Studies in the field theory - quantum gravity correspondence

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the requirements for the degree of Master of Science.

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

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of February 2005

Abstract

The giant graviton part of the AdS/CFT dictionary is expanded from consideration of $\mathcal{N} = 4$ super Yang-Mills theory with U(N) gauge group to the case of an SU(N) gauge group. Candidate duals to giant gravitons for the U(N) case were found by Corley, Jevicki and Ramgoolam [4] to be Schur polynomials of the Lie algebra. In this dissertation, computational generalisation of the U(N) result is achieved, and a set of linearly independent operators in one-to-one correspondence with the half-BPS representations of the SU(N) gauge theory given. These tools allow the first elucidation of bulk and boundary degrees of freedom via the dual field theory, exploiting the usefulness of giant gravitons as probes of the geometry.

Acknowledgements

The various obstacles I have encountered during the course of this degree have tested both my capabilities and my resolve. Many people have helped me to complete it in spite of this. I am extremely grateful to my mother for the time and advice she has given me, and to friends whose belief in me has substituted for my own at times.

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Contents

Chapter 1

Introduction

1.1 An overview

The AdS/CFT correspondence proposed by Maldacena [1, 2, 3] allows insights about string theory to be gained in the dual field theory and vice versa. In particular, it includes a duality between Type IIB string theory on $AdS_5 \times S^5$ and $3 + 1$ dimensional $\mathcal{N} = 4$ super Yang-Mills theory. Machinery developed by Corley, Jevicki and Ramgoolam [4] to compute exactly correlators of a class of operators in one-to-one correspondence with half-BPS representations of this SYM theory with U(N) (or SU(N)) gauge group has allowed investigation of duals to giant gravitons in the conformal field theory. The aim of this work was to extend these results to the case of an SU(N) gauge group, and in so doing to examine the roles of bulk and boundary degrees of freedom in the dual. This chapter contains a brief review of the background to the AdS/CFT correspondence as it is realised here and a description of the problem treated in this dissertation and why it is of interest. Overall references for the topics discussed in this chapter are [1, 2, 3, 5, 6, 7].

1.2 String theory

Often described as the two pillars of twentieth century physics, quantum mechanics and Einstein's general theory of relativity have provided us with beautiful theoretical descriptions of the physics at both ends of the spectrum of distances, many orders of magnitude on either side of the human scale. Quantum mechanics has successfully been combined with the strong, weak and electromagnetic forces, but never with the force of gravity in a unified theoretical framework. The quest for such unification has engaged the greatest modern minds, and towards the end of the twentieth century became the "Holy Grail" of modern physics. In this quest, many believe that string theory furnishes the brightest guiding light.

Of the forces listed above, electromagnetism and the weak force have been successfully unified in the electroweak force, and the standard model encompasses the strong force in the form of quantum chromodynamics. Quantum chromodynamics, or QCD, arose from the integration of quantum mechanics with special relativity in quantum field theory or QFT. The problem with unifying these theories with gravity is that general relativity and quantum mechanics prove to be mutually incompatible in limiting situations where they both apply, such as black holes and gravitational interactions at the quantum level. The hope is that there exists an underlying theory reducing to general relativity and quantum mechanics at their respective limits, and resolving the existing difficulties when they coincide. With various degrees of magic, string theory seems to offer such an underlying theory. We shall now review its history and development, and the reasons it provides such a beacon of hope.

In the late 1960s it was observed that the theory being used to explain particle spectra was equivalent to a theory of vibrating strings. Each particle could be viewed as a different oscillation mode of a string, so that the possible excitations of the string resulted in a spectrum of energies or masses corresponding to different particles. Thinking of particles as objects extended in one dimension instead of geometrical points has some unexpected consequences. Supersymmetric theories constructed on this premise need 10 or 11 dimensions to be consistent. There are various theories to explain how some of these dimensions might be 'curled' up to leave the 3+1 dimensions of our world. More significantly, it is not possible to eliminate the appearance of massless spin 2 bosons in such a theory. Gravity is the only consistent mode of interaction for these bosons, called gravitons. They appear spontaneously in string theory. It is this fact - that gravity is not put into string theory by hand - that makes string theory such a promising approach to quantum gravity.

1.3 Dualities in string theory

Five distinct consistent string theories were found, called Type I, Type IIA, Type IIB, SO(32) heterotic, and $E_8 \times E_8$ heterotic. Each of these requires spacetime supersymmetry in $9 + 1$ dimensions to be consistent. Each can be formulated in terms of a given background. However, different backgrounds can be shown to be equivalent. The five string theories possess duality symmetries which allow transformations between equivalent backgrounds. Such duality symmetries were found to relate all five string theories and supergravity, a low energy limit of string theory.

These duality transformations led to the conjectured master M-theory of which each of the five original string theories would be a limiting case. The duality symmetries thus provide us with a unified string theoretical framework. Further, they allow in some cases investigation of a strong coupling regime in a suitable dual where the corresponding regime is not strongly coupled. Such dualities are termed strong-weak coupling dualities.

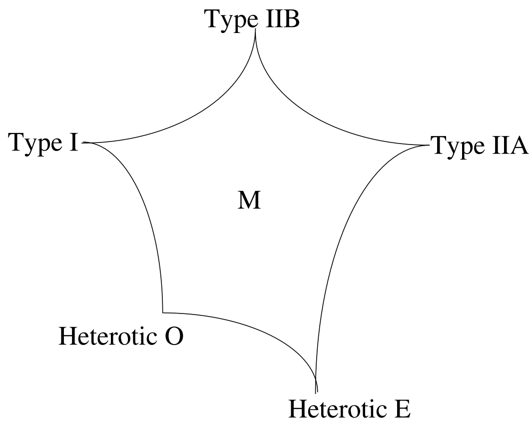


Figure 1.1: *A version of the usual string theory duality map, showing how each string theory is a limit of a conjectural master or M-theory.*

Among the string dualities is one relating open and closed strings. Open string theories reduce to gauge field theories at low energies, while closed string theories are described by theories of gravity. Thus in particular the open/closed string duality leads to a field theory-quantum gravity correspondence, as will be seen below.

1.4 The Anti-de Sitter/Conformal Field Theory correspondence

1.4.1 General idea

The AdS/CFT correspondence proposed by Maldacena [1] is a conjectured duality or equivalence between string theory on a given space, and a field theory on the boundary of that space. Before discussing in more detail what these concepts mean, it is worth summarising the most important and striking aspects of this correspondence.

Firstly, it is a conjectured equivalence between string theory – a theory of gravity – and a field theory in which gravity does not appear. The equivalence includes a map between gravity states and operators in the field theory. It therefore allows calculations in one theory to be done equivalently in the dual, via a 'dictionary' relating quantities in the different theories. Secondly, it is a strong-weak coupling correspondence. In the 't Hooft large N limit the correspondence relates the weakly coupled regime of the field theory to the strongly coupled one of the string theory, and vice versa [5]. This means that calculations that cannot be done perturbatively on one side of the duality, *can* be treated perturbatively on the other. The significance of this is clear, especially considering how little we know of string theory at strong coupling. Thirdly, the correspondence is a holographic one, since it relates a theory on a certain space to another on the boundary of that space. The information for the string theory dynamics in ten dimensions is encoded in fewer dimensions. This suggests that the bulk dynamics are somehow a holographic projection from the dynamics on the boundary. It is for this reason that the AdS/CFT correspondence is referred to as being holographic. In fact, the AdS/CFT correspondence is a concrete realisation of the suggestion in [8, 9] that theories of quantum gravity should be holographic.

1.4.2 Development

A revealing limit

QCD is a field theory with gauge group SU(3). It is strongly coupled at low energies, so a perturbative approach is not valid. 't Hooft [10] suggested that a theory with gauge group SU(N) might simplify in the large N limit, with the hope that an expansion in $\frac{1}{N}$ could facilitate QCD calculations. The large N expansion of the field theory appeared to be equivalent to a perturbative string theory expansion, with the string coupling constant identified with $\frac{1}{N}$. This was a strong indicator of the link between field and string theories, and further suggested that this link would be most obvious in the large N limit.

The Maldacena conjecture

Maldacena was led to his conjecture [1] by considering a configuration of N parallel D3 branes in Type IIB string theory. A D3 brane is a (3+1)-dimensional hyperplane in (9+1)-dimensional spacetime, and provides a non-perturbative solution of the theory.

There are two possible kinds of excitation of this configuration: open strings ending on the branes, which are excitations of the branes themselves, and closed strings in the bulk (or the space not occupied by the branes).

At low enough energies we can consider only massless excitations. The closed string massless states reduce to those of type IIB supergravity (the low energy limit of string theory). On the other hand the theory of the open string massless states reduces to $\mathcal{N} = 4$ super Yang-Mills U(N) gauge theory. Essentially the

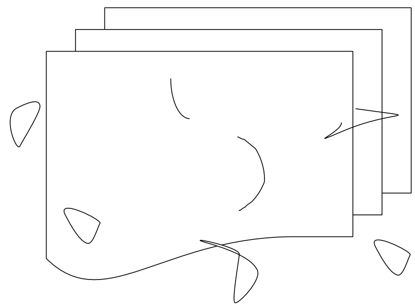


Figure 1.2: *An artist's (if not an artistic) impression of open and closed string excitations. Open strings can end on a single brane, or stretch between two. When the branes are coincident, all these possibilities correspond to massless excitations.*

dynamics on the brane decouple from the dynamics in the bulk. In this limit we are left with a free theory of gravity in the bulk and a field theory describing the dynamics on the brane.

Alternatively we can consider D-branes as massive charged objects, or sources of closed strings. This entails considering the geometry of the D-branes. In the low energy limit of this description we again have two decoupled theories: one is supergravity on flat Minkowski space, far from the branes, while near the branes the dynamics are described by string theory on $AdS_5 \times S^5$, which is the near horizon geometry of the branes. Note that the near-horizon string theory is complete, including massive states whose energies are redshifted by the gravitational potential and so seem very low to an outside observer.

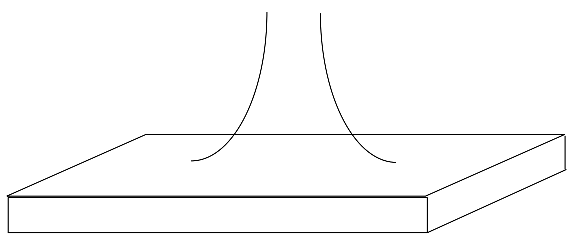


Figure 1.3: *Far from the brane the geometry is flat Minkowski space. The near-horizon geometry in the throat limit and near the branes is Anti-de Sitter.*

From both points of view of the D-branes we obtain two decoupled theories in the low energy limit. In both of these, the description far from the branes is that of supergravity. The descriptions near the branes can be identified: leading us to the Maldacena conjecture that string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ (3+1)-dimensional super Yang-Mills theory are equivalent. The coupling constant relation is $g_{YM}^2 = 4\pi g_s$, where g_{YM} is the Yang-Mills coupling constant and g_S is the string theory coupling constant. The duality is a strong/weak duality in the 't Hooft coupling $\lambda = g_{YM}^2 N = g_S N$. The 't Hooft limit consists of holding this constant and taking N to infinity. In the large N limit, field theory is weakly coupled when λ is small while curvature corrections in supergravity are small when λ is large.

1.4.3 Type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ $d = 3+1$ super Yang-Mills theory

The realisation of the correspondence we are interested in is that between Type IIB string theory on an $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ $d = 3+1$ super Yang-Mills theory. The jargon in this sentence demands some unravelling. We note immediately the dimensional relation between the five-dimensional AdS space and the 4-dimensional field theory on its boundary. The full string theory background is a product space of AdS_5 with the compact five-dimensional spherical manifold S^5. This compactification corresponds to the maximally supersymmetric case, the importance of which observation will become clear shortly. AdS_5 is a five dimensional Anti-de Sitter space – a negatively curved maximally supersymmetric solution of the Einstein equations. Anti-de Sitter space can be visualised as being saddle-shaped, and has a boundary at spatial infinity. It is on this 4-dimensional boundary that dynamics are described by the superconformal Yang-Mills theory: a field theory with supersymmetric conformal symmetry, or superconformal symmetry. The isometry group of AdS_5 can be identified with the conformal symmetry of the boundary. A closer look reveals that the symmetries of the two theories match exactly, as they must for this identification to hold.

The conformal group is SO(2,4) – it is isomorphic to SU(2,2) and generated by translations, Lorentz transformations (these two sets of transformations together make up the Poincaré group), dilations D (essentially scaling transformations) and special conformal transformations K^μ.[1] This is extended by Poincaré supersymmetries which add the supercharges Q^a_α and $\bar{Q}_{\dot{\alpha}a} = (Q^a_\alpha)^\dagger$, $a = 1, 2$ and $\alpha = 1, ..., \mathcal{N}$ to the algebra. $\mathcal{N}$ is the number of independent supersymmetries of the algebra - in this case there are four. The Poincaré supersymmetries and the special conformal transformations K_μ do not commute. Their commutators must also be symmetries and they generate what are called the conformal supersymmetries. Together with the R-symmetry $SO(6)_R \approx SU(4)_R$ (which is a symmetry under which the supercharges can be rotated into each other), the global symmetry group of $\mathcal{N} = 4$ SYM is thus given by the supergroup $SU(2,2|4)$.

The isometry group of AdS_5 is $SO(2,4)$, so the symmetry group of $AdS_5 \times S^5$ is $SO(2,4) \times SO(6)$. Because $AdS_5 \times S^5$ corresponds to a maximally supersymmetric

[1]See [6], for example, for a comprehensive discussion and references.

vacuum of IIB supergravity, the full symmetry group is $SU(2,2|4)$. Thus the supersymmetries of the two theories are the same [11].

1.5 Motivation for the current work

1.5.1 Giant gravitons and their duals

Giant gravitons

Giant gravitons are gravitons which are dilated to a macroscopic size by a background tensor field. This process is analogous to dipole stretching in an electromagnetic field.[2] They have been found in $AdS \times S$-type backgrounds to expand in both the AdS and sphere spaces, and are then called AdS and sphere giants respectively. The expansion of the sphere giants is limited to the radius of the sphere, offering a mechanism to explain the stringy exclusion principle, which states that the tower of Kaluza-Klein states (a discretely labelled spectrum of massive graviton states) has a certain momentum cut-off related to the radius of the space.

A precise correspondence between CFT observables and those of supergravity was proposed by Witten [3]. The local operators in the boundary field theory correspond to the quantum states on the AdS space. The dimension of the field theory operator is related to the energy or mass of the AdS state.

For the case of Type IIB supergravity and $\mathcal{N} = 4$ $(3+1)$-dimensional super Yang-Mills theory, each operator of conformal dimension k in the field theory corresponds to a scalar field in supergravity with mass related to k. Specifically, we are interested in a special class of representations of states on the gravity side, and the dual class of operators in the field theory. States in IIB supergravity are written in terms of spherical harmonics on S^5 and scalar fields in AdS_5 with definite SO(6) representations. Certain massive particle representations have a lower bound on their mass, called the BPS bound after Bogomolnyi, Prasad and Sommerfield [12, 13]. When the BPS bound is saturated, a supercharge vanishes and the supersymmetric representation is shortened, i.e. the dimension of the supermultiplet to which the representation belongs is decreased. Since the mass of a BPS state is completely determined by an analysis of the supersymmetry algebra, it is protected from quantum corrections. D3 branes are BPS states that preserve half of the supersymmetries. These are half-BPS representations. Giant gravitons can be thought of as types of D3 branes (see Chapter 2 for details).

The next point to address is to what operators in the field theory these half-BPS representations are dual. Since both the string theory modes and the con-

[2] See the discussion in the following chapter.

formal operators in SYM fall into representations of $SU(2,2|4)$ it is natural to try to associate modes to operators transforming in the same representation. Furthermore, since IIB strings might not know about the gauge symmetry of SYM, the SYM operators should be gauge invariant. A bound similar to the BPS bound discussed above exists for dimensions of operators in representations of the superconformal algebra. For those for which one or more supercharges commutes with the primary or lowest dimensional operator, the representation is shortened. Furthermore, the scale dimension of such primary operators is said to be protected: it is given for all g_{YM} by its free field value. These representations are called chiral or BPS by analogy with the case of massive particle states. Thus, the half-BPS representations of interest in $AdS_5 \times S^5$ are dual to chiral primary operators in $\mathcal{N} = 4$ SYM. In particular, the single trace half-BPS operators in the SYM theory match in a one-to-one way with the canonical fields of super gravity, compactified on $AdS_5 \times S^5$ [3].

With this correspondence, Corley, Jevicki and Ramgoolam [4] were able to propose candidate operators dual to giant gravitons. By studying the zero coupling limit of $\mathcal{N} = 4$ SYM, they suggested different types of representations of chiral primary operators in 3+1 SYM with U(N) gauge group as dual to AdS and sphere giants respectively. This will be discussed further in Chapter 3.

Correlators in $3+1$ $\mathcal{N} = 4$ U(N) super Yang-Mills theory

In addition, Corley, Jevicki and Ramgoolam [4] were able to develop a powerful machinery for computing exactly a related class of correlators at all N. Previous calculations had only been possible at large N. Corley, Jevicki and Ramgoolam calculated exactly correlation functions of half-BPS operators in the $\mathcal{N} = 4$ SYM theory with U(N) gauge group in the zero coupling limit. As will be discussed in Chapter 3, they used a map to a Schur polynomial basis in U(N) to diagonalise these correlators. This map hinges on the Frobenius-Schur duality between U(N) and S_n the permutation group.[3] The extension of results to finite N is especially significant since the string coupling constant is inversely proportional to N, and this therefore implies an extension in the dual to a regime where the string coupling constant is large.

[3]See Appendix A for details.

1.5.2 U(N) vs SU(N)

Bulk and boundary

Because a U(N) gauge theory is equivalent to a theory of a free U(1) vector multiplet times an SU(N) gauge theory factored by Z_N, it is an SU(N) gauge group which is of interest in the bulk, where these gravitons exist [5]. There are no decoupled modes in the string theory. The U(1) vector multiplet is related to the centre of mass motion of all the branes[14]. These modes live at the d-dimensional boundary M_d and are called singleton or doubleton representations of SO(2,d)[3].[4]

At large N, the results of a theory with U(N) gauge group will almost always coincide with the results of a theory with SU(N) gauge group. There are only a limited number of questions one could ask that would distinguish between the two in the large N limit. The extension to finite N meant that questions distinguishing the two were now sensible. To differentiate the roles of bulk and boundary degrees of freedom in the dual string theory, the analogous calculational tools for the SU(N) gauge theory were needed.

Adding to the AdS/CFT dictionary

The aim of the present work was to extend the results of Corley, Jevicki and Ramgoolam [4] to the case of an SU(N) gauge group, so as to contribute to the dictionary of giant gravitons and chiral primaries. Our goal was to provide a one-to-one correspondence between the half-BPS representations and a set of operators with diagonal two-point functions. These operators (or ones related to them by a unitary transformation and hence preserving the two-point functions) would then be natural candidates for particle states in the supergravity.

However, the generalisation from gauge group U(N) to SU(N) could not be achieved via minor modification of the results of [4]. In the U(N) case, a map to the space of Schur polynomials was used. The Schur polynomials diagonalise the two-point functions and are in one-to-one correspondence with the half-BPS representations of the theory with U(N) gauge group. Schur polynomials of su(N) fields[5] are in contrast no longer linearly independent. This means that orthogonality of the two-point function is lost.

The central difference between the two cases is the tracelessness condition obeyed by the su(N) fields. This results in a modified correlator which suggests

[4]Those of SO(2,3) were first discovered by Dirac. See [15] who cite Dirac [16]. The singletons discovered by Dirac are a pair called Di (spin $\frac{1}{2}$) and Rac (spinless). See [17] for a discussion and comprehensive set of references.

[5]Fields in the Lie algebra of SU(N), denoted su(N). Similarly the Lie algebra of U(N) is denoted u(N). The construction of these fields will be made clearer in Chapter 3.

the introduction of a ghost (a field with negative norm states) to convert between u(N) and su(N) fields. Taylor expansion of Schur polynomials in terms of this ghost allows us to express SU(N) correlators in terms of U(N) correlators which are known from the technology of Corley, Jevicki and Ramgoolam. Thus we show that the Schur polynomials remain a useful set of operators to consider. Further, we are able to give a set of linearly independent operators that are in one-to-one correspondence with the half-BPS representations of the SU(N) gauge theory.

These results are used to extend the giant graviton analysis to the case of the dual operators – this time in SU(N). They constitute the first steps towards providing a dictionary between half-BPS representations in the $\mathcal{N} = 4$ SYM theory with gauge group SU(N) and states in the dual supergravity.

1.6 Organisation of this dissertation

This dissertation is organised as follows: Chapter 2 provides a review of giant gravitons and their properties. Chapter 3 deals with the U(N) gauge symmetry case which provided the starting point for the original work in this dissertation. Chapter 4 covers the extension of the U(N) results to SU(N), while Chapter 5 comprises a discussion of the physical implications of our results. Chapter 6 gives a summary and some concluding comments, including ideas for future study. The appendices cover the group theory used in the main body of the text and tabulate the U(N) and SU(N) calculation results. Note that the original work presented in this dissertation has been published in the Journal of High Energy Physics: see [18].

Chapter 2

Giant gravitons

2.1 Introduction

The expressively named giant gravitons of $AdS \times S$-type spaces were first discovered by McGreevy, Susskind and Toumbas [19]. These are gravitons which expand in the spherical part of the geometry, and have been termed 'sphere giants'. Their AdS space counterparts were discovered in [20] and [21]. As has been explained, the operators of U(N) SYM studied by Corley, Jevicki and Ramgoolam [4], and extended to the SU(N) case in this dissertation, were of interest in the context of the AdS/CFT correspondence because of their natural suggestion of field theory duals to giant gravitons. This chapter is intended as a brief review of giant gravitons. References for the results presented in some detail here are [19, 20] and [21]. Other works of interest on giant gravitons can be found in [28, 59, 60, 61, 29, 62, 63] and [22, 23, 24].

2.2 Sphere giants and the stringy exclusion principle

The stringy exclusion principle, predicted by the AdS/CFT correspondence, places a limit on the number of single particle states propagating on the sphere of $AdS \times S$-type geometries. The giant gravitons discovered by McGreevy, Susskind and Toumbas [19] offer a mechanism to explain this principle. McGreevy, Susskind and Toumbas found sphere giants which expand in the spherical part of the geometry being considered. This is a qualitative extension of the Myers' effect [25] for dielectric branes. The gravitons expand with increasing momentum and can reach macroscopic dimensions. However, this growth is limited to the size of the space in which the gravitons are propagating. When the expansion extends to the dimension of the space, the tower of Kaluza-Klein states is terminated. This cut-off matches that of the stringy exclusion principle, thus providing a mechanism to explain it.

Particles with spatial extension proportional to their angular momentum have been seen before, in the context of non-commutative field theories [26, 27]. The basic object of non-commutative field theory is the system of an electric dipole moving in a magnetic field. The dipole analogy is instructive, and we review it here, following [19] and [27].

2.2.1 Dipoles in magnetic fields

Consider a pair of unit charges with opposite signs moving on a plane in a constant magnetic field B. In the regime where the Coulomb and radiation terms are negligible, the Lagrangian is

$$\mathcal{L} = \frac{m}{2}\left(\dot{x_1}^2 + \dot{x_2}^2\right)^2 + \frac{B}{2}\epsilon_{ij}\left(\dot{x_1}^i x_1{}^j - \dot{x_2}^i x_2{}^j\right) - \frac{K}{2}(x_1 - x_2)^2. \tag{2.1}$$

The first term is the kinetic energy, the second gives the interaction of the charges with the magnetic field and the last term is the harmonic potential between the charges. If we assume that the mass is so small that we can ignore the kinetic energy term, we can focus on the remaining two terms. Introducing centre of mass and relative co-ordinates,

$$\begin{aligned} X &= \frac{x_1 + x_2}{2}; \\ \Delta &= \frac{x_1 - x_2}{2}, \end{aligned}$$

the Lagrangian becomes

$$\mathcal{L} = 2B\epsilon_{ij}\dot{X}^i\Delta^j - 2K\Delta^2. \tag{2.2}$$

From (2.2) we see that X and Δ are non-commuting variables such that

$$\left[X^i, \Delta^j\right] = \frac{\imath\epsilon^{ij}}{2B}. \tag{2.3}$$

Further, the centre of mass momentum conjugate to X is

$$\frac{\partial\mathcal{L}}{\partial\left(\dot{X}^i\right)} = 2B\epsilon_{ij}\Delta^j = P_i, \tag{2.4}$$

and

$$\begin{aligned}\mathcal{H} &= \dot{X}^i \frac{\partial \mathcal{L}}{\partial \dot{X}^i} \\ &= 2K\Delta^2 \\ &= \frac{K}{2B^2}P^2.\end{aligned}$$

This is the Hamiltonian of a non-relativistic particle with mass $m = \frac{B^2}{K}$. Note that Δ gives the relative co-ordinates of the two unit charges. It is the dimension of the dipole, or this particle.

$$|\Delta| = \frac{|P|}{2B}. \tag{2.5}$$

So the dipole is stretched in the perpendicular direction according to a linear dependence on the momentum.

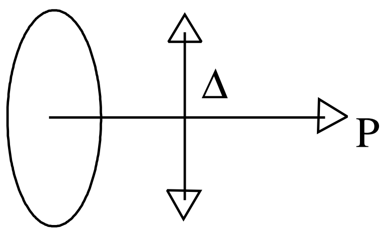


Figure 2.1: *The dipole is stretched in the perpendicular direction.*

Now we allow the dipole to move on the surface of a sphere of radius R, with magnetic flux N. In other words we place a magnetic 'monopole' of strength $2\pi N = \Omega_2 BR^2$ at the centre of the sphere. A rough analysis of (2.5) tells us that when the momentum of the dipole is about 2BR, the dipole will be as big as the sphere. This corresponds to an angular momentum $L = PR \approx BR^2$, i.e. of the order of N the total magnetic flux. If the dipole cannot get any bigger than the sphere that contains it, the angular momentum cannot exceed $\approx$ N.

We can do a more precise analysis and see that the maximum angular momentum is exactly N: Parametrise the sphere by two angles θ and ϕ. ϕ measures angular distance from the equator while θ is the azimuthal angle running from 0 to 2π. We work in a gauge in which the θ component of the vector potential is non-zero:

$$A_\theta = N\frac{1 - sin\phi}{2Rcos\phi}. \tag{2.6}$$

For a unit charged point particle moving on the sphere, the term coupling the velocity to the vector potential is

$$\begin{aligned}\mathcal{L}_A &= A_\theta R cos\phi\dot{\theta}\\ \mathcal{L}_A &= NR\frac{1-sin\phi}{2R}\dot{\theta}.\end{aligned}$$

Now consider a dipole with its centre of mass moving on the equator. The positive charge is at position (θ,ϕ) and the negative charge is at position $(\theta,-\phi)$. For the motion we consider, ϕ is time-independent. In other words, we are neglecting motion that involves vibrational modes of the dipole itself. Then we find that

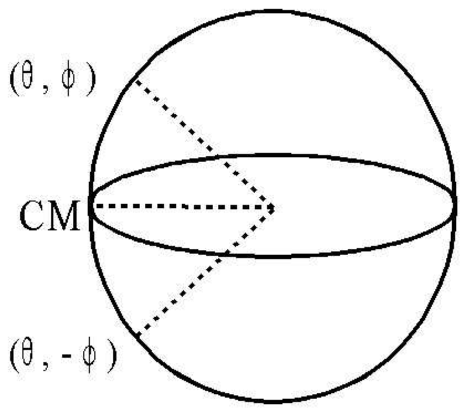


Figure 2.2: *A dipole with its centre of mass moving on the equator of a sphere.*

$$\begin{aligned}\mathcal{L}_A &= \frac{N}{2}(1-sin\phi)\dot{\theta}-\frac{N}{2}(1+sin\phi)\dot{\theta}\\ \Rightarrow \mathcal{L}_A &= -Nsin\phi\,\dot{\theta}.\end{aligned}$$

Once more we consider a slow-moving dipole for which the kinetic term is negligible compared to the coupling to the magnetic field. We add in a spring coupling term

$$\mathcal{L}_S = -\frac{k}{2}R^2 sin^2\phi, \tag{2.7}$$

where we have used the chordal distance R $sin\phi$ for simplicity. This leaves us with

$$\mathcal{L} = -\frac{k}{2}R^2 sin^2\phi - N sin\phi\dot{\theta}, \tag{2.8}$$

which yields the angular momentum

$$\begin{aligned} L &= \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \\ L &= -N sin\phi. \end{aligned}$$

This is at its maximum value $|L_{max}| = N$ when $\phi = \frac{\pi}{2}$. The fact that the angular momentum of a single field quantum in non-commutative field theory is bounded by N is well-known in the context of such a theory on a sphere.

