# Symmetry Analysis and Invariant Properties of some Partial Differential Equations

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# DECLARATION

I declare that the contents of this dissertation are original, except where due references have been made. It is submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It was not submitted before for any degree or examination to any other institution.

AS Mathebula

Signed at Johannesburg on the $30^{th}$ day of May 2017.

**Abstract**

This dissertation contains evolutionary partial differential equations (PDEs). The PDEs are used to investigate ecological phenomena. The main goal is to determine Lie point symmetries, perform Lie reduction, obtain analytical solutions and visualize the solutions in 3D plots using the help of Mathematica. Drift diffusion, biased diffusion and the Kierstead, Slobodkin and Skellam (KiSS) models arising in population ecology are discussed. The importance of these PDEs in ecology is to analyse the movements of organisms and their long-term existence especially in heterogeneous environments.

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# Chapter 1

# Introduction

In the nineteenth century, the theory of group symmetries were discovered by Sophus Lie (1842-1899) which are now also referred to as Lie groups. Lie was inspired by Abel and Galois' theory [1] which stated that the procedure in all exceptional cases of a universal integration on differential equations (DEs), is centred on the invariance of the DE under continuous symmetries. Lie wanted to create a theory of integrating ordinary DEs equivalent to the Abelian theory of computing algebraic equations.

A transformation of solutions of differential equation to other solutions can be performed using a symmetry group. Hermann Weyl (1885-1955) in [2], remarked that an object must be subjected to a certain operation which will result in it appearing precisely the same. This object is invariant with respect to the operation. One can easily understand symmetries of DEs by considering symmetries of an object. The symmetries admitted by DEs leads to solutions which can be interpreted physically.

In this investigation we focus on analysing Lie point symmetries for PDEs arising in ecology. Noether's famous Theorem [3] which is used to determine conservation laws for given Euler equations once their group of symmetries are identified [4], is important in the study of non-linear PDEs [2].

The work by Sophus Lie has an impact in fields of applied mathematics, physical sciences and engineering. The admitted symmetries are used to decrease the quantity of independent variables [1, 5]. Hence, the application of symmetry methods to differential equations is to discover related DEs of a simpler form, thereby reducing complex non-linear problems to simpler linear ones.

Numerous studies have been dedicated to studying DEs that admit Lie point symmetries [5, 6]. Another method for obtaining conservation laws is the multiplier approach. Here, every multiplier (integrating factor) provides a conserved quantity. The aim is to study conservation laws, symmetries and multipliers for ecological models.

The expansion of muskrat populations in Europe was modelled by [7], this was the first mathematical attempt to model reaction-diffusion equations in ecology. A correlated random walk [8] is usually followed by animals if they depart from simple Brownian motion but instead continue onward in the direction of their existing path. This notion gives a telegraph PDE model (e.g. [9]):

$$\frac{1}{2\lambda}u_{tt} + u_t = \frac{s^2}{2\lambda}(u_{xx} + u_{yy}), \tag{1}$$

where $u$ is the population size, $2\lambda$ is the degree of the relationship between directions of travel from one step to the subsequent and $s$ is the speed of the animals. Additional importance is placed on reaction-diffusion models in which a combination of population dynamics such as movement and multi-species interactions is considered [8]. The following is a form of a reaction-diffusion model;

$$u_t = (\mathcal{D}(u)u_x)_x + (\mathcal{D}(u)u_y)_y + f(u), \tag{2}$$

the diffusive movement is $\mathcal{D}(u)$ while reaction and population dynamics is represented by $f(u)$. The Fisher model [10] is the most important reaction-diffusion model, which represents Brownian random dispersal and logistic population growth:

$$u_t = ru\left(1 - \frac{u}{K}\right) + \mathcal{D}(u_{xx} + u_{yy}), \tag{3}$$

where the population's growth rate is represented by $r$ and the capacity is represented by $K$.

## 1.1 Outline of Chapters

Chapter 2 provides the notation and theory that we utilize to determine Lie point symmetries and Noether point symmetries for ordinary DEs and PDEs, such as the second order Ermakov Pinney equation, second order time independent Transonic Gas equation, Simple Harmonic Oscillator equation and $1 + 1$ wave equation. Also, we provide an example involving flows of the KdV equation using a recursion operator.

In Chapters 3,4 and 5, Lie point symmetries, Lie reductions, analytical solutions and 3D plots of some diffusion equations are obtained. Multipliers are derived through the multiplier approach and lead to conservation laws. Lastly, Mathematica is used to obtain 3D plots of the solutions to the equation. We explore classical and nonclassical Lie point symmetries of the diffusive KiSS model. Lastly, higher-order symmetries of the KiSS model are computed using a recursion operator.

Some of the contents of this dissertation has been published at an appropriate journal [11].

# Chapter 2

# Notation and Theory

## 2.1 Differential Functions

Intrinsic to a Lie algebraic treatment of DEs is the universal space $\mathcal{A}$ [5]. A locally analytic function $f(x, u, u_{(1)}, \ldots, u_{(k)})$ of a finite amount of variables is said to be a *differential function of order* $k$. The vector space $\mathcal{A}$ of all differential functions of all finite orders forms an algebra. A total derivative transforms every differential function of order $k$ to a differential function of order $k+1$. Therefore, the space $\mathcal{A}$ is closed under total derivations $D_i$. There are also additional operators on $\mathcal{A}$ and some of the significant ones which we make use of are presented below in definitions 1 to 4 and can be found in the books [12] , [13], [5] and [14].

## 2.2 Symmetry Methods

This section outlines the universal method for obtaining point symmetries for an arbitrary system of DEs. The nonlinear system with $q$ indefinite functions $u^a$ which depends on $s$ independent variables $x^i$, is studied. In this system $u = (u^1, \dots, u^q)$ and $x = (x_1, \dots, x_s)$, respectively. Let

$$G_\alpha\left(x, u^{(k)}\right) = 0, \quad \alpha = 1, \dots, q, \tag{4}$$

be a system of $q$ nonlinear DEs, where $u^{(k)}$ and $x$ are the independent vectors. A one-parameter Lie group of transformations ($\epsilon$ is the group parameter) that is invariant, in (4) it is defined by

$$\bar{x} = \Xi(x, u, \epsilon) \quad \bar{u} = \Phi(x, u, \epsilon), \tag{5}$$

where $\Xi$ and $\Phi$ depends on $x$, $u$ and $\epsilon$. The Invariance of (4) under the transformation (5) implies that every solution $u = \Theta(x)$ of (4) maps into a different solution $v = \Psi(x; \epsilon)$ of (4). Expanding (5) around the identity $\epsilon = 0$, the transformations below are generated;

$$\begin{aligned} \bar{x}^i &= x^i + \epsilon\xi^i(x, u) + \mathcal{O}(\epsilon^2), \quad i = 1, \dots, s, \\ \bar{u}^\alpha &= u^\alpha + \epsilon\eta^\alpha(x, u) + \mathcal{O}(\epsilon^2), \quad \alpha = 1, \dots, q. \end{aligned} \tag{6}$$

Where both $\xi^i$ and $\eta^\alpha$ depend on $x$ and $u$, $\mathcal{O}$ depends on $\epsilon^2$. The action of the Lie group can be derived from that of its generators acting on the vector space of dependent and independent variables. Hence, the following vector field will be considered

$$X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}. \tag{7}$$

The infinitesimal criterion for invariance is represented below by

$$XG_\alpha(x, u^k) \mid_{Eq.(4)} = 0. \tag{8}$$

The above Eq. (8) provides an overdetermined system of linear homogeneous equations, more details can be found in [5] among other texts.

