \phi(x, u)\frac{\partial}{\partial u} + \phi_1(x, u^{(1)})\frac{\partial}{\partial u_x} + \phi_2(x, u^{(2)})\frac{\partial}{\partial u_{xx}}$$

where

$$\phi_1 = \phi_x + (\phi_u - \xi_x)u_x - \xi_u u_x^2 \tag{2.3}$$

$$\begin{aligned}\phi_2 =& \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 \\ &+ (\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_x u_{xx}.\end{aligned} \tag{2.4}$$

We wish to determine all the possible coefficient functions ξ and ϕ, such that the corresponding symmetry group leaves invariant the heat equation. Applying $\Gamma^{(2)}$ to the PDE, we get that according to Theorem 1, the infinitesimal criterion $\phi_2 = 0$ which must be satisfied whenever $u_{xx} = 0$. Substituting the equation of ϕ_2, we get that:

$$\phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 + (\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_x u_{xx} = 0$$

From here, we then equate the different derivatives of u to zero with the purpose of obtaining the determining equations:

$$\begin{aligned}
\phi_{xx} &= 0 \\
2\phi_{xu} - \xi_{xx} &= 0 \\
\phi_{uu} - 2\xi_{xu} &= 0 \\
-\xi_{uu} &= 0 \\
\phi_u - 2\xi_x &= 0 \\
3\xi_u &= 0.
\end{aligned}$$

Solving these equations using integration, we get the following general equations:

$$\begin{aligned}
\xi(x,u) =& Ax^2 + Bxu + Cx + Du + E \\
\phi(x,u) =& Axu + Bu^2 + Fx + Gu + H,
\end{aligned}$$

where the capital letters are arbitrary constants. To get the first symmetry we set $A = 1$ and the rest equal zero and we get that $\Gamma^{(2)} = x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}$. To get the second symmetry we set $B = 1$ and the rest equal zero and so on. Notice that we have eight constants which means that we will have eight different symmetries, namely:

$$\begin{aligned}
\Gamma_1 =& x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u} \\
\Gamma_2 =& xu\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u} \\
\Gamma_3 =& x\frac{\partial}{\partial x} \\
\Gamma_4 =& u\frac{\partial}{\partial x} \\
\Gamma_5 =& \frac{\partial}{\partial x} \\
\Gamma_6 =& x\frac{\partial}{\partial u} \\
\Gamma_7 =& u\frac{\partial}{\partial u} \\
\Gamma_8 =& \frac{\partial}{\partial u}
\end{aligned}$$

That is how one can find the Lie Symmetries of a differential equation and once you have found these, you can use them to either solve the differential equation or use them to reduce the order of the differential equation. We need to note that the infinitesimal symmetry generator can have coefficients ξ, η, ϕ dependent on (x, t, u), i.e.

$$\Gamma = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \phi(x, t, u)\frac{\partial}{\partial u}.$$

The process of finding the symmetries in this case is also similar, the only difference may be the prolongation formulas. Differential forms and Lie derivatives may also be used for computing Lie symmetries as an alternative method. In the following section we see how these symmetries can be used to actually solve differential equations and obtain a solution.

2.2 Application of Symmetries to Differential Equations

In this section, we illustrate how symmetries may be used to solve differential equations. We are going to achieve this by means of an example.

Example 2: Consider Burgers' differential equation

$$u_t + uu_x = au_{xx} \tag{2.5}$$

where a is a constant. It is a fundamental equation used in mathematics, applied mathematics and physics to model fluid dynamics, gas dynamics and traffic flow. The constant a is normally used to denote viscosity. Using the method in the above subsection, we can determine the Lie symmetries of

Burgers' equation. Below is a list of some of the symmetries:

$$\begin{aligned}\Gamma_1 &= \frac{\partial}{\partial x}\\ \Gamma_2 &= \frac{\partial}{\partial t}\\ \Gamma_3 &= \frac{\partial}{\partial u}\\ \Gamma_4 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}\end{aligned}$$