2.2.2 $AdS_5 \times S^5$

The treatment of BPS particles moving on the sphere of $AdS \times S$-type spaces is very similar to the example of the dipole considered above. Analogous to the background magnetic field on the sphere is a background tensor field. Once again the maximum angular momentum of these gravitons is found to be N. We see that the graviton has a maximum angular momentum agreeing with the stringy exclusion principle [19].

For $AdS_5 \times S^5$, the radius of the 5-sphere is given by

$$R = (4\pi g_s N)^{\frac{1}{4}} l_s, \tag{2.9}$$

where g_s is the string coupling constant, l_s is the string length scale, and N is the number of units of 5-form flux on the sphere. We consider the 't Hooft limit, taking $N \to \infty$ while keeping $\lambda = g_S N$ fixed and large. If we consider the exact classical analysis of a D3 brane mapping an S^3 moving in S^5 (again following [19]) we have the Lagrangian

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{DBI} + \mathcal{L}_{CS} \\ \mathcal{L} &= -T_{D3}\Omega_3 r^3 \sqrt{1 - (R^2 - r^2)\,\dot{\phi}^2} + \dot{\phi} N \frac{r^4}{R^4}. \end{aligned}$$

Here $\mathcal{L}_{DBI}$ is the Dirac-Born-Infeld Lagrangian, giving the kinetic energy of the (spherical) membrane. $\mathcal{L}_{CS}$ is the Chern-Simons term corresponding to the background field coupling. T_{D3} gives the tension of the D3-brane

$$T_{D3} = \frac{1}{(2\pi)^3 l_s^4 g_s} = \frac{N}{\Omega_3 R^4}, \tag{2.10}$$

and the angular momentum in terms of $\dot{\phi}$ is

$$L = \frac{m\dot{\phi}\left(R^2 - r^2\right)}{\sqrt{1 - \dot{\phi}^2\left(R^2 - r^2\right)}} + N\frac{r^4}{R^4}, \tag{2.11}$$

where $m = T_{D3}\Omega_3 r^3 = \left(\frac{N}{R^4}\right) r^3$. Again we see that L is bounded by N since $0 \leq r \leq R$ and $0 \leq \dot{\phi}R \leq 1$ ($\dot{\phi}R$ cannot exceed the speed of light). In type IIB string theory on $AdS_5 \times S^5$ with $N >> 1$, the maximum angular momentum of a BPS particle on the S^5 is N [28]. These results provide a mechanism to understand this: the particle moving on the sphere expands into a spherical D3 brane. The energy is given by

$$E = \sqrt{m^2 + \frac{\left(L - Nr^4/R^4\right)^2}{R^2 - r^2}}. \tag{2.12}$$

Varying this with respect to r while keeping L fixed, we find a stable minimum when $r^2 = \frac{L}{N}R^2$. The energy at this minimum matches the BPS bound $E = \frac{L}{R}$ when L is large and $N >> 1$. This is strong evidence that Kaluza-Klein gravitons are described by spherical branes. Of course, there is also a minimum at $r = 0$. This corresponds to the normal point-like graviton solution. Because of the stringy exclusion principle and the absence of any upper bound on the momentum of this graviton, these results suggested that it was the expanded graviton which provided the more valid description at large momenta.

2.3 AdS giants

Grisaru, Myers and Tafjørd [20], and Hashimoto, Hirano and Itzhaki [21] independently showed that there also exist giant gravitons consisting of spherical branes expanding into the AdS space. Consider a spherical D3-brane in AdS_5, also moving along the equator of S^5 with angular velocity $\dot{\phi}$. The Lagrangian of this configuration is

$$\mathcal{L} = -T\Omega_3 R^4 \left(tan^3\rho\sqrt{sec^2\rho - \dot{\phi}^2} - tan^4\rho\right) \tag{2.13}$$

and the energy is given by

$$E = N\left(sec\rho\sqrt{\frac{L^2}{N^2} + tan^6\rho} - tan^4\rho\right), \tag{2.14}$$

where we have used the fact that $T\Omega_3 R^4 = N$. The energy has local minima at $tan\rho = 0$ and $tan\rho = \sqrt{\frac{L}{N}}$ where $E = \frac{L}{R}$.

The quantum numbers of the two kinds of giant graviton match and are equal to those of the point-like graviton solution. Further, an analysis of the supersymmetry properties of these expanded branes shows that they preserve exactly the same supersymmetries as a point-like graviton. The two kinds of giant gravitons can be thought of as dual to one another[1] in the sense that the AdS giant can be thought of as a dielectric brane that couples electrically to the background field while the sphere giant behaves like a dimagnetic brane.

Since $tan\rho$ is not bounded, there is no upper limit on L for these AdS giants. The explanation of the stringy exclusion principle offered by the sphere giants made it natural to suggest that these giant gravitons were the description valid at high angular momenta. However, the AdS giants–although they also have a radius that is proportional to their angular momentum–have no upper limit on their expansion because Anti-de Sitter space is not compact. They therefore pose a problem with regards to the stringy exclusion principle mechanism so neatly put forward by the angular momentum cut-off on the sphere giants.

2.4 Instanton transitions

A possible solution to this problem was put forward in [20]. If quantum tunnelling between the three different states were taken into account, there might be no supersymmetric ground state for angular momenta larger than the exclusion principle bound. This supersymmetry-breaking scenario is one which would leave intact the original explanation of the origin of the stringy exclusion principle from the point of view of sphere giants.

Analytic instanton solutions describing tunnelling between either of the expanded graviton configurations and the pointlike graviton were found in [21, 20] and [29]. These instantons are $\frac{1}{4}$-BPS states, while the three gravitons are known to be $\frac{1}{2}$-BPS. An investigation of solutions describing instantons tunnelling directly between the two types of giant gravitons can be found in [22]. Numerical simulation showed that there is no direct tunnelling solution: the only way to tunnel from the AdS to the sphere giant state is via the point-like graviton. The problem is therefore one-dimensional from a tunnelling point of view. Moving between the giant graviton states is essentially a two-instanton effect linking the outermost minima of a one-dimensional triple potential well.

This observation leads to a supersymmetry argument which puts the SUSY-

[1]Sometimes sphere giants are referred to simply as giant gravitons and AdS giants are referred to as dual giant gravitons.

breaking scenario of [20] on a firmer footing. The one-dimensionality of the problem as discussed above allows us to draw from the analogous one-dimensional supersymmetric quantum mechanics.[2]

Qualitatively, the question of supersymmetry breaking and restoration is identical in the two systems (up to number of fermionic zero modes). Lee was able to show that there is no supersymmetric state for the instanton configuration linking the AdS giant and the point-like graviton. If there were such a state for this double minimum case, we would have a possible graviton configuration which did not include an angular momentum cut-off to explain the stringy exclusion principle. As it is though, we only have a suitable symmetric state for the triple potential well instanton configuration. Thus, there is no supersymmetric state of gravitons without the desired bound. This argument supports the conjecture of Grisaru, Myers and Tafjørd [20]. Note that it makes critical use of the fact that there is no apparent instanton solution linking the AdS and sphere giant configurations.

[2]Lee [22] cites [30, 31, 32, 33, 34] and [35].

Chapter 3

U(N) correlators

3.1 Overview

Corley, Jevicki and Ramgoolam [4] were able to propose candidate duals to giant gravitons in U(N) $\mathcal{N} = 4$ super Yang-Mills. These were suggested by the Schur polynomial basis they used to diagonalise the two-point function for a class of correlation functions of half-BPS operators in the field theory. In this chapter a detailed review of [4] is given. This work was the starting point for the research presented in this dissertation. The main reference for this chapter is [4]; see [6] and [36] for more detailed presentation of the introductory comments.

3.2 Operators corresponding to half-BPS representations

As discussed in Chapter 1, operators in the conformal field theory are mapped by the AdS/CFT correspondence into quantum states in $AdS_5 \times S^5$. The energy of the state is related to the conformal dimension of the field theory operator. We are interested in states or operators for which this quantity is protected from quantum corrections. This is so for a class of representations which satisfy a BPS-bound and which are consequently shortened. These are called BPS states or representations on either side of the duality, and are also referred to as chiral operators in field theory. Because of the (lower) BPS bound on their dimensions, it suffices to consider the lowest-dimensional such operators, called primary operators. The half-BPS representations in IIB string theory on $AdS_5 \times S^5$ we are interested in (BPS representations that preserve half the supersymmetries) are dual to chiral primary operators in $\mathcal{N} = 4$ SYM.

Observables in a gauge theory must be gauge invariant quantities such as correlation functions of gauge invariant operators. Corley, Jevicki and Ramgoolam's work dealt with correlation functions of chiral primary operators. Extremal correlators of these half-BPS operators are protected by non-renormalisation theorems [37, 38], so it is sufficient to calculate them for $g_{YM} = 0$.

States in Type IIB string theory can be written in terms of spherical harmonics on S^5 and scalar fields in AdS_5 with definite SO(6) representations. These are indexed by n and correspond to states with well-defined angular momentum n. Both the gravitational modes and the conformal SYM operators fall into representations of $SU(2,2|4)$, so it is natural to associate modes and operators which transform in the same representation. We would therefore like to investigate gauge invariant operators in the field theory which have a well-defined SO(6) representation.

The elementary fields of the SYM theory are represented by $N \times N$ Hermitian matrices. The problem of studying these operators therefore reduces to the problem of matrix model quantum mechanics with matrices in the U(N) Lie algebra, i.e. $N \times N$ Hermitian matrices. The half-BPS operators are in a one-to-one correspondence with the states of this matrix model quantum mechanics. There are six chiral primary operators which transform as SO(6) vectors. The operators considered in [4] are half-BPS chiral primary operators built from a single complex combination of any two of the six scalars appearing in the theory:

$$\hat{\phi} = X^j + \imath X^k \tag{3.1}$$

where the index on the X^is is an SO(6) vector index.

The lowest weight states are approximately given by single trace operators. These are of the form $T_{i_1....i_6} tr\left(X^{i_1}....X^{i_n}\right)$ where the X^is, $i = 1, ..., 6$ are the scalars of the theory transforming in the vector representation of the SO(6) R-symmetry group and the coefficients $T_{i_1....i_6}$ are symmetric and traceless on the SO(6) indices. They correspond to single particle states in AdS, while multiple trace states are then interpreted as bound states of single particle states.

A generic chiral primary of R-charge n will be of the form

$$\left(tr\left(\hat{\phi}^{l_1}\right)\right)^{k_1}\left(tr\left(\hat{\phi}^{l_2}\right)\right)^{k_2}...\left(tr\left(\hat{\phi}^{l_m}\right)\right)^{k_m},$$

where

$$n = \sum_{i=1}^{m} l_i k_i; \tag{3.2}$$

the integers l_i and k_i form a partition of n. We have a one-to-one correspondence between half-BPS representations of charge n and partitions of n. These partitions are represented by Young diagrams of n boxes.[1] They can be used to characterise the permutation group S_n as well as the unitary group U(N).[2]

Schur polynomials are the characters of the unitary group in their irreducible representations, and are given in terms of characters of S_n in the same representations, both of which can be labelled by Young diagrams.

$$\chi_R(U) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)tr(\sigma U) \tag{3.3}$$

where U $\in$ U(N) the unitary group. In [4] Schur polynomials of $\hat{\phi} \in$ u(N) the Lie algebra were considered. The space of half-BPS representations can be mapped to the space of Schur polynomials of $\hat{\phi}$, or equivalently to the space of Young diagrams characterising representations of U(N). The Schur polynomials provide an orthogonal basis for this space, as will be shown in the bulk of this chapter.

3.3 Computation of correlators

3.3.1 Brute force computation

The correlators we will find follow from the propagator

$$\left\langle \hat{\phi}^i_j(x_1)\hat{\phi}^{*k}{}_l(x_2)\right\rangle = \frac{\delta^i_l\delta^k_j}{(x_1-x_2)^2}. \tag{3.4}$$

The spacetime dependence is trivial and can easily be reinstated. We therefore suppress it and use the correlator

$$\left\langle \hat{\phi}^i_j\hat{\phi}^{*k}{}_l\right\rangle = \delta^i_l\delta^k_j \tag{3.5}$$

to compute correlation functions of products of $\hat{\phi}$s and $\hat{\phi}^*$s. By brute force such calculations rapidly become unwieldy.

[1] See Appendix A.

[2] Note that different Young diagrams with n boxes are distinct representations of S_n and U(N), but not of SO(6) which depends only on n.

For example, consider

$$
\begin{aligned}
& \left\langle tr\left(\hat{\phi}\right)^4 tr\left(\hat{\phi}^*\right)^4\right\rangle \\
= & \left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^k_l\hat{\phi}^l_i\hat{\phi}^{*m}{}_n\hat{\phi}^{*n}{}_p\hat{\phi}^{*p}{}_q\hat{\phi}^{*q}{}_m\right\rangle \\
= & \delta^l_n\delta^m_i\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^k_l\hat{\phi}^{*n}{}_p\hat{\phi}^{*p}{}_q\hat{\phi}^{*q}{}_m\right\rangle \\
& +\delta^l_p\delta^n_i\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^k_l\hat{\phi}^{*n}{}_p\hat{\phi}^{*p}{}_q\hat{\phi}^{*q}{}_m\right\rangle \\
& +\delta^l_q\delta^p_i\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^k_l\hat{\phi}^{*m}{}_n\hat{\phi}^{*n}{}_p\hat{\phi}^{*q}{}_m\right\rangle \\
& +\delta^l_m\delta^q_i\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^k_l\hat{\phi}^{*m}{}_n\hat{\phi}^{*n}{}_p\hat{\phi}^{*p}{}_q\right\rangle \\
= & \delta^l_n\delta^m_i\left[\delta^k_p\delta^n_l\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^{*p}{}_q\hat{\phi}^{*q}{}_m\right\rangle + \delta^k_q\delta^p_l\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^{*n}{}_p\hat{\phi}^{*q}{}_m\right\rangle\right. \\
& \left.+\delta^k_m\delta^q_l\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^{*n}{}_p\hat{\phi}^{*p}{}_q\right\rangle\right] \\
& +\delta^l_p\delta^n_i\left[\delta^k_p\delta^n_l\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^{*p}{}_q\hat{\phi}^{*q}{}_m\right\rangle + \delta^k_q\delta^p_l\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^{*n}{}_p\hat{\phi}^{*q}{}_m\right\rangle\right. \\
& \left.+\delta^k_m\delta^q_l\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^{*n}{}_p\hat{\phi}^{*p}{}_q\right\rangle\right] \\
& +\delta^l_q\delta^p_i\left[\delta^k_n\delta^m_l\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^{*n}{}_p\hat{\phi}^{*q}{}_m\right\rangle + \delta^k_p\delta^n_l\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^{*m}{}_n\hat{\phi}^{*q}{}_m\right\rangle\right. \\
& \left.+\delta^k_m\delta^q_l\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^{*m}{}_n\hat{\phi}^{*n}{}_p\right\rangle\right] \\
& +\delta^l_m\delta^q_i\left[\delta^k_n\delta^m_l\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^{*n}{}_p\hat{\phi}^{*p}{}_q\right\rangle + \delta^k_p\delta^n_l\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^{*m}{}_n\hat{\phi}^{*p}{}_q\right\rangle\right. \\
& \left.+\delta^k_q\delta^p_l\left\langle \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^{*m}{}_n\hat{\phi}^{*n}{}_p\right\rangle\right] \\
= & .... \\
= & 4N^4+20N^2
\end{aligned}
$$

3.3.2 Diagrammatic computation

Calculation can be simplified by a diagrammatic evaluation, using the fact that the calculation consists essentially of adding all the different possible ways of contracting the indices in the function. Each traced product is represented by a double line vertex where a given line corresponds to a single index, as seen in the following example. Given one or more of these vertices, closed loop diagrams are obtained by joining these fat lines in different ways corresponding to the different possible index contractions. The co-efficients of different terms are then given by the combinatorics [10] of counting in how many ways each distinct diagram can be obtained, while counting the number of closed loops in a diagram gives the factor of N associated with that term in the sum.

In fact this is just the diagrammatic expansion of 't Hooft [10]. In the 't Hooft limit where $\lambda = g_{YM}^2 N = g_S N$ is held constant and N is taken to infinity, this expansion is well-defined in Yang-Mills theory and appears to correspond to a perturbative expansion in some string theory, with $g_S = \frac{\lambda}{N} \to 0$ [5].

These diagrams therefore simplify calculation and also make explicit which terms correspond to classical and which to quantum gravity contributions in a perturbative expansion of a string theory. Diagrams which can be drawn flat (without intersections) on a sphere are known as planar diagrams and correspond to classical gravity contributions, while non-planar diagrams correspond to quantum gravity contributions. In the 't Hooft limit the planar diagrams dominate the expansion.

$\left\langle tr\left(\hat{\phi}\right)^4 tr\left(\hat{\phi}^*\right)^4\right\rangle$ can be evaluated diagrammatically. $tr\left(\hat{\phi}\right)^4 = \hat{\phi}^i_j\hat{\phi}^j_k\hat{\phi}^k_l\hat{\phi}^l_i$ is represented by a "fat" quartic vertex made up of lines equating indices.

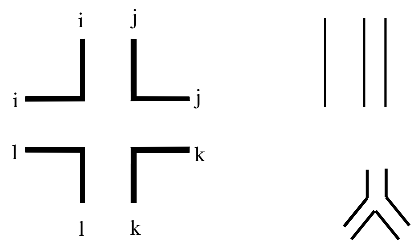


Figure 3.1: *$tr\left(\hat{\phi}\right)^4$, $tr\left(\hat{\phi}\right)^2 tr\left(\hat{\phi}\right)^2$ and $tr\left(\hat{\phi}\right)^3$ as double line diagram vertices.*

The different ways of contracting indices – written out above – is represented diagrammatically by the joining of branches. For our example, two types of fat diagrams are obtained: flat or planar diagrams, and twisted or non-planar diagrams.

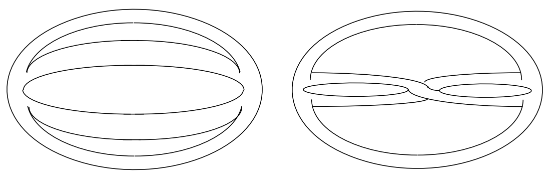


Figure 3.2: *"Fat" diagrams.*

Closed loops in these diagrams correspond to sums of the form δ^i_i which give rise to factors of N since i is an index of an $N \times N$ matrix. Here, planar diagrams give factors of N^4 and twisted or non-planar diagrams give rise to factors of N^2. The coefficients are found via combinatorics. In our example there are 4! ways to join four branches to four branches. Clearly there are four possible ways to draw the planar diagram. This diagram gives us the $4N^4$ term of the correlation function computed in laborious detail above. The remaining 24 - 4 non-planar diagrams give us the 20 N^2 term, since they have two closed loops. These contributions to the expansion differ by a factor of $\frac{1}{N^2}$.

3.4 U(N) Correlators as computed in [4]

3.4.1 Group theoretical tools

Denote by V the fundamental representation of U(N). The space $Sym(V^{\otimes n})$ is a representation of U(N) and also admits a commuting action of S_n. The action of U(N) and S_n can thus be simultaneously diagonalised. This allows Young diagrams to be associated with both U(N) and S_n representations.[3]

To find the correlation functions of traced products of the fields $\hat{\phi}$, Corley, Jevicki and Ramgoolam [4] used these powerful results from group theory, hinging on the Frobenius-Schur duality between U(N) and the permutation group S_n.[4] The space of half-BPS representations can be mapped to the space of Schur polynomials of u(N), or equivalently to the space of Young diagrams characterising representations of U(N), as discussed above. The correlators take a very simple form when a Young diagram basis is used [40]: the two-point functions are diagonal, and the three point functions are proportional to fusion coefficients of the unitary groups ie Littlewood Richardson coefficients.

Schur polynomials of $\hat{\phi} \in$ u(N)

$$\chi_R\left(\hat{\phi}\right) = \sum_{\sigma \in S_n} \chi_R(\sigma) tr\left(\sigma\hat{\phi}\right) \tag{3.6}$$

where

$$tr\left(\sigma\hat{\phi}\right) = \sum_{i_1,\ldots i_n} \hat{\phi}^{i_1}_{i_{\sigma(1)}} \ldots \hat{\phi}^{i_n}_{i_{\sigma(n)}}, \tag{3.7}$$

(i.e. we take characters of polynomials of the Lie algebra instead of characters of U(N)), turn out to be an orthogonal basis for the fully traced states of the system in terms of representations of S_n.

3.4.2 Results

The two-point function is given by

$$\left\langle \chi_R\left(\hat{\phi}\right) \chi_S\left(\hat{\phi}^*\right) \right\rangle = \delta_{RS} \frac{Dim(R) n_R!}{d_R}, \tag{3.8}$$

[3]See, for example, [39]. A brief discussion is also given in Appendix A.

[4]These are discussed in Appendix A. Only the central results and their relevance to this problem will be presented here.

where Dim(R) is the dimension of R as a representation of the unitary group[5] i.e. it is given by the U(N) labelling factor f_R of the representation R, divided by the product of box hook lengths. d_R is the dimension of R as a representation of the permutation groups: it is given by

$$d_R = \frac{n_R!}{\prod(\text{hook lengths})} \tag{3.9}$$

where n_R is the number of boxes in the Young diagram. The $n_R!$ factors and the hook lengths cancel in (3.8) so that the two-point function reduces to

$$\left\langle \chi_R\left(\hat{\phi}\right)\chi_S\left(\hat{\phi}^*\right)\right\rangle = \delta_{RS} f_R. \tag{3.10}$$

The orthogonality of the two-point function suggests a natural particle interpretation for the Schur polynomials. The orthogonality will hold even when the coupling is non-zero. Note also that the Krønecker delta ensures the two-point function is only non-zero if $n_S = n_R = n$. Arbitrary correlators can also be computed. The results for the three- and multi- point functions are stated here, and in the next section derivations of the two- and multi-point functions are given.

The three-point function is given by

$$\left\langle \chi_R\left(\hat{\phi}\right)\chi_S\left(\hat{\phi}\right)\chi_T\left(\hat{\phi}^*\right)\right\rangle = g(R,S;T)\frac{Dim(T)n_T!}{d_T}, \tag{3.11}$$

while the multipoint function is

$$\begin{aligned}&\left\langle \chi_{R_1}\left(\hat{\phi}\right)...\chi_{R_l}\left(\hat{\phi}\right)\chi_{S_1}\left(\hat{\phi}^*\right)...\chi_{S_k}\left(\hat{\phi}^*\right)\right\rangle \\ &= \sum_S g(R_1,...,R_l;S)\frac{Dim(S)\ n_S!}{d_S}g(S_1,...S_k;S),\end{aligned} \tag{3.12}$$

[5]For example, the dimension of $\yng(2,1)$ as a representation the unitary group is given by

$$\frac{\begin{ytableau} N & N+1 \\ N-1 \end{ytableau}}{\begin{ytableau} 3 & 1 \\ 1 \end{ytableau}} = \frac{N(N+1)(N-1)}{3.1.1} = \frac{N(N^2-1)}{3}:$$

the U(N) labelling factor $N(N^2-1)$ of the representation over the hook length product (see Appendix A for details).

where $g(S_1, ..., S_k; S)$ is the Littlewood-Richardson coefficient.

3.5 Derivation of U(N) correlators

3.5.1 The two-point function

The result for the two-point function is (3.8). Note that the correlators will only be non-zero when there are as many $\hat{\phi}s$ as there are $\hat{\phi}^*s$, i.e. when the number of boxes in R is the same as the number of boxes in S : $n_R = n_S = n$.

Using the Schur polynomials (3.6), we find that

$$\left\langle \chi_R\left(\hat{\phi}\right)\chi_S\left(\hat{\phi}^*\right)\right\rangle = \left\langle \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)tr\left(\sigma\hat{\phi}\right)\frac{1}{n!}\sum_{\tau\in S_n}\chi_S(\tau)tr\left(\tau\hat{\phi}^*\right)\right\rangle, \quad (3.13)$$

where $tr\left(\sigma\hat{\phi}\right)$ is given by (3.7).

These expressions are best explained by means of examples. The simplest possible non-trivial example is S_2, the permutation group of order $2! = 2$. The elements of S_2 are (12) and (1) the identity. The representations are

$$\square\square \text{ and } \begin{array}{c}\square\\ \square\end{array}.$$

The row is the completely symmetric representation and the column is the completely antisymmetric representation. Then we have that the characters of the representations of S_2 (which can be read off the character table given in Appendix B or found by inspection) are

$$\begin{aligned}\chi_{\square\square}((1)) &= 1,\\ \chi_{\square\square}((12)) &= 1,\\ \chi_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}((1)) &= 1,\\ \chi_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}((12)) &= -1.\end{aligned}$$

The traced products appearing in the Schur polynomial (3.6) are

$$\begin{aligned} tr\left((1)\hat{\phi}\right) &= \sum_{i,j} \hat{\phi}^i_i \hat{\phi}^j_j \\ &= \left(tr\hat{\phi}\right)^2; \\ tr\left((12)\hat{\phi}\right) &= \sum_{i,j} \hat{\phi}^i_j \hat{\phi}^j_i \\ &= tr\left(\hat{\phi}^2\right), \end{aligned}$$

so that the Schur Polynomials are given by

$$\begin{aligned} \chi_{\square\square}(\hat{\phi}) &= \frac{1}{2}[(tr\hat{\phi})^2 + tr(\hat{\phi}^2)], \\ \chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(\hat{\phi}) &= \frac{1}{2}[(tr\hat{\phi})^2 - tr(\hat{\phi}^2)]. \end{aligned}$$

By a simple substitution of the expressions for the Schur polynomials (3.6) and $tr(\sigma\hat{\phi})$ (3.7) into (3.13), we obtain that for the two-point function

$$\begin{aligned} \left\langle \chi_R\left(\hat{\phi}\right) \chi_S\left(\hat{\phi}^*\right) \right\rangle &= \sum_{\sigma,\tau} \frac{\chi_R(\sigma)}{n!} \frac{\chi_S(\tau)}{n!} \\ &\times \sum_{i_1 \ldots i_n} \sum_{j_1,\ldots,j_n} \left\langle \hat{\phi}^{i_1}_{i_{\sigma(1)}} \ldots \hat{\phi}^{i_n}_{i_{\sigma(n)}} \hat{\phi}^{*j_1}{}_{j_{\tau(1)}} \ldots \hat{\phi}^{*j_n}{}_{j_{\tau(n)}} \right\rangle . \end{aligned} \tag{3.14}$$

From (3.5) we have that:

$$\begin{aligned} \left\langle \hat{\phi}^{i_1}_{i_{\sigma(1)}} \ldots \hat{\phi}^{i_n}_{i_{\sigma(n)}} \hat{\phi}^{*j_1}{}_{j_{\tau(1)}} \ldots \hat{\phi}^{*j_n}{}_{j_{\tau(n)}} \right\rangle &= \delta^{i_1}_{j_{\tau(1)}} \delta^{j_1}_{j_{\sigma(1)}} \ldots \delta^{i_n}_{j_{\tau(n)}} \delta^{j_n}_{j_{\sigma(n)}} \\ &+ \text{all other ways of contracting.} \end{aligned} \tag{3.15}$$