## 2.3 Fundamental Operators

**Definition 1.** The Euler-Lagrange operator is given by

$$\frac{\delta}{\delta u^\beta} = \frac{\partial}{\partial u^\beta} + \sum_{r\geq 1} (-1)^r D_{i_1} \cdots D_{i_r} \frac{\partial}{\partial u^\beta_{i_1 \cdots i_r}} , \quad \beta = 1, \ldots, q \, . \tag{9}$$

**Definition 2.** The Lie-Bäcklund operator is defined by the infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\beta \frac{\partial}{\partial u^\beta} + \sum_{r \geq 1} \zeta^\beta_{i_1 \ldots i_r} \frac{\partial}{\partial u^\beta_{i_1 \ldots i_r}}, \tag{10}$$

the extra coefficients are obtained exclusively by the prolongation formulae

$$\begin{aligned} \zeta^\beta_i \quad &= \quad D_i(W^\beta) + \xi^j u^\beta_{ij} \, , \\ \zeta^\beta_{i_1 \ldots i_r} \quad &= \quad D_{i_1} \ldots D_{i_r}(W^\beta) + \xi^j u^\beta_{j i_1 \ldots i_r}, \quad r > 1, \end{aligned} \tag{11}$$

where the total differentiation operator $D_i$ with respect to $x^i$ is defined by

$$D_i = \frac{\partial}{\partial x^i} + u^\beta_i \frac{\partial}{\partial u^\beta} + u^\beta_{ij} \frac{\partial}{\partial u^\beta_j} + \ldots, \quad i = 1, \ldots, m. \tag{12}$$

In (11), $W^\beta$ is the Lie characteristic function given by

$$W^\beta = \eta^\beta - \xi^j u^\beta_j \, . \tag{13}$$

The Lie-Bäcklund operator (10) can be represented in the characteristic form

$$X = \xi^i D_i + W^\beta \frac{\partial}{\partial u^\beta} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^\beta) \frac{\partial}{\partial u^\beta_{i_1 \ldots i_s}} \tag{14}$$

**Definition 3**. The generalized operators $\tilde{X}$ and $X$ are *equal* if

$$X - \tilde{X} = \lambda^i D_i, \quad \lambda^i \in \mathcal{A} \, .$$

In particular, a generalized operator of the form $\tilde{X} = \eta^\beta \partial / \partial u^\beta + \cdots$ is said to be a *canonical* or *evolutionary* representation of $X$.

**Definition 4.** The Noether operator associated with a generalized operator $X$ is defined by

$$N^i = \xi^i + W^\beta \frac{\delta}{\delta u_i^\beta} + \sum_{r\geq 1} D_{i_1}\cdots D_{i_r}(W^\beta)\,\frac{\delta}{\delta u_{ii_1\cdots i_r}^\beta}\,, \quad i = 1,\ldots,m, \tag{15}$$

where the *Euler* operators with respect to derivatives of $u^\beta$ are determined from (4) by substituting $u^\beta$ by the consistent derivatives, i.e.,

$$\frac{\delta}{\delta u_i^\beta} = \frac{\partial}{\partial u_i^\beta} + \sum_{r\geq 1}(-1)^r D_{j_1}\cdots D_{j_r}\,\frac{\partial}{\partial u_{ij_1\cdots j_r}^\beta} \quad i = 1,\ldots,n, \quad \beta = 1,\ldots,n\,. \tag{16}$$

The *Euler*, Lie-Bäcklund, and Noether operators are associated by the operator identity

$$X + D_i(\xi^i) = W^\beta\,\frac{\delta}{\delta u^\beta} + D_i N^i\,. \tag{17}$$

**Definition 5.** If there exists a function $L = L(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}) \in \mathcal{A}; \quad s \leq r$, $r$ being the order of (4), such that

$$\frac{\delta L}{\delta u^\beta} = 0, \qquad \beta = 1,\ldots,q$$

then $L$ is called a Lagrangian of (4) and $\dfrac{\delta}{\delta u^\beta}$ is the corresponding Euler operator. $\dfrac{\delta L}{\delta u^\beta} = 0$ are known as the Euler-Lagrange equations.

**Definition 6.** The Lie-Bäcklund operator $X$ is said to be a Noether symmetry corresponding to a Lagrangian $L \in \mathcal{A}$, if there exists a vector $B^i = (B^1,\ldots,B^m)$, $B^i \in \mathcal{A}$, such that,

$$X(L) + LD_i(\xi^i) = D_i(B^i). \tag{18}$$

If $B^i = 0\,(i = 1,\ldots,m)$, then $X$ is said to be a strict Noether symmetry corresponding to a Lagrangian $L \in \mathcal{A}$.

## 2.4 Noether's Theorem

Emmy Noether [4] studied the fascinating connection amongst symmetries and conservation laws, presenting that for all transformation admitted by the action integral of a system, there exists a conservation law. That is, for every Noether symmetry $X$ corresponding to a given Lagrangian $L \in \mathcal{A}$, there exists a current $\Phi^i = (\Phi^1, \ldots, \Phi^m), \quad \Phi^i \in \mathcal{A}$, given by

$$\Phi^i = B^i - N^i(L), \qquad i = 1, \ldots, m, \tag{19}$$

where

$$\sum_{i \geq 1} D_i \, \Phi^i = 0 \tag{20}$$

along the solutions of (4).

## 2.5 Invariant Solutions

The operator in Eq.(7) can be used to define the Lagrange system

$$\frac{dx^i}{\xi^i} = \frac{du^\alpha}{\eta^\alpha}$$

whose solution provides the zero-order invariants

$$W^{[0]}(x^i, u^\alpha). \tag{21}$$

These invariants can be utilized in order to decrease the order of the PDE. Further information of the appropriate equations and formulae can be found in, inter alia, [15].

## 2.6 Multipliers

The multipliers $\Lambda^\alpha$ of (4) satisfies the relation

$$D_i T^i = \Lambda^\alpha G_\alpha \tag{22}$$

for the function $u^\alpha$ [5, 16]. The overdetermined equations for $\Lambda^\alpha$ are

$$\frac{\delta}{\delta u^\alpha}[\Lambda^\alpha G_\alpha] = 0. \tag{23}$$

Eq.(22) is satisfied for the functions $u^\alpha$ and not only for the solutions of (4). Conserved quantities may be derived methodically using (22) as the determining equation, however, in various cases it is simple to construct the conserved quantities by elementary manipulations once the multiplier has been obtained. Alternatively, the conserved quantities are determined by a homotopy operator, see [17] for details.

## 2.7 Nonclassical Symmetries

An extra condition on the infinite criterion for invariance is the difference in obtaining nonclassical symmetries from classical symmetries, that is, to achieve invariance on the invariant surface condition(ISC) and solution of the equation. In this case we can add the following condition in equation (8), which can be noted below as

$$X^{[\mathrm{k}]}G \mid_{G=0,ISC} = 0, \tag{24}$$

where ISC is given by

$$\xi^i(x^i, u)\tfrac{\partial u^\alpha}{\partial x^i} = \eta^\alpha(x^i, u). \tag{25}$$

This is a different case to classical Lie point symmetries, equation (24) provides a system of nonlinear determining equations [18]. The set of all nonclassical symmetries include all Lie symmetries, and do not form a vector space (no associated Lie algebra).