Let us take for an example Γ_4 and use it to try and find a solution to (2.5). The symmetry generator:

$$\Gamma_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$$

implies the following system

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du}{0}.$$

Taking $\frac{dx}{x} = \frac{dt}{2t}$ and using integration to solve we get that

$$x = I_1\sqrt{t}$$

where I_1 is a constant of integration. Then $\frac{du}{0}$ tells us that

$$u = I_2$$

where I_2 is a constant and I_2 is a function of I_1. This implies that

$$u(x,t) = f(I_1),$$

which implies

$$u(x,t) = f\left(\frac{x}{\sqrt{t}}\right).$$

Substituting $u(x,t)$ into partial differential equation (2.5) using the chain rule, we get the following ordinary differential equation:

$$\frac{x}{2t}f'(I_1) + ff'(I_1) = af''(I_1) \tag{2.6}$$

We notice that (2.6) is now an ODE which is more easier to solve using the well known methods. When we have found the function $f(I_1)$, we have our solution to equation (2.5). We find that the solution is:

$$f(I_1) = \frac{ax - \sqrt{a^2x^2 - 8a^2t^2C_1}\,\mathrm{Tanh}\left[\frac{\sqrt{ax^2-8at^2C_1}\left(\frac{x}{\sqrt{t}}+C_2\right)}{4at\sqrt{a}}\right]}{2at},$$

where C_1 and C_2 are arbitrary constants.

At this point we can safely say that we have the ability to find Lie Symmetries and use them to find a solution to that particular differential equation. But the problems we are really interested in are boundary-valued problems, in the next section we discuss the technique we will use to find symmetries that leave the boundary condition invariant and we also derive the Fundamental

solution.

Chapter 3

Symmetries and Boundary Value Problems

The following problem is referred to as the *Cauchy Problem*:

$$\begin{aligned} u_t &= ku_{xx}, \mid x \mid < \infty, t > 0 \\ u(x,0) &= F(x), \mid x \mid < \infty \end{aligned} \tag{3.1}$$

It is an Initial Value Problem (IVP), with a general initial condition and we will show how one can solve such a problem using the fundamental solution of the heat equation, sometimes referred to as the heat kernel. We will first

derive the heat kernel then show how it can be used to solve such problems as (3.1).

3.1 Deriving the Fundamental Solution

The one dimensional heat equation can be defined as follows:

$$u_t = u_{xx}, \tag{3.2}$$

for $t > 0$, $x \in \mathbb{R}$. One can use the Fundamental solution to obtain the general solution of the heat equation corresponding to the initial condition of an initial point source of the heat. Before we start the derivation, we are going to introduce the invariance properties of the heat equation which can be found in [5]. The heat equation is invariant under the following properties:

- If $u(x,t)$ is a solution of the heat equation, then so is the function $u(x-y,t)$ for a fixed y.
- If u is a solution of the heat equation, then so are u_x, u_{xx}, u_t and so on.
- If $u_1, u_2 ... u_n$ are solutions of the heat equation, then so is their linear combination $c_1u_1 + c_2u_2 + ... + c_nu_n$

- If $S(x,t)$ is a solution of the heat equation, then so is the integral

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy$$

  for any function $g(y)$.

- If $u(x,t)$ is a solution of the heat equation, then so is $u(\lambda x, \lambda^2 t)$.