The correlator of our two Schur Polynomials for S_2 is therefore

$$\begin{aligned}
\left\langle \chi_{\square\square}\left(\hat{\phi}\right)\chi_{\substack{\square\\\square}}\left(\hat{\phi}^*\right)\right\rangle &= \frac{1}{4}\left[\left\langle\left(tr\hat{\phi}\right)^2\left(tr\hat{\phi}^*\right)^2\right\rangle + \left\langle tr\left(\hat{\phi}^2\right)\left(tr\hat{\phi}^*\right)^2\right\rangle\right.\\
&\quad \left. -\left\langle\left(tr\hat{\phi}\right)^2 tr\left(\hat{\phi}^{*2}\right)\right\rangle - \left\langle tr\left(\hat{\phi}^2\right)tr\left(\hat{\phi}^{*2}\right)\right\rangle\right]\\
&= \frac{1}{4}\left[\left\langle \hat{\phi}^i_i\hat{\phi}^j_j\hat{\phi}^{*k}{}_k\hat{\phi}^{*l}{}_l\right\rangle + \left\langle \hat{\phi}^i_j\hat{\phi}^j_i\hat{\phi}^{*k}{}_k\hat{\phi}^{*l}{}_l\right\rangle\right.\\
&\quad \left. -\left\langle \hat{\phi}^i_i\hat{\phi}^j_j\hat{\phi}^{*k}{}_l\hat{\phi}^{*l}{}_k\right\rangle - \left\langle \hat{\phi}^i_j\hat{\phi}^j_i\hat{\phi}^{*k}{}_l\hat{\phi}^{*l}{}_k\right\rangle\right]\\
&= \frac{1}{4}[\delta^j_k\delta^k_j\delta^i_l\delta^l_i + \delta^j_l\delta^l_j\delta^i_k\delta^k_i + \delta^j_k\delta^k_i\delta^i_l\delta^l_j + \delta^j_l\delta^l_i\delta^i_k\delta^k_j\\
&\quad -\delta^j_l\delta^k_j\delta^i_k\delta^l_i - \delta^j_k\delta^l_j\delta^i_l\delta^k_i - \delta^j_l\delta^k_i\delta^i_k\delta^l_j - \delta^j_k\delta^l_i\delta^k_j\delta^i_l]\\
&= \frac{1}{4}\left[\delta^j_j\delta^i_i + \delta^j_j\delta^i_i + \delta^j_i\delta^i_j + \delta^j_i\delta^i_j - \delta^j_i\delta^i_j - \delta^j_i\delta^i_j - \delta^j_j\delta^i_i - \delta^j_j\delta^i_i\right]\\
&= \frac{1}{4}[N^2 + N^2 + N + N - N - N - N^2 - N^2]\\
&= \frac{1}{4}[2N^2 + 2N - 2N - 2N^2]\\
&= 0,
\end{aligned}$$

verifying the important property of orthogonality. The non-zero correlators are

$$\begin{aligned}
\left\langle \chi_{\square\square}\left(\hat{\phi}\right)\chi_{\square\square}\left(\hat{\phi}^*\right)\right\rangle &= \frac{1}{4}\left[\left\langle\left(tr\hat{\phi}\right)^2\left(tr\hat{\phi}^*\right)^2\right\rangle + \left\langle tr\left(\hat{\phi}^2\right)\left(tr\hat{\phi}^*\right)^2\right\rangle\right.\\
&\quad \left. +\left\langle\left(tr\hat{\phi}\right)^2 tr\left(\hat{\phi}^{*2}\right)\right\rangle + \left\langle tr\left(\hat{\phi}^2\right)tr\left(\hat{\phi}^{*2}\right)\right\rangle\right]\\
&= \frac{1}{4}[2N^2 + 2N + 2N + 2N^2]\\
&= N(N+1),
\end{aligned}$$

and

$$\left\langle \chi_{\substack{\square\\\square}}\left(\hat{\phi}\right)\chi_{\substack{\square\\\square}}\left(\hat{\phi}^*\right)\right\rangle = N(N-1).$$

Thus $\left\langle \hat{\phi}^{i_1}_{i_{\sigma(1)}}...\hat{\phi}^{i_n}_{i_{\sigma(n)}}\hat{\phi}^{*j_1}{}_{j_{\tau(1)}}...\hat{\phi}^{*j_n}{}_{j_{\tau(n)}}\right\rangle$ can be written as a sum running over an element of the permutation group of n indices

$$\sum_\alpha \delta^{i_1}_{j_{\alpha\tau(1)}}...\delta^{i_n}_{j_{\alpha\tau(n)}}\delta^{j_1}_{i_{\alpha^{-1}\sigma(1)}}...\delta^{j_n}_{i_{\alpha^{-1}\sigma(n)}},$$

which implies that our two point correlator is given by

$$\sum_{i_1,\ldots i_n}\sum_{j_1,\ldots j_n}\sum_{\alpha,\sigma,\tau}\frac{\chi_R(\sigma)}{n!}\frac{\chi_S(\tau)}{n!}\delta^{i_1}_{j_{\alpha\tau(1)}}\ldots\delta^{i_n}_{j_{\alpha\tau(n)}}\delta^{j_1}_{i_{\alpha^{-1}\sigma(1)}}\ldots\delta^{j_n}_{i_{\alpha^{-1}\sigma(n)}}.$$

The next step is to perform a contraction over the js: $\delta^{j_q}_{i_{\alpha^{-1}\sigma(q)}}$ will contract with some $\delta^{i_k}_{j_q}$. The form of the first n deltas means that this $\delta^{i_k}_{j_q}$ will be of the form

$$\delta^{i_k}_{j_q} = \delta^{i_k}_{j_{\alpha\tau(k)}}, \tag{3.16}$$

allowing us to solve for k:

$$\begin{aligned} j_q &= j_{\alpha\tau(k)}; \\ k &= \tau^{-1}\alpha^{-1}(q). \end{aligned}$$

Then $\delta^{j_q}_{i_{\alpha^{-1}\sigma(q)}}$ contracts with $\delta^{i_{\tau^{-1}\alpha^{-1}(q)}}_{j_q}$, leaving us with $\delta^{i_{\tau^{-1}\alpha^{-1}(q)}}_{i_{\alpha^{-1}\sigma(q)}} = \delta^{i_{\sigma^{-1}\alpha\tau^{-1}\alpha^{-1}(q)}}_{i_q}$. This gives us

$$\sum_{i_1,\ldots i_n}\sum_{j_1,\ldots j_n}\sum_{\alpha,\sigma,\tau}\frac{\chi_R(\sigma)}{n!}\frac{\chi_S(\tau)}{n!}\delta^{i_{\sigma^{-1}\alpha\tau^{-1}\alpha^{-1}(1)}}_{i_1}\ldots\delta^{i_{\sigma^{-1}\alpha\tau^{-1}\alpha^{-1}(n)}}_{i_n}.$$

In general,

$$\sum_{i_1,\ldots,i_n}\delta^{i_1}_{i_{\sigma(1)}}\ldots\delta^{i_n}_{i_{\sigma(n)}} = N^{C(\sigma)}, \tag{3.17}$$

where $C(\sigma)$ is the number of cycles in the permutation group element σ. So when σ is the identity of n indices, $N^{C(\sigma)} = N^n$. This is easily seen since

$$\sum_{i_1,\ldots,i_n}\delta^{i_1}_{i_1}\ldots\delta^{i_n}_{i_n} = N\times N\times N\ldots.\times N. \tag{3.18}$$

As a simple example consider S_3 : this has $3! = 6$ elements. There are three classes of S_3 :
$\sigma = (1)$ is the identity. It is made up of three one-cycles;

$$\sum_{i_1,i_2,i_3} \delta^{i_1}_{i_1}\delta^{i_2}_{i_2}\delta^{i_3}_{i_3} = N^3,$$

$\sigma = (12)(3)$ contains one one-cycle and a two-cycle;

$$\sum_{i_1,i_2,i_3} \delta^{i_1}_{i_2}\delta^{i_2}_{i_1}\delta^{i_3}_{i_3} = N^2,$$

and $\sigma = (123)$ is an example of a single three-cycle:

$$\sum_{i_1,i_2,i_3} \delta^{i_1}_{i_2}\delta^{i_2}_{i_3}\delta^{i_3}_{i_1} = N.$$

The expression (3.17) above leads to

$$\left\langle \chi_R\left(\hat{\phi}\right)\chi_S\left(\hat{\phi}^*\right)\right\rangle = \sum_{\alpha,\sigma,\tau} \frac{\chi_R(\sigma)}{n!}\frac{\chi_S(\tau)}{n!} N^{C(\sigma^{-1}\alpha\tau^{-1}\alpha^{-1})}. \tag{3.19}$$

Now we introduce a sum over p (which will allow us to perform the sum over τ):

$$\left\langle \chi_R\left(\hat{\phi}\right)\chi_S\left(\hat{\phi}^*\right)\right\rangle = \sum_{\alpha,\sigma,\tau}\sum_p \frac{\chi_R(\sigma)}{n!}\frac{\chi_S(\tau)}{n!} N^{C(p)}\delta(p^{-1}\sigma^{-1}\alpha\tau^{-1}\alpha^{-1})$$
$$\left\langle \chi_R\left(\hat{\phi}\right)\chi_S\left(\hat{\phi}^*\right)\right\rangle = \sum_{\alpha,\sigma}\sum_p \frac{\chi_R(\sigma)}{n!}\frac{\chi_S(\alpha^{-1}p^{-1}\sigma^{-1}\alpha)}{n!} N^{C(p)}.$$

$\chi_S(\alpha^{-1}p^{-1}\sigma^{-1}\alpha) = \chi_S(p^{-1}\sigma^{-1})$ so the sum over α just contributes a factor of n! ($\alpha \in S_n$ and S_n has n! elements). Then

$$\left\langle \chi_R\left(\hat{\phi}\right)\chi_S\left(\hat{\phi}^*\right)\right\rangle = \frac{1}{n!}\sum_{\sigma,p} \chi_R(\sigma)\chi_S(p^{-1}\sigma^{-1})N^{C(p)}. \tag{3.20}$$

Using a group theory result which follows from the fundamental orthogonality relation:[6]

$$\sum_\sigma \chi_R(\sigma^{-1})\chi_S(\sigma\alpha) = \frac{\delta_{RS}n!}{d_S}\chi_S(\alpha), \tag{3.21}$$

[6]See Appendix A for a derivation.

we find that

$$\begin{aligned}\frac{1}{n!}\sum_{\sigma,p}\chi_R(\sigma)\chi_S(p^{-1}\sigma^{-1})N^{C(p)} &= \frac{1}{n!}\sum_{p}\frac{\delta_{RS}n!}{d_S}N^{C(p)}\chi_S(p^{-1}) \\ &= \sum_{p}\frac{\delta_{RS}}{d_S}N^{C(p)}\chi_S(p^{-1}).\end{aligned}$$

Finally, using the Schur Polynomial (3.6) in the specific case when $\hat{\phi} = \mathbf{1}$ so $\chi_R\left(\hat{\phi}\right) = Dim(R)$ is the dimension of the representation R as a representation of the unitary group, and $tr\left(\sigma\hat{\phi}\right) = tr\left(\hat{\phi}\right) = N^{C(\sigma)}$ we obtain

$$\frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)N^{C(\sigma)} = Dim(R). \tag{3.22}$$

Since $C(p) = C(p^{-1})$, this leads us to the desired result:

$$\left\langle \chi_R\left(\hat{\phi}\right)\chi_S\left(\hat{\phi}^*\right)\right\rangle = \frac{n!}{d_R}\delta_{RS}Dim(R). \tag{3.23}$$

Our explicit results for S_2 were

$$\begin{aligned}\left\langle \chi_{\yng(2)}\left(\hat{\phi}\right)\chi_{\yng(1,1)}\left(\hat{\phi}^*\right)\right\rangle &= 0; \\ \left\langle \chi_{\yng(2)}\left(\hat{\phi}\right)\chi_{\yng(2)}\left(\hat{\phi}^*\right)\right\rangle &= N(N+1).\end{aligned}$$

Using the new result we see this immediately:

$$\left\langle \chi_{\yng(2)}\left(\hat{\phi}\right)\chi_{\yng(1,1)}\left(\hat{\phi}^*\right)\right\rangle = \frac{n!}{d_{\yng(2)}}\delta_{\yng(2)\,,\,\yng(1,1)}\,Dim(\yng(2)).$$

$d_{\yng(2)} = \frac{2!}{2} = 1$ and $n! = 2! = 2$. The dimension of $\yng(2)$ as a representation of U(N) is $\frac{N(N+1)}{2.1} = \frac{N(N+1)}{2}$. This is an explicit verification of the closed form result.

3.5.2 The (l,k) multipoint function

The correlator we are interested in in this case is

$$\left\langle \chi_{R_1}\left(\hat{\phi}\right)...\chi_{R_l}\left(\hat{\phi}\right)\chi_{S_1}\left(\hat{\phi}^*\right)...\chi_{S_k}\left(\hat{\phi}^*\right)\right\rangle. \tag{3.24}$$

Substituting the Schur polynomial expression (3.6) we have

$$\begin{aligned}
&\left\langle \chi_{R_1}\left(\hat{\phi}\right)...\chi_{R_l}\left(\hat{\phi}\right)\chi_{S_1}\left(\hat{\phi}^*\right)...\chi_{S_k}\left(\hat{\phi}^*\right)\right\rangle \\
=&\left\langle \sum_{\sigma_1}\frac{\chi_{R_1}(\sigma_1)}{n_{R_1}!}tr\left(\sigma_1\hat{\phi}\right)...\sum_{\sigma_l}\frac{\chi_{R_l}(\sigma_l)}{n_{R_l}!}tr\left(\sigma_l\hat{\phi}\right)\right. \\
&\left.\times\sum_{\tau_1}\frac{\chi_{S_1}(\tau_1)}{n_{S_1}!}tr\left(\tau_1\hat{\phi}^*\right)...\sum_{\tau_k}\frac{\chi_{S_k}(\tau_k)}{n_{S_k}!}tr\left(\tau_k\hat{\phi}^*\right)\right\rangle \\
=&\sum_{\sigma_i\in S_{n_{R_i}}}\sum_{\tau_j\in S_{n_{S_j}}}\left\langle\prod_i\frac{\chi_{R_i}(\sigma_i)}{n_{R_i}!}tr\left(\sigma_i\hat{\phi}\right)\prod_j\frac{\chi_{S_j}(\tau_j)}{n_{S_j}!}tr\left(\tau_j\hat{\phi}^*\right)\right\rangle.
\end{aligned}$$

By unravelling the trace using the notation $I(\sigma(n))$ for a set of n indices with their labels shuffled by a permutation σ (I(n) is a multi-index which is shorthand for a set of n indices) we find that the expansion above is equal to

$$\begin{aligned}
&\sum\sum_{I_1,...,I_l}\sum_{J_1,...,J_k} \\
&\left\langle\hat{\phi}\begin{pmatrix} I_1(n_{R_1}) \\ I_1(\sigma_1(n_{R_1}))\end{pmatrix}...\hat{\phi}\begin{pmatrix} I_l(n_{R_l}) \\ I_l(\sigma_l(n_{R_l}))\end{pmatrix}\hat{\phi}^*\begin{pmatrix} J_1(n_{S_1}) \\ J_1(\tau_1(n_{S_1}))\end{pmatrix}...\hat{\phi}^*\begin{pmatrix} J_k(n_{S_k}) \\ J_k(\tau_k(n_{S_k}))\end{pmatrix}\right\rangle,
\end{aligned}$$

with the upfront factor abbreviated as $\sum$

$$\sum=\sum_{\sigma_i\in S_{n_{R_i}}}\sum_{\tau_j\in S_{n_{S_j}}}\prod_i\frac{\chi_{R_i}(\sigma_i)}{n_{R_i}!}\prod_j\frac{\chi_{S_j}(\tau_j)}{n_{S_j}!}.$$

After contracting we obtain a sum over permutations in S_{n_T} where

$$n_T=\sum_{i=1}^{l}n_{R_i}=\sum_{j=1}^{k}n_{S_j}. \tag{3.25}$$

Then the multipoint function is given by

$$\sum\sum_I\sum_J\sum_{\alpha\in S_{n_T}}\delta\begin{pmatrix} I(n_T) \\ J(\alpha\prod_j\tau_j(n_T))\end{pmatrix}\delta\begin{pmatrix} J(n_T) \\ I(\alpha^{-1}\prod_i\sigma_i(n_T))\end{pmatrix}. \tag{3.26}$$

To perform the sum over the J multi-index, we have that 1 will contract with some $\alpha\tau_1\tau_2...\tau_k(k)$. This is the same as setting $\tau_k^{-1}...\tau_1^{-1}\alpha^{-1}(1) = k$. Upon collapse of the J sum then, we have

$$\begin{aligned} &\sum\sum_I \sum_{\alpha\in S_{n_T}} \delta\left(\begin{array}{c} I(\prod_j \tau_j^{-1}\alpha^{-1}(n_T)) \\ J(\alpha^{-1}\prod_i \sigma_i(n_T)) \end{array}\right) \\ &= \sum\sum_{\alpha\in S_{n_T}} N^{C(\prod_i \sigma_i^{-1}\alpha \prod_j \tau_j^{-1}\alpha^{-1})}. \end{aligned} \tag{3.27}$$

This follows from the multi-index version of

$$\begin{aligned} N^{C(\sigma)} &= \sum_{i_1,...i_n} \delta^{i_1}_{1_{\sigma(1)}}\delta^{i_2}_{1_{\sigma(2)}}...\delta^{i_n}_{1_{\sigma(n)}} \\ &= \sum_I \delta\left(\begin{array}{c} I(n) \\ I(\sigma(n)) \end{array}\right), \end{aligned}$$

which generalises to

$$\sum_I \delta\left(\begin{array}{c} I(\alpha(n)) \\ I(\beta(n)) \end{array}\right) = N^{C(\beta\alpha^{-1})} = N^{C(\alpha^{-1}\beta)}. \tag{3.28}$$

(3.28) follows from acting on the pair (α,β) with α^{-1} from the left. This leads to (3.27). Now, as we did for the two-point case, we introduce a sum over $p \in S_{n_T}$ and constrain it via a delta function. This yields

$$\sum\sum_{\alpha\in S_{n_T}}\sum_{p\in S_{n_T}} N^{C(p)}\delta\left(p^{-1}\prod_i \sigma_i^{-1}\alpha\prod_j \tau_j^{-1}\alpha^{-1}\right). \tag{3.29}$$

The relation (3.21) above for orthogonality of characters, follows from the fundamental orthogonality relation. When applied to results arising from Cayley's theorem, it gives rise[7] to the following expression in terms of characters for a delta function on the group algebra of S_n:

$$\sum_R d_R\chi_R(\rho) = n!\delta(\rho), \tag{3.30}$$

where the delta function above is one when the argument is the identity and zero otherwise [4]. This yields the following expression for our multi-point function

[7]See Appendix A.

(where only the relevant factors are given for reasons of clarity of presentation):

$$\sum_{\alpha,p\in S_{n_T}} N^{C(p)}\delta\left(p^{-1}\prod_i \sigma_i^{-1}\alpha\prod_j \tau_j^{-1}\alpha^{-1}\right) \tag{3.31}$$
$$= \sum_{\alpha,p\in S_{n_T}} \sum_{T\in Rep(S_{n_T})} \frac{d_T}{n_T!} N^{C(p)}\chi_T\left(p^{-1}\prod_i \sigma_i^{-1}\alpha\prod_j \tau_j^{-1}\alpha^{-1}\right).$$

Next we make use of the fact that

$$\chi_R(C\sigma) = \frac{\chi_R(C)\chi_R(\sigma)}{d_R}, \tag{3.32}$$

where C is an element of the centre of the group and σ is an arbitrary element of the symmetric group. Recall that an element of the centre is an element of the group in question which commutes with all other elements. We invoke (3.32) for both

$$\sum_{p\in S_{n_T}} N^{C(p)} \text{ and } \sum_{\alpha\in S_{n_T}} \alpha\prod_j \tau_j^{-1}\alpha^{-1}.$$

$\sum_{p\in S_{n_T}} N^{C(p)}p$ is in the centre of S_{n_T}(i.e. it commutes with everything) because C is a class function. Consider $\sum_{p\in S_{n_T}} N^{C(p)}p\tau$ and $\sum_{p\in S_{n_T}} N^{C(p)}\tau p$. Since C is a class function (this can be seen from the definition), $N^{C(\tau^{-1}p\tau)} = N^{C(p)}$. If we choose to sum over $\tau^{-1}p\tau \in S_{n_T}$ then we have

$$\begin{aligned}\sum_{p\in S_{n_T}} N^{C(p)}\tau p &= \sum_{\tau^{-1}p\tau\in S_{n_T}} N^{C(\tau^{-1}p\tau)}\tau\tau^{-1}p\tau \\ &= \sum_{p\in S_{n_T}} N^{C(p)}p\tau.\end{aligned}$$

Thus $\sum_{p\in S_{n_T}} N^{C(p)}\tau p = \sum_{p\in S_{n_T}} N^{C(p)}p\tau$ and since τ was an arbitrary element of S_{n_T}, $\sum_{p\in S_{n_T}} N^{C(p)}p$ commutes with everything in S_{n_T} and is therefore in the centre of the group.

Using (3.32) we find that the correlator under consideration is given by

$$\begin{aligned}
&\sum\sum_{\alpha,p\in S_{n_T}}\sum_{T\in Rep(S_{n_T})}\frac{d_T}{n_T!}N^{C(p)}\chi_T\left(\prod_i\sigma_i^{-1}\alpha\prod_j\tau_j^{-1}\alpha^{-1}p\right)\\
=&\sum\sum_{\alpha,p\in S_{n_T}}\sum_{T\in Rep(S_{n_T})}\frac{d_T}{n_T!}\frac{1}{d_T}N^{C(p)}\chi_T\left(\prod_i\sigma_i^{-1}\alpha\prod_j\tau_j^{-1}\alpha^{-1}\right)\\
=&\sum\sum_{\alpha,p\in S_{n_T}}\sum_{T\in Rep(S_{n_T})}\frac{1}{n_T!}N^{C(p)}\chi_T(p)\chi_T\left(\prod_i\sigma_i^{-1}\alpha\prod_j\tau_j^{-1}\alpha^{-1}\right),
\end{aligned}$$

It is trivial to see that $\sum_\alpha \alpha\prod_j\tau_j^{-1}\alpha^{-1}$ is also in the centre. Now,

$$\sum_{\alpha\in S_{n_T}}\alpha\prod_j\tau_j^{-1}\alpha^{-1}\kappa \;=\; \sum_{\alpha\in S_{n_T}}\kappa\alpha\prod_j\tau_j^{-1}\alpha^{-1} \tag{3.33}$$

for an arbitrary $\kappa\in S_{n_T}$. This can be seen by setting $\beta=\kappa\alpha$.
Then $\beta^{-1}=\alpha^{-1}\kappa^{-1}$ so $\beta^{-1}\kappa=\alpha^{-1}$ and

$$\sum_\alpha\kappa\alpha\prod_j\tau_j^{-1}\alpha^{-1} \;=\; \sum_\beta\beta\prod_j\tau_j^{-1}\beta^{-1}\kappa. \tag{3.34}$$

We can therefore use (3.32) once more to arrive at

$$\begin{aligned}
&\sum\sum_{\alpha,p\in S_{n_T}}\sum_{T\in Rep(S_{n_T})}\frac{1}{n_T!}N^{C(p)}\chi_T(p)\chi_T(\prod_i\sigma_i^{-1}\alpha\prod_j\tau_j^{-1}\alpha^{-1})\\
=&\sum\sum_{\alpha,p\in S_{n_T}}\sum_{T\in Rep(S_{n_T})}\frac{1}{n_T!}N^{C(p)}\chi_T(p)\frac{\chi_T(\alpha\prod_j\tau_j^{-1}\alpha^{-1})\chi_T(\prod_i\sigma_i^{-1})}{d_T}.
\end{aligned}$$

$\chi_T(\alpha\prod_j\tau_j^{-1}\alpha^{-1})=\chi_T(\prod_j\tau_j^{-1})$ since characters are class functions, so we can perform the sum over $\alpha\in S_{n_T}$. Because S_{n_T} has $n_T!$ elements this sum contributes a factor of $n_T!$ which cancels with $\frac{1}{n_T!}$, leaving us with

$$\sum\sum_{p\in S_{n_T}}\sum_{T\in Rep(S_{n_T})}\frac{1}{d_T}N^{C(p)}\chi_T(p)\chi_T(\prod_j\tau_j^{-1})\chi_T(\prod_i\sigma_i^{-1}).$$

From the expansion

$$\chi_S(\sigma_1.\sigma_2) \;=\; \sum_{R_1\in Rep(S_{n_1})}\sum_{R_2\in Rep(S_{n_2})}g(R_1,R_2;S)\chi_{R_1}(\sigma_1)\chi_{R_2}(\sigma_2), \tag{3.35}$$

where $g(R_1, R_2; S)$ is the multiplicity of the representation S in the tensor product of the representations R_1 and R_2, we have that

$$\begin{aligned} \chi_T\left(\prod_i \sigma_i^{-1}\right) &= \chi_T(\sigma_1^{-1}\sigma_2^{-1}...) \\ &= \sum_{R_1',...,R_l'} g(R_1',...,,R_l';T)\prod_i \chi_{R_i'}(\sigma_i^{-1}). \end{aligned}$$

We can expand $\chi_T(\prod_j \tau_j^{-1})$ similarly. Then the multipoint function can be seen to be

$$\begin{aligned} &\sum\sum_{p\in S_{n_T}}\sum_{T\in \mathrm{Rep}(S_{n_T})} \frac{N^{C(p)}\chi_T(p)}{d_T} \\ &\times \sum_{R_1',R_2',...,R_l'}\sum_{S_1',S_2',...,S_k'} g(R_1',...,R_l';T)g(S_1',...,S_k';T)\prod_i \chi_{R_i'}(\sigma_i^{-1})\prod_j \chi_{S_j'}(\tau_j^{-1}). \end{aligned}$$

Now we use the orthogonality of characters (3.21) with the identity

$$\sum_\sigma \chi_R(\sigma^{-1})\chi_S(\sigma) = \delta_{RS}n!, \tag{3.36}$$

and find that

$$\sum_{\sigma_i\in S_{n_{R_i}}}\prod_i \frac{\chi_{R_i}(\sigma_i)}{n_{R_i}!}\prod_i \chi_{R_i'}(\sigma_i^{-1}) = \prod_i \delta_{R_i,R_i'}. \tag{3.37}$$

$\prod_j \delta_{S_j,S_j'}$ is similarly obtained from the sum over $\tau_j \in S_{n_{S_j}}$. The resulting deltas allow us to collapse the sums over R_i' and S_j', leaving us with

$$\sum_{p\in S_{n_T}}\sum_{T\in Rep(S_{n_T})} \frac{1}{d_T}N^{C(p)}\chi_T(p)g(R_1,...,R_l;T)g(S_1,...,S_k;T).$$

Finally, the sum $\sum_p N^{C(p)}\chi_T(p)$ is given by

$$\sum_p N^{C(p)}\chi_T(p) = n_T!Dim(T), \tag{3.38}$$

so that we see the multipoint function must be:

$$\begin{aligned} &\sum_{\sigma_i\in S_{n_{R_i}}}\sum_{\tau_j\in S_{n_{S_j}}}\left\langle \prod_i \frac{\chi_{R_i(\sigma_i)}}{n_{R_i}!}\, tr\left(\sigma_i\hat\phi\right)\prod_j \frac{\chi_{S_j(\tau_j)}}{n_{S_j}!}\, tr\left(\tau_j\hat\phi^*\right)\right\rangle \\ &= \sum_T \frac{n_T!\ Dim(T)}{d_T} g(R_1,R_2,...,R_l;T)g(S_1,S_2,...,S_k;T). \end{aligned} \tag{3.39}$$

3.6 Giant gravitons

So far in this chapter we have presented in detail the machinery developed by Corley, Jevicki and Ramgoolam [4] to allow computation of a class of correlators in U(N) $\mathcal{N} = 4$ SYM at all N. The significance of this extension in terms of the AdS/CFT correspondence and the 't Hooft limit has already been discussed. Here we review the giant graviton interpretation of the results of [4].

Young diagrams with a small number of boxes, i.e. those associated with small R-charge, correspond to Kaluza-Klein states. This can be seen by comparing these Young diagram correlators with their Kaluza-Klein counterparts: the correlators have been found to agree in the large N limit.[8]

Corley, Jevicki and Ramgoolam identified two classes of Young diagrams with a large number of boxes: mostly antisymmetric representations (diagrams with mostly long columns of boxes) and mostly symmetric representations (diagrams with mostly long rows of boxes). Young diagrams characterising representations of U(N) have a restriction on the length of their columns, which cannot have more than N boxes. This cut-off was identified with the angular momentum cut-off obeyed by sphere giants, since the number of boxes in the Young diagram of the representation corresponds to the angular momentum of the gravitational counterpart. With no cut-off on the row length, the large symmetric representations have the natural interpretation of being dual to AdS giants. Thus the map to Young diagrams naturally suggests candidates for giant graviton duals.

[8]See [4] where [41] is cited.

Chapter 4

A "special" case: fields in su(N)

4.1 Motivation for extending the U(N) results to SU(N)

A U(N) gauge theory is equivalent to a $U(1) \times SU(N)$ gauge theory up to Z_N identifications, as discussed in Chapter 1. The U(1) vector multiplet is related to the centre of mass motion of all the branes [14]. It is an SU(N) gauge group which is therefore of interest in the bulk [5].