## 2.8 Recursion Operator

Polynomial systems arising from evolution equations are given by

$$u_t = G(u), \tag{26}$$

where $G$ is simply expressed as $G(u)$ even though $G$ depends on $u$ and it's $x$-derivatives up to order $n$. Assuming that all parameters in the system are non-zero. A higher-order symmetry, $H(u)$, leaves the partial differential equation invariant under the substitution $u \to u + \epsilon H$ within order $\epsilon$ [19]. A linear integro-differential operator which is a recursion operator, $\mathcal{R}$, links higher-order symmetries [19]

$$H^{(p+q)} = \mathcal{R}H^{(p)}, p = 1, 2, 3, \ldots \tag{27}$$

where $q = 1$ in the most cases and $H^{(p)}$ is the $p - th$ higher-order symmetry.

## 2.9 Illustrative Examples

Next, some well-known classical examples from the literature are presented to illustrate the theoretical approach. The Lie symmetry methods are adopted to compute the Lie point symmetries of the Ermakov-Pinney and the time independent Transonic Gas equations, to show how one can decrease the order of the equations. Further, Noether point symmetries of a second order Simple Harmonic Oscillator equation and the wave equation are evaluated. The results are used to construct some conserved vectors. Finally some flows of the KdV equation are presented.

### 2.9.1 Example 1. Lie point symmetries and invariant solutions of ODEs

The first task is to find Lie symmetries and reduction of second order Ermakov-Pinney (EP) equation:

$$y'' - \tfrac{\alpha}{y^3} = 0. \tag{28}$$

The physical context in which the EP equation plays an important role includes nonlinear optics, nonlinear elasticity, quantum cosmology, quantum field theory, to name a few [20]. Considering the following vector field $X = \xi\partial_x + \phi\partial_y$ from Eq.(7) with Lie point symmetry condition based on formula (8):

$$X^{[2]}\left(y'' - \frac{\alpha}{y^3}\right)\Big|_{y'' - \frac{\alpha}{y^3} = 0} = 0.$$

After prolongation and simplification this leads to:

$$\frac{3\alpha}{y^4}\phi + \phi^{xx} = 0.$$

Making a substitution of $y'' = \frac{\alpha}{y^3}$ we find:

$$\frac{3\alpha}{y^4}\phi + \phi_{xx} + 2y'\phi_{yx} - y'\xi_{xx} - 2(y')^2\xi_{yx} + (y')^2\phi_{yy} - (y')^3\xi_{yy} + \frac{\alpha}{y^3}(\phi_y - 2\xi_x - 3y'\xi_y) = 0,$$

separation by derivatives of $y$ provides the following determining system:

$$\xi_{yy} = 0,\ \phi_{yy} - 2\xi_{yx} = 0,\ 2\phi_{yx} - \xi_{xx} - \frac{3\alpha}{y^3}\xi_y = 0,\ \phi_{xx} + \frac{\alpha}{y^3}\phi_y + \frac{3\alpha}{y^4}\phi - \frac{2\alpha}{y^3}\xi_x = 0, \tag{29}$$

solving this system, the following is obtained

$$\begin{aligned} \phi(x,y) = &\quad (c_0 x + c_1)y, \\ \xi(x,y) = &\quad c_0 x^2 + 2c_1 x + c_2, \end{aligned} \tag{30}$$

where the $c_i\,(i = 0, 1, 2)$ are arbitrary constants. Hence, the Lie point symmetries of second-order equation (28) are:

$$X_1 = x^2\partial_x + xy\partial_y, \quad X_2 = 2x\partial_x + y\partial_y, \quad X_3 = \partial_x. \tag{31}$$

Reducing the EP equation (28) using $X_2$ prolonged

$$X_2 = 2x\partial_x + y\partial_y + \phi^x\partial_{y'}, \tag{32}$$

where $\phi^x = -y'$ and, solving

$$\frac{dx}{2x} = \frac{dy}{y} = \frac{-dy'}{y'}, \tag{33}$$

to find invariants $\kappa = \frac{\sqrt{x}}{y}, \beta = \sqrt{x}y'$. The differential invariant is given by the equation:

$$\frac{d\beta}{d\kappa} = \frac{d\beta}{dx}\frac{dx}{d\kappa}, \tag{34}$$

where

$$\frac{d\beta}{dx} = \frac{y' + 2xy''}{2\sqrt{x}}, \quad \frac{dx}{d\kappa} = \frac{2\sqrt{x}y^3}{y(y - 2xy')}. \tag{35}$$

This leads to:

$$y'' = \frac{y - 2xy'}{2xy^2}\beta_\kappa - \frac{y'}{2x},$$

and substituting into equation (28) the reduced first-order ODE is obtained

$$\left(\frac{1}{2\kappa} - \beta\right)\beta_\kappa - \frac{\beta}{2\kappa^2} - \alpha\kappa = 0. \tag{36}$$

### 2.9.2 Example 2. Lie point symmetries and invariant solutions of PDEs

Here Lie point symmetries and reduction of second order time independent Transonic Gas(TG) equation are obtained. The TG equation is

$$u_x u_{xx} + u_{yy} = 0, \tag{37}$$

and is found in the theory of transonic flow. According to [21], the transonic flow involves a transition from the subsonic to the supersonic region through the sonic curve. This is a very interesting phenomena in the field of aerodynamics and hydrodynamics. Assuming $X = \xi\partial_x + \phi\partial_y + \eta\partial_u$, from Eq.(10) the Lie point symmetry condition based on Eq.(8) is given by:

$$X^{[2]}\left(u_x u_{xx} + u_{yy}\right)\Big|_{u_x u_{xx}+u_{yy}=0} = 0,$$

which leads to

$$\eta^x u_{xx} + \eta^{xx} u_x + \eta^{yy} = 0. \tag{38}$$

Substitutions of $\eta^x$,$\eta^{xx}$,$\eta^{yy}$ and $u_x u_{xx} = -u_{yy}$ into equation (38) provides the following monomials and coefficients after separating by powers of derivatives of $u$:

$$\begin{aligned}
&\eta_x = \xi_u = \phi_x = \phi_u = \phi_{ux} = \eta_{uu} = \xi_{uu} = \phi_{uu} = \xi_y = \eta_{yy} = 0,\\
&3\xi_x - 2\phi_y - \eta_u = 0, \eta_{xx} - \xi_{yy} = 0, 2\eta_{ux} - \xi_{xx} - 2\xi_{ux} = 0,\\
&\phi_{xx} + 2\xi_{uy} = 0, 2\eta_{uy} - \phi_{yy} = 0, \eta_{uu} - 2\phi_{uy} = 0.
\end{aligned} \tag{39}$$

Solving this system, the results are

$$\begin{aligned}
\xi(x, y) &= c_1 x + c_2,\\
\eta(x, y) &= (3c_1 - 2c_4)u + c_5 y + c_6,\\
\phi(x, y) &= c_4 y + c_3,
\end{aligned} \tag{40}$$

where the $c_i$ $(i = 0, 1, 2, 3, 4, 5, 6)$ are arbitrary coefficients. Therefore, the Lie point symmetries of second-order time independent TG equation (37) are given by:

$$X_1 = x\partial_x + 3u\partial_u, X_2 = \partial_x, \quad X_3 = \partial_y, \quad X_4 = y\partial_y - 2u\partial_u, \quad X_5 = y\partial_u, \quad X_6 = \partial_u, \tag{41}$$

we reduce the TG equation using $X_3$, followed by $X_1$, and after some lengthy calculations, the reduced first-order ODE is:

$$\left(\frac{-3\alpha^2}{\beta^2} + \frac{\alpha^3}{\beta^3}\right)\beta_\alpha + \frac{2\alpha}{\beta} = 0.$$