Consider the following particular Initial Value Problem i.e IVP:

$$\begin{aligned} M_t - kM_{xx} &= 0 \\ M(x,0) &= H(x) \end{aligned} \tag{3.3}$$

for $x \in \mathbb{R}$, $t > 0$ where $H(x)$ is the *Heaviside step function* defined as

$$H(x) = \begin{cases} 1, & \text{if } x > 0. \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

The first step is to reduce the PDE (3.3) to an ODE. One needs to note that the Heaviside function is invariant under the scaling given by $x \to \sqrt{a}x$. This means that $H(\sqrt{a}x) = H(x)$. We also know from the last invariance property mentioned earlier that $M(\sqrt{a}x, at)$ also solves the heat equation. But $M(\sqrt{a}x, 0) = H(\sqrt{a}x) = H(x)$, therefore $M(x,t)$ and $M(\sqrt{a}x, at)$ both solve the IVP above. The scaling property comes from one of the finite dimensional symmetries of the heat equation (i.e. $X = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$). The

scaling ($(x,t) \to (\sqrt{a}x, at)$) implies that M can only depend on the ratio $\frac{x}{\sqrt{t}}$ or $\frac{x^2}{t}$, that is we can define $M(x,t) = q(\frac{x}{\sqrt{t}})$. To show that this is true, we can use the scaling symmetry $X = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$, to come up with a one variable function q that completely defines M. The scaling symmetry implies the following system:

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du}{0},$$

where we saw previously in example 2 that this leads to the solution

$$u(x,t) = f\left(\frac{x}{\sqrt{t}}\right).$$

Now that we have found our function $q(\frac{x}{\sqrt{t}})$, for the purposes of future calculations, we define the function $g(z) = q(\sqrt{4k}z)$ so that, $M(x,t) = q(\frac{x}{\sqrt{t}}) = g(\frac{x}{\sqrt{4kt}}) = g(p)$ where $p = \frac{x}{\sqrt{4kt}}$. We then use the chain rule to compute the derivatives of M in terms of g and we substitute them into the PDE (3.3) to get an ODE for g.

$$M_t = \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{4k}{2}\frac{x}{(\sqrt{4kt})^3}g'(p) = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p),$$

$$M_x = \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p),$$

$$M_{xx} = \frac{dM_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4kt}g''(p).$$

Then the PDE (3.3) implies that

$$0 = M_t - kM_{xx} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p) - k\frac{1}{4kt}g''(p) =,$$

which then simplifies to the ODE;

$$g''(p) + 2pg'(p) = 0. \tag{3.5}$$

The first step has now been achieved which was transforming our PDE into an ODE. Our second step would be to solve (3.5) which is a simple linear ODE and can be solved easily. The solution to (3.5) is:

$$g(p) = c_1 \int e^{-p^2} dp + c_2,$$

where c_1 and c_2 are arbitrary constants.

Next, we move on to the third step which is to check whether the initial condition of (3.3) is satisfied. We recall that $p = \frac{x}{\sqrt{4kt}}$ and $M(x,t) = g(p)$ hence

$$M(x,t) = c_1 \int_0^{\frac{x}{4kt}} e^{-p^2} dp + c_2. \tag{3.6}$$

The presence of arbitrary constants enables us to choose a limit of our choice. Also to check the initial condition we need to take the limit $t \to 0^+$ because the above equation is only valid for $t > 0$. Using the initial condition of (3.3),

we have that:

$$\lim_{t\to 0^+} M(x,t) = \begin{cases} c_1 \int_0^{\infty} e^{-p^2} dp + c_2 = c_1 \frac{\sqrt{\pi}}{2} + c_2 = 1, & \text{if } x > 0, \\ c_1 \int_0^{-\infty} e^{-p^2} dp + c_2 = -c_1 \frac{\sqrt{\pi}}{2} + c_2 = 0, & \text{if } x < 0 \end{cases} \tag{3.7}$$

Where we used the fact that $\int_0^{\infty} e^{-p^2} dp = \frac{\sqrt{\pi}}{2}$ to compute the improper integrals. From the computations (3.7) above, we get that:

$$\begin{aligned} c_1 \frac{\sqrt{\pi}}{2} + c_2 &= 1 \\ -c_1 \frac{\sqrt{\pi}}{2} + c_2 &= 0. \end{aligned} \tag{3.8}$$

