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Convex functions, their extensions and extremal structure of their epigraphs

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September 10, 2012

Convex functions, their extensions and extremal structure of their epigraphs

by
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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own, unaided work, and that I have not previously in its entirety or partly submitted it at any university for a degree. It is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg.

Signature:.....

On the..... day of..... 20.....in.....

Abstract

Let f be a real valued function with the domain $\mathrm{dom}(f)$ in some vector space X and let $\mathfrak{C}$ be the collection of convex subsets of X. The following two questions are investigated;
1. Do there exist maximal convex restrictions g of f with $\mathrm{dom}(g) \in \mathfrak{C}$?
2. If f is convex with $\mathrm{dom}(f) \in \mathfrak{C}$, do there exist maximal convex extension g of f with $\mathrm{dom}(g) \in \mathfrak{C}$?
We will show that the answer to both questions is positive under a certain condition on $\mathfrak{C}$.

We also show that the extreme points of the epigraph of a real continuous strictly convex function are dense in the graph of such a function, and the set of such extreme points of an epigraph may be equal to the graph.
Moreover we show that a set of extreme points of an epigraph may be equal to a graph of such a convex function under certain conditions. We also discuss conditions under which an epigraph of a real convex function on a Banach space X may, and may not, have extreme points, denting points and/or strongly exposed points.

One of the interesting results in this discussion is that boundary points, extreme points, denting points and the graphs in an closed epigraph of a strictly convex function coincide. Moreover, we show that there is relationship between the extremal structure of an epigraph of a convex function and a point in a domain on which such a function attains its minimum.

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Johannes M. T. Nthebe

Historical background

The theory of convex functions has been studied mostly due to its usefulness and applicability in the Optimisation or optimal value analysis. It originates from Convex Analysis where the structure of convex sets and convex functions is the core of the subject. Applications of convex sets were discovered particularly in the field of Optimisation in 1950s. The importance of these application has in turn sparked a renewed interest in the theory of convex sets.

Convex functions and Optimisation meet in the sense that the graph of a convex function has at least one global optimal point and it is almost guaranteed to get such point hence makes it easier to work within the field of Optimisation.

During the twentieth century, there was an intense research activity and significant results were obtained in geometric functional analysis, mathematical economics, convex analysis and non-linear optimisation, see [14, Preface]. The development of convexity as a subject during the last fifty years was mostly due to W. Frechel (1905-1988), J.-J. Moreau (1923-) and R.T. Rockafellar (1935-). Frechel dealt more with geometrical aspects of convexity, and Moreau applied Mechanics to Mathematics, while Rockafellar is associated with the concept of 'dual problem'. Besides mechanics, and economics as discussed in the Application section, convexity comes naturally in thermodynamic branch of science, see [10, Bibliographical background, p.245].
It is universally understood that convex functions are continuous at least on the interior of their domains but not necessarily differentiable. The reason why convex functions are not necessarily differentiable sparked a lot of debate and hence the weaker form of differentiability was introduced to suit convex functions, that was called *subdifferentiability* denoted by most as ∂f. Differentiability of a function is closely related to the slope and the tangent

to the graph of a function at a given point. Even in the case of convex function, subdifferentiability brought about the new characterisation of convex functions using a tangent at any given point on the graph of such convex functions. It was found, as expected, that the tangent at any point will not be anywhere above the graph. Meaning that if T is a function of a tangent to the graph of f and f is a function whose tangent at the point $(y, f(y))$ is T, then $f(y) \geq T(y)$ for all $y \in \text{dom}(f) \cap \text{dom}(T)$. This became one of the most useful characterisations of convex functions across the board. We intend using these characterisation to construct some convex extension of a given convex function.

This subdifferentiability characterisation was carried over to extended real valued convex functions on vector spaces, Euclidean spaces and partially on Banach spaces. For convex functions on an infinite dimensional vector space it is not easy to find their derivatives, hence it is much more convenient to explore first the subderivatives of such functions as they exists due to the fact that these functions are convex, and then build from there and check as to whether it is differentiable at each point in its domain. The theory on the vector valued convex functions on vector spaces has not had much attention relatively but there is a good paper we used in which they were discussed, see [9].
Recently the study of convex function has evolved into a larger theory about functions which are adapted to other geometries of the domain and/or obey other laws of comparison of means. Examples are log-convex functions, multiplicative convex function, subharmornic functions, and functions which are convex with respect to a subgroup of the linear group, see [14, Preface, p.VIII]

Convex functions have many applications, ranging from those found in the field of Optimisation to those found in Business Mathematics. These applications are useful in the field of Applied Mathematics and many results

have been published on these applications. It would be worthwhile to develop theory to better apply convex functions to solving practical problems in Business setting, such as maximising the profit intake and minimising profit loss.

Moreover, on the structure of convex set, we say a point is an extreme point of a set if it is not an inner point of any line segment contained in that particular set. For instance, the extreme points of a closed triangular region are its vertices, while those of the closed solid ball are its surface points. Note that in convex functions, the domain of such a function is convex and its epigraph is also convex.

The notion of extreme point goes back to H. Minkowski, who proved that if C is a compact convex set in $\mathbb{R}^3$, then each point of C can be expressed as a convex combination of extreme point of C, that is, as a sum, $\Sigma_{i=1}^{m}\lambda_i x_i$, where $\lambda_i > 0$, $\Sigma_{i=1}^{m}\lambda_i = 1$, and each x_i is an extreme point of C. This result is known (in $\mathbb{R}^n$) as Minkowski's theorem, was sharpened by Caratheodory, who showed that if C is a compact convex subset of $\mathbb{R}^n$, then each point of C can be expressed as a convex combination of at most $n+1$ extreme points of C.

In 1940, M. Krein and D. Milman extended Minkowski's theorem to infinite-dimensional spaces by proving that if C is a compact convex subset of a locally convex Hausdorff linear space, then C is the closure of the set of all convex combinations of extreme points of C. The Krein-Milman theorem then served as the starting point for virtually all modern research into the extremal structure of convex sets in infinite-dimensional spaces. See [19] for more background information of extremal structure of convex sets.

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Introduction

Convex functions play an important role in many fields of mathematics and they have applications in other fields of science such as convex and global optimization, differential inclusion theory, and mathematical economic and risk analysis.

Moreover, certain authors in trying to explain the importance of convex functions in Mathematical Analysis stated the following: 'Convexity appears like an octopus, tentacles reaching far and wide, its shape and color changing as it roams from one area to the next. It is quite clear that research opportunity abounds', see [14].

This is the kind of statement that brought onto us the interest in this far-reaching and interesting field of mathematics, and hence the main purpose of this research.

In the first chapter we discuss convex functions and the structure of their domains. Moreover, we discuss extensions of such functions which ensures keeping the convexity property(ies) and determine the largest of the domains on which the function could be extended and still be convex. 'The existence of the convex extension of a set-valued map arises in the solutions of some inverse problems of differential inclusion theory, where it is required to construct a differential inclusion with prescribed attainable sets of integral funnel', see [9, p.674].

In our result we prove the conditional existence of such extension(s) as such extensions do not always occur naturally and we also construct

and define a suitable extension for a $\mathfrak{C}$-convex function from some maximal domain to a bigger domain. The uniqueness, if any, would also be discussed together with conditions of their existence. Most of the results in chapter one, at least the abridged version, have already been published in our paper, see [13].

In chapter two we look at the graphs of convex function and their characteristics. More importantly, we show the relation between the graph and the epigraph of a convex function with respect to the extreme points, denting points and consequently strongly exposed points of the convex (and sometimes closed) epigraph. Moreover, we look at the relation between extremal structure of convex (and closed) epigraphs, and the set minimizer(s) of convex functions. Furthermore we discuss the convexity, or lack of, of such set of minimizers of convex functions along with the extreme points preserving maps.
One cannot have a complete discussion of extremal structure of closed and convex sets without making mention of the Krein-Milman Property and the closed convex hull of extreme points. Hence one of the important results in chapter two is the charaterisation of closed convex hull of extreme points of convex epigraph.

In chapter three we rehearse some of the interesting result on the differentiability of convex functions, and most importantly the subdifferentiability of convex function. We show that that even though a convex function might be non-differentiable, it might be subdifferentiable. Moreover we show the relationship between subdifferentiability, global minimizers and extremal structure of epigraphs.

0.1 Preliminaries and basic concepts

Henceforth X denotes a real normed vector space unless otherwise stated, $f : \text{dom}(f) \subseteq X \to \mathbb{R}$ denotes a real valued function with non-empty domain in X and

$$\mathfrak{C} \subseteq \{C \subseteq A : \emptyset \neq C \text{ is convex } \}$$

denotes a non-empty class of convex non-empty subsets of A, for some fixed non-empty subset A in X.

The following are well-known definitions of both the convex function and the convex set;

Definition 0.1.1 A set C in X is said to be *convex* if for any $x, y \in C$ and $\alpha \in [0, 1]$ we have $\alpha x + (1 - \alpha)y \in C$. Moreover, a real valued function $f : A \subseteq X \to \mathbb{R}$ is *convex* if A is convex and for any $x, y \in A$ and $\lambda \in [0, 1]$ we have $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$.

Clearly if a function $f : \text{dom}(f) \to \mathbb{R}$ is convex on a convex set $\text{dom}(f)$, then $\text{dom}(f)$ is the maximal subset of $\text{dom}(f)$ on which f is convex. Hence in trying to determine the maximal subset(s) of any $\text{dom}(f)$, if any, where f is not necessarily convex on $\text{dom}(f)$ but has a restriction convex on some subset of $\text{dom}(f)$, we make the following definition;

Definition 0.1.2 Let A be a non-empty subset of a vector space X.
(1) A real function $f : A \subseteq X \to \mathbb{R}$ is $\mathfrak{C}$-*convex* if there is $C \in \mathfrak{C}$ such that $f|_C$ is a convex restriction of f.
(2) Let $f : A \subseteq X \to \mathbb{R}$ be a $\mathfrak{C}$-convex function. Then a non-empty subset $M \in \mathfrak{C}$ satisfying
(a) f is convex on M, and
(b) there exists no convex set $P \in \mathfrak{C}$ such that $M \subset P \subseteq A$ and f is convex on P,
is called *a* $\mathfrak{C}$-*maximal domain of convexity* ($\mathfrak{C}$-MDC) for f.

If $\mathfrak{C}$ contains a singleton, then every function $f : A \subseteq X \to \mathbb{R}$ is $\mathfrak{C}$-convex and that would defeat the purpose of this discussion. Our objective is to discuss non-trivial $\mathfrak{C}$-convex functions, hence excluding those defined on a singleton, and thus we make the following assumption:

Assumption 0.1.3 $\emptyset \neq \mathfrak{C} \subseteq \{C \subseteq A : \emptyset \neq C \text{ is convex and infinite}\} = \mathfrak{C}_0$.

Moreover, since it is our aim to discuss the maximal of those convex and infinite subsets C, looking at chains of convex sets C would help, hence there is one additional and vital condition we impose on $\mathfrak{C}$ we shall discuss subsequently.

One cannot discuss the convex functions on convex domain without mentioning the convex epigraphs, as the two concepts coincide. Hence the epigraph and, its important subset, the graphs are defined as follows;

Definition 0.1.4 An *epigraph* of a real convex function f is the set of points lying on and above the graph of f. $\text{epi}(f)$ and $\text{gr}(f)$ will denote the epigraph and the graph of f respectively, and they are defined as follows:

$$\text{epi}(f) = \{(x, \lambda) : f(x) \leq \lambda,\ x \in \text{dom}(f)\} \text{ and}$$
$$\text{gr}(f) = \{(x, \lambda) : f(x) = \lambda,\ x \in \text{dom}(f)\}$$

0.2 Basic concepts of convex sets and convex functions

In this section we discuss some of the well-known and sometimes trivial concepts associated with convex sets and convex functions as the foundation to the subsequent topics we shall be looking at henceforth.

The following is the result outlining properties of convex sets

Proposition 0.2.1 *Let A and B in a normed vector space X be convex and $\lambda \in \mathbb{R}$ be a real number. Then the following operations preserve convexity*

1. *Intersection*
2. *Scalar multiplication*
3. *Closure*
4. *Interior*
5. *Coordinate Projection*
6. *Translate of a set*
7. *Sum of sets*
8. *Direct sum*

Proof

Suppose A and B in X are convex and $\alpha \in [0,1]$.

1. Suppose $A \cap B \neq \emptyset$. Take any $a, b \in A \cap B$ and $\alpha \in [0,1]$. It follows that $a, b \in A$ and $a, b \in B$. Since A and B are convex, we have $\alpha a + (1-\alpha)b \in A$ and $\alpha a + (1-\alpha)b \in B$, and thus $\alpha a + (1-\alpha)b \in A \cap B$.

2. Let $\lambda \in \mathbb{R}$ be a scalar and $\lambda A = \{\lambda a : a \in A\}$. Take any λa and λb in λA with $a, b \in A$ and $\alpha \in [0,1]$. Since A is convex and hence $\alpha a + (1-\alpha)b \in A$, we have $\alpha(\lambda a) + (1-\alpha)(\lambda b) = \lambda(\alpha a + (1-\alpha)b) \in \lambda A$. This follows from the fact that A is convex and $\alpha a + (1-\alpha)b \in A$.

3. Let $\mathrm{cl}(A)$ be the closure of A and take any $x, y \in \mathrm{cl}(A)$ and $\alpha \in [0,1]$. Then there exists sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ in A such that $(x_n)_{n\geq 1} \to x$ and $(y_n)_{n\geq 1} \to y$. Since A is convex $\alpha x_n + (1-\alpha)y_n \in A$ for each $n \geq 1$. Hence $m = \alpha x + (1-\alpha)y = \alpha \lim_{n\to\infty} x_n + (1-\alpha) \lim_{n\to\infty} y_n = \lim_{n\to\infty} [\alpha x_n + (1-\alpha)y_n]$ Hence $(\alpha x_n + (1-\alpha)y_n)_{n\geq 1} \in A$ is a convergent sequence and converges to m for each $\alpha \in [0,1]$. Hence $m \in \mathrm{cl}(A)$ and it follows that $\mathrm{cl}(A)$ is convex.

4. Let $\mathrm{int}(A)$ be the interior of a convex set A and take $x, y \in \mathrm{int}(A)$ with $x \neq y$, $\alpha \in (0,1)$ and $m = \alpha x + (1-\alpha)y$. Choosing $\beta > 0$ such that $B_\beta(y) \subset A$ we show that $B_{(1-\alpha)\beta}(m) \subset A$. Clearly $m - x = \alpha x - x + (1-\alpha)y = (1-\alpha)y - (1-\alpha)x = (1-\alpha)(y-x)$ and hence $\frac{\|m-x\|}{\|y-x\|} = 1-\alpha$ and con-

sequently $B_{(1-\alpha)\beta}(m) = \alpha x + (1-\alpha)B_\beta(y)$. Clearly $\alpha x + (1-\alpha)\tau \in \text{int}(A)$ for each $\tau \in B_\beta(y)$ hence $B_{(1-\alpha)\beta}(m) = \alpha x + (1-\alpha)B_\beta(y) \subseteq \text{int}(A)$ and thus $m \in \text{int}(A)$ and consequently $\text{int}(A)$ is convex.

5. Let $C = \{x_1 : (x_1, x_2) \in A \text{ for some } x_2\}$ be coordinate projection. Take $y_1, z_1 \in C$. It follows that $(y_1, y_2), (z_1, z_2) \in A$ for some y_2, z_2 and $\alpha(y_1, y_2) + (1-\alpha)(z_1, z_2) = (\alpha y_1 + (1-\alpha)z_1, \alpha y_2 + (1-\alpha)z_2) \in A$, $\alpha \in [0, 1]$, as A is convex. Hence $\alpha y_1 + (1-\alpha)z_1 \in C$ and consequently C is convex.

6. For A convex, it follows from [18, p.16] that the translate $A + x = \{a + x : a \in A, x \in \mathbb{R}^n\}$ is also convex.

7. It follows from [18, Theorem 3.1, p.16] that the sum of sets $A + B = \{a + b : a \in A, b \in B\}$ is convex.

8. It follows from [18, Theorem 3.5, p.19] that direct sum $A \oplus B = \{x = (y, z) : y \in A, z \in B\}$ is convex. □

Remark 0.2.2 (a) Convex functions are continuous on $\text{int}(\text{dom}(f))$ where $\text{dom}(f) \subseteq X$ and X a normed vector space, and if ever they are not continuous that would be at the boundary $\partial(\text{dom}(f))$ of its domain, see [14, Proposition 3.5.2, p.119]

(b) It is also clear in subsequent sections that a continuous real (convex) function f is not neccessarily differentiable on $\mathbb{R}$.

The following example illustrates this assertion:
Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a real (convex) function defined by $f(x) = |x|$. This function f is not differentiable at 0, but it is **continuous** there. This follows from the fact that $\lim\limits_{x \to 0^-} \frac{f(x)-f(0)}{x-0} = -1 \neq 1 = \lim\limits_{x \to 0^+} \frac{f(x)-f(0)}{x-0}$. Yet $\lim\limits_{x \to 0} f(x) = f(0)$.
Clearly continuity at a point does not imply differentiability at a point even for convex function.

(c) *Conversely*,
Differentiable real functions of one variable are not necessarily convex as in the following example;

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be real function defined by $f(x) = x^3$. This function f is differentiable but not convex, though convex on some subset of its domain, that is it has a convex restriction.

(d) One of the ways in which on one can test for convexity is;

f is convex if its second derivative f'' exists and is non-negative everywhere.

Note that, using the example above we have $f''(x) = 6x$ and hence $f''(x) < 0$ for all $x < 0$. Consequently f is not convex. Yet $f''(x) = 6x \geq 0$ for each $x \in [0, \infty)$. Hence $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^3$ is convex.

Lemma 0.2.3 *Any real linear continuous function $f : \text{int}(I) \subset \mathbb{R} \to \mathbb{R}$ is differentiable and convex.*

Proof

Let $f(x) = ax + c$ for any $a, c \in \mathbb{R}$. Obviously $f'(x) = a$ for all $x \in I$, hence the limit exists at c for all $c \in \text{int}(I)$, that is

$$\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c},$$

and thus f is differentiable.

Moreover $f'' = 0 \geq 0$ for all $x \in I$. Hence f is convex. □

Differentiability might be tricky at the endpoints and a convex function f is not always continuous at the endpoints.
Contrapositively; lack of continuity clearly implies lack of differentiability. This problem will be explored later in this discussion along with the weaker form of differentiability that seem to work on convex functions.

On convergence of convex functions, let us look at the following result;

Theorem 0.2.4 *[10, Theorem 3.1.4, p.105] Let the convex functions $f_k : \mathbb{R}^n \to \mathbb{R}$ converge pointwise for $k \to +\infty$ to $f : \mathbb{R}^n \to \mathbb{R}$. Then f is convex and for each compact set S in $\mathbb{R}^n$ the convergence $f_k \to f$ is uniform on S*

Example 0.2.5 1. Consider the following convex real function $f = x^2$ and the following sequence of its translations $(f_i(x))_{i\geq 1}$ defined by $f_i(x) = x^2 + n_i$ where $n_i \to 0$ as $i \to \infty$, with $n_i \in \mathbb{R}$ for all $i \in \mathbb{N}$. Hence $(n_i)_{i\geq 1}$ is a sequence of real numbers converging to zero.

(a) It is easy to see that each f_i is a real convex continuous function.

(b) It is also easy to see that $\lim_{i\to\infty} f_i = f$ and the convergence is pointwise. Furthermore f, f_i have convex epigraphs for each i, hence we have a sequence $(\text{epi}(f_i))_{i\geq 1}$ of convex epigraphs.
We say $(\text{epi}(f_i))_{i\geq 1}$ converges to $\text{epi}(f)$ if for each $x \in \text{dom}(f)$ there exists a sequence $(x_i)_i \in \text{dom}(f_i)$ converging to $x \in \text{dom}(f)$ and $\lim_{i\to\infty} f_i(x_i) = f(x)$.

Clearly $\lim_{i\to\infty} \text{epi}(f_i) = \text{epi}(f)$ for the f_i and f as defined above.

2. Consider the sequence of convex continuous functions $(f_i)_{i\geq 1}$ defined by $f_i = x^{2i}$, $i \in \mathbb{N}$. Then the following hold: $\bigcap_{i\geq 1}\text{epi}(f_i) = \text{epi}(h)$ where h is defined by $h(x) = x^2$, and $\text{dom}(h) = [-1, 1]$.

Below are well-known properties of convex functions we shall be using and further exploring in subsequent chapter we deem important;

Properties of convex functions

1. A continuously differentiable function of one variable is convex on an interval if and only if it lies above all of its tangents. In other words $f(y) \geq f(x) + f'(x)(y - x)$ for all x and y in the interval, see [14, p.30].

The same will be explored in the subsequent chapters especially for continuous convex functions which are not necessarily differentiable.

2. A convex function f satisfies the following: $\sum_{i=1}^n \lambda_i x_i \in C$ and $f(\sum_{i=1}^n \lambda_i x_i) \leq \sum_{i=1}^n \lambda_i f(x_i)$ whenever $x_i \in C$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, where C is in the domain of f, see [14, Lemma 8, p.8].

This is clearly the generalised definition of convex functions from just two elements in the set.

Lemma 0.2.6 *A continuous extended-real-valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is closed and convex if* $\text{epi}(f)$ *is closed and convex, respectively.*

The closure of an epigraph is important and is integral to our subsequent discussion on extremal structure of epigraphs, especially due to the boundary of such epigraphs being contained in the epigraph itself.

Proposition 0.2.7 *If $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is a convex function, then $\{x \in C : f(x) < a\}$, $\text{cl}(\{x \in C : f(x) < a\})$ and $\{x \in C : f(x) \leq a\}$ are convex, where $a \in \mathbb{R}$ and C is a convex subset of $\mathbb{R}^n$.*

Proof
It is noteworthy that if $A = \{x \in C : f(x) < a\}$ is convex then so is $\text{cl}(\{x \in C : f(x) < a\})$ as per Proposition 0.2.1 no 3.
We show $A = \{x \in C : f(x) < a\}$ is convex:
Suppose f is convex and take $y, z \in A$ and $\lambda \in [0, 1]$. It follows that $y, z \in C$ and hence $\lambda y + (1 - \lambda)z \in C$ as C is convex, and $f(z) < a$ and $f(y) < a$. Since f is convex, we have

$$\begin{aligned} f(\lambda y + (1 - \lambda)z) &\leq \lambda f(y) + (1 - \lambda)f(z) \\ &< \lambda a + (1 - \lambda)a \\ &= \lambda a + a - \lambda a = a \end{aligned}$$

Hence $\lambda y+(1-\lambda)z \in C$ and $f(\lambda y+(1-\lambda)z) < a$ or each $\lambda \in [0, 1]$ and thus A is convex. Moreover, appealing to Proposition 0.2.1, $\text{cl}(\{x \in C : f(x) < a\})$ is convex.

Let $B = \{x \in C : f(x) \leq a\}$ and take $y, z \in B$. It follows that $f(y) \leq a$ and $f(z) \leq a$ and thus $\max\{f(y), f(z)\} \leq a$. Moreover, since $B \subseteq C$, we have $y, z \in C$ and hence $\lambda y + (1 - \lambda)z \in C$ for $\lambda \in [0, 1]$ as C is convex. Since f is convex, we have
$f(\lambda y + (1 - \lambda)z) \leq \max\{f(y), f(z)\} \leq a$. It follows that $\lambda y + (1 - \lambda)z \in B$

and for any $y, z \in B$, and thus B is convex. $\square$

There is another characterisation of convex function that is worth mentioning and it is as follows;

Corollary 0.2.8 *Let $f : X \to \mathbb{R}$ a convex function on a normed linear space X, then the mapping $\phi : X \to \mathbb{R}$ defined by*

$$\phi(x) = \inf\{\lambda : (x, \lambda) \in \text{epi}(f)\}$$

is a convex function. Moreover $f = \phi$.

Proof

Take any $y, x \in X$ and $\alpha \in [0, 1]$ such that $\alpha x + (1 - \alpha)y \in X$. Then

$$\begin{aligned}
\phi(\alpha x + (1-\alpha)y) &= \inf\{\lambda : (\alpha x + (1-\alpha)y, \lambda) \in \text{epi}(f)\} \\
&= \inf\{\lambda : f(\alpha x + (1-\alpha)y) \leq \lambda\} \\
&= \{\lambda : f(\alpha x + (1-\alpha)y) = \lambda\} \text{ since } f \text{ is convex} \\
&= \{\lambda : \lambda \leq \alpha f(x) + (1-\alpha)f(y),\ \alpha \in [0,1]\} \\
&\leq \{\alpha f(x) + (1-\alpha)f(y) : (\alpha x + (1-\alpha)y, \lambda) \in \text{epi}(f), \alpha \in [0,1]\} \\
&= \alpha\{f(x) : (x, \lambda) \in \text{epi}(f)\} + (1-\alpha)\{f(y) : (y, \lambda) \in \text{epi}(f)\} \\
&= \alpha(\inf\{\lambda : f(x) \leq \lambda\} + (1-\alpha)\{\lambda : f(y) \leq \lambda\},\ \alpha \in [0,1]. \\
&= \alpha(\inf\{\lambda : (x, \lambda) \in \text{epi}(f)\} + (1-\alpha)\{\lambda : (y, \lambda) \in \text{epi}(f)\} \\
&= \alpha\phi(x) + (1-\alpha)\phi(y).
\end{aligned}$$

Hence ϕ is convex.

Moreover,

$\phi(x) = \inf\{\lambda : (x, \lambda) \in \text{epi}(f)\} = \inf\{\lambda : f(x) \leq \lambda\} = \{\lambda : f(x) \leq \lambda\} = f(x)$. Hence $f(x) = \phi(x)$ for each $x \in X$, and hence $f = \phi$. $\square$

Chapter 1

Real convex functions and maximal domains

In this chapter we discuss convex domains of convex functions and we also identify maximal domains on which convex functions are defined, especially maximal domains of a given (not necessarily convex) function with convex restriction. Furthermore we discuss extensions of convex functions, or convex restrictions, and condition(s) of their existence. Note that some of the results in this chapter are published in our paper [13].

In this chapter we address two questions associated with convex functions. Namely;

1. Firstly, given an arbitrary real-valued function f and $\mathfrak{C}$ the collection of convex subsets of a vector space X, do there exist or can we find a maximal convex restrictions of f whose domain is in the collection $\mathfrak{C}$?

2. Secondly, what can be said about extensions and maximal extensions of convex functions?

The first question is encouraged by considering non-convex functions and wanting to take advantage of convexity property. We aim to find conditions, if any, under which restiction of such a function to some subset(s) of its

domain will be convex. One might also want to restrict the domains to certain classes of convex sets, such as closed or open convex sets.

Real non-convex functions have trivial convex restrictions especially when restricted to singleton subset(s) of its domain. It turns out that an additional condition is needed to guarantee the existence of maximal convex restrictions under such general conditions, and this condition will be discussed subsequently.

The simple example is that of a non-convex function $f(x) = x^3$ on $\mathbb{R}$ whose restriction $g(x) = x^3$ on $[0, \alpha)$ is convex, for any $\alpha > 0$. Note that the sets $[n, \infty)$ for each $n \in \mathbb{N}$ with $n < \infty$ would be a domain of convexity of g for each n, and that $[0, \infty)$ would be the maximal of such domains on which g would be convex. The uniqueness of such a domain, if it exists, will be discussed, yet it is easy to see that in some cases such maximal domain does not exist, i.e, where $g(x) = \cos x$, on $\mathbb{R}$. It is worth noting though that it is not always easy to determine the maximal domain on which a function is convex.

Conditional existence of convex extensions have been discussed by few other authors, and below we make mention of the few of those extensions and also identify the difference between previous publications and our results.

In [9], the authors considered set-valued functions $V : [t_0, \theta] \to \text{comp}(\mathbb{R}^n)$, whose epigraphs are convex. The authors give necessary and sufficent conditions for the existence of extensions of V to larger compact intervals in terms of upper and lower derivatives.

They characterised the existence of convex extensions as follows;

Denoting by $\text{comp}(\mathbb{R}^n)$ the family of non-empty compact subsets of $\mathbb{R}^n$, that is $\text{comp}(\mathbb{R}^n) = \{C \in \mathbb{R}^n : \emptyset \neq C \text{ compact }\}$,

Proposition 1.0.9 *[9, Existence Proposition 2.1, p.675] Let $\alpha > 0$, $V_\alpha(.) : [t_0 - \alpha, \theta] \to comp(\mathbb{R}^n)$ and $V(.) : [t_0 - \alpha, \theta] \to comp(\mathbb{R}^n)$ be set-valued maps. Suppose that $V_\alpha(t_0) = V(t_0)$ and $V(t) \subset V_\alpha(t)$ for all $t \in [t_0, \theta]$. Then the set valued map $W(.) : [t_0 - \alpha, \theta] \to comp(\mathbb{R}^n)$ defined by $W(t) = V_\alpha(t)$ on $[t_0 - \alpha, t_0)$ and $W(t) = V(t)$ on $[t_0, \theta]$ is a left-hand convex α-extension*

of $V(.)$.

Furthermore, in [9, Maximal Extension Theorem 4.4, p.682] the authors prove the existence of maximal extension of the set-valued functions and in [9, No Extension Example 3.4, p.680] they show that convex extensions do not always exist, for any convex function.

On the other hand, we are concerned with convex functions in the usual sense, i. e., functions whose epigraphs are convex. This corresponds to convex set-valued functions mapping to semi-infinite intervals, and consequently, the results in [9] are disjoint from our results on convex extensions.

Moreover, in [20] an explicit necessary and sufficient condition is given such that a real-valued function from the boundary of a nonempty bounded open convex set $\Omega \subset \mathbb{R}^n$ has a Lipschitz continuous extension to a function on $\mathbb{R}^n$. It is well-known that Lipschitz continuous functions have similar characteristics to those of convex functions.

The main result is as follows;
Denote by $\partial(\Omega)$ the boundary of the set Ω, and hence we have;

Theorem 1.0.10 *[20, Conditional Existence Theorem 1, p.30] Let* $\Omega \subset \mathbb{R}^n$ *be a non-empty bounded open and convex set and let* $f : \partial(\Omega) \to \mathbb{R}$ *be a function (not necessarily convex). The function* f *admits a convex extension* $W_f : \mathbb{R}^n \to \mathbb{R}$ *satisfying the Lipschitz condition (equivalently convexity condition) with constant* L *if and only if the following condition (call it condition (C)) is met,*

$f(z) - \frac{f(x)+f(y)}{2} \leq L\|z - \frac{x+y}{2}\|$ *(C)*

for all $x, y, z \in \partial(\Omega)$.

Put differently;
Let $f : \partial(\Omega) \to \mathbb{R}$ be a function and Ω be as defined above. If f fulfills condition (C), then there exists a convex extension $W_f : \mathbb{R}^n \to \mathbb{R}$ of f.

In [22] the authors construct a convex extension and consider a condition which is necessary and sufficient for the construction of such a convex extension. This construction may be related to global optimisation theory in that, the condition asserts and imposes upper bounds for the images of the constructible convex extension.

The following is a definition of an extension and the main theorem on the construction of convex extension and the condition(s) under which such a construction could be possible, both stated in [22].

Definition 1.0.11 *[22, Definition 1, p.249]* Let $C \subseteq X$ be a convex set in a normed vector space X and $Y \subseteq C$. A convex extension of a function $\phi : Y \to \mathbb{R} \cup \{\infty\}$ over C is any convex function $\eta : C \to \mathbb{R} \cup \{\infty\}$ such that $\eta(x) = \phi(x)$ for all $x \in Y$.

Theorem 1.0.12 *[22, Theorem 2, p.249] A convex extension of* $\phi(x) : Y \to \mathbb{R}$ *over a convex set* $C \supseteq Y$ *may be constructed if and only if*

$$\phi(x) \leq \min\{\Sigma_i \lambda_i \phi(x_i) : \Sigma_i \lambda_i x_i = x, \Sigma_i \lambda_i = 1, \lambda_i \in (0,1), x_i \in Y\}$$

for all $x \in X$ *where the above summation consists of finite terms.*

Another extension theorem was obtained in [16, Theorem 1], in which functions on non-convex domains in a vector space V are considered and the existence of convex extensions to all of V are discussed. Here the definition of convexity allows convex functions to take the values $\pm\infty$, hence extended real valued (convex) functions are considered.

Their main result is as follows;

Theorem 1.0.13 *[16, Extension Theorem 1, p.252] Let* V *be a linear space over the reals* $\mathbb{R}$*, and* $T \subset V$ *an arbitrary subset. Moreover, let* $f : T \to \mathbb{R} \cup \{\pm\infty\}$ *be convex. Then there exists a convex function* $g : V \to \mathbb{R} \cup \{\pm\infty\}$ *which extends* f*.*

The authors in [16, p.254] proclaim that it is worthwhile to define convexity of a function also on non-convex domains since these may occur in economics in a natural way, in particular in risk aversion problems.

Our main results on the other hand prove that the existence of a maximal epigraph extension is realised under certain conditions, namely, CUP (chain union property) along with the pseudo-arbsorbing property. For convex functions defined on intervals in $\mathbb{R}$, this maximal epigraph extension may be unique, under certain stated conditions.

1.1 Maximal domains of convexity

In this section we consider real and not necessarily convex functions on subsets of a vector space X and discuss their maximal convex restrictions.

For any $x, y \in A$, denote by $[x, y]$ the line segment connecting x and y, that is, $[x, y] = \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$.

Recall the following definition of a chain;

Definition 1.1.1 A collection $\mathfrak{B}$ of subsets, or family of sets, in $\mathfrak{C}$ is called a *chain* in $\mathfrak{C}$ if for each $B_1, B_2 \in \mathfrak{B}$, we have $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

Furthermore, an element $F \in \mathfrak{C}$ is called *an upper bound* for $\mathfrak{B}$ if $B \subseteq F$ for each $B \in \mathfrak{B}$.

Henceforth we shall denote by $\max\{B_1, B_2\}$ the larger of the two elements $B_1, B_2 \in \mathfrak{B}$.

Lemma 1.1.2 *Let $\mathfrak{B}$ be a chain in $\mathfrak{C}$ and $D = \bigcup_{B \in \mathfrak{B}} B$ be union of all $B \in \mathfrak{B}$. Then D is convex.*

Proof

Take $a, b \in D$. Then $a \in B_i$ and $b \in B_j$ for some $B_i, B_j \in \mathfrak{B}$. Since $\mathfrak{B}$ is a chain (and partially ordered by inclusion), we have $B_i \subseteq B_j$ and

hence $a, b \in B_j$. Since $\mathfrak{B}$ is a chain in $\mathfrak{C}$ and each $B \in \mathfrak{C}$ is convex, we have that B_j is convex and thus $\lambda a + (1-\lambda)b \in B_j$ in $\mathfrak{B}$. It follows that $\lambda a + (1-\lambda)b \in B_j \subseteq \bigcup_{B \in \mathfrak{B}} B = D$ for each $\lambda \in [0,1]$, and thus D is convex. $\square$

Definition 1.1.3 *[13, Definition 1.4, p.653]* If for any chain $\mathfrak{B} \subset \mathfrak{C}$ we have $\bigcup_{B \in \mathfrak{B}} B \in \mathfrak{C}$, then we say that $\mathfrak{C}$ satisfies the *Chain Union Property* (CUP).

Note that $\mathfrak{C}$ satisfying the CUP means that $\mathfrak{C}$ is chain-complete with every least upper bound of a chain being the union of the sets in the chain. A set is said to be chain complete if each chain of its subsets has a least upper bound.

Let $f : A \subseteq X \to \mathbb{R}$ be $\mathfrak{C}$-convex. Then we denote by

$$\mathfrak{C}_f = \{F \in \mathfrak{C} : f|_F \text{ is convex}\}$$

the collection of $\mathfrak{C}$-domains in X on which f is convex. Clearly $\mathfrak{C}_f$ is a subset of $\mathfrak{C}$, and since f is $\mathfrak{C}$-convex, $\mathfrak{C}_f \neq \emptyset$.

Since $\mathfrak{C}_0 \neq \emptyset$, A contains at least one non-trivial convex set C. It will become clear that the CUP is important for the collection $\mathfrak{C}_f$ in order to obtain the maximal elements in $\mathfrak{C}_f$, especially in the context of the well-known Zorn's lemma and chain completeness.

The CUP in $\mathfrak{C}$ therefore requires that any chain $\mathfrak{B}$ is closed under union, thus CUP would be a useful assumption in subsequent results. Consider the following examples;

Example 1.1.4 (1) The collection $\mathfrak{C}_0 = \{C \subseteq A : C \text{ convex and infinite}\}$ satisfies the CUP. Clearly for any chain $\mathfrak{K}$ in $\mathfrak{C}_0$, $D = \bigcup_{F \in \mathfrak{K}} F \in \mathfrak{C}_0$ as D is convex.

The convexity of D follows from the similar reasoning as the one given in the proof of Lemma 1.1.2. Moreover the infiniteness of D follows from

the assumption that each $F \in \mathfrak{K} \subseteq \mathfrak{C}_0$ is infinite. Thus $D = \bigcup_{F \in \mathfrak{K}} F \in \mathfrak{C}_0$.

(2) Let X be a Banach space, and consider the collection

$$\mathfrak{C}' = \{C \in \mathfrak{C}_0 : C \text{ open}\}.$$

Clearly $\mathfrak{C}' \neq \emptyset$ if and only if $\text{int}(A) \neq \emptyset$ (and $\mathfrak{C}_0 \neq \emptyset$), and then $\mathfrak{C}'$ satisfies the CUP.

Clearly each $C \in \mathfrak{C}'$ is a convex infinite and open set, and hence for each $x \in C$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset C$.
$\Rightarrow$ If $\mathfrak{C}' \neq \emptyset$ then $B_\epsilon(x) \subset C \subseteq A$ for some $C \in \mathfrak{C}'$. It follows that $x \in \text{int}(A)$ and hence $\text{int}(A) \neq \emptyset$.
$\Leftarrow$ If $\text{int}(A) \neq \emptyset$ then there is $x \in \text{int}(A)$ and thus $B_\epsilon(x) \subset A$. Moreover, since $\mathfrak{C}_0 \neq \emptyset$ then A is infinite as each $C \subseteq A$ is infinite and so is $\text{int}(A)$. It follows that $\text{int}(C)$ is also infinite and convex, see Proposition 0.2.1 4. and hence $\mathfrak{C}' \neq \emptyset$.
To show that $\mathfrak{C}'$ satisfies the CUP, take $D = \bigcup_{M \in \mathfrak{O}} M$ for any chain $\mathfrak{O}$ in $\mathfrak{C}'$. The convexity of D follow from Lemma 1.1.2, the infiniteness of D follows from the infiniteness of each of $M \in \mathfrak{O}$. The openness of D follows from the fact that any union of open set is open. That is, if we take any $M = \text{int}(M)$ for each $M \in \mathfrak{O}$ and hence $\text{int}(D) \subseteq D = \bigcup_{M \in \mathfrak{O}} \text{int}(M)$. If we take any $x \in D$ then $x \in \text{int}(M)$ for some $M \in \mathfrak{O}$. It follows that $x \in \text{int}(D)$ as $\text{int}(M) \subseteq \text{int}(D)$. Hence $\text{int}(D) = D$.