### 2.9.3 Example 3. Noether point symmetries and conservation laws of ODEs

The Simple Harmonic Oscillator (SHO) equation

$$y'' + y = 0, \tag{42}$$

describes a simple harmonic motion. One can consider an example such as a spring attached to a mass $m$ at a distance $x$. Another example would be a simple pendulum involving an object of a quantity $m$ dangling at the close of a cord of dimension $L$. According to [22] the simple pendulum experiments by Galileo Galilei (1564-1642) in the early 1600s are regarded as the beginning of experimental physics. Below is a Lagrangian of the SHO equation

$$L = \frac{1}{2}(y')^2 - \frac{1}{2}y^2.$$

The determining equation of a Noether symmetry is given by

$$XL + LD_x\xi = D_x f, \tag{43}$$

from Eq.(18), where $X = \xi(x,y)\partial_x + \phi(x,y)\partial_y$ and after making substitutions and simplifying Eq.(43) then:

$$y'\phi_x + (y')^2(\phi_y - \xi_x) - (y')^3\xi_y - y\phi + \frac{1}{2}(y')^2\xi_x + \frac{1}{2}(y')^3\xi_y - \frac{1}{2}y^2\xi_x - \frac{1}{2}y^2y'\xi_y = f_x + y'f_y.$$

Separation by powers of derivatives of $y$ leads to:

$$\xi_y = 0, 2\phi_y - \xi_x = 0,\ 2\phi_x - y^2\xi_y - 2f_y = 0,\ 2y\phi + y^2\xi_x + 2f_x = 0, \tag{44}$$

solving (44) yields the following results.

$$f = \frac{1}{2}\sin(2x)c_2y^2 + \frac{1}{2}\cos(2x)c_3y^2 + \sin(x)c_4y + cos(x)c_5y + c_1,$$

$$\phi = \frac{1}{2}\sin(2x)c_3 y - \frac{1}{2}\cos(2x)c_2 y + \sin(x)c_5 - \cos(x)c_4,$$
$$\xi = -\frac{1}{2}c_3\cos(2x) - \frac{1}{2}c_2\sin(2x) + c_6.$$

Lastly, calculating a conserved vector using formula (19) for the Noether symmetry $X = \partial_x$, that is,( let $c_6 = 1$)

$$\Phi = -\frac{1}{2}(y^2 + (y')^2),$$

and testing this result, the observation is made that, along the solutions of Eq.(42)

$$D_x\Phi = -\frac{1}{2}(2yy' + 2y'y'') = 0.$$

### 2.9.4 Example 4. Noether point symmetries and Conservation laws of PDEs

In physics, waves such as sound waves and water waves may be described by a second order PDE. It was d'Alembert who discovered a one-dimensional wave equation after studying the problem of a vibrating string. Considering the $1+1$ dimensional wave equation:

$$u_{tt} - u_{xx} = 0, \tag{45}$$

with the following Lagrangian :

$$L = \frac{1}{2}((u_t)^2 - (u_x)^2). \tag{46}$$

A similar computation as in the previous example, leads us to the following determining equation:

$$\begin{aligned}
&u_t\phi_t + (u_t)^2(\phi_u - \tau_t) - u_x u_t(\xi_t - \tau_x) - u_x(u_t)^2\xi_u - (u_t)^3\tau_u \\
&-u_x\phi_x + (u_x)^2(\xi_x - \phi_u) + (u_x)^3\xi_u + (u_x)^2u_t\tau_u + \tfrac{1}{2}(u_t)^2\tau_t + \\
&\tfrac{1}{2}(u_t)^3\tau_u + \tfrac{1}{2}(u_t)^2\xi_x + \tfrac{1}{2}(u_t)^2u_x\xi_u - \tfrac{1}{2}(u_x)^2\tau_t - \\
&\tfrac{1}{2}(u_x)^2u_t\tau_u - \tfrac{1}{2}(u_x)^2\xi_x - \tfrac{1}{2}(u_x)^3\xi_u = f_t + u_t f_u + g_x + u_x g_u.
\end{aligned} \tag{47}$$

Separating the powers of derivatives of $u$ gives:

$$\begin{aligned}
&\phi_t - f_u = 0, 2\phi_u - \tau_t + \xi_x = 0, \xi_t - \tau_x = 0,\\
&\phi_x + g_u = 0, \xi_x - 2\phi_u - \tau_t = 0, f_t + g_x = 0,\\
&\xi_u = \tau_u = 0.
\end{aligned} \tag{48}$$

Solving the determining equations in (48) and setting $f = g = 0$ yields the following strict Noether symmetries:

$$\begin{aligned}
&X_1 = \partial_u,\ X_2 = \partial_x,\ X_3 = H(t+x)\partial_x + H(t+x)\partial_t,\\
&X_4 = K(t-x)\partial_t - K(t-x)\partial_x.
\end{aligned} \tag{49}$$

Considering $X_2 = \partial x$, to find the associated conserved quantity using formula (19) yields:

$$\Phi^t = -u_x u_t$$
$$\Phi^x = \frac{1}{2}\Big((u_t)^2 - (u_x)^2\Big) + (u_x)^2.$$

### 2.9.5 Example 5. Flows of the KdV Equation

In this example, flows of the KdV equation using a recursion operator defined by Olver are discussed. The flows of the KdV equation represent infinitely many higher-order symmetries. Considering the KdV equation,

$$u_t = C(u) = u_{xxx} + uu_x. \tag{50}$$

Using the operator

$$\mathcal{R} = D^2 + \tfrac{2}{3}u + \tfrac{1}{3}u_x D^{-1}, \tag{51}$$

where $D = u_x\partial_u + u_{xx}\partial_{u_x} + u_{xxx}\partial_{u_{xx}} + .....$ and $D^{-1}$ is an integration with respect to $x$, is a recursion operator for $C$. We denote $R^j C$ by $C^{(j)}(u)$ which gives the infinite series of symmetries, that is,

$$
\begin{aligned}
&C^{(0)}(u) = \mathcal{R}^0 C = u_{xxx} + uu_x,\\
&C^{(1)}(u) = \mathcal{R}^1 C = \mathcal{R}C^{(0)}(u) = u_{xxxxx} + \tfrac{10}{3}u_x u_{xx} + \tfrac{5}{3}uu_{xxx} + \tfrac{5}{6}u^2 u_x,\\
&C^{(2)}(u) = \mathcal{R}^2 C = \mathcal{R}C^{(1)}(u) = u_{xxxxxxx} + \tfrac{35}{18}u_x^3 + \tfrac{70}{9}uu_x u_{xx} + 7u_x u_{xxxx} + \tfrac{35}{3}u_{xx}u_{xxx}\\
&+\tfrac{35}{18}u^2 u_{xxx} + \tfrac{7}{3}uu_{xxxxx} + \tfrac{35}{54}u^3 u_x, etc.
\end{aligned}
\tag{52}
$$

The infinite series of symmetries verifies the existence of infinitely many higher-order symmetries.

# Chapter 3

# Drift Diffusion Equation

This study contains numerous PDEs that are utilized to model a range of ecological occurrences, see [8] and references therein. Many of these PDEs are dispersive models. In many of the early PDE models of population ecology, one expects organisms to have brownian arbitrary movement, the rate of which is invariant in space and time [8]. The conjecture yields a simplistic diffusion model [23],

$$u_t = B(u_{xx} + u_{yy}), \tag{53}$$

where $u(x, y, t)$ is the density of animals at spatial coordinates $x$, $y$, and time $t$, and $B$ is the diffusion constant that measures the rate of dispersal. This equation has had success with analysing movement of numerous organisms in mark-recapture studies (e.g. [24]). Convection or drift terms are added to (53) when organisms adapt to external provocations or are moved by rainwater or air-stream currents, resulting in the model [25]

$$u_t = B(u_{xx} + u_{yy}) - w_x u_x - w_y u_y, \tag{54}$$

where $w_x$ and $w_y$ are convection speeds.