A field $\hat{\phi}$ in the U(N) Lie Algebra is an $\mathrm{N} \times \mathrm{N}$ hermitian matrix. It has N^2 co-ordinates. For a field ϕ in the SU(N) Lie algebra we have the added condition of tracelessness of the matrix, so that there are only $\mathrm{N}^2 - 1$ independent co-ordinates. At large enough N therefore, there are a limited number of tests one could perform to distinguish between the two cases. Thus, until the analogous calculational tools for the SU(N) gauge theory found in this work were obtained, there was no clear way to distinguish between the roles of bulk and boundary degrees of freedom in the field theory dual. This explains why it was desirable to generalise or extend the computational techniques of [4] to the SU(N) case. This chapter covers that extension, while in the following we examine its interpretation in terms of giant gravitons and the disentangling of bulk and boundary degrees of freedom. The contents of these two chapters were published in [18].

4.2 The differences between the SU(N) and U(N) cases

4.2.1 An easy example

The crucial difference between the U(N) and SU(N) cases is best seen by considering their Schur polynomials for a given case. A simple example is that of S_2, the permutation group of 2 elements :

$$S_2 \quad = \quad \{(1), (12)\}\,. \tag{4.1}$$

The representations of S_2 are $\square\square$ and $\begin{array}{c}\square\\[-4pt]\square\end{array}$: the completely symmetric and the completely antisymmetric representations respectively.

Schur polynomials of a field $\hat{\phi} \in$ u(N) are given by

$$\chi_R\left(\hat{\phi}\right) \quad = \quad \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)tr\left(\sigma\hat{\phi}\right), \tag{4.2}$$

where R is the representation being considered - of the unitary group on the left hand side of the expression and of the permutation group on the right hand side. Both can be characterised by the same Young diagram. $\chi_R(\sigma)$ is the character of $\sigma \in S_n$. The trace $tr\left(\sigma\hat{\phi}\right)$ is given by

$$tr\left(\sigma\hat{\phi}\right) \quad = \quad \sum_{i_1,\ldots,1_n}\hat{\phi}^{i_1}_{i_{\sigma(1)}}\ldots\hat{\phi}^{i_n}_{i_{\sigma(n)}}. \tag{4.3}$$

So, for n = 2,

$$\begin{aligned} tr\left((1)\hat{\phi}\right) &= \sum_{i,j}\hat{\phi}^i_i\hat{\phi}^j_j \\ &= \left(tr\hat{\phi}\right)^2; \\ tr\left((12)\hat{\phi}\right) &= \sum_{i,j}\hat{\phi}^i_j\hat{\phi}^j_i \\ &= tr\left(\hat{\phi}^2\right). \end{aligned}$$

From (4.2) then, the Schur polynomials for S_2 can be constructed using the character table of the permutation group:[1]

$$\begin{aligned}
\chi_{\ydiagram{2}}\left(\hat{\phi}\right) &= \frac{1}{2!}\sum_{\sigma\in S_2}\chi_{\ydiagram{2}}(\sigma)tr\left(\sigma\hat{\phi}\right)\\
&= \frac{1}{2}\left[\chi_{\ydiagram{2}}((1))tr\left((1)\hat{\phi}\right)+\chi_{\ydiagram{2}}((12))tr\left((12)\hat{\phi}\right)\right]\\
&= \frac{1}{2}\left[\left(tr\hat{\phi}\right)^2+tr\left(\hat{\phi}^2\right)\right];
\end{aligned}$$

$$\begin{aligned}
\chi_{\ydiagram{1,1}}\left(\hat{\phi}\right) &= \frac{1}{2!}\sum_{\sigma\in S_2}\chi_{\ydiagram{1,1}}(\sigma)tr\left(\sigma\hat{\phi}\right)\\
&= \frac{1}{2}\left[\chi_{\ydiagram{1,1}}((1))tr\left((1)\hat{\phi}\right)+\chi_{\ydiagram{1,1}}((12))tr\left((12)\hat{\phi}\right)\right]\\
&= \frac{1}{2}\left[\left(tr\hat{\phi}\right)^2-tr\left(\hat{\phi}^2\right)\right].
\end{aligned}$$

However, for an su(N) field the full trace $tr\left(\phi\right)$ is zero so the Schur polynomials of $\phi\in$ su(N) for n = 2 are

$$\begin{aligned}
\chi_{\ydiagram{2}}(\phi) &= \frac{1}{2}\left[tr\left(\phi^2\right)\right],\\
\chi_{\ydiagram{1,1}}(\phi) &= -\frac{1}{2}\left[tr\left(\phi^2\right)\right].
\end{aligned}$$

These are clearly not independent. There is only one Schur polynomial for $n = 2$ and $\phi\in$ su(N), whereas there are 2 for $\hat{\phi}\in$ u(N). A less trivial example is the n = 4 case. S_4 has 4! = 24 elements and 5 representations: (4), (3,1), (2,2), (2,1^2),(1^4) or

$$\ydiagram{4}\,,\ \ydiagram{3,1}\,,\ \ydiagram{2,2}\,,\ \ydiagram{2,1,1}\,,\ \ydiagram{1,1,1,1}\,.$$

[1]See Appendix B.

So we can read off from the character tables that for $\hat{\phi} \in$ u(N):

$$\begin{aligned}
\chi_{\yng(4)}\left(\hat{\phi}\right) &= \frac{1}{24}\left[\left(tr\hat{\phi}\right)^4 + 6tr\left(\hat{\phi}^2\right)\left(tr\hat{\phi}\right)^2 + 3tr\left(\hat{\phi}^2\right)tr\left(\hat{\phi}^2\right)\right.\\
&\qquad \left. + 8tr\left(\hat{\phi}^3\right)tr\hat{\phi} + 6tr\left(\hat{\phi}^4\right)\right]\\
\chi_{\yng(3,1)}\left(\hat{\phi}\right) &= \frac{1}{24}\left[3\left(tr\hat{\phi}\right)^4 + 6tr\left(\hat{\phi}^2\right)\left(tr\hat{\phi}\right)^2 - 3tr\left(\hat{\phi}^2\right)tr\left(\hat{\phi}^2\right) - 6tr\left(\hat{\phi}^4\right)\right]\\
\chi_{\yng(2,2)}\left(\hat{\phi}\right) &= \frac{1}{24}\left[2\left(tr\hat{\phi}\right)^4 + 6tr\left(\hat{\phi}^2\right)tr\left(\hat{\phi}^2\right) - 8tr\left(\hat{\phi}^3\right)tr\hat{\phi}\right]\\
\chi_{\yng(2,1,1)}\left(\hat{\phi}\right) &= \frac{1}{24}\left[3\left(tr\hat{\phi}\right)^4 - 6tr\left(\hat{\phi}^2\right)\left(tr\hat{\phi}\right)^2 - 3tr\left(\hat{\phi}^2\right)tr\left(\hat{\phi}^2\right) + 6tr\left(\hat{\phi}^4\right)\right]\\
\chi_{\yng(1,1,1,1)}\left(\hat{\phi}\right) &= \frac{1}{24}\left[\left(tr\hat{\phi}\right)^4 - 6tr\left(\hat{\phi}^2\right)\left(tr\hat{\phi}\right)^2 + 3tr\left(\hat{\phi}^2\right)tr\left(\hat{\phi}^2\right)\right.\\
&\qquad \left. + 8tr\left(\hat{\phi}^3\right)tr\hat{\phi} - 6tr\left(\hat{\phi}^4\right)\right].
\end{aligned}$$

The Schur polynomials for $\phi \in$ su(N) are therefore

$$\begin{aligned}
\chi_{\yng(4)}(\phi) &= \frac{1}{24}\left[3tr\left(\phi^2\right)tr\left(\phi^2\right) + 6tr\left(\phi^4\right)\right]\\
\chi_{\yng(3,1)}(\phi) &= \frac{1}{24}\left[-3tr\left(\phi^2\right)tr\left(\phi^2\right) - 6tr\left(\phi^4\right)\right]\\
\chi_{\yng(2,2)}(\phi) &= \frac{1}{24}\left[6tr\left(\phi^2\right)tr\left(\phi^2\right)\right]\\
\chi_{\yng(2,1,1)}(\phi) &= \frac{1}{24}\left[-3tr\left(\phi^2\right)tr\left(\phi^2\right) + 6tr\left(\phi^4\right)\right]\\
\chi_{\yng(1,1,1,1)}(\phi) &= \frac{1}{24}\left[3tr\left(\phi^2\right)tr\left(\phi^2\right) - 6tr\left(\phi^4\right)\right],
\end{aligned}$$

i.e. there are fewer independent Schur polynomials of $\phi \in$ su(N) for a given n.

4.2.2 At the drop of a hat: loss of orthogonality

The deeper reason

The orthogonality of the two-point function in the U(N) gauge group case is an orthogonality in the representations. For an su(N) field, however, the full trace of ϕ is zero, so - as we have just seen - the Schur polynomials are no longer independent. The condition of tracelessness therefore means that the orthogonality of these correlators for the u(N) field does not survive for the correlators of the su(N) field. The orthogonal two-point function has been lost.

A modified correlator

From the discussion above we know that Schur polynomials of su(N) fields ϕ are not orthogonal. This is because of the tracelessness of the fields. When considering correlators, it can be seen that this condition is contained in the basic correlator that we are forced to modify for the SU(N) gauge group case.

We have that

$$\left\langle \hat{\phi}^i_j \hat{\phi}^{*k}_l \right\rangle = \delta^i_l \delta^k_j \tag{4.4}$$

for $\hat{\phi} \in$ u(N). If we allow for the only other possible term in the correlator and solve for its coefficient c in the su(N) field case,

$$\left\langle \phi^i_j \phi^{*k}_{\ l} \right\rangle = \delta^i_l \delta^k_j + c\delta^i_j \delta^k_l, \tag{4.5}$$

by using the condition of tracelessness of the su(N) field,

$$\left\langle \phi^i_i \phi^{*j}_{\ j} \right\rangle = 0, \tag{4.6}$$

we find that

$$\begin{aligned} 0 &= \delta^i_j \delta^j_i + c\delta^i_i \delta^j_j \\ 0 &= \delta^i_i + cN^2 \\ 0 &= N + cN^2; \\ c &= -\frac{1}{N}. \end{aligned}$$

The second term is clearly <u>not</u> zero for $\phi \in$ su(N) and the correlator is modified:

$$\left\langle \phi^i_j \phi^{*k}_{\ l} \right\rangle = \delta^i_l \delta^k_j - \frac{1}{N}\delta^i_j \delta^k_l. \tag{4.7}$$

Note that this modified correlator reduces to the U(N) correlator when N is large.

4.3 A computational rule

4.3.1 Starting point: a modified correlator

The modified correlator (4.7) means that the introduction of a ghost v with the following properties

$$\begin{aligned} < vv^* > &= -\frac{1}{N}, \\ < vv > &= < v^* v^* >= 0, \\ < v^n, v^{*n} > &= \left(-\frac{1}{N}\right)^n n! \end{aligned} \tag{4.8}$$

will allow us to transform between u(N) and su(N) fields. The ghost is so-called because its norm is negative. It counts as a negative degree of freedom. The ghost is used to subtract the trace of a u(N) field to produce an su(N) field, as can be seen in the explicit transformation which replaces $\phi \in$ su(N) with $\hat{\phi} \in$ u(N) + v:

$$\phi^i_j \quad \Rightarrow \quad \hat{\phi}^i_j + v\delta^i_j. \tag{4.9}$$

Using this transformation, we can write the expectation value of an arbitrary function of su(N) fields in terms of u(N) fields and the ghost v. The basic relation is

$$\langle F[\phi, \phi^*] \rangle = \left\langle F\left[\hat{\phi} + v\mathbf{1}, \hat{\phi}^* + v^*\mathbf{1}\right]\right\rangle. \tag{4.10}$$

4.3.2 A Schur polynomial derivative

Because the results for the U(N) gauge group case hinge on the Schur polynomials of the fields, we next consider how the transformation (4.9) applies to the Schur polynomial. Via (4.9), a Taylor expansion allows us to express Schur polynomials in ϕ as sums of derivatives of Schur polynomials of $\hat{\phi}$:

$$
\begin{aligned}
\chi_R(\phi) &= \chi_R\left(\hat{\phi}+\mathbf{1}v\right) \\
\chi_R(\phi) &= \sum_{m=0}^{n} \frac{v^m \delta_{ij}}{m!} \frac{d^m}{d\phi_{ij}^m} \chi_R\left(\hat{\phi}\right) \\
\chi_R(\phi) &= \sum_{m=0}^{n} \frac{v^m}{m!} \frac{d^m}{d\phi_{ii}^m} \chi_R\left(\hat{\phi}\right) \\
\chi_R(\phi) &= \sum_{m=0}^{n} \frac{v^m}{m!} D^m \chi_R\left(\hat{\phi}\right) \\
\chi_R(\phi) &= \chi_R\left(\hat{\phi}\right) + v\frac{d}{d\phi_{ii}}\chi_R\left(\hat{\phi}\right) + ...
\end{aligned}
$$

We are therefore interested in how the derivative

$$
D = \frac{d}{d\phi_{ii}} \tag{4.11}
$$

(where the repeated index indicates a summation) acts on the Schur polynomials. First we consider the action on the traced products that make up these polynomials.
For $n > 1$:

$$
\begin{aligned}
\frac{d}{d\hat{\phi}_{ij}} tr\left(\hat{\phi}^n\right) &= n\left(\hat{\phi}^{n-1}\right)_{ji} \\
\Rightarrow \delta_{ij}\frac{d}{d\hat{\phi}_{ij}} tr\left(\hat{\phi}^n\right) &= n\left(\hat{\phi}^{n-1}\right)_{ii} \\
\delta_{ij}\frac{d}{d\hat{\phi}_{ij}} tr\left(\hat{\phi}^n\right) &= ntr\left(\hat{\phi}^{n-1}\right) \\
\frac{d}{d\hat{\phi}_{ii}} tr\left(\hat{\phi}^n\right) &= ntr\left(\hat{\phi}^{n-1}\right).
\end{aligned}
$$

For n = 1:

$$
\begin{aligned}
\frac{d}{d\hat{\phi}_{ij}} tr\left(\hat{\phi}\right) &= \delta_{ij} \\
\Rightarrow \delta_{ij}\frac{d}{d\hat{\phi}_{ij}} tr\left(\hat{\phi}\right) &= \delta_{ii} \\
\frac{d}{d\hat{\phi}_{ii}} tr\left(\hat{\phi}\right) &= N.
\end{aligned}
$$

If we introduce the notation $(n) \leftrightarrow tr\left(\hat{\phi}^n\right)$, then the action of the above derivative is

$$
\begin{aligned}
(n) &\rightarrow n(n-1) \qquad n>1; \\
(1) &\rightarrow N \qquad n=1,
\end{aligned}
$$

which looks like a usual derivative. The N arises from the fact that the fields are matrix-valued. We can now investigate the Schur polynomial of ϕ as a sum of derivatives of Schur polynomials of $\hat{\phi}$ which we know how to evaluate:

$$
\chi_R(\phi) \;=\; \sum_{m=0}^{n} \frac{v^m}{m!} \chi_R^{(m)}\left(\hat{\phi}\right), \tag{4.12}
$$

where

$$
\chi_R^{(n)}\left(\hat{\phi}\right) \;=\; \frac{d}{d\phi_{ii}^n} \chi_R\left(\hat{\phi}\right) = D^n \chi_R\left(\hat{\phi}\right). \tag{4.13}
$$

4.3.3 A Young diagram derivative

In applying the derivative defined in the previous section, a pattern emerges which allows us to write this derivative immediately from the Young diagram of the representation. Consider the following example:

$$
\begin{aligned}
\chi_{\square\square\square}\left(\hat{\phi}\right) &= \frac{1}{6}\left[\left(tr\hat{\phi}\right)^3 + 3tr\hat{\phi}tr\hat{\phi}^2 + 2tr\hat{\phi}^3\right] \\
\chi_{\square\square\square}^{(1)}\left(\hat{\phi}\right) &= \frac{1}{6}\left[3N\left(tr\hat{\phi}\right)^2 + 3Ntr\left(\hat{\phi}^2\right) + 6\left(tr\hat{\phi}\right)^2 + 6tr\left(\hat{\phi}^2\right)\right] \\
\chi_{\square\square\square}^{(1)}\left(\hat{\phi}\right) &= (N+2)\frac{1}{2}\left[\left(tr\hat{\phi}\right)^2 + tr\left(\hat{\phi}^2\right)\right] \\
\chi_{\square\square\square}^{(1)}\left(\hat{\phi}\right) &= (N+2)\chi_{\square\square}\left(\hat{\phi}\right).
\end{aligned}
$$

It turns out that in general,

$$
\chi_R^{(1)}\left(\hat{\phi}\right) \;=\; \sum_S c_S \chi_S\left(\hat{\phi}\right), \tag{4.14}
$$

where we sum over all Young diagrams S that can be obtained by removing a single box from R such that what remains is still a valid Young diagram, and

c_S is the factor of the removed box.[2] In other words, the derivative turns out to have a simple expansion in terms of reduced Young diagrams. This process of reduction can be repeated until there are no more boxes to be removed, as demonstrated in the following examples:

Example 1

N	N + 1	N + 2

$$
\begin{aligned}
\chi^{(0)}_{\square\square\square}\left(\hat{\phi}\right) &= \chi_{\square\square\square}\left(\hat{\phi}\right)\\
\chi^{(1)}_{\square\square\square}\left(\hat{\phi}\right) &= (N+2)\chi_{\square\square}\left(\hat{\phi}\right)\\
\chi^{(2)}_{\square\square\square}\left(\hat{\phi}\right) &= (N+2)(N+1)\chi_{\square}\left(\hat{\phi}\right)\\
\chi^{(2)}_{\square\square\square}\left(\hat{\phi}\right) &= (N+2)(N+1)N.
\end{aligned}
$$

Or the more complicated

[2]This factor is the U(N) (or, more correctly, SU(N)) labelling factor of the removed box. The two-point function for $\hat{\phi}$ is $\left\langle \chi_R\left(\hat{\phi}\right)\chi_S\left(\hat{\phi}^*\right)\right\rangle = f_R\delta_{RS}$ where f_R is the same factor being referred to here. For example, for R given by

N	N + 1	N + 2
N -1	N	
N - 2		

we have $f_R = N(N+1)(N+2)(N-1)N(N-2) = N^2(N^2-1)(N^2-4)$.

Example 2

N	N + 1	N + 2
N - 1	N	

$$
\begin{aligned}
\chi^{(0)}_{(3,2)}\left(\hat{\phi}\right) &= \chi_{(3,2)}\left(\hat{\phi}\right)\\
\chi^{(1)}_{(3,2)}\left(\hat{\phi}\right) &= (N+2)\chi_{(2,2)}\left(\hat{\phi}\right) + N\chi_{(3,1)}\left(\hat{\phi}\right)\\
\chi^{(2)}_{(3,2)}\left(\hat{\phi}\right) &= N(N+2)\chi_{(2,1)}\left(\hat{\phi}\right) + N(N-1)\chi_{(3)}\left(\hat{\phi}\right) + N(N+2)\chi_{(2,1)}\left(\hat{\phi}\right)\\
\chi^{(2)}_{(3,2)}\left(\hat{\phi}\right) &= 2N(N+2)\chi_{(2,1)}\left(\hat{\phi}\right) + N(N-1)\chi_{(3)}\left(\hat{\phi}\right)\\
\chi^{(3)}_{(3,2)}\left(\hat{\phi}\right) &= 2N(N+2)(N+1)\chi_{(1,1)}\left(\hat{\phi}\right) + 2N(N+2)(N-1)\chi_{(2)}\left(\hat{\phi}\right)\\
&\quad +N(N-1)(N+2)\chi_{(2)}\left(\hat{\phi}\right)\\
\chi^{(4)}_{(3,2)}\left(\hat{\phi}\right) &= 5N(N+2)(N+1)(N-1)\chi_{(1)}\left(\hat{\phi}\right)\\
\chi^{(5)}_{(3,2)}\left(\hat{\phi}\right) &= 5N^2(N+2)(N+1)(N-1).
\end{aligned}
$$

Note that if there is more than one way to arrive at the same reduced Young diagram, a term is included for every possible route.

4.3.4 A computational rule for su(N) correlators

Putting together the expansion (4.12) of the su(N) Schur polynomial in terms of derivatives of the u(N) Schur polynomial (for which we have the reduction formulae (4.13)), and the contraction (4.8) of the expectation value of v, we find that

$$
\begin{aligned}
\langle \chi_R(\phi)\chi_S(\phi^*)\rangle &= \sum_{m=0}^{n}\sum_{p=0}^{n}\left\langle \frac{v^m}{m!}\frac{v^{*p}}{p!}\right\rangle \left\langle \chi_R^{(m)}\left(\hat{\phi}\right)\chi_S^{(p)}\left(\hat{\phi}^*\right)\right\rangle\\
&= \sum_{m=0}^{n}\frac{1}{m!}\sum_{p=0}^{n}\frac{1}{p!}\delta_{mp}\left(-\frac{1}{N}\right)^m m!\left\langle \chi_R^{(m)}\left(\hat{\phi}\right)\chi_S^{(p)}\left(\hat{\phi}^*\right)\right\rangle\\
&= \sum_{m=0}^{n}\frac{1}{m!}\left(-\frac{1}{N}\right)^m \left\langle \chi_R^{(m)}\left(\hat{\phi}\right)\chi_S^{(m)}\left(\hat{\phi}^*\right)\right\rangle.
\end{aligned}
$$

We have obtained a formula,

$$\langle \chi_R(\phi)\,\chi_S(\phi^*)\rangle = \sum_{m=0}^{n} \frac{1}{m!}\left(-\frac{1}{N}\right)^m \left\langle \chi_R^{(m)}\left(\hat{\phi}\right)\chi_S^{(m)}\left(\hat{\phi}^*\right)\right\rangle , \quad (4.15)$$

expressing su(N) correlators in terms of known u(N) correlators. The expectation value of su(N) Schur polynomials can easily be expressed as a sum over expectation values of products of multiple u(N) Schur polynomials. This is therefore a complete computational generalisation of the two-point function result of Corley, Jevicki and Ramgoolam [4].[3]

It is clear that analogous computation will lead to the corresponding generalisations of the higher-point results in [4]. For example, consider the three-point function (higher point functions follow similarly):

$$\begin{aligned}
&\langle \chi_R(\phi)\,\chi_S(\phi)\,\chi_T(\phi^*)\rangle \\
&= \sum_{m=0}^{n_R}\sum_{p=0}^{n_S}\sum_{q=0}^{n_T=n_R+n_S=n} \left\langle \frac{v^m}{m!}\frac{v^p}{p!}\frac{v^{*q}}{q!}\right\rangle \left\langle \chi_R^{(m)}\left(\hat{\phi}\right)\chi_S^{(p)}\left(\hat{\phi}\right)\chi_T^{(q)}\left(\hat{\phi}^*\right)\right\rangle \\
&= \sum_{m=0}^{n_R}\sum_{p=0}^{n_S}\sum_{q=0}^{n}\left(\frac{-1}{N}\right)^q q!\,\delta_{q,m+p}\frac{1}{m!}\frac{1}{p!}\frac{1}{q!}\left\langle \chi_R^{(m)}\left(\hat{\phi}\right)\chi_S^{(p)}\left(\hat{\phi}\right)\chi_T^{(q)}\left(\hat{\phi}^*\right)\right\rangle \\
&= \sum_{m=0}^{n_R}\sum_{p=0}^{n_S}\left(\frac{-1}{N}\right)^{m+p}\frac{1}{m!}\frac{1}{p!}\left\langle \chi_R^{(m)}\left(\hat{\phi}\right)\chi_S^{(p)}\left(\hat{\phi}\right)\chi_T^{(m+p)}\left(\hat{\phi}^*\right)\right\rangle .
\end{aligned}$$

4.3.5 Graphical simplifications

The visual nature of the reduction formula found above means that these correlators lend themselves to diagram-based calculation. Some observations on visual rules are presented here. Reflection in 45 degrees of a Young diagram R to give its conjugate $\bar{\mathrm{R}}$ simply corresponds to a change in sign in the factor brackets of f_R appearing in the respective Schur polynomial. In addition, a rule can be found for the correlator of a completely symmetric and a completely antisymmetric representation, because their reductions only overlap at order n and n-1.

$$\left\langle \chi_{(n)}(\phi)\,\chi_{(1^n)}(\phi^*)\right\rangle = \sum_{i=0}^{n}\frac{1}{i!}\left(-\frac{1}{N}\right)^i\left\langle \chi_{(n)}^{(i)}\left(\hat{\phi}\right)\chi_{(1^n)}^{(i)}\left(\hat{\phi}^*\right)\right\rangle \quad (4.16)$$

[3]In [40] Corley and Ramgoolam used projection operators to extend the U(N) computation to the SU(N) case. Our results agree.

where (n) is the completely symmetric and (1^n)'is the completely antisymmetric representation of m boxes:

$\begin{ytableau} \; & & & & & \cdots & \end{ytableau}$ and $\begin{ytableau} \; \\ \\ \\ \\ \\ \\ \vdots \\ \\ \end{ytableau}$. The only terms that will contribute to this sum are $i = n-1$ and $i = n$. These correspond to the $\square$ and no box steps. So,

$$
\begin{aligned}
&\left\langle \chi_{(n)}\left(\phi\right)\chi_{(1^n)}\left(\phi^*\right)\right\rangle \\
&= (N+(n-1))...(N+1)(N-(n-1))...(N-1) \\
&\times \left[\left\langle \chi_{\square}\left(\hat{\phi}\right)\chi_{\square}\left(\hat{\phi}^*\right)\right\rangle \frac{1}{(n-1)!}\left(-\frac{1}{N}\right)^{n-1} + \frac{1}{n!}\left(-\frac{1}{N}\right)^{n} N\right] \\
&= \frac{(N+n-1)!}{N(N-n)!\left(-\frac{1}{N}\right)^{n-1}}\frac{1}{n!}N(n-1) \\
&= \frac{(-1)^{n-1}(n-1)}{n!N^{n-1}}\frac{(N+n-1)!}{(N-n)!}.
\end{aligned}
$$

An additional simplification which cuts calculation of different Schur polynomials and correlators down considerably, is the existence of a visual block rule, explained in Appendix C. This block rule contains once more the information that the Schur polynomials are no longer independent, this time in a diagrammatic format.

4.4 The SU(N) constraints

4.4.1 The tensor product

The tensor product of two representations can be written as a combination of the corresponding Young diagrams as shown in this example:

$$\ydiagram{2,1} \times \ydiagram{1} = \ydiagram{2,1,1} + \ydiagram{2,2} + \ydiagram{3,1}. \tag{4.17}$$

The character of a group element T in a direct product representation is given by

$$\chi_{R\times S}(T) \quad = \quad \chi_R(T)\chi_S(T) \tag{4.18}$$

so Schur polynomials combine in the same way:

$$\chi_{\yng(2,1)} \times \chi_{\yng(1)} = \chi_{\yng(2,1,1)} + \chi_{\yng(2,2)} + \chi_{\yng(3,1)}. \tag{4.19}$$

Evaluating both sides as a function of $\phi \in su(N)$ with $tr(\phi) = 0$ and using $\chi_{\yng(1)}(\phi) = tr(\phi) = 0$ we find that the sum above is zero. This is the same information as is contained in the observation that the Schur polynomials are no longer independent in the SU(N) case, and can be used to find the zero eigenvectors of the su(N) correlators. This is discussed in the following section.

4.4.2 Zero eigenvectors

The conjugacy classes of S_n can be labelled by the partitions of n (to each of which we have been associating a Young diagram). Each partition of n that includes a cycle of length 1 can be turned into a partition of n-1 by dropping the cycle of length 1.