Remark 1.1.5 Henceforth we consider $\mathfrak{C}'$ only if $\text{int}(A)$ is non-empty, and thus $\mathfrak{C}'$ satisfies Assumption 0.1.3.

Not all collections $\mathfrak{C}$ satisfy the CUP, hence we subsequently give examples of those collections $\mathfrak{C}$ with, and those lacking, this property:

Example 1.1.6 Let $A = \mathbb{R}$, $\mathfrak{C} = \{(a, b) : a, b \in \mathbb{Q}, a < b\}$ be a collection of convex subsets in $\mathbb{R}$, and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x$.

(a) Let $B_n = (a_n, b_n) \in \mathfrak{C}$ with $a_n, b_n \in \mathbb{Q}$ $(n \in \mathbb{N})$ be such that $a_n \searrow \pi$ and $b_n \nearrow 2\pi$. Consequently $\mathfrak{B} = \{B_n : n \in \mathbb{N}\}$ is a chain in $\mathfrak{C}$ as $(a_k, b_k) \subseteq (a_{k+1}, b_{k+1}) \subseteq (\pi, 2\pi)$ for each $k \leq k+1 \in \mathbb{N}$.

Clearly $B_n = (a_n, b_n) \subseteq (\pi, 2\pi)$ for each $n \in \mathbb{N}$ and hence $\bigcup_{n\in\mathbb{N}} B_n \subseteq (\pi, 2\pi)$. Conversely, for each $x \in (\pi, 2\pi)$ there exists an open interval $I = (a, b) \subset \mathbb{R}$ such that $x \in I \subseteq (\pi, 2\pi)$ and thus $\bigcup_{n\in\mathbb{N}} B_n = (\pi, 2\pi)$. Yet $(\pi, 2\pi) \notin \mathfrak{C}$ since $\pi \notin \mathbb{Q}$. Hence $\mathfrak{C}$ does not satisfy the CUP.

(b) $\mathfrak{C}_f$ contains no maximal element and fails the CUP. This follows from the fact that $(\pi, 2\pi) \notin \mathfrak{C}$ and hence $\mathfrak{C}$ contains no maximal element. This means that $f : (\pi, 2\pi) \to \mathbb{R}$ is not relevant for $\mathfrak{C}_f$ as $\pi \notin \mathbb{Q}$.

(c) Observe that $\mathfrak{C}_0 = \{I \subseteq \mathbb{R} : \exists a \in I \text{ with } a \notin \partial(I)\}$ is the collection of non-trivial intervals I in $\mathbb{R}$. Clearly, for each $I \supset [(2k-1)\pi, 2k\pi], k \in \mathbb{Z}$, we have $f|_I$ not convex as $f''(x) = -\sin x$ and $f''(\alpha) < 0$ for some $\alpha \in [(2k-1)\pi - m, [(2k-1)\pi] \cup [2k\pi, 2k\pi + m]$ where $m \in (0, \pi)$ and each fixed $k \in \mathbb{Z}$. It follows that $[(2k-1)\pi, 2k\pi], k \in \mathbb{Z}$ are maximal elements in $\mathfrak{C}_{0,f}$.

(d) Let $\mathfrak{C}_{\text{int}}$ be the collection of open intervals. Hence we have;
$\mathfrak{C}_{\text{int}} = \{I \subseteq \mathbb{R} : I \text{ open}\} = \{I \subseteq \mathbb{R} : \text{int}(I) \neq \emptyset\}$
$\subseteq \{I \subseteq \mathbb{R} : \exists a \in I \text{ with } a \notin \partial(I)\}$
$= \mathfrak{C}_0$.
Then $A = \mathbb{R}$ satisfies Assumption 0.1.3.

Moreover, take any chain $\mathfrak{B}$ in $\mathfrak{C}_{\text{int}}$. Clearly $M = \bigcup_{V\in\mathfrak{B}} V$ is open as union of open sets is an open set. Hence M is in $\mathfrak{C}_{\text{int}}$ and thus $\mathfrak{C}_{\text{int}}$ satisfies the CUP. Moreover $f_{((2k-1)\pi, 2k\pi)}$ is convex for each $k \in \mathbb{Z}$, and since $((2k-1)\pi, 2k\pi]$, $[(2k-1)\pi, 2k\pi) \notin \mathfrak{C}_{\text{int}}$ as they are not open, we have that $((2k-1)\pi, 2k\pi), k \in \mathbb{Z}$ are maximal elements in $\mathfrak{C}_{\text{int},f}$.

The above example shows amongst other things that in general $\mathfrak{C}$-maximal domains of convexity may or may not exist. Below we will show that the

CUP will guarantee the existence of $\mathfrak{C}$-MDC.

Proposition 1.1.7 *For any $\mathfrak{C}$-convex function $f : A \subseteq X \to \mathbb{R}$, if $\mathfrak{C}$ satisfies the CUP, so does $\mathfrak{C}_f$.*

Proof
Let $\mathfrak{B}$ be a chain in $\mathfrak{C}_f$. Then, since $\mathfrak{C}$ satisfies the CUP and $\mathfrak{C}_f \subseteq \mathfrak{C}$, $D = \bigcup_{B \in \mathfrak{B}} B \in \mathfrak{C}$. Since $f|_B$ is convex for each $B \in \mathfrak{B}$, also $f|_D$ is convex and therefore $D \in \mathfrak{C}_f$. It follows that $\mathfrak{C}_f$ satisfies the CUP. □

Can the converse of the Proposition 1.1.7 above hold? It is clear that it might hold if $\mathfrak{C} \backslash \mathfrak{C}_f = \emptyset$, and clearly not every chain of convex set K is such that $K \subseteq \text{dom}(f)$ and hence the converse might not be true in general.

Proposition 1.1.8 *If $\mathfrak{C}$ satisfies the CUP, then $\mathfrak{C}$ contains a maximal element.*

Proof
Let $\mathfrak{B}$ be a chain in $\mathfrak{C}$. It follows that $\bigcup_{B \in \mathfrak{B}} B$ is an element of $\mathfrak{C}$ as $\bigcup_{B \in \mathfrak{B}} B$ is convex and $\mathfrak{C}$ satisfies the CUP. Thus each chain $\mathfrak{B}$ in $\mathfrak{C}$ has an upper bound $\bigcup_{B \in \mathfrak{B}} B$. Appealing to Zorn's lemma, it follows that $\mathfrak{C}$ contains a maximal element. □

Propositions 1.1.7 and 1.1.8 immediately lead to the following result;

Theorem 1.1.9 *Assume $\mathfrak{C}$ satisfies the CUP and let $f : A \subseteq X \to \mathbb{R}$ be a $\mathfrak{C}$-convex real function. Then $\mathfrak{C}_f$ contains a maximal element, that is, f has a $\mathfrak{C}$-MDC.*

Proof
Let $\mathfrak{B}$ be a chain in $\mathfrak{C}_f$. It follows that $\bigcup_{F \in \mathfrak{B}} F$ is an element of $\mathfrak{C}_f$ since $\mathfrak{C}_f$ satisfies the CUP as per Proposition 1.1.7. Thus each chain $\mathfrak{B}$ in $\mathfrak{C}_f$ has an

upper bound $\bigcup_{F\in\mathfrak{B}} F$. Appealing to Zorn's lemma and Proposition 1.1.8, it follows that $\mathfrak{C}$ contains a maximal element. □

Note that if $\mathfrak{C}$ satisfies the CUP and $C \in \mathfrak{C}$, then also the set $\mathfrak{C}_C = \{B \in \mathfrak{C} : C \subseteq B\}$ satisfies the CUP. Clearly, any maximal element in $\mathfrak{C}_C$ is also a maximal element in $\mathfrak{C}$. Hence the following two results:

Theorem 1.1.10 *Assume $\mathfrak{C}$ satisfies the CUP. Then the following hold.*
(a) *If C is an element of $\mathfrak{C}$, then there exists a maximal element, say M, in $\mathfrak{C}$ such that $C \subseteq M$.*
(b) *If C is an element of $\mathfrak{C}_f$ for some $\mathfrak{C}$-convex function $f : A \subset X \to \mathbb{R}$, then there exists a maximal element, say M, in $\mathfrak{C}_f$ such that $C \subseteq M$.*

Proof
(a) Appealing to Proposition 1.1.7, $\mathfrak{C}$ contains a maximal element, say M, and $D = \bigcup_{F\in\mathfrak{B}} F$ for some $\mathfrak{B}$ a chain in $\mathfrak{C}$. Take any $C \in \mathfrak{C}$. Then there exists a chain $\mathfrak{B}^* \in \mathfrak{C}$ such that $D^* = \bigcup_{C\in\mathfrak{B}^*} C \in \mathfrak{C}$. Clearly $C \subseteq D^*$ and hence $C \subseteq M$.
(b) Appealing to Theorem 1.1.9, $\mathfrak{C}_f$ contains a maximal element, say M, and $D = \bigcup_{F\in\mathfrak{B}} F$ for some $\mathfrak{B}$ a chain in $\mathfrak{C}_f$. Take any $C \in \mathfrak{C}_f$. Then there exists a chain $\mathfrak{B}^* \in \mathfrak{C}_f$ such that $D^* = \bigcup_{C\in\mathfrak{B}^*} C \in \mathfrak{C}_f$. Clearly $C \subseteq D^*$ and hence $C \subseteq M$. □

Remark 1.1.11 Let $C \in \mathfrak{C}_f$ be fixed for some $\mathfrak{C}$-convex real function $f : A \subseteq X \to \mathbb{R}$. Then there may exist a maximal element M in $\mathfrak{C}_f$ such that C may not be contained in M.
(a) For instance, if $\mathfrak{C} = \{I \subseteq \mathbb{R} : I \text{ an interval}\}$, $f(x) = \sin x$ and we choose $C \subseteq (\pi, 2\pi)$, then $f|_C$ is convex, $C \subset M = [\pi, 2\pi]$ and M is maximal in $\mathfrak{C}_f$. Furthermore, $M_0 = [-\pi, 0]$ is also a maximal element in $\mathfrak{C}_f$ with $C \nsubseteq M_0$.
(b) On the other hand if $f : A \subseteq X \to \mathbb{R}$ is convex and $A \in \mathfrak{C}$, then A is a unique maximal domain of f in $\mathfrak{C}_f$ and f is also $\mathfrak{C}$-convex. Observe that

$\bigcup_{C\in\mathfrak{C}_f} C = A \in \mathfrak{C}_f$.
(c) Suppose conversely that f is $\mathfrak{C}$-convex and has a unique maximal domain of convexity M. Then every $C \in \mathfrak{C}_f$ is contained in M, hence $M = \bigcup_{C\in\mathfrak{C}_f} C$ provided $\mathfrak{C}$ satisfies the CUP.

We therefore have the following result:

Proposition 1.1.12 *Suppose $\mathfrak{C}$ satisfies the CUP.*
(a) $\mathfrak{C}$ has a unique maximal element if and only if $\bigcup_{C\in\mathfrak{C}} C \in \mathfrak{C}$.
(b) Let $f : A \subseteq X \to \mathbb{R}$ be $\mathfrak{C}$-convex, then $\mathfrak{C}_f$ has a unique maximal element if and only if $\bigcup_{C\in\mathfrak{C}_f} C \in \mathfrak{C}_f$.

Proof
(a) The existence of the CUP and the result of Proposition 1.1.8 together imply that $\mathfrak{C}$ contains a maximal element. Let M be a unique maximal element in $\mathfrak{C}$. Then for all $B \in \mathfrak{C}$, $B \subseteq M$. Consequently, $\bigcup_{C\in\mathfrak{C}} C \subseteq M$, and thus $\bigcup_{C\in\mathfrak{C}} C \in \mathfrak{C}$.
Conversely, suppose $\bigcup_{C\in\mathfrak{C}} C \in \mathfrak{C}$ and let $M = \bigcup_{C\in\mathfrak{C}} C$. Clearly, for all $B \in \mathfrak{C}$, $B \subseteq M$, hence M is maximal in $\mathfrak{C}$. Suppose K is also maximal in $\mathfrak{C}$. Thus $K \subseteq \bigcup_{C\in\mathfrak{C}} C = M$ since $K \in \mathfrak{C}$, and $M \subseteq K$ since K is maximal. Hence $K = M$, and it follows that M is unique.

(b) The existence of the CUP and the result of Theorem 1.1.9 together imply that $\mathfrak{C}_f$ contains a maximal element. Let M be a unique maximal element in $\mathfrak{C}_f$. Then for all $B \in \mathfrak{C}_f$, $B \subseteq M$. Consequently, $\bigcup_{C\in\mathfrak{C}_f} C \subseteq M$, and thus $\bigcup_{C\in\mathfrak{C}_f} C \in \mathfrak{C}_f$.
Conversely, suppose $\bigcup_{C\in\mathfrak{C}_f} C \in \mathfrak{C}_f$ and let $M = \bigcup_{C\in\mathfrak{C}_f} C$. Clearly, for all $B \in \mathfrak{C}_f$, $B \subseteq M$, hence M is maximal in $\mathfrak{C}_f$. Suppose K is also maximal in $\mathfrak{C}_f$. Thus $K \subseteq \bigcup_{C\in\mathfrak{C}_f} C = M$ since $K \in \mathfrak{C}_f$, and $M \subseteq K$ since K is maximal. Hence $K \subseteq \bigcup_{C\in\mathfrak{C}_f} C = M \subseteq K$ and it follows that $K = M$, and that M is

unique. □

It is not difficult to see that for any $C \in \mathfrak{C}$, we have $C \subset \mathrm{clco}(C) = \mathrm{cl}\, C \subset D$ where $\mathrm{clco}(C)$ denotes the closed convex hull of C and clC the closure of C. The convex set $\mathrm{clco}(C)$ could be a maximal convex subset of D if and only if $\mathrm{clco}(C) = M$.

1.2 Convex extensions

In this section we consider convex functions on subsets of a vector space X and discuss their maximal convex extensions with respect to two different orderings on the set of all convex extensions.

Definition 1.2.1 Let g, h be real functions with $\text{dom}(g), \text{dom}(h) \subseteq X$. Then g is *an extension* of h, denoted by $g \succeq_{\text{ext}} h$, if $\text{dom}(h) \subseteq \text{dom}(g)$ and $g|_{\text{dom}(h)} = h$. Moreover, if $f : Y \subseteq X \to \mathbb{R}$ is convex and real, then the set

$$\mathfrak{X}_f = \{g : \text{dom}(g) \subseteq X, g \succeq_{\text{ext}} f, g \text{ convex}\}$$

is the collection of convex extensions of f.

Lemma 1.2.2 *If $f : Y \subseteq X \to \mathbb{R}$ is convex then the set $\mathfrak{X}_f = \{g : \text{dom}(g) \subseteq X, g \succeq_{ext} f, g$ convex$\}$ is partially ordered by $\succeq_{ext}$*

Proof
(a) Take any $g \in \mathfrak{X}_f$. Clearly $g \succeq_{\text{ext}} g$ and hence the reflexive property is satisfied.
(b) Take any $g_1, g_2 \in \mathfrak{X}_f$, and let $g_1 \succeq_{\text{ext}} g_2$ and $g_2 \succeq_{\text{ext}} g_1$. It follows that $\text{dom}(g_1) \subseteq \text{dom}(g_2)$ and $\text{dom}(g_2) \subseteq \text{dom}(g_1)$ and hence $\text{dom}(g_1) = \text{dom}(g_2)$. Since $g_1 \succeq_{\text{ext}} f$ and $g_2 \succeq_{\text{ext}} f$ it follows that $g_1 = g_2$ and hence the antisymmetric property is satisfied.
(c) Take any $g_1, g_2, g_3 \in \mathfrak{X}_f$ and assume $g_1 \succeq_{\text{ext}} g_2$ and $g_2 \succeq_{\text{ext}} g_3$. It follows that $\text{dom}(g_3) \subseteq \text{dom}(g_2) \subseteq \text{dom}(g_2)$ and $g_1(x) = g_2(x) = g_3(x) = f(x)$ for each $x \in \text{dom}(f)$, $g_2(y) = g_3(y)$ for each $y \in \text{dom}(g_3)$, $g_1(z) = g_2(z)$ for each $z \in \text{dom}(g_2)$. Clearly $g_1 \succeq_{\text{ext}} g_3$ and hence the transitive property is satisfied. This completes the proof. □

Clearly not all convex functions have convex extension other than themselves, that is non-trivial extension. Look at the following examples;
(i) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = ax + c$ with $a, c \in \mathbb{R}$ fixed. There is no proper extension of f other than f itself.

(ii) Let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function defined by $g(x) = \sec x$ where $I = (-\frac{\pi}{2}, \frac{\pi}{2})$. Clearly g is continuous on $I \backslash \partial(I)$ and does not have a proper convex extension.

Remark 1.2.3 These functions above do not have proper convex extension, is it because they are real and $\lim\limits_{x \to \partial(\mathrm{dom}(f))} f(x) = \pm\infty$

Henceforth if a continuous and convex function $f : A \subseteq X \to \mathbb{R}$ satisfies $\lim\limits_{x_n \to \partial(A)} f(x_n) = \infty$ then any of the following holds;

- A is open.
  This follows from the fact that if A is closed, the $\partial(A) \subseteq A$. Since f is continuous, it would be continuous on each $x \in \partial(A)$ and hence $\lim\limits_{x_n \to x \in \partial(A)} f(x_n) = f(x) \neq \infty$ and leads to a contradiction.

- $\partial(\mathrm{epi}(f)) \backslash \mathrm{gr}(f) = \emptyset$.
  If $(x, k) \in \partial(\mathrm{epi}(f)) \backslash \mathrm{gr}(f)$, this would means than $\mathrm{epi}(f)$ is not closed and thus f is not continuous. Hence it would lead to a contradiction as f is continuous.

- f (and hence $\mathrm{gr}(f)$) is not bounded above.
  This follows from the fact that $f(x_n)$ approaches ∞ by assumption.

Obviously, $\mathfrak{X}_f \neq \emptyset$ since $f \in \mathfrak{X}_f$. Our aim is to prove the existence of maximal extensions for any given convex f. If $\mathrm{dom}(f) \in \mathfrak{C}$, we also consider $\mathfrak{X}_{f,\mathfrak{C}} = \{g \in \mathfrak{X}_f : \mathrm{dom}(g) \in \mathfrak{C}\}$.

Note that a chain $\mathfrak{B}$ in $\mathfrak{X}_{f,\mathfrak{C}}$ is such that $g \succeq_{\mathrm{ext}} h \succeq_{\mathrm{ext}} f|_M$ or $h \succeq_{\mathrm{ext}} g \succeq_{\mathrm{ext}} f|_M$ for each pair $g, h \in \mathfrak{X}_{f,\mathfrak{C}}$ and M a $\mathfrak{C}$-MDC.

Theorem 1.2.4 *Let $\mathfrak{C}$ satisfy the CUP. For any convex function $f : Y \subseteq X \to \mathbb{R}$ with $\mathrm{dom}(f) \in \mathfrak{C}$, there exists a maximal convex extension in $\mathfrak{X}_{f,\mathfrak{C}}$ with respect to the ordering $\succeq_{ext}$.*

Proof

Let $\mathfrak{B}$ be a chain in $\mathfrak{X}_{f,\mathfrak{C}}$. Hence for each pair $g, h \in \mathfrak{B}$ such that $h \succeq_{\text{ext}} g$ we have $\text{dom}(f) \subseteq \text{dom}(g) \subseteq \text{dom}(h)$. It follows that for each chain $\mathfrak{B}$ in $\mathfrak{X}_f$ there exists a chain $Dom(\mathfrak{B}) \in X$ of domains of functions in $\mathfrak{B}$. By the definition of $\mathfrak{X}_{f,\mathfrak{C}}$ we have $B \in \mathfrak{C}$ for all $B \in \text{Dom}(\mathfrak{B})$, and thus $\text{Dom}(\mathfrak{B})$ is a chain in $\mathfrak{C}$. Moreover $\bigcup_{B \in \text{Dom}(\mathfrak{X}_f)} B = D \in \mathfrak{C}$ since $\mathfrak{C}$ satisfies the CUP, and D is also an upper bound for $\text{Dom}(\mathfrak{B})$ in X.

We define $g^* : D \to \mathbb{R}$ as follows:
For any $x \in D$ there exists $g \in \mathfrak{B}$ such that $x \in \text{dom}(g)$ and $g^*(x) = g(x)$. Clearly, since $\mathfrak{B}$ is a chain, g^* is well defined and convex, and $g^* \succeq_{\text{ext}} g \succeq_{\text{ext}} f$ for all $g \in \mathfrak{B}$. Therefore $g^* \in \mathfrak{X}_{f,\mathfrak{C}}$ is an upper bound of $\mathfrak{B}$.

Since $\mathfrak{X}_{f,\mathfrak{C}}$ is partially ordered and each chain $\mathfrak{B}$ in $\mathfrak{X}_{f,\mathfrak{C}}$ has an upper bound g^* in $\mathfrak{X}_{f,\mathfrak{C}}$, it follows from Zorn's lemma that $\mathfrak{X}_{f,\mathfrak{C}}$ has a maximal element. □

Convex functions and convex epigraphs coincide and since we have been discussing extensions of convex function, we subsequently discuss their convex epigraphs and the ways in which one might determine epigraph extension of convex functions and still preserve their convexity.

Recall that for a function $f : A \subseteq X \to \mathbb{R}$ its epigraph is defined and denoted by $\text{epi}(f) = \{(x, \lambda) \in A \times \mathbb{R} : f(x) \leq \lambda,\ \lambda \in \mathbb{R}\}$.

It is well known and easily seen that the function $f : A \subseteq X \to \mathbb{R}$ is convex if and only if its epigraph is a convex subset of $X \times \mathbb{R}$.

Considering the collection of convex extensions of f as in Definition 1.2.1 above, we define the ordering $\sqsupseteq_{\text{epi}}$ on $\mathfrak{X}_f$ as follows:

Definition 1.2.5 For any $g, h \in \mathfrak{X}_{f,\mathfrak{C}}$, $g \sqsupseteq_{\text{epi}} h$ if and only if $\text{epi}(h) \subseteq \text{epi}(g)$. Equivalently, $\text{dom}(h) \subseteq \text{dom}(g)$ and $g(x) \leq h(x)$, $x \in \text{dom}(h)$.

Clearly $\mathfrak{X}_{f,\mathfrak{C}}$ is partially ordered by $\sqsupseteq_{\text{epi}}$, and $g \sqsupseteq_{\text{epi}} f$ for all $g \in \mathfrak{X}_{f,\mathfrak{C}}$. We denote by $\text{Epi}(\mathfrak{X}_{f,\mathfrak{C}}) = \{\text{epi}(g) : g \in \mathfrak{X}_{f,\mathfrak{C}}\}$ the collection of convex epigraphs $\text{epi}(g)$ containing (or which are extensions of) $\text{epi}(f)$.

There is one additional condition we need to prove the existence of maximal epigraph, and we state it as follows;

Definition 1.2.6 A subset A of X is said to be a *pseudo-absorbing* subset of X if for each $x \in X$ there exists $a, b \in A$ and $\alpha \in \mathbb{R}$ such that $x = a + \alpha(b - a)$.

Clearly, if $\alpha \in [0, 1]$ then the pseudo-absorbing property is reduced to convexity of A as it would mean $x \in [a, b]$. Furthermore every absorbing subset of X is pseudo-absorbing. It turns out that, assuming pseudo-absorbing and CUP together bring about the existence of the maximal epigraphs, but before we state and prove this assertion, we need the following lemma;

Lemma 1.2.7 *Let $f : A \subseteq X \to \mathbb{R}$ be $\mathfrak{C}$-convex and* $\mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$ *be nonepmpty. Then for each chain $\mathfrak{B}$ in* $\mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$*, $\mathfrak{K} = \{\mathrm{dom}(g) : \mathrm{epi}(g) \in \mathfrak{B}\}$ is a chain in $\mathfrak{C}$.*

Proof
Take a chain $\mathfrak{B}$ in $\mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$. Then each $B \in \mathfrak{B}$ is such that $B = \mathrm{epi}(g)$ for $g \in \mathfrak{X}_{f,\mathfrak{C}}$. Clearly $B = \mathrm{epi}(g)$ implies $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$. Since $\mathfrak{B}$ is a chain, for any $B_1, B_2 \in \mathfrak{B}$ we have, without loss of generality, $B_1 = \mathrm{epi}(g_1) \subseteq \mathrm{epi}(g_2) = B_2$ for some $g_1, g_2 \in \mathfrak{X}_{f,\mathfrak{C}}$. Hence $\mathrm{dom}(g_1) \subseteq \mathrm{dom}(g_2)$. It follows that for $B_1 \subseteq B_2 \subseteq ... \subseteq B_n$ in $\mathfrak{B}$ there exist $\mathrm{dom}(g_1) \subseteq \mathrm{dom}(g_2) \subseteq ... \subseteq \mathrm{dom}(g_n)$, $g_i \in \mathfrak{X}_{f,\mathfrak{C}}$ for each $i = 1, ..., n$. It follows from the definition of $\mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$ that $\mathrm{dom}(g_i)$ is convex and $g_i \in \mathfrak{X}_{f,\mathfrak{C}}$ for each $i = 1, ..., n$ and hence $\mathrm{dom}(g_i) \in \mathfrak{C}$. Consequently $\mathfrak{K} = \{\mathrm{dom}(g) : \mathrm{epi}(g) \in \mathfrak{B}\}$ is a chain in $\mathfrak{C}$. □

Theorem 1.2.8 *Let $f : A \subseteq X \to \mathbb{R}$ be $\mathfrak{C}$-convex and suppose that $\mathfrak{C}$ satisfy the CUP and that A is pseudo-absorbing in X. Then there exists a maximal epigraph extension of* $\mathrm{epi}(f)$ *in $X \times \mathbb{R}$, equivalently,* $\mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$ *has a maximal element.*

Proof

Let $\mathfrak{B}$ be any chain in $\mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$. Then $\mathrm{epi}(f) \subseteq \widehat{B} = \bigcup_{B\in\mathfrak{B}} B \in X \times \mathbb{R}$. Moreover, for each $B \in \mathfrak{B}$ there exists $g \in \mathfrak{X}_{f,\mathfrak{C}}$ such that $B = \mathrm{epi}(g)$, $g \sqsupseteq_{\mathrm{epi}} f$ and $\mathrm{dom}(f) \subseteq \mathrm{dom}(g) \in \mathfrak{C}$. Then $\mathfrak{K} = \{\mathrm{dom}(g) : \mathrm{epi}(g) \in \mathfrak{B}\}$ is a chain in $\mathfrak{C}$, and it follows that $D = \bigcup_{K\in\mathfrak{K}} K \in \mathfrak{C}$ since $\mathfrak{C}$ satisfies the CUP.

Obviously, $\widehat{B}$ is a convex subset of $X \times \mathbb{R}$. We are going to show that it is contained in the epigraph of some real convex function g^* in $\mathfrak{X}_{f,\mathfrak{C}}$. Indeed, we define the function $g^* : D \to \mathbb{R}$ as

$$g^*(x) = \inf\{g(x) \in \mathbb{R} : \mathrm{epi}(g) \in \mathfrak{B},\ x \in \mathrm{dom}(g)\} \quad (x \in D).$$

In order to show that $g^*(x) \in \mathbb{R}$ for all $x \in D$, we first observe that $g^*(x) = f(x)$ if $x \in \mathrm{dom}(f)$. Now fix $x \in D \setminus \mathrm{dom}(f)$. Appealing to the pseudo-absorbing property of A, it follows that there exist $a, b \in A = \mathrm{dom}(f)$ and $\alpha \in \mathbb{R}$ such that $x = a+\alpha(b-a)$. Since A is convex, $x \notin [a, b]$, and therefore we may assume without loss of generality that $a \in [x, b]$. Thus there exists $\lambda \in (0, 1)$ such that $a = \lambda x + (1 - \lambda)b$ and hence

$$g(a) = g(\lambda x + (1 - \lambda)b) \leq \lambda g(x) + (1 - \lambda)g(b)$$

for each $g \in \mathfrak{X}_{f,\mathfrak{C}}$ with $\mathrm{epi}(g) \in \mathfrak{B}$ and $x \in \mathrm{dom}(g)$, which leads to

$$g(x) \geq \frac{1}{\lambda}(g(a) - (1 - \lambda)g(b)) = \frac{1}{\lambda}(f(a) - (1 - \lambda)f(b)).$$

It follows that $g^*(x) \geq \frac{1}{\lambda}(g(a)-(1-\lambda)g(b))$ for each $g \in \mathfrak{X}_{f,\mathfrak{C}}$ with $\mathrm{epi}(g) \in \mathfrak{B}$ and $x \in \mathrm{dom}(g)$. Consequently, $g^*(x) > -\infty$ for each $x \in D$.

Now we are going to show that g^* is convex. Let $x, y \in D$ and $\lambda \in [0, 1]$. By definition of g^*, for any $\varepsilon > 0$ there exist $g_1, g_2 \in \mathfrak{X}_{f,\mathfrak{C}}$ such that $\mathrm{epi}(g_1), \mathrm{epi}(g_2) \in \mathfrak{B}$, $x \in \mathrm{dom}(g_1)$, $y \in \mathrm{dom}(g_2)$, $g_1(x) \leq g^*(x) + \varepsilon$ and $g_2(y) \leq g^*(y)+\varepsilon$. Since $\mathfrak{B}$ is a chain, $g_1 \sqsupseteq_{\mathrm{epi}} g_2$ or $g_2 \sqsupseteq_{\mathrm{epi}} g_1$, and we denote by g the maximum of g_1 and g_2. Then $g(x) \leq g_1(x)$ and $g(y) \leq g_2(y)$.

Therefore,

$$\begin{aligned} g^*(\lambda x + (1-\lambda)y) &\leq g(\lambda x + (1-\lambda)y) \\ &\leq \lambda g(x) + (1-\lambda)g(y) \\ &\leq \lambda(g^*(x) + \varepsilon) + (1-\lambda)(g^*(y) + \varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\begin{aligned} g^*(\lambda x + (1-\lambda)y) &\leq \lim_{\varepsilon \to 0} \lambda(g^*(x) + \varepsilon) + (1-\lambda)(g^*(y) + \varepsilon) \\ &= \lambda g^*(x) + (1-\lambda)g^*(y). \end{aligned}$$

Thus we have shown that g^* is convex. Together with $D \in \mathfrak{C}$ and $g^*|_{\mathrm{dom}(f)} = f$ this leads to $g^* \in \mathfrak{X}_{f,\mathfrak{C}}$ and hence $\mathrm{epi}(g^*) \in \mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$.

Finally, we show that $\mathrm{epi}(g^*)$ is an upper bound of $\mathfrak{B}$. To this end take $(x, \alpha) \in \widehat{B}$. Then there exists $g \in \mathfrak{X}_{f,\mathfrak{C}}$ such that $\mathrm{epi}(g) \in \mathfrak{B}$ and $x \in \mathrm{dom}(g)$. It follows that $\alpha \geq g(x) \geq g^*(x)$ and hence $(x, \alpha) \in \mathrm{epi}(g^*)$. Consequently $\widehat{B} \subseteq \mathrm{epi}(g^*) \in \mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$, and hence $\mathrm{epi}(g^*)$ is an upper bound of $\mathfrak{B}$ in $\mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$. By Zorn's lemma, $\mathrm{Epi}(\mathfrak{X}_{f,\mathfrak{C}})$ has a maximal element. □

Proposition 1.2.9 *Let $f : A \subseteq X \to \mathbb{R}$ be $\mathfrak{C}$-convex, $\mathfrak{C}$ satisfy the CUP and $f_i \in \mathfrak{X}_{f,\mathfrak{C}}, i \geq 1$. Then*

$$\mathrm{epi}(\sup_{i \geq 1} f_i) = \bigcap_{i \geq 1} \mathrm{epi}(f_i)$$

Proof

Take $(x, \lambda) \in \mathrm{epi}(\sup_{i \geq 1} f_i)$. Then $\sup_{i \geq 1} f_i(x) \geq f_i(x)$ for all $x \in \mathrm{dom}(f_i)$. Clearly, $f_i(x) \leq \sup_{i \geq 1} f_i(x) \leq \lambda$ and hence $(x, \lambda) \in \mathrm{epi}(f_i)$ for each $i \geq 1$, and consequently $(x, \lambda) \in \bigcap_{i \geq 1} \mathrm{epi}(f_i)$. This implies that $\mathrm{epi}(\sup_{i \geq 1} f_i) \subseteq \bigcap_{i \geq 1} \mathrm{epi}(f_i)$ and the other inclusion follows easily, and hence $\mathrm{epi}(\sup_{i \geq 1} f_i) = \bigcap \mathrm{epi}(f_i)$ □

Corollary 1.2.10 *Let $f : A \subseteq X \to \mathbb{R}$ be $\mathfrak{C}$-convex, $\mathfrak{C}$ satisfy the CUP and $f_i \in \mathfrak{X}_{f,\mathfrak{C}}, i \geq 1$. Then*

$$\text{epi}(\inf_{i\geq 1} f_i) = \bigcup_{i\geq 1} \text{epi}(f_i)$$

Proof
Clearly is we consider $f_i \in \mathfrak{X}_{f,\mathfrak{C}}, i \geq 1$, $\inf_{i\geq 1} = f_k \in \mathfrak{X}_{f,\mathfrak{C}}$ such that $f_k(x) \leq f_i$ for each $x \in A$. Thus $\text{epi}(\inf_{i\geq 1} f_i) \subseteq \bigcup_{i\geq 1} \text{epi}(f_i)$. The other inclusion also follows easily as if $(x, \lambda) \in \bigcup_{i\geq 1} \text{epi}(f_i)$, then the exists $f_m \in \mathfrak{X}_{f,\mathfrak{C}}$ such that $(x, \lambda) \in \text{epi}(f_m)$ and $\inf_{i\geq 1}(x) \leq f_m(x)$. It follows that $(x, \lambda) \in \text{epi}(\inf_{i\geq 1} f_i)$. $\square$

Corollary 1.2.10 above together with the additional assumption that A is pseudo-absorbing in X may lead to the existence of maximal epigraphs as stated in Theorem 1.2.8 and we might have $\inf_{i\geq 1} f_i = g^*$ where g^* is defined in the proof of Theorem 1.2.8.

Remark 1.2.11 Note that $f_i \leq f$ for all $i \geq 1$ and $\bigcap_i \text{dom}(f_i) \supseteq \text{dom}(f)$ for f convex on $\text{dom}(f) = M$ the $\mathfrak{C}$-MDC. It follows that $\inf f_i(x) \leq f_i(x)$ for all $f_i \in \mathfrak{X}_{f,\mathfrak{C}}$ and hence $\inf f_i(x) = \inf\{f_i(x) : x \in \text{dom}(f_i)\} = g^*$

Theorem 1.2.12 *[14, Theorem 1.3.3, p.21] Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be convex. Then f is continuous on the interior int(A) of A and has finite left and right derivatives at each point of int(A). Moreover, $x < y$ in int(A) implies*

$$f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$$

Consequently $f'(x) \leq f'(y)$ provided they exists.

Conversely,

Proposition 1.2.13 *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function on an open interval A, and assume that f has finite left and right derivatives at each point of A satisfying*

$$f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$$

for all $x < y$ in A. Then f is convex.

Proof

Suppose f is not convex. Then there are $\alpha, \beta \in A$ such that $\alpha < \beta$ and such that the chord $\lambda(\alpha, f(\alpha)) + (1-\lambda)(\beta, f(\beta))$, $\lambda \in [0,1]$, is not entirely contained in epi(f). Consider the function

$$h(x) = f(x) - f(\alpha) - \tfrac{f(\beta)-f(\alpha)}{\beta-\alpha}(x-\alpha)$$

defined on $[\alpha, \beta]$. Clearly $h(\alpha) = h(\beta) = 0$, and h is continuous, due to the continuity of f. Let $k(x) = f(\alpha) + \frac{f(\beta)-f(\alpha)}{\beta-\alpha}(x-\alpha)$ be a function on $[\alpha, \beta]$. Hence the graph of k is the chord mentioned above, and there exists $x \in [\alpha, \beta]$ such that $h(x) = f(x) - k(x) > 0$. Therefore $\max\{h(x) : x \in [\alpha, \beta]\} > 0$. Choose $c \in (\alpha, \beta)$ such that $h(c) \geq h(x), x \in [\alpha, \beta]$. Thus $h(c) > 0$,

$$h'_-(\alpha) = \lim_{x \nearrow c} \tfrac{h(x)-h(c)}{x-c} \geq 0 \text{ and } h'_+(\alpha) = \lim_{x \searrow c} \tfrac{h(x)-h(c)}{x-c} \leq 0.$$

It follows that $0 \leq h'_-(c) = f'_-(c) - \frac{f(\beta)-f(\alpha)}{\beta-\alpha}$ and $f'_+(c) - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} = h'_+(c) \leq 0$. Since $f'_-(c) \leq f'_+(c)$ by assumption, we have $0 \leq h'_-(c) \leq h'_+(c) \leq 0$, and hence $h'_+(c) = h'_-(c) = 0$.

Moreover define the function g on $[\alpha, c]$ as

$$g(x) = h(x) - \tfrac{h(c)}{c-\alpha}(x-\alpha)$$

Assume $g(x) \leq 0$ for all $x \in (\alpha, c)$. Obviously $g(\alpha) = 0 = g(c)$. Thus for $x \in (\alpha, c)$, $g(c) = 0 \geq g(x)$ and thus $g(x) - g(c) \leq 0$, and $x < c$ and thus $x - c < 0$. Consequently $\frac{g(x)-g(c)}{x-c} \geq 0$, and hence

$$\lim_{x \to c^-} \tfrac{g(x)-g(c)}{x-c} = \lim_{x \nearrow c} \tfrac{g(x)-g(c)}{x-c} = g'_-(c) \geq 0$$

But $g'_-(x) = h'_-(x) - \frac{h(c)}{c-\alpha} < h'_-(x)$. Thus $0 \leq g'_-(c) < h'_-(c) = 0$. This leads to a contradiction and hence $g(x) > 0$ for some $x \in (\alpha, c)$. Thus g takes its maximum on $[\alpha, c]$ at some point $t \in (\alpha, c)$.