## 3.1 Lie point symmetries of Drift Diffusion Equation

Considering

$$X = \xi^1\partial_t + \xi^2\partial_x + \xi^3\partial_y + \phi^1\partial u$$

to be the symmetry generator for Eq.(54), The Lie symmetry condition from (8), solves to provide the determining equations:

$$\begin{aligned}
&\xi^1_u = \xi^2_u = \xi^3_u = \xi^1_y = \xi^1_x = 0, \\
&2B\xi^2_y + 2B\xi^3_x = 0,\ -2\xi^2_y - 2\xi^3_x = 0,\ -B\phi^1_{yy} - B\phi^1_{xx} + \phi^1_t = 0, \\
&\phi^1_u - B\phi^1_{uu} - 2\xi^3_y + \xi^1_t = 0,\ \phi^1_u - B\phi^1_{uu} - 2\xi^2_x + \xi^1_t = 0, \\
&2B\xi^3_y - B\xi^1_t = 0,\ 2B\xi^2_x - B\xi^1_t = 0, \\
&B\xi^2_{yy} + 2\phi^1_x - 2B\phi^1_{xu} + B\xi^2_{xx} - \xi^2_t = 0, \\
&2\phi^1_y - 2B\phi^1_{yu} + B\xi^3_{yy} + B\xi^3_{xx} - \xi^3_t = 0.
\end{aligned} \tag{55}$$

The lie point symmetries of (54) are obtained by solving the above system, namely, a 10-dimensional algebra :

$$\begin{aligned}
&X_1 = \partial_y,\ X_2 = \partial_x,\ X_3 = \partial_t,\ X_4 = u\partial_u, \\
&X_5 = t\partial_x + \frac{1}{2}\frac{u\Big(tw_y - x\Big)}{B}\partial_u, X_6 = t\partial_y + \frac{1}{2}\frac{u\Big(tw_x - y\Big)}{B}\partial_u, \\
&X_7 = y\partial_x - x\partial_y - \frac{1}{2}\frac{u\Big(w_x x - w_y y\Big)}{B}\partial_u, \\
&X_8 = \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y + t\partial_t - \frac{1}{4}\frac{u\Big(\Big((w_x)^2 + (w_y)^2\Big)t - w_x y - w_y x\Big)}{B}\partial_u, \\
&X_9 = \frac{1}{2}tx\partial_x + \frac{1}{2}ty\partial_y + \frac{1}{2}t^2\partial_t \\
&-\frac{1}{8}\frac{u\Big(t^2(w_x)^2 + t^2(w_y)^2 - 2tw_x y - 2tw_y x + 4Bt + x^2 + y^2\Big)}{B}\partial_u, \\
&X_{10} = F(t,x,y)\partial_u,
\end{aligned} \tag{56}$$

where $F(t,x,y)$ a solution to the original equation, giving rise to an infinite-dimensional symmetry.

## 3.2 Lie Reduction of Drift Diffusion Equation

Suppose the symmetries $\{X_3 + aX_4, X_2\}$ are used, then the solution to Eq.(54) is

$$u(t,x,y) = \mathrm{e}^{at+bx} F_1(y), \tag{57}$$

where $F_1(y)$ satisfies

$$a + b(-bB + w_y)F_1 + w_x F_1' - BF'' = 0. \tag{58}$$

Solving the second-order ordinary differential equation (58), results in:

$$\begin{aligned} F_1(y) = \; & \tfrac{a}{b^2 B - b w_y} + \mathrm{e}^{\frac{\left(w_x - \sqrt{-4b^2B^2 + w_x^2 + 4bBw_y}\right)y}{2B}} C_1 \\ & + \mathrm{e}^{\frac{\left(w_x + \sqrt{-4b^2B^2 + w_x^2 + 4bBw_y}\right)y}{2B}} C_2. \end{aligned} \tag{59}$$

Figure 3.1 represents the population density $u(t,x,y)$ over space $x$ and $y$ when time $t$ is fixed at 2.

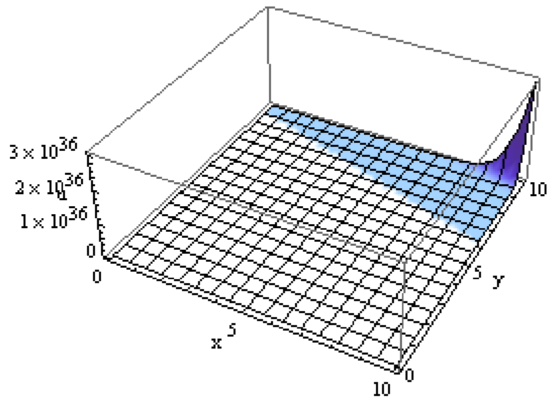


Figure 3.1: 3D Plot of equation(57), when $t = 2$.

## 3.3 Discussion and Conclusion

In this Chapter, Lie point symmetries of drift diffusion equation were presented. Group symmetries were used to decrease the PDE to second-order ordinary DE and finally to obtain exact solutions. The solution of Eq.(57) was plotted in a 3D plot using Mathematica, where $t$ was fixed at 2. The model shown above has been generated by the interaction between the organisms and external stimuli, such as, wind or water currents. It relies on the size of the environment or area in which the organisms live and also on the particulars of the diffusion rates. A point of particular interest is the nature of the exact solutions, they are exponential functions which leads to the shape of the surface plane. The population density increases when space is large and this could mean the animals survive when there is no competition for space.

# Chapter 4

# Symmetries, Multipliers and Conservation Laws of a Biased Diffusion Equation

Some PDEs can also model interactions between conspecifics. For instance, if animals are either repelled from each other or attracted to each other. In such cases, the simple diffusion equation can be substituted by a biased arbitrary movement equation [8, 26]:

$$u_t = Bu_{xx} + (kuu_x)_x, \tag{60}$$

where $u$ is the density of population, and $k$ is a measure of the tendency to travel away from conspecifics ($k > 0$) or to travel near conspecifics ($k < 0$).

## 4.1 Lie point symmetries of Biased Diffusion Equation

Suppose that

$$X = \xi^1 \partial_t + \xi^2 \partial_x + \phi^1 \partial_u$$

is the symmetry for Eq.(60), The Lie symmetry criterion from (8) yields the determining equations

$$\begin{aligned}
&\xi^1_u = \xi^2_u = \xi^1_x = 0,\\
&(-B - ku)\phi^1_{xx} + \phi^1_t = 0,\ -k\phi^1_u + (-B - ku)\phi^1_{uu} + 2k\xi^2_x - k\xi^1_t = 0,\\
&-k\phi^1 - 2(-B - ku)\xi^2_x + (-B - ku)\xi^1_t = 0,\\
&-2k\phi^1_x + 2(-B - ku)\phi^1_{xu} + (B + ku)\xi^2_{xx} - \xi^2_t = 0.
\end{aligned} \tag{61}$$

The above determining equations leads to the Lie symmetries

$$\begin{aligned}
&X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = \tfrac{1}{2}x\partial_x + t\partial_t,\\
&X_4 = \tfrac{1}{2}kx\partial_x + \Big(ku + B\Big)\partial_u,
\end{aligned} \tag{62}$$

| $[X_i, X_j]$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|
| $X_1$ | 0 | 0 | $X_1$ | 0 |
| $X_2$ | 0 | 0 | $\frac{X_2}{2}$ | 0 |
| $X_3$ | $-X_1$ | $\frac{-X_2}{2}$ | 0 | $\frac{-kxX_2}{4}$ |
| $X_4$ | 0 | 0 | $\frac{kxX_2}{4}$ | 0 |