For example: the number of classes of S_4 is 5, while the number of classes of S_3 is 3. If we take the partitions of S_4 that contain cycles of length 1, and drop these cycles, we are left with the partitions of S_3:

$$(4),\ (3)(1),\ (2)(2),\ (2)(1)(1),\ (1)^4 \quad \xrightarrow{\text{1 cycles}} \quad (3)(1),\ (2)(1)(1),\ (1)^4$$

$$(3)(1),\ (2)(1)(1),\ (1)^4 \quad \xrightarrow{\text{drop the 1 cycles}} \quad (3),\ (2)(1) \text{ and } (1)^3.$$

It is these dropped partitions, the ones that contain cycles of length 1, which vanish for fields in su(N); i.e. it is these partitions which yield terms containing $tr\phi$ in the Schur polynomial. Thus the number of partitions that don't vanish is given by the difference between the number of classes of S_n and the number of classes of S_{n-1} (there are 2 partitions of 4 which contain no cycles of length 1 and so cannot be reduced to partitions of 3 by dropping a 1-cycle, as we saw for S_4 above). This is the number of non-zero eigenvectors of $\langle \chi_R(\phi)\chi_S(\phi^*)\rangle$. The zero eigenvectors can therefore be found by 'adding a box' to each representation of S_{n-1} i.e. taking the tensor product of each representation of S_{n-1} with $\yng(1)$. In our example there are two non-zero eigenvectors and three zero-eigenvectors, found by adding a box to each representation of S_3:

$$\begin{aligned}
\ydiagram{3} \times \ydiagram{1} &= \ydiagram{3,1} + \ydiagram{4} \\
\ydiagram{2,1} \times \ydiagram{1} &= \ydiagram{3,1} + \ydiagram{2,2} + \ydiagram{2,1,1} \\
\ydiagram{1,1,1} \times \ydiagram{1} &= \ydiagram{1,1,1,1} + \ydiagram{2,1,1}.
\end{aligned}$$

Then the zero eigenvectors of the two-point function are:

$$\begin{aligned}
v^{(1)} &= \chi_{\ydiagram{3,1}} + \chi_{\ydiagram{4}} \\
v^{(2)} &= \chi_{\ydiagram{3,1}} + \chi_{\ydiagram{2,2}} + \chi_{\ydiagram{2,1,1}} \\
v^{(3)} &= \chi_{\ydiagram{1,1,1,1}} + \chi_{\ydiagram{2,1,1}}.
\end{aligned}$$

It turns out that the constraints implied by these zero eigenvectors make up the full set of constraints. This can be seen by considering the partition function in more detail, as seen in the following section.

4.4.3 The partition function and the full set of constraints

The number of classes or representations of S_n is given by the partition function:

$$\begin{aligned}
\prod_{n=1}^{\infty} \frac{1}{1-(n)} &= \frac{1}{1-(1)} \frac{1}{1-(2)} \frac{1}{1-(3)} \cdots \\
&= (1 + (1) + (1)^2 + ...)(1 + (2) + (2)^2 + ...).... \\
&= 1 + (1) + [(1)^2 + (2)] + [(1)^3 + (2)(1) + (3)] + \\
&\quad [(1)^4 + (1)(3) + (1)^2(2) + (2)^2 + (4)] + ...
\end{aligned}$$

Because the u(N) Schur polynomials are orthogonal in the representations, there are as many independent u(N) Schur polynomials as there are representations (for a given n). Grouping the numbers of boxes together by using x^n instead of (n) we have :

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = 1 + x + 2x^2 + 3x^3 + 5x^4 + ... \tag{4.20}$$

from which we can read the number of u(N) Schur polynomials for a given n. For example, there are three u(N) Schur polynomials for $n = 3$. More concisely, we can write

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} c_n x^n \tag{4.21}$$

where c_n is the number of ways to partition n.

To find the number of independent su(N) Schur polynomials for a given n we must drop all the cycles of length 1 (note the lower limit of the product):

$$\begin{aligned}
\prod_{n=2}^{\infty} \frac{1}{1-(n)} &= \frac{1}{1-(2)} \frac{1}{1-(3)} \cdots \\
&= 1 + (2) + (3) + (2)^2 + (4) + (2)(3) + (5) + \ldots
\end{aligned}$$

or, following the U(N) analysis above,

$$\prod_{n=2}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} d_n x^n. \tag{4.22}$$

We can rewrite this:

$$\begin{aligned}
\prod_{n=2}^{\infty} \frac{1}{1-x^n} &= (1-x) \prod_{n=1}^{\infty} \frac{1}{1-x^n} \\
&= (1-x) \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} \\
&= \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_{n-1} x^n \\
&= \sum_{n=0}^{\infty} \left(c_n - c_{n-1}\right) x^n \\
&= \sum_{n=0}^{\infty} d_n x^n \text{ from above.}
\end{aligned}$$

We read off that

$$d_n = c_n - c_{n-1} \tag{4.23}$$

i.e. the number of independent su(N) Schur polynomials is less than that for U(N) by c_{n-1}. c_{n-1} is the number of degrees of freedom lost when one moves from

U(N) to SU(N). It corresponds to the number of representations of S_{n-1}, which is exactly the number of zero eigenvectors for n - the number of constraints we found above. The constraints given by "adding a box" to get the zero eigenvectors are therefore the full set of constraints for the su(N) field.

4.5 A linearly independent basis of operators

We have seen that the Schur polynomials no longer provide us with an orthogonal basis for half-BPS operators in $\mathcal{N} = 4$ SYM theory with SU(N) gauge group. Attempts to find such a basis using other types of symmetric polynomials failed. However, Schur polynomials do still prove useful in the SU(N) case. They allow us to obtain a set of linearly independent operators in one-to-one correspondence with the representations of interest in the SU(N) gauge theory.

4.5.1 A useful class of polynomials

We introduce the normalised Schur polynomial $\tilde{\chi}_R$:

$$\tilde{\chi}_R(\phi) = \frac{d_R}{D_R n_R!} \chi_R(\phi) . \tag{4.24}$$

$\tilde{\chi}_R\left(\hat{\phi}\right)$ (recall that $\hat{\phi} \in$ u(N), $\phi \in$ su(N)), when acted on by the derivative defined in (4.11), has the simplified form

$$D\tilde{\chi}_R\left(\hat{\phi}\right) = \sum_S \tilde{\chi}_S\left(\hat{\phi}\right) , \tag{4.25}$$

where, as before, we sum over all Young diagrams S that can be obtained by removing a single box from R. The factor of that box has been normalised out of this expansion.

To illustrate this explicitly consider D acting on $\chi_{\yng(2,1)}$ and $\tilde{\chi}_{\yng(2,1)}$ respectively.

$$D\left(\chi_{\yng(2,1)}\right) = (N-1)\chi_{\yng(2)} + (N+1)\chi_{\yng(1,1)}$$

and

$$\tilde{\chi}_{\yng(2,1)} = \frac{\chi_{\yng(2,1)}}{N(N+1)(N-1)},$$

from (4.24) above. So

$$\begin{aligned} D\left(\tilde{\chi}_{\yng(2,1)}\right) &= \frac{(N-1)\chi_{\yng(2)}}{N(N+1)(N-1)} + \frac{(N+1)\chi_{\yng(1,1)}}{N(N+1)(N-1)} \\ &= \frac{\chi_{\yng(2)}}{N(N+1)} + \frac{\chi_{\yng(1,1)}}{N(N-1)} \\ &= \tilde{\chi}_{\yng(2)} + \tilde{\chi}_{\yng(1,1)}. \end{aligned}$$

Note that D maps the space of Schur polynomials associated with Young diagrams with n boxes to the space of Schur polynomials associated with Young diagrams with n-1 boxes. The latter space has dimension c_{n-1} and the former has dimension c_n, with c_n the number of ways to partition n. Thus D maps from a c_n-dimensional space to a c_{n-1}-dimensional space. Clearly, it has $c_n - c_{n-1} = d_n$ zero eigenvectors, from (4.23). As we have seen, this is equal to the number of linearly independent Schur polynomials of $\phi \in$ su(N). Thus the basis of the null space of D is in a one-to-one correspondence with the half-BPS representations of the super Yang-Mills theory with gauge group SU(N). For any null vector $\hat{\phi}_0$ of D, i.e. any vector $\hat{\phi}_0$ such that $Df(\hat{\phi}_0) = 0$, we have the identity

$$f(\phi_0) \to f(\hat{\phi}_0 + v\mathbf{1}) = f(\hat{\phi}_0). \tag{4.26}$$

Since the derivative terms in the expansion of Section 3.3.2 are all zero, calculations for su(N) fields ϕ will be identical to those for u(N) fields $\hat{\phi}$ for this class of operators.

4.5.2 An algorithm for finding a linearly independent basis

Top block Young diagrams

Let us call the rightmost uppermost block in a Young diagram the 'top block'. Either the box in this position can be removed to leave a valid Young diagram, or it is fixed and we call the diagram a 'fixed top block diagram'. In the former case we can call the diagram a moveable top block diagram.

These classifications are useful because, as we will now show, fixed top block diagrams can be put in a one-to-one correspondence with the null vectors of D. To see this we construct a basis for the null vectors of D with a given $\mathcal{R}$ charge n, i.e. those corresponding to Young diagrams with n boxes which characterise representations dual to those of $\mathcal{R}$ charge n in the corresponding string theory.

Note that every moveable top block Young diagram of n boxes can be obtained by ‘adding’ $\square$ to a Young diagram with n-1 boxes such that the added box is in the top block position. For example,  is a moveable top block Young diagram with four boxes. It can be obtained by taking the tensor product of D with a 3-block Young diagram such that the added block is in the top block position, as shown in (4.19). These products give the relations between the Schur polynomials of $\phi \in$ su(N), as we saw in section 4.4.2, and can be used to eliminate the moveable top block Young diagrams. We are left with the fixed top block Young diagrams. Having used the relations between Schur polynomials of su(N) to obtain these, we know that there are as many fixed top block diagrams as there are independent Schur polynomials. This could also be seen by considering the fact that the Schur polynomial corresponding to a given Young diagram is proportional to the Schur polynomial corresponding to its conjugate, and the conjugate of a moveable top block diagram is a diagram corresponding to a representation containing a 1-cycle. The linearly independent Schur polynomials of ϕ are in one-to-one correspondence with the null vectors of D so we can also map fixed top block diagrams to the null space of D in a one-to-one way.

The algorithm

Before stating our algorithm for constructing a null vector of D from each fixed top block diagram, we define a reduction operation on the Schur polynomial. We refer to the Schur polynomial associated with a representation given by a certain Young diagram by referring to the Young diagram directly, since the reduction operation is defined most simply on Young diagrams. The reduction of a Schur polynomial is obtained by taking minus one times the sum of Schur polynomials corresponding to diagrams that can be obtained by all moves which move a box into the first row, such that after the move we obtain a valid Young diagram.

For example (with a prime denoting this reduction):

$$\begin{aligned}\chi'_{\yng(3,2,1)}(\phi) &= -\chi_{\yng(4,1,1)}(\phi) - \chi_{\yng(4,2)}(\phi)\\ \chi'_{\yng(4)}(\phi) &= 0.\end{aligned}$$

We can iterate these reductions: reducing at most n times an n-block Young diagram (or the Schur polynomial associated with that Young diagram). Our algorithm is this: we obtain a null vector of D for a given $\mathcal{R}$ charge n by taking the Schur polynomials corresponding to the fixed top block diagrams of n boxes and adding all possible reductions of that polynomial.

As an example consider the construction of the null vectors of D with $\mathcal{R}$ charge 4. The Young diagrams with 4 boxes are

$$\yng(4)\;,\;\yng(3,1)\;,\;\yng(2,2)\;,\;\yng(2,1,1)\;,\;\yng(1,1,1,1)\;.$$

Two of these are fixed top block diagrams: $\yng(2,2)$ and $\yng(1,1,1,1)$. Our algorithm gives us a null vector from each of these:

$$\begin{aligned}\chi_1(\phi) &= \chi_{\yng(2,2)}(\phi) - \chi_{\yng(3,1)}(\phi) + \chi_{\yng(4)}(\phi)\\ &\text{and}\\ \chi_2(\phi) &= \chi_{\yng(1,1,1,1)}(\phi) - \chi_{\yng(2,1,1)}(\phi) + \chi_{\yng(3,1)}(\phi) - \chi_{\yng(4)}(\phi)\,.\end{aligned}$$

It is easy to check that

$$\langle \chi_1(\phi)\,\chi_2(\phi^*)\rangle = \left\langle \chi_1\left(\hat{\phi}\right)\chi_2\left(\hat{\phi}^*\right)\right\rangle \neq 0. \tag{4.27}$$

Thus, the basis constructed using our algorithm is not an orthogonal one. Its linear independence can be seen by considering its two-point functions. When computing these, we can replace all ϕs by $\hat{\phi}$s, since the vectors are null vectors of D. Then we have that the Schur polynomials of the u(N) fields $\hat{\phi}$ that correspond to distinct Young diagrams are linearly independent. Each fixed top block diagram appears in a unique operator belonging to our basis, so its linear independence follows.

Chapter 5

Bulky giants

5.1 Introduction

The objective of this research was to extend a set of calculational rules for correlation functions in $\mathcal{N} = 4$ SYM with gauge group U(N) to the case of an SU(N) gauge symmetry. This was motivated by the role of certain of these correlators as duals to giant gravitons in the AdS/CFT correspondence. Corley, Jevicki and Ramgoolam [4] found an orthogonal basis of operators in a one-to-one correspondence with the half-BPS representations of $\mathcal{N} = 4$ SYM with U(N) gauge group. This was given by the Schur polynomials of $\hat{\phi} \in \mathrm{u(N)}$, which are labelled by Young diagrams characterising representations of the gauge group. The orthogonal basis these formed suggested natural field theory candidates for duals to giant gravitons, both AdS and sphere giants. The extension to SU(N) is of interest because an SU(N) gauge theory is believed to be the relevant dual to the theory in the AdS bulk, as discussed in Chapter 4.

A further goal of the research presented here was to elucidate the roles of supergravity bulk and boundary degrees of freedom in the dual $\mathcal{N} = 4$ super Yang-Mills theory. One way to do this is via the investigation of the corresponding candidate duals in the SU(N) case. Once the machinery of the last chapter had been developed, it was possible to study the physical implications of the extension to SU(N), including the operators of the SU(N) theory dual to giant gravitons in the bulk. This chapter deals with that investigation. Note that the results of this chapter were published in [18].

5.2 Schur polynomials and giant gravitons

5.2.1 Why Schur polynomials?

A toy model of the AdS/CFT correspondence was investigated in [42] and provides an explanation of the connection between Schur polynomials of (anti-)symmetric representations and the corresponding giant gravitons. Berenstein studied the large N gauge quantum mechanics for a single Hermitian matrix in the harmonic oscillator potential well, relating it to a decoupling limit of $\mathcal{N} = 4$ super Yang-Mills theory. With this identification one can relate BPS states in the SYM theory and states in the matrix model.

Three a priori distinct descriptions of the spectrum of this model were given. The first employs single trace operators and is related to closed string states in the dual. The second is found by integrating out the angular degrees of freedom to obtain a description in terms of eigenvalues and corresponding to an open string, D-brane picture. A similarity transform leads to the Hamiltonian for N free particles in the harmonic oscillator potential well. The eigenvalues are fermions, with the relevant antisymmetry arising from a Van der Monde determinant resulting from the integration. The third way to describe the spectrum is via a Schur polynomial basis. The Schur polynomials give another basis for the closed string description. Surprisingly, they are also shown to coincide with the dynamics of the eigenvalue basis. This equivalence is essentially an open-closed string duality.

Further, we know from the $c = 1$ matrix model that single eigenvalue excitations correspond to D-brane states. The map between the Schur polynomials and the description in terms of fermions therefore makes it clearer why the Schur polynomials corresponding to completely (anti-)symmetric representations are duals to giant gravitons, as suggested in [4]. From the relation of the eigenvalue basis to a description in terms of fermions, it can be seen that the ground state corresponds to a Fermi sea of eigenvalues with its level determined by N. We can view the spectrum as arising from excitations of the Fermi sea. One can describe a state in this spectrum with a Young diagram. The AdS and sphere giants can then be interpreted in terms of eigenvalues in the Fermi sea. The giant graviton in the AdS space corresponds in the matrix model to taking an eigenvalue from the top of the Fermi sea and exciting it by a large amount, while the sphere giant corresponds to creating a hole deep in the Fermi sea. The diagram (a column of n boxes) representing such a state has a limit on its number of boxes, putting a limit on the depth of the hole. This limit relates to the angular momentum cut-off obeyed by sphere giants.

5.2.2 Large N

When the Higgs fields (of which ϕ and $\hat{\phi}$ are built) can be simultaneously diagonalised, their eigenvalues can be interpreted as the transverse co-ordinates of the branes. $tr\,(\phi) = 0$ implies that the centre of mass of all the branes is fixed. The linearly independent basis constructed in the previous chapter consists of operators that are annihilated by D. D is essentially the centre of mass momentum, so in light of the tracelessness condition we see that the basis obtained is a natural one.

Strictly speaking, this constraint on the centre of mass motion implies that we cannot have single eigenvalue dynamics. We would like to identify natural candidate duals for giant gravitons in the SU(N) gauge theory. Without single eigenvalue dynamics, it is still possible in the large N limit to recover the relevant Schur polynomials as the operators dual to giant graviton states. Consider changing a single eigenvalue by an amount Δ, and then compensating for this by changing each of the remaining N-1 eigenvalues by $-\frac{\Delta}{N-1}$. This preserves the condition of tracelessness. In a systematic large N expansion, at leading order, we can ignore the $-\frac{\Delta}{N-1}$ effect. In this limit we therefore recover single eigenvalue dynamics and can expect the Schur polynomials to provide giant graviton duals as before.

As a test of this idea, we should recover orthogonality between Schur polynomials corresponding to totally symmetric representations (dual to AdS giants) and Schur polynomials corresponding to totally anti-symmetric representations (dual to sphere giants) in this limit.

We can check this explicitly using the techniques of Chapter 4. Note first that orthogonality in the large N limit is not obvious. Indeed, ratios like

$$\frac{\left\langle \chi_{\yng(5,1)}\left(\phi\right)\chi_{\yng(6)}\left(\phi^*\right)\right\rangle}{\left\langle \chi_{\yng(6)}\left(\phi\right)\chi_{\yng(6)}\left(\phi^*\right)\right\rangle},$$

and

$$\frac{\left\langle \left(\chi_{\yng(4,1,1)}\left(\phi\right)+\chi_{\yng(4,2)}\left(\phi\right)\right)\chi_{\yng(6)}\left(\phi^*\right)\right\rangle}{\left\langle \chi_{\yng(6)}\left(\phi\right)\chi_{\yng(6)}\left(\phi^*\right)\right\rangle},$$

are of order 1 independent of N. In fact,

$$\chi_{\yng(6)}\left(\phi\right) \quad = \quad -\chi_{\yng(5,1)}\left(\phi\right), \tag{5.1}$$

which is easily seen from the tensor product discussion in Chapter 4 or the (consequent) block rule set out in Appendix C. It will be true for similar ratios of

Schur polynomials labelled by Young diagrams with an arbitrarily large number of boxes.

To check the large N orthogonality between su(N) Schur polynomials in symmetric and antisymmetric representations we calculate

$$\frac{\mathcal{O}}{\mathcal{N}_S \mathcal{N}_A}.$$

$\mathcal{O}$ is the overlap between the states corresponding to completely symmetric and completely antisymmetric representations, and the denominator is the product of normalisations for the symmetric (S) and antisymmetric (A) representations:

$$\begin{aligned}
\mathcal{N}_S^2 &= \left\langle \chi_{(n)}\left(\phi\right), \chi_{(n)}\left(\phi^*\right)\right\rangle ; \\
\mathcal{N}_A^2 &= \left\langle \chi_{(1^n)}\left(\phi\right), \chi_{(1^n)}\left(\phi^*\right)\right\rangle ; \\
\mathcal{O} &= \left\langle \chi_{(n)}\left(\phi\right), \chi_{(1^n)}\left(\phi^*\right)\right\rangle .
\end{aligned}$$

where (n) is a row and (1^n) is a column of n boxes.

In the six box representation, these quantities would be

$$\begin{aligned}
\mathcal{O} &= \left\langle \chi_{\yng(6)}\left(\phi\right) \chi_{\yng(1,1,1,1,1,1)}\left(\phi^*\right)\right\rangle , \\
\mathcal{N}_S^2 &= \langle \chi_{\yng(6)}\left(\phi\right) \chi_{\yng(6)}\left(\phi^*\right)\rangle , \\
\mathcal{N}_A^2 &= \left\langle \chi_{\yng(1,1,1,1,1,1)}\left(\phi\right) \chi_{\yng(1,1,1,1,1,1)}\left(\phi^*\right)\right\rangle .
\end{aligned}$$

We are interested in computing these quantities as functions of the number of boxes n in the Young diagrams characterising the representations. The expression for the correlator of a completely symmetric and a completely antisymmetric representation $\mathcal{O}$ has already been given in Section 4.3.5. It is particularly simple because the reductions only overlap at order n and order $n-1$.

$$\mathcal{O} = \frac{(-1)^{n-1}(n-1)}{n!N^{n-1}} \frac{(N+n-1)!}{(N-n)!}. \tag{5.2}$$

For both $\mathcal{N}_A$ and $\mathcal{N}_S$, contributions need to be summed from all $n+1$ terms in the Taylor expansion. We find

$$\begin{aligned}
\mathcal{N}_S^2 &= \left\langle \chi_{(n)}\left(\phi\right)\chi_{(n)}\left(\phi^*\right)\right\rangle \\
&= \sum_{i=0}^{n}\frac{1}{i!}\left(-\frac{1}{N}\right)^i \left\langle \chi_{(n)}^{(i)}\left(\hat{\phi}\right)\chi_{(n)}^{(i)}\left(\hat{\phi}^*\right)\right\rangle \\
&= \sum_{i=0}^{n}\frac{1}{i!}\left(-\frac{1}{N}\right)^i (N+n-1)^2(N+n-2)^2...(N+n+i)^2 \left\langle \chi_{(n-i)}\left(\hat{\phi}\right)\chi_{(n-i)}\left(\hat{\phi}^*\right)\right\rangle \\
&= \sum_{i=0}^{n}\frac{1}{i!}\left(-\frac{1}{N}\right)^i (N+n-1)^2(N+n-2)^2...(N+n-i)^2 \frac{Dim(n-i)(n-i)!}{d_{(n-i)}} \\
&= \sum_{i=0}^{n}\frac{1}{i!}\left(-\frac{1}{N}\right)^i \frac{(N+n-1)!}{(N+n-i-1)!}\frac{(N+n-1)!}{(N+n-i-1)!}\frac{(N+n-i-1)!}{(N-1)!} \\
&= \frac{(N+n-1)!}{(N-1)!}\sum_{i=0}^{n}\frac{(-1)^i}{N^i}\frac{1}{i!}\frac{(N+n-1)!}{(N+n-i-1)!},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{N}_A^2 &= \left\langle \chi_{(1^n}\left(\phi\right)\chi_{(1^n)}\left(\phi^*\right)\right\rangle \\
&= \sum_{i=0}^{n}\frac{1}{i!}\left(-\frac{1}{N}\right)^i (N-(n-1))^2...(N-(n-i))^2 \left\langle \chi_{(1^n)}\left(\hat{\phi}\right)\chi_{(1^n)}\left(\hat{\phi}^*\right)\right\rangle \\
&= \sum_{i=0}^{n}\frac{1}{i!}\left(-\frac{1}{N}\right)^i (N-n+1)^2...(N-n+i)^2 \\
&= \sum_{i=0}^{n}\frac{1}{i!}\left(-\frac{1}{N}\right)^i \frac{(N-n+i)!}{(N-n)!}\frac{N!}{(N-n+i)!}\frac{(N-n+i)!}{(N-n)!} \\
&= \frac{N!}{(N-n)!}\sum_{i=0}^{n}\frac{(-1)^i}{N^i}\frac{(N+i-n)!}{i!(N-n)!}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\mathcal{O}}{\mathcal{N}_S\mathcal{N}_A} &= \frac{(-1)^{n-1}(n-1)(N+n-1)!}{n!N^{n-1}(N-n)!}\sqrt{\frac{(N-1)!(N-n)!}{N!(N+n-1)!}} \\
&\quad\times\left(\sum_{i,j=0}^{n}\frac{(-1)^{i+j}}{N^{i+j}}\frac{1}{i!j!}\frac{(N+n-1)!(N+j-n)!}{(N+n-i-1)!(N-n)!}\right)^{-\frac{1}{2}} \\
&= \frac{(-1)^{n-1}(n-1)}{n!N^{n-1}}\sqrt{\frac{(N-1)!}{N!}}\left(\sum_{i,j=0}^{n}\frac{(-1)^{i+j}}{N^{i+j}}\frac{1}{i!j!}\frac{(N+j-n)!}{(N+n-i-1)!}\right)^{-\frac{1}{2}}
\end{aligned}$$

so

$$\frac{\mathcal{O}}{\mathcal{N}_S\mathcal{N}_A} = \frac{(-1)^{n-1}(n-1)}{n!N^{n-1}}\left[\sum_{i,j=0}^{n}\frac{(-1)^{i+j}(N-n+j)!}{N^{i+j-1}i!j!(N+n-i-1)!}\right]^{-\frac{1}{2}}. \quad (5.3)$$

We do not have a closed form answer for this quantity. It is, however, easy to compute numerically. The result is shown in the figure below.

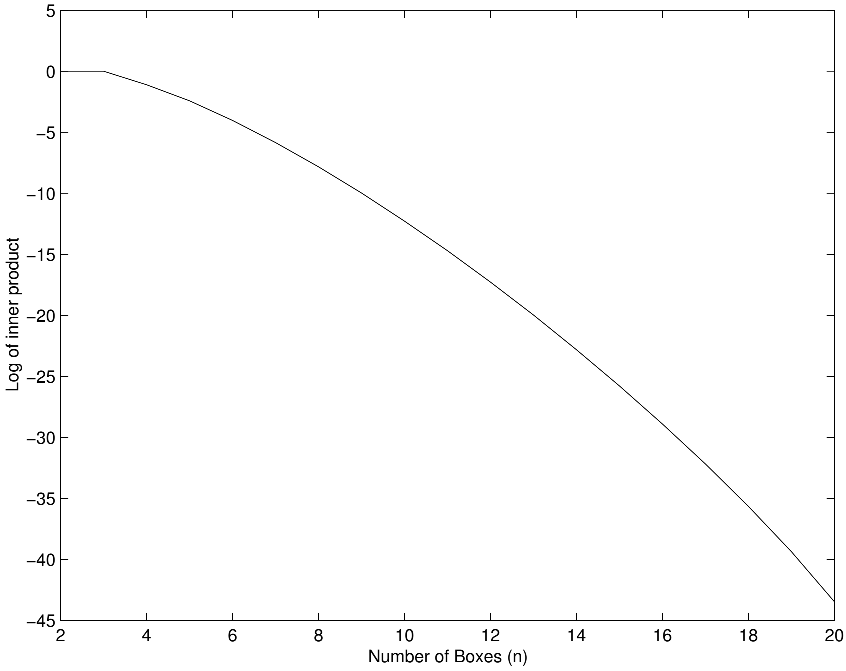


Figure 5.1: *log* $\left|\frac{\mathcal{O}}{\mathcal{N}_S\mathcal{N}_A}\right|$ *as a function of the number of boxes n, for* $N = 20$.

We see clearly that the mixing between the AdS and sphere giant states is not suppressed for small n, but rapidly goes to zero as n is increased. Note that for any value of N, $\left|\frac{\mathcal{O}}{\mathcal{N}_S\mathcal{N}_A}\right| = 1$ for $n = 2, 3$. This is the maximum value that the overlap can attain. It indicates that these two states are identical up to a sign, which is easily seen directly from the su(N) Schur polynomials for 2 and 3 boxes.

The physical interpretation follows. For small values of n we have a maximum overlap of symmetric and antisymmetric representations. This is as far removed as possible from the orthogonality recovered at large n. This implies that the difference between operators in the SU(N) theory and those in the U(N) theory is greatest at small n. Because the SU(N) theory does not include the boundary degrees of freedom that the U(N) theory does, we can interpret this as indicating that the boundary degrees of freedom have a larger role at small n. The supergravity counterparts to the operators we are considering in this limit are states with a small mass. These are described by wavefunctions with large fluctuations in position, and are therefore not well-localised. This is the reason for the mixing of bulk and boundary degrees of freedom that means we cannot expect to inherit the U(N) theory's orthogonality in these operators, even at large N.

When n is large, of the order $\sim$ N, our operators are dual to states with a large mass. These will have well-localised wavefunctions in the bulk of the AdS space. For these states we expect much less mixing of the bulk and boundary degrees of freedom. As a consequence, the boundary degrees of freedom do not play a great role and the orthogonality of the u(N) operators is inherited by the su(N) operators.

5.2.3 Finite N

At finite N, there is not even an approximate notion of single eigenvalue dynamics. In this case, one can employ the Gram-Schmidt algorithm to obtain from the linearly independent operators of Chapter 4 a set of operators which will diagonalise the inner product. This process can be used to produce many different sets of operators diagonalising the two-point function. We have no way as yet to determine which of these would be most relevant in the AdS/CFT correspondence. Also, the Gram-Schmidt algorithm does not provide a practical way to diagonalise the two-point function for operators corresponding to Young diagrams with a large number of boxes. An elegant result for all n has proved elusive. One possibility arose from considering the fact that the orthogonality of Schur polynomials in $\hat{\phi}$ was proved via the Frobenius-Schur duality between the symmetric and unitary groups[4]. It is natural to ask if there is a generalisation of this for the Schur polynomials in ϕ. Irreducible representations of the unitary group are obtained by taking contractions of objects transforming in the fundamental representation with tensors that have a definite symmetry under interchange of indices. Irreducible representations of the orthogonal groups are obtained by taking contractions with *traceless* tensors that have a specific symmetry under interchange of indices. One might guess that this tracelessness constraint is exactly what is needed to solve the problem studied here.