Therefore $\max\{g(x) : x \in [\alpha, c]\} > 0$ and $g(t) > 0$. It was shown above that $h'_+(c) = h'_-(c) = 0$ where $c \in (\alpha, \beta)$ such that $h(c) \geq h(x), x \in [\alpha, \beta]$

and $h(c) > 0$. A similar argument holds for g, that is, $g'_-(t) = g'_+(t) = 0$. Hence

$$h'_-(t) = g'_-(t) + \tfrac{h(c)}{c-\alpha} = \tfrac{h(c)}{c-\alpha} > 0 = h'_-(c)$$

that is, $h'_-(t) > h'_-(c)$. Then,

$$f'_-(t) = h'_-(t) + \tfrac{f(\beta)-f(\alpha)}{\beta-\alpha} > h'_-(c) + \tfrac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'_-(c)$$

and hence $f'_-(t) > f'_-(c), t < c$. This leads to a contradiction, and hence f is convex. □

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be convex and $\beta \in A = \text{dom}(f)$ be an endpoint. Then define by $T_{f,\beta}$ a tangent to the curve f at the point $\beta \in \mathbb{R}$. Moreover, domain A can be open, closed or neither and hence its endpoints are important and influence the results in that, they can take any of the following forms:
That is, for α an endpoint of A, we have either one of the following

1. $\alpha \in A$ and $\alpha \in \mathbb{R}$
2. $\alpha \notin A$ and $\alpha \in \mathbb{R}$
3. $\alpha \notin \mathbb{R}$

Henceforth denote by $\text{gr}(f)$ the graph of f and, by $[x, y]$ the line segment joining $x, y \in \mathbb{R}^2$.

Proposition 1.2.14 *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be convex and $\text{dom}(f)$ be non-trivial. If $\mathfrak{C}$ satisfy the CUP and A is pseudo-absorbing in X, then $\text{Epi}(\mathfrak{X}_f)$ has a unique maximal element.*

Proof
Appealing to Theorem 1.2.8, $\text{Epi}(\mathfrak{X}_f)$ contains a maximal element, hence there is some $g \in \mathfrak{X}_f$ such that $\text{epi}(g)$ is maximal. We construct a function g^* in two steps as follows, and will show that it is a maximal element:

Let α be an endpoint of $\text{dom}(f)$. If $\alpha \in \mathbb{R}$, denote by $\lim_{x\to\alpha^{\pm}} f(x)$ a one-sided limit where $\lim_{x\to\alpha^{-}} f(x)$ is considered if $\alpha \geq b$ for each $b \in \text{dom}(f)$, and

$\lim_{x\to\alpha^+} f(x)$ is considered otherwise. Furthermore, both will be considered provided α represents different endpoints, hence not fixed. Take a, b the endpoints of $\mathrm{dom}(f)$, with $a < b$, and let

$$\mathfrak{L} = \{\alpha \in \{a, b\} : \alpha \in \mathbb{R}\backslash\, \mathrm{dom}(f),\ \lim_{x\to\alpha^{\pm}} f(x) \text{ exists } \}.$$

Moreover define the function $\hat{g}$ as follows

$$\hat{g}(x) = \begin{cases} f(x) & \text{if} \quad x \in \mathrm{dom}(f) \\ \lim_{t\to x^{\pm}} f(t) & \text{if} \quad x \in \mathfrak{L} \end{cases}$$

Furthermore, we define the function g^* as follows

$$g^*(x) = \begin{cases} \hat{g}(x) & \text{if} \quad x \in \mathrm{dom}(\hat{g}) \\ T_{\hat{g},a}(x) & \text{if} \quad x < a, a \in \mathrm{dom}(\hat{g}), (\hat{g})'_+(a) \text{ exists} \\ T_{\hat{g},b}(x) & \text{if} \quad x > b, b \in \mathrm{dom}(\hat{g}), (\hat{g})'_-(b) \text{ exists} \end{cases}$$

Take $t, y, z, k \in \mathbb{R}$ such that $t < a < y < z < b < k$. Moreover,

$$(g^*)'(x) = \begin{cases} (\hat{g})'(x) & \text{if} \quad x \in \mathrm{dom}(\hat{g}) \\ C_a & \text{if} \quad x < a, a \in \mathrm{dom}(\hat{g}), (\hat{g})'_+(a) \text{ exists} \\ C_b & \text{if} \quad x > b, b \in \mathrm{dom}(\hat{g}), (\hat{g})'_-(b) \text{ exists} \end{cases}$$

where $(T_{\hat{g},a})'(x) = C_a \in \mathbb{R}, (T_{\hat{g},b})'(x) = C_b \in \mathbb{R}$ for each x is respective domains.

Then $t \in \mathrm{dom}(T_{\hat{g},a})$ and $(g^*)'(t) = \lim_{x\to t} \frac{f(x)-f(t)}{x-t} = C_a$. Hence $(g^*)'_-(t) = (g^*)'_+(t)$. Similarly $(g^*)'_-(k) = (g^*)'_+(k)$.

Moreover, if $\hat{g}$ is not right-continuous at a, then $\hat{g}$ is not differentiable at a, hence $T_{\hat{g},a}$ is not defined. That is $\hat{g}$ would not be extendable by $T_{\hat{g},a}$. Similarly if $\hat{g}$ is not left-continuous at b, $\hat{g}$ would not be extendable by $T_{\hat{g},b}$.

Henceforth suppose $\hat{g}$ is right-differentiable at a. Then,

$$(g^*)'_-(a) = (g^*)'_+(a) = (T_{\hat{g},a})'(a) = C_a = (g^*)'_-(t) = (g^*)'_+(t).$$

That is, $(g^*)'(x) = (T_{\hat{g},a})'(a)$ for each $x \leq a$. Similarly, if $\hat{g}$ is left-differentiable at b, then,

$$(g^*)'_-(b) = (g^*)'_+(b) = (T_{\hat{g},b})'(b) = C_b = (g^*)'_-(k) = (g^*)'_+(k).$$

Moreover

$$(g^*)'_+(a) = (\hat{g})'_+(a) \leq (\hat{g})'_-(y)$$

due to the convexity of $\hat{g}$ and the result of Proposition 1.2.12. Furthermore, due to the same reason, we have

$$(\hat{g})'_-(y) = (\hat{g})'_+(y) \leq (\hat{g})'_-(z) \leq (\hat{g})'_+(z)$$

and consequently

$$(g^*)'_+(a) \leq (g^*)'_-(y) \leq (g^*)'_+(y) \leq (g^*)'_-(z) \leq (g^*)'_+(z)$$

Since $\hat{g}$ is left-differentiable at b and convex we have

$$(g^*)'_+(z) = (\hat{g})'_+(z) \leq (\hat{g})'_-(b) = (g^*)'_-(b).$$

In addition, $T'_{\hat{g},b}(b) = (\hat{g})'_-(b) = (\hat{g})'_+(b)$ and hence $(g^*)'(x) = C_b$ for each $x \geq b$. It follows that $(g^*)'_+(k) = (g^*)'(k) = (g^*)'(k)$ and consequently

$$(g^*)'(\beta) \leq (g^*)'_-(y) \leq (g^*)'_+(y) \leq (g^*)'_-(z) \leq (g^*)'_+(z) \leq (g^*)'(\gamma)$$

for each $\beta \leq a$ and each $\gamma \geq b$. It follows from Proposition 1.2.13 that g^* is convex

We show that g^* is maximal in $\mathfrak{X}_f$, that is $\text{epi}(g^*)$ is maximal in $\text{Epi}(\mathfrak{X}_f)$: Let $g \in \mathfrak{X}_f$. Then $g^*(\beta) = g(\beta) = f(\beta), \beta \in \text{dom}(f)$, and $g(\varepsilon) = \hat{g}(\varepsilon)$ for each $\varepsilon \in \{a, b\} \cap \text{dom}(\hat{g})$.

Consider the case where $\text{dom}(g^*) = \mathbb{R}$, other cases have similar results, and assume $\text{epi}(g^*) \subset \text{epi}(g)$. Then there exists $x \in \text{dom}(g)$ such that $g(x) < g^*(x)$. Since $g^*(\beta) = g(\beta) = \hat{g}(\beta), \beta \in \text{dom}(\hat{g})$, then $x \notin \text{dom}(\hat{g})$, and hence $x < a$ or $x > b$. Without a loss of generality, take $x > b$. It follows that $x - b > 0$ and hence $\frac{g(x)-g(b)}{x-b} < \frac{g^*(x)-g^*(b)}{x-b}$. Moreover $g'_+(b) = \lim\limits_{h \searrow 0} \frac{g(b+h)-g(b)}{h}$.
But $h+b = \lambda x + (1-\lambda)b = \lambda(x-b)+b, \lambda \in [0,1]$ and thus $\lambda = \frac{h}{x-b}$. Hence $g(h+b) \leq \lambda g(x)+(1-\lambda)g(b) = \lambda(g(x)-g(b))+g(b) = \frac{h}{x-b}(g(x)-g(b))+g(b)$. Consequently,

$$
\begin{aligned}
g'_+(b) &\leq \lim_{h \searrow 0} \tfrac{1}{h}(\tfrac{h}{x-b}(g(x) - g(b))) \\
&= \lim_{h \searrow 0} \tfrac{g(x)-g(b)}{x-b} = \tfrac{g(x)-g(b)}{x-b} \\
&< \tfrac{g^*(x)-g^*(b)}{x-b} = (\hat{g})'_-(b) = g'_-(b)
\end{aligned}
$$

This shows that $g'_+(b) < g'_-(b)$, and since g is convex, contradicts Proposition 1.2.13. Thus $g^*(x) \leq g(x), g \in \mathfrak{X}_f, x \in \mathrm{dom}(g)$, and consequently $\mathrm{epi}(g^*)$ is maximal and unique in $\mathrm{Epi}(\mathfrak{X}_f)$. □

Remark 1.2.15 1. Even for convex functions $f : A \subseteq \mathbb{R} \to \mathbb{R}$ it depends on $\mathfrak{C}$ if $\mathfrak{X}_{f,\mathfrak{C}}$ has a unique maximal element. For example, if $\mathfrak{C}$ is the collection of all finite intervals (a, b) with $0 < b-a \leq 10$ and $f(x) = x^2$ with $\mathrm{dom}(f) = (0, 1)$, then $\mathfrak{C}$ satisfies the CUP, and with f^* being the maximal extension of f in $\mathfrak{X}_f$, every function $f^*|_{(-c,10-c)}$ with $0 \leq c \leq 9$ is a maximal element in $\mathfrak{X}_{f,\mathfrak{C}}$ with respect to $\succeq$.

2. We are not aware of any uniqueness result if $f : A \subseteq X \to \mathbb{R}$ is convex with $\dim X \geq 2$, and it may be the case that there are f such that there is more than one maximal element in $\mathrm{Epi}(\mathfrak{X}_f)$.

1.3 Hahn-Banach theorem and convex extensions

In this section we discuss the relation between convex extensions and the Hahn-Banach Theorem. We also discuss sublinear functionals and their convexity characteristics, and also the role sublinear functionals play in the Hahn-Banach Theorem and convex extension.

Definition 1.3.1 Let E be a real linear space. A functional $p : E \to \mathbb{R}$ is sublinear if the following are satisfied;

1. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in E$, that is, subadditive property.
2. $p(\lambda x) = \lambda p(x)$ for each $\lambda \geq 0$ and each $x \in E$, that is, positively homogeneous property.

Clearly if p sublinear the p is convex on any convex set A in a real linear space E:
Take any $x, y \in A$, hence
$p(\lambda x + (1 - \lambda)y) \leq p(\lambda x) + p((1 - \lambda)y)$ (since subadditive)
$= \lambda p(x) + (1 - \lambda)p(y)$ (since positively homogeneous).

Theorem 1.3.2 *[14, The Hahn-Banach theorem, p.203] Let p be a sublinear functional on E, E_0 be a linear subspace of E, and $f_0 : E_0 \to \mathbb{R}$ be a linear functional dominated by p, that is, $f_0(x) \leq p(x)$ for all $x \in E_0$. Then f_0 has a linear extension $f : E \to \mathbb{R}$ which is also dominated by p.*

Clearly f is a convex extension of f_0 since f is linear and it extends f_0 from the linear subspace E_0 to E.
Moreover, The Hahn-Banach theorem above can be extended from real linear space to the normed linear space as in the next result.

Theorem 1.3.3 *[14, The Hahn-Banach theorem, p.203] Let E_0 be a linear subspace of normed linear space E, and $f_0 : E_0 \to \mathbb{R}$ be a continuous linear functional. Then f_0 has a continuous linear extension $f : E \to \mathbb{R}$ with $\|f\| = \|f_0\|$.*

Moreover, below is the result in [18] about the convex extension of a convex function with a particular form of a convexity in $\mathbb{R}^n$. Let first define such a convex function as follows;

Definition 1.3.4 *[18, p.12,103]* A set $S \subseteq \mathbb{R}^n$ is called a *simplex* if it is a convex hull of linearly (affinely) independent finite elements in $\mathbb{R}^n$, and *locally simplicial* if for each $x \in S$ there exists a finite collection of simplices $S_1, ..., S_m$ contained in S such that, for some neighbourhood U if x

$$U \bigcap (S_1 \cup ... \cup S_m) = U \bigcap S$$

Note that not all locally simplicial sets are convex, hence;

Theorem 1.3.5 *[18, Theorem 10.3, p.103] Let C be a locally simplicial convex set, and let f be a finite convex function on the relative interior $ri(C)$ of C which is bounded above on every bounded subset of $ri(C)$. Then f can be extended in one and only way to a continuous finite convex function on the whole of C.*

Clearly this results states that there is a unique convex extension from the relative interior of a locally simplical set to the whole sets, see [18] for more details.

Chapter 2

Extremal properties of convex epigraphs

In this chapter we discuss the extremal structure of convex epigraphs, along with the relation between extreme points and denting points of an epigraph.

A point is an extreme point of a set A if it is not an interior point of any line segment contained in that particular set A. For instance, extreme points of a closed triangular region are its vertices, while those of the closed solid ball are its surface points.

In [4] the author discusses the mapping whose image is defined as the convex hull of its extreme points or similarly, the sets of extreme points of the convex hull of the image of a function. Moreover, the result in [4, Lemma 1, p.17] ensures the connection between an extreme point $x \in \mathrm{dom}(f) \subseteq \mathbb{R}^n$ of the domain $\mathrm{dom}(f)$ of a real convex Lipschitz function f, and an extreme point $(x, \lambda) \in \mathrm{dom}(f) \times \mathbb{R} \subset \mathbb{R}^{n+1}$ such that $f(x) \leq \lambda$ or $f(x) > \lambda$.

The interesting question to investigate is whether a convex function can be extreme point preserving map. If not, are there conditions under which this might happen? In essence, we need to discuss the structure of the image of such a function in order to gain understanding of this extreme-point-preserving map.

In [17] again the author discusses the mapping of extreme points using a real convex bounded lower semicontinuous function $f : K \subseteq \Omega \to \mathbb{R}$ defined on a compact convex set K contained in some locally convex space Ω. Obviously, K contains at least one extreme point according to the well-known Krein Milman theorem.

In [1] the authors discuss functions whose epigraphs have extreme points, that is, they study the relationship between the geometrical property of an extended real convex function and the extremal properties of epi(f).

Henceforth X denotes a real normed vector space, $f : \operatorname{dom} f \subseteq X \to \mathbb{R}$ denotes a real-valued function with non-empty domain in X, and

$$\mathfrak{C} \subseteq \{C \subseteq A : \emptyset \neq C \text{ is convex}\}$$

denotes a non-empty class of convex non-empty subsets of A, for some fixed non-empty subset $A = \operatorname{dom}(f)$ unless otherwise stated.

Definition 2.0.6 *[5, Theorem 10, p.138]* Let D be a subset of a Banach space X.

1. A point $x \in D$ is called an *extreme point* of D if $x = \lambda y + (1 - \lambda)z$, for some $\lambda \in [0, 1]$ and for some $y, z \in D$, then $y = x$ or $z = x$.

2. A point $x \in D$ is called an *exposed point* of D if there is a functional $f^* \in X^*$ such that $f^*(x) > f^*(y)$ for all $y \in D\backslash\{x\}$, where X^* is the dual space of X.

3. A point $x \in D$ is called a *strongly exposed point* of D if there is a functional $f^* \in X^*$ such that $f^*(x) > f^*(y)$ for all $y \in D\backslash\{x\}$, and such that $f^*(x_n) \to f^*(x)$ for $(x_n)_{n\geq 1}$ in D implies that $x_n \to x$.

4. A point $x \in D$ is called a *denting point* of D if $x \notin \operatorname{clco}(D\backslash B_\varepsilon(x))$ for any $\varepsilon > 0$.

Note that, if f is convex and $(x, \alpha) \in \text{ext}(\text{epi}(f))$, then $x \in A$ is not necessarily an extreme point of A. For example, take $f(x) = \|x\|$ on $\mathbb{R}^n$ and $(x_0, f(x_0)) \in \text{ext}(\text{epi}(f))$. Then $x_0 = 0 \in \mathbb{R}^n$ is not an extreme point.

Moreover, recall that an epigraph of a real convex function f is convex. Conversely, if and epigraph of a real function f is convex, then f is also convex.

2.1 Extremal properties of convex epigraphs

In this section we discusss the existence, or perhaps the lack thereof, of extreme points, denting points and strongly exposed points in the epigraph of a real convex continuous function f.

We shall use a well-known convention that a convex set is *dentable* if it contains a denting point.

Lemma 2.1.1 *Let $f : A \subseteq X \to \mathbb{R}$ be a real function. Then*

$$\text{gr}(f) = \{(x, \lambda) \in \text{epi}(f) : f(x) = \lambda\} \subseteq \partial(\text{epi}(f)).$$

Proof
Take $(x, \lambda) \in \text{gr}(f) \subseteq \text{epi}(f)$. Then $x \in A$ and $f(x) = \lambda$. Moreover, there exists no $\epsilon > 0$ such that a ball $B_\epsilon(x, \lambda) \subset \text{epi}(f)$ and hence $(x, \lambda) \notin \text{int}(\text{epi}(f))$. It follows that $(x, \lambda) \in \partial(\text{epi}(f))$, and consequently $\text{gr}(f) \subseteq \partial(\text{epi}(f))$ □

Lemma 2.1.2 *If $f : A \subseteq X \to \mathbb{R}$ is a strictly convex function and $x, y \in A$, then $(\lambda x + (1 - \lambda)y, f(\lambda x + (1 - \lambda)y)) \notin [(x, f(x)), (y, f(y))]$ for each $\lambda \in (0, 1)$.*

Proof
Clearly $x, y, \lambda x + (1 - \lambda)y \in A$ as A is convex, and $f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$ for each $\lambda \in (0, 1)$ as f is strictly convex. Hence

$f(\lambda x + (1-\lambda)y) \neq \lambda f(x) + (1-\lambda)f(y)$ for each $\lambda \in (0,1)$. If $(\lambda x + (1-\lambda)y, f(\lambda x + (1-\lambda)y)) \in [(x, f(x)), (y, f(y))]$ for some $\lambda \in (0,1)$, then $(\lambda x + (1-\lambda)y, f(\lambda x + (1-\lambda)y)) = \alpha(x, f(x)) + (1-\alpha)(y, f(y)) = (\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y))$ for some $\alpha \in (0,1)$. Hence it would follow that $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$ for some $\lambda \in (0,1)$ and thus lead to a contradiction. It follows that $(\lambda x + (1-\lambda)y, f(\lambda x + (1-\lambda)y)) \notin [(x, f(x)), (y, f(y))]$ for each $\lambda \in (0,1)$. □

Proposition 2.1.3 *Let $f : X \to \mathbb{R}$ be strictly convex. Then*

1. $\text{ext}(\text{epi}(f)) = \text{gr}(f)$
2. $\text{ext}(\text{epi}(f)) \subseteq \partial(\text{epi}(f))$.

Proof

1. Take $(x, \lambda) \in \text{ext}(\text{epi}(f))$. Clearly $(x, \lambda) \notin \text{int}(\text{epi}(f))$ and $f(x) \leq \lambda$.

   If $f(x) < \lambda$, then take any $\delta > \lambda$ in $\mathbb{R}$ and consider $(x, \delta) \in \text{epi}(f)$. Clearly $(x, \lambda) \in \text{int}[(x, f(x)), (x, \delta)]$ and hence $(x, \lambda) \notin \text{ext}(\text{epi}(f))$. This would lead to a contradiction and hence $f(x) = \lambda$, that is $(x, \lambda) \in \text{gr}(f)$.

   For the other inclusion, take $(x_0, f(x_0)) \in \text{gr}(f) \subseteq \text{epi}(f)$ and take any $(x, \alpha), (y, \beta) \in \text{epi}(f)$ such that $(x_0, f(x_0)) = \lambda(x, \alpha) + (1-\lambda)(y, \beta)$ with $\lambda \in [0,1]$. Since $\text{epi}(f)$ is convex, $\lambda(x, \alpha) + (1-\lambda)(y, \beta) \in \text{epi}(f)$ for all $\lambda \in [0,1]$, that is $[(x, \alpha), (y, \beta)] \subseteq \text{epi}(f)$. Clearly $\text{gr}(f) \bigcap \text{int}[(x, \alpha), (y, \beta)] = \emptyset$ since f is strictly convex, and hence $\text{gr}(f) \bigcap [(x, \alpha), (y, \beta)] \subseteq \{(x, \alpha), (y, \beta)\}$. Thus $(x_0, f(x_0)) = (x, \alpha)$ or $(x_0, f(x_0)) = (y, \beta)$, that is $(x_0, f(x_0)) \in \text{ext}(\text{epi}(f))$.

2. Take $(x, \lambda) \in \text{ext}(\text{epi}(f))$. It follows from 1. above that $(x, \lambda) \in \text{gr}(f)$. Appealing to Lemma 2.1.1 it follows that $(x, \lambda) \in \partial(\text{epi}(f))$. □

Clearly a boundary point is not always an extreme point and that is illustrated in the following example;

Example 2.1.4 Let $f : I \to \mathbb{R}$ be a convex function defined by $f(x) = ax^3$ on $I = [0, n)$, for any $a, n \in \mathbb{N}$. Then $(0, k) \in \text{epi}(f)$ for each $k \in \mathbb{N}$ with $(0, k) \in \partial(\text{epi}(f))\backslash\ \text{gr}(f)$ as $(0, k) \in \text{int}[(0, 0), (0, k + 1)] \in \text{epi}(f)$. Moreover, since $\text{ext}(\text{epi}(f)) \subseteq \text{gr}(f)$ we have $\partial(\text{epi}(f))\backslash\ \text{gr}(f) \subseteq \partial(\text{epi}(f))\backslash\ \text{ext}(\text{epi}(f))$. Hence $\text{int}[(0, 0), (0, k + 1)] \subseteq \partial(\text{epi}(f))\backslash\ \text{ext}(\text{epi}(f))$.

Hence a boundary point may be an extreme point under certain conditions, some of which being an unbounded domain and non-linear boundary or non-linear graph. See the Corollary below.

Corollary 2.1.5 *Let $f : X \to \mathbb{R}$ be strictly convex. If* $\text{epi}(f)$ *is closed, then* $\text{ext}(\text{epi}(f)) = \partial(\text{epi}(f))$.

Proof

Take $(x, \alpha) \in \partial(\text{epi}(f))$. Since $\text{epi}(f)$ is closed, we have $(x, \alpha) \in \text{epi}(f)$ and $(x, \alpha) \notin \text{int}(\text{epi}(f))$ with $f(x) < \alpha$ or $f(x) = \alpha$. Since X is open, we have $x \in \text{int}(X)$ and hence there is $\epsilon > 0$ such that $B_\epsilon(x) \subset X$. Clearly f is defined on each $a \in B_\epsilon(x)$ and hence $K_a = \{(a, \mu) : f(a) < \mu\} \subseteq \text{epi}(f)$ for each $a \in \text{cl}(B_\epsilon(x))$.

Suppose $(x, \alpha) \in \partial(\text{epi}(f))\backslash(\text{gr}(f) \cup K_x)$. This would mean that $(x, \alpha) \notin \text{gr}(f)$ and $(x, \alpha) \notin K_x$, and hence $(x, \alpha) \in \partial(\text{epi}(f))$ such that $f(x) > \alpha$. However, since $\text{epi}(f)$ is closed, we have $(x, \alpha) \in \partial(\text{epi}(f)) \subseteq \text{epi}(f)$ and thus $f(x) \leq \alpha$. This leads to a contradiction and thus $(x, \alpha) \in K_x$ or $(x, \alpha) \in \text{gr}(f)$ for $x \in X$ with $(x, \alpha) \in \partial(\text{epi}(f))$.

If $(x, \alpha) \in K_x \cap \partial(\text{epi}(f))$, then it would mean that $\partial[(x, f(x)), (x, \alpha)] \subseteq \partial(\text{epi}(f))$. Since f and $\text{epi}(f)$ are convex, it follows that $K_x \subseteq \partial(\text{epi}(f))$. But since f is continuous on $\text{cl}(B_\epsilon(x))$ and $K_a \subseteq \text{epi}(f)$ for each for each $a \in \text{cl}(B_\epsilon(x))$, it follows that $K_x \subseteq \text{int}(\text{epi}(f))$ and hence lead to a contradiction. It follows that $(x, \alpha) \notin K_x$ and consequently $(x, \alpha) \in \text{gr}(f) = \text{ext}(\text{epi}(f))$, see Proposition 2.1.3. This shows that $\partial(\text{epi}(f)) \subseteq \text{ext}(\text{epi}(f))$. Furthermore, the other inclusion follows from Proposition 2.1.3 (no 2.). $\square$

Theorem 2.1.6 *[14, Theorem 1.3.3, p.21] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be convex. Then f is continuous on int(I).*

Corollary 2.1.7 *Let $f : I \to \mathbb{R}$ be convex. If f is not continuous at $x_0 \in I$ then $x_0 \in \partial(I)$.*

Remark 2.1.8 If $f : \mathbb{R} \to \mathbb{R}$ is convex, then f is continuous as each $x_0 \in \mathbb{R}$ is such that $x_0 \in \text{int}(I)$ for some (open) $I \subseteq \mathbb{R}$. This follows from the fact that $\mathbb{R}$ is open (and also closed).

Consider the following example of a convex function not continuous at some $x_0 \in \partial(I) \cap I$;

$$g(x) = \begin{cases} x^2 & \text{if} \quad x \in (-\infty, 0) \\ x^3 & \text{if} \quad x \in [0, n), 0 < n \in \mathbb{N} \\ x^3 + 1 & \text{if} \quad x = n \end{cases}$$

It follows that $g : (-\infty, n] \to \mathbb{R}$ is convex and continuous on $(-\infty, n)$ and discontinuous at some $x_0 = n \in \mathbb{N}$, for any choice of $n \in \mathbb{N}$ fixed. Clearly $n \in \partial(I) \cap I$.

More generally;

Proposition 2.1.9 *[14, Proposition 3.5.2, p.119] Let $f : \mathfrak{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ be convex and $\mathfrak{U}$ be open. Then f is continuous on $\mathfrak{U}$.*

Furthermore;

Theorem 2.1.10 *[23, Theorem 2.2.9, p.64] Let X be a separable locally convex space, $f : \text{dom}(f) \subseteq X \to \mathbb{R}$ be convex and bounded above on a neighborhood of any point $x_0 \in \text{dom}(f)$. Then f is continuous on* int$(\text{dom}(f))$.

Consequently;

Corollary 2.1.11 *Let $f : A \subseteq X \to \mathbb{R}$ be convex and not continuous at some $x \in A$, X a separable Banach space. If f is bounded above then $x \in \partial(A)$.*

Proof
By assumption f is convex and bounded above on a separable Banach space X. Clearly f is bounded above on a neighborhood of any point $x_0 \in A$, and thus follows from Theorem 2.1.10 above that f is continuous on $\operatorname{int} A$. Since f is not continuous at $x \in A$ and $\partial(A) \neq \emptyset$, it follows that $x \in \partial(A)$, and this completes the proof. □

Still on the continuity of convex functions one should make mention of *a closed epigraph theorem* which is an analogue of the classical Banach closed graph theorem. It simply states that;

If $f : D \subset X \to \mathbb{R}$ is convex and $\operatorname{epi}(f)$ is closed, then f is continuous.

This is true under certain conditions imposed on both X and the function f itself, see [7].

Note that one can find a convex function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ not defined on its boundary points, that is, on some $x_0 \in \partial(A) \backslash A$, in which case the sided-limit is not a real number. For example, $f(x) = \sec x$ on $A = (-\pi/2, \pi/2)$ is convex and continuous on $\operatorname{int}(A)$ but not continuous at $\partial(A)$. Moreover

$$\lim_{x_n \to -(\pi/2)^+} f(x_n) = \lim_{x_n \to (\pi/2)^-} f(x_n) = \infty,$$

and consequently $\partial(\operatorname{epi}(f)) \backslash \operatorname{gr}(f) = \emptyset$. Hence $\lim_{x \to \partial(A)} f(x) = \infty$ would be integral in the subsequent sections.

Henceforth X denotes a Banach space unless otherwise stated.

Definition 2.1.12 *[14, Definition 3.5.7, p.121]* An extended real (convex) function $f : A \subseteq X \to \overline{\mathbb{R}}$ is *lower-semicontinuous*, l.s.c for brevity, at $y \in A$ if

$$f(y) \leq \liminf_{x \to y} f(x),\ x \in A$$

Such a function f is said to be l.s.c if f is l.s.c at each point of $x \in A$.

There is a relation between continuity of f and closure of its epigraph and it is stated in the well known result about l.s.c real function as follows;

Theorem 2.1.13 *[18, Theorem 7.1, p.51] If $f : A \subseteq X \to \overline{\mathbb{R}}$ is a extended real function, then the following are equivalent;*

1. *f is lower-semicontinuous*
2. *the level sets $\{x \in A : f(x) \leq \lambda\}$ are closed for each $\lambda \in \mathbb{R}$.*
3. *The epigraph* $\text{epi}(f)$ *is a closed subset of $A \times \mathbb{R}$*

Since continuity implies lower semicontinuity, we have the following result as a consequence of Theorem 2.1.13;

Proposition 2.1.14 *Let $f : A \subseteq X \to \mathbb{R}$ be convex and continuous. If A is closed, then* $\text{epi}(f)$ *is closed.*

Proof
Let f be continuous on A. Then it follows that f is lower semi-continuous on A and appealing to Theorem 2.1.13 $\text{epi}(f)$ is closed. □

Corollary 2.1.15 *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be convex. If $A = \mathbb{R}^n$ then f is continuous and* $\text{epi}(f)$ *is closed.*

Proof
If $A = \mathbb{R}^n$ then A is both open and closed. If A is open then f is continuous as per Proposition 2.1.9. Moreover, since A is closed it follows from Proposition 2.1.14 that $\text{epi}(f)$ is closed. □

In what follows we discuss the extreme points of epigraphs using convex extension theory we developed in the preceeding sections. Recall that $\mathfrak{C}_0$ is a non-empty class of convex, infinite and non-empty subsets of A in X.

We denote by

$$\text{Epi}(\mathfrak{X}_{f,\mathfrak{C}}) = \{\text{epi}(g) : g \in \mathfrak{X}_{f,\mathfrak{C}}\}$$

the collection of convex epigraphs $epi(g)$ containing (or which are extensions of) epi(f). Moreover, we denote by

$$\text{Ext}_c(\mathfrak{X}_{f,\mathfrak{C}}) = \{\text{ext}(\text{epi}(g)) \subseteq \text{epi}(g)\text{: epi}(g) \in \text{Epi}(\mathfrak{X}_{f,\mathfrak{C}}) \text{ is convex}\}$$

the collection of sets of extreme points of **closed** epigrahs of convex extensions of f.

Lemma 2.1.16 *Let $f : A \subseteq X \to \mathbb{R}$ be continuous $\mathfrak{C}$-convex function and $\mathfrak{C}$ be non-empty. Then for each closed set $F \in \mathfrak{C}_f$ with $f|_F$ strictly convex, we have* $\text{Ext}_c(\mathfrak{X}_{f,\mathfrak{C}}) \ni \text{ext}(\text{epi}(f|_F)) \neq \emptyset$.

Proof
Since F is closed and $f|_F$ is convex and continuous, it follows from Proposition 2.1.14 that epi$(f|_F)$ is closed. Moreover, it follows from Proposition 2.1.3 (1) that $\text{ext}(\text{epi}(f|_F)) = \text{gr}(f|_F)$. Since $\text{gr}(f|_F) \neq \emptyset$ it follows that $\text{ext}(\text{epi}(f|_F)) \neq \emptyset$. □

Example 2.1.17 Consider $f : A \subseteq \mathbb{R} \to \mathbb{R}$ a continuous convex function defined by $f(x) = x^2$ on $A = [-2, 2]$.
1. $C_x = \{(x, \lambda) : \lambda \in \mathbb{R},\ (x, \lambda) \in \text{epi}(f)\}$ is a convex subset of epi(f) for each fixed $x \in \{-2, 2\}$.
2. $\{(x, f(x))\} = C_x \bigcap \text{ext}(\text{epi}(f))$ for each fixed $x \in \{-2, 2\}$.
3. $\mathfrak{C}_x = \{(x, \lambda) : \lambda \in \mathbb{R},\ (x, \lambda) \in \text{epi}(f) \setminus \text{gr}(f)\}$ is convex for each fixed $x \in \{-2, 2\}$, and it follows that $\mathfrak{C}_x \bigcap \text{ext}(\text{epi}(f)) = \emptyset$ for each fixed $x \in \{-2, 2\}$.
4. $\partial(\text{epi}(f)) \setminus \text{gr}(f) = \mathfrak{C}_{-2} \cup \mathfrak{C}_2 = \{(x, \lambda) : f(x) < \lambda,\ x \in \partial(A)\} \subseteq \text{epi}(f)$.
5. Clearly $\partial(\text{epi}(f)) \setminus \text{gr}(f) \neq \emptyset$ and $\text{ext}(\text{epi}(f)) \bigcap \partial(\text{epi}(f)) \setminus \text{gr}(f) = \emptyset$, as $\text{ext}(\text{epi}(f)) \subseteq \text{gr}(f) \subset \partial(\text{epi}(f))$, see Lemma 2.1.1.

Under what condition would $f : A \subseteq X \to \mathbb{R}$ convex satisfy the following property(ies);
1. **Boundary Preserving Property:**

For which $\lambda \in \mathbb{R}$ would $x \in \partial(A)$ imply $(x, \lambda) \in \partial\ (\text{epi}(f))$?
2. **Extreme Preserving Property:** For which $\lambda \in \mathbb{R}$ would $x \in \text{ext}(A)$ imply $(x, \lambda) \in \text{ext}(\text{epi}(f))$?

Clearly the converse of 1. and 2. above does not hold on $\mathbb{R}^n$. Consider the case where $n = 1$, $f(x) = x^2$ and $x = 0 \in \text{int}(A)$ with $A \subseteq \mathbb{R}$. It follows that $(0, f(0)) \in \text{ext}(\text{epi}(f)) \cap \partial\ (\text{epi}(f))$, but $x \notin \text{ext}(A)$.

Lemma 2.1.18 *Let $f : A \subseteq X \to \mathbb{R}$ be convex and continuous, A be closed and bounded. Then*

1. $K = \partial(\text{epi}(f)) \backslash \text{gr}(f) \neq \emptyset$
2. $\text{ext}(\text{epi}(f)) \bigcap K = \emptyset$

Proof

1. Since A is bounded and closed we have $\emptyset \neq \partial(A) \subseteq A$. Clearly $\mathfrak{C}_x = \{(x, \lambda) : \lambda \in \mathbb{R},\ (x, \lambda) \in \text{epi}(f) \backslash\ \text{gr}(f)\}$ is convex for each fixed $x \in \partial(A)$ and $\text{gr}(f) \bigcap \mathfrak{C}_x = \emptyset$. Moreover $(x, \lambda) \in \text{epi}(f) \backslash\ \text{gr}(f)$ for each $(x, \lambda) \in \mathfrak{C}_x$. Take $(x, \lambda) \in \mathfrak{C}_x$ with $x \in \partial(A)$. Since $x \in \partial(A) \subset A$ there exists no $\varepsilon > 0$ such that $B_\varepsilon(x, \lambda) \subset \text{epi}(f)$. Hence $(x, \lambda) \notin \text{int}(\text{epi}(f))$ and consequently $\partial(\text{epi}(f)) \backslash\ \text{gr}(f) \neq \emptyset$.

2. Take $(x, \lambda) \in \text{ext}(\text{epi}(f))$. Then $(x, \lambda) \in \partial(\text{epi}(f))$ as per Proposition 2.1.3 and hence $(x, \lambda) \in \text{gr}(f) \cup \mathfrak{C}_x$ for some $x \in \partial(A)$. If $(x, \lambda) \in \mathfrak{C}_x$ for some $x \in \partial(A)$, then $(x, \lambda) \in \text{int}[(x, f(x)), (x, \lambda + 1)] \subseteq \text{epi}(f)$. Thus $(x, \lambda) \notin \text{ext}(\text{epi}(f))$ and it would lead to a contradiction.

   It follows that $(x, \lambda) \in \text{gr}(f)$ and hence $(x, \lambda) \notin \partial(\text{epi}(f)) \backslash\ \text{gr}(f)$. Thus $\text{ext}(\text{epi}(f)) \bigcap \partial(\text{epi}(f)) \backslash\ \text{gr}(f)) = \emptyset$ and hence $\text{ext}(\text{epi}(f)) \bigcap K = \emptyset$. □

The following is the simple illustration of Lemma 2.1.18 above;

Example 2.1.19 Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be convex and continuous. If A is closed and bounded, with $A = [a, b], a, b \in \mathbb{R}$, then $(a, \lambda) \in \text{epi}(f)$ for all $\lambda \geq f(a), \lambda \in \mathbb{R}$. Choose $\alpha > \lambda, \alpha \in \mathbb{R}$. It follows that $(a, \alpha) \in \text{epi}(f) \backslash \text{gr}(f) \neq \emptyset$, and thus $(a, \alpha) \notin \text{ext}(\text{epi}(f)) \subseteq \text{gr}(f)$.

That is, if $a = 2$ and $f(x) = x^2$, then $f(a) \neq 5$ and hence $(a, \alpha) = (2, 5) \in \text{epi}(f) \backslash \text{gr}(f) \subseteq \text{epi}(f) \backslash \text{ext}(\text{epi}(f))$. More generally, we have

$\text{int}(\text{epi}(f)) = \{(c, \mu) \in \text{epi}(f) : a < c < b \text{ and } f(c) < \mu\}$. Let
$B = \{(c, \mu) \in \text{epi}(f) : f(c) = \mu \text{ and } c \in [a, b]\}$ and
$D = \{(c, \mu) \in \text{epi}(f) : f(c) < \mu,\ c \in \{a, b\}\}$. Hence $\text{epi}(f) \backslash \text{int}(\text{epi}(f)) = B \bigcup D$.

Consequently $\text{gr}(f) = B$ and $\text{ext}(\text{epi}(f)) \subseteq \text{gr}(f) \subseteq \partial(\text{epi}(f)) = B \bigcup D$. Hence $\partial(\text{epi}(f)) \backslash \text{gr}(f) = D \neq \emptyset$ since $\partial(A) \neq \emptyset$. Moreover, since
$B = \text{ext}(\text{epi}(f)) \subseteq \text{gr}(f)$ we have $\partial(\text{epi}(f)) \backslash \text{gr}(f) \bigcap \text{ext}(\text{epi}(f)) = B \bigcap D = \emptyset$.

