

**MAINTENANCE MODELS FOR
REPAIRABLE SYSTEMS WITH
GENERAL DEGREE OF REPAIR.**

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1 Introduction

Systems used in the production of goods and delivery of services constitute the vast majority of most industry's capital. These systems are subject to deterioration with usage and age. System breakdowns can lead to massive financial loss or dangerous situations especially in the case with systems involved in military operation. Hence there is an urgent need to have maintenance strategies in place to avoid such unwanted scenarios. Over the last 50 years, many maintenance models (policies) have been developed to meet the needs of various industries. In this research paper we will consider three classical ones that form the basis of many maintenance policies and also analyse some of the extensions made to them. Maintenance policies based on the concept of general repair as defined below are also analysed. We consider a system that can be subject to failure or breakdown at any time. Maintenance on such a system can be done either after its failure, known as corrective maintenance or during its operation to avoid or reduce future failures, known as preventive maintenance. Corrective maintenance is also known as repair or replacement (unplanned maintenance) since we are dealing with a failed unit. Preventive maintenance is planned maintenance carried out when the system is working properly in order to reduce the likelihood of future failures. Furthermore corrective and preventive maintenance can be classified into three categories: perfect, minimal and imperfect. During a corrective maintenance the three options are : a perfect repair implying that the failed unit is replaced by a new one, also known as replacement and the system is said to be "as good as new" after repair(Barlow and Hunter[1]); a minimal repair means that the system is fixed so that it carries on functioning with the same failure rate value it had

immediately before failure and the system is said to be “as bad as old” after repair (Barlow and Hunter[1]) and Shizuo[41]; Unfortunately, the two above repairs are extreme responses to failure. However, the type of repair required may be one which is in between a replacement and a minimal repair, where the state of the system is restored to a level between new and prior to failure. In other words the value of the failure rate is in between that of a new system and one just before failure.

This leads to the concept of a general repair or simply imperfect repair. Various maintenance models under general repairs have been proposed and the ones that use the “Virtual Age” of the system are investigated here with particular attention to the two Kijima types.

The three classical maintenance policies are:

- (i) Age-replacement policy
- (ii) Block replacement policy
- (iii) Repair cost limit policy.

Under each of the above-mentioned policies, various classes exist as well depending on whether maintenance falls into the perfect, minimal or imperfect categories. Before giving a brief introduction to each policy, the following assumptions are necessary: the lifetime distribution of a unit has increasing failure rate (IFR) and the time taken to carry out either a preventive or corrective maintenance is negligible.

The age-replacement policy was first introduced by Shizuo [41]. Under this policy, a unit is replaced upon failure or at a specified age T , whichever comes first. T is fixed and is measured from the time of the last replacement. If a failure occurs, the corrective maintenance actions can either be perfect, minimal or imperfect repairs depending on the state of the unit and the cost involved in implementing such repair. Similarly when maintenance is

planned at time T , it can be either perfect, minimal or imperfect, also taken into account the state of the system and cost factors, etc.... Various extensions to this policy have been made. In this paper we considered the following three modifications:

(i) Nakagawa ([37]) suggested that the unit is replaced at fixed time T or when the number of failures reaches N (say), whichever comes first. And any failure that occurs between planned maintenance is fixed by a minimal repair.

(ii) The second extensions we consider are a combination of both the age replacement policy and the “periodic replacement with minimal repair at failure” policy. In such policy, two types of failures can occur at age t . A Type I failure occurs with probability $1 - p(t)$ and is removed by minimal repair. And a Type II failure occurs with probability $p(t)$, and is fixed by a replacement. The unit is replaced at the n th Type I failure or the first Type II failure or at the planned age T , whichever occurs first. As will be shown, if $p(t) = 1$ and $n = \infty$, we get the original age replacement policy. The “periodic replacement with minimal repair at failure” policy is introduced below in section 3.2.1. So if $p(t) = 0$ and $n = \infty$, we will get such policy. This second extensions (which we called Policy 6) is a mixture of the one by Sheu et al ([40]) and Beichelt [9]

(iii) Wang and Pham ([46]) called their modification the mixed “age” preventive maintenance model. They assume that after the first $(k - 1)$ imperfect repairs, two types of failures can occur at time t : Type I failure which is fixed by a minimal repair having probability of occurrence $1 - p(t)$; and Type II failure which is fixed by a replacement having probability of occurrence $p(t)$. Hence after the first $(k - 1)$ imperfect

repairs the unit will be subject to a perfect maintenance at age T or at the first Type II failure whichever occurs first.