Solving these two equations from (3.8) simultaneously, we get that $c_1 = \frac{1}{\sqrt{\pi}}$ and $c_2 = \frac{1}{2}$. Substituting these into equation (3.6) we get that

$$M(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{4kt}} e^{-p^2} dp, \tag{3.9}$$

where $t > 0$. The last step is to solve the general IVP i.e. the *Cauchy problem* which we defined earlier in the beginning of chapter 3 as equation (3.1).

$$u_t - k u_{xx} = 0,$$

subject to

$$u(x,0) = \phi(x).$$

But firstly, we need to define the function

$$V(x,t) = \frac{\partial M}{\partial x}(x,t), \tag{3.10}$$

where $M(x,t)$ is a solution to our particular IVP equation (3.3) and is given by equation (3.9). By the invariance properties discussed above, the function $V(x,t)$ solves the heat equation and so does

$$u(x,t) = \int_{-\infty}^{\infty} V(x-y,t)\phi(y)dy, \tag{3.11}$$

for $t > 0$. We will show that $u(x,t)$ is a unique solution to our general IVP. In order to show this, we need to check the initial condition of equation (3.1). Since $V = M_x$, we rewrite $u(x,t)$ in the following way;

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\partial M}{\partial x}(x-y,t)\phi(y)dy = -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[M(x-y,t)]\phi(y)dy. \tag{3.12}$$

Using integration by parts on the last integral in the above equation, we get that:

$$u(x,t) = -M(x-y,t)\ \phi(y)|_{y=-\infty}^{y=\infty} + \int_{-\infty}^{\infty} M(x-y,t)\phi'(y)dy. \tag{3.13}$$

Assuming that $M(x,0)$ has the *Heaviside* function as stated in equation (3.4) and substituting in $t=0$ in equation (3.13), we have:

$$u(x,0)=\int_{-\infty}^{\infty}M(x-y,t)\phi'(y)dy=\int_{-\infty}^{x}\phi'(y)dy=\phi(y)|_{y=-\infty}^{y=x}=\phi(x). \tag{3.14}$$

In the calculation above we assumed that, $\phi(y)$ vanishes as $|y|$ becomes large, which guarantees the vanishing of our boundary terms. We also have to note that the calculation above satisfies the initial condition of our general IVP. At this point we can explicitly define our function $V(x,t)$ which is given by:

$$V(x,t)=\frac{1}{\sqrt{4\pi kt}}e^{\frac{-x^2}{4kt}}. \tag{3.15}$$

Using equation (3.15) we can rewrite equation (3.11) for $t>0$ as follows:

$$u(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}e^{\frac{-(x-y)^2}{4kt}}\phi(y)dy, \tag{3.16}$$

for $t>0$. The equation

$$V(x,t)=\frac{1}{\sqrt{4\pi kt}}e^{\frac{-x^2}{4kt}},$$

is known as the Fundamental solution of the heat equation. So to obtain the general solution of the heat equation with initial condition $u(x,0)=g(x)$,

one has to apply the convolution:

$$u(x,t) = \int_{-\infty}^{\infty} \phi(x-y,t)g(y)dy,$$

where

$$\phi(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{\left(\frac{-x^2}{4kt}\right)}$$

is the Fundamental solution.

One of the interesting parts of this paper is how we use the Fundamental solution. Instead of using it to find the general solution to the heat equation as stated above, we are going to use it to construct a symmetry that will incorporate the boundary condition of the heat equation. The same symmetry constructed will then be used to solve the heat equation. We talk about how such a symmetry can be constructed in the next subsection.