Lemma 2.1.20 *Let $f : X \to \mathbb{R}$ be convex. If $(x, \lambda) \in \text{int}(\text{epi}(f))$, then there exists $\epsilon > 0$ such that $(x, \lambda) \in \text{clco}(\text{epi}(f) \backslash B_\epsilon(x, \lambda))$.*

Proof
Take any $(x, \lambda) \in \text{int}(\text{epi}(f))$. If $f(x) = \lambda$, then
$(x, \lambda) \in \text{gr}(f) \subseteq \partial(\text{epi}(f)) \backslash \text{int}(\text{epi}(f))$, and would lead to a contradiction.

It follows that $f(x) < \lambda$ and thus $\lambda - f(x) > 0$. Choose $0 < \epsilon \leq \lambda - f(x) > 0$, then $(x, f(x)) \notin B_\epsilon(x, \lambda)$ and $(x, \lambda + \epsilon + 1), (x, f(x)) \in \text{epi}(f) \backslash B_\epsilon(x, \lambda)$. Hence $(x, \lambda) \in [(x, f(x)), (x, \lambda + \epsilon + 1)] \subseteq \text{clco}(\text{epi}(f) \backslash B_\epsilon(x, \lambda))$.
□

This show that an interior of an epigraph $\text{epi}(f)$ is disjoint from the set of denting points of such epigraph.

Proposition 2.1.21 *Let $f : X \to \mathbb{R}$ be a strictly convex (and $\text{epi}(f)$ closed). Then*

$$\text{dent}(\text{epi}(f)) \subseteq \text{ext}(\text{epi}(f)) \subseteq \text{gr}(f).$$

Proof
Take $(x,\lambda) \in \text{dent}(\text{epi}(f))$. Then $(x,\lambda) \notin \text{clco}(\text{epi}(f)\backslash B_\epsilon(x,\lambda))$ for any $\epsilon > 0$. Hence $(x,\lambda) \notin \text{co}(\text{epi}(f)\backslash B_\epsilon(x,\lambda))$ for any $\epsilon > 0$ and it follows that $(x,\lambda) \notin [(y,\beta),(m,\alpha)]$ for any $(y,\beta),(m,\alpha) \in \text{epi}(f)\backslash B_\epsilon(x,\lambda)$ for any $\epsilon > 0$. Since $\epsilon > 0$, for any $(y,\beta),(m,\alpha) \in \text{epi}(f)\backslash B_\epsilon(x,\lambda)$, we have $(x,\lambda) \neq (y,\beta)$ and $(x,\lambda) \neq (m,\alpha)$.

Take any distinct $(a,b),(c,d) \in \text{epi}(f)$ such that $(x,\lambda) \in [(a,b),(c,d)]$. If $(x,\lambda) \in \text{int}[(a,b),(c,d)]$, then choose $\epsilon > 0$ such that $(a,b),(c,d) \in \text{epi}(f)\backslash B_\epsilon(x,\lambda)$. Clearly $(x,\lambda) \in [(a,b),(c,d)] \subseteq \text{co}(\text{epi}(f)\backslash B_\epsilon(x,\lambda)) \subseteq \text{clco}(\text{epi}(f)\backslash B_\epsilon(x,\lambda))$ and this leads to a contradiction. Hence $(x,\lambda) \notin \text{int}[(a,b),(c,d)]$ for any distinct $(a,b),(c,d) \in \text{epi}(f)$ and thus $(x,\lambda) \in \text{ext}(\text{epi}(f))$.

The other inclusion follows from Proposition 2.1.3 no.1. □

It has been proven by Edelstein in [6, Proposition 1, p.111] that dentability in general is not related to extrema structure, in the following result;

Proposition 2.1.22 *[6, Proposition 1, p.111] There is a dentable closed bounded convex body in the Banach space c_0 which has no extreme points.*

Moreover, we know that it might not always be true that extreme points are denting points hence we pose the following questions;
Under what condition would $\text{dent}(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = \text{gr}(f)$?

If $f : X \to \mathbb{R}$ is convex, it follows from Proposition 2.1.14 that $\text{epi}(f)$ is closed as X is closed and open. Hence it is clear from Lemma 2.1.20 that an interior point of a closed epigraph of a convex function f is not a denting point in such an epigraph, and cannot be an extreme point either.

Definition 2.1.23 *[12, p.526]* Let A be a bounded closed and convex subset of a Banach space X and $x \in A$. Then

1. x is a *point of continuity (PC)* for A if the identity mapping $id : (A, weak) \to (A, norm)$ is continuous at such x.

2. x is a *strong extreme point* of A if for any sequences $(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ in A, $\lim_{n\to\infty} \|\frac{1}{2}(y_n + z_n) - x\| = 0$ implies $\lim_{n\to\infty} \|y_n - x\| = 0$.

3. x is a *weak*-extreme point* of A if x is an extreme point of $\bar{\text{A}}$, where $\bar{\text{A}}$ is the weak*-closure of A in X^{**}.

4. $x \in A$ is a *very strong extreme point* of A if for every sequence $(f_n)_{n\geq 1}$ of A-valued Bochner integrable functions on [0,1], the condition $\lim_{n\to\infty} \| \int_0^1 f_n(t)dt - x\| = 0$ implies $\lim_{n\to\infty} \int_0^1 \|f_n(t) - x\|dt = 0$.

The following result by Troyanski et al., suggests that a denting point is an extreme point in a closed bounded and convex set in a Banach space X.

Theorem 2.1.24 *[12, p.526] Let x be an element in a bounded closed convex set A of a Banach space. Then the following are equivalent:*

1. *x is a denting point of A.*

2. *x is a very strong extreme point of A.*

3. *x is a PC for A, and x is an extreme point of A, (respectively, strong extreme point, weak*-extreme point of A).*

If a set A in Theorem 2.1.24 above is an epigraph of a convex function, then we would have a good characterisation of extreme points and denting points of an epigraph. The challenge however would be brought about by a fact that an epigraph of a convex function is never bounded.

This leads to asking the following question;
Does there exist a closed convex subset C of epi(f) of a (strictly) convex function such that dent(C) = dent(epi(f)) and ext(C) = dent(epi(f))?
Under what condition, if any, would an extreme point of an epigraph be a denting point?

Remark 2.1.25 Let $f : A \subseteq X \to \mathbb{R}$ be a strictly convex function and $\epsilon \in \mathbb{R}$. Denote by $L_\epsilon = \{(x, \lambda) \in \text{epi}(f)\colon\ x \in A \text{ and } \lambda < \epsilon\}$ a special convex

subset of epi(f). Moreover, assume there is some $m \in A$, a minimizer of f, such that $f(m) \leq f(y)$, for all $y \in A$.

Clearly for each $\epsilon > \lambda$ with $(x, \lambda) \in \text{epi}(f)$ or $(m, \lambda) \in \text{epi}(f)$ and for $m \in A$ a minimizer of f, we have $f(m) \leq \epsilon$ and hence $(m, f(m)) \in L_\epsilon \neq \emptyset$.

Clearly L_ϵ is convex and we show it as follows:
Take any $(y, \mu), (z, \nu) \in L_\epsilon$. It follows that $y, z \in A$ and $\mu, \nu, \max\{\mu, \nu\} < \epsilon$. Hence for any $\alpha \in [0, 1]$, we have $\alpha(y, \mu) + (1 - \alpha)(z, \nu) = (\alpha y + (1 - \alpha)z, \alpha\mu + (1 - \alpha)\nu) \in L_\epsilon$ as $\alpha y + (1 - \alpha)z \in A$ (since A convex) and $\alpha\mu + (1 - \alpha)\nu \leq \max\{\mu, \nu\} < \epsilon$.

Furthermore, denote by $A_\epsilon = \{x \in A : (x, \lambda) \in L_\epsilon\}$ a subset of A depended on the choice of $\epsilon \in \mathbb{R}$.

Clearly A_ϵ is convex and we show it as follows:
Take any $y, z \in A_\epsilon$. It follows that $y, z, \alpha y + (1-\alpha)z \in A$ for $\alpha \in [0, 1]$ since A is convex, and $(y, \mu), (z, \nu) \in \text{epi}(f) \cap L_\epsilon$ with $\epsilon > \mu$ and $\epsilon > \nu$. Thus $\alpha(y, \mu) + (1-\alpha)(z, \nu) = (\alpha y + (1-\alpha)z, \alpha\mu + (1-\alpha)\nu) \in \text{epi}(f)$ as $\text{epi}(f) \cap L_\epsilon$ is convex, and $\alpha\mu + (1 - \alpha)\nu \leq \max\{\mu, \nu\} < \epsilon$. Hence $\alpha y + (1 - \alpha)z \in A_\epsilon$ and hence A_ϵ is convex.

Theorem 2.1.26 *Let $f : A \subseteq X \to \mathbb{R}$ be strictly convex and epi(f) be closed, and $L_\epsilon \neq \emptyset$ be defined as above for $\epsilon \in \mathbb{R}$, then the following hold;*
1. $\text{clco}(L_\epsilon) \subseteq \text{epi}(f)$.
2. $\partial(\text{clco}(L_\epsilon)) \setminus \text{int}(\text{epi}(f)) \subseteq \partial(\text{epi}(f))$.
3. *If $A = X$, then for each $(x, \lambda) \in \text{ext}(\text{epi}(f))$, there exists $\epsilon \in \mathbb{R}$ such that $(x, \lambda) \in \text{clco}(L_\epsilon) \neq \emptyset$.*
4. *Let $A = X$, then $(x, \lambda) \in \text{ext}(\text{epi}(f))$ if and only if $(x, \lambda) \in \text{ext}(\text{clco}(L_\epsilon))$ for some ϵ.*

Proof
1. Clearly $L_\epsilon \subseteq \text{epi}(f)$, and since epi($f$) is closed and convex we have $\text{epi}(f) = \text{clco}(\text{epi}(f))$. Thus $\text{clco}(L_\epsilon) \subseteq \text{clco}(\text{epi}(f)) = \text{epi}(f)$.

2. Take $(y, \beta) \in \partial(\text{clco}(L_\epsilon))$. It follows that $(y, \beta) \in \text{epi}(f)$. Hence if

$(y,\beta) \in \partial(\text{clco}(L_\epsilon))\backslash \text{int}(\text{epi}(f))$, then $(y,\beta) \in \text{epi}(f)\backslash \text{int}(\text{epi}(f)) \subseteq \partial(\text{epi}(f))$.

3. Take any $(x,\lambda) \in \text{ext}(\text{epi}(f))$. It follows from Proposition 2.1.3 that $(x,\lambda) \in \text{gr}(f) \subseteq \text{epi}(f)$ and hence $f(x) = \lambda < +\infty$. Choose $\epsilon \in \mathbb{R}$ such that $f(x) = \lambda < \epsilon$ and $(x,\lambda) \in L_\epsilon$. Thus there exists $\epsilon \in \mathbb{R}$ such that $(x,\lambda) \in \text{clco}(L_\epsilon) \neq \emptyset$.

4. Take any $(x,\lambda) \in \text{ext}(\text{clco}(L_\epsilon))$. Hence $(x,\lambda) \in \partial(\text{clco}(L_\epsilon))\backslash \text{int}(\text{epi}(f))$ and thus $(x,\lambda) \notin \text{int}(\text{clco}(L_\epsilon)) \cup \text{int}(\text{epi}(f))$. It follows from 2. above that $(x,\lambda) \in \partial(\text{epi}(f))$. Appealing to Corollary 2.1.5, we have $(x,\lambda) \in \text{ext}(\text{epi}(f))$.
Conversely, take any $(y,\beta) \in \text{ext}(\text{epi}(f))$. It follows from 3. above that $(y,\beta) \in \text{clco}(L_\epsilon)$ for some ϵ. Clearly $(y,\beta) \in \partial(\text{clco}(L_\epsilon))$ as if we assume that $(y,\beta) \in \text{int}(\text{clco}(L_\epsilon))$, we would have $(y,\beta) \in \text{int}(\text{epi}(f))$ since $\text{clco}(L_\epsilon) \subseteq \text{epi}(f)$. This would lead to a contradiction.

Take any $(m,\tau),(z,\mu) \in \text{clco}(L_\epsilon) \subseteq \text{epi}(f)$ and $\alpha \in [0,1]$ such that $(y,\beta) = \alpha(m,\tau) + (1-\alpha)(z,\mu)$. It follows that $(y,\beta) = (m,\tau)$ or $(y,\beta) = (z,\mu)$ as $(y,\beta) \in \text{ext}(\text{epi}(f))$. Hence $(x,\lambda) \in \text{ext}(\text{clco}(L_\epsilon))$. □

Corollary 2.1.27 *Let X be finite dimensional Banach space, $f : X \to \mathbb{R}$ be strictly convex (and* epi(f) *be closed). If f achieves its minimum on X then* $\text{ext}(\text{epi}(f)) = \text{dent}(\text{epi}(f))$

Proof
Appealing to Proposition 2.1.21 $\text{dent}(\text{epi}(f)) \subseteq \text{ext}(\text{epi}(f)$. Morever, since X, and $X \times \mathbb{R}$, are finite dimensional normed vector spaces, weak and norm topologies are equivalent and hence each $(x,\lambda) \in \text{ext}(\text{epi}(f))$ is a PC. Morever, it follows from Theorem 2.1.26 that $(x,\lambda) \in \text{ext}(\text{clco}(L_\epsilon))$ for some ϵ. Since $\text{clco}(L_\epsilon))$ is closed convex and bounded, we have $(x,\lambda) \in \text{dent}(\text{epi}(f))$, as per Theorem 2.1.24.

Remark 2.1.28 1. Let X be a Banach space and X' its topological dual. A function $f : X \to (-\infty, +\infty]$ is *rotund* at $x_0 \in X$ if there exists $x' \in X'$ which verifies the following property

$$\forall \epsilon > 0 \ \exists r > 0, \text{ such that}$$
$$\langle x', x_0 \rangle - f(x_0) < \langle x', x \rangle - f(x) + r \Rightarrow \|x - x_0\| \leq \epsilon,$$

$x \in X$, see [1]
2. Functions f is rotund at x_0 implies that the subdifferential
$\partial f(x_0) = \{x' \in X' : \langle x', x_0 \rangle - f(x_0) \geq \langle x', x \rangle - f(x), \forall x \in X\} \neq \emptyset$,
and $\sup\{\langle x', x \rangle - f(x), x \in X\}$ reached at x_0, see [1, Remark (2), p.15].
3. If f is rotund at x_0 then f is strictly convex at x_0, [1, Remark (3), p.16].

Consider the following result about the equivalence between denting points and extreme points of a convex epigraph.

Corollary 2.1.29 *[1, Corollary 3.5, p.20] Let f be a convex lower semi-continuous (extended) real valued function on Banach space X. Then, f is rotund at each $x \in X$ if and only if*

$$\text{dent}(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = \partial(\text{epi}(f)) = \text{gr}(f).$$

Corollary 2.1.30 *Let X be a Banach space and $f : X \to \mathbb{R}$ be convex. If (epi(f) are closed and) f is rotund on X then*

$$\text{dent}(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = \partial(\text{epi}(f) = \text{gr}(f).$$

Proof
Since epi(f) is closed, it follows from Theorem 2.1.13 that f is lower semi-continuous and rotund. Moreover, it follows from Corollary 2.1.29 that $\text{dent}(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = \partial(\text{epi}(f) = \text{gr}(f)$. □

Proposition 2.1.31 *Let X be Banach, $f : X \to \mathbb{R}$ be convex (and* epi(f) *be closed). Then the following holds;*

1. *For each* $\alpha \in [0,1]$, $\alpha(\text{epi}(f)) + (1-\alpha)(\text{epi}(f)) = \text{epi}(f)$.

2. *For any* $\alpha \in \mathbb{R}_+$, $\alpha(\text{epi}(f)) \subseteq \text{epi}(f)$ *provided* f *is positively homogeneous, that is* $f(\alpha x) = \alpha f(x)$ *for all* $x \in X$

3. *For each* $\alpha \in \partial[0,1]$, $\alpha(\text{ext}(\text{epi}(f)))+(1-\alpha)(\text{ext}(\text{epi}(f))) \subseteq \text{ext}(\text{epi}(f))$.

Proof

1. $\alpha(\text{epi}(f)) + (1-\alpha)(\text{epi}(f)) = \{\alpha(x,\beta) + (1-\alpha)(z,\mu) : (x,\beta),(z,\mu) \in \text{epi}(f), \alpha \in [0,1]\}$. Since f is convex, so is $\text{epi}(f)$ and hence $\alpha(x,\beta) + (1-\alpha)(z,\mu) \in \text{epi}(f)$ for any $(x,\beta),(z,\mu) \in \text{epi}(f)$ It follows that $\alpha(\text{epi}(f)) + (1-\alpha)(\text{epi}(f)) \subseteq \text{epi}(f)$.

   For the other inclusion, take $(y,\omega) \in \text{epi}(f)$. Since $\text{epi}(f)$ is convex there exist $(b,\beta),(d,\delta) \in \text{epi}(f)$ such that $(y,\omega) = \alpha(b,\beta) + (1-\alpha)(d,\delta) \in \alpha(\text{epi}(f)) + (1-\alpha)(\text{epi}(f))$. It follows that $(y,\omega) \in \alpha(\text{epi}(f)) + (1-\alpha)(\text{epi}(f))$ and hence $\alpha(\text{epi}(f)) + (1-\alpha)(\text{epi}(f)) = \text{epi}(f)$.

2. $\alpha(\text{epi}(f)) = \{\alpha(x,\tau) : (x,\tau) \in \text{epi}(f)\}$. Take any $\alpha(x,\tau) \in \alpha(\text{epi}(f))$. Clearly $(x,\tau) \in \text{epi}(f)$ and hence $f(x) \leq \tau$. Since f is positively homogeneous and $\alpha x \in X$ for each $x \in X$, we have $f(\alpha x) = \alpha f(x) \leq \alpha\tau$ hence $(\alpha x, \alpha\tau) \in \text{epi}(f)$.

3. This result follows from letting $\alpha = 0$ and $\alpha = 1$. That is, take any $(w,\lambda) \in \alpha(\text{ext}(\text{epi}(f))) + (1-\alpha)(\text{ext}(\text{epi}(f)))$. It follows that $(w,\lambda) = \alpha(x,\beta)+(1-\alpha)(z,\mu)$ for $(x,\beta),(z,\mu) \in \text{ext}(\text{epi}(f))$. If $\alpha = 0$, then $(w,\lambda) = (z,\mu) \in \text{ext}(\text{epi}(f))$, and if $\alpha = 1$, then $(w,\lambda) = (x,\beta) \in \text{ext}(\text{epi}(f)$. This proves that $(w,\lambda) \in \text{ext}(\text{epi}(f)$. □

It is clear that a function $f : A \subseteq X \to \mathbb{R}$ is convex linear if $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$ for any $x, y \in A$. We say $f : A \subseteq X \to \mathbb{R}$ is *affine linear* if there exist $g : X \to \mathbb{R}$ linear and $b \in \mathbb{R}$ such that $f(x) = g(x) + b$ for all $x \in A$.

Lemma 2.1.32 *Let $f : A \subseteq X \to \mathbb{R}$ be convex and continuous, then* $\mathrm{gr}(f)$ *is convex if and only if f is convex linear function.*

Proof

Suppose $\mathrm{gr}(f)$ is convex, and take any $x, y \in A$ such that $(x, f(x)), (y, f(y)) \in \mathrm{gr}(f)$, with $\lambda x + (1-\lambda)y \in A, \lambda \in [0,1]$, since A is convex.

Convexity of $\mathrm{gr}(f)$ imposes that

$\lambda(x, f(x)) + (1-\lambda)(y, f(y)) = (\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y)) \in \mathrm{gr}(f)$.

Moreover, since $\lambda x + (1-\lambda)y \in A$ for any $x, y \in A$ and all $\lambda \in [0,1]$ we have $(\lambda x + (1-\lambda)y, f(\lambda x + (1-\lambda)y) \in \mathrm{gr}(f)$ for each $\lambda \in [0,1]$. Since f is well defined, we have

$(\lambda x + (1-\lambda)y, f(\lambda x + (1-\lambda)y)) = (\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y)) \in \mathrm{gr}(f)$. Hence $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$ for any $x, y \in A$ and all $\lambda \in [0,1]$. It follows that f is linear.

Conversely, let f be linear and $(\alpha, f(\alpha)), (\beta, f(\beta)) \in \mathrm{gr}(f)$, $\alpha, \beta \in A$, A convex. Thus for $\lambda \in [0,1]$,

$$\begin{aligned}\lambda(\alpha, f(\alpha)) + (1-\lambda)(\beta, f(\beta)) &= (\lambda\alpha + (1-\lambda)\beta, \lambda f(\alpha) + (1-\lambda)f(\beta)) \\ &= (\lambda\alpha + (1-\lambda)\beta, f(\lambda\alpha + (1-\lambda)\beta) \\ &\in \mathrm{gr}(f),\end{aligned}$$

as $\lambda\alpha + (1-\lambda)\beta \in A$. Consequently $\mathrm{gr}(f)$ is convex. □

Still on the relation between the continuity of convex functions and extreme points of its epigraph, one considers linear functions on a vector space and their behavior on the boundary of its domain, see below;

Lemma 2.1.33 *Let $f : A \subseteq X \to \mathbb{R}$ be linear convex function and X be normed linear space. The following holds*

1. *If* $(x, \lambda) \in \mathrm{ext}(\mathrm{epi}(f))$*, then* $(x, \lambda) \in \mathrm{gr}(f)$ *and* $x \notin \mathrm{int}(A)$
2. *If* $\partial(A) \cap A = \emptyset$ *(and hence A open) then* $\mathrm{ext}(\mathrm{epi}(f)) = \emptyset$

Proof

1. Take $(x,\lambda) \in \text{ext}(\text{epi}(f)) \subseteq \text{epi}(f)$. Then $(x,\lambda) \in \partial(\text{epi}(f))$. Moreover, if $(x,\lambda) \in \partial(\text{epi}(f)) \setminus \text{gr}(f)$, then $(x,\lambda) \in \text{int}[(x,f(x)),(x,\lambda+1)]$ for $(x,f(x)),(x,\lambda+1) \in \text{epi}(f)$. This would contradict the fact that $(x,\lambda) \in \text{ext}(\text{epi}(f))$, hence $(x,\lambda) \in \partial(\text{epi}(f)) \cap \text{gr}(f)$, that is $(x,\lambda) = (x,f(x)) \in \text{ext}(\text{epi}(f))$.

Morever, take any $(m,\mu),(n,\nu) \in \text{epi}(f)$ such that $(x,\lambda) \in [(m,\mu),(n,\nu)]$. It follows $(x,\lambda) = (\alpha m + (1-\alpha)n, \alpha\mu + (1-\alpha)\nu)$, $\alpha \in [0,1]$.

Hence, $x = \alpha m + (1-\alpha)n \in A$, and since $(x,\lambda) \in \text{ext}(\text{epi}(f))$ we have $(x,\lambda) = (m,\mu)$ or $(x,\lambda) = (n,\nu)$. Thus $x = \alpha m + (1-\alpha)n \in A$ and $x = m$ or $x = n$. It follows that $x \notin \text{int}[m,n]$.

Assume $x \in \text{int}(A)$. It follows from the convexity of A that there exists some $a,b \in A$ with $x \in \text{int}[a,b]$. Since f is linear we have

$$(x,f(x)) = \alpha((a,f(a)) + (1-\alpha)(b,f(b)), \alpha \in (0,1).$$

Moreover, since $(x,\lambda) = (x,f(x)) \in \text{ext}(\text{epi}(f))$, we have $(x,f(x)) \in \partial[((a,f(a)),(b,f(b))]$, that is

$$(x,f(x)) = \alpha((a,f(a)) + (1-\alpha)(b,f(b)), \alpha \notin (0,1),$$

and this leads to a contradiction. It follows that $x \notin \text{int}(A)$.

2. If $\partial(A) \cap A = \emptyset$ then A is open, and $x \in \text{int}(A)$ for each $x \in A$. Hence the contraposition of 1. above implies that $(x,\lambda) \notin \text{ext}(\text{epi}(f))$ for each $x \in \text{int}(A) = A$. Hence $\text{ext}(\text{epi}(f)) = \emptyset$. □

Definition 2.1.34 *[2, Definition 2.3.1, p.27]* Let D be a bounded set in X and $f^* \in X^*$, $f^* \neq 0$ and let $M(D,f^*) = \sup\{f^*(x) : x \in D\}$. If $\alpha > 0$, then the set $S(D,f^*,\alpha) = \{x \in D : f^*(x) > M(D,f^*) - \alpha\}$ is called the *slice of* D determined by f^* and α.

Lemma 2.1.35 *[2, Proposition 2.3.21, p.28] A bounded subset D of a Banach space X is dentable if and only if D has a slice of arbitrarily small diameter*

There is a relation between denting points and stronngly exposed points of a closed unit ball and it is stated in [8] as follows;

Theorem 2.1.36 *[8, Theorem 8, p.232] A Banach space X where every point of a unit sphere $S(X)$ is denting points of a closed unit ball $B(X)$ in X has the set of strongly exposed points of $B(X)$ dense in $S(X)$.*

On strongly exposed points of an epigraph we have the following result;

Lemma 2.1.37 *[1, Proposition 3.4, p.18] Let $f : X \to (-\infty, +\infty]$ be lower semicontinuous convex function and $x_0 \in \text{dom}(f)$. If f is rotund then $(x_0, f(x_0) \in$ s-exp$(\text{epi}(f))$. The converse also holds.*

Clearly it follows from Lemma 2.1.37 that rotundity of l.s.c extended real convex f implies $\text{gr}(f) = \text{s-exp}(\text{epi}(f))$. Hence, appealling to Proposition 2.1.21 we have the following result;

Lemma 2.1.38 *Let $f : X \to \mathbb{R}$ be rotund convex (and* epi(f) *closed). Then*

$$\text{dent}(\text{epi}(f)) \subseteq \text{ext}(\text{epi}(f)) \subseteq \text{gr}(f) \subseteq \text{s-exp}(\text{epi}(f)).$$

Proposition 2.1.39 *Let $f : X \to \mathbb{R}$ be rotund strictly convex, X a Banach space (and* epi(f) *be closed). Then*

$$\text{dent}(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = \partial(\text{epi}(f) = \text{gr}(f) = \text{s-exp}(\text{epi}(f)).$$

Proof
It follows from Corollary 2.1.30 that $\text{dent}(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = \text{gr}(f) = \partial(\text{epi}(f)$. Since epi$(f)$ closed implies that f is lower semi-continuous (see Theorem 2.1.13), appealing to Lemma 2.1.37 we have $\text{gr}(f) = \text{s-exp}(\text{epi}(f))$ and the result follows. □

2.2 Piece-wise linear convex functions

In this section we give a special attention to piece-wise linear (non-linear) convex functions which elicit interesting characteristics of convex functions different from those we find using the general convex functions.

Henceforth by piece-wise linear (affine) convex function $f : A \subseteq X \to \mathbb{R}$ we would mean a function $f : A \subseteq X \to \mathbb{R}$ not linear (affine) on A but linear (affine) on some convex proper subsets A_i of A, with $\text{int}(A_i) \cap \text{int}(A_k) = \emptyset$ for each $A_i, A_k \subseteq A$ convex with $\bigcup_i A_i = A$.

Example 2.2.1 Consider the following piece-wise linear convex function $f : [-5, 4] \to \mathbb{R}$ define on a closed convex and bounded set as follows.

$$f(x) = \begin{cases} g_1(x) = -3x - \frac{5}{2} & \text{if} \quad x \in [-5, -1) \\ g_2(x) = -\frac{1}{4}x + \frac{1}{4} & \text{if} \quad x \in [-1, 1) \\ g_3(x) = x - 1 & \text{if} \quad x \in [1, 3) \\ g_4(x) = 2x - 4 & \text{if} \quad x \in [3, 4] \end{cases}$$

a) Each g_i is linear and each $\text{dom}(g_i)$ is a maximal domain of linearity for f.

b) f is not linear though it has linear restrictions, namely $f|_{(0,1)}$ is linear.

c) Note that $\text{dom}(g_3) = [1, 3)$ is a maximal domain of linearity for g_3 but $[1, 3]$ is a maximal domain for linearity for f. However, $\text{dom}(g_4) = [3, 4]$ is a maximal domain for linearity for both g_4 and f. In general $\text{cl}(\text{dom}(g_i))$ is a maximal domain of linearity for each $i = 1, .., 4$

d) Moreover, if $x \in \text{int}(\text{dom}(g_i))$ for some i, then there is $y, z \in \partial(\text{dom}(g_i))$ such that $x \in \text{int}[y, z]$ and $(x, f(x)) \in [(y, f(y)), (z, f(z))]$. Hence $(x, f(x)) \notin \text{ext}(\text{epi}(f))$ for any $x \in \text{int}(\text{dom}(g_i))$ for some i.

e) Suppose $z \in \text{int}(\text{dom}(g_i))$ and $y \in \text{int}(\text{dom}(g_k))$ with $\text{dom}(g_k) \cap \text{dom}(g_i) = \emptyset$ and $[y, z] \subseteq A$. Since f is convex, we have $f(\lambda y + (1-\lambda)z) \leq \lambda f(y) + (1-\lambda)f(z)$ for $\lambda \in [0, 1]$. Moreover, $f(\lambda y + (1-\lambda)z) \neq \lambda f(y) + (1-\lambda)f(z)$ for some $\lambda \in [0, 1]$. This follows from the fact that $f|_{[z,y]}$ is not linear as $cl(\text{dom}(g_i)), cl(\text{dom}(g_k))$ are maximal domains of linearity for f.

Morever, for any $x \in \text{int}[y,z]$ with y,z as above, we have $f(x) < f(\lambda y + (1-\lambda)z)$ since f is convex and not linear, and thus $(x,f(x)) \notin [(y,f(y)),(z,f(z))]$. It follows that for any $x \in \partial(\text{dom}(g_m)) \cap \text{int}[y,z]$ we have $(x,f(x)) \notin [(y,f(y)),(z,f(z))]$ for some $i = 1,..,4$. Consequently, if $(x,f(x)) \in \text{ext}(\text{epi}(f))$ and $x \in [y,z]$ for some $z \in \text{dom}(g_i)$ and $y \in \text{dom}(g_k)$, then the following hold;

$\bullet y \in \partial(\text{dom}(g_k))$ or $z \in \partial(\text{dom}(g_i))$.

$\bullet x \in \partial(\text{dom}(g_k))$ or $x \in \partial(\text{dom}(g_i))$, provided $k = i+1$.

Proposition 2.2.2 *Let $f : A \subseteq X \to \mathbb{R}$ be a piece-wise linear convex function on a closed convex and bounded set A, and $A_i \subseteq A$ be a maximal domain of linearity for each $i \in \mathbb{N}$. Then* $\text{ext}(\text{epi}(f)) \subseteq \text{gr}(f|_Q)$ *for* $Q = \bigcup_{i\in\mathbb{N}} \partial(A_i)$.

Proof

Take $(x,\lambda) \in \text{ext}(\text{epi}(f))$. Then $(x,\lambda) \in \text{epi}(f)$ and for all $(z,\beta),(y,\tau) \in \text{epi}(f)$ such that $(x,\lambda) \in [(z,\beta),(y,\tau)]$ we have $(x,\lambda) = (z,\beta)$ or $(x,\lambda) = (y,\tau)$ as $(x,\lambda) \in \text{ext}(\text{epi}(f))$. Clearly $(x,\lambda) \in \text{gr}(f)$ as $\text{ext}(\text{epi}(f)) \subseteq \text{gr}(f)$ and hence $\{(z,\beta),(y,\tau)\} \cap \text{gr}(f) \neq \emptyset$.

Then either $(y,\tau) \in \text{gr}(f)$ or $(z,\beta) \in \text{gr}(f)$ or $(z,\beta),(y,\tau) \in \text{gr}(f)$. Moreover, if $y \in \text{int}(A_j)$ and $z \in \text{int}(A_k)$ then since $f|_{A_j}$ and $f|_{A_k}$ are linear, there exists $b,c \in \partial(A_j)$ and $d,e \in \partial(A_k)$ such that $y \in \text{int}[b,c]$, $z \in \text{int}[d,e]$ with $(y,\tau) \in \text{int}[(b,f(b)),(c,f(c))]$, $(z,\beta) \in \text{int}[(d,f(d)),(e,f(e))]$ and $(b,f(b)),(c,f(c)),(e,f(e)),(d,f(d)) \in \text{epi}(f)$. This would would mean $(x,\lambda) \in \text{int}[(b,f(b)),(c,f(c))]$ or $(x,\lambda) \in \text{int}[(d,f(d)),(e,f(e))]$ and would contradict $(x,\lambda) \in \text{ext}(\text{epi}(f))$. Hence $y \in \partial(A_j)$ or $z \in \partial(A_k)$

Furthermore, since $(x,\lambda) \in \text{ext}(\text{epi}(f))$ and thus $(x,\lambda) = (z,\beta)$ or $(x,\lambda) = (y,\tau)$, we have $x = y$ or $x = z$ and $y \in \partial(A_j)$ or $z \in \partial(A_k)$ for some $j,k \in \mathbb{R}$. If $(x,\lambda) = (z,\beta) \in \text{gr}(f)$, then $x = z$ and thus $z \in \partial(A_k)$. Similarly, if $(x,\lambda) = (y,\tau) \in \text{gr}(f)$, then $x = y$ and thus $y \in \partial(A_j)$. Hence

$x \in Q = \bigcup_{i \in \mathbb{R}} \partial(A_i)$ and thus $(x, \lambda) \in \text{gr}(f|_Q)$. $\square$

The converse holds provided f is continuous on A and $Q \subset A$.

Corollary 2.2.3 *Let $f : A \subseteq X \to \mathbb{R}$ be a piece-wise linear convex function and $A_i \subseteq A$ be a maximal domain of linearity for each $i \in \mathbb{N}$ with $\bigcup_{i \in \mathbb{N}} A_i = A$. If A is closed and bounded and f is continuous, then $\text{ext}(\text{epi}(f)) = \text{gr}(f|_Q)$ for $Q = \bigcup_{i \in \mathbb{N}} \partial(A_i) \subset A$.*

Proof

Take any $x \in Q$ with $(x, f(x)) \in \text{gr}(f|_Q) \subseteq \text{epi}(f)$. Clearly there is $(m, \mu), (n, \nu) \in \text{epi}(f)$ with $(x, f(x)) \in [(m, \mu), (n, \nu)]$ and $x \in \partial(A_i)$ for some $A_i \subseteq A$ a maximal domain. Assume without loss of generality that $(m, \mu), (n, \nu) \in \text{epi}(f) \cap \text{gr}(f)$. If $(x, f(x)) \in \text{int}[(m, \mu), (n, \nu)]$ it would follow that $[(m, \mu), (n, \nu)] \subseteq A_m$ for some A_m a maximal domain. This would lead to a contradiction as it would mean $x \in \partial(A_i) \cap \text{int}(A_m)$. Consequently $(x, f(x)) \in \partial[(m, \mu), (n, \nu)]$ and hence $(x, f(x)) \in \text{ext}(\text{epi}(f))$. $\square$

Proposition 2.2.4 *Suppose $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a linear convex function, $\text{epi}(f)$ closed, A bounded and closed, and $|f(x)| < \varepsilon$ for some $\varepsilon > 0$. The following hold;*

1. *If $(z, \alpha) \in \text{ext}(\text{epi}(f))$, then $z \in \text{ext}(A)$.*
2. *$\text{ext}(\text{epi}(f)) = \{(x, \alpha) : x \in \partial(A)\}$ and hence has 2 elements.*

Proof

1. Clearly $A = [x, y]$ is a closed interval and $\text{gr}(f)$ is a finite line segment including its endpoints, $(x, f(x)), (y, f(y)) \in \text{gr}(f)$. Consequently $\text{gr}(f) = [(x, f(x)), (y, f(y))] \subset \text{epi}(f)$. Take $(z, \alpha) \in \text{ext}(\text{epi}(f))$. It follows that $(z, \alpha) \in \text{gr}(f) = [(x, f(x)), (y, f(y))]$. It follows that either $(z, \alpha) = (x, f(x))$ or $(z, \alpha) = (y, f(y))$. It follows that $x, y, \lambda x + (1 - \lambda)y \in A$ for $\lambda \in [0, 1]$ since $A = [x, y]$ is convex. Clearly $x, y \in \text{ext}(A)$.

2. Clearly, $\mathrm{gr}(f) = [(x, f(x)), (y, f(y))]$ as f is linear, and hence $\mathrm{ext}(\mathrm{epi}(f)) \subseteq [(x, f(x)), (y, f(y))]$. But since $\mathrm{int}[(x, f(x)), (y, f(y))] \cap \mathrm{ext}(\mathrm{epi}(f)) = \emptyset$, we have $\mathrm{ext}(\mathrm{epi}(f)) = \{(x, f(x)), (y, f(y))\}$. $\square$

2.3 Convexifiable functions

In this section we discuss the (closed) convex hulls of epigraphs of real-valued functions and the functions, namely convexifications, representing them. It is clear that not all functions have convex epigraphs, as only convex function may have convex epigraphs and not all functions are convex.

Hence we aim to define functions whose epigraphs are convex hulls of epigraphs of other given (and mostly non-convex) functions, and those functions are defined as follows;

In this section X denotes a Banach space unless otherwise stated.

Definition 2.3.1 A real-valued function $g : A \subseteq X \to \mathbb{R}$ is called a *convexification* of a real-valued (usually non-convex) function $f : A \subseteq X \to \mathbb{R}$ if $\text{epi}(g) = \text{co}(\text{epi}(f))$.

Lemma 2.3.2 *Let $f : A \subseteq X \to \mathbb{R}$ and $g : A \subseteq X \to \mathbb{R}$ be real functions and A be a convex set. Then the following hold;*
(a) *If $\text{epi}(f) = \text{clco}(\text{epi}(f))$, then f is convex and lower-semicontinuous. Moreover, f is continuous on $\text{int}(A)$, provided $X = \mathbb{R}^n$.*
(b) *If $\text{epi}(f) = \text{clco}(\text{epi}(g))$, then f is a lower semi-continuous convexification of g.*

Proof
(a) If $\text{epi}(f) = \text{clco}(\text{epi}(f))$ then $\text{epi}(f)$ is closed and convex. Moreover, the convexity of $\text{epi}(f)$ implies the convexity of f, and the closure of $\text{epi}(f)$ implies that f is lower semi-continuous, see Theorem 2.1.13. Furthermore, the continuity of f on $\text{int}(A)$ follows from Proposition 2.1.9.

(b) Clearly $\text{epi}(f)$ is closed and hence f is lower semi-continuous as shown in (a) above. Moreover, the fact that f is a convexification of g follows from the Definition 2.3.1 above. □

In addition to Definition 2.3.1 above, we shall call such a function f *convexifiable* provided it has at least one real convexification g.