Table 4.1: Lie Brackets

## 4.2 Higher-order symmetries of Biased Diffusion Equation

If a vector field with components $\xi^j, \eta^\delta$ rely on the following derivatives

$$\begin{aligned}(x^*)^j &= x^j + \epsilon\xi^j(x, u, u_{(1)}, \ldots, u_{(q)}) + \mathcal{O}(\epsilon^2), j = 1, \ldots, m,\\ (x^*)^\delta &= u^\delta + \epsilon\eta^\delta(x, u, u_{(1)}, \ldots, u_{(q)}) + \mathcal{O}(\epsilon^2), \delta = 1, \ldots, \delta\end{aligned} \tag{63}$$

then the conforming symmetries are said to be higher-order symmetries. Higher-order symmetries are calculated the same way as point symmetries. Without loss of generality, the symmetry generators for higher-order symmetries are usually sought in the evolutionary(characteristic) form

$$\bar{X} = \zeta^\delta(x, u, \partial_u, \ldots, \partial_u^q)\partial_{u^\delta}. \tag{64}$$

Let

$$X = \phi^1(x, t, u, u_x, u_{xx})\partial_u \tag{65}$$

be the higher-order symmetry generator of Eq.(60). The higher-order symmetries of Eq.(60) are given by the determining equations ( let $B = k = 1$ for simplicity)

$$\begin{aligned}&\phi^1_{u_{xx},u_{xx}} + \phi^1_{u_{xx},u_{xx}u} = 0,\\ &2\phi^1_{u_{xx},u_{u_x}}u_{xx} + 2\phi^1_{u_{xx},u}uu_x + 2\phi^1_{u_{xx},u_x}uu_{xx} - 2\phi^1_{u_{xx}}u_x + 2\phi^1_{x,u_{xx}}u\\ &+2\phi^1_{x,uxx} + 2\phi^1_{u_{xx},u}u_x = 0\\ &2\phi^1_{u_x,u}uu_xu_{xx} - \phi^1_{u_x}u_xu_{xx} + \phi^1_{u,u}u_x^2 - 3\phi^1_{u_{xx}}u_{xx}^2\\ &+\phi^1_{x,x} + \phi^1u_{xx} + 2\phi^1_{x,u}uu_x + \phi^1_{u,u}uu_x^2 + \phi^1_{u_x,u_x}uu_{xx}^2\\ &+2\phi^1_{u_x,u}u_xu_{xx} + 2\phi^1_xu_x + 2\phi^1_{x,u_x}uu_{xx} + \phi^1_{x,x}u + \phi^1_uu_x^2\\ &-\phi^1_t + 2\phi^1_{x,u}u_x + \phi^1_{u_x,u_x}u_{xx}^2 + 2\phi^1_{x,u_x}u_{xx} = 0.\end{aligned} \tag{66}$$

Solving the above determining equations gives the following higher-order symmetries:

$$\begin{aligned}&X_1 = u_x\partial_u,\ X_2 = (u_{xx} - 2u - 2)\partial_u,\ X_3 = (uu_{xx} + u_x^2 + u_{xx})\partial_u,\\ &X_4 = (uu_{xx}t + u_x^2t + u + 1)\partial_u,\end{aligned} \tag{67}$$

## 4.3 Lie Reduction of Biased Diffusion Equation

The symmetry $X_4 = \frac{1}{2}kx\partial_x + \left(ku + B\right)\partial_u$ from Eq.(62) yields the following exact solution form:

$$u(t,x) = \frac{-B}{k} + x^2 G(t), \tag{68}$$

where $G(t)$ satisfies

$$-6k(G(t))^2 + G' = 0. \tag{69}$$

Solving the first-order ordinary differential equation (69) yields:

$$G(t) = \frac{1}{-6kt - C_1}. \tag{70}$$

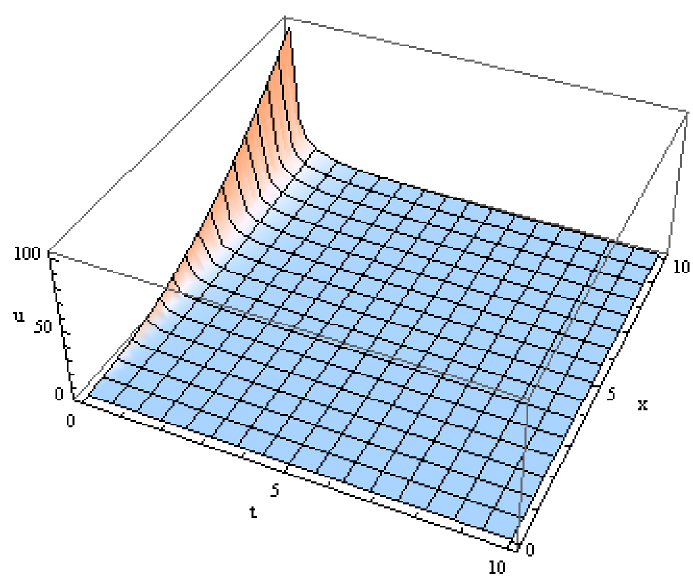


Figure 4.1: 3D Plot of equation (68)

Figure 4.1 demonstrates the population density $u(t,x)$ over time $t$ and space $x$.

## 4.4 Multipliers and Conservation Laws

Information about conservation laws is important to any symmetry study of a PDE. These conservation laws are of paramount importance and it is well known that they show a vital part in mathematical physics as they define critical physical properties of the modelled process. Conservation laws are also applicable to eliminating numerical errors of PDEs [27]. The multiplier approach is a famous technique used in the derivation of conservation laws. Here, conservation laws are obtained by applying the multiplier approach. To apply this approach, consider a multiplier containing the dependent variable, the independent variables and derivatives of dependent variables up to some fixed order, i.e., let $\Lambda = \Lambda(t, x, u)$ and $u_{(1),\ldots,u_{(q)}}$ of Eq.(60) have the property that

$$\Lambda\left[u_t - Bu_{xx} - (kuu_x)_x\right] = D_x T^x + D_t T^t \tag{71}$$

for all functions $u(t, x)$, where the total derivative operative is defined as in Eq.(12). The right-hand side of this expression is a divergence expression and the conserved quantity $T = (T^x, T^t)$ has components $T^j$ $(j = x, t)$. The determining equations for the multipliers $\Lambda$ are obtained from the expressions

$$\frac{\delta}{\delta u}[\Lambda(u_t - Bu_{xx} - (kuu_x)_x)] = 0, \tag{72}$$

where $\dfrac{\delta}{\delta u}$ are the Euler-Lagrange operators given in (9), which annihilate divergence expressions. Solving Eq.(72) yields,

$$u_x^2(ku+B)\Lambda_{uu} + 2u_x(ku+B)\Lambda_{ux} + (ku+B)\Lambda_{xx} + \Big(ku_x^2 + 2u_{xx}(ku+B)\Big)\Lambda_u + \Lambda_t$$
$$= 0. \tag{73}$$

Solving for $\Lambda$, the following results are found:

$$\Lambda = C_1 x + C_2 \tag{74}$$

The two multipliers are

$$\Lambda_1 = 1$$

$$\Lambda_2 = x.$$

As mentioned in section 2.6, conserved vectors can be calculated. For $\Lambda_1$, the components of the conserved vector are :

$$T_1^t = -u$$

and

$$T_1^x = (B+ku)u_x$$

For $\Lambda_2$, the components are:

$$T_2^t = -xu$$

and

$$T_2^x = -\frac{1}{2}ku^2 + Bxu_x + u(kxu_x - B).$$

We now check, for the case $\Lambda_1 = 1$, that

$$D_t T_1^t + D_x T_1^x = 0.$$

The left hand side gives:

$$\begin{aligned} & D_t(-u) + D_x(Bu_x + kuu_x) \\ = \; & (\partial_t + u_t\partial_u)(-u) + (\partial_x + u_x\partial_u + u_{xx}\partial_{u_x})(Bu_x + kuu_x) \\ = \; & 2(-u_t + Bu_{xx} + kuu_{xx} + ku_x^2) = 0, \end{aligned} \tag{75}$$

along the solutions of Eq.(60). Similarly this is also true for the case $\Lambda_2 = x$.