3.2 Non-invariant Boundary Conditions

In the beginning of chapter 2 we discussed the classical method which may be used to find symmetries of partial differential equations. The classical method as stated in [3] can be very restrictive when applied to IVP's and BVP's. This is because it requires that the given initial or boundary condition, as well as the governing PDE remain invariant under the one-parameter Lie group of

infinitesimal transformations. In this subsection, we try to find conditions which are less restrictive. These conditions must be satisfied by infinitesimal generators and need to leave the governing PDE invariant, but do not leave the given boundary or initial condition invariant. It is required that this be done in such a way that they can still be used to solve the IVP or BVP.
This methodology explained in [3], does not only apply to second-order PDEs of the form:

$$u_t = K(x, t, u, u_x, u_{xx}), \tag{3.17}$$

but it also equally applies well to higher-order PDEs of more independent variables. Before we commence with our analysis, we need to introduce the Invariant Surface Condition (ISC):

$$\Omega = \sum_i \xi_i(x, u) u_{x_i} - \phi(x, u) = 0, \tag{3.18}$$

which will be solved once a suitable generator is determined.
Now suppose one wished to solve (3.17) subject to the initial condition

$$u(x, 0) = F(x) \tag{3.19}$$

and the generator

$$\Gamma = \xi(x, t, u)\frac{\partial}{\partial x} + \eta(x, t, u)\frac{\partial}{\partial t} + \phi(x, t, u)\frac{\partial}{\partial u}, \tag{3.20}$$

is admitted by (3.17). The invariant solutions produced from (3.20) will satisfy the ISC (3.18) hence

$$\xi(x,t,u)\frac{\partial u}{\partial x}+\eta(x,t,u)\frac{\partial u}{\partial t}=\phi(x,t,u). \tag{3.21}$$

The initial condition (3.19) says that when $t=0$, $u=F(x)$ and invariant solutions that satisfy this condition will also satisfy

$$\xi(x,0,F(x))F'(x)+\eta(x,0,F(x))u_t(x,0)-\phi(x,0,F(x))=0. \tag{3.22}$$

If the generator (3.20) leaves the initial condition (3.19) invariant, then

$$\eta(x,0,u)=0 \tag{3.23}$$

and

$$\phi(x,0,F(x))-\xi(x,0,F(x))F'(x)=0 \tag{3.24}$$

and hence (3.22) will be satisfied.

On the other hand, if $t=0$ is not left invariant such that $\eta(x,0,u)\neq 0$, then an invariant solution that satisfies initial condition (3.19) needs also to satisfy

$$u_t(x,0)=\frac{\phi(x,0,F(x))-\xi(x,0,F(x))F'(x)}{\eta(x,0,F(x))}, \tag{3.25}$$

from (3.22) above. Now using the governing PDE (3.17), we require that:

$$\frac{\phi(x,0,F(x)) - \xi(x,0,F(x))F'(x)}{\eta(x,0,F(x))} = K(x,0,F(x),F'(x),F''(x)). \quad (3.26)$$

Equation (3.26) is an ordinary differential equation which can be solved using the well known techniques.

At this point, we have most of the tools we are going to need to solve the main problem, which is obtaining a solution to Black-Scholes equation. In the following chapter we discuss how one can transform from Black-Scholes equation to the heat equation and we obtain the solution to Black-Scholes equation.

Chapter 4

Solving the Black-Scholes equation

For convenience, we state the Black-Scholes equation again, along with its terminal condition:

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{4.1}$$

with

$$V(S,T) = \begin{cases} S - K, & \text{if } S > K \\ 0, & \text{if } S < K \end{cases} \tag{4.2}$$

We are going to denote Black-Scholes as B-S. The B-S equation (4.1) along with its terminal condition defines the price of a Europian call option where:

- V is the option value.
- τ is the time until expiration of the option.
- T is the time of maturity of the option.
- σ is the volatility of the underlying stock. It is assumed to be known and constant
- r is the risk-free interest rate. It is also assumed to be known and constant
- S is the current stock price.
- K is the option strike price.