Remark 2.3.3 The following statements hold true:
1) There exists no real valued convexification of a function f if $f(x) = -\infty$ for some $x \in \text{dom}(f)$.
2) The epigraph of the convexification of f is the smallest convex epigraph containing $\text{epi}(f)$. This follows from the fact that the convex hull $\text{co}(A)$ of any set A is the smallest convex set containing A.
3) If f is (not convex but) $\mathfrak{C}$-convex and g^* a convex extension of $f|_C$, then $\text{epi}(f) \subset \text{epi}(g^*)$ does not always hold especially for $f(x) = x^3$, a $\mathfrak{C}$-convex function. This is due to the fact that f is convex on $C = [0, \infty)$ and g^* extends $f|_C$ and not f on $\mathbb{R}$. That is $\text{epi}(f|_C) \subseteq \text{epi}(g^*)$ but $(-1, -1) \in \text{epi}(f) \setminus \text{epi}(g^*)$. This follows from the fact that g^* is a convex extension of a convex restriction $f|_M$ of f on the $\mathfrak{C}$-MDC $M \subset \text{dom}(f)$. Hence $f(x) < g^*(x)$ for some $x \in \text{dom}(f) \setminus M$ and $\text{epi}(g^*) \supseteq \text{epi}(f|_M)$. Moreover $\text{epi}(g^*) \subseteq \text{epi}(f)$ provided $f(x) \leq g^*(x)$ for any $x \in \text{dom}(g^*) \cap (\text{dom}(f) \setminus M)$ and $\text{dom}(g^*) \subseteq \text{dom}(f)$.
4) If g is a convexification of $f : A \subseteq X \to \mathbb{R}$ and $\text{epi}(f) \subset \text{epi}(g)$, that is $\text{epi}(g) \setminus \text{epi}(f) \neq \emptyset$, then f is not convex.

Our aim is to compare the epigraphs of the convexification and that of the convex extension g^* as defined above.

Lemma 2.3.4 *Let $f : A \subseteq X \to \mathbb{R}$ be a real convex function and g its convexification. Then $f(x) = g(x)$ for all $x \in \text{dom}(f) \cap \text{dom}(g)$ and $f = g$. Consequently,* $\text{dom}(f) = \text{dom}(g)$.

Proof
Clearly $\text{epi}(f) = \text{co}(\text{epi}(f))$ as f is convex and $\text{co}(\text{epi}(f) = \text{epi}(g)$ as g is a convexification of f, by assumption. Hence $\text{epi}(f) = \text{epi}(g)$ and thus $f = g$. It follows that $f(x) = g(x)$ for all $x \in \text{dom}(f) \cap \text{dom}(g)$. Clearly $\text{dom}(f) \cap \text{dom}(g) \subseteq \text{dom}(f)$ and $\text{dom}(f) \cap \text{dom}(g) \subseteq \text{dom}(g)$, and for each for each $z \in \text{dom}(f)$ and each $y \in \text{dom}(g)$, we have $f(z) = g(z), f(y) = g(y)$. This implies that $z, y \in \text{dom}(f) \cap \text{dom}(g)$ and thus $\text{dom}(g) = \text{dom}(f) \cap$

$\text{dom}(g) = \text{dom}(f)$ □

Note that the real function f is proper if $f(x) \in (-\infty, \infty)$ for all $x \in \text{dom}(f)$. Clearly if g is a convexification of f, then $g(x) \leq f(x)$ for all $x \in \text{dom}(f) \cap \text{dom}(g)$.

Lemma 2.3.5 *Every real proper $\mathfrak{C}$-convex function $f : A \subseteq X \to \mathbb{R}$ is convexifiable and has a unique convexification, provided f is bounded below.*

Proof

Clearly $\text{co}(\text{epi}(f))$ is convex for each $\mathfrak{C}$-convex function f. Consider a function $g : \text{co}(\text{dom}(f)) \to \mathbb{R}$ such that

$g(x) = \inf\{\lambda \in \mathbb{R} : (x, \lambda) \in \text{co}(\text{epi}(f))\}$ for each $x \in \text{co}(\text{dom}(f))$.

We show that g is convex:

Take $x, y, \alpha x + (1-\alpha)y \in \text{co}(\text{dom}(f))$, $\alpha \in [0, 1]$

$$\begin{aligned} g(\alpha x + (1-\alpha)y) &= \inf\{k \in \mathbb{R} : (\alpha x + (1-\alpha)y, k) \in \text{co}(\text{epi}(f))\} \\ &= \inf\{\alpha\lambda_x + (1-\alpha)\lambda_y \in \mathbb{R} : (\alpha x + (1-\alpha)y, \alpha\lambda_x + (1-\alpha)\lambda_y) \in \text{co}(\text{epi}(f))\} \\ &= \inf\{\alpha\lambda_x + (1-\alpha)\lambda_y \in \mathbb{R} : \alpha(x, \lambda_x) + (1-\alpha)(y, \lambda_y) \in \text{co}(\text{epi}(f))\} \\ &\leq \inf\{\alpha\lambda_x \in \mathbb{R} : (x, \lambda_x) \in \text{co}(\text{epi}(f))\} \\ &\qquad + \inf\{(1-\alpha)\lambda_y \in \mathbb{R} : (y, \lambda_y) \in \text{co}(\text{epi}(f))\} \\ &= \alpha g(x) + (1-\alpha)g(y) \end{aligned}$$

Consequently g is convex.

Clearly $\text{epi}(g)$ is the smallest convex epigraph containing $\text{epi}(f)$ due to definition of g. Moreover due to the uniqueness of $\text{co}(\text{epi}(f)) = \text{epi}(g)$ there follows the uniqueness of g the convexification. □

Proposition 2.3.6 *Let $f : A \subset X \to \mathbb{R}$ be a real proper non-convex $\mathfrak{C}$-convex function, C be $\mathfrak{C}$-MDC and g a convexification of f. Then the following hold;*

(1) $\text{epi}(f|_C) \subset \text{co}(\text{epi}(f)) = \text{epi}(g)$ if $C \subset A$ and

$g(x) = \min\{\lambda \in \mathbb{R} : (x, \lambda) \in \text{co}(\text{epi}(f))\}$ for each $x \in \text{co}(\text{dom}(f))$.

(2) If $\text{epi}(f) \subset \text{epi}(g)$ then f is not convex.

Proof
(1) This follows easily from the fact that $\text{epi}(f|_C) \subset \text{epi}(f) \subseteq \text{co}(\text{epi}(f))$ and the fact that g is the convexification of f.

(2) Suppose $\text{epi}(f) \subset \text{epi}(g)$, that is $\text{epi}(g) \setminus \text{epi}(f) \neq \emptyset$. Since g is the convexification of f, we have $\text{co}(\text{epi}(f)) = \text{epi}(g) \supset \text{epi}(f)$. It follows that $\text{co}(\text{epi}(f)) \setminus \text{epi}(f) \neq \emptyset$ and hence $\text{epi}(f)$ is not convex. Since the convexity of f coincides with the convexity of its epigraph $\text{epi}(f)$, it follows that f is not convex. □

Consider the following example

Example 2.3.7 Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = a \sin x, x, a \in \mathbb{R}$. Then the following hold;
(1) $C_i = [(2i-1)\pi, 2i\pi]$ for each $i \in \mathbb{Z}$ is a $\mathfrak{C}$-MDC for f and are mutually disjoint, provided $a \neq 0$.
(2) $\text{epi}(f) \subset \text{co}(\text{epi}(f))$ as f is not convex on $\mathbb{R}$, provided $a \neq 0$.
(3) f is convexifiable with a convexification

$$g(x) = \inf\{a \sin x : x \in \mathbb{R}\} = -|a|$$

Moreover, it follows from Lemma 2.3.5 that the following is true;

Proposition 2.3.8 *Let $f : X \to \mathbb{R}$ be a function and $\mathfrak{B}$ be a chain of closed subset C of X. Then, for each domain $C_i \in \mathfrak{B}$ of a restriction $f|_{C_i}$ there exists a unique convexification g_i for each i.*

2.4 Extremal-mapping convex function

In this section we discuss a special kind of a vector valued functions on the epigraph of some real convex (strictly or linear) function f. We shall also discuss its properties and explore which of the well-known properties of convex functions it satisfies, if any.

Definition 2.4.1 Let $f : A \subset X \to \mathbb{R}$ be a real continuous convex function and $\text{epi}(f)$ be its epigraph. A continuous function $\phi : \text{epi}(f) \subset X \times \mathbb{R} \to \text{gr}(f)$ defined by $\phi(x, \lambda) = (x, f(x))$, for any fixed $x \in A$ and for any $(x, \lambda) \in \text{epi}(f)$, is called *a minimizer of the epigraph of* f, *epi-min* for brevity.

Furthermore, a restriction $\phi|_{\text{gr}(f)} : \text{epi}(f) \subset X \times \mathbb{R} \to \text{gr}(f)$ is an identity map. Moreover, define an ordering $\leq$ on $X \times \mathbb{R}$, where X finite dimensional Banach space, as follows;
For any $(a, \alpha), (b, \beta) \in X \times \mathbb{R}$, we say $(a, \alpha) \leq (b, \beta)$ if $\Sigma_{i=1}^{n} a_i \leq \Sigma_{i=1}^{n} b_i$ and $\alpha \leq \beta$.

Henceforth X will denote a finite dimensional Banach space, unless otherwise stated.

Theorem 2.4.2 *Let $f : A \subset X \to \mathbb{R}$ be a real valued convex function. Then epi-min $\phi : \text{epi}(f) \subset X \times \mathbb{R} \to \text{gr}(f)$ is a vector valued convex function on* $\text{epi}(f)$.

Proof
Let (x, λ), $(y, \beta) \in \text{epi}(f)$ with $x, y \in A$ and $\alpha \in [0, 1]$. Hence $f(x) \leq \lambda, f(y) \leq \beta$ and $\alpha(x, \lambda) + (1 - \alpha)(y, \beta) = (\alpha x + (1 - \alpha)y, \alpha\lambda + (1 - \alpha)\beta) \in \text{epi}(f)$ since $\text{epi}(f)$ and A are convex and $\alpha x + (1 - \alpha)y \in A$. Thus

$$f(\alpha x + (1 - \alpha)y) \leq \alpha\lambda + (1 - \alpha)\beta = \varepsilon \in \mathbb{R},$$

for each $\lambda \in [0, 1]$, $\phi(x, \lambda) = (x, f(x))$ and $\phi(y, \beta) = (y, f(y))$. Hence

$$\begin{aligned}\phi(\alpha(x, \lambda) + (1 - \alpha)(y, \beta)) &= \phi(\alpha x + (1 - \alpha)y, \alpha\lambda + (1 - \alpha)\beta) \\ &= (\alpha x + (1 - \alpha)y, f(\alpha x + (1 - \alpha)y))\end{aligned}$$

$$\leq (\alpha x + (1-\alpha)y, \alpha f(x) + (1-\alpha)f(y))$$
$$= \alpha(x, f(x)) + (1-\alpha)(y, f(y))$$
$$= \alpha\phi(x, \lambda) + (1-\alpha)\phi(y, \beta)$$

This proves that ϕ is a convex function. □

Remark 2.4.3 1) Clearly the continuity of the epi-min depends on the continuity of the underlying convex function f.
2) Morever $R_\phi = \{\phi(x, \lambda) : (x, \lambda) \in \text{epi}(f)\} = \{(x, f(x)) : f(x) \leq \lambda \text{ for all } x \in A\} = \text{gr}(f)$.
3) Furthermore, the epi-min ϕ of an epigraph of f is unique for each convex function f. Clearly $\text{epi}(f)$ and $\text{gr}(f)$ are unique for real-valued function f.

Lemma 2.4.4 *Let $f : A \subseteq X \to \mathbb{R}$ be continuous and convex and ϕ be an epi-min of f. The following holds;*
1. *If $A = X$ and R_ϕ is closed, then $R_\phi = \partial(\text{epi}(f)) \subseteq \text{epi}(f)$.*
2. *If $A \subset X$ and R_ϕ is closed, then $R_\phi \subseteq \partial(\text{epi}(f))$.*

Proof
1. Clearly $R_\phi \subseteq \text{gr}(f)$ follows from the definition of the epi-min ϕ. Moreover the definition of an epigraph and the fact that $\text{gr}(f) \subseteq \text{epi}(f))$ we have $\text{gr}(f) \cap \text{int}(\text{epi}(f)) = \emptyset$. Furthermore, since $\text{epi}(f)$ be closed, see Proposition 2.1.14, we have $\partial(\text{epi}(f)) \subseteq \text{epi}(f)$, and consequently $\text{gr}(f) \subseteq \partial(\text{epi}(f))$. Hence $R_\phi = \partial(\text{epi}(f))$, see Proposition 2.1.3 and Corollary 2.1.5.
2. Clearly $R_\phi = \text{gr}(f)$ by definition of epi-min and $\text{gr}(f) \subseteq \partial(\text{epi}(f))$ since f is convex. □

Remark 2.4.5 Suppose $f : A \subset X \to \mathbb{R}$ is a real valued convex function, and $\text{epi}(f)$ its convex epigraph. Hence for each fixed $x \in A$ recall the following subset of $\text{epi}(f)$, that is,

$$C_x = \{(x, \lambda) : \lambda \in \mathbb{R}, (x, \lambda) \in \text{epi}(f)\}.$$

It is easy to see that, for any $x, y \in A$;

- $x = y$ if and only if $C_x = C_y$
- $x \neq y$ if and only if $C_x \bigcap C_y = \emptyset$

Proposition 2.4.6 *Let C_x be as defined above, $\phi|_{C_x} : \text{epi}(f) \to R_\phi$ be the restriction of an epi-min ϕ of a convex function $f : A \subseteq X \to \mathbb{R}$. Then the following holds;*

1. *for each fixed $x \in A$, $\{\phi|_{C_x}(x, \alpha) : (x, \alpha) \in C_x\} = \{(x, f(x))\} = R_{\phi|_{C_x}} \subset X \times \mathbb{R}$ is a singleton.*
2. *$(x, f(x)) \in \text{ext}(\text{epi}(f)) \bigcap C_x$ for each $x \in A$, provided epi(f) is closed and f is strictly convex and continuous on X.*

Proof
1. As per Definition 2.4.1 above, $\phi(x, \alpha) = (x, f(x))$ for any fixed $x \in A$. Hence $\phi|_{C_x}(x, \alpha) = (x, f(x))$ for any fixed $x \in A$ and any $(x, \alpha) \in C_x \subseteq \text{epi}(f)$. It follows that $\{\phi|_{C_x}(x, \alpha) : (x, \alpha) \in C_x\} = \{(x, f(x))\}$ for any fixed $x \in A$. Moreover, since $R_\phi = \text{gr}(f)$, we have $R_{\phi|_{C_x}} = \text{gr}(f|_{\{x\}}) = \{(x, f(x))\}$ for any fixed $x \in A$.

2. Suppose f is strictly convex and epi(f) is closed. Appealing to Proposition 2.1.3, we have $\text{ext}(\text{epi}(f)) = \text{gr}(f)$. Consequently $\{(x, f(x))\} = \text{gr}(f|_{\{x\}}) \subseteq \text{ext}(\text{epi}(f))$. Thus $(x, f(x)) \in \text{ext}(\text{epi}(f)) \bigcap C_x$ for each $x \in A$. □

Remark 2.4.7 (a) Let C_x be as defined above, $\phi|_{C_x} : \text{epi}(f) \to R_\phi$ be the restriction of an epi-min ϕ of a convex function $f : A \subseteq X \to \mathbb{R}$. Then

$$\phi(x, f(x)) \in \text{ext}(\text{epi}(f)) \subseteq R_\phi \text{ for each } x \in A,$$

provided $A = X$, see Proposition 2.1.3. Consequently ϕ is extreme point preserving.
(b) Let $f : A \subseteq X \to \mathbb{R}$ be convex and continuous. Then an epigraph of f has a different characterisation;

$$\text{epi}(f) = \text{co}(\bigcup_{x \in A} C_x).$$

Clearly this would not hold if f is not convex nor continuous, see the example below.

• Let f be a real function defined on $I = [0, 3]\backslash\{1\}$ by $f(x) = 0$ whenever $x \in [0, 1)$ and $f(x) = -x + 1$ whenever $x \in (1, 3]$. Clearly $C_1 = \{(1, \lambda) \in \mathbb{R}^2 : \lambda \geq 0\} \cap \text{epi}(f) = \emptyset$ as $1 \notin \text{dom}(f) = I$, however $C_1 \subseteq \text{co}(\bigcup_{x \in I} C_x)$. Hence $\text{epi}(f) \subset \text{co}(\bigcup_{x \in I} C_x)$ as f is not continuous.

• Moreover, $\{(x, \lambda) \in \mathbb{R}^2 : 3\lambda = -2x\} \subseteq \partial(\text{co}(\text{epi}(f)))$, and $(\frac{3}{2}, -\frac{3}{4}) \in \text{co}(\text{epi}(f))\backslash \text{epi}(f)$, as f is not convex.

Theorem 2.4.8 *Let $f : X \to \mathbb{R}$ be strictly convex (and $\text{epi}(f)$ be closed) and ϕ be its epi-min. Then the restriction $\phi^* : \partial(\text{epi}(f)) \to R_\phi$ of the epi-min is an extreme (denting) points preserving map.*

Proof
Clearly $\text{dom}(\phi^*) = \partial(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = R_\phi$ and ϕ^* is an identity map. Hence for each $(x, \lambda) \in \partial(\text{epi}(f)) = \text{ext}(\text{epi}(f))$ we have $\phi^*(x, \lambda) \in \text{ext}(\text{epi}(f))$. □

It follows that the epi-min $\phi : \text{epi}(f) \subset X \times \mathbb{R} \to \text{epi}(f)$ of an epigraph of a convex continuous function f has a convex epigraph $\text{epi}(\phi)$. Furthermore, the epi-min is convex if and only if its base function f is convex. Henceforth, we consider $\mathfrak{C}_{\text{epi}} \subset \{C \subseteq \text{epi}(f)\colon \emptyset \neq C \text{ is convex}\}$.

Note that a $\mathfrak{C}_{\text{epi}}$-convex function need not be convex but is assumed to have a convex restriction. An epi-min τ is $\mathfrak{C}_{\text{epi}}$-convex if the underlying function f is $\mathfrak{C}$-convex and $\text{epi}(f|_C)$ is a convex domain of τ with some $C \in \mathfrak{C}$.

Proposition 2.4.9 *Let $f : A \subseteq X \to \mathbb{R}$ be $\mathfrak{C}$-convex and ϕ be a $\mathfrak{C}_{\text{epi}}$-convex epi-min on* epi$(f)$. *Then f is convexifiable if and only if ϕ is convexifiable.*

In other words, there exists a real continuous function g on domf such that epi$(g) =$ co(epi(f)) *if and only if ϕ is convex on* co(epi(f)).

Proof

If f is convex, then epi$(f) =$ co(epi(g)) for some $g = f$ on A. Hence f is convexifiable. Thus ϕ is convex, epi$(f) =$ co(epi(χ)) for some $\chi = \phi$, and $\phi :$ epi$(f) \to$ co(epi(f)). Hence ϕ is convex on co(epi(f)). Similarly, the converse follows.

On the other hand, suppose f is not convex on A; $\Rightarrow$ If f is convexifiable then and epi$(f) \subset$ co(epi(f)) = epi(g) for some convexification g on X and $\phi :$ epi$(f) \to$ epi(f) is not convex as epi(f) is not convex. Hence epi$(\phi) \subset$ co(epi(f)). Since f is convexifiable, consider $\chi :$ epi$(g) \to$ epi(g) an epi-min of g a convex convexification of f. Since g is convex and dom$(\chi) =$ epi(g) are convex, it follows from Theorem 2.4.2 that χ is convex. Moreover, epi$(\phi) \subseteq$ epi$(\chi) =$ co(epi(χ)) and hence ϕ is convexifiable.

$\Leftarrow$ If ϕ is convexifiable, it follows that epi$(\phi) \subseteq$ epi$(\mu) =$ co(epi(chi)) for some convexification μ of ϕ. Clearly epi(f) is not convex and hence epi$(f) \subset$ co(epi(f)). Hence $\mu :$co(epi(f)) $\to \partial$(co(epi(f))) is convex, as each convexification of convex. Hence $g(x) = \{\lambda \in \mathbb{R} : (x, \lambda) \in \partial(\text{co}(\text{epi}(f)))\}$, that is gr$(g) = \partial$(co(epi$(f)$)), is a convexification of f. Clearly for each $(a, b) \in$ epi$(f) \subset$ co(epi(f)), $g(a) \leq f(a) \leq b$. □

It follows from the Proposition 2.4.9 above that any $\mathfrak{C}$-convex real function has some real convex extension, hence our subsequent discussion shall be on the way(s) in which such convex extension(s) can be contructed. We shall try and show that if f is a $\mathfrak{C}$-convex function on $A \subset X$ and epi(f) is not necessarily convex epigraph on which the epi-min ϕ is defined, then there exists an extension χ of an epi-min such that if f has a global minimum $(c, f(c))$, then $\chi(x, \lambda) = (x, f(c))$. The range of χ denoted R_χ is the graph of χ whose epigraph is convex.

Example 2.4.10 Let $f : [0, \infty) \to \mathbb{R}$ be $\mathfrak{C}$-convex defined by $f(x) = \sin x$. Clearly $f|_{C_i} : [0, \infty) \to \mathbb{R}$ is convex for each $C_i = [(2i-1)\pi, 2i\pi]$, $i \in \mathbb{N}$. Moreover, for each $i \in \mathbb{N}$, there is a convex extension $k_i : \mathbb{R} \to \mathbb{R}$, such that $f(x) = k_i(x) = f|_{C_i}(x)$ for each $x \in C_i$ and $f(x) \leq k_i(x)$ for each $x \notin C_i$. Furthermore, for g a convexification of f, we have $\text{epi}(f) \subseteq \text{co}(\text{epi}(f)) = \text{epi}(g)$.

Lemma 2.4.11 *Let $f : A \subseteq X \to \mathbb{R}$ be $\mathfrak{C}$-convex, $g : X \to \mathbb{R}$ be convexification of f, C a $\mathfrak{C}$-MDC for f and $K : X \to \mathbb{R}$ a convex extensions of $f|_C$, the restriction of f to C. Then $\text{epi}(f|_C) \subseteq \text{epi}(K) \subseteq \text{epi}(g)$ provided $f(x) \leq K(x)$ for all $x \notin C$.*

Proof
Clearly $f(x) = f|_C(x) = K(x)$ for each $x \in C$ and hence $\text{epi}(f|_C) \subseteq \text{epi}(K)$. Since $f(x) \leq K(x)$ for all $x \notin C$ and $f(x) \leq K(x)$ for all $x \in C$, we have $\text{epi}(K) \subseteq \text{epi}(f) \subseteq \text{co}(\text{epi}(f)) = \text{epi}(g)$. It follows that $\text{epi}(f|_C) \subseteq \text{epi}(K) \subseteq \text{epi}(g)$. □

Clearly, for each $\mathfrak{C}$-convex function f there exists a convex epigraph that contains $\text{epi}(f)$, and the smallest of those epigraphs is that of its convexification.

An interesting questions is:
If $\text{gr}(f) \cap \text{gr}(g)$ is a singleton say $\{(m, \mu)\}$, what is the characteristic of $m \in \text{dom}(f)$?

2.5 Minima and maxima of convex functions

Convex functions exhibit nice properties related to maxima and minima, which makes them (convex functions) important in theoretical and applied mathematics. Be reminded of the conventional definition of a minimum point of a convex function f as

$$\text{a point } a \in \text{dom}(f) \text{ such that } f(a) \leq f(x) \text{ for all } x \in \text{dom}(f).$$

The maximum point would mean the opposite. Moreover, a point $a \in \text{dom}(f)$ may be both a minimum and maximal point for a convex function f under certain conditions, one of which being that f is linear. Clearly, the maximum point (that is, maximizer) does not always exists for convex functions, especially if the graph $\text{gr}(f)$ is not bounded.

In this discussion though we would go a step further and consider the minimum of an epigraph of such a function which is invariably related to the minimun point as defined above. That is, for $a \in \text{dom}(f)$ a minimum point of convex f, we have $(a, f(a)) \in \text{gr}(f) \subseteq \text{epi}(f)$ is a minimum of an epigraph of f.

If an epigraph of a function f is bounded from below, especially on a convex and bounded domain, or if $f(x) > -\infty$ for all $x \in \text{dom}(f)$, then such a function is a proper function. Moreover, real strictly convex function defined on a bounded closed set has a minimum point provided it is bounded from below (or it is proper), and its minimum point is not always unique.

For instance, for $f(x) = \|x\|$ the minimum point of its epigraph is unique and it is also the extreme point of such epigraph, that is

$$\text{ext}(\text{epi}(f)) = \{(0, f(0))\} = \{(a, f(a)) : f(a) \leq f(y), y \in \mathbb{R}\}.$$

Moreover, for $f(x) = e^x$ on $[\mu, \tau]$, the minimum point is attains at $\partial[\mu, \tau]$, that is

$$\text{ext}(\text{epi}(f)) = \{(a, f(a)) : f(a) \leq f(y), y \in \mathbb{R}\} = \{(\mu, f(\mu))\}.$$

However, if f is linear convex and non-constant, then it may not have a minimizer, at least not on the interior of its domain.

Hence in this section we discuss the relation, if any, between the minima/maxima and the extremal structure of convex epigraphs. Henceforth E would denote normed linear space unless otherwise stated;

Theorem 2.5.1 *[14, Theorem 3.4.4, p.114] Assume $U \subseteq E$ is a convex subset. Then for a convex function $f : U \to \mathbb{R}$ the following hold;*

1. *Local minimum of f is also a global minimum*
2. *A set of global minima of f is convex*
3. *If f is strictly convex in a neigbourhood N_a of $a \in U$ of a minimum point, then the minimum point is unique.*

Corollary 2.5.2 *If $f : U \subseteq E \to \mathbb{R}$ is strictly convex and $x \in U$ is a local minimizer of f, then x is a global minimizer of f and it is unique.*

Proof

Suppose $x \in U$ is a local minimizer of a convex function f. It follows from Theorem 2.5.1 (no.1) that $x \in U$ is a global minimizer of f. Since f is strictly convex and hence strictly convex in any neigbourhood N_a of each $a \in U$ it follows from Theorem 2.5.1 (no.3) that $x \in U$ is unique. □

Remark 2.5.3 1) The converse of Corollary 2.5.2 does not hold especially when one considers the function $f(x) = \|x\|$ convex with unique global minimizer $x = 0$ yet not strictly convex.

2) Consider the set $\mathfrak{M}_f = \{a \in \text{dom}(f) : f(a) \leq f(x) \text{ for all } x \in \text{dom}(f)\}$ of minimizers of a convex function f. Clearly $\mathfrak{M}_f$ is not always a singleton for f convex. For example $f(x) = \alpha$ for each $x \in \text{dom}(f)$ and $\alpha \in \mathbb{R}$ fixed is such that $\mathfrak{M}_f = \text{dom}(f)$ with $\text{dom}(f)$ non-trivial.

Moreover, since f is convex, so is $\text{dom}(f)$ and hence $\mathfrak{M}_f$ is also convex just as Theorem 2.5.1 stated. Clearly f is constant, and this leads to the following result about constant functions

Theorem 2.5.4 *[14, The Maximum Principle Theorem 3.4.6, p.115] If $f : U \subseteq E \to \mathbb{R}$ is a convex function on a convex U and attains a global maximum at $a \in \text{int}(U)$, then f is constant.*

Corollary 2.5.5 *Let $f : U \subseteq E \to \mathbb{R}$ be a convex function on a convex U, $a \in \text{int}(U)$ and N_a a (convex) neighborhood of a. If $f(x) \leq f(a)$ for all $x \in N_a \cap U$, then f is constant on N_a.*

Corollary 2.5.6 *Let $f : U \subseteq E \to \mathbb{R}$ be a convex function on a convex U. If f attains a global minimum for each $a \in U$, then f is constant.*

Proof
Take any $x, y \in U$ and $\lambda \in [0, 1]$. Since U is convex, it follows that $\lambda x + (1 - \lambda)y \in U$. Morever, $x, y, \lambda x + (1 - \lambda)y \in U$ are global minima and hence $f(x) \leq f(y)$ as $x \in U$ is a global minimum and $f(y) \leq f(x)$ as $y \in U$ is a global minimum. It follows that $f(x) = f(y)$ for all $x, y \in U$. Hence
$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y) = f(y) = f(x) = f(\lambda x+(1-\lambda)y = \alpha$ for some $\alpha \in \mathbb{R}$. Clearly $f(x) = \alpha$ for each $x \in U$ and thus f is constant. □

Theorem 2.5.7 *[14, Theorem 3.4.7, p.114] If $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is a continuous convex function on a compact convex set K, then f attains a global maximum at an extreme point.*

As a consequence of Theorem 2.5.4 and Theorem 2.5.7, we have the following results;

Corollary 2.5.8 *Suppose $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is a continuous convex function on a compact convex set K. If f attains a unique global minimum at $a \in \text{int}(K)$, then $(a, f(a)) \in \text{ext}(\text{epi}(f))$.*

Proof
Assume f attains a unique global minimum at $a \in \text{int}(K)$. Clearly f is not constant on K and on some neighborhood N_a of $a \in \text{int}(K)$.

Moreover, $f(a) < f(x)$ for each $x \neq a$ in K. Take any $(y, \mu), (z, \nu) \in \text{epi}(f)$ and $\lambda \in [0, 1]$ such that $(a, f(a)) = \lambda(y, \mu) + (1 - \lambda)(z, \nu) \in \text{epi}(f)$. It follows that $a = \lambda y + (1 - \lambda)z \in K$ and $f(a) = \lambda\mu + (1 - \lambda)\nu$ for some $\lambda \in [0, 1]$, and $f(y) \leq \mu$ and $f(z) \leq \nu$. Moreover, $f(a) = \min\{f(y), f(z)\} = \min\{\mu, \nu\}$ and hence $f(a) = \mu$ or $f(a) = \nu$. It follows that $(a, f(a)) = (a, \mu)$ or $(a, f(a)) = (a, \nu)$ for $a = \lambda y + (1 - \lambda)z \in K$ and $\lambda \in [0, 1]$. Hence $(a, f(a)) = (y, \mu)$ or $(a, f(a)) = (z, \nu)$ and the result follows. □

Proposition 2.5.9 *If $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is a continuous convex function on a compact convex set K, then f attains a global maximum at $a \in \partial(K)$, that is, $a \in K$ is a boundary point.*

Proof

It follows from Theorem 2.5.7 that a global maximum $a \in K$, with $f(a) \geq f(x)$ for all $x \in K$, is such that $a \in \text{ext}(K)$. If $a \in \text{int}(K)$ then it follows from Theorem 2.5.4 that f is constant (or linear) and there exists $b, c \in K$ and $\lambda \in (0, 1)$ such that $a = \lambda x + (1 - \lambda)y \in K$ since K is convex and $\partial(A) \subset A$. Thus $a \in \text{int}[x, y]$ and hence $a \neq x$ nor $a \neq y$. This contradicts the fact that $a \in \text{ext}(K)$ and hence $a \in \partial(K)$. □

Remark 2.5.10 Note that if $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is continuous convex function on a compact K and if f attains a global maximum at $x \in K$, then the following holds;

(1) if $x \in \text{int}(K)$ then f is constant and consequently $a \in \partial(K)$ is also a global maximum (also global minimum). That is, a global maximum is not unique.

(2) if f is strictly convex then $a \notin \text{int}(K)$. This follows from the observation that as for any $b, c \in K$ and $\lambda \in (0, 1)$ with $a = \lambda b + (1 - \lambda)c$, we have $f(a) = f(\lambda b + (1 - \lambda)c) < \lambda f(b) + (1 - \lambda)f(c) \leq \max\{f(b), f(c)\}$. Clearly $f(a) < \max\{f(b), f(c)\}$ would lead to a contradiction.

(3) A convex and continuous function $f : A \to \mathbb{R}$ attains both maximum and minimum points on A provided A is compact and convex.

Corollary 2.5.11 *[14, Corollary 1.3.6, p.23] Let $f : I \to \mathbb{R}$ be a convex function not monotonic on* $\text{int}(I)$*. Then f has an interior global minimum.*

Consider the following class of convex functions;

Definition 2.5.12 *[14, Exercise 8, p.117] [15, Definition 1.17, p.7]* A function $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is quasi-convex if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(y), f(x)\} \text{ (or } \leq \sup\{f(y), f(x)\})$$

for all $x, y \in K$ and all $\lambda \in [0, 1]$.

Moreover f is strictly quasi-convex if

$$f(\lambda x + (1-\lambda)y) < \max\{f(y), f(x)\}$$

and quasi-linear if

$$f(\lambda x + (1-\lambda)y) = \max\{f(y), f(x)\}$$

for all $x, y \in K$ and all $\lambda \in (0, 1)$.

The following is an example is quasi-convex function;

Example 2.5.13 1. All real monotonic increasing functions are quasi-convex, and are also convex provided they are continuous.
2. Logarithmic functions on positive real line is quasi-convex, and never convex.
3. All linear convex and linear affine functions are quasi-convex.
4. A constant real function is quasi-convex but not strictly quasi-convex.
5. However, a real function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is not quasi-convex if $-f$ is strictly convex and the maximiser $x \in A$ of f is an interior point. For example $f(x) = -x^2$ for $n = 1$ and $A = [-2, 2]$ and $x^* = 0 \in \text{int}(A)$.

Clearly $f(0) \geq f(\alpha)$, for each $\alpha \in A$. Morever, if $a = -1, b = 1$, then $\max\{f(a).f(b)\} = -1$ and $f(\lambda a + (1-\lambda)b) = 0$ for some $x^* = 0 \in \lambda a + (1-\lambda)b$ and some $\lambda \in (0,1)$.

Hence, $f(\lambda a + (1-\lambda)b) > \max\{f(a).f(b)\}$ for some $\lambda = \frac{1}{2}$.

Remark 2.5.14 Clearly quasi-convex functions are forms of pointwise maximum functions, hence we discuss some of the pointwise maximum convex functions as follows;

1. A componentwise maximum function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = \max\{x_1, ..., x_n\}$ is convex. Clearly, if $f(x) = \max_i x_i$, hence let $\lambda \in [0,1]$ and take $x, y \in \mathbb{R}^n$. It follows that;
$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \max\{\lambda x_i + (1-\lambda)y_i\} \\ &\leq \lambda \max x_i + (1-\lambda)\max y_i \\ &= \lambda f(x) + (1-\lambda)f(y). \end{aligned}$$
Hence f is convex.

2. A pointwise maximum function $g : \mathbb{R}^n \to \mathbb{R}$ defined as $g(x) = \max\{f_1(x), f_2(x)\}$ with f_1, f_2 convex functions such that $\text{dom}(g) = \text{dom}(f_1) \bigcap \text{dom}(f_2)$. Let $\lambda \in [0,1]$ and take $x, y \in \text{dom}(g)$;
$$\begin{aligned} g(\lambda x + (1-\lambda)y) &= \max\{f_1(\lambda x + (1-\lambda)y), f_2(\lambda x + (1-\lambda)y)\} \\ &\leq \max\{\lambda f_1(x) + (1-\lambda)f_1(y), \lambda f_2(x) + (1-\lambda)f_2(y)\} \\ &\leq \lambda \max\{f_1(x), f_2(y)\} + (1-\lambda)\max\{f_1(x), f_2(y)\} \\ &= \lambda g(x) + (1-\lambda)g(y) \text{ and hence } g \text{ is convex.} \end{aligned}$$

Lemma 2.5.15 *If $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex, then f is also quasi-convex.*

Proof

Take any $x, y, \lambda x + (1-\lambda)y \in K$ for $\lambda \in [0,1]$. Clearly $f(x), f(y)$, $\lambda f(x) + (1-\lambda)f(y) \in \mathbb{R}$. Clearly $\lambda f(x) + (1-\lambda)f(y) \in [f(y), f(x)]$ and thus $\lambda f(x) + (1-\lambda)f(y) \leq \max\{f(y), f(x)\}$ for all $\lambda \in [0,1]$. Moreover, since f is convex, we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq \max\{f(y), f(x)\}$. It follows that f is quasi-convex. □

Clearly quasi-convexity is a generalisation of convexity as there are more quasi-convex functions than there are convex functions. For example $f(x) = \log(x)$ on $\mathbb{R}_+$ is quasi-convex but not convex. Hence the converse of Lemma 2.5.15 is invalid.

Moreover, Theorem 2.5.7 can be extended to quasi-convex functions and appealing to Lemma 2.5.15 and Minkowski's result that each compact convex set $K \subseteq \mathbb{R}^n$ is a closed convex hull of its extreme points, see [14, Minkowski Theorem, P.110], see the following result.

Proposition 2.5.16 *Let $f : K \subseteq \mathbb{R}^n$ be continuous and quasi-convex and K be compact convex. Then f attains global maximum at an extreme point.*

Proof

Take any $x \in K$ a global maximizer of f, that is, $f(x) \geq f(a)$ for all $a \in K$. It follows that $x = \Sigma_{i=1}^m \lambda_i y_i$ for $y_i \in \text{ext}(K)$ for each i and $\Sigma_{i=1}^m \lambda_i = 1$. Hence $f(x) = f(\Sigma_{i=1}^m \lambda_i y_i) \leq \max\{f(y_i)\}$ as f is quasi-convex and $f(y_i) \leq f(x)$ for all $y_i \in K$. Consequently $f(x) = f(y_i)$ for some $i = 1, .., m$ with $y_i \in \text{ext}(K)$. It follows that f attains a global maximum at an extreme point. □

Lemma 2.5.17 *If $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is strictly (or constant) convex, then f is also strictly (respectively linear) quasi-convex.*

Proof

Take any $x, t \in K$, with $\lambda x + (1-\lambda)y \in K$ as K is convex. If f is strictly convex then $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y) \leq \max\{f(x), f(y)\}$. Consequently f is strictly quasi-convex. Similarly if f is constant, then $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y) \leq \max\{f(x), f(y)\}$. But since $f(x) = f(y) = \lambda f(x) + (1-\lambda)f(y)$ for each $\lambda \in [0, 1]$, we have $f(\lambda x + (1-\lambda)y) = \max\{f(x), f(y)\} = \min\{f(x), f(y)\}$. It follows that f is quasi-linear, that is linear quasi-convex. □

Clearly, there are some useful forms of convexity as discussed above which elicit interesting properties for their epigraphs. Hence for completeness we mention other forms of convexity and relevant results in the literature on those forms of convexity, if any, as follows;

Definition 2.5.18 *[10, Definition 1.1.1, p.73]* Let $C \in \mathbb{R}^n$ be a non-empty convex set in $\mathbb{R}$. Then $f : C \to \mathbb{R}$ is *strongly convex* if there exists the modulus $c > 0$ such that

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) - \tfrac{1}{2}c\alpha(1-\alpha)\|x-y\|^2$$

for all $x, y \in C$ and all $\alpha \in [0,1]$.