## 4.5 Discussion and Conclusion

In this Chapter, Lie point symmetries of a biased diffusion equation were obtained and the multiplier approach was used to derive conservation laws. Analytical solutions of the PDE were found by first reducing to a nonlinear ordinary DE using the Lie reduction algorithm. The population density in Eq.(68) has been plotted using Mathematica. The model illustrated the interaction of the animals and $k$ which is the tendency to move away or towards each other, that is, the animals are repelled from one another or attracted to one another. The model also depends on the patch size in which the animals live and the diffusion rates.

# Chapter 5

# A symmetry classification of the Diffusive KiSS Model

Reaction-diffusion problems have been studied extensively. These models have appeared in theoretical work by engineers, mathematicians, biologists, and population ecologists. In this study, the linear Kierstead-Slobodkin [28] and Skellam [7] problem, commonly known as the KiSS model is explored, and the study is extended to the non-linear model. Such an equation is known as the basic critical patch equation.

Determination of critical patch size to guarantee the sustenance of the population is an important study. The rate at which animals exit an area, the population dynamics in the patch, the area geometry and the degree to which the area outside the patch is disastrous, are all the factors which the critical patch magnitude depends on.

A generalized KiSS model is given by:

$$u_t = B\Bigg(u_{xx} + u_{yy}\Bigg) + rF(u)^{\rho}, \tag{76}$$

here, $u(t,x,y)$ is the species density, $B > 0$ is the diffusion coefficient, $\rho > 0$ is the critical exponent parameter that determines whether the model is linear ($\rho = 1$) or nonlinear ($\rho \geq 2$), and $r$ is the growth rate.

Several special cases that produce interesting and extra symmetries are investigated. Case A. is done in some detail. It is well known that whenever Eq.(76) is a linear model, it will always admit the linear symmetry, $X_u = u\partial_u$ and the infinite-dinensional abelian sub-algebra of solutions, $X_\infty = F(t,x,y)\partial_u$, where function $F(t,x,y)$ , is a solution of the original Eq.(76) [29].

## 5.1 Classical symmetries of KiSS Model

First, considering that

$$X = \xi^1\partial t + \xi^2\partial x + \xi^3\partial y + \phi^1\partial u$$

is the symmetry generator for Eq.(76).

### 5.1.1 Case A. $F(u) = u$, $\rho = 1$.

In this case, from Eq.(8) the Lie point symmetries of Eq.(76) are given by the determining equations:

$$\begin{aligned}
&\xi^1_u = \xi^2_u = \xi^3_u = \phi^1_{uu} = \xi^1_y = \xi^1_x = 0,\\
&2B\xi^2_y + 2B\xi^3_x = 0,\ 2B\xi^3_y - B\xi^1_t = 0,\ 2B\xi^2_x - B\xi^1_t = 0,\\
&-nru^{-1+n}\phi^1 + ru^n\phi^1_u - B\phi^1_{yy} - B\phi^1_{xx} + \phi^1_t - ru^n\xi^1_t = 0,\\
&B\xi^2_{yy} - 2B\phi^1_{xu} + B\xi^2_{xx} - \xi^2_t = 0,\ -2B\phi^1_{yu} + B\xi^3_{yy} + B\xi^3_{xx} - \xi^3_t = 0.
\end{aligned} \tag{77}$$

The Lie point symmetries are

$$\begin{aligned}
&X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = \partial_y,\ X_4 = u\partial_u,\ X_5 = y\partial_x - x\partial_y,\\
&X_6 = t\partial_x - \tfrac{1}{2}\tfrac{ux}{B}\partial_u,\ X_7 = t\partial_y - \tfrac{1}{2}\tfrac{uy}{B}\partial_u,\ X_8 = t\partial_t + \tfrac{1}{2}x\partial_x + \tfrac{1}{2}y\partial_y + urt\partial_u,\\
&X_9 = \tfrac{1}{2}t^2\partial_t + \tfrac{1}{2}xt\partial_x + \tfrac{1}{2}yt\partial_y + \tfrac{1}{8}\frac{u\left(Brt^24 - 4Bt - x^2 - y^2\right)}{B}\partial_u.
\end{aligned} \tag{78}$$

### 5.1.2 Case B. $F(u) = u$, $\rho = n(n \neq 1)$.

The Lie point symmetries are

$$X_1, X_2, X_3, X_5, t\partial_t + \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y - \frac{u}{n-1}\partial_u$$

### 5.1.3 Case C.$\rho = 1$.

The Lie point symmetries are

$$X_1,\ X_2$$

### 5.1.4 Case D. $F(u) = \mathrm{e}^{bu}$, $\rho = 1$.

The Lie point symmetries are ($b$ is a non-zero constant)

$$X_1,\, X_2,\, t\partial_t + \frac{x}{2}\partial_x - \frac{1}{b}\partial_u$$

### 5.1.5 Case E. $F(u) = 0$.

The Lie point symmetries are

$$X_1,\, X_2,\, X_4,\, t\partial_t + \frac{x}{2}\partial_x - 2Bt\partial_x + xu\partial_u,\, \frac{1}{2}t^2\partial_t + \frac{1}{2}tx\partial_x + \left(\frac{t}{4} - \frac{x^2}{8B}\right)u\partial_u$$

This is equivalent to Eq.(53)

## 5.2 Nonclassical symmetries of KiSS Model

After studying the diffusion equation, Cole and Bluman [18] presented the nonclassical method. The outcomes of both nonclassical and classical procedures are equivalent most of the time. Therefore, this means that the nonclassical procedure may fail to yield new solutions.

Considering Eq.(76) with $F(u) = u$ and $\rho = 1$, and with

$$X = \theta^1\partial_t + \theta^2\partial_x + \theta^3\partial_y + \eta^1\partial_u.$$

Without a loss of generality, letting $\theta^1 = 1$. The $\theta^i$ are functions of $x, y, t$ and $u$.