We are going to transform (4.1) to the heat equation using the following transformation steps:

1. $w = e^{-r\tau}V$
2. $y = \ln(S)$

3. $x = y - \tau(r - \frac{\sigma^2}{2}) - \ln(K) + T(r - \frac{\sigma^2}{2})$, $t = \frac{\sigma^2}{2}(T - \tau)$

4. $u = \frac{1}{K}e^{rT}w$ (scaling transformation)

The 1st transformation changes our equation to

$$w_\tau + \frac{1}{2}\sigma^2 S^2 w_{SS} + rSw_S = 0. \tag{4.3}$$

The 2nd transformation changes (4.3) to:

$$w_\tau + \frac{1}{2}\sigma^2 w_{yy} + (r - \frac{1}{2}\sigma^2)w_y = 0. \tag{4.4}$$

The 3rd transformation leaves our (4.4) as:

$$\frac{\sigma^2}{2}w_{xx} = \frac{\sigma^2}{2}w_t. \tag{4.5}$$

We will see that we have already reached our desired one dimensional heat equation without applying the 4th transformation, the reason we apply it to reach the following equation will be clear when we apply the transformations to the terminal condition:

$$u_{xx} = u_t \tag{4.6}$$

Now using the same transformation steps, we transform the B-S boundary condition (4.2) to an initial condition of the heat equation. We note that the terminal condition can also be written in the form $V = \max\{S - K, 0\}$. The

1st transformation step changes our terminal condition to:

$$we^{r\tau} = \max\{S - K, 0\}, \tag{4.7}$$

the 2nd transformation leads to the following equation:

$$we^{r\tau} = \max\{e^{y} - K, 0\}. \tag{4.8}$$

The 3rd step from (4.8) implies that

$$we^{r\tau} = \max\{e^{x+\tau(r-\frac{\sigma^2}{2})+\ln(K)-T(r-\frac{\sigma^2}{2})} - K, 0\}, \tag{4.9}$$

which implies that

$$w = \max\{K(e^{x+\frac{\sigma^2}{2}(T-\tau)-Tr} - e^{-r\tau}), 0\}. \tag{4.10}$$

We need to note that transformation step no.3 says that $t = \frac{\sigma^2}{2}(T - \tau)$ and yet at the terminal condition $t = 0$ this then implies that $T = \tau$, which leaves our equation as;

$$w = \max\{K(e^{x-Tr} - e^{-r\tau}), 0\}.$$

The 4th and last step gives us the following equation:

$$u = \max\{e^{x} - 1, 0\}, \tag{4.11}$$

when $t = 0$. We note that (4.11) can also be written as

$$u(x, 0) = H(x)(e^x - 1).$$

Since we have transformed both the B-S and its terminal condition, we are currently faced with the problem of solving the following IVP:

$$\begin{aligned} u_t &= u_{xx} \\ u(x, 0) &= H(x)(e^x - 1) \end{aligned} \tag{4.12}$$

The methodology explained in section 3.2 found in [3] paper can now be used to solve the IVP (4.12) Consider the following well known symmetries of the heat equation taken from [4]:

$$\begin{aligned} \Gamma_1 =& \frac{\partial}{\partial x} \\ \Gamma_2 =& \frac{\partial}{\partial t} \\ \Gamma_3 =& u\frac{\partial}{\partial u} \\ \Gamma_4 =& x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} \\ \Gamma_5 =& 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} \\ \Gamma_6 =& 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2 + 2t)u\frac{\partial}{\partial u} \\ \Gamma_\alpha =& \alpha\frac{\partial}{\partial u} \end{aligned}$$

The last symmetry Γ_α is called the infinite-dimensional sub-algebra where α is an arbitrary solution of the heat equation meaning that it satisfies:

$$\alpha_t = \alpha_{xx}. \tag{4.13}$$

According to the *Invariance Surface Condition* stated in Section 3.2, the symmetry generator given by:

$$\Gamma = \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u} \tag{4.14}$$

becomes:

$$\frac{\phi(x,0,F(x)) - \xi(x,0,F(x))F'(x)}{\eta(x,0,F(x))} = u_{xx} \tag{4.15}$$

The determining equations used to generate the symmetries of the heat equation stated above are as follows:

- $\xi = c_6 + c_2 x + 2c_3 t + 4c_1 xt,$
- $\eta = c_4 + 2c_2 t + 4c_1 t^2,$
- $\phi = (c_5 - c_3 x - 2c_1 t - c_1 x^2)u + \alpha(x,t)$

and from these, one can deduce that the most general finite-dimensional

symmetry for the heat equation is given by:

$$\Gamma = \left\{ \left[c_1(-2t - x^2) - c_3 x + c_5 \right] u + \alpha \right\} \frac{\partial}{\partial u} + (c_1 4t^2 + 2c_2 t + c_6)\frac{\partial}{\partial t} + (4c_1 xt + c_2 x + 2c_3 t + c_4)\frac{\partial}{\partial x} \tag{4.16}$$

and hence from (4.15) when $t = 0$ we get that:

$$\alpha(x, 0) = (c_1 x^2 + c_3 x - c_5)F(x) + (c_2 x + c_4)F'(x) + c_6 F''(x), \tag{4.17}$$

where the $c's$ are arbitrary constants and $F(x) = H(x)(e^x - 1)$. We now solve $\alpha_t = \alpha_{xx}$ subject to initial condition (4.17). We solve this IVP using the fundamental solution explained above in section 3.1 i.e. we are going to take the convolution $\phi(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ and $\alpha(x, 0) = g(x)$, where $\phi(x, t)$ is the fundamental solution:

$$\alpha(x, t) = \int_{-\infty}^{\infty} \phi(x - y, t) g(y) dy.$$

We used Mathematica to solve this rather complicated integral and we get that:

$$\alpha(x, t) = \frac{1}{2} e^{x+t} \left(1 + \operatorname{erf} \left[\frac{x + 2t}{2\sqrt{t}} \right] \right). \tag{4.18}$$

The solution above is then substituted back into (4.16) in order to obtain a much more detailed general symmetry of the heat equation. For simplicity purposes, we can set the constants found in (4.16) as follows; $c_1 = 0, c_2 =$

$0, c_3 = 0, c_4 = 1, c_5 = 0, c_6 = 0$ such the (4.16) becomes:

$$\Gamma = \alpha \frac{\partial}{\partial u} + \frac{\partial}{\partial x}. \tag{4.19}$$

Note that in (4.19), the coefficient of ∂_t is zero.

As we saw in example 2, (4.19) implies that

$$\frac{du}{\alpha} = \frac{dt}{0} = \frac{dx}{1}.$$

From the above system, we get that $t = I_1$ implying that t is invariant and we also get that $u = \int \alpha dx + I_2$, where $I_2 = f(I_1)$. Using Mathematica to solve the integral over α, we get that:

$$u(x,t) = \frac{1}{2}\left(-\operatorname{erf}\left[\frac{x}{2\sqrt{t}}\right] + e^{x+t}\left(1 + \operatorname{erf}\left[\frac{2t+x}{2\sqrt{t}}\right]\right)\right) - \frac{1}{2}. \tag{4.20}$$

The equation above is the solution of the IVP (4.12), but what we really want is a solution to B-S equation which will also satisfy its corresponding terminal condition. So what we need to do next is to transform (4.20) backwards using the four transformation steps into a solution to B-S equation. We also used Mathematica to help with the backward transformation because doing it by

hand would have been tedious and messy. We finally got that:

$$\begin{aligned} V(S,\tau) = \frac{1}{2}(S + S\,\mathrm{erf}[\frac{(2r+\sigma^2)(T-\tau) - 2\,\mathrm{Log}[K] + 2\,\mathrm{Log}[S]}{2\sqrt{2\sigma^2(T-\tau)}}] \\ + e^{r(-T+\tau)}k(-2 + \mathrm{erfc}[\frac{(2r-\sigma^2)(T-\tau) - 2\,\mathrm{Log}[k] + 2\,\mathrm{Log}[S]}{2\sqrt{2\sigma^2(T-\tau)}}]. \end{aligned} \tag{4.21}$$