Definition 2.5.19 *[15, Definition 1.8, p.5]* A function $f : [a,b] \to \mathbb{R}$ is *Jensen convex* on $[a,b]$ if for all $x, y \in [a,b]$ and $\alpha = \frac{1}{2}$ then $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$.

Note that Jensen convexity is also known as midpoint-convexity, and is called strict Jensen convex if the inequality is strict.

Definition 2.5.20 *[15, Definition 1.81, p.40]* A function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ on convex S is called *strongly Jensen convex* if for all $x, y \in S$ we have

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) - c\alpha(1-\alpha)\|x-y\|^2$$

for $c > 0$ and for all $x, y \in S$ with $\alpha = \frac{1}{2}$

Definition 2.5.21 *[15, Definition 1.81, p.40]* Let $C \in \mathbb{R}^n$ be a non-empty convex set in $\mathbb{R}$. Then $f : C \to \mathbb{R}$ is *strongly quasi-convex* if there exists the modulus $c > 0$ such that

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\} - c\alpha(1-\alpha)\|x-y\|^2$$

for all $x, y \in C$ and all $\alpha \in [0,1]$ with $c > 0$.

Definition 2.5.22 *[15, Definition 1.15, p.7]* A function $f : [a,b] \to \mathbb{R}$ is *Wright convex* if for each $x \leq y, z \geq 0$, $x, y+z \in [a,b]$ we have

$$f(x+z)+f(y)\leq f(y+z)+f(x).$$

Definition 2.5.23 *[15, Definition 1.15, p.7]* A function $f : I \to \mathbb{R}$ is *multiplicatively convex* if for each $x, y \in I$ and all $\alpha \in [0,1]$ we have $f(\alpha x+(1-\alpha)y) \leq f(x)^{\alpha} f(y)^{1-\alpha}$. This is also called log-convex as $\log(f)$ is convex.

Clearly convexity implies Wright convex which in turn implies multiplicatively convex, see [15, Remark 1.14].

Lastly we look at the form of convexity related to the concept called 'majorisation' of convex functions.

Definition 2.5.24 *[15, Definition 12.1, p.319]* Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be n-tuples and assume the components are ordered in the form $x_j \geq x_{j+1}$ and $y_j \geq y_{j+1}$. Then y is said to *majorize* x, denotes $y \succ x$, if

$$\Sigma_{i=1}^{m} x_i \leq \Sigma_{i=1}^{m} y_i \text{ holds for } m = 1, 2, ...n-1, \text{ and}$$
$$\Sigma_{i=1}^{n} x_i = \Sigma_{i=1}^{n} y_i$$

Clearly each n-tuple self-majorize as per Definition 2.5.24, hence for each fixed n-tuple k, the collection $\mathfrak{M}_x = \{y = (y_i, ..., y_n) : y \succ x\} \neq \emptyset$ as $x \in \mathfrak{M}_x$. Note that each n-tuple in the majorisation definition should have at least two components, and if for example $x = (5, 3)$, then $(5,3), (6,2), (7,1), (8,0), (9,-1) \in \mathfrak{M}_x \subset \{(a,b) : b-3 = 5-a\}$.

Lemma 2.5.25 *Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, and $\mathfrak{M}_x$ be the collection of majorizers of $x \in \mathbb{R}^n$. Then $\mathfrak{M}_x$ is a convex in $\mathbb{R}^n$.*

Proof

Take any $\tau = (\tau_1, ..., \tau_n)$, $\nu = (\nu_1, ..., \nu_n) \in \mathfrak{M}_x$, $\lambda \in [0,1]$ and consider $k_i = \lambda\tau_i + (1-\lambda)\nu_i$ and $k = \lambda(\tau_1, ..., \tau_n) + (1-\lambda)(\nu_1, ..., \nu_n) \in \mathbb{R}^n$. Clearly $\Sigma_{i=1}^{n} x_i = \Sigma_{i=1}^{n} \tau_i = \Sigma_{i=1}^{n} \nu_i$, $\Sigma_{i=1}^{m} x_i \leq \Sigma_{i=1}^{m} \tau_i$ and $\Sigma_{i=1}^{m} x_i \leq \Sigma_{i=1}^{m} \nu_i$ hold for $m = 1, 2, ...n-1$. It follows that

$$\begin{aligned}\Sigma_{i=1}^{m}(\lambda\tau_i + (1-\lambda)\nu_i) &= \lambda\Sigma_{i=1}^{m}\tau_i + (1-\lambda)\Sigma_{i=1}^{m}\nu_i \\ &\geq \lambda\Sigma_{i=1}^{m}x_i + (1-\lambda)\Sigma_{i=1}^{m}x_i \\ &\geq \Sigma_{i=1}^{m}x_i.\end{aligned}$$

Hence $\Sigma_{i=1}^{m}x_i \leq \Sigma_{i=1}^{m}k_i$.

Moreover,

$$\begin{aligned}\Sigma_{i=1}^{n}(\lambda\tau_i + (1-\lambda)\nu_i) &= \lambda\Sigma_{i=1}^{n}\tau_i + (1-\lambda)\Sigma_{i=1}^{n}\nu_i \\ &= \lambda\Sigma_{i=1}^{n}x_i + (1-\lambda)\Sigma_{i=1}^{n}x_i \\ &= \Sigma_{i=1}^{n}x_i.\end{aligned}$$

Hence $\Sigma_{i=1}^{m}x_i = \Sigma_{i=1}^{m}k_i$.

It follows that $k = (k_i, ..., k_n) \in \mathbb{R}^n$ majorizes $x \in \mathbb{R}^n$, that is $k \in \mathfrak{M}_x$, and hence $\mathfrak{M}_x$ is convex. □

One other form of convexity is defined as follows;

Definition 2.5.26 *[15, Definition 12.23, p.332]* A function $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is said to be *Schur-convex* on A if $y \succ x$ implies $f(y) \geq f(x)$ for all $x, y \in \mathbb{R}^n$. Moreover, if $y \succ x$ implies $f(y) > f(x)$ for all $x, y \in \mathbb{R}^n$ then f is said to be *strictly Schur-convex* on A.

2.6 Krein-Milman property and its characteristics

A Banach space X over the field of real numbers $\mathbb{R}$ has the *Radon-Nikodým property (RNP)* if for each finite positive measure space (Ω, Σ, μ) and each X-valued, μ-continuous measure ν on Σ, with bounded variation $|\nu|$, there exists a Bochner integrable function $f : \Omega \to X$ such that $\nu(E) = \int_E f \, \mathrm{d}\mu$ for $E \in \Sigma$.

The RNP has become a geometrical property when the following result was introduced:
A Banach space X has the RNP if and only if each non-empty bounded subset of X is dentable.

Furthermore, Banach space X has the *Krein-Milman property (KMP)* if each closed bounded convex subset of X is the closed convex hull of its extreme points.
Lindenstrauss proved that if each nonempty closed bounded convex subset of a Banach space X contains an extreme point, then X has the Krein-Milman property. In particular, a Banach space with the RNP has the KMP. The converse remains an open question.

Clearly, the RNP is more general than the KMP in that, KMP considers the extremal structure of closed bounded and convex sets and the RNP consider the bounded sets, not necessarily closed and convex. The fact that the RNP implies the KMP follow from the fact that denting points are in most cases extreme points, that is, most dentable sets have extreme points, more so in the closure of bounded sets.

Proving the converse is quite a challenge in that assuming the KMP means you consider the smaller collection of sets, that is, closed convex and bounded sets and try and prove the extremal and dents in the bigger collection. The equivalence has been since proven in the dual Banach spaces

but is still an open question in the underlying Banach spaces themselves.

In this section we discuss the Krein Milman Theorem and Krein Milman property as they are characterised by extreme points. Moreover, the closed convex hull of extreme points of a set, and extreme points of a closed convex hull of a set will also be compared to one another.

Clearly extreme points are landmarks of compact convex sets in $\mathbb{R}^n$, as suggested in the following theorem;

Theorem 2.6.1 *[14, Minkowski Theorem, P.110] Every nonempty convex and compact subset $K \in \mathbb{R}^n$ is the convex hull of its extreme points.*

It is clear though that epigraphs of real convex functions are not compact as they are not bounded above, hence the direct application of Theorem 2.6.1 would be invalid.

The results of Theorem Theorem 2.6.1 can be extended to Hausdorff spaces as follows;

Corollary 2.6.2 *[14, Krein Milman Theorem, p.210] Let K be a nonempty convex susbset of a locally convex Hausdorff space E. Then K is the closed convex hull of its extreme points.*

Clearly $\text{ext}(\text{epi}(f)) \neq \emptyset$ if f is nonlinear and $\text{epi}(f)$ is convex, amongst other conditions, hence we discuss convex hull of such extreme points. Note that in the Krein Milman Theorem, compactness of K is vital for the existence of extreme points, see Theorem 2.6.1, and since a convex epigraph is not necessarily compact, in what follows we characterize the closed conved hull of extreme points of and epigraphs.

Example 2.6.3 Consider the following statement: Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly convex function satisfying $\lim_{|x|\to\infty} f(x) = \infty$. For any $(a, b) \in \text{int}(\text{epi}(f))$, there exists $\epsilon = \frac{|d-c|}{2}$ and $(c, f(c)), (d, f(d)) \in \text{gr}(f)$ such that
1. $c, d \in \partial[a - \epsilon, a + \epsilon]$, and
2. $(a, b) \in \text{int}[(c, f(c)), (d, f(d))] \subseteq \text{co}(\text{gr}(f))$ and $\text{int}[(c, f(c)), (d, f(d))] \subseteq$

$\{(x,y) \in \mathbb{R}^2 : y = \frac{f(d)-f(c)}{d-c}(x-a) + f(c)\}$.

This example shows that each elements in an interior of and epigraphs of a strictly convex function may be contained in the convex hull $\text{co}(\text{gr}(f))$ of the graph of f.

Let us choose $f(x) = x^2$ with $\text{dom}(f) = \mathbb{R}$, and look at the following choices of interior points of $\text{epi}(f)$.

(a) Choose $(1, 10) \in \text{int}(\text{epi}(f))$. It follows that $y = 2x + 8$ passes through $(1, 10)$ and with slope $2 = f'(1)$. If $x^2 = 2x + 8$ then $x = 4 = c$ and $x = -2 = d$. Hence $\epsilon = 3$ and $4, -2 \in \partial[1 - \epsilon, 1 + \epsilon]$. Consequently $(1, 10) \in \text{int}[(-2, f(-2)), (4, f(4))] \subseteq \text{co}(\text{gr}(f))$ and $\frac{f(4)-f(-2)}{4-(-2)} = 2 = f'(1)$.

(b) Secondly if we take $(-2, 13) \in \text{int}(\text{epi}(f))$, then $(-2, 13) \in \text{int}[(1, f(1)), (-5, f(-5))] \subseteq \text{co}(\text{gr}(f))$, with $\epsilon = 3$ and $1, -5 \in \partial[-2 - \epsilon, -2 + \epsilon]$.

(c) Lastly, if we take $(4, 17) \in \text{int}(\text{epi}(f))$, then $(4, 17) \in \text{int}[(3, f(3)), (5, f(5))] \subseteq \text{co}(\text{gr}(f))$, with $\epsilon = 1$ and $3, 5 \in \partial[4 - \epsilon, 4 + \epsilon]$.

Proposition 2.6.4 *Let X be Banach space and $f : X \to \mathbb{R}$ be convex (and* $\text{epi}(f)$ *be closed). Then* $\text{epi}(f)$ *is a convex hull of its extreme points, that is* $\text{epi}(f) = \text{clco}(\text{ext}(\text{epi}(f)))$ *if the following holds;*

• f be strictly convex (and $\lim_{\|x_n\| \to \infty} f(x_n) = \infty$) and for each $(x, \lambda) \in \text{int}(\text{epi}(f))$ there exists $(y_1, f(y_1)), (y_2, f(y_2)) \in \text{gr}(f)$ with $y_1 \neq y_2$ such that line segments $L_1 = [(x, \lambda), (y_1, f(y_1))]$ and $L_2 = [(x, \lambda), (y_2, f(y_2))]$ are parallel to each other.

Proof

Clearly $\text{ext}(\text{epi}(f)) \subseteq \text{epi}(f)$ and since $\text{epi}(f)$ is closed and convex, we have $\text{clco}(\text{ext}(\text{epi}(f))) \subseteq \text{epi}(f)$.

For the reverse inclusion take $(x, \lambda) \in \text{epi}(f)$. It follows that $(x, \lambda) \in \partial(\text{epi}(f))$ or $(x, \lambda) \in \text{int}(\text{epi}(f))$. If $(x, \lambda) \in \partial(\text{epi}(f))$, then appealling to Corollary 2.1.5 we have $(x, \lambda) \in \text{ext}(\text{epi}(f)) \subseteq \text{clco}(\text{ext}(\text{epi}(f)))$.

On the other hand, if $(x, \lambda) \in \text{int}(\text{epi}(f))$ then there exist parallel lines $L_1 = [(x, \lambda), (y, f(y))]$ and $L_2 = [(x, \lambda), (z, f(z))]$. It follows that $(x, \lambda) \in L_1 \cup L_2 = [(z, f(z)), (y, f(y))] \subseteq \text{clco}(\text{gr}(f)) \subseteq \text{epi}(f)$ since $\text{epi}(f)$ is convex. Since $\text{gr}(f) = \text{ext}(\text{epi}(f))$, see Proposition 2.1.3, it follows that $\text{clco}(\text{gr}(f)) = \text{clco}(\text{ext}(\text{epi}(f)))$. Consequently $(x, \lambda) \in \text{clco}(\text{ext}(\text{epi}(f)))$. □

Clearly the result above does not hold if f is not strictly convex as in $f(x) = \|x\|$ where $\text{clco}(\text{ext}(\text{epi}(f))) = \text{ext}(\text{epi}(f)) = \{(0, f(0))\}$ is a singleton and hence strictly contained in $\text{epi}(f)$.

Lemma 2.6.5 *If* $f : X \to \mathbb{R}$ *is strictly convex and* $\text{clco}(\text{ext}(\text{epi}(f))) = \text{epi}(f)$*, then* $\text{int}(\text{epi}(f)) \subseteq \text{clco}(\text{gr}(f))$.

Proof
Clearly $\text{int}(\text{epi}(f)) \subseteq \text{epi}(f) = \text{clco}(\text{ext}(\text{epi}(f)))$. Moreover, it follows from Proposition 2.1.3 that $\text{gr}(f) = \text{ext}(\text{epi}(f))$ and hence $\text{clco}(\text{gr}(f)) = \text{clco}(\text{ext}(\text{epi}(f)))$. Consequently, $\text{int}(\text{epi}(f)) \subseteq \text{clco}(\text{gr}(f))$. □

In the following example we show that a continuous function on a compact subset of the real line $\mathbb{R}$ might have an epigraph which is not contained in the closed convex hull of its graph. We also show that the slopes of the lines passing through any interior point of such epigraph might be used to show that $\text{epi}(f) \subseteq \text{clco}(\text{gr}(f))$.

Example 2.6.6 Consider the function $f : I \to \mathbb{R}$ defined by $f(x) = x^2$ with $I = [m, n] = [-3, 4]$. Clearly f is strictly convex and $\text{epi}(f)$ is closed and $|f(x)| \leq \epsilon = 16$.
(1) Take $(\alpha, \beta) = (1, 17) \in \text{epi}(f)$ and consider the slope(s), $M_{(\alpha,\beta,f(x))} = \frac{f(x)-\beta}{x-\alpha}$ for each $x > \alpha$ and $(x, f(x)) \in \text{gr}(f)$, of any line segment L through (α, β) and cutting $\text{gr}(f)$ at $(x, f(x))$, that is $L(x) = f(x)$ for each L. Clearly $M_{(\alpha,\beta,f(x))} = (-\infty, -\frac{1}{3}]$ for $x \in (\alpha, n]$ and $M_{(\alpha,\beta,f(x))} = [2, \infty)$ for $x \in [m, \alpha)$, and $M_{(\alpha,\beta,f(x))}$ is undefined for $x = 1$. It follows that

$$M_{(\alpha,\beta,f(x_i))} \bigcap M_{(\alpha,\beta,f(x_j))} = \emptyset$$

for all $x_j \in (\alpha, n]$ and all $x_i \in [m, \alpha)$, that is for $x_i < \alpha < x_j$ with $x_i, x_j \in [m, n]$. It follows that there exist no slope $m \in \mathbb{R}$ such that $y = mx + c$ through (α, β) cuts $\text{gr}(f)$ more than once. Thus $(\alpha, \beta) \notin \text{clco}(\text{gr}(f)) \supseteq \text{clco}(\text{ext}(\text{epi}(f)))$, and hence $(\alpha, \beta) \notin \text{clco}(\text{ext}(\text{epi}(f)))$.
(2) Take $(\alpha, \beta) = (1, 13) \in \text{epi}(f)$, instead, and consider the slopes as in (1) above. Clearly $M_{(\alpha,\beta,f(x))} = (-\infty, 1]$ for $x \in (\alpha, n]$ and $M_{(\alpha,\beta,f(x))} = [1, \infty)$ for $x \in [m, \alpha)$, and $M_{(\alpha,\beta,f(x))}$ is undefined for $x = 1$. It follows that

$$M_{(\alpha,\beta,f(x_i))} \bigcap M_{(\alpha,\beta,f(x_j))} = \{1\} \neq \emptyset$$

Clearly there exists a line $y = mx + c = x + c$ through $(\alpha, \beta) = (1, 13) \in \text{epi}(f)$ cutting $\text{gr}(f)$ in more that one point. Hence $(\alpha, \beta) \in \text{clco}(\text{gr}(f))$ and since f is strictly convex we have $(\alpha, \beta) \in \text{clco}(\text{gr}(f)) = \text{clco}(\text{ext}(\text{epi}(f)))$.
(3) If $I = \mathbb{R}$ for $f(x) = x^2$ we show that
(i) $\text{epi}(f) \subseteq \text{clco}(\text{gr}(f))$ and consequently
(ii) $\text{epi}(f) = \text{clco}(\text{gr}(f))$;

(i) Since $I = \mathbb{R}$, $\text{epi}(f)$ closed and f is strictly convex, it follows from Corollary 2.1.5 and Proposition 2.1.3 that $\partial(\text{epi}(f)) \subset \text{gr}(f) \subseteq \text{clco}(\text{gr}(f))$.

To show that the interior $\text{int}(\text{epi}(f))$ is also in $\text{clco}(\text{gr}(f))$, take any $(x, \lambda) \in \text{int}(\text{epi}(f))$. As in (2) above there exist slopes $M_\beta = \frac{f(\beta)-\lambda}{\beta-x}$ for each $\beta > x$ and $M_\alpha = \frac{f(\alpha)-\lambda}{\alpha-x}$ for $\alpha < x$ with $(\alpha, f(\alpha)), (\beta, f(\beta)) \in \text{gr}(f)$, such that
$(x, \lambda) \in [(\alpha, f(\alpha)), (\beta, f(\beta))] \subseteq \text{clco}(\text{gr}(f))$.

Clearly $f(x) \leq \lambda$, and for a fixed $\lambda \in \mathbb{R}$ we have
$C_\lambda = \{(k, \lambda) \in \text{epi}(f)\colon\ k \in \text{dom}(f)\} \neq \emptyset$ as $(x, \lambda) \in C_\lambda$. Since
$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} x^2 = \infty$ there are $\alpha, \beta \in \mathbb{R}$ such that $(\alpha, \lambda), (\beta, \lambda) \in C_\lambda$. Hence we have $0 \in A_x \cap B_x \neq \emptyset$ for $A_x = \{M_\beta : x < \beta \in \mathbb{R}\}$ and $B_x = \{M_\alpha : x > \alpha \in \mathbb{R}\}$.

Furthermore choose $m, n \in \mathbb{R}$ such that $m \leq \alpha < x < \beta \leq n$ with $f(m) = f(n) = \lambda$). It follows that $(m, f(m)), (n, f(n)) \in C_\lambda \cap \text{gr}(f)$ and

$(x, \lambda) \in \text{int}[(m, f(m)), (n, f(n))] \subseteq \text{clco}(\text{gr}(f))$. This shows that $\text{int}(\text{epi}(f))$ is also in $\text{clco}(\text{gr}(f))$ and consequently $\text{epi}(f) \subseteq \text{clco}(\text{gr}(f))$.
(ii) Since $\text{gr}(f) \subseteq \text{epi}(f)$ and $\text{epi}(f)$ is closed and convex, we have $\text{clco}(\text{gr}(f)) \subseteq \text{epi}(f)$ and hence $\text{epi}(f) = \text{clco}(\text{gr}(f))$.

Lemma 2.6.7 *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a convex constant function. Then*

$$\text{epi}(f) \neq \text{clco}(\text{ext}(\text{epi}(f)))$$

Proof
If $A = \mathbb{R}^n$, then $\text{ext}(\text{epi}(f)) = \emptyset = \text{clco}(\text{ext}(\text{epi}(f))) \subset \text{epi}(f) \neq \emptyset$. Moreover, if $A \subset \mathbb{R}^n$, then each $(x, \lambda) \in \text{epi}(f)$ with $f(x) < \lambda$ satisfies the fact that $f(y) < \lambda$ for each $(y, f(y)) \in \text{gr}(f)$. Clearly $(x, \lambda) \notin [(m, f(m)), (n, f(n))]$ for each $(m, f(m)), (n, f(n)) \in \text{gr}(f)$ and consequently for each $(m, f(m)), (n, f(n)) \in \text{ext}(\text{epi}(f)) \subseteq \text{gr}(f)$. It follows that $(x, \lambda) \in \text{epi}(f) \setminus \text{clco}(\text{ext}(\text{epi}(f)))$. $\square$

Proposition 2.6.8 *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a convex function and* $\text{epi}(f)$ *be closed. Then* $\text{epi}(f) \neq \text{clco}(\text{ext}(\text{epi}(f)))$ *if one of the following hold;*
(1) *If f is linear*
(2) *f is bounded above, that is there is $\epsilon > 0$ such that $|f(x)| \leq \epsilon$ for each $x \in A$*

Proof
(1) Let f be linear. Clearly $\text{epi}(f) \neq \emptyset$ and
$\text{ext}(\text{epi}(f)) = \{(x, f(x)) \in \text{gr}(f) : x \in A \bigcap \partial(A)\}$. If $A = \mathbb{R}^n$, then all $(x, f(x)) \in \text{gr}(f)$ is such that $(x, f(x)) \in \text{int}[(a, f(a)), (b, f(b))]$ for some $(a, f(a)), (b, f(b)) \in \text{gr}(f)$. It follows that $\text{ext}(\text{epi}(f)) = \emptyset$ and hence $\text{epi}(f) \neq \text{clco}(\text{ext}(\text{epi}(f)))$.

Furthermore, if $A \subset \mathbb{R}^n$ then either (a)$A \cap \partial(A) = \emptyset$, or, (b) $A \cap \partial(A) \neq \emptyset$. If (a) holds, or $\partial(A)$ is empty, then $\text{ext}(\text{epi}(f)) = \emptyset$ and clearly $\text{epi}(f) \neq \text{clco}(\text{ext}(\text{epi}(f)))$.

Moreover, if (b) holds, then $f(x) \in \mathbb{R}$ for some $x \in \partial(A)$. Then $\mathfrak{C}_x = \{(x,\lambda) \in \text{epi}(f) : x \in \partial(A)\} \neq \emptyset$ as $(x, f(x)) \in \mathfrak{C}_x$, Clearly $(x,\lambda) \in \mathfrak{C}_x \cap \partial(\text{epi}(f))$ for all $f(x) < \lambda$.

Take any $(x,\beta) \in \mathfrak{C}_x$ such that $f(x) < \beta$. Since $\text{epi}(f)$ is not bounded above $\mathfrak{C}_x$ is also not bounded above and thus there exists $k \in \mathbb{R}$ such that $f(x) < \beta < k$ and $(x,k) \in \mathfrak{C}_x$. It follows that $(x,\beta) \in \text{int}[(x, f(x)), (x,k)]$ and thus $(x,\beta) \notin \text{ext}(\text{epi}(f))$ and $(x,\beta) \notin \text{ext}(\mathfrak{C}_x)$. Since $\text{ext}(\mathfrak{C}_x) \subseteq \text{ext}(\text{epi}(f))$ it follows that $(x,\beta) \notin \text{clco}(\text{ext}(\text{epi}(f)))$ and thus $\text{epi}(f) \neq \text{clco}(\text{ext}(\text{epi}(f)))$.

(2) Assume f is bounded above and $f(x) \leq \epsilon$ for each $x \in A$. Then either (i) $A \cap \partial(A) \neq \emptyset$, or (ii) $A \cap \partial(A) = \emptyset$.

(i) If $A \cap \partial(A) \neq \emptyset$ then there exists $a \in A \cap \partial(A)$ with $f(a) \in \mathbb{R}$. Take $a \in \partial(A)$ such that $f(x) \geq f(a)$ for all $x \in \partial(A)$. Clearly $(a, f(a)) \in \text{gr}(f)$ and $(a,\lambda) \in \text{epi}(f)$ for all $f(a) \leq \lambda$. Consider $\mathfrak{C}_x = \{(a,\lambda) \in \partial(\text{epi}(f)) : f(x) \leq \lambda\} \neq \emptyset$, with $(a, f(a)) \in \mathfrak{C}_x$. Clearly
$(a,b) \in \mathfrak{C}_x \backslash \{(a, f(a))\} \subseteq \text{epi}(f) \backslash \text{ext}(\text{epi}(f))$ for any $b > f(a)$. Moreover, since $\text{epi}(f)$ is not bounded above, so is $\mathfrak{C}_x$ and hence for each $b \in \mathbb{R}$ there exist $k \in \mathbb{R}$ such that $k > b$ and $(a,b) \neq (a,k) \in \text{epi}(f)$. Clearly $(a,b) \in [(a, f(a)), (a,k)]$ there exists $(a,k) \in \mathfrak{C}_x \backslash \text{ext}(\text{epi}(f)) \subseteq \text{epi}(f)$. Consequently, there exists no $(a,\mu), (a,\nu) \in \text{ext}(\text{epi}(f))$ such that $(a,b) \in [(a,\mu), (a,\nu)]$, and hence $(a,b) \notin \text{clco}(\text{ext}(\text{epi}(f)))$.

(ii) Moreover, if $A \cap \partial(A) = \emptyset$ and hence A open and we have $(a, f(a)) \notin \text{gr}(f)$ for $a \in \partial(A)$ as $f(a)$ is not defined. Clearly $\partial(\text{epi}(f)) \neq \emptyset$ as $(a,\beta) \in \partial(\text{epi}(f))$ for all $\beta > \epsilon > 0$ and $a \in \partial(A)$. Clearly $(a,\beta) \in \partial(\text{epi}(f)) \subseteq \text{epi}(f)$ for aach $\beta > \epsilon$ since $\text{epi}(f)$ is closed. Similarly, each $(a,n) \in \partial$ $(\text{epi}(f))$ with $n > \beta$ is such that $(a,n) \notin \text{ext}(\text{epi}(f))$. Since $\text{epi}(f)$ is not bounded above, so is ∂ $(\text{epi}(f))$ and hence for each $(a,n) \in \partial$ $(\text{epi}(f))$ there exists $(a,n+1) \in \partial$ $(\text{epi}(f))$ such that $(a,n) \in [(a,\beta), (a,n+1)] \subseteq \text{epi}(f)$. It follows that $(a,n+1) \in \text{epi}((f)) \backslash$ clco$(\text{ext}(\text{epi}(f)))$. □

Remark 2.6.9 Note that if $f : I \to \mathbb{R}$ is a convex function, the bound-

edness of I does not imply the boundedness of f. Consider the function $f(x) = \frac{1}{x}$ on $I = (0, n]$, $n \in \mathbb{R}_+$. As $n \searrow 0$ then $f(x) \to \infty$ and hence there is no $\epsilon > 0$ such that $|f(x)| < \epsilon$ for all $x \in I$

Example 2.6.10 Consider a strictly convex function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ with

$$\text{co}(\text{gr}(f)) = \{\Sigma_{i=1}^n \alpha_i (x_i, f(x_i)) : \Sigma_{i=1}^n \alpha_i = 1 \text{ and } (x_i, f(x_i)) \in \text{gr}(f)\}.$$

Clearly, $(0, 2) \in \text{co}(\text{gr}(f))$ as $(0, 2) = \frac{2}{3}(-1, 1) + \frac{1}{3}(2, 4)$ with $(-1, 1), (2, 4) \in \text{gr}(f)$. Hence
$(0, 2) \in [(-1, 1), (2, 4)] \subseteq \text{co}(\text{gr}(f))$. Suppose $(0, 2)$ can be written as a convex combination of three distinct elements in $\text{gr}(f)$, that is
$(0, 2) = \alpha_1(x_1, f(x_1)) + \alpha_2(x_2, f(x_2)) + \alpha_3(x_3, f(x_3))$, with $\alpha_1 + \alpha_2 + \alpha_2 = 1$;

(1) Assume $\alpha_i = \frac{1}{3}$ for each $i = 1, 2, 3$, $(x_1, f(x_1)) = (-1, 1)$ and $(x_1, f(x_1)) = (2, 4)$. Hence $(0, 2) = \alpha_1(-1, 1) + \alpha_2(2, 4) + \alpha_3(x_3, f(x_3))$ and consequently $0 = -\frac{1}{3} + \frac{2}{3} + \frac{x_3}{3}$ and $2 = -\frac{1}{3} + \frac{4}{3} + \frac{f(x_3)}{3}$. Hence $\frac{1+x_3}{3} = 0$ and $\frac{5+f(x_3)}{3} = 2$. It follows that $x_3 = -1$ and $f(x_3) = 1$ and consequently $(x_1, f(x_1)) = (x_3, f(x_3))$ and it leads to a contradiction as the combination is not of three distinct elements.

(2) On the other hand, assume that $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$ and $\alpha_3 = \frac{1}{4}$.
Then we have $(0, 2) = \alpha_1(-1, 1) + \alpha_2(2, 4) + \alpha_3(x_3, f(x_3))$ and thus $x_3 = -3$ and $f(x_3) = -1$. Clearly $(-3, -1) \notin \text{gr}(f) \cup \text{epi}(f)$.

(3) Lastly assume $(0, 2) = \alpha_1(-1, 1) + \alpha_2(2, 4) + \alpha_3(1, 1)$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Clearly for each combination of α_1 and α_2 we choose, α_3 should be unique. Moreover,

(i) If $\alpha_i = \frac{1}{3}$ for $i = 1, 2$ then $\alpha_3 = -\frac{1}{3}$ and $\alpha_3 = \frac{1}{3}$. This leads to a contadiction as $\alpha_3 = -\frac{1}{3}$ implies $\alpha_1 + \alpha_2 + \alpha_3 \neq 1$, and α_3 should have a unique value.

(ii) If $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{1}{3}$ then $\alpha_3 = -\frac{1}{6}$ and $\alpha_3 = \frac{1}{6}$. This leads to a contadiction as $\alpha_3 = -\frac{1}{6}$ implies $\alpha_1 + \alpha_2 + \alpha_3 \neq 1$, and $alpha_3$ should have a unique value.

Clearly this example shows that each $(x, \mu) \in \text{co}(\text{gr}(f))$ cannot always be a convex combination three element in $\text{gr}(f)$, for f a non-linear non-affine convex function.

Lemma 2.6.11 *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be strictly convex and* epi(f) *be closed. Then;*

1. $\text{co}(\text{gr}(f)) \supseteq \{\Sigma_{i=1}^2 \alpha_i (x_i, f(x_i)) : \Sigma_{i=1}^2 \alpha_i = 1, (x_i, f(x_i)) \in \text{gr}(f)\} = K.$
$= \{[(x, f(x)), (y, f(y))]; x, y \in A\}.$
2. $\text{co}(\text{gr}(f)) = \text{co}(\{[(x, f(x)), (y, f(y))]; x, y \in A\}) = Q$

Proof

1. Clearly $K \subseteq \text{epi}(f)$ as epi(f) is convex and $\text{gr}(f) \subseteq \text{epi}(f)$. Moreover, $(a, b) = \alpha(x, f(x)) + (1 - \alpha)(y, f(y)) \in \text{co}(\text{gr}(f))$ for $(a, b) \in K$. Moreover, $K = \{[(x, f(x)), (y, f(y))]; x, y \in A\}$ follows easily.

2. Clearly $\{[(x, f(x)), (y, f(y))]; x, y \in A\} \subseteq \text{co}(\text{gr}(f))$ from 1. above. Hence $\text{co}(\{[(x, f(x)), (y, f(y))]; x, y \in A\}) \subseteq \text{co}(\text{co}(\text{gr}(f))) = \text{co}(\text{gr}(f))$. For the other inclusion, if $(d, \delta) \in \text{co}(\text{gr}(f))$ then $(d, \delta) = \Sigma_{i=1}^n \alpha_i (x_i, f(x_i))$, with $\Sigma_{i=1}^n \alpha_i = 1$ and $(x_i, f(x_i)) \in \text{gr}(f)$. Since for each $(a, f(a)) \in \text{gr}(f)$ there is $(c, f(c)) \in \text{gr}(f)$ such that $[(a, f(a)), (c, f(c))] \subseteq Q$, we have that $(d, \delta) = \Sigma_{i=1}^n \alpha_i (x_i, f(x_i)) = \Sigma_{i=1}^n \alpha_i [(x_i, f(x_i), (c, f(c))] \subseteq Q$. □

Lemma 2.6.12 *If $f : X \to \mathbb{R}$ be strictly convex and continuous, then* $\text{clco}(\text{gr}(f)) = \text{co}(\text{gr}(f))$.

Proof

Clearly $\text{co}(\text{gr}(f)) \subseteq \text{clco}(\text{gr}(f))$. For the reverse inclusion take $(x, \lambda) \in \text{clco}(\text{gr}(f))$. Since $\text{gr}(f) \subseteq \text{epi}(f)$ and epi(f) is closed and convex, we have $\text{clco}(\text{gr}(f)) \subseteq \text{clco}(\text{epi}(f)) = \text{epi}(f)$. Hence $f(x) < \lambda$ or $f(x) = \lambda$ as $(x, \lambda) \in \text{epi}(f)$.

If $f(x) = \lambda$ then $(x, \lambda) \in \text{gr}(f) \subseteq \text{co}(\text{gr}(f))$ and hence the result follows. Moreover if $f(x) < \lambda$ then $(x, \lambda) \in \text{int}(\text{epi}(f)) \cap \text{clco}(\text{gr}(f))$. Since $(x, \lambda) \in \text{gr}(f) \subseteq \text{epi}(f)$, it follows that $(x, \lambda) = \Sigma_{i=1}^n \alpha_i (x_i, \lambda_i)$ for $(x_i, \lambda_i) \in \text{epi}(f)$

and $\Sigma_{i=1}^n \alpha_i = 1$ such that $(x_i, \lambda_i) \in \text{co}(\text{gr}(f)) \cup \partial(\text{epi}(f))$. It follows that $(x, \lambda) = \Sigma_{i=1}^n \alpha_i (x_i, \lambda_i)$ for $(x_i, \lambda_i) \in \text{co}(\text{gr}(f))$ and $\Sigma_{i=1}^n \alpha_i = 1$ as $\partial(\text{epi}(f)) \subseteq \text{gr}(f)$, see Proposition 2.1.3 and Corollary 2.1.5. Hence $(x, \lambda) \in \text{co}(\text{gr}(f))$. $\square$

Clearly $\partial(\text{epi}(f)) \backslash \text{gr}(f) = \emptyset$ if one of the following holds
1) X is Euclidean, $f : X \to \mathbb{R}$ is strictly convex and $\text{epi}(f)$ are closed,
2) $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ strictly convex and $\text{epi}(f)$ is closed and $\lim_{x \to \partial(A)} f(x) = \infty$.
3) Moreover, if 2) holds, then $\text{clco}(\text{ext}(\text{epi}(f))) = \text{epi}(f)$.
4) The converse of 2) above does not hold in that: If $f : A = \mathbb{R} \to \mathbb{R}$ is strictly convex defined by $f(x) = e^x$, then $\partial(\text{epi}(f)) \backslash \text{gr}(f) = \emptyset$ and $\text{epi}(f)$ is closed, yet $\lim_{x \to -\infty} f(x) = 0 \neq \infty$.

Corollary 2.6.13 *Let $f : X \to \mathbb{R}$ be strictly convex (and $\text{epi}(f)$ be closed). If f satisfies this condition*
• *f for each $(x, \lambda) \in \text{int}(\text{epi}(f))$ there exists $(y_1, f(y_1)), (y_2, f(y_2)) \in \text{gr}(f)$ with $y_1 \neq y_2$ such that line segments $L_1 = [(x, \lambda), (y_1, f(y_1))]$ and $L_2 = [(x, \lambda), (y_2, f(y_2))]$ are parallel to each other (and $\lim_{\|x_n\| \to \infty} f(x_n) = \infty$), then the following hold.*
1. $\text{epi}(f) = \text{clco}(\text{gr}(f))$.
2. $\text{epi}(f) = \text{clco}(\text{dent}(f))$.
3. $\text{ext}(\text{epi}(f)) = \text{ext}(\text{clco}(\text{gr}(f))) = \text{ext}(\text{clco}(\text{ext}(\text{epi}(f))))$.

Proof
1. Clearly $\text{gr}(f) = \text{ext}(\text{epi}(f)$ (see Proposition 2.1.3) and $\text{clco}(\text{ext}(\text{epi}(f)) = \text{epi}(f)$ (see Proposition 2.6.4), and thus $\text{clco}(\text{gr}(f)) = \text{clco}(\text{ext}(\text{epi}(f)) = \text{epi}(f)$.
2. It follows from Theorem 2.1.30 and Proposition 2.6.4 that $\text{dent}(\text{epi}(f)) = \text{ext}(\text{epi}(f))$ and $\text{clco}(\text{ext}(\text{epi}(f)) = \text{epi}(f)$, respectively. Hence $\text{clco}(\text{dent}(\text{epi}(f))) = \text{clco}(\text{ext}(\text{epi}(f))) = \text{epi}(f)$.
3. From 1. above we have $\text{epi}(f) = \text{clco}(\text{gr}(f))$, hence $\text{ext}(\text{epi}(f)) = \text{ext}(\text{clco}(\text{gr}(f)))$. Morever, since $\text{clco}(\text{ext}(\text{epi}(f))) = \text{epi}(f)$, see Propo-

sition 2.6.4, we have $\text{ext}(\text{epi}(f)) = \text{ext}(\text{clco}(\text{ext}(\text{epi}(f))))$. Consequently, $\text{ext}(\text{epi}(f)) = \text{ext}(\text{clco}(\text{gr}(f))) = \text{ext}(\text{clco}(\text{ext}(\text{epi}(f))))$. □

Clearly, $\text{ext}(\text{epi}(f))$ is convex if and only if it is singleton, e.g where $f(x) = \|x\|$.