The Lie invariance condition is

$$X^{[2]}[u_t - B\left(u_{xx} - u_{yy}\right) - rF(u)^{\rho}] \mid_{u_t - B\left(u_{xx}-u_{yy}\right) - rF(u)^{\rho}, ISC} = 0. \tag{79}$$

Here the ISC is given by,

$$\begin{aligned} &\phi^i(x^i, u)\frac{\partial u}{\partial x^i} = \eta(x^i, u), \\ &\phi^1 u_t + \phi^2 u_x + \phi^3 u_y = \eta, \\ &\phi^1 = 1 \Rightarrow u_t = \eta - \phi^2 u_x - \phi^3 u_y. \end{aligned} \tag{80}$$

The determining equations for the nonclassical symmetries are given below:

$$\begin{aligned} &uB^2\eta^1_{uu} = \theta^2_u = \theta^3_u = 0, \\ &-2B^2u\theta^2_y - 2u\theta^3_x B^2 = 0,\ 2uB^2\theta^2_x - 2B^2u\theta^3_y = 0, \\ &B\theta^3_t u + B\theta^3_u \eta^1 u + 2Bu\theta^2_x\theta^3 + 2B^2u\eta^1_{yu} - u\theta^3_{xx}B^2 - B^2u\theta^3_{yy} = 0, \\ &2B\theta^2\theta^2_x u + B\theta^2_t u + B\theta^2_u\eta^1 u + 2u\eta^1_{xu}B^2 - u\theta^2_{xx}B^2 - B^2u\theta^2_{yy} = 0, \\ &-Bu\eta^1_u u^n r - B\eta^1_t u - 2Bu\theta^2_x\eta^1 + u\eta^1_{xx}B^2 + B^2u\eta^1_{yy} + 2Bu\theta^2_x u^n r + Bru^n n\eta^1 = 0, \end{aligned} \tag{81}$$

The general solution or symmetry generator for the KiSS model is:

$$\begin{aligned} \eta^1(u,t,x,y) &= \tfrac{1}{4}\Bigg( -4BC_{11}C_{12}e^{rt}\left(C_2^2 + 2C_2t + t^2 - C_1\right)\left(C_{14} + C_{15}\right) \\ &\quad +4u\left(-C_{10}^2 - 2C_{10}t - t^2 + C_9\right)\Bigg(Brt^2 + B\left(2C_2r - 1\right)t \\ &\quad +BC_8 - \tfrac{x^2}{4} + \tfrac{C_6x}{2} - \tfrac{y(y-2C_4)}{4}\Bigg)\Bigg) \\ &\quad - \frac{4BC_{11}C_{13}e^{rt}\left(C_2^2+2C_2t+t^2-C_1\right)\left(C_{14}+C_{15}\right)}{\left(\left(C_2^2+2C_2t+t^2-C_1\right)B\left(-C_{10}^2-2C_{10}t-t^2+C_9\right)\right)}, \\ \theta^2(u,t,x,y) &= \frac{\left(x-C_6\right)t+xC_2-C_3y-C_7}{C_2^2+2C_2t+t^2-C_1}, \\ \theta^3(u,t,x,y) &= \frac{\left(y-C_4\right)t+C_3x+yC_2-C_5}{C_2^2+2C_2t+t^2-C_1}. \end{aligned} \tag{82}$$

Based on this solution, no nonclassical symmetries are admitted.

## 5.3 Lie Reduction of KiSS Model

### 5.3.1 Case A. $F(u) = u$, $\rho = 1$.

Reducing by $X_6$ and $X_7$ leads to the exact solution

$$u(t, x, y) = \mathrm{e}^{\frac{-x^2}{4Bt} - \frac{y^2}{4Bt}} G(t), \tag{83}$$

where $G(t)$ satisfies

$$(1 - rt)G + tG' = 0. \tag{84}$$

Solving Eq.(84), then:

$$G(t) = \frac{\mathrm{e}^{rt} C_1}{t}. \tag{85}$$

Figure 5.1 demonstrate the population density $u(t, x, y)$ when time $t$ is fixed at 2.

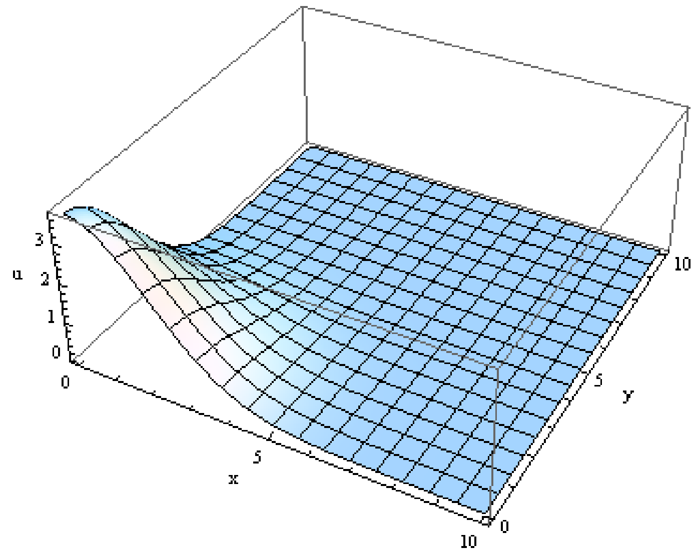


Figure 5.1: 3D Plot of equation (83), when $t = 2$

## 5.4 Flows of the diffusive KiSS model when $F(u) = u$ and $B = r = \rho = 1$

In this section an effort is made in determining an infinite system of higher-order symmetries by creating and analysing a recursion operator. The technique makes available a mechanism for producing infinite hierarchies of higher-order symmetries in a single step but it cannot deliver a complete grouping of all possible higher-order symmetries without further analysis. The deduction of the form of the recursion operator involves a certain amount of inspired conjecture [19]. The recursion operator is significant in verifying the presence of infinitely many higher-order symmetries. Fokas [32] studied higher-order symmetries as the basic feature of completely integrable equations, and provided the existence of an infinite number of higher-order symmetries as a definition of complete integrability. In the diffusive KiSS model, the PDE is studied in $(1+1)$ dimensions where $u$ only depends on $t$ and $x$. The flows of the diffusive KiSS model are determined.

The $(1+1)$ dimensions diffusive KiSS model,

$$u_t = S(u) = u_{xx} + u. \tag{86}$$

Using the operator

$$\mathcal{R} = D, \tag{87}$$

where $D = u_x \partial_u + u_{xx} \partial_{u_x} + u_{xxx} \partial_{u_{xx}} + .....$ , is a recursion operator for $S$. The infinite series of symmetries are $S^{(j)}(u) = \mathcal{R}^j S$, that is

$$\begin{aligned}
&S^{(0)}(u) = \mathcal{R}^0 S = u_{xx} + u, \\
&S^{(1)}(u) = \mathcal{R}^1 S = \mathcal{R} S^{(0)}(u) = u_{xxx} + u_x, \\
&S^{(2)}(u) = \mathcal{R}^2 S = \mathcal{R} S^{(1)}(u) = u_{xxxx} + u_{xx}, \\
&S^{(3)}(u) = \mathcal{R}^3 S = \mathcal{R} S^{(2)}(u) = u_{xxxxx} + u_{xxx}, etc.
\end{aligned} \tag{88}$$

## 5.5 Discussion and Conclusion

In this Chapter, both classical and nonclassical symmetries of the KiSS model were explored. Lie reduction algorithm was performed to obtain exact solutions. A corresponding graph for Eq.(83) was provided. Examining the surface plane it is evident that the exponential functions from the exact solutions give the resulting shape. Finally, it has been shown that the KiSS model has infinitely many symmetries using the recursion operator in a special case.

# Conclusion

In this study, symmetries, exact solutions and their corresponding 3D plots, multipliers and conservation laws have been provided. An analysis of drift diffusion equation, a biased diffusion equation and the KiSS model were performed. The drift diffusion model was reduced to a second-order differential equation, which was solved and a corresponding graph for the population was plotted using Mathematica. Four Lie point symmetries and corresponding Lie brackets for the biased diffusion equation were found, then the multiplier approach was applied to find additional symmetries which were then used to construct the conservation laws. The biased diffusion equation was reduced to a nonlinear ordinary differential equation. After finding the solution of the first-order differential equation a graph of the population was plotted. Classical and nonclassical methods were applied to the KiSS model followed by a reduction. This work provides a survey of the use of PDEs in population ecology that can model the interactions of animals and ecological processes. The results herein are new and the symmetry analysis of these PDEs have not been done before.

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