The next step would be to check if our solution $V(S,\tau)$ satisfies the B-S equation (1.1) and its terminal condition. One way to check if (4.21) satisfies the B-S equation is compute the individual terms; V_τ, $\frac{1}{2}\sigma^2 S^2 V_{SS}$, rSV_S, rV and combine them to see if the left-hand side equals the right-hand side. One may use any preferred method to do this but Mathematica was used in this paper and indeed we found that (4.21) satisfies the B-S equation. Now to check if the solution satisfies the terminal condition we take the limit of $V(S,\tau)$ as $\tau \to T$ i.e. as the option reaches maturity. The term inside the error function (erf) and the compliment error function (erfc) goes to infinity as we take this limit. We note that

$$\lim_{x\to\infty} \mathrm{erf}(x) = 1$$

$$\lim_{x\to\infty} \mathrm{erfc}(x) = 0,$$

therefore taking these two above into account, we get that $V(S,\tau) = \frac{1}{2}(2S - sK)$ which equals $S-K$. So our solutions satisfies both (1.1) and its terminal condition, which brings us to our next chapter.

Discussion and Conclusions

It is important to point out that our main objective of solving a boundary valued problem using symmetries has been achieved. The method used in this research report is powerful because it shows that, even though symmetries generally favour smooth boundary conditions, we can still use them to solve BVPs with cornered boundary conditions (i.e. boundary conditions which are not smooth) like the B-S terminal condition.

In summary, we transformed (1.1) and its terminal condition (4.2) to the one dimensional heat equation and initial condition ($u(x,0) = H(x)(e^x - 1)$) respectively. Then we found a general symmetry generator that satisfies the Invariant Surface Condition according to [3]. After which the most general finite-dimensional symmetry of the heat equation was computed using the symmetries of the heat equation. We then constructed equation (4.17) and

substituting in $t = 0$, making this equation an initial condition to the equation $\alpha_t = \alpha_{xx}$. We note that α comes from the infinite-dimensional symmetry of the heat equation and it is an arbitrary solution of the heat equation. We then solved this IVP using the Fundamental solution to obtain α. This solution was then substituted back into the finite-dimensional symmetry. We then came up with our choice of arbitrary constants to make the finite-dimensional symmetry simpler to solve. The symmetry ended looking thus; $\Gamma = \alpha\frac{\partial}{\partial u} + \frac{\partial}{\partial x}$. We then solved the resulting system using integration to obtain the solution $u(x,t)$ of the heat equation. This solution was then transformed backwards into a solution of the B-S equation $V(S,\tau)$.

However, there are questions that may arise such as could this method be applied to other BVPs? Before this question is answered, we note that we used the fact that the heat equation has the infinite-dimensional symmetry as part of its symmetry group and that is where we got our α. So, this method can be applied to any BVP as long one can construct the most general finite-dimensional symmetry from the symmetries of the governing PDE, the symmetry group does not necessarily have to contain an infinite-dimensional symmetry. Another question that may arise is that does a different choice of the arbitrary constants affect or change the resulting solution? A different choice of constants does not affect the solution, but it may affect the difficulty in the process of obtaining the solution. So if one chooses a complicated set of constants, obtaining the final solution may also be complicated process.

One may also be asking themselves that why not apply the fundamental solution directly after transforming the BVP into an IVP? The truth is, one would most likely get to the same solution if the Fundamental solution was applied directly but in actual sense, the Fundamental solution is more restrictive as it requires that when $t = 0$, all other variables must be unrestricted. It does not work if for instance we want a general solution for $x > 0$, $x < 0$, $x < k$. The symmetry method is advantageous in the sense that knowing the symmetries opens doors to finding the conserved quantities of the given PDE. But it must be noted that not all PDEs have conserved quantities, however, conserved quantities is one of the questions that arise in this paper deserving further research. For an example, the research would answer questions such as, does the B-S equation have conserved quantities? and if it does, how can they be computed.

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