The relevance of the permutation group to Frobenius-Schur duality comes from the fact that the permutation and unitary groups are centralisers of each other. The centralisers of the orthogonal group are the Brauer algebras.[1] This might lead one to hope that su(N) polynomials built with Brauer algebra characters instead of permutation group characters would have the desired diagonal two-point functions. We have explicitly checked that this is not the case.

[1]A good reference on the Brauer algebras is [43].

5.3 Giant gravitons as probes of the dual geometry

Natural probes of the dual geometry are heavy objects which can be treated semiclassically. The scale to which one wishes to probe the geometry determines what the masses of these objects must be. To resolve on a scale of γ, a probe's fluctuations in position need to be less than γ. This is equivalent to demanding that the mass of the probe be greater than $\frac{1}{\gamma}$. There is a lower limit to the scale at which we can resolve. At the Planck scale the probe's gravitational radius will exceed γ, and the background metric will be significantly deformed by the probe itself.

In the strongly coupled regime of the gauge theory, the radius of the AdS space is much larger than the string scale. Consequently we expect giant gravitons to be suitable probes of the geometry in this limit and at geometric distances larger than the string scale. In the gravity dual to the U(N) gauge theory there are both bulk and boundary degrees of freedom. To resolve features of the bulk geometry to a scale γ, it will not be simply sufficient to require a probe with mass greater than $\frac{1}{\gamma}$. For example, consider a composite state made up of a heavy excitation of boundary modes and a light excitation of the bulk gravity. To find good probes of the gravity, it is necessary in general to be able to distinguish between the roles of bulk and boundary contributions. The separation of bulk and boundary degrees of freedom can be addressed, as we have seen, in the dual field theory.

5.3.1 Disentangling bulk and boundary degrees of freedom

In the last chapter, we used a ghost field to subtract away the mode corresponding to the trace of $\hat{\phi}$. It is also possible to express $\hat{\phi}$ in terms of ϕ by adding the mode corresponding to the trace. In terms of this b field, with properties

$$\begin{aligned} \langle bb \rangle &= \langle b^* b^* \rangle = 0; \\ \langle bb^* \rangle &= \frac{1}{N}, \end{aligned}$$

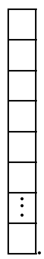
we have the identity

$$\left\langle F\left(\hat{\phi}_{ij}, \hat{\phi}^*_{kl}\right)\right\rangle = \left\langle F\left(\phi_{ij} + b\delta_{ij}, \phi^*_{kl} + b^*\delta_{kl}\right)\right\rangle, \tag{5.4}$$

for an arbitrary function $F(.,.)$. In contrast to our previous computations, where the ghost had no physical meaning, b and ϕ do have natural physical interpretations. We have seen that b describes the overall centre of mass motion of all the branes; in the gravity dual it is localised on the boundary of $AdS_5 \times S^5$. ϕ describes the bulk string theory.

SU(N) duals to sphere giants

We can now use the technology already developed to disentangle bulk and boundary degrees of freedom in the dual super Yang-Mills theory. Consider first the operator dual to a sphere giant. Sphere giants are heavy states so we expect them to provide good probes of the geometry of the gravitational theory. Their field theory counterparts are Schur polynomials in completely antisymmetric representations with $n \sim N$ boxes. These Schur polynomials will be labelled by Young diagrams consisting of a single column of n boxes, which we denote $\mathcal{R}_n$:



If n $= 0$, $\mathcal{R}_n = 1$. We have that

$$\chi_{\mathcal{R}_n}\left(\hat{\phi}\right) \rightarrow \chi_{\mathcal{R}_n}\left(\phi + b\mathbf{1}\right) \quad = \quad \sum_{i=0}^{n} \frac{b^i}{i!} \prod_{k=1}^{i} (N - n + k) \chi_{\mathcal{R}_{n-i}}\left(\phi\right). \qquad (5.5)$$

Terms in this sum that correspond to low values of i are the product of a Schur polynomial in a representation with a large number of boxes and a boundary state with a small $\mathcal{R}$ charge. These terms therefore correspond to objects in the bulk gravity which are heavy and hence well-localised times light boundary excitations which will not be well-localised. Terms corresponding to large values of i are the product of a Schur polynomial in a representation with a small number of boxes and a boundary state with a large $\mathcal{R}$ charge. Thus, these terms correspond to objects in the bulk gravity which are light and hence not well-localised times heavy boundary excitations which will be well-localised. In spite of this, the Schur polynomials still provide localised probes. To understand why, we need to work with normalised operators. This will allow us to see how large the different contributions to $\chi_{\mathcal{R}_n}\left(\hat{\phi}\right)$ are. We want to consider these correlators in terms of $\frac{b'^n}{\sqrt{n!}}$ and $\bar{\chi}_{\mathcal{R}_n}\left(\phi\right)$, with

$$\left\langle \frac{b'^i}{\sqrt{i!}} \bar{\chi}_{\mathcal{R}_{n-i}}\left(\phi\right) \frac{b'^{j*}}{\sqrt{j!}} \bar{\chi}_{\mathcal{R}_{n-j}}\left(\phi^*\right) \right\rangle \quad = \quad \delta_{ij}. \qquad (5.6)$$

It is easy to see that

$$\begin{aligned}
&\left\langle \frac{b'^i}{\sqrt{i!}}\chi_{\mathcal{R}_{n-i}}(\phi)\,\frac{b'^{j*}}{\sqrt{j!}}\chi_{\mathcal{R}_{n-j}}(\phi^*)\right\rangle \\
&= \frac{1}{\sqrt{i!}\sqrt{j!}}\left\langle b^i b^{*j}\right\rangle\left\langle \chi_{\mathcal{R}_{n-i}}(\phi)\,\chi_{\mathcal{R}_{n-j}}(\phi^*)\right\rangle \\
&= \frac{1}{\sqrt{i!}\sqrt{j!}}\delta_{ij}\frac{1}{N^i}\mathcal{N}_A^2(n-i) \\
&= \frac{1}{i!N^i}\frac{N!}{(N-(n-i))!}\sum_{h=0}^{n-i}\frac{(-1)^h}{N^h}\frac{(N+h-(n-i))!}{h!(N-(n-i))!}\delta_{ij}.
\end{aligned}$$

So the new normalisation is given by

$$\begin{aligned}
\mathcal{N}_{new}^2 &= \frac{N!}{i!N^i}\frac{1}{(N-(n-i))!(N-(n-i))!}\sum_{h=0}^{n-i}\frac{(-1)^h}{N^h}\frac{(N+h-(n-i))!}{h!} \\
&= \frac{1}{i!}\frac{1}{N^i}\delta_{ij}\mathcal{N}_A^2(n-i).
\end{aligned}$$

Then we have

$$\begin{aligned}
\chi_{\mathcal{R}_n}\left(\hat{\phi}\right) &\rightarrow \chi_{\mathcal{R}_n}(\phi+b\mathbf{1}) \\
&= \sum_{i=0}^{n}\frac{b^i}{i!}\prod_{k=1}^{i}(N-n+k)\chi_{\mathcal{R}_{n-i}}(\phi) \\
&= \sum_{i=0}^{n}\frac{b'^i}{\sqrt{i!}}\prod_{k=1}^{i}(N-n+k)\frac{1}{\sqrt{N^i i!}}\mathcal{N}_A(n-i)\bar{\chi}_{\mathcal{R}_{n-i}}(\phi) \\
&= \sum_{i=0}^{n}\alpha_i\frac{b'^i}{\sqrt{i!}}\bar{\chi}_{\mathcal{R}_{n-i}}(\phi)
\end{aligned}$$

with

$$\begin{aligned}
\alpha_i &= \frac{(N-n+i)!}{(N-n)!}\frac{\mathcal{N}_A(n-i)}{\sqrt{N^i i!}}; \\
\mathcal{N}_A^2(n) &= \frac{N!}{(N-n)!}\sum_{i=0}^{n}\frac{(-1)^i}{N^i}\frac{(N+i-n)!}{i!(N-n)!}.
\end{aligned}$$

The dependence of the wave function coefficient α_i on i is plotted below. This expansion of the operator $\chi_{\mathcal{R}_n}\left(\hat{\phi}\right)$ is dominated by terms corresponding to low values of i. Consequently, the contribution from the boundary degrees of freedom to those operators is exponentially suppressed. This reveals how the $\chi_{\mathcal{R}_n}\left(\hat{\phi}\right)$ provide localised probes of the dual geometry. Essentially, Schur polynomials for

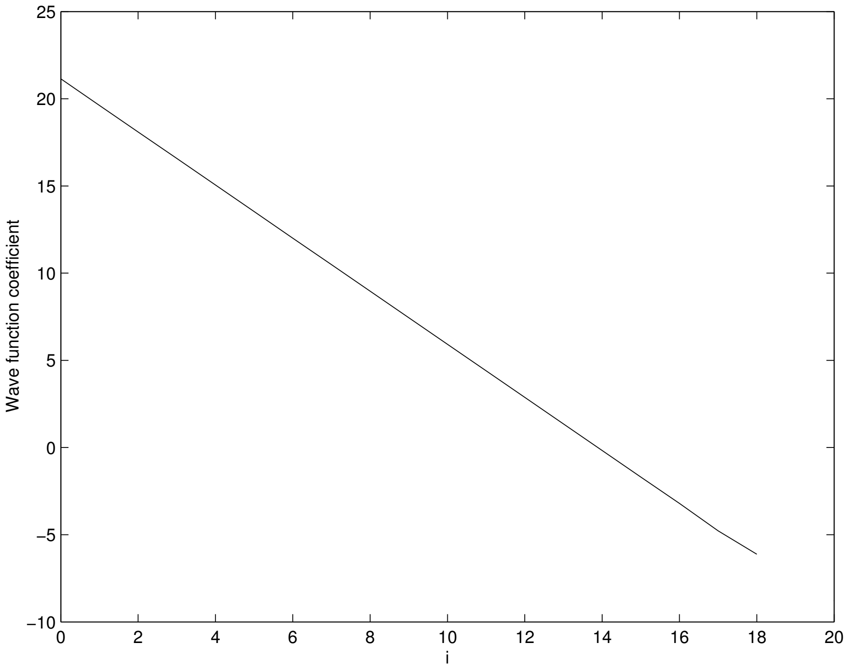
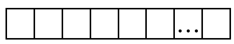


Figure 5.2: *The components of the operator dual to a sphere giant.* $log|\alpha_i|$ *is plotted as a function of* i*, for* $N = n = 20$*. Note that the contribution from the boundary degrees of freedom is exponentially suppressed.*

the totally antisymmetric representations in $\hat{\phi}$ coincide at large N and for $n \sim N$ with those for the totally antisymmetric representations in ϕ. This means that su(N) Schur polynomials corresponding to the totally antisymmetric representations with $n \sim N$ boxes again provide the duals to sphere giants.

Duals to AdS giants

Similarly we can consider the roles of bulk and boundary degrees of freedom for operators dual to AdS giants. We look at Schur polynomials of completely symmetric representations i.e. those labelled by Young diagrams $\mathcal{S}_n$



consisting of a single row of n boxes. Then

$$\chi_{\mathcal{S}_n}\left(\hat{\phi}\right) \to \chi_{\mathcal{S}_n}\left(\phi + b\mathbf{1}\right) = \sum_{i=0}^{n} \kappa_i \frac{b'^i}{\sqrt{i!}} \bar{\chi}_{\mathcal{S}_{n-i}}\left(\phi\right), \tag{5.7}$$

with

$$\kappa_i = \frac{(N+n-1)!}{(N+n-i-1)!}\frac{\mathcal{N}_S(n-i)}{\sqrt{N^i i!}}; \tag{5.8}$$

$$\mathcal{N}_S^2(n) = \frac{(N+n-1)!}{(N-1)!}\sum_{i=0}^{n}\frac{(-1)^i}{N^i}\frac{(N+n-1)!}{i!(N+n-i-1)!}. \tag{5.9}$$

The results for the AdS giants are plotted below:

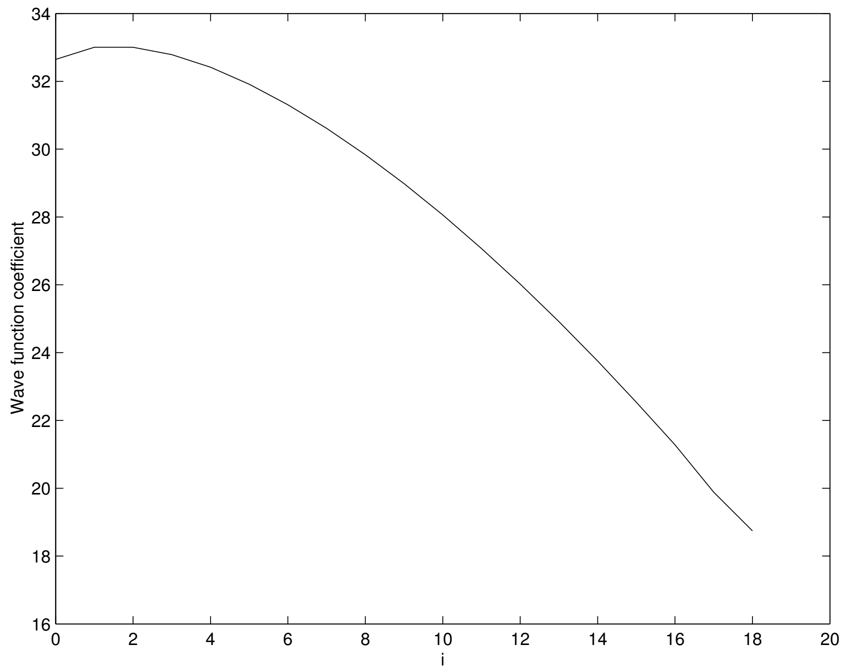


Figure 5.3: *The components of an operator dual to an AdS giant. log$|\kappa_i|$ is plotted as a function of i, for $N = n = 20$. Note that once again the terms corresponding to low values of i dominate.*

It is interesting to note that the coefficients κ_i have a maximum at $i = 1, 2$. For these terms there is a very small contribution coming from the boundary degrees of freedom. Clearly, however, the boundary contribution is still significantly suppressed. It has a much slower fall-off than in the sphere giant case though. This is because of the fact that when the AdS giants expand they grow in the AdS space taking them closer to the boundary, while the sphere giants expand in S^5, remaining at the origin of AdS_5.

5.4 Conclusions

We have discussed the importance of the AdS/CFT correspondence in studying string theory, the most promising line of attack to the investigation of quantum gravity. It is in the light of this correspondence that giant gravitons, and the mechanism to explain the stringy exclusion principle that they provide, are of interest. Further, giant gravitons serve as useful probes of the supergravity geometry. In this context, U(N) SYM duals to giant gravitons were proposed [4]. A set of gauge theory operators which diagonalise the two-point function and are in a one-to-one correspondence with the half-BPS representations of the theory was found.

The object of the research presented in this dissertation was to extend these results to the case of an SU(N) gauge theory and in so doing to shed light on the roles of bulk and boundary degrees of freedom in the context of giant gravitons. We have been able to develop the technology necessary to study Schur polynomials in the SU(N) gauge theory. In addition, we have obtained a complete set of relations between the Schur polynomials and provided an algorithm with which to select a unique linearly independent basis for the half-BPS representations. Using the Gram-Schmidt algorithm this allows one to obtain a new set of operators with diagonal two-point functions. Although this answer is not very explicit, it does provide the generalisation we were seeking. It means that the SU(N) gauge theory which is of interest as being dual to the dynamics in the AdS bulk can now be dealt with directly.

Further, we have been able to see that for a large number of boxes, su(N) Schur polynomials corresponding to the totally symmetric or totally antisymmetric representations are still approximately dual to giant graviton states (we don't have strict duality because the strict orthogonality of the u(N) operators has been lost). Our extension of the results of [4] allows separation of the bulk and boundary degrees of freedom of these operators. Analysis of their differentiated roles in the dual U(N) gauge theory revealed that for both sphere and AdS giants, the boundary contribution is sufficiently suppressed for Schur polynomials of the SU(N) gauge theory to remain good candidate duals to giant gravitons.

Chapter 6

Discussion

6.1 Summary

We have seen how the conjectured correspondence between string theory in Anti-de Sitter space and conformal field theory offers a powerful analytical tool for studying string theory. The strong-weak coupling nature of this duality makes it exceptionally useful as it allows one to study a strong coupling string theory problem via the corresponding weak coupling situation in the dual field theory. In particular we have looked at the duality between Type IIB string theory on $AdS_5 \times S^5$ and 3+1 $\mathcal{N} = 4$ super Yang-Mills theory. The boundary of AdS_5 is a 4-dimensional Minkowskian spacetime where we can think of the full string theory dynamics as being encoded into the lower dimensional field theory dynamics of the SYM. In this sense the AdS/CFT correspondence conjectured by Maldacena [1] is a realisation of the holographic principle first suggested by 't Hooft and Susskind [8, 9].

In the AdS/CFT dictionary, states in the string theory are mapped to field theory operators. One useful group of states to consider in the string theory is that of states satisfying a particular (lower) bound on their masses. This is called the Bogomolnyi-Prasad-Sommerfield or BPS bound and is significant because it arises from supersymmetry considerations alone, and does not depend on the scale of the problem. Thus, the masses (and energies) of these types of states are protected from quantum corrections. Their counterparts in the field theory are the chiral primary operators, which look like traces of monomials of fields in the Lie algebra of the theory, and which are built from the six scalars of $\mathcal{N} = 4$ SYM.

Gravitons are half-BPS states, and can therefore be studied via the corresponding operators in the field theory, provided one is able to choose the correct ones to consider. The discovery of giant gravitons by McGreevy, Susskind and Toumbas [19]: graviton states which are able to expand linearly with momentum in either the AdS or spherical spaces, led Corley, Jevicki and Ramgoolam [4] to consider certain operators as candidate duals to these giant graviton states. The sphere giants have an upper bound on their dimension provided by the radius of the spherical space itself. This amounts to an angular momentum cut-off, and so the discovery of these particular giant gravitons offered a compelling mechanism to explain the stringy exclusion principle (a result from the AdS/CFT correspondence which amounted to demanding just such a bound).

With this in mind, Corley, Jevicki and Ramgoolam considered Schur polynomials of u(N) fields. The Schur polynomials are a class of symmetric polynomials which when expressed as characters of U(N) encapsulate the Frobenius-Schur duality between the unitary and symmetric groups. The Schur polynomials are indexed by Young diagrams that can characterise (among other things) both the irreducibles representations of U(N) and the irreducible representations of S_n.

As representations of the unitary group, Young diagrams can be thought of as tensors summarising the symmetry properties of the irreducible representation in question. In particular, a column of more than N boxes in a Young diagram of a U(N) representation corresponds to an antisymmetric tensor with two indices the same. All tensors of this form are zero, so Young diagrams of this type have an upper limit on their number of boxes. The number of boxes in the Young diagram indexing a Schur polynomial relates to the power of the products of $\hat{\phi} \in$ u(N) appearing in the operators being considered. This is mapped to the angular momentum of the dual string theory states. Thus, Schur polynomials of u(N) fields and indexed by completely antisymmetric Young diagrams with a large number of boxes are natural candidate duals to sphere giants. On the other hand, there is no upper bound on the length of a row of a Young diagram, so Schur polynomials indexed by completely symmetric Young diagrams with a large number of boxes provide the natural duals to AdS giants. These are the giant gravitons expanding in the Anti-de Sitter space, and have no bound on their angular momentum.

Corley, Jevicki and Ramgoolam's study of u(N) Schur polynomials thus revealed natural candidates for duals to giant gravitons in U(N) $\mathcal{N} = 4$ SYM in 3+1 dimensions. Further, they established computational machinery for exact calculation of correlators of these polynomials at all N. The possibility of calculation at all N allows for investigation of strongly coupled string theory ($g_S = \frac{1}{N}$) in the field theory and also brought us much closer to being able to ask sensible questions about the bulk and boundary contributions in the dual.

A U(N) gauge theory is equivalent to an SU(N) gauge theory times a free U(1) vector multiplet (up to Z_N identifications). The U(1) dynamics are known as singleton dynamics, and live on the boundary of AdS. Thus it is the SU(N) gauge theory which is expected to be of relevance in the bulk. U(N) and SU(N) are not always easy to distinguish at large N, so with the computational results of [4] it made sense to try to elucidate the SU(N) correlators. With the introduction of a ghost field we were able to rewrite the su(N) Schur polynomials in terms of a Taylor expansion of naturally defined derivatives of the u(N) Schur polynomials, and achieve a computational generalisation of the U(N) results.

6.2 Results discussion

The diagonal basis we sought to further generalise was not found, but we were able to give a linearly independent basis in one-to-one correspondence with the half-BPS representations of interest. Moreover, our computational generalisation allowed us to investigate the roles of bulk and boundary degrees of freedom in the string theory, using giant gravitons as probes of the geometry.

As shown in Chapter 5, su(N) Schur polynomials corresponding to the totally symmetric or totally antisymmetric representations are still approximately dual to giant graviton states in the large N limit and for a large number n of boxes.

Our analysis of the 'bulky giants' dual to these su(N) Schur polynomials and comparison to the orthogonality between the Schur polynomials in the U(N) case amounted to a differentiation of bulk and boundary degrees of freedom. This ubiquitous phrase refers to the identification of situations where the SU(N) and U(N) results differ most. Where this is the case, one can conclude that the boundary degrees of freedom (included in the U(N) but not in the SU(N) theory) play the most significant role. Thus we were able to see when bulk and boundary degrees of freedom are mixed and when they can be disentangled.

Specifically we were able to look at the role of boundary degrees of freedom for the AdS and sphere giants. For both the boundary contribution is sufficiently suppressed for Schur polynomials of the SU(N) gauge theory to remain good candidate duals.

6.3 Outlook

Many questions on the giant graviton dictionary of AdS/CFT remain, and several are suggested by the developments presented here. In general, there are many ways one could now use the SU(N) results put forward here or the U(N) results of [4] to investigate graviton interactions and excitations. Another general area of inquiry would be to examine the corresponding operators in other cases of the AdS/CFT correspondence. Considering different geometries and gauge theories might lead to general results about the roles of bulk and boundary degrees of freedom, or provide us with sharper tools with which to probe the relevant geometries.

Three specific questions that arise from the SU(N) extension covered in this dissertation are the following. Firstly, the question of finding a basis to diagonalise the su(N) Schur polynomials has not been satisfactorily resolved. The desired operators might also provide quantitative clues as to the differentiation of contributions from bulk and boundary. We have seen that orthogonality of different states is lost to varying degrees depending on the number of boxes n and the Young diagrams. It is possible that suitable operators might be obtained when one restricts to large n, where it appears easier to separate boundary contributions. It would be interesting to see the implications of this restriction.

This leads us to a second avenue of possible future study. The only explicit parameter of the map between states in string theory and operators in SYM is that the angular momentum is mapped to the number of boxes in a Young diagram. The different Young diagrams for a fixed number of boxes label orthogonal operators in the U(N) gauge theory, as found by Corley, Jevicki and Ramgoolam. This orthogonality suggests that the different representations correspond to different states, and indeed we have strong reasons for believing that certain types of representations can be identified with certain types of gravitons. The giant gravitons we can map to totally (anti-)symmetric Young diagrams are clearly distinguishable: the AdS giants expand in AdS and the sphere giants expand in S^d. However, the remaining Young diagrams are clearly distinguishable and we do not yet have a dictionary for what these map to, or even what quantities might distinguish the corresponding states in the string theory. The strict orthogonality between operators labelled by different Young diagrams is lost to varying degrees in the SU(N) case, leading one to surmise that further comparisons of the U(N) and SU(N) correlators could further elucidate these composite states, for example. Major steps to clarify this point were taken in [44] (see also [45]) with a correspondence between half-BPS states and free fermions.

A third interesting question arises from considering the duality between sphere giants which source a magnetic dipole field and AdS giants which source an electric dipole field. This electro-magnetic duality is understood in IIB string theory as being implemented by $SL(2,\mathbb{Z})$. The same duality is conjectured to hold in $\mathcal{N}=4$ SYM. Some insight to finding an orthogonal basis of states in the SU(N) case might be gained by trying to trace this duality via the Schur polynomials believed to correspond to these giant gravitons. Using the free fermion description of [44] one might be able to find the ('electromagnetic') dual to the Schur polynomial labelled by a column via the understood duality in Type IIB string theory, for example.

To conclude, there is no end to the interesting questions posed by the AdS/CFT correspondence with a view to learning more about string theory and even QCD. In particular we also see that the tools and insights developed in this work are the first steps to a specific area of inquiry in the correspondence. The elucidation of operators in SU(N) SYM dual to giant gravitons shows strong signs of adding to the set of clues that will hopefully allow physicists to put a few more pieces of a very interesting and fundamental puzzle in place.

Appendix A

Representation theory

This appendix aims to provide some of the background to the group theoretical results used in the body of the dissertation. In particular, some explanation of the orthogonality relations used is required, as well as some background to the Frobenius-Schur duality and the use of Young diagrams to characterise representations via symmetrisation considerations. The treatment given here is by no means comprehensive. Useful references for group and representation theory include [39, 46] and [47], which are variously followed here.

A.1 Some definitions

A.1.1 Groups

Groups

A group consists of a set G, together with a rule for combining any two elements of G to form another element of G. This rule must satisfy the following axioms $\forall\ g, h \in G$ (where the combination or product of g and h is written gh):

1. $\forall\ \ g, h, k \in G$,
$$(gh)k = g(hk),$$
i.e. the product operation must be associative;

2. $\exists\ e \in G$ such that $\forall\ g \in G$,
$$eg = ge = g,$$
which amounts to requiring that a group include the identity of its product operation;

3. $\forall\, g \in G$, $\exists\, g^{-1} \in G$ such that

$$gg^{-1} = g^{-1}g = e.$$

   This axiom guarantees the existence of an inverse in the group for every element of the group.

The number of elements in a group G is called the order of the group. The centre of a group G is the set Z(G) of all elements in G which commute with all other elements:

$$Z(G) \quad = \quad \{z \in G : zg = gz \;\forall\, g \in G\} \tag{A.1}$$

Homomorphisms

A function $\theta : G \to H$, G and H both groups, is a rule which assigns a unique element of H to each element of G. Homomorphisms are functions which preserve the group structure. Explicitly, a function $\phi : G \to H$ is a homomorphism if

$$(g_1 g_2)\phi \quad = \quad (g_1\phi)(g_2\phi) \;\forall\, g_1, g_2 \in G. \tag{A.2}$$

Note that functions are usually applied on the right in group theory texts. An invertible homomorphism is called an isomorphism.

A.1.2 Vector spaces

Vector spaces

A vector space over a field F ($\mathbb{C}$ or $\mathbb{R}$ for example) is a set V, together with a rule for adding any $u, v \in V$ to form $u + v \in V$ and a scalar multiplication, such that

1. V is an Abelian group under addition;
2. $\forall\, u, v \in V, \lambda, \mu \in F$:
   (a) $\lambda(u + v) = \lambda u + \lambda v$;
   (b) $(\lambda + \mu)v = \lambda v + \mu v$;
   (c) $(\lambda\mu)v = \lambda(\mu v)$;
   (d) $1v = v$.

The vector space FG with multiplication defined by

$$\left(\sum_{g \in G} \lambda_g g\right)\left(\sum_{h \in G} \mu_h h\right) \quad = \quad \sum_{g,h \in G} \lambda_g \mu_h (gh), \tag{A.3}$$

where λ_g, $\mu_h \in F$, is called the group algebra of G over the field F. This provides a definition for the concept of a group algebra, used in section 3.5.2.