Denote by $\text{hyp}(f) = \{(x, \lambda) \in \text{dom}(f) \times \mathbb{R} : f(x) \geq \lambda\}$ the hypograph of a real function $f : \text{dom}(f) \to \mathbb{R}$, and consider the following example;

Example 2.6.14 Take a (strictly) convex function $f : [0, 5] \to \mathbb{R}$ defined by $f(x) = x^2$ with $\text{epi}(f)$ closed, and consider a continuous functions $h_i : \mathbb{R} \to \mathbb{R}$ such that, for each $i \in \mathbb{N}$;

1. h_i is not necessarily convex
2. $h_i(x) \geq f(x)$ for all $x \in [0, 5]$ (that is, the hypograph of h_i contains $\text{gr}(f)$)
3. there exists $\beta \in [0, 5]$, such that $h_i(\beta) \neq f(\beta)$ for each i

In addition, denote by H_f the collection of real functions h_i satisfying property (1),(2) and (3) above.

Clearly $H_f \neq \emptyset$ as $h_1, h_2 \in H_f$ where $h_1(x) = 5x$ and $h_2(x) = 25$. Moreover, define by $E(h_i) = \{(x, \lambda) \in \text{epi}(f) : h_i(x) \geq \lambda \geq f(x)$ and $x \in [0, 5]\}$ the region above the graph of f and below the graph of h_i, for each $i \in \mathbb{N}$. That is, the $E(h_i)$ is the intersection $\text{epi}(f) \bigcap \text{hyp}(h_i)$.

Consider $K = \bigcap_{h_i \in H_f} E(h_i)$, the smallest region undernearth the graph of h_i, for each $i \in \mathbb{N}$, containing $\text{gr}(f)$. It follows that $\text{gr}(f) \subseteq K \subseteq \text{epi}(f)$ and there exists $g \in H_f$ (probably piece-wise defined) such that $h_i(x) \geq g(x) \geq f(x)$ for all $x \in [0, 5]$ for each i. According to property (3), $f \neq g$. Such function g is not necessarily convex according to property (1).

Hence denote by $H_{f,c} = \{h \in H_f : h$ convex $\}$ and consider $K_c = \bigcap_{h_i \in H_{f,c}} E(h_i)$. Note that $h_1(x) = 5x$ and $h_2(x) = 25$ are convex and thus $h_1, h_2 \in H_{f,c}$. Clearly $K_c \subseteq E(h_1)$.

Denote by $H_{f,c,\partial} = \{h \in H_f : h \text{ convex, and } f(x) = h(x), \text{ where } x \in \partial[0,5]\}$. It follows that $K_{c,\partial} = \bigcap_{h_i \in H_{f,c,\partial}} E(h_i) = E(h_1)$. Clearly $E(h_1) \subseteq \text{epi}(f)$ and since $\text{epi}(f)$ is convex $\text{co}(E(h_1)) \subseteq \text{epi}(f)$, since $\text{epi}(f)$ is closed $\text{clco}(E(h_1)) \subseteq \text{epi}(f)$. Moreover, $\partial(\text{clco}(E(h_1))) \subseteq \text{epi}(f)$ and $\partial(\text{clco}(E(h_1)))\setminus \text{gr}(h_1) \subseteq \partial\text{epi}(f)$. Since f is strictly conces and $\text{epi}(f)$ is closed, it follows from Theorem 2.1.30 that $\partial(\text{clco}(E(h_1)))\setminus \text{gr}(h_1) \subseteq \text{ext}(\text{epi}(f))$ and $\text{ext}(\text{clco}(E(h_1))) = \text{ext}(\text{epi}(f))$. Clearly $\text{clco}(E(h_1))$ is closed and bounded, but assume $E(h_1)$ is compact, and note that $\partial E(h_1) \bigcap \partial(\text{epi}(f) \neq \emptyset$. Clearly $E(h_1)$ is closed and hence $E(h_1) = \text{clco}(E(h_1))$. Moreover, $\text{ext}(E(h_1)) = \text{ext}(\text{epi}(f))$ and $E(h_1) = \text{clco}(\text{ext}(E(h_1)))$.

Clearly one can construct a convex compact subset of a convex epigraph which equals to the closed convex hull of its extreme points, and hence there could be a relationship between the KMP and epigraphs.

What about in the case where $\text{dom}(f) \subset \mathbb{R}^n$, $\mathbb{N} \ni n > 1$? Can we find convex compact subset C of an epigraph in $\mathbb{R}^{n+1}$ such that $\text{ext}(C) = \text{dent}(C) \subset \text{epi}(f)$?

Recall a set $L_\epsilon = \{(x,\lambda) : f(x) \leq \lambda < \epsilon\}$ and consider the closed convex bounded subset $\text{clco}(L_\epsilon)$ of an epigraph and a graph $\text{gr}(f_\epsilon) = \{(x, f(x)) \in \text{gr}(f) : f(x) < \epsilon\}$

Lemma 2.6.15 *Let $f : X \to \mathbb{R}$ be strictly convex, $X \times \mathbb{R}$ have the KMP and (*epi(f) *be closed). Then the following holds;*
1. $C = \text{clco}(\text{ext}(C))$, for any closed convex and bounded subset $C \subseteq \text{epi}(f)$.
2. L_ϵ is convex, $\text{clco}(L_\epsilon) \cap \text{gr}(f) \subseteq \text{ext}(\text{epi}(f))$ and $\text{clco}(\text{ext}(\text{clco}(L_\epsilon))) \subseteq \text{clco}(\text{gr}(f_\epsilon))$ for each fixed $\epsilon \in \mathbb{R}$.

Proof
1. Since $X \times \mathbb{R}$ has the KMP and C is closed convex and bounded, it follows that $C = \text{clco}(\text{ext}(C))$.
2. L_ϵ **is convex:**

Take any $(x,\lambda),(y,\mu) \in L_\epsilon$ and consider $\beta(x,\lambda)+(1-\beta)(y,\mu) \in X \times \mathbb{R}$ for $\beta \in [0,1]$. It follows that $f(x) \leq \lambda < \epsilon$ and $f(y) \leq \mu < \epsilon$ and $\beta x+(1-\beta)y \in X$ and thus

$$f(\beta x + (1-\beta)y) \leq \beta f(x) + (1-\beta) f(y) \leq \max\{\lambda,\mu\} < \epsilon.$$

Hence $\beta(x,\lambda) + (1-\beta)(y,\mu) = (\beta x + (1-\beta)y, \beta\lambda + (1-\beta)\mu) \in L_\epsilon$ and thus L_ϵ is convex.

We show that clco(L_ϵ) $\cap$ gr(f) $\subseteq$ ext(epi(f)):
For any $\epsilon \in \mathbb{R}$ satisfying $L_\epsilon \neq \emptyset$ and for $x \in X$ a minimizer for f we have $(x, f(x)) \in L_\epsilon \cap \operatorname{gr}(f) \subseteq \operatorname{clco}(L_\epsilon) \cap \operatorname{gr}(f) \neq \emptyset$.

Moreover, appealing for Proposition 2.1.3 we have $\operatorname{gr}(f) = \operatorname{ext}(\operatorname{epi}(f))$ and thus $\operatorname{clco}(L_\epsilon) \cap \operatorname{gr}(f) \subseteq \operatorname{gr}(f) = \operatorname{ext}(\operatorname{epi}(f))$.

We show that clco(ext(**clco(**L_ϵ))) = **clco**(gr(f_ϵ))
Take any $(a,b) \in \operatorname{ext}(\operatorname{clco}(L_\epsilon)$ and $\alpha \in [0,1]$. It follows that for any $(c,\nu),(d,\delta) \in \operatorname{clco}(L_\epsilon)$ such that $(a,b) = \alpha(c,\nu) + (1-\alpha)(d,\delta) \in \operatorname{clco}(L_\epsilon)$, we have $(a,b) = (c,\nu)$ or $(a,b) = (d,\delta)$. Clearly $(a,b) \in \operatorname{epi}(f)$ and $a < \epsilon$.

If $(a,b) \in \operatorname{int}(\operatorname{epi}(f))$, then there is $k > 0$ such that $B_k(a,b) \subset \operatorname{epi}(f)$. Moreover, there are $(e,n),(g,h) \in \partial B_k(a,b)$ with $\max\{n,h\} < \epsilon$ and $(a,b) \in [(e,n),(g,h)] \subset \operatorname{clco}(L_\epsilon)$ as $(a,b),(e,n),(g,h) \in L_\epsilon$. It follows that $(a,b) \in \operatorname{int}(\operatorname{clco}(L_\epsilon)) \setminus \operatorname{ext}(\operatorname{clco}(L_\epsilon))$ and thus leads to a contradiction. It follows that $(a,b) \in \partial(\operatorname{epi}(f))$ with $b < \epsilon$. Hence $\operatorname{ext}(\operatorname{clco}(L_\epsilon) \subseteq \operatorname{gr}(f_\epsilon)$.

For the reverse inclusion, take $(\beta, f(\beta)) \in \operatorname{gr}(f_\epsilon)$. It follows that $f(\beta) < \epsilon$ and clearly $(\beta, f(\beta)) \in L_\epsilon \subseteq \operatorname{clco}(L_\epsilon)$. Since $(\beta, f(\beta)) \in \partial(\operatorname{epi}(f))$, it follows that $(\beta, f(\beta)) \in \partial(L_\epsilon)$ and hence $(\beta, f(\beta)) \in \partial(\operatorname{clco}(L_\epsilon))$. Furthermore, take any $(w,\phi),(v,\tau) \in \operatorname{clco}(L_\epsilon)$ such that $(\beta, f(\beta)) \in [(w,\phi),(v,\tau)] \subseteq \operatorname{clco}(L_\epsilon) \subseteq \operatorname{epi}(f)$. Since f is strictly convex, we have $(\beta, f(\beta)) = (w,\phi)$ or $(\beta, f(\beta)) = (v,\tau)$ and hence $(\beta, f(\beta)) \in \operatorname{ext}(\operatorname{clco}(L_\epsilon))$. Consequently $\operatorname{ext}(\operatorname{clco}(L_\epsilon)) = \operatorname{gr}(f_\epsilon)$ and thus $\operatorname{clco}(\operatorname{ext}(\operatorname{clco}(L_\epsilon))) = \operatorname{clco}(\operatorname{gr}(f_\epsilon))$. □

Lemma 2.6.16 *Let $f : X \to \mathbb{R}$ be strictly convex and continuous (and* epi(f) *be closed). Then* gr(f) $\subseteq$ ext(clco(gr(f))). *Moreover, the reverse*

inclusion holds if the following condition holds

• *for each* $(x, \lambda) \in \text{int}(\text{epi}(f))$ *there exists* $(y_1, f(y_1)), (y_2, f(y_2)) \in \text{gr}(f)$ *with* $y_1 \neq y_2$ *such that line segments* $L_1 = [(x, \lambda), (y_1, f(y_1))]$ *and* $L_2 = [(x, \lambda), (y_2, f(y_2))]$ *are parallel to each other (and* $\lim_{\|x_n\| \to \infty} f(x_n) = \infty$*).*

Proof

Take any $(x, \lambda) \in \text{gr}(f)$. Then $(x, \lambda) \in \text{ext}(\text{epi}(f)) \cap \text{clco}(\text{gr}(f))$. Clearly $\text{clco}(\text{gr}(f)) \subseteq \text{clco}(\text{epi}(f)) = \text{epi}(f)$ since $\text{epi}(f)$ is closed and convex.

If $(x, \lambda) \in \text{int}(\text{clco}(\text{gr}(f)))$ then since $\text{int}(\text{clco}(\text{gr}(f))) \subseteq \text{clco}(\text{gr}(f)) \subseteq \text{epi}(f)$, it follows that $(x, \lambda) \in \text{int}(\text{epi}(f))$. This contradicts the fact that $(x, \lambda) \in \text{ext}(\text{epi}(f))$ and thus $(x, \lambda) \in \partial(\text{clco}(\text{gr}(f)))$.

Take any $(y, \mu), (z, \delta) \in \text{clco}(\text{epi}(f))$ and $\alpha \in [0, 1]$ such that $\alpha(y, \mu) + (1 - \alpha)(z, \delta)$. Since $(x, \lambda) \in \text{ext}(\text{epi}(f))$ and $(y, \mu), (z, \delta) \in \text{clco}(\text{gr}(f)) \subseteq \text{epi}(f)$, it follows that $(x, \lambda) = (y, \mu)$ or $(x, \lambda) = (z, \delta)$. Hence $(x, \lambda) \in \text{ext}(\text{clco}(\text{gr}(f)))$.

For the reverse inclusion assume $\text{clco}(\text{ext}(\text{epi}(f))) = \text{epi}(f)$. Appealing to Corollary 2.6.13 we have $\text{ext}(\text{clco}(\text{gr}(f))) = \text{ext}(\text{epi}(f))$. Moreover, since $\text{gr}(f) = \text{ext}(\text{epi}(f)$, see Proposition 2.1.3, the result follows. □

One might be tempted to ask if the results of Lemma 2.6.16 above would hold if the domain is a proper subset of X. Hence we state the following Theorem.

Theorem 2.6.17 *Let* $f : A \subset X \to \mathbb{R}$ *be strictly convex and continuous and* $\text{epi}(f)$ *be closed. Then* $\text{gr}(f) \subseteq \text{ext}(\text{clco}(\text{gr}(f)))$*. Moreover, the reverse inclusion holds if the following condition holds*

• *for each* $(x, \lambda) \in \text{int}(\text{epi}(f))$ *there exists* $(y_1, k_1), (y_2, k_2) \in \partial(\text{epi}(f))$ *with* $y_1 \neq y_2$ *such that line segments* $L_1 = [(x, \lambda), (y_1, k_1)]$ *and* $L_2 = [(x, \lambda), (y_2, k_2)]$ *are parallel to each other.*

Proof

Take any $(x, \lambda) \in \text{gr}(f)$. Then $(x, \lambda) \in \partial(\text{epi}(f)) \cap \text{clco}(\text{gr}(f))$ since $\text{gr}(f) \subseteq$

$\partial(\text{epi}(f))$. Clearly $\text{clco}(\text{gr}(f)) \subseteq \text{clco}(\text{epi}(f)) = \text{epi}(f)$ since $\text{epi}(f)$ is closed and convex.

However, if $(x, \lambda) \in \text{int}(\text{clco}(\text{gr}(f)))$ then, since $\text{int}(\text{clco}(\text{gr}(f))) \subset \text{clco}(\text{gr}(f)) \subseteq \text{epi}(f)$, it follows that $(x, \lambda) \in \text{int}(\text{epi}(f))$. This contradicts the fact that $(x, \lambda) \in \partial(\text{epi}(f))$. Hence $(x, \lambda) \in \partial(\text{clco}(\text{gr}(f)))$ and thus $\text{gr}(f) \subseteq \partial(\text{clco}(\text{gr}(f)))$.

Moreover, since $\text{clco}(\text{gr}(f))$ is closed, we have $(x, \lambda) \in \text{gr}(f) \subseteq \partial(\text{clco}(\text{gr}(f))) \subseteq \text{clco}(\text{gr}(f)) \subseteq \text{epi}(f)$. Take $(y, \mu), (z, \delta) \in \text{clco}(\text{gr}(f))$ such that $(x, \lambda) \in [(y, \mu), (z, \delta)] \in \text{clco}(\text{gr}(f)) \subseteq \text{epi}(f)$. Since f is strictly convex, we have $(x, \lambda) \in \text{gr}(f) \cap [(y, \mu), (z, \delta)] = \{(y, \mu), (z, \delta)\}$. It follows that $(x, \lambda) = (y, \mu)$ or $(x, \lambda) = (z, \delta)$ and hence $(x, \lambda) \in \text{ext}(\text{clco}(\text{gr}(f)))$. Consequently, $\text{gr}(f) \subseteq \text{ext}(\text{clco}(\text{gr}(f)))$.

For the reverse inclusion, take $(b, \beta) \in \text{ext}(\text{clco}(\text{gr}(f))) \subseteq \text{epi}(f)$ and any $(n, \nu), (e, \varepsilon) \in \text{epi}(f)$ such that $(b, \beta) \in [(n, \nu), (e, \varepsilon)]$. Clearly $(b, \beta) \in \partial(\text{clco}(\text{gr}(f)))$ and since $\text{gr}(f) \subseteq \partial(\text{epi}(f))$, we have $\partial(\text{clco}(\text{gr}(f))) \subseteq \partial(\text{clco}(\partial(\text{epi}(f))))$. But since f is continuous and $\text{epi}(f)$ is closed, $epi(f) = \text{clco}(\partial(\text{epi}(f)))$, see the assumption above, and thus $\partial(\text{clco}(\text{gr}(f))) \subseteq \partial(\text{clco}(\partial(\text{epi}(f)))) \subseteq \partial(\text{epi}(f))$. Consequently, $\text{ext}(\text{clco}(\text{gr}(f))) \subseteq \partial(\text{clco}(\text{gr}(f))) \subseteq \partial(\text{epi}(f))$.

Moreover, suppose $(b, \beta) \in \partial(\text{epi}(f)) \setminus \text{gr}(f)$. This would imply that $(b, \beta) \in \partial(\text{epi}(f)) \cap \{(b, \alpha) \in \text{epi}(f) : b \in \partial(A), f(x) < \alpha\}$. Then $(b, \beta) \in \text{int}[(b, f(b)), (b, \beta + n)]$, $n \geq 1$, with $(b, \beta + n) \in \text{epi}(f)$. Clearly $(b, \beta + n) \notin \text{clco}(\text{gr}(f))$ for each $n \geq 1$ as $\text{epi}(f))$ (and hence $\text{gr}(f)$) is closed, and hence leads to a contradiction. It follows that $(b, \beta) \in \partial(\text{epi}(f)) \cap \text{gr}(f)$ and hence $\text{ext}(\text{clco}(\text{gr}(f))) \subseteq \text{gr}(f)$. □

Theorem 2.6.18 *Let $f : X \to \mathbb{R}$ be strictly convex and continuous (and $\text{epi}(f)$ be closed). Then the following hold*

1. $\partial(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = \text{gr}(f) \subseteq \text{ext}(\text{clco}(\text{gr}(f))) \subseteq \text{epi}(f)$.
2. $\partial(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = \text{gr}(f) = \text{ext}(\text{clco}(\text{gr}(f)))$, provided $\text{int}(\text{epi}(f)) \subseteq \text{int}(\text{clco}(\text{gr}(f)))$.
3. Consequently, $\text{ext}(\text{clco}(\partial(\text{epi}(f)))) = \partial(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = \text{ext}(\text{clco}(\text{ext}(\text{epi}(f)))) = \text{ext}(\text{clco}(\text{gr}(f))) = \text{gr}(f)$, provided $\text{int}(\text{epi}(f)) \subseteq \text{int}(\text{clco}(\text{gr}(f)))$.

Proof

1. Since $\text{epi}(f)$ is closed and $\text{gr}(f) \subseteq \text{epi}(f)$, we have $\text{clco}(\text{gr}(f)) \subseteq \text{clco}(\text{epi}(f)) = \text{epi}(f)$, and hence

$$\text{ext}(\text{clco}(\text{gr}(f))) \subseteq \text{clco}(\text{gr}(f)) \subseteq \text{epi}(f).$$

Moreover, since $\text{gr}(f) \subseteq \text{ext}(\text{clco}(\text{gr}(f)))$, see lemma 2.6.16 we have

$$\text{gr}(f) \subseteq \text{ext}(\text{clco}(\text{gr}(f))) \subseteq \text{epi}(f).$$

Furthermore, appealing to Proposition 2.1.3 and Corollary 2.1.5, we have $\text{gr}(f) = \text{ext}(\text{epi}(f)) = \partial(\text{epi}(f))$. Hence

$$\partial(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = \text{gr}(f) \subseteq \text{ext}(\text{clco}(\text{gr}(f))) \subseteq \text{epi}(f).$$

2. Take any $(m, \mu) \in \text{ext}(\text{clco}(\text{gr}(f))) \subseteq \text{epi}(f)$. It follows that $(m, \mu) \in \text{ext}(\text{clco}(\text{gr}(f))) \subseteq \partial(\text{clco}(\text{gr}(f))) \subseteq \text{clco}(\text{gr}(f)) \subseteq \text{epi}(f)$. If $(m, \mu) \in \text{int}(\text{epi}(f))$, then $(m, \mu) \in \text{int}(\text{clco}(gr(f)))$ by assumption and hence constradicts the fact that $(m, \mu) \in \partial(\text{clco}(\text{gr}(f)))$. It follows that $(m, \mu) \in \partial(\text{epi}(f))$. Hence $\text{ext}(\text{clco}(\text{gr}(f))) \subseteq \partial(\text{epi}(f))$ and thus follows from 1. above that

$$\partial(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = \text{gr}(f) \subseteq \text{ext}(\text{clco}(\text{gr}(f))) \subseteq \partial(\text{epi}(f)).$$

Hence we have

$$\partial(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = \text{gr}(f) = \text{ext}(\text{clco}(\text{gr}(f))).$$

3. It follows from 2. above that $\text{gr}(f) = \text{ext}(\text{clco}(\text{gr}(f)))$, and since $\partial(\text{epi}(f)) = \text{ext}(\text{epi}(f)) = \text{gr}(f)$, we have

$$\text{ext}(\text{clco}(\partial(\text{epi}(f)))) = \partial(\text{epi}(f)),\ \text{ext}(\text{epi}(f)) = \text{ext}(\text{clco}(\text{ext}(\text{epi}(f)))),$$
$$\text{ext}(\text{clco}(\text{gr}(f))) = \text{gr}(f),$$

and the result follows. □

Proposition 2.6.19 *Let $f : A \subset X \to \mathbb{R}$ be strictly convex and continuous, A be bounded and closed and* $\mathrm{epi}(f)$ *be closed. If $|f| < \varepsilon$ for some $\varepsilon > 0$ then* $\mathrm{clco}(\mathrm{gr}(f)) \subseteq \mathrm{epi}(f)$ *is closed, convex and bounded.*

Proof

Clearly $\mathrm{clco}(\mathrm{gr}(f))$ is closed and convex. Its boundedness follows from boundedness of A and $|f| < \varepsilon$. Moreover, $\mathrm{gr}(f) \subseteq \mathrm{epi}(f)$ and hence $\mathrm{co}(\mathrm{gr}(f)) \subseteq \mathrm{co}(\mathrm{epi}(f)) = \mathrm{epi}(f)$ since $\mathrm{epi}(f)$ is convex for f convex. Since $\mathrm{epi}(f)$ is closed, we have $\mathrm{clco}(\mathrm{gr}(f)) \subseteq \mathrm{cl}(\mathrm{epi}(f)) = \mathrm{epi}(f)$. □

Chapter 3

Differentiability of real-valued convex functions

In this chapter we discuss the real-valued convex functions, their differentiability and integrability in convex subset of their domain. First we look at the real convex funtions defined on the real line and discuss all the characteristics of such a function in relation to the derivatives and then in the following section discuss the differentiability of convex function on Euclidean space.

In the preceeding sections we discussed continuity of convex function and found out that they are continuous, at least on the interior of their convex domains. Hence it is but logical to determine conditions under which convex functions would be differentiable at the each point of $\text{int}(\text{dom}(f))$.

We note though that there are convex functions not differentiable at some point $x \in \text{int}(\text{dom}(f))$, one of them being $f(x) = \|x\|$.

3.1 Derivatives of convex functions on intervals

Not all convex functions are differentiable, and since we figured out earlier that convex functions are continuous on the interior of their domain, we find out conditions under which convex functions would be differentiable. Moreover, we characterise points at which convex functions are differentiable, if any.

Theorem 3.1.1 *[14, Theorem 1.3.3, p.21] Let $f : I \to \mathbb{R}$ be a convex function. Then f has finite left and right derivatives at each points of* $\text{int}(I)$. *Moreover, if $x < y$ for $x, y \in \text{int}(I)$ then*

$$f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$$

Corollary 3.1.2 *[14, Corollary 1.3.8, p.23] If $f_n : I \to \mathbb{R}(n \in \mathbb{N})$ is a pointwise converging sequence of convex functions, then*

1. *the limit f is also convex*
2. *the convergence is uniform on any compact subinterval included in* $\text{int}\, I$
3. *$(f'_n)_n$ converges to f' except possibly at countably many points.*

Proposition 3.1.3 *[14, The Second Derivative Test 1.3.10, p.24] Suppose $f : I \to \mathbb{R}$ is twice differentiable, then f is convex if and only $f'' \geq 0$.*

Note that; at the points where a differentiable (strictly) non-linear convex function f on $\mathbb{R}$ attains the global mimimum, the derivative is equal to zero if it exists.
Hence we state the following Rolle's theorem;

Theorem 3.1.4 (Rolle's Theorem) *If a real convex function f is continuous on a closed interval $I = [a, b]$ and differentiable on the open interval (a, b) such that $f(a) = f(b)$, then there exists some number $c \in (a, b)$ such that $f'(c) = 0$.*

Clearly Rolle's theorem confirms the existence of the global mimimum of non-monotonic convex functions and the fact that the tangent, if any, has a slope equals to zero at such a global minimum.
Note however that the function has to be differentiable at each point in the interior of the domain. See the next example for illustration;

Example 3.1.5 (1) Even though $f = |x|$ is a convex function which is continuous on I, with $f(a) = f(b)$ for some $a, b \in I$, especially where $a = -b$, there exists no $c \in (a, b)$ such that $f'(c) = 0$ as $f'(x) \in \{-1, 1\}$ for $x \in (a, b)$. Moreover, $(0, 0) \in \text{ext}(\text{epi}(f))$ for $f(x) = |x|$. Hence this brings about the interplay between a global minimizer, differentiability and extremal structure of an epigraph $\text{epi}(f)$ of a convex function f. This leak would be explored later in this section.

Hence Rolle's result is applicable to convex functions that are **differentiable** on the interior of their domains, as they are continuous there.

There is another condition under which Rolle's theorem would not and would apply and we identify some of them in the following;

(2) Let $f : I \to \mathbb{R}$ be continuous non-linear convex on a closed interval $I = [a, b]$ and differentiable on (a, b). If f attains its global minimum at some $c \in \partial[a, b]$, then $f'(c) = 0$ does not hold in general.

Consider $f(x) = e^x$ on $I = [1, 10]$. Clearly for all $x \in \partial(I)$ we have $f(x) = f'(x) = e^x \neq 0$. Moreover, if $f(x) = x^2$ on $[0, \infty)$ then $f'(c) = 0$ for c a minimizer of f.

Hence we state the result as follows;

Lemma 3.1.6 *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable strictly convex function. Then $f'(c) = 0$ if and only if $c \in \mathbb{R}$ is the global minimizer for f.*

Proof
Since f is differentiable, we have

$f'(x) = \lim_{\beta \to x} \frac{f(\beta)-f(x)}{\beta - x} = \lim_{\beta \to x^-} \frac{f(\beta)-f(x)}{\beta - x} = \lim_{\beta \to x^+} \frac{f(\beta)-f(x)}{\beta - x}$
for each $x \in \mathbb{R}$. Morever, since $c \in \mathbb{R}$ is the global minimizer for f, we have $f(c) \leq f(\beta)$ and thus $f(\beta) - f(c) \geq 0$ for any $\beta \in \mathbb{R}$.

It follows that if $\beta > c$ then $\beta - c > 0$, and since $f(c) \leq f(\beta)$, we have $\lim_{\beta \to c^+} \frac{f(\beta)-f(c)}{\beta - c} \geq 0$. Morever, if $\beta < c$ then $\beta - c < 0$, and since $f(c) \leq f(\beta)$ we have $\lim_{\beta \to c^-} \frac{f(\beta)-f(c)}{\beta - c} \leq 0$. Since f is differentiable and $\lim_{\beta \to c^-} \frac{f(\beta)-f(c)}{\beta - c} = \lim_{\beta \to c^+} \frac{f(\beta)-f(c)}{\beta - c}$, for each $c \in \mathbb{R}$, it follows that
$f'(c) = \lim_{\beta \to c^-} \frac{f(\beta)-f(c)}{\beta - c} = \lim_{\beta \to c^+} \frac{f(\beta)-f(c)}{\beta - c} = 0$ and this complete the proof.

Conversely, since f is convex and differentiable at $c \in \mathbb{R}$ we have $f(y) \geq f(c) + f'(c)(y - c)$ for all $y, c \in \mathbb{R}$. If we assume that $f'(c) = 0$ then $f(y) \geq f(y)$ for all $y \in \mathbb{R}$ and thus $c \in \mathbb{R}$ is a global minimizer. □

Proposition 3.1.7 *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable strictly convex function and $f|_K : \mathbb{R} \to \mathbb{R}$ its convex monotonic restriction on a compact convex interval $K \subset \mathbb{R}$. Then the following hold;*
1. *$f|_K$ attains a global minimum at $\mu \in \partial(K)$.*
2. *$f'|_K(a) \neq 0$ for each $a \in \text{int}(K)$.*

Proof 1. Suppose $K = [a, b]$ with $a \geq \beta \geq b$ for each $\beta \in K$. Since $f|_K$ is monotonic we have $f|_K(a) \leq f|_K(\beta) \leq f|_K(b)$ or $f|_K(b) \leq f|_K(\beta) \leq f|_K(a)$ for each $\beta \in K$. Consequently $f|_K(\mu) \leq f|_K(\beta)$ for each $\beta \in K$ and $\mu \in \{a, b\} \subseteq \partial(K)$. It follows that $f|_K$ attains a global minimum at $\mu \in \partial(K)$.

2. It follows from lemma 3.1.6 that $f'|_K(a) = 0$ if and only if $a \in K$ is a global minimizer of $f|_K$. Hence, if follows from 1. above that $f'|_K(a) = 0$ if and only if $a \in \partial(K)$. Thus, if $a \in \text{int}(K)$, then $f'|_K(a) \neq 0$. □

Below is an example of piece-wise linear convex function;

Example 3.1.8 Let $f : I \to \mathbb{R}$ be a convex piece-wise linear convex function defined by

$$f(x) = \begin{cases} -x & \text{if} \quad x < 0 \\ x & \text{if} \quad x \in [0,1) \\ 3x-2 & \text{if} \quad x \in [1,2) \\ 5x-6 & \text{if} \quad x \geq 2 \end{cases}$$

whose derivative is,

$$f'(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \in (0,1) \\ 3 & \text{if} \quad x \in (1,2) \\ 5 & \text{if} \quad x > 2 \end{cases}$$

and hence $f'(x)$ does not exists for each $x \in \{0,1,2\}$.

If f is strictly convex then appealing to Proposition 2.1.3 we have $(a, f(a)) \in \text{ext}(\text{epi}(f))$ for each $a \in \text{dom}(f)$ and hence

Proposition 3.1.9 *Let I be closed and $f : I \to \mathbb{R}$ be a differentiable strictly convex function that attains its global minimum at $a \in \text{int}(I)$, then $(a, f(a)) \in$ ext$(\text{epi}(f))$.*

As an analogue to rotundity and that of translations of line segments with non-empty intersection with an epigraph of a convex function we have the following results;

For $f : A \subseteq \mathbb{R} \to \mathbb{R}$ convex function denote by

$$T = \{g : g(x) = f'(x_1)(x - x_1) + f(x_1), x \in \text{int}(A)\}$$

a collection of tangent to the graph of f at $(x_1, f(x_1)) \in \text{gr}(f)$, provided $f'(x_1)$ exists. Moreover, $\text{gr}(T) = \{\text{gr}(g) : g \in T\}$, and for each $r > 0$ denote by

$T_r = \{g_r : g(x) = f'(x_1)(x - x_1) + f(x_1) + r\}$

a collection of upward translates of tangents $g \in T$ to the graph of f at $(x_1, f(x_1)) \in \text{gr}(f)$, with $\text{gr}(T_r) = \{\text{gr}(g_r) : g_r \in T_r\}$ the collection of their graphs.

Lemma 3.1.10 *For $f : \mathbb{R} \to \mathbb{R}$ convex and differentiable, the following holds;*
(1) $\emptyset \neq \text{gr}(g) \cap \text{gr}(f) \subseteq \text{ext}(\text{epi}(f))$ *for any $g \in T$ if and only if f is strictly convex.*
(2) *For each $(x, \lambda) \in \text{int}(\text{epi}(f))$ and $r > 0$, there exists $g_r \in T_r$ such that $(x, \lambda) \in \text{gr}(g_r)$ provided f is strictly convex.*

Proof
(1) Take any $g \in T$ such that $(x_1, f(x_1)) \in \text{gr}(g) \cap \text{gr}(f)$ for some $g(x) = f'(x_1)(x - x_1) + f(x_1)$. Hence $\emptyset \neq \text{gr}(g) \cap \text{gr}(f)$.

Moreover, if f is strictly convex it follows from Proposition 2.1.3 that $\text{gr}(f) = \text{ext}(\text{epi}(f))$. Hence $\text{gr}(g) \cap \text{gr}(f) = \text{gr}(g) \cap \text{ext}(\text{epi}(f)) \subseteq \text{ext}(\text{epi}(f))$.

Conversely, assume $\text{gr}(g) \cap \text{gr}(f) \subseteq \text{ext}(\text{epi}(f))$. If f is linear on some non-trivial convex $K \subseteq \mathbb{R}$, then there exists $a, c \in K$ with $a \neq c$, $\alpha b + (1 - \alpha)c \in K$ such that $f(\alpha b + (1 - \alpha)c) = \alpha f(b) + (1 - \alpha)f(c)$. Hence for $\alpha \in [0, 1]$,

$$\begin{aligned}(\alpha b + (1 - \alpha)c, f(\alpha b + (1 - \alpha)c)) &= (\alpha b + (1 - \alpha)c, \alpha f(b) + (1 - \alpha)f(c)) \\ &= \alpha(a, f(a)) + (1 - \alpha)(c, f(c)), \\ &\in [(a, f(a)), (c, f(c))].\end{aligned}$$

Clearly $\emptyset \neq \text{int}[(a, f(a)), (c, f(c))] \subseteq \text{gr}(f)$ and $g(x) = \frac{f(b)-f(a)}{b-a}(x-a)+f(a)$ with $g \in T$ is a tangent to the graph of f at $\alpha(a, f(a)) + (1 - \alpha)(c, f(c))$ for each $\alpha \in [0, 1]$. Thus each $(m, f(m)) \in \text{int}[(a, f(a)), (c, f(c))]$ is such that $(m, f(m)) \in \text{gr}(g) \cap \text{gr}(f) \setminus \text{ext}(\text{epi}(f))$. This leads to a contradiction since $\text{gr}(g) \cap \text{gr}(f) \subseteq \text{ext}(\text{epi}(f))$, and it follows that there exists no $K \in \mathbb{R}$ such that $f|_K$ is linear. Hence f is strictly convex.

(2) Take $(x, \lambda) \in \text{int}(\text{epi}(f))$ and consider $(x, f(x)) \in \text{gr}(f)$ for each $x \in \mathbb{R}$ fixed. Clearly $g(y) = f'(x)(y - x) + f(x)$ exists as f is differentiable for $x \in \mathbb{R}$ fixed and for all $y \in \mathbb{R}$ ($f'(x)$ may be equal to zero at some x).

For any $(x, \lambda) \in \text{int}(\text{epi}(f))$, we have $f(x) < \lambda$ and thus $r = \lambda - f(x)$. It follows that
$g_r(y) = f'(x)(y-x)+f(x)+r = f'(x)(y-x)+\lambda$, $y \in \mathbb{R}$. Clearly $g_r(x) = \lambda$ for $x \in \mathbb{R}$ such that $(x, \lambda) \in \text{int}(\text{epi}(f))$. Hence $(x, \lambda) = (x, g_r(x)) \in \text{gr}(g_r)$

and thus $g_r \in T_r$, and this completes the proof. $\square$

Remark 3.1.11 Let $f : A \subseteq X \to \mathbb{R}$ be convex and denote by

$$\mathrm{Gr}'(f) = \{(x_0, f(x_0)) \in \mathrm{gr}(f) : f \text{ not differentiable at } x_0\}$$

the graph of restriction of f defined on the non-differentiability points of f.

In [21], the authors discuss the concept similar to the one above. They state that, for any countable subset $B \subseteq \mathbb{R}$, a real convex function g not differentiable at every element of B where B is called a set of *bad points*, can be constructed, see [21, Theorem 1, p.726].
An interesting question is; For any such function g, is $(x, g(x)) \in \mathrm{ext}(\mathrm{epi}(g))$ for each $x \in B$? That is, is $\mathrm{gr}(g|_B) \subseteq \mathrm{ext}(\mathrm{epi}(g))$?

Clearly $f(x) = \|x\|$ is not differentiable at $x = 0$ and $(0, f(0)) \in \mathrm{ext}(\mathrm{epi}(f))$. See below the example on the existing interplay between the bad points and the extreme points;

Example 3.1.12 Consider the real piece-wise linear convex function $f(x) = |x|$ on $\mathbb{R}$. It is clear that;

- $\mathrm{dom}(f) = D_1 \cup D_2$ such that $D_1 = (-\infty, 0]$ and $D_2 = [0, \infty)$
- each D_i, $i = 1, 2$, is a maximal domain of linearity for each linear restriction of f.
- $\mathrm{Gr}'(f) = \{(0, f(0))\} \subseteq \mathrm{ext}(\mathrm{epi}(f))$.

i) Conversely, if $(x, y) \in \mathrm{ext}(\mathrm{epi}(f))$, then $(x, y) \in \mathrm{gr}(f)$, and since f is piece-wise linear (so is $\mathrm{gr}(f)$) we have $x \notin \mathrm{int}(D_i)$ for each i.
Note that, if $x \in \mathrm{int}(D_i)$, there exist $y, z \in D_i$ such that $x \in [y, z]$ and $f|_{[y,z]}$ is linear. Thus $(x, f(x)) \in \mathrm{int}([y, z], f|_{[y,z]}) = \{(\alpha, f(\alpha) : \alpha \in [y, z]\}\ \mathrm{int}([(y, f(y)), (z, f(z))])$.

ii) $(x, y) = \{(0, f(0))\} = \mathrm{Gr}'(f)$ and thus $\mathrm{Gr}'(f) = \mathrm{ext}(\mathrm{epi}(f))$. Hence a real piece-wise convex function f can have an epigraph whose extreme points (if any) are related to the points of non-differentiability of f.

Proposition 3.1.13 *Let $f : I \to \mathbb{R}$ be a piece-wise linear convex function function and $I = \bigcup_{i=1}^{n} I_i$ with $f : I_i \to \mathbb{R}$ linear for each $i = 1, ..., n$ and each I_i be a maximal domain of linearity.*

1. *If f is not differentiable at $x \in I$ then $(x, f(x)) \in \mathrm{ext}(\mathrm{epi}(f))$. Consequently*

$Gr'(f) = \{(x, f(x)) \in \mathrm{gr}(f) : f'(x) \notin \mathbb{R}\} \subseteq \mathrm{ext}(\mathrm{epi}(f))$.