The exterior powers $\Lambda^n V$ of a vector space V are equipped with an alternating multilinear map[1]

$$V \times ... \times V \quad \to \quad \Lambda^n V; v_1 \times ... \times v_n \mapsto v_1 \wedge ... \wedge v_n, \tag{A.4}$$

that is universal, i.e. for an alternating multilinear map $\beta : V \times ... \times V \to U$ there is a unique linear map from $\Lambda^n V$ to U which takes $v_1 \wedge ... \wedge v_n$ to $\beta(v_1, ..., v_n)$. A multilinear map β is said to be alternating if $\beta(v_1, ..., v_n) = 0$ if any two of the vectors v_i are equal. This makes $\beta(v_1, ..., v_n)$ antisymmetric under interchange of any two vectors. A symmetric multilinear map will be unchanged under such an interchange. The symmetric powers $\mathrm{Sym}^n V$ come with a universal symmetric multilinear map

$$V \times ... \times V \quad \to \quad \mathrm{Sym}^n V; \;\; v_1 \times ... \times v_n \mapsto v_1.....v_n. \tag{A.5}$$

Linear transformations

A linear transformation $\theta : V \to W$ mapping the vector space V over F to the vector space W over F is a function which satisfies

1. $(u + v)\theta = u\theta + v\theta \;\forall\; u, v \in V$;
2. $(\lambda v)\theta = \lambda(v\theta) \;\forall\; \lambda \in F, v \in V$.

A linear transformation from a vector space V to itself is called an endomorphism.

Tensor products

The tensor product of two vector spaces V and W over a field is a vector space $V \otimes W$ equipped with a bilinear map $V \times W \to V \otimes W$ which is universal: for any bilinear map $\beta : V \times W \to U$ to a vector space U there is a unique linear map from V $\otimes$ W to U that takes $v \otimes w$ to $\beta(v, w)$.

A.1.3 Representations

Let G be a group over F. $GL(n, F)$ is the group of invertible $n \times n$ matrices in F. A representation of G over F is then a homomorphism $\rho : G \to GL(n, F)$ for some n. n gives the dimension of the representation. Every group has an infinite number of representations. We focus on irreducible representations (representations which contain no proper subrepresentations. A subrepresentation of a representation V

[1] This discussion is taken from [39].

is a vector subspace W of V which is invariant under G.) Any representation is a direct sum of irreducible representations.

A.2 Young diagrams

A.2.1 The symmetric group

Permutations

A permutation of a set X is a one-to-one mapping of X onto itself. The set of all permutations of $\{1, ..., n\}$ (n a positive integer) is a group under the product operation of composition: the product $\sigma\tau$ of a pair of permutations σ and τ is defined by $(\sigma\tau)x = \sigma(\tau x))$. It is called the symmetric group of degree n, and is denoted S_n. The order of S_n is n!.

Conjugacy classes

Let $x, y \in G$. Then x is conjugate to y if $y = g^{-1}xg$ for some $g \in G$. Conjugacy is easily shown to be an equivalence relation. The set of all elements in G conjugate to x is called the conjugacy class of x in G. Every group is a union of conjugacy classes, and distinct conjugacy classes are disjoint. Thus, a permutation group G on X gives rise to a partitioning of X into disjoint classes. For $x \in S_n$, the conjugacy class of x in S_n consists of all permutations in S_n with the same cycle shape as x.

A.2.2 Young diagrams

The number of irreducible representations of S_d is the same as the number of conjugacy classes of S_d. We can associate in a well-defined way a collection of positive integers whose sum is n to each element ρ of S_n : this partition of n consists of the lengths of the cycles that appear in a disjoint cycle decomposition of ρ and is called the cycle structure of ρ. The number of disjoint cycle decompositions is given by the number of partitions of $d = \lambda_1 + ... + \lambda_k, \lambda_1 \geq ... \geq \lambda_k \geq 1$, which can be found from the partition function p(d):

$$\begin{aligned}\sum_{d=0}^{\infty} p(d)t^d &= \prod_{n=1}^{\infty}\left(\frac{1}{1-t^n}\right) \\ &= (1+t+t^2+...)(1+t^2+t^4+...)(1+t^3+...) \\ &= 1+t+2t^2+3t^3+5t^4+...,\end{aligned}$$

so there are five ways to partition 4, for example. There is a useful way to "draw" representations of S_d, suggested by the link to partitions of d. To a partition $\lambda = (\lambda_1, ..., \lambda_k)$ where $\sum \lambda_i = d$ we can associate a Young diagram made of rows of λ_i boxes aligned on the left.

$$\lambda = \ydiagram{4,2,1}\,, \quad \lambda' = \ydiagram{3,2,1,1}\,.$$

There are λ_1 boxes in the first row, λ_2 boxes in the second and so on. The conjugate partition λ' is obtained by interchanging rows and columns. A Young tableau is defined to be a numbering of the boxes by the integers 1, ..., d.

$$\begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \\ 6 \\ 7 \end{ytableau}$$

The usefulness of the Young diagrams or tableaux in the context of symmetric groups comes from the symmetry properties we can assign to these shapes. The following discussion aims to provide some of the background to the way in which Young diagrams are used to label irreducible representations, as in the main body of the text. Given a tableau, we can define two subgroups of the symmetric group:

$$\begin{aligned} P = P_\lambda &= \{g \in S_d : \text{g preserves each row}\}\,; \\ Q = Q_\lambda &= \{g \in S_d : \text{g preserves each column}\}\,. \end{aligned}$$

In the group algebra $\mathbb{C}S_d$ we introduce two elements corresponding to these subgroups: set

$$\begin{aligned} a_\lambda &= \sum_{g \in P} e_g; \\ b_\lambda &= \sum_{g \in Q} sgn(g) e_g. \end{aligned}$$

To see what a_λ and b_λ do, observe that if V is any vector space and S_d acts on the dth tensor power $V^{\otimes d}$ by permuting factors, then the image of the elements $a_\lambda, b_\lambda \in \mathbb{C}S_d \to \mathrm{End}(V^{\otimes d})$ are

$$\begin{aligned} Im(a_\lambda) &= Sym^{\lambda_1} V \otimes Sym^{\lambda_2} V \otimes Sym^{\lambda_3} V \otimes ... \otimes Sym^{\lambda_k} V \subset V^{\otimes d}; \\ Im(b_\lambda) &= \Lambda^{\mu_1} V \otimes \Lambda^{\mu_2} V \otimes ... \otimes \Lambda^{\mu_k} V \subset V^{\otimes d}, \end{aligned}$$

where μ is the conjugate partition to λ, and $\Lambda^n V$ are the exterior or alternating powers of V.

Young symmetrisers are given by $c_\lambda = a_\lambda . b_\lambda \in \mathbb{C}S_d$. When $\lambda = (d)$, the corresponding Young diagram is a row of d boxes, and

$$c_{(d)} = a_{(d)} \quad = \quad \sum_{g \in S_d} e_g, \tag{A.6}$$

so the image of $c_{(d)}$on $V^{\otimes d}$ is $Sym^d V$. Thus such Young diagrams correspond to completely symmetric representations. When $\lambda = (1, ..., 1)$,

$$c_{(1,...,1)} = b_{(1,...,1)} \quad = \quad \sum_{g \in S_d} sgn(g) e_g, \tag{A.7}$$

and the image of $c_{(1,...,1)}$ on $V^{\otimes d}$ is $\Lambda^d V$. Young diagrams which are just columns therefore correspond to completely antisymmetric representations. The image of the symmetrisers c_λ in $V^{\otimes d}$ provide all the finite-dimensional irreducible representations of GL(V). This means there is a direct correspondence between conjugate classes in S_d and irreducible representations of S_d.

A.3 Characters and orthogonality relations

A.3.1 Characters

Suppose that $\rho : G \to GL(n, \mathbb{C})$ is a representation of the finite group G. With each $n \times n$ matrix $g\rho (g \in G)$ we associate the number given by adding all the diagonal entries of the matrix (i.e. taking the trace) and call this number $\chi(g)$. $\chi : G \to \mathbb{C}$ is called the character of the representation ρ. Note from the definition of conjugacy above that if x and y are conjugate to each other, $\chi(x) = \chi(y)$. A class function on G is a function $\psi : G \to \mathbb{C}$ such that $\psi(x) = \psi(y)$ whenever x and y are conjugate. The irreducible characters of a finite group G are therefore class functions and the number of them is equal to the number of conjugacy classes of G. It is convenient to record all the values of all the irreducible characters of G in a square matrix. This matrix is called the character table of G. Character tables for S_n, $n = 1$ to $n = 7$, are given in Appendix B.

A.3.2 Schur's Lemmas

Lemma I

If Γ and Γ' are two irreducible representations (of different dimensions) of a group G then if the matrix A satisfies

$$\Gamma(T)A = A\Gamma'(T), \tag{A.8}$$

for all T in G, it follows that A = 0.

Lemma I a

If Γ and Γ' are irreducible representations (of the same dimension) of the group G and if the matrix A satisfies

$$\Gamma(T)A = A\Gamma'(T), \tag{A.9}$$

for all T in G, then either Γ and Γ' are equivalent or A = 0.

Lemma II

If the matrices $\Gamma(T)$ are an irreducible representation of a group G and

$$\Gamma(T)A = A\Gamma(T) \tag{A.10}$$

for all T in G, then $A = \lambda \mathbf{1}$ where λ is a constant. In other words, if a matrix commutes with all the matrices of an irreducible representation, that matrix must be a multiple of the unit matrix.

A.3.3 Orthogonality relations

The fundamental orthogonality relations

Consider an irreducible representation of degree n for the group G of order g. Construct the matrix

$$A = \sum_S \Gamma(S) X \Gamma(S^{-1}), \tag{A.11}$$

where X is an arbitrary matrix and the sum over S runs through all of the elements of the group G. A satisfies the conditions of Lemma II, since

$$\begin{aligned}
\Gamma(T)A &= \sum_S \Gamma(T)\Gamma(S)X\Gamma(S^{-1}) \\
&= \sum_S \Gamma(T)\Gamma(S)X\Gamma(S^{-1})\Gamma(T^{-1}).\Gamma(T) \\
&= \left[\sum_S \Gamma(TS)X\Gamma(\{TS\}^{-1})\right]\Gamma(T).
\end{aligned}$$

As S runs through the elements of the group, so does TS (for fixed T) and therefore

$$\sum_S \Gamma(TS)X\Gamma(\{TS\}^{-1}) = \sum_S \Gamma(S)X\Gamma(S^{-1}), \tag{A.12}$$

and

$$\Gamma(T)A = A\Gamma(T). \tag{A.13}$$

Then, according to the lemma, $A = \lambda\mathbf{1}$. We choose X to have all its elements zero, except $X_{lm} = 1$, ,and let the constant λ be λ_{lm}. Then, from (A.11),

$$\sum_S \Gamma_{il}(S)\Gamma_{mj}(S^{-1}) = \lambda_{lm}\delta_{ij}, \tag{A.14}$$

or, if Γ is unitary,

$$\sum_S \Gamma_{il}(S)\Gamma^*_{jm}(S) = \lambda_{lm}\delta_{ij}. \tag{A.15}$$

To evaluate λ_{lm}, we set $i = j$, and sum over i:

$$\begin{aligned}
\sum_S \sum_i \Gamma_{il}(S)\Gamma_{mi}(S^{-1}) &= n\lambda_{lm} \\
n\lambda_{lm} &= \sum_S \Gamma_{ml}(E) \\
n\lambda_{lm} &= \sum_S \delta_{ml} \\
n\lambda_{lm} &= g\delta_{ml}.
\end{aligned}$$

So,

$$\lambda_{lm} = \frac{g}{n}\delta_{lm}, \tag{A.16}$$

and

$$\sum_S \Gamma_{il}(S)\Gamma_{mj}(S^{-1}) = \frac{g}{n}\delta_{lm}\delta_{ij}, \tag{A.17}$$

or, if Γ is unitary,

$$\sum_S \Gamma_{il}(S)\Gamma^*_{jm}(S) = \frac{g}{n}\delta_{lm}\delta_{ij}. \tag{A.18}$$

Similarly we can construct a matrix A satisfying the conditions of Lemma I. Given any two nonequivalent irreducible representations of G, $\Gamma^{(a)}$ and $\Gamma^{(b)}$ (of dimensions n_a and n_b), let

$$A = \sum_S \Gamma^{(b)}(S)X\Gamma^{(a)}(S^{-1}), \tag{A.19}$$

where X is an arbitrary matrix. Then

$$\begin{aligned}
\Gamma^{(b)}(T)A &= \sum_S \Gamma^{(b)}(T)\Gamma^{(b)}(S)X\Gamma^{(a)}(S^{-1}) \\
&= \sum_S \Gamma^{(b)}(T)\Gamma^{(b)}(S)X\Gamma^{(a)}(S^{-1})\Gamma^{(a)}(T^{-1}).\Gamma^{(a)}(T) \\
&= \sum_S \Gamma^{(b)}(TS)X\Gamma^{(a)}(\{TS\}^{-1}).\Gamma^{(a)}(T) \\
&= A\Gamma^{(a)}(T).
\end{aligned}$$

Thus, according to Lemma I, A = 0. Choosing X as before, we find

$$\sum_S \Gamma^{(b)}_{il}(S)\Gamma^{(a)}_{mj}(S^{-1}) = 0, \tag{A.20}$$

for all i, j, l, m; or, if $\Gamma^{(a)}$ and $\Gamma^{(b)}$ are unitary,

$$\sum_S \Gamma^{(b)}_{il}(S)\Gamma^{(a)*}_{jm}(S) = 0. \tag{A.21}$$

Together, these results give us the following: If we consider all nonequivalent irreducible representations of a group G, then the quantities $\Gamma^{(\mu)}_{ij}(T)$ for fixed μ, i, j, form a g-dimensional vector space, such that

$$\sum_T \Gamma^{(\mu)}_{il}(T)\Gamma^{(\nu)}_{mj}(T^{-1}) = \frac{g}{n_\mu}\delta_{\mu\nu}\delta_{ij}\delta_{lm}, \tag{A.22}$$

or, if the representation is unitary,

$$\sum_T \Gamma_{il}^{(\mu)}(T)\Gamma_{mj}^{(\nu)*}(T) \;=\; \frac{g}{n_\mu}\delta_{\mu\nu}\delta_{ij}\delta_{lm}. \tag{A.23}$$

These are known as the fundamental orthogonality relations.

Orthogonality of characters

Starting from the fundamental orthogonality relations we can derive similar orthogonality relations for the characters. If we set $i = l$ and $j = m$ we obtain

$$\begin{aligned}\sum_T \Gamma_{ii}^{(\mu)}(T)\Gamma_{jj}^{(\nu)}(T^{-1}) &= \frac{g}{n_\mu}\delta_{\mu\nu}\delta_{ij}^2 \\ &= \frac{g}{n_\mu}\delta_{\mu\nu}\delta_{ij}\end{aligned}$$

Now sum over all i and j:

$$\sum_T \chi^{(\mu)}(T)\chi^{(\nu)}(T^{-1}) \;=\; g\delta_{\mu\nu}, \tag{A.24}$$

or

$$\sum_T \chi^{(\mu)}(T)\chi^{(\nu)*}(T) \;=\; g\delta_{\mu\nu}. \tag{A.25}$$

if the representations are unitary.

Recall that all the elements in a given class in G have the same character. Let K_1, ..., K_k be the k classes of G and let the number of elements in K_i be g_i. Then for all the elements of K_i, the character in the μ-representation is the same: $\chi^{(\mu)}(T) = \chi_i^{(\mu)}$. Then we have

$$\begin{aligned}\sum_i \chi_i^{(\mu)}\chi_i^{(\nu)*}g_i &= g\delta_{\mu\nu} \\ \sum_i \chi_i^{(\mu)}\chi_i^{(\nu)*} &= \frac{g}{g_i}\delta_{\mu\nu}.\end{aligned}$$

Irreducibility

Consider an arbitrary representation Γ. Γ can be expressed in terms of irreducible representations as

$$\Gamma(T) \;=\; \sum_\nu a_\nu \Gamma^{(\nu)}(T), \tag{A.26}$$

where the a_ν are integers ≥ 0. Now take the trace of this equation for an element T in the class K_i of the group G:

$$\chi_i = \sum_\nu a_\nu \chi_i^{(\nu)}. \tag{A.27}$$

Multiplying by $\chi_i^{(\mu)*} g_i$ and summing over i:

$$\begin{aligned} \sum_i \chi_i^{(\mu)*} \chi_i g_i &= \sum_\nu a_\nu \sum_i g_i \chi_i^{(\mu)*} \chi_i^{(\nu)} \\ &= \sum_\nu a_\nu g \delta_{\mu\nu} \\ &= g a_\mu, \end{aligned}$$

or

$$\begin{aligned} a_\mu &= \frac{1}{g} \sum_i \chi_i^{(\mu)*} \chi_i; \\ a_\mu &= \frac{1}{g} \sum_{T \in G} \chi(T) \chi^{(a)}(T^{-1}). \end{aligned}$$

which gives us the number of times a given irreducible representation is contained in D.

Cayley's theorem and a group algebra delta function

Let G be a group of order n. Label its elements $g_1, ..., g_n$. Cayley's theorem states that every group G of order n is isomorphic to a subgroup of the symmetric group S_n. This is shown by considering that multiplication by any $g_\nu \in G$ permutes the g_i. This permutation can be achieved by permutation of the indices. Considering g_1, ..., g_n as co-ordinates in an n-dimensional space, we can represent the element g_ν by a permutation of the n co-ordinates.

$$\{g_1, ..., g_n\} \quad \overset{\sigma}{\longrightarrow} \quad \{g_{\sigma(1)}, ..., g_{\sigma(n)}\} \tag{A.28}$$

$$\sigma g_i = g_{\sigma(i)} = \Gamma(\sigma)_{ij} g_j$$

where the $\Gamma(\sigma)_{ij}$ are $n \times n$ matrices providing an n-dimensional representation of σ. Permutations which are subgroups of order n of S_n and which (except for the identity) leave no symbol unchanged are called regular permutations. For this representation, the diagonal elements of all matrices are zero except for the element g_ν such that $g_\nu g_i = g_i$ i.e. except for the unit element E. So, in the regular representation,

$$\chi(T) = \begin{cases} 0 & \text{for} \quad T \neq E, \\ g & \text{for} \quad T = E. \end{cases} \tag{A.29}$$

Thus the only non-zero term in the sum of a_ν corresponds to the identity and

$$\begin{aligned} a_a &= \frac{1}{g}\sum_{T\in G}\chi(T)\chi^{(a)}(T^{-1}) \\ a_a &= \frac{1}{g}\chi(E)\chi^{(a)}(E) \\ a_a &= d_a, \end{aligned}$$

since $\chi(E) = g$ for a group of order g in the regular representation, and where d_a is the dimension of $\Gamma^{(a)}(E)$. Thus the number of times each irreducible representation is contained in the regular representation is equal to the dimension of the representation. (A.26) becomes

$$\begin{aligned} \chi(T) &= \sum_{a=1}^{r} d_a\chi^{(a)}(T) \\ &= \sum_{a=1}^{r}\chi^{(a)}(E)\chi^{(a)}(T); \end{aligned}$$

and $\chi(T) = g\delta_{T,E}$ so

$$\sum_{a=1}^{r}\chi^{(a)}(E)\chi^{(a)}(T) = g\delta_{T,E}. \tag{A.30}$$

For the symmetric group, with $g = n!$, this gives us (3.30).

The centre of the group and another character relation

For an element C in the centre of the group we have by definition that C commutes with all the elements of the group. By Schur's lemma, this means $C = \lambda\mathbf{1}$. So,

$$\begin{aligned} \chi_R(C\sigma) &= \chi_R(\lambda\sigma) \\ \chi_R(C\sigma) &= \lambda\chi_R(\sigma). \end{aligned}$$

Also,

$$\begin{aligned}\chi_R(C) &= \chi_R(\lambda\mathbf{1})\\ \chi_R(C) &= \lambda\chi_R(\mathbf{1})\\ \chi_R(C) &= \lambda d_R.\end{aligned}$$

This gives us

$$\begin{aligned}\lambda &= \frac{\chi_R(C)}{d_R};\\ \Rightarrow \chi_R(C\sigma) &= \frac{\chi_R(C)\chi_R(\sigma)}{d_R},\end{aligned}$$

which proves (3.32). Now let

$$A = \sum_\sigma \chi_R\left(\sigma^{-1}\right)\Gamma_S(\sigma), \tag{A.31}$$

where S is some irreducible representation. $A\Gamma_S(\tau) = \Gamma_S(\tau)A$, so $A = \lambda\mathbf{1}$. Tracing both sides of this equation, we find

$$\lambda = \sum_\sigma \frac{\chi_R\left(\sigma^{-1}\right)\chi_S(\sigma)}{d_S}. \tag{A.32}$$

Also, $A\Gamma_S(\alpha) = \lambda\Gamma_S(\alpha)$, so

$$\begin{aligned}\sum_\sigma \chi_R\left(\sigma^{-1}\right)\Gamma_S(\sigma\alpha) &= \sum_\sigma \frac{\chi_R\left(\sigma^{-1}\right)\chi_S(\sigma)\Gamma_S(\alpha)}{d_S} &\text{(A.33)}\\ &= \frac{\delta_{RS}n!}{d_S}\Gamma_S(\alpha), &\text{(A.34)}\end{aligned}$$

from the orthogonality relation. Therefore

$$\sum_\sigma \chi_R\left(\sigma^{-1}\right)\chi_S\left(\sigma\alpha\right) = \frac{\delta_{RS}n!}{d_S}\chi_S(\alpha), \tag{A.35}$$

which is just (3.21).

Dimensionality of the unitary group

The expression for Schur polynomials in terms of U(N) characters,

$$\chi_R(U) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)tr(\sigma U) \tag{A.36}$$

gives an expression for $Dim(R)$ – the dimension of R as a representation of the unitary group – when evaluated at $U = \mathbf{1}$.

$$\begin{aligned} tr(\sigma U) &= \sum_{i_1,...,i_n} U^{i_1}_{i_{\sigma(1)}}...U^{i_n}_{i_{\sigma(n)}} \\ tr(\sigma\mathbf{1}) &= \sum_{i_1,...,i_n} \delta^{i_1}_{i_{\sigma(1)}}...\delta^{i_n}_{i_{\sigma(n)}} \\ tr(\sigma\mathbf{1}) &= N^{C(\sigma)}, \end{aligned}$$

so

$$Dim_N(R) = \frac{1}{n!}\sum_{\sigma\in S_n} \chi_R(\sigma)N^{C(\sigma)}. \tag{A.37}$$

A.4 Schur polynomials

A.4.1 Frobenius's character formula

Let C_i denote the conjugacy class of S_d consisting of permutations with i_1 1-cycles, i_2 2-cycles, ..., i_d d-cycles. This conjugacy class can be labelled by the sequence $i = (i_1, ..., i_d)$ with $\sum \alpha i_\alpha = d$. Now we introduce independent variables $x_1, ..., x_k$ with k at least as large as the number of rows in the Young diagram of λ. We define the power sums $P_j(x), 1 \le j \le d$ and the discriminant $\Delta(x)$ by

$$\begin{aligned} P_j(x) &= x_1^j + x_2^j + ... + x_k^j; \\ \Delta(x) &= \prod_{i<j}(x_i - x_j). \end{aligned}$$

Further, we define a notation for the coefficient of $x_1^{l_1}...x_k^{l_k}$ in a formal power series $f(x) = f(x_1, ..., x_k)$, where $(l_1, ..., l_k)$ is a k-tuple of non-negative integers:

$$[f(x)]_{(l_1,...,l_k)}$$

Now, if we set $l_1 = \lambda_1 + k - 1, l_2 = \lambda_2 + k - 2....l_k = \lambda_k$ (given a partition λ) we have the Frobenius character formula for the character of V_λ evaluated on $g \in C_i$: (V_λ is the irreducible representation of S_d corresponding to the partition λ.)

$$\chi_\lambda(C_i) = \left[\Delta(x).\prod_j P_j(x)^{i_j}\right]_{(l_1,...,l_k)} \tag{A.38}$$

For example, if $d = 5$, $\lambda = (3, 2)$ and C_i is the conjugacy class of (12)(345) (i.e. $i_1 = 0$ is the number of 1-cycles, $i_2 = 1$, $i_3 = 1$) then

$$\begin{aligned}
\chi_{(3,2)}(C_i) &= [(x_1 - x_2).(x_1^1 + x_2^1)^{i_1}(x_1^2 + x_2^2)^{i_2}(x_1^3 + x_2^3)^{i_3}]_{(\lambda_1+2-1,\lambda_2+2-2)} \\
&= [(x_1 - x_2).(x_1^2 + x_2^2)(x_1^3 + x_2^3)]_{(4,2)} \\
&= [(x_1^3 + x_1x_2^2 - x_2x_1^2 - x_2^3)(x_1^3 + x_2^3)]_{(4,2)} \\
&= [x_1^6 + x_2^3x_1^3 + x_1^4x_2^2 + x_1x_2^5 - x_1^5x_2 - x_2^4x_1^2 - x_1^3x_2^3 - x_2^6]_{(4,2)} \\
&= 1 \text{ (the coefficient of } x_1^4x_2^2)
\end{aligned}$$

Frobenius's formula can be used to find the dimension of V_λ: it is given by the character of the identity. The conjugacy class of the identity corresponds to $i = (d)(i = (i_1), i_1 = d$ i.e. d 1-cycles) so

$$\prod_j P_j(x) = P_1(x)^{i_1} \tag{A.39}$$

$$\prod_j P_j(x) = (x_1 + ... + x_k)^d, \tag{A.40}$$

and

$$dimV_\lambda = \chi_\lambda(C_{(d)}) = [\Delta(x).(x_1 +. .. + x_k)^d]_{(l_1,,...,l_k)}. \tag{A.41}$$

Now $\Delta(x)$ is the Van der Monde determinant:

$$\begin{vmatrix} 1 & x_k & ... & x_k^{k-1} \\ \vdots & & & \vdots \\ 1 & x_1 & ... & x_1^{k-1} \end{vmatrix} = \sum_{\sigma\in S_k}(sgn\sigma)x_k^{\sigma(1)-1}...x_1^{\sigma(k)-1}, \tag{A.42}$$

and

$$(x_1 + ... + x_k)^d = \sum \frac{d!}{r_1!...r_k!}x_1^{r_1}....x_k^{r_k}. \tag{A.43}$$

The sum above is over k-tuples $(r_1, ..., r_k)$ that sum to d. To find the coefficient of $x_1^{l_1}...x_k^{l_k}$ in the product, we pair off corresponding terms in these two sums, getting

$$\sum_\sigma sgn(\sigma)\frac{d!}{(l_1 - \sigma(k) + 1)!...(l_k - \sigma(1) + 1)!},$$

where now we are summing over those σ in S_k such that $l_{k-i+1} - \sigma(i) + 1 \geq 0 \;\forall\; 1 \leq i \leq k$. This sum can be written as

$$\frac{d!}{l_1!...l_k!} \sum_{\sigma \in S_k} sgn(\sigma) \prod_{j=1}^{k} l_j(l_j-1)...(l_j - \sigma(k-j+1)+2)$$
$$= \frac{d!}{l_1!...l_k!} \begin{vmatrix} 1 & l_k & l_k(l_k-1) & ... \\ \vdots & \vdots & \vdots & \vdots \\ 1 & l_1 & l_1(l_1-1) & ... \end{vmatrix}. \quad \text{(A.44)}$$

By column reduction this determinant reduces to the Van der Monde determinant, so

$$dimV_\lambda \;=\; \frac{d!}{l_1!...l_k!} \prod_{i<j} (l_i - l_j), \quad \text{(A.45)}$$

with $l_i = \lambda_i + k - i$. This is much easier to understand and work with when expressed in terms of 'hook lengths', where the hook length of a box in a Young diagram is the number of boxes in the right-hand hook of the diagram whose corner is defined by the box in question. For example,

$$\begin{array}{|c|c|} \hline 3 & 1 \\ \hline 1 \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|c|} \hline 5 & 3 & 2 \\ \hline 4 & 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array}\,,$$

where each box is labelled by its hook length. In terms of these quantities,

$$\text{dimV}_{\lambda} \;=\; \frac{d!}{\prod(\text{hook lengths})}. \quad \text{(A.46)}$$

A.4.2 Frobenius-Schur duality

Let $V = \mathbb{C}^n$. The vector space $V^{\otimes d}$ admits commuting actions of U(n) and S_d: a left action of U(n)($g \in U(n)$, $\sigma \in S_d$):

$$g.(v_1 \otimes ... \otimes v_d) \;=\; gv_1 \otimes ... \otimes gv_d, \quad \text{(A.47)}$$

and a right action of S_d:

$$(v_1 \otimes ... \otimes v_d).\sigma \;=\; v_{\sigma(1)} \otimes ... \otimes v_{\sigma(d)}, \quad \text{(A.48)}$$

Given a representation R_θ of S_d, $U_\theta = \left(V^{\otimes d}\right) \otimes \mathbb{C}S_d R_\theta$ is a representation of U(n). The vector space of endomorphisms of $V^{\otimes d}$ which commute with the linear transformations of S_d can be shown to be spanned by the linear transformations of U(n). It follows that (see [48])

$$\otimes^r V \cong \otimes_\theta U_\theta \otimes R_\theta, \tag{A.49}$$

where θ runs through the irreducible representations of S_d such that U_θ is nonzero. This will be all irreducible representations of S_d if $n \geq d$. This fact is known as Frobenius-Schur duality.[2] It arises crucially from the commuting actions of U(N) and S_n; the two commute and therefore can be simultaneously diagonalised. In addition, there is no degeneracy, so the irreducible representations of U(N) are the irreducible representations of S_n. Let η_θ be the character of U_θ and χ_θ be the character of θ.

Now let λ be a partition of d. Thus $\lambda = (\lambda_1, ..., \lambda_l)$. Let c_λ be a representation of the conjugacy class of S_d of cycle type λ. A fundamental formula known to Frobenius asserts that if $g \in$ U(n) then

$$\prod tr(g^{\lambda_i}) \quad = \quad \sum_\theta \chi_\theta(c_\lambda)\eta_\theta(g). \tag{A.50}$$

A.4.3 Schur polynomials

The vector space of homogeneous symmetric polynomials of degree d in k variables x_1, ..., x_k has several important bases, usually indexed by partitions λ of d into at most k parts or by Young diagrams with at most k rows. One important basis is given by the Schur polynomials

$$S_\lambda = \frac{|x_j^{\lambda_i+k-i}|}{|x_j^{k-i}|} = \frac{|x_j^{\lambda_i+k-i}|}{\Delta}, \tag{A.51}$$

where

$$\Delta \quad = \quad \prod_{i<j}(x_i - x_j) \tag{A.52}$$

is the discriminant, and $|a_j^i|$ denotes the determinant of a $k \times k$ matrix.

In terms of the Schur polynomials, Frobenius's formula can be expressed by

$$\prod_j P_j(x)^{i_j} \quad = \quad \sum \chi_\lambda(C_i)S_\lambda, \tag{A.53}$$

the sum over all partitions λ of d in at most k parts. This, together with (A.50), gives us the expression of [4] for Schur polynomials in terms of characters that is

[2] References are [39] and [48].

used in the body of the text.

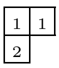
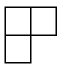
We have some useful rules for products of Schur polynomials:

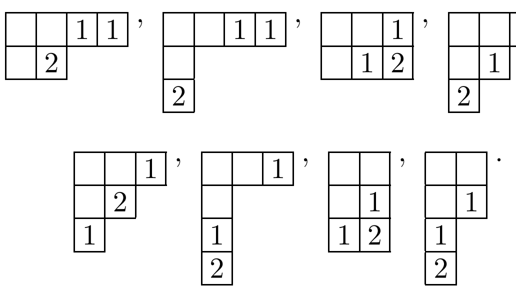
$$S_\lambda S_{(m)} \quad = \quad \sum S_\nu, \tag{A.54}$$

the sum over all ν whose Young diagram can be obtained from that of λ by adding a total of m boxes to the rows, but with no two boxes in the same row. More generally there is a Littlewood-Richardson rule that gives a combinatorical formula for the coefficient $N_{\lambda\mu\nu}$ in the expansion of a product as a linear combination of Schur polynomials.

$$S_\lambda S_\mu \quad = \quad \sum N_{\lambda\mu\nu} S_\nu \tag{A.55}$$

Here λ is a partition of d, μ is a partition of m, and the sum is over all partitions ν of d+m (each with at most k parts). The Littlewood-Richardson rule says that $N_{\lambda\mu\nu}$ is the number of ways the Young diagram for λ can be expanded to the Young diagram for ν by a strict μ-expansion. If $\mu = (\mu_1, ..., \mu_k)$, a μ-expansion of a Young diagram is obtained by first adding μ_1 boxes (with no two boxes in the same column) and labelling these boxes '1', then adding μ_2 boxes labelled 2 in the same way and so on. The expansion is strict if, when the integers in the boxes are listed from right to left, starting with the top row and working down, and one looks at the first t entries in this list, each integer p between 1 and k-1 occurs at least as many times as the next integer p+1.

For example, a strict  expansion of  yields :



Appendix B

Character tables for the symmetric group

n = 2

Class / Partition	1 (1^2)	1 (2)
(2)	1	1
(1^2)	1	-1

n = 3

Class / Partition	1 (1^3)	3 (1,2)	2 (3)
(3)	1	1	1
(2,1)	2	0	-1
(1^3)	1	-1	1

n = 4

Partition \ Class	1 (1^4)	6 $(1^2,2)$	8 $(1,3)$	3 (2^2)	6 (4)
(4)	1	1	1	1	1
$(3,1)$	3	1	0	-1	-1
(2^2)	2	0	-1	2	0
$(2,1^2)$	3	-1	0	-1	1
(1^4)	1	-1	1	1	-1

n = 5

Partition \ Class	1 (1^5)	10 $(1^3,2)$	20 $(1^2,3)$	15 $(1,2^2)$	30 $(1,4)$	20 $(2,3)$	24 (5)
(5)	1	1	1	1	1	1	1
$(4,1)$	4	2	1	0	0	-1	-1
$(3,2)$	5	1	-1	1	-1	1	0
$(3,1^2)$	6	0	0	-2	0	0	1
$(2^2,1)$	5	-1	-1	1	1	-1	0
$(2,1^3)$	4	-2	1	0	0	1	-1
(1^5)	1	-1	1	1	-1	-1	1

n = 6

Partition \ Class	1 (1^6)	15 $(1^4,2)$	40 $(1^3,3)$	45 $(1^2,2^2)$	90 $(1^2,4)$	120 $(1,2,3)$	144 $(1,5)$	15 (2^3)	90 $(2,4)$	40 (3^2)	120 (6)
(6)	1	1	1	1	1	1	1	1	1	1	1
$(5,1)$	5	3	2	1	1	0	0	-1	-1	-1	-1
$(4,2)$	9	3	0	1	-1	0	-1	3	1	0	0
$(4,1^2)$	10	2	1	-2	0	-1	0	-2	0	1	1
(3^2)	5	1	-1	1	-1	1	0	-3	-1	2	0
$(3,2,1)$	16	0	-2	0	0	0	1	0	0	-2	0
(2^3)	5	-1	-1	1	1	-1	0	3	-1	2	0
$(3,1^3)$	10	-2	1	-2	0	1	0	2	0	1	-1
$(2^2, 1^2)$	9	-3	0	1	1	0	-1	-3	1	0	0
$(2, 1^4)$	5	-3	2	1	-1	0	0	1	-1	-1	1
(1^6)	1	-1	1	1	-1	-1	1	-1	1	1	-1

n = 7

Class	1	21	70	105	210	420	504	105	630	280	840	210	504	420	720
Partition	(1^7)	(1^5,2)	(1^4,3)	(1^3,2^2)	(1^3, 4)	(1^2, 2, 3)	(1^2,5)	(1,2^3)	(1,2,4)	(1, 3^2)	(1,6)	(2^2, 3)	(2,5)	(3,4)	(7)
(7)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
(6,1)	6	4	3	2	2	1	1	0	0	0	0	-1	-1	-1	-1
(5,2)	14	6	2	2	0	0	-1	2	0	-1	-1	2	1	0	0
(5,1^2)	15	5	3	-1	1	-1	0	-3	-1	0	0	-1	0	1	1
(4,3)	14	4	-1	2	-2	1	-1	0	0	2	0	-1	-1	1	0
(4,2,1)	35	5	-1	-1	-1	-1	0	1	1	-1	1	-1	0	-1	0
(3^2,1)	21	1	-3	1	-1	1	1	-3	-1	0	0	1	1	-1	0
(4,1^3)	20	0	2	-4	0	0	0	0	0	2	0	2	0	0	-1
(3, 2^2)	21	-1	-3	1	1	-1	1	3	-1	0	0	1	-1	1	0
(3, 2, 1^2)	35	-5	-1	-1	1	1	0	-1	1	-1	-1	-1	0	1	0
(2^3, 1)	14	-4	-1	2	2	-1	-1	0	0	2	0	-1	1	-1	0
(3, 1^4)	15	-5	3	-1	-1	1	0	3	-1	0	0	-1	0	-1	1
(2^2, 1^3)	14	-6	2	2	0	0	-1	-2	0	-1	1	2	-1	0	0
(2, 1^5)	6	-4	3	2	-2	-1	1	0	0	0	0	-1	1	1	-1
(1^7)	1	-1	1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1

Appendix C

Results

The su(N) Schur polynomials for up to $n = 5$ boxes and there two-point correlators are set out below. Note that there are fewer independent Schur polynomials for a given n than there are representations of S_n. Only distinct correlators are given here.

One box

$$
\begin{aligned}
\chi_{\yng(1)}(\phi) &= 0; \\
\langle \chi_{\yng(1)}(\phi), \chi_{\yng(1)}(\phi^*) \rangle &= 0.
\end{aligned}
$$

Two boxes

$$
\begin{aligned}
\chi_{\yng(2)}(\phi) &= \frac{1}{2} tr\left(\phi^2\right); \\
\chi_{\yng(1,1)}(\phi) &= -\frac{1}{2} tr\left(\phi^2\right); \\
\langle \chi_{\yng(2)}(\phi), \chi_{\yng(2)}(\phi^*) \rangle &= \frac{1}{2}\left(N^2 - 1\right); \\
\left\langle \chi_{\yng(2)}(\phi), \chi_{\yng(1,1)}(\phi^*) \right\rangle &= -\frac{1}{2}\left(N^2 - 1\right); \\
\left\langle \chi_{\yng(1,1)}(\phi), \chi_{\yng(1,1)}(\phi^*) \right\rangle &= \frac{1}{2}\left(N^2 - 1\right).
\end{aligned}
$$

As we have already seen, the su(N) Schur polynomials are not linearly independent. A block rule which simplifies calculation arises from the tensor product discussion in 4.4.1:

$$
\begin{aligned}
\chi_{\ydiagram{2,1}} \times \chi_{\ydiagram{1}} &= \chi_{\ydiagram{2,1,1}} + \chi_{\ydiagram{2,2}} + \chi_{\ydiagram{3,1}} \\
\chi_{\ydiagram{2,1}} \times \chi_{\ydiagram{1}} &= 0,
\end{aligned}
$$

because of the tracelessness of $\phi \in \mathrm{su(N)}$. This gives us the identities we need to calculate only distinct correlators. In the two block example, $\chi_{\ydiagram{2}}(\phi) = -\chi_{\ydiagram{1,1}}(\phi)$, which allows deduction of two of the correlators given above from $\langle \chi_{\ydiagram{2}}(\phi), \chi_{\ydiagram{2}}(\phi^*) \rangle$.

Three boxes

$$
\begin{aligned}
\chi_{\ydiagram{3}}(\phi) &= \frac{1}{3} tr\left(\phi^3\right); \\
\chi_{\ydiagram{2,1}}(\phi) &= -\frac{1}{3} tr\left(\phi^3\right); \\
\chi_{\ydiagram{1,1,1}}(\phi) &= \frac{1}{3} tr\left(\phi^3\right); \\
\langle \chi_{\ydiagram{3}}(\phi), \chi_{\ydiagram{3}}(\phi^*) \rangle &= \frac{1}{3N}\left(N^2-1\right)\left(N^2-4\right).
\end{aligned}
$$

Four boxes

$$
\begin{aligned}
\chi_{\ydiagram{4}}(\phi) &= \frac{1}{8}\left[tr\left(\phi^2\right) tr\left(\phi^2\right) + 2tr\left(\phi^4\right)\right]; \\
\chi_{\ydiagram{3,1}}(\phi) &= -\frac{1}{8}\left[tr\left(\phi^2\right) tr\left(\phi^2\right) + 2tr\left(\phi^4\right)\right]; \\
\chi_{\ydiagram{2,2}}(\phi) &= \frac{1}{4} tr\left(\phi^2\right) tr\left(\phi^2\right); \\
\chi_{\ydiagram{2,1,1}}(\phi) &= -\frac{1}{8}\left[tr\left(\phi^2\right) tr\left(\phi^2\right) - 2tr\left(\phi^4\right)\right]; \\
\chi_{\ydiagram{1,1,1,1}}(\phi) &= \frac{1}{8}\left[tr\left(\phi^2\right) tr\left(\phi^2\right) - 2tr\left(\phi^4\right)\right].
\end{aligned}
$$

Thus we need only consider the correlators of Schur polynomials labelled by $\yng(4)$, $\yng(1,1,1,1)$, $\yng(2,2)$:

$$
\begin{aligned}
\left\langle \chi_{\yng(4)}(\phi), \chi_{\yng(4)}(\phi^*) \right\rangle &= \frac{1}{8N^2}\left(N^2-1\right)(N+2)(N+3)\left(3N^2-7N+6\right); \\
\left\langle \chi_{\yng(4)}(\phi), \chi_{\yng(2,2)}(\phi^*) \right\rangle &= \frac{1}{4N}\left(N^2-1\right)(N-1)(N+2)(N+3); \\
\left\langle \chi_{\yng(4)}(\phi), \chi_{\yng(1,1,1,1)}(\phi^*) \right\rangle &= -\frac{1}{8N^2}\left(N^2-1\right)\left(N^2-4\right)\left(N^2-9\right); \\
\left\langle \chi_{\yng(2,2)}(\phi), \chi_{\yng(2,2)}(\phi^*) \right\rangle &= \frac{1}{2}\left(N^2-1\right)\left(N^2+1\right); \\
\left\langle \chi_{\yng(2,2)}(\phi), \chi_{\yng(1,1,1,1)}(\phi^*) \right\rangle &= \frac{1}{4N}\left(N^2-1\right)(N+1)(N-2)(N-3); \\
\left\langle \chi_{\yng(1,1,1,1)}(\phi), \chi_{\yng(1,1,1,1)}(\phi^*) \right\rangle &= \frac{1}{8N^2}\left(N^2-1\right)(N-2)(N-3)\left(3N^2+7N+6\right).
\end{aligned}
$$

Five boxes

$$
\begin{aligned}
\chi_{\yng(5)}(\phi) &= \frac{1}{120}\left[24tr\left(\phi^5\right)+20tr\left(\phi^2\right)tr\left(\phi^3\right)\right]; \\
\chi_{\yng(4,1)}(\phi) &= -\frac{1}{120}\left[24tr\left(\phi^5\right)+20tr\left(\phi^2\right)tr\left(\phi^3\right)\right]; \\
\chi_{\yng(3,2)}(\phi) &= \frac{1}{120}\left[20tr\left(\phi^2\right)tr\left(\phi^3\right)\right]; \\
\chi_{\yng(3,1,1)}(\phi) &= \frac{1}{120}\left[24tr\left(\phi^5\right)\right]; \\
\chi_{\yng(2,2,1)}(\phi) &= -\frac{1}{120}\left[20tr\left(\phi^2\right)tr\left(\phi^3\right)\right]; \\
\chi_{\yng(2,1,1,1)}(\phi) &= -\frac{1}{120}\left[24tr\left(\phi^5\right)-20tr\left(\phi^2\right)tr\left(\phi^3\right)\right]; \\
\chi_{\yng(1,1,1,1,1)}(\phi) &= \frac{1}{120}\left[24tr\left(\phi^5\right)-20tr\left(\phi^2\right)tr\left(\phi^3\right)\right].
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \chi_{(5)}(\phi)\,,\chi_{(5)}(\phi^*)\right\rangle &= \frac{1}{30N^3}(N+1)(N+2)(N+3)(N+4) \\
&\quad \times \left(11N^4-50N^3+85N^2-70N+24\right); \\
\left\langle \chi_{(5)}(\phi)\,,\chi_{(3,2)}(\phi^*)\right\rangle &= \frac{1}{6N^2}\left(N^2-1\right)\left(N^2-4\right)(N-1)(N+3)(N+4); \\
\left\langle \chi_{(5)}(\phi)\,,\chi_{(3,1,1)}(\phi^*)\right\rangle &= \frac{1}{5N^3}\left(N^2-1\right)\left(N^2-4\right)(N+3)(N+4) \\
&\quad \times \left(N^2-2N+2\right); \\
\left\langle \chi_{(5)}(\phi)\,,\chi_{(1^5)}(\phi^*)\right\rangle &= \frac{1}{30N^3}\left(N^2-1\right)\left(N^2-4\right)\left(N^2-9\right)\left(N^2-16\right); \\
\left\langle \chi_{(3,2)}(\phi)\,,\chi_{(3,2)}(\phi^*)\right\rangle &= \frac{1}{6N}\left(N^2-1\right)(N+2)\left(N^3-2N^2+5N-10\right); \\
\left\langle \chi_{(3,2)}(\phi)\,,\chi_{(3,1,1)}(\phi^*)\right\rangle &= \frac{1}{N^2}\left(N^2-1\right)\left(N^2-2\right)\left(N^2-4\right); \\
\left\langle \chi_{(3,2)}(\phi)\,,\chi_{(1^5)}(\phi^*)\right\rangle &= -\frac{1}{6N^2}\left(N^2-1\right)(N+1)\left(N^2-4\right)(N-3)(N-4); \\
\left\langle \chi_{(3,1,1)}(\phi)\,,\chi_{(3,1,1)}(\phi^*)\right\rangle &= -\frac{1}{5N^3}\left(N^2-1\right)\left(N^2-4\right)\left(N^4+12\right); \\
\left\langle \chi_{(3,1,1)}(\phi)\,,\chi_{(1^5)}(\phi^*)\right\rangle &= \frac{1}{5N^3}\left(N^2-1\right)\left(N^2-4\right)(N-3)(N-4) \\
&\quad \times \left(N^2+2N+2\right); \\
\left\langle \chi_{(1^5)}(\phi)\,,\chi_{(1^5)}(\phi^*)\right\rangle &= \frac{1}{30N^3}(N-1)(N-2)(N-3)(N-4) \\
&\quad \times \left(11N^4+50N^3+85N^2+70N+24\right).
\end{aligned}
$$

Bibliography

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[3] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[4] S. Corley, A. Jevicki and S. Ramgoolam, "Exact correlators of giant gravitons from dual $\mathcal{N} = 4$ SYM theory," Adv. Theor. Math. Phys. **5**, 809 (2002) [arXiv:hep-th/0111222].

[5] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[6] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS/CFT correspondence," arXiv:hep-th/0201253.

[7] A. Sen, "An introduction to non-perturbative string theory," arXiv:hep-th/9802051.

[8] L. Susskind, "The world as a hologram," J. Math. Phys. **36**, 6377 (1995) [arXiv:hep-th/9409089].

[9] G. 't Hooft, "Dimensional reduction in quantum gravity," arXiv:gr-qc/9310026.

[10] G. 't Hooft, "A Planar Diagram Theory For Strong Interactions," Nucl. Phys. B **72**, 461 (1974).

[11] R. Haag, J. T. Lopuszanski and M. Sohnius, "All Possible Generators Of Supersymmetries Of The S Matrix," Nucl. Phys. B **88**, 257 (1975).

[12] M. K. Prasad and C. M. Sommerfield, "An Exact Classical Solution For The 'T Hooft Monopole And The Julia-Zee Dyon," Phys. Rev. Lett. **35**, 760 (1975).

[13] E. B. Bogomolny, "Stability Of Classical Solutions," Sov. J. Nucl. Phys. **24**, 449 (1976) [Yad. Fiz. **24**, 861 (1976)].

[14] G. W. Gibbons and P. K. Townsend, "Vacuum interpolation in supergravity via super p-branes," Phys. Rev. Lett. **71**, 3754 (1993) [arXiv:hep-th/9307049].

[15] M. Flato and C. Fronsdal, "Quantum Field Theory Of Singletons: The Rac," J. Math. Phys. **22**, 1100 (1981).

[16] P. A. M. Dirac, "A Remarkable Respresentation Of The 3+2 De Sitter Group," J. Math. Phys. **4**, 901 (1963).

[17] M. Flato, C. Fronsdal and D. Sternheimer, "Singleton physics," arXiv:hep-th/9901043.

[18] R. de Mello Koch and R. Gwyn, "Giant graviton correlators from dual SU(N) super Yang-Mills theory," JHEP **0411**, 081 (2004) [arXiv:hep-th/0410236].

[19] J. McGreevy, L. Susskind and N. Toumbas, "Invasion of the giant gravitons from anti-de Sitter space," JHEP **0006**, 008 (2000) [arXiv:hep-th/0003075].

[20] M. T. Grisaru, R. C. Myers and O. Tafjord, "SUSY and Goliath," JHEP **0008**, 040 (2000) [arXiv:hep-th/0008015].

[21] A. Hashimoto, S. Hirano and N. Itzhaki, "Large branes in AdS and their field theory dual," JHEP **0008**, 051 (2000) [arXiv:hep-th/0008016].

[22] J. Lee, "Tunneling between the giant gravitons in AdS(5) x S(5)," Phys. Rev. D **64**, 046012 (2001) [arXiv:hep-th/0010191].

[23] D. Sadri and M. M. Sheikh-Jabbari, "Giant hedge-hogs: Spikes on giant gravitons," Nucl. Phys. B **687** (2004) 161 [arXiv:hep-th/0312155].

[24] M. M. Sheikh-Jabbari, "Tiny graviton matrix theory: DLCQ of IIB plane-wave string theory, a conjecture," JHEP **0409** (2004) 017 [arXiv:hep-th/0406214].

[25] R. C. Myers, "Dielectric-branes," JHEP **9912**, 022 (1999) [arXiv:hep-th/9910053].

[26] M. M. Sheikh-Jabbari, "Open strings in a B-field background as electric dipoles," Phys. Lett. B **455**, 129 (1999) [arXiv:hep-th/9901080].

[27] D. Bigatti and L. Susskind, "Magnetic fields, branes and noncommutative geometry," Phys. Rev. D **62**, 066004 (2000) [arXiv:hep-th/9908056].

[28] J. M. Maldacena and A. Strominger, "AdS(3) black holes and a stringy exclusion principle," JHEP **9812**, 005 (1998) [arXiv:hep-th/9804085].

[29] S. R. Das, A. Jevicki and S. D. Mathur, "Giant gravitons, BPS bounds and noncommutativity," Phys. Rev. D **63**, 044001 (2001) [arXiv:hep-th/0008088].

[30] E. Witten, "Dynamical Breaking Of Supersymmetry," Nucl. Phys. B **188** (1981) 513.

[31] P. Salomonson and J. W. van Holten, "Fermionic Coordinates And Supersymmetry In Quantum Mechanics," Nucl. Phys. B **196** (1982) 509.

[32] D. Lancaster, "Supersymmetry Breakdown In Supersymmetric Quantum Mechanics," Nuovo Cim. A **79** (1984) 28.

[33] F. Cooper, A. Khare and U. Sukhatme, "Supersymmetry and quantum mechanics," Phys. Rept. **251** (1995) 267 [arXiv:hep-th/9405029].

[34] I. I. Balitsky and A. V. Yung, "Instanton Molecular Vacuum In $\mathcal{N} = 1$ Supersymmetric Quantum Mechanics," Nucl. Phys. B **274** (1986) 475.

[35] R. K. Kaul and L. Mizrachi, "Instanton - Anti-Instanton Induced Vacuum Energy In Supersymmetric QCD," J. Phys. G **15** (1989) 1633.

[36] V. Balasubramanian, D. Berenstein, B. Feng and M. x. Huang, "D-branes in Yang-Mills theory and emergent gauge symmetry," arXiv:hep-th/0411205.

[37] B. U. Eden, P. S. Howe, A. Pickering, E. Sokatchev and P. C. West, "Four-point functions in $\mathcal{N} = 2$ superconformal field theories," Nucl. Phys. B **581**, 523 (2000) [arXiv:hep-th/0001138].

[38] B. U. Eden, P. S. Howe, E. Sokatchev and P. C. West, "Extremal and next-to-extremal n-point correlators in four-dimensional SCFT," Phys. Lett. B **494**, 141 (2000) [arXiv:hep-th/0004102].

[39] W. Fulton and J. Harris, " Representation Theory," Springer Verlag, 1991.

[40] S. Corley and S. Ramgoolam, "Finite factorization equations and sum rules for BPS correlators in $\mathcal{N} = 4$ SYM theory," Nucl. Phys. B **641**, 131 (2002) [arXiv:hep-th/0205221].

[41] S. M. Lee, S. Minwalla, M. Rangamani and N. Seiberg, "Three-point functions of chiral operators in D = 4, $\mathcal{N} = 4$ SYM at large N," Adv. Theor. Math. Phys. **2**, 697 (1998) [arXiv:hep-th/9806074].

[42] D. Berenstein, "A toy model for the AdS/CFT correspondence," JHEP **0407** (2004) 018 [arXiv:hep-th/0403110].

[43] A. Ram, "Characters of Brauer's Centralizer Algebras," Pacific Journal of Mathematics, Vol. 169, No. 1, 1995.

[44] H. Lin, O. Lunin and J. Maldacena, "Bubbling AdS space and 1/2 BPS geometries," JHEP **0410** (2004) 025 [arXiv:hep-th/0409174].

[45] G. Mandal, "Fermions from Half-BPS Supergravity," arXiv:hep-th/0502104.

[46] G. James and M. Liebeck, " Representations and Characters of Groups," Cambridge University Press, 1993.

[47] M. Hamermesh, "Group Theory and its Application to Physical Problems," Addison-Wesley, 1962.

[48] D. Bump and D. Ginzburg, "Generalised Frobenius-Schur numbers," 2004 [sporadic.stanford.edu/bump/gfsnumbers.ps]

Other useful references not explicitly cited in the text are:

[49] C. M. Hull and P. K. Townsend, "Unity of superstring dualities," Nucl. Phys. B **438**, 109 (1995) [arXiv:hep-th/9410167].

[50] H. J. Boonstra, B. Peeters and K. Skenderis, "Branes and anti-de Sitter spacetimes," Fortsch. Phys. **47**, 109 (1999) [arXiv:hep-th/9801076].

[51] K. Sfetsos and K. Skenderis, "Microscopic derivation of the Bekenstein-Hawking entropy formula for non-extremal black holes," Nucl. Phys. B **517**, 179 (1998) [arXiv:hep-th/9711138].

[52] E. Witten, "String theory dynamics in various dimensions," Nucl. Phys. B **443**, 85 (1995) [arXiv:hep-th/9503124].

[53] H. J. Kim, L. J. Romans and P. van Nieuwenhuizen, "The Mass Spectrum Of Chiral $\mathcal{N} = 2$ D = 10 Supergravity On S^5," Phys. Rev. D **32**, 389 (1985).

[54] M. Gunaydin and N. Marcus, "The Spectrum Of The S^5 Compactification Of The Chiral $\mathcal{N} = 2$, D = 10 Supergravity And The Unitary Supermultiplets Of U(2, 2/4)," Class. Quant. Grav. **2**, L11 (1985).

[55] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, "Correlation functions in the CFT(d)/AdS($d+1$) correspondence," Nucl. Phys. B **546**, 96 (1999) [arXiv:hep-th/9804058].

[56] J. Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges," Phys. Rev. Lett. **75**, 4724 (1995) [arXiv:hep-th/9510017].

[57] O. Lunin, S. D. Mathur and A. Saxena, "What is the gravity dual of a chiral primary?," Nucl. Phys. B **655**, 185 (2003) [arXiv:hep-th/0211292].

[58] R. Grimm, M. Sohnius and J. Wess, "Extended Supersymmetry And Gauge Theories," Nucl. Phys. B **133**, 275 (1978).

[59] B. Janssen, Y. Lozano and D. Rodriguez-Gomez, "Giant gravitons in $AdS(3) \times S^3 \times T^4$ as fuzzy cylinders," arXiv:hep-th/0406148.

[60] N. V. Suryanarayana, "Half-BPS giants, free fermions and microstates of superstars," arXiv:hep-th/0411145.

[61] S. R. Das, A. Jevicki and S. D. Mathur, "Vibration modes of giant gravitons," Phys. Rev. D **63**, 024013 (2001) [arXiv:hep-th/0009019].

[62] S. Prokushkin and M. M. Sheikh-Jabbari, "Squashed giants: Bound states of giant gravitons," JHEP **0407** (2004) 077 [arXiv:hep-th/0406053].

[63] V. Balasubramanian, M. Berkooz, A. Naqvi and M. J. Strassler, "Giant gravitons in conformal field theory," JHEP **0204**, 034 (2002) [arXiv:hep-th/0107119].

[64] H. Bohr, E. Katznelson and K. S. Narain, "On Supersymmetry Breaking By Instanton Effects," Nucl. Phys. B **238** (1984) 407.

[65] J. M. Leinaas and K. Olaussen, "On The Interpretation Of Fermionic Zero Modes," Nucl. Phys. B **239** (1984) 209.