2. *Moreover, if I is closed and f is continuous and not differentiable at each $x \in \partial(I)$, then $Gr'(f) = \mathrm{ext}(\mathrm{epi}(f))$.*

Proof

1. Clearly for each $x \in \mathrm{int}(I_i)$, $i = 1, ..., n$, we have $f'_-(x) = f'_+(x)$, see Theorem 3.1.1, and hence f is differentiable on $\mathrm{int}(I_i)$ for each $i = 1, ..., n$. Moreover, if $x \in \partial(I_i) \cap \mathrm{int}(I) \cap I_i$ for each fixed i, then since each I_i is the maximal domain of linearity, we have

$f|'_+(x) = \lim\limits_{\mu \to x^+} \frac{f(\mu)-f(x)}{\mu-x} \neq \lim\limits_{\beta \to x^-} \frac{f(\beta)-f(x)}{\beta-x} = f|'_-(x)$ with $\beta \in I_k$ and $\mu \in I_{k+1}$ such that $\beta, \mu \notin I_k \cap I_{k+1}$. Thus $f'(x)$ would not exist for $x \in \partial(I_i) \cap \mathrm{int}(I)$ for each $i = 1, ..., n$.

Hence, by assumption take any $(x, f(x)) \in \mathrm{gr}(f)$ for each $x \in \partial(I_i) \cap I_i \cap \mathrm{int}(I)$ for some $i = 1, ..., n$. It follows that $(x, f(x)) \in \mathrm{gr}(f|_{\partial(I_i)})$, and $(x, f(x)) \in \partial[(a, f(a)), (b, f(b))]$ if $a, b \in I_i$ and $x \in \partial(I_i) \cap \mathrm{int}(I) \cap I_i$ for some i.

Moreover, take any $(\tau, f(\tau)), (\nu, f(\nu)) \in \mathrm{gr}(f)$ such that $(x, f(x)) \in [(\tau, f(\tau)), (\nu, f(\nu))]$. If $\tau, \nu \in I_i$ for some i then $(x, f(x)) \in \partial[(\tau, f(\tau)), (\nu, f(\nu))]$ and hence $(x, f(x)) \in \mathrm{ext}(\mathrm{epi}(f))$.

Moreover, suppose on the other hand that $\tau \in \mathrm{int}(I_k)$ and $\nu \in \mathrm{int}(I_m)$. Then $(x, f(x)) \in \partial[(\tau, f(\tau)), (\nu, f(\nu))]$ would imply that $x \in \mathrm{int}(I_k) \cup \mathrm{int}(I_m)$ and hence $f'(x)$ exists and leads to a contradiction. It follows that $\tau \in \partial(I_k)$ or $\nu \in \partial(I_m)$ and hence $x = \tau \in \partial(I_k)$ or $x = \nu \in \partial(I_m)$. Conse-

quently $(x, f(x)) = (\tau.f(\tau))$ or $(x, f(x)) = (\nu, f(\nu))$ and hence $(x, f(x)) \in \text{ext}(\text{epi}(f))$. Hence $Gr'(f) \subseteq \text{ext}(\text{epi}(f))$.

2. If I is closed and f is continuous, then $\text{epi}(f)$ is closed. It follows from 1. above that $Gr'(f) \subseteq \text{ext}(\text{epi}(f))$. For the other inclusion take any $(y, f(y)) \in \text{ext}(\text{epi}(f)) \subseteq \text{gr}(f)$. It follows that $(y, f(y)) \in \text{gr}(f_{I_i})$ for some i and $y \in \partial(I_i) \cap I$. Clearly f is not differentiable at each $y \in \partial(I_i) \cap I$ and hence $(y, f(y)) \in \text{Gr}'(f)$. Hence $\text{Gr}'(f) = \text{ext}(\text{epi}(f))$. □

3.2 Subdifferential of functions on interval

Subdifferential discussion is borne out of the lack of tangent lines to the graphs of non-smooth (of non-differentiable) convex functions. The so-called support line at a particular point of the domain of some convex function is some form of a quasi-tangent as such a point, and the collection of slopes of such supporting lines is loosely called the subdifferential of f at such a point.

Definition 3.2.1 *[14, p.30]* Given a function $f : I \to \mathbb{R}$, the subdifferential $\partial f(x)$ of f at $x \in I$ is defined as:

$$\partial f(x) = \{\lambda \in \mathbb{R} : f(y) \geq f(x) + \lambda(y - x) \text{ for all } y \in I\}$$

That is; the slopes of the supporting lines for $\text{gr}(f)$.

For each supporting line there is a unique *subderivative* and hence the subdifferential is the collection of subderivatives. That is, the subderivative of a real function $f : I \subset \mathbb{R} \to \mathbb{R}$ at a point $(x, f(x))$ is the slope of the line L touching the graph of f such that the line passing through this point is below the graph, where I is an open interval.

If f is convex, then L is a tangent to the graph of f. In other words, $c \in \mathbb{R}$ is a subderivative of a convex function f at y if $f(x) - f(y) \geq c(x - y)$

for all $x \in I$. Hence $c \leq \frac{f(x)-f(y)}{x-y}$ for all $x > y$ in I and $c \geq \frac{f(x)-f(y)}{x-y}$ for all $x < y$ in I. Therefore $c = \lim\limits_{x \to y^+} c \leq \lim\limits_{x \to y^+} \frac{f(x)-f(y)}{x-y} = f'_+(y)$ and $c = \lim\limits_{x \to y^-} c \geq \lim\limits_{x \to y^-} \frac{f(x)-f(y)}{x-y} = f'_-(y)$ for all $y \in I$. Hence $f'_-(y) \leq c \leq f'_+(y)$, and thus $c = f'(y)$ provided $f'(y)$ exists.

Lemma 3.2.2 *If a real convex function $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$ then $\partial f(x_0)$ is a singleton.*

Proof
Since f is convex, $\partial f(x_0)$ contains slopes of tangents to the graph of f at $(x_0, f(x_0))$. Moreover, since f is differentiable, each slope is unique as $f'(x_0)$ is unique for each $x_0 \in I$. It follows that $\partial f(x_0)$ is a singleton at each $x_0 \in I$. □

Example 3.2.3 1) If $f(x) = |x|$, then the subdifferential of f at $(0, 0)$ (or just $y = 0$) is $[-1, 1]$. At any $y < 0$ it is a singleton $\{-1\}$ and at $y > 0$ it is a singleton $\{1\}$.
2) If $f(x) = x^2$, with $f : I \subset \mathbb{R} \to \mathbb{R}$, then the subdifferential of f at 1 and at 0, are $\partial f(1) = \{2\}$ and $\partial f(0) = \{0\}$ respectively. Moreover
$\partial f(y) = \{c \in \mathbb{R} : f(x) \geq c(x - y) + f(y)\} = \{c \in \mathbb{R} : c = 2y, y \in I\}$.
3) If $f : I \subset \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3$, where $I = (-10, 0)$, then the subdifferential of f at -1 and at -10 are empty, that is, $\partial f(-1) = \emptyset$ and $\partial f(-10) = \emptyset$ respectively. This is because the tangent L passsing through $y = -1$ *or* -10 is above the graph of f. Moreover $\partial f(y) = \emptyset$ for all $y \in I$.
4) If $f : I \subset \mathbb{R} \to \mathbb{R}$ is defined by $f = \sin x$, where $I = (\pi, 2\pi)$, then the subdifferential $\partial f(y)$ of f at y is $\partial f(y) = \{\cos y\}$ for all $y \in I$ because the tangent passing through any $y \in I$ is below the graph of f since f is convex.

Example 3.2.4 1. The real functions $f : I \to \mathbb{R}$ defined by $f(x) = -x^2$ (and $f = \sqrt{x}$) are continuous on $\mathbb{R}$ (respectively on $\mathbb{R}_+$), and nowhere

convex with empty subdifferential at any point $y \in I$, that is $\partial f(y) = \emptyset$ for all $y \in I$.

2. Consider a convex function $f : I \to \mathbb{R}$ defined $f(x) = 4 - \sqrt{1 - x^2}$ on $I = [-1, 1]$. Clearly this function is not subdifferentiable at each point $x \in \{-1, 1\}$ hence $\partial f(-1) = \partial f(1) = \emptyset$.

Lemma 3.2.5 *[14, Lemma 1.5.1, p.30] Let $f : I \to \mathbb{R}$ be convex, then*

1. *$\partial f(x) \neq \emptyset$ for all $x \in \text{int}(I)$*

2. *$\partial f(x) = \emptyset$ if $x \in \partial(I)$ and f is not continuous at $x \in \partial(I)$.*

This result and its converse can be extended to domains in higher dimension as follows;

Definition 3.2.6 *[14, Theorem 3.7.1, p.128]* Let $f : U \subseteq E \to \mathbb{R}$ be convex function, U be open set and E be normed linear space. Then

$$\partial f(a) = \{h : f(x) \geq f(a) + h(x - a) \text{ for all } x \in U\}$$

the collection of linear functionals $h : E \to \mathbb{R}$ constitutes the subdifferential of f at the point $a \in U$. Moreover, those linear functionals are also called supporting hyperplanes of f at each point $a \in U$.

Moreover, if E is finite dimensional, then those linear functionals $h : E \to \mathbb{R}$ are defined as $h(x) = \langle x, z \rangle$ for $x \in E$ and hence we have the following definition;

Definition 3.2.7 *[8, p.116, 118]* For X a linear space, a real function $f : A \subseteq X \to \mathbb{R}$ has a *right-hand (respectively left-hand) Gateaux derivative* at $x \in \text{int}(A)$ in the direction $y \in X$ if

$$f'_+(x; y) = \lim_{\lambda \to 0^+} \frac{f(x+\lambda y) - f(x)}{\lambda} \text{ (resp. } f'_-(x; y) = \lim_{\lambda \to 0^-} \frac{f(x+\lambda y) - f(x)}{\lambda})$$

exists.

Moreover, $f'(x;y) = \lim_{\lambda \to 0} \frac{f(x+\lambda y)-f(x)}{\lambda}$ is a *Gateaux derivative* and exists if and only if $f'_+(x;y)$ and $f'_-(x;y)$ exists and $f'_+(x;y) = f'_-(x;y)$. Furthermore, f is Gateaux differentiable at $x \in \text{int}(A)$ in the direction $y \in X$ if $f'(x;y)$ exists.

Definition 3.2.8 *[10, Subdifferential I Definition 1.1.4, p.165]* Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. The subdifferential $\partial f(x)$ of f at x is the non-empty compact convex set in $\mathbb{R}^n$ whose support function is $f'(x, \cdot)$, that is

$$\partial f(x) = \{s \in \mathbb{R}^n : \langle s, d\rangle \leq f'(x, d) \text{ for all } y \in \mathbb{R}^n\},$$

Moreover, each vector $s \in \partial f(x)$ is called a subgradient of f at x.

Theorem 3.2.9 *[14, Theorem 3.7.1, p.128] Let U be an open convex set in a normed linear space E. Then $f : U \subseteq E \to \mathbb{R}$ is convex if and only if $\partial f(x) \neq \emptyset$ for all $x \in U$.*

This gives an interesting characterisation of convex function.

Proposition 3.2.10 *Let $f : I \to \mathbb{R}$ be convex and I be open. Then f attains the global minimum at $a \in I$ if $0 \in \partial f(a) \neq \emptyset$.*

Proof
Since $\partial f(a) \neq \emptyset$ then $f(y) \geq f(a) + \lambda(y - a)$ for all $y \in I$ with $\lambda \in \partial f(a)$. Since $0 \in \partial f(a)$, we have $f(a) \leq f(y)$ for all $y \in I$ if $\lambda = 0$. Hence $a \in I$ is a global minimizer of f. □

Corollary 3.2.11 *Let $f : I \to \mathbb{R}$ be convex and I be open. If $0 \in \partial f(a)$ for all $a \in I$ then f is constant.*

Proof
Suppose $0 \in \partial f(a)$ for all $a \in I$. It follows from Proposition 3.2.10 that f attains a global minimum at each $a \in I$. Consequently $f(a) \leq f(x)$ for all $x \in I$, and since each $x \in I$ is also a global minimum, we have $f(x) \leq f(a)$ for each $x \in I$. Thus $f(a) = f(x) = \alpha \in \mathbb{R}$ for each $x \in I$ and some α. It follows that f is constant. □

Remark 3.2.12 Take a strictly convex function $f(x) = x^2$ and a piece-wise convex function $g(x) = |x|$ on $K = [-3, 4]$ and consider $\partial f(0)$ and $\partial g(0)$.
1. Clearly $\partial f(0) = \{0\}$ and $\partial g(0) = [-1, 1]$.
2. It follows that $\partial f(0)$ and $\partial g(0)$ are convex sets.
3. Note that since f is differentiable at $0 \in K$ we have $\partial f(0) = \{f'(0)\}$ is a singleton, or unique. However, g is not differentiable at $0 \in K$ hence $\partial g(0)$ is a not singleton.

Henceforth a function $f : \mathbb{R}^n \to [-\infty, \infty]$ will be called *extended real-valued* function, and $[-\infty, \infty]$ will be denoted by $\overline{\mathbb{R}}$ where necessary.

The following theorem shows that the existence of partial derivatives at a point in the domain of f may imply differentiability of f at such a point;

Theorem 3.2.13 *[14, Theorem 3.8.1, p.135] Suppose $U \in \mathbb{R}^n$ is open and convex and $f : U \to \mathbb{R}$ is convex and possess all its partial derivatives $\frac{\partial f}{\partial x}, ..., \frac{\partial f}{\partial x_n}$ at some point $a = (x, ...x_n) \in U$. Then f is differentiable at point a*

Clearly such a function f is continuous as it is convex on the open set U. Moreover differentiablity is equivalent to the uniqueness of support function as the following example illustrate;

Example 3.2.14 Let $f : I \to \mathbb{R}$ be a convex function, I be open and $h : \mathbb{R} \to \mathbb{R}$ be convex linear. Then

1. Consider a piece-wise linear function $f(x) = |x|$ defined on I with $0 \in I$. Hence we have the collection $S_0 = \{h : h(x) = mx, m \in (-1, 1)$ for all $x \in \mathbb{R}\}$ of support functions of f at $0 \in I$ (similarly at $(0, f(0))$). Since m is not unique, it follows that support function of f at $0 \in I$ is not unique.
2. Consider a strictly convex function $f(x) = x^2$ on an open interval. Clearly f is differentiable on I and $f'(a)$ is unique for each $a \in I$. It follows that, for each $a \in I$,
$S_a = \{h : h(x) = mx + c, m = f'(a)$ for all $x \in \mathbb{R}$ and some $c \in \mathbb{R}\}$
$= \{h : h(x) = f'(a)x + c$ for all $x \in \mathbb{R}$ and some $c \in \mathbb{R}\}$
is a collection of support functions of f at $a \in I$. Since $f'(a)$ exists and is unique for each $a \in I$, $h(x) = f'(a)x + c$, $c \in \mathbb{R}$ is a unique support function for each $a \in I$ and for all $x \in \mathbb{R}$. Hence we have the following result on the interplay between support functions and derivatives at a point in the domain.

Theorem 3.2.15 *[14, Theorem 3.8.2, p.136] Suppose $U \subseteq \mathbb{R}^n$ is open and convex and $f : U \to \mathbb{R}$ is convex. Then f is differentiable at a if and only if f has a unique support function.*

The following results classify the points of differentiability for a convex function f and it states thus;

Theorem 3.2.16 *[18, Theorem 25.5, p.246] Let $f : M \subseteq \mathbb{R}^n \to \mathbb{R}$ be a convex function and D be a set of points where f is differentiable. Then D is a dense subset of* $\text{int}(M)$

Moreover,

Lemma 3.2.17 *[18, Theorem 25.5, p.246] Let $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ be a convex function and C be an open convex set. If f is differentiable on C, then f is actually continuously differentiable on C.*

Clearly piece-wise defined linear functions are vital in this discussion of bad points mapping by convex functions to the extreme points of epigraphs. Hence we have the following;

Definition 3.2.18 Define $f : A \subseteq X \to \mathbb{R}$ a piece-wise (affine) linear function as follow;

- There are $D_i \subseteq A$ such that $\bigcup_i D_i = A$ and $i \in \mathbb{N}$,
- $f|_{D_i}$ is affine linear restriction of f,
- There exists $\varphi_i \in X^*$ for each D_i such that $f(x) = \varphi_i(x) + \alpha_i$, where $\alpha_i \in \mathbb{R}$, $x \in D_i$ for each fixed i.

It follows from Theorem 3.2.15 that a convex function on an open set U is differentiable at $x \in U$ if and only if it has a unique support function there. Hence, considering the fact that any linear convex function on an open convex set has a unique support function, then it is differentiable on each of the interior point in its domain, see below.

Lemma 3.2.19 *If $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is linear convex and A is open, then f is differentiable at each $x \in A$.*

Proposition 3.2.20 *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be piece-wise linear convex and $A_i \subseteq A$ be maximal domain of linearity for each $i \in \mathbb{N}$ with $A = \bigcup_{i\in\mathbb{N}} A_i$. Moreover, if A be bounded, $Q = \bigcup_{i\in\mathbb{N}} \partial(A_i) \subset A$ and f is not differentiable at $x \in A$, then $x \in Q$ and $(x, f(x)) \in \text{ext}(\text{epi}(f))$.*

Proof
Take any $x \in A$ on which f is not differentiable. It follows that $x \in A_i$ for some $i \in \mathbb{N}$ and A_i a maximal domain of linearity, that is $f|_{A_i}$ is linear. Since a linear function is differentiable on the interior of its domain, see to Lemma 3.2.19, it follows that $x \in \partial(A_i) \subseteq Q$. Moreover, appealing to Corollary 2.2.3, $(x, f(x)) \in \text{gr}(f|_Q) = \text{ext}(\text{epi}(f))$, the the result follows. □

Corollary 3.2.21 *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a piece-wise linear convex function on bounded subset A in a Banach space X. If* epi(f) *is closed and $Gr'(f) \neq \emptyset$ then*

$$\text{dent}(\text{epi}(f)) \subseteq \text{ext}(\text{epi}(f)) \subseteq \text{Gr}'(f).$$

Proof

Take $(x,\lambda) \in \text{dent}(\text{epi}(f))$. Then $(x,\lambda) \notin \text{clco}(\text{epi}(f)\backslash B_\epsilon(x,\lambda))$ for any $\epsilon > 0$. Hence $(x,\lambda) \notin \text{co}(\text{epi}(f)\backslash B_\epsilon(x,\lambda))$ for any $\epsilon > 0$ and it follows that $(x,\lambda) \notin [(y,\beta),(m,\alpha)]$ for any $(y,\beta),(m,\alpha) \in \text{epi}(f)\backslash B_\epsilon(x,\lambda)$ for any $\epsilon > 0$. Since $\epsilon > 0$, for any $(y,\beta),(m,\alpha) \in \text{epi}(f)\backslash B_\epsilon(x,\lambda)$, we have $(x,\lambda) \neq (y,\beta)$ and $(x,\lambda) \neq (m,\alpha)$.

Take any distinct $(a,b),(c,d) \in \text{epi}(f)$ such that $(x,\lambda) \in [(a,b),(c,d)]$. If $(x,\lambda) \in \text{int}[(a,b),(c,d)]$, then choose $\epsilon > 0$ such that $(a,b),(c,d) \in \text{epi}(f)\backslash B_\epsilon(x,\lambda)$. Clearly $(x,\lambda) \in [(a,b),(c,d)] \subseteq \text{co}(\text{epi}(f)\backslash B_\epsilon(x,\lambda)) \subseteq \text{clco}(\text{epi}(f)\backslash B_\epsilon(x,\lambda))$ and this leads to a contradiction. Hence $(x,\lambda) \notin \text{int}[(a,b),(c,d)]$ for any distinct $(a,b),(c,d) \in \text{epi}(f)$ and thus $(x,\lambda) \in \text{ext}(\text{epi}(f))$.

Moreover, a piece-wise linear convex function has maximal domain of linearity say $A_i \subseteq A$ with $A = \bigcup_{i\in\mathbb{N}} A_i$. If $(x,\lambda) \in \text{ext}(\text{epi}(f)) \subseteq \text{gr}(f)$, then suppose $x \in \text{int}(A_i) \subseteq A_i \subseteq A$ for some i. Then there is $z,w \in A \cap \partial(A_i)$ such that $x \in [z,w]$ as A_i is convex for each i. Since f is linear on A_i, then $(x,\lambda) \in [(z,f(z)),(w,f(w))]$ and it would follow that $(x,\lambda) = (z,f(z))$ or $(x,\lambda) = (w,f(w))$ in which case $x = z$ of $x = w$. This would mean $x \in \text{int}(A_i) \cap A \cap \partial(A_i)$, but since $\text{int}(A_i) \cap A \cap \partial(A_i) = \emptyset$ this leads to a contradiction. Hence if $(x,\lambda) \in \text{ext}(\text{epi}(f)) \subseteq \text{gr}(f)$, then $x \in A \cap \partial(A_i)$. Hence f is not differentiable on $x \in A$ and thus $(x,\lambda) \in \text{Gr}'(f)$. $\square$

3.3 Equivalence between Frechet and Gateaux derivatives of convex functions

Clearly Gateaux and Frechet derivatives of a convex function at a fixed point are not the same in general. Hence in this section we recall results on conditions under which these forms of derivatives of convex functions, if they exist, would be equivalent.

Recall that the directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at x in the direction d is

$$f'(x,d) = \lim\{q(t) : t \searrow 0\} = \inf\{q(t) : t > 0\}$$

where $q(t) = \frac{f(x+td)-f(x)}{t}$ for $t > 0$.

Note that if $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $x \in \mathbb{R}^n$, the directional derivative $f'(x,\cdot)$ exists, see [10, Introduction, p.163].

It is easy to see that, the directional derivative is the same as the right derivative of f at x.

Proposition 3.3.1 *[10, Proposition 1.1.2, p.164] For a fixed x, the function $f'(x,\cdot)$ is finite sublinear, provided $f : \mathbb{R}^n \to \mathbb{R}$ is convex.*

Corollary 3.3.2 *For $f : \mathbb{R}^n \to \mathbb{R}$ convex, $x \in \mathbb{R}^n$ fixed, the function $f'(x,\cdot)$ is convex.*

Definition 3.3.3 *[8, p.116, 118]* For X a linear space, a real function $f : A \subseteq X \to \mathbb{R}$ has a *right-hand (respectively left-hand) Gateaux derivative* at $x \in \text{int}(A)$ in the direction $y \in X$ if

$$f'_+(x;y) = \lim_{\lambda \to 0^+} \frac{f(x+\lambda y)-f(x)}{\lambda} \text{ (resp. } f'_-(x;y) = \lim_{\lambda \to 0^-} \frac{f(x+\lambda y)-f(x)}{\lambda})$$

exists.

Moreover, $f'(x;y) = \lim_{\lambda \to 0} \frac{f(x+\lambda y)-f(x)}{\lambda}$ is a *Gateaux derivative* and exists if and only if $f'_+(x;y)$ and $f'_-(x;y)$ exists and $f'_+(x;y) = f'_-(x;y)$. Futhermore, f is Gateaux differentiable at $x \in \text{int}(A)$ in the direction $y \in X$ if $f'(x;y)$ exists.

Theorem 3.3.4 *[8, Theorem 1, p.117] Let X be a Banach space and $\phi : A \subseteq X \to \mathbb{R}$ be a convex function with A convex in a linear space X and $\text{int}(A) \neq \emptyset$. For $x \in \text{int}(A)$ and $y \in X$, $\phi'_+(x;y)$ and $\phi'_-(x;y)$ both exist, and $\phi'_+(x)$ is a sublinear functional on X.*

Lemma 3.3.5 *[8, p.120] Let $\phi : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be convex and assume that partial derivatives of ϕ exist at $x \in A$. Then ϕ is Gateaux differentiable at $x \in A$.*

Definition 3.3.6 *[8, p.141]* Let X be normed linear space and A be open. A function $\psi : A \subseteq X \to \mathbb{R}$ is *Frechet differentiable* at $x \in A$ if there exists a linear functional $f : X \to \mathbb{R}$ such that given $\varepsilon > 0$ there is a $\delta(\varepsilon, x) > 0$ such that

$$|\psi(x+y) - \psi(x) - f(y)| < \varepsilon\|y\| \text{ for all } \|y\| \ < \delta$$

Clearly, if ψ is Frechet differentiable at $x \in A$, then $\lim_{\lambda \to 0} \frac{\psi(x+\lambda y)-\psi(x)}{\lambda}$ exists and is approached uniformly for all $y \in X$ where $\|y\| = 1$. Then ψ is Gateaux differentiable at x and a linear functional f is the Gateaux derivative $\psi'(x)$ which is in this case called Frechet derivative of ψ at x, see [8, p.141]. This shortly means that Frechet differentiable at a point implies Gateaux differentiable at that point, and that Gateaux differentiability is more general than the Frechet differentiability. Subsequently we discuss condition(s) under which the converse hold.

Proposition 3.3.7 *[8, Corollary, p.142] If a real function $\varphi : A \subseteq X \to \mathbb{R}$ is Frechet differentiable for an open subset A of a finite dimensional normed linear space X, then φ is continuous at $x \in A$.*

Clearly Frechet differentiability on an interior point implies continuity at such a point adn the converse is not always true. As an analogue of Lemma 3.3.5 above we have the following result;

Lemma 3.3.8 *[8, p.143] Let $g : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be convex and assume partial derivatives of g exist in A and are continuous at $x \in A$, then g is Frechet differentiable at $x \in A$.*

Theorem 3.3.9 *[8, Theorem 2, p.144] For A an open convex set and X a finite dimensional normed linear space, if a convex function $\phi : A \subseteq X \to \mathbb{R}$ is Gateaux differentiable at $x \in A$, then ϕ is Frechet differentiable at $x \in A$.*

Corollary 3.3.10 *For A an open convex set and X a finite dimensional normed linear space, if a function $\phi : A \subseteq X \to \mathbb{R}$ if convex and $x \in A$, then Gateaux derivative and Frechet derivative of ϕ at $x \in A$ are equivalent.*

The equivalence between Gateaux differentiability and Frechet differentiability holds if a function ϕ is convex and defined on a finite dimensional space as stated in the following result;

Theorem 3.3.11 *[8] Let E be a Banach space such that for each continuous convex function $f : E \to \mathbb{R}$, every point of Gateaux differentiability is also a point of Frechet differentiability. Then E is finite dimensional.*

3.4 Subdifferentials of convex functions on vector space

As stated in the preceeding chapters, convex functions are not necessarily differentiable (but can be locally differentiable). Hence, we turn to subdifferentiability to discuss the derivatives of convex functions. As a reminder, we consider a *subgradient* of a function $f : D \subset X \to \overline{\mathbb{R}}$ at a point x_0 as the gradient of the tangent T passing the graph of f through the point x_0 or $(x_0, f(x_0))$, such that T is always below the graph of f. The *subdifferential* of f at x_0 is the set of all subgradients of f at x_0, that is, the set of the gradients m_i of all the tangents T_i passing the graph of f through the point x_0 or $(x_0, f(x_0))$, such that T_i is always below the graph of f. This is

formally stated as follows;

There are few definitions of the subdifferentials of a function, some with reference to differentiation of a function and others not. We state both versions here as follows;

Definition 3.4.1 *[10, Subdifferential I Definition 1.1.4, p.165]* Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. The subdifferential $\partial f(x)$ of f at x is the non-empty compact convex set in $\mathbb{R}^n$ whose support function is $f'(x, \cdot)$, that is

$$\partial f(x) = \{s \in \mathbb{R}^n : \langle s, d\rangle \leq f'(x, d), \text{ for all } d \in \mathbb{R}^n\},$$

Moreover, each vector $s \in \partial f(x)$ is called a subgradient of f at x.

The following makes no reference to differentiation of f;

Definition 3.4.2 *[10, Subdifferential II Definition 1.2.1, p.167]* Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. The subdifferential $\partial f(x)$ of f at x is the non-empty set of vectors $s \in \mathbb{R}^n$ satisfying

$$f(y) \geq f(x) + \langle s, y - x\rangle, \text{ for all } y \in \mathbb{R}^n,$$

This second definition is similar to the one given in [14, p.128] though for the function on normed linear space as follows;
If U is an open convex subset of a normed linear space E, and $f : U \subseteq E \to \mathbb{R}$ is a function, then the set $\partial f(a) = \{h : E \to \mathbb{R}$ continuous linear functional: $f(x) \geq f(a) + h(x - a), \forall x \in U\}$

The easiest version of the definition of subdifferentiable is as follows;

Definition 3.4.3 For A a convex subset of a linear space X, a real convex function $f : A \to \mathbb{R}$ is said to be subdifferentiable at $x_0 \in A$ if there exists a real affine functional $h : X \to \mathbb{R}$ such that h satisfies the following

1. $h(x_0) = f(x_0)$

2. $h(x) \leq f(x)$ for all $x \in A \backslash \{x_0\}$

If f is strictly convex the inequality in 2. change to $f(x) > h(x)$, and if f is a linear functional then $h(x) = f(x_0) + f(x - x_0)$ for all $x \in A$.

Henceforth we would say the function f is subdifferentiable if $\partial f(x) \neq \emptyset$ for each $x \in \text{dom}(f)$. The following is the example of a subdifferentiable function on $\mathbb{R}^n$:

Example 3.4.4 A real function $f(x) = \|x\|$, the Euclidean norm, is subdifferentiable at every $x \in \mathbb{R}^n$:
This function is differentiable everywhere but not on $x = 0$. The set $\partial f(x)$ of subdifferentials (subderivatives) of f for $x \neq 0$, $x \in \mathbb{R}^n$, is equal to $\partial f(x) = \{\|x\|^{-1}x : \ x \in \mathbb{R}^n\}$.
If f is an **extended** real function then these hold provided $-\infty < f(x) < +\infty$, that is, f is finite on $\mathbb{R}$.

Proposition 3.4.5 *[14, Lemma 3.7.2, p.129] Let U be an open convex set in $\mathbb{R}^n$ and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be convex. Then $z \in \partial f(a) \neq \emptyset$ if and only if $f'_+(a, v) = \lim_{t \to 0^+} \frac{f(a+tv)-f(a)}{t} \geq \langle z, u \rangle$ for all $v \in \mathbb{R}^n$*

Theorem 3.4.6 *[10, Theorem 2.2.1, p.177] For $f : \mathbb{R}^n \to \mathbb{R}$ convex, the following three property are equivalent;*

1. *f is minimized at $x \in \mathbb{R}^n$*
2. *$0 \in \partial f(x)$*
3. *$f'(x, d) \geq 0$ for all $d \in \mathbb{R}^n$*

Below is an example of a a convex function with points of non-subdifferentiability.

Example 3.4.7 Consider a convex function $f : \mathbb{R}^n \to \mathbb{R}$ define by

$$f(x) = \begin{cases} -\sqrt{(1 - \|x\|^2)} & \text{if} \quad \|x\| \leq 1 \\ +\infty & \text{if} \quad \|x\| > 1 \end{cases}$$

This function f is subdifferentiable (an also differentiable) at each x satisfying $\|x\| < 1$, yet $\partial f(x) = \emptyset$ whenever $\|x\| \geq 1$

It is therefore clear that the lack of subdifferentiability of a convex function may be attained around the boundary of the domain. This is the analogue of the following result from [8]:

Theorem 3.4.8 *[8, The Differentiability Theorem, p.27] Let X be normed linear space and A its convex subset. If $f : A \to \mathbb{R}$ is convex, then f is subdifferential at each point $x \in \text{int}(A)$.*

Moreover, on the non-subdifferentiability of f at a point, we have the following result from [18]:

Theorem 3.4.9 *[18, Theorem 23.3, p.216] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function (and let $x \in \mathbb{R}^n$ be a point where f is finite). If f is not subdifferentiable at x, there must be some infinite two-sided directional derivative at x, i.e. there must exists some y such that*

$$f'(x; y) = -f'(x, -y) = -\infty$$

Conclusion

Clearly convex functions reach far and wide as they feature in different branches of Mathematics, both in Pure and Applied, including Optimisation theory, study of minimizers and maximizers of a function as discussed in Chapter 2, and also Mathematics of Economics.

Convex extensions have interesting characteristics as seen in Chapter 1, and assuming a suitable ordering on the chain of these extensions, amongst other conditions, one is able to determine the maximal of these extensions. Clearly one cannot accurately define the convex extension of a function f without looking at the bigger domain containing the domain $\mathrm{dom}(f)$ of the underlying function f. Hence this consideration brought about the discussion we undertook on the maximal of those domains $\mathrm{dom}(g^*)$ containing the domain $\mathrm{dom}(f)$ of the underlying function f, that is $\mathrm{dom}(f) \subseteq \mathrm{dom}(g^*)$.

In Chapter 1, the following two questions were asked and addressed;

1. Firstly, given an arbitrary real-valued function f and the collection $\mathfrak{C}$ of convex subsets of a vector space X, do there exist a maximal convex restrictions of f whose domain is in the collection $\mathfrak{C}$?

2. Secondly, what can be said about extensions and maximal extensions of convex functions?

In an attempt to answer the first question, we first looked for a convex subset C of X on which the restriction of f is convex. One can always find such a subset even though it could be a trivial singleton set. If there is only one set on which the restriction of f is convex, then such a set would also be the maximal convex domain of that particular restriction of f. However, we chose to consider only those restrictions of f whose domain is non-trivial in $\mathfrak{C}$ and that is when the discussion got a bit more interesting.

We realised that before we could look at the maximal convex restriction of f, we needed to look first at a maximal domain in $\mathfrak{C}$, and hence in X, on which a restriction of f would be convex, that is $\mathfrak{C}$-MDC for f. It was clear we needed to assume that the collection $\mathfrak{C}$ satisfies the chain union property, that is CUP, meaning that any union of elements in the chain $\mathfrak{B} \subset \mathfrak{C}$ is also a convex set in $\mathfrak{C}$.

The assumption of the CUP in $\mathfrak{C}$ enables us to prove the existence of maximal element in $\mathfrak{C}$, and hence the existence of $\mathfrak{C}$-MDC for f. Moreover, it was clear that the maximal convex restriction of f has to be defined on $\mathfrak{C}$-MDC, a subset of $\mathfrak{C}$. This answered question one above, however, we do not know of any other condition under which one may have the existence of a maximal convex restrictions of f whose domain is in the collection $\mathfrak{C}$, hence this leaves room for more research to be done on this topic. An interesting question might be; If the upper bound of such chain of domains of convex extensions does not exist, under what conditions if any, would there be maximal convex extension?

Turning our attention to the second question above, it was clear that not every extension of a convex function is convex. Hence we needed to define the collection $\mathfrak{X}_f$ of a convex extensions of a convex function f, and look for the maximal elements in such a collection. Considering the fact that convex epigraphs and convex functions coincide, we had to look for convex epigraphs $\text{epi}(g)$ of function g such that $\text{epi}(f) \subseteq \text{epi}(g)$. It turned out that, if $\text{gr}(f) \subset \text{gr}(g)$ and g is convex then g might be a convex extension of a convex function f, that is $g \in \mathfrak{X}_f$. Moreover, the largest of those convex epigraphs $\text{epi}(g)$ would likely be the epigraph of a maximal convex extension of convex function f, under certain assumptions, including the CUP in $\mathfrak{C}$ and the $\text{dom}(f) \subset X$ pseudo-absorbing in X, amongst others.

Moreover, uniqueness of convex extension of any convex function (and a convex restriction of a convex function) does not always exist. Moreover, we are not aware of any uniqueness result if $f : A \subseteq X \to \mathbb{R}$ is convex with

$\dim X \geq 2$, and it may be the case that there are f such that there is more than one maximal element in $\mathrm{Epi}(\mathfrak{X}_f)$, the collection of convex epigraphs $\mathrm{epi}(g)$ containing (or which are extensions of) $\mathrm{epi}(f)$.

Existence, and lack of, extreme points or denting points of an epigraph of a convex functions was an interesting discussion in Chapter 2. It is clear that extreme points and denting points are not necessarily equivalent in a convex epigraph. Hence conditions of their equivalence was discussed and what became apparent was that both extreme points and denting points of a convex epigraph are found on the boundary of such a convex epigraph, together with a graph of an underlying convex function. However, not every epigraph of a convex function has an extreme point, and this is apparent when we look at a real linear (hence convex) function defined on the whole space X. Hence, assuming the function is strictly (at least non-linear, and) convex would elicit the desired characteristics of an epigraph, that is, its extremal structure and its dentability.

Convexifiable functions were also discussed in Chapter 2, and these are functions f whose epigraph has a convex hull equal to an epigraph of another function g, that is $\mathrm{epi}(g) = \mathrm{co}(\mathrm{epi}(f))$, for some real function g. This other function, g, is named convexification and has interesting characteristics in that, for each convexifiable function f there is only one convexification g. Clearly convex functions are convexifiable as their epigraphs are convex and hence equal to their convex hull. That is, for each f convex and real, we have $\mathrm{epi}(f) = \mathrm{co}(\mathrm{epi}(f))$. Moreover, since a convex hull is unique for each set, it is clear that a real convex function is its unique convex convexification.

Moreover, in the latter sections of Chapter 2, we were reminded of the fact that one cannot discuss the extremal structure of convex sets without looking at the well known Krein Milman theorem and Krein Milman property. Closed convex hull of extreme points of a convex epigraph was

observed, its characteristics, and condition(s) of its existence in an epigraph of a convex function. It turned out that assuming a function is strictly convex would not be enough to get its epigraph equal to, or at least contained into, its closed convex hull of its extreme points, and the domain on which the function is defined plays a vital role as well. The lack of boundedness of a convex epigraph was the let down as we could not apply neither the Krein Milman Theorem nor the Krein Milman property accurately to get the equivalence between a convex closed epigraph and a closed convex hull of its extreme points.

However, we could construct a closed convex bounded subset of a convex epigraph equal to a closed convex hull of its extreme points, whose extreme points are also extreme points of the epigraph it is contained in.

Moreover, it was interesting to find the correlation between the point at which a strictly convex function attains its maximal or minimal point and the extreme point of an epigraph of such a strictly convex function.

In Chapter 3 we went a step further and it was clear that some convex functions have points in their domain(s) on which their are not differentiable. Consequently the relation between those points of non-differentiability of a convex function f and extremal structure of an epigraph $\text{epi}(f)$ of such convex function was established. This in turn tied together the differentiability of convex functions, extremal structure of their epigraphs and their maximizers and minimizers.

We noticed that if we were to look for extreme points of an epigraph of a convex function f, we have to look at the following first;

• the points at which the function f is differentiable whenever the function in question is strictly convex, or

• the points at which the function f is not differentiable whenever the function is linear.

• the points $(x, f(x)) \in \text{gr}(f)$, graph of f, such that the function f attains its global minimum at $x \in \text{dom}(f)$, as most extreme points of an epigraph of a convex function f are contained in these sets of points.

Though this entire discussion was extensive and detailed at time, it just scratched the surface of this interesting topic and hence much is yet to be discovered and published on this topic. The equivalence, or lack of it, between the Krein Milman and Radon Nykodym Properties is still the question to answer, especially in Banach space as in dual Banach spaces the equivalence have since been established.

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