The second classical maintenance policy is the block replacement policy introduced by Barlow and Hunter ([1]). Under such policy a unit is replaced at set times kT ($k = 1, 2, \dots$), independent of failure history. And replacement (perfect repairs) also takes place at failures. This policy is easy to apply since only the calendar times since the last planned replacement need to be recorded. But the drawback is that almost new items might be replaced at planned replacement times. Barlow and Hunter ([1]) also introduce the concept of “periodic replacement with minimal repair at failures”. They called it ‘Policy II’. It’s similar to the block replacement policy except that at failure, minimal repair (as described earlier) is done. We look at three modifications made to the above policies.

(i) Another modification to the block replacement policy is the one by Tango [60]. In his model, preventive maintenance is done at times kT ($k = 1, 2, \dots$). Within each preventive maintenance cycle of length T , if a unit fails at time less than $T - s$ ($0 < s < T$), it is replaced by a new one. And if it fails after time $T - s$ in the interval $(0, T)$, it is replaced by a used one. Used units are assumed to have been collected during previous schedule maintenance.

(ii) Nakagawa ([38]) modified the “periodic replacement with minimal repair at failure” policy by keeping track of the number of planned maintenances done. If that number reaches N , then the unit is replaced by a new one, otherwise no action is taken.

(iii) Wang and Pham ([46]) also modified the block replacement policy to a general case. They suggested that at each failure a unit undergoes

an imperfect repair and the number of imperfect repairs performed is recorded. Once $(k - 1)$ th imperfect repairs have been done, then as in the original policy, preventive maintenance is carried out at times $kT (k = 1, 2, \dots)$. Such preventive maintenance is perfect. They assume that after the first $(k - 1)$ imperfect repairs, two types of failures can occur at time t : Type I failure which is fixed by a minimal repair having probability of occurrence $p(t)$; and Type II failure which is fixed by a replacement having probability of occurrence $1 - p(t)$.

The third classical maintenance policy is the repair cost limit policy introduced by Gardent and Nonant ([22]) and Drinkwater and Hastings [19]. In such policy, the variable of interest is the cost of repair. Once a unit failed, the cost of repair is determined and if it is less than a certain amount called the repair cost limit, the unit undergoes a repair. Otherwise it is replaced.

Beichelt ([5], [6], [7]) introduced other policies based on the repair cost rate (instead on the cost of a single repair) to decide whether a unit need to be replaced or repaired. His idea was to replace the unit once the repair cost rate is at least a predetermined level else it is minimally repaired. The repair cost rate function can either be linear or general as shown in section 6.1-6.3. In section 6.4 we look at a repair cost limit policy which is age dependent. The repair cost limit policy can be incorporated with the other classical policies. For example we will look at a policy where the first $(k - 1)$ th failures of a unit is fixed by imperfect repairs, and any failures after that will either be minimally fixed or replaced base on cost and the type of failures.

We start with the necessary mathematical and statistical theory required for this study in section 2. In section 3, the three main and first classical maintenance policies are presented in details and the cost functions are derived. In section 4 we look at the three mentioned extensions made to the age replacement policy. In section 5 and 6, extensions made to the block replacement and repair cost limit policies are studied respectively. Maintenance policies under general repair (base on the virtual age) are analysed in section 7. We looked at the models proposed by Kijima ([31], [32]) and give two applications

And finally section 8 gives the conclusion. For convenience, the policies are numbered as policy 1, 2, etc...

2. Theoretical Foundations

2.1 The Poisson Process (Beichelt [9], Feller [21], Ross [39])

A stochastic process $\{N(t), t \geq 0\}$ is called a counting process if $N(t)$ counts the number of occurrences of an event in the interval $(0, t]$ and satisfied the following conditions:

- (i) $N(t) \geq 0$.
- (ii) $N(t)$ is integer valued.
- (iii) If $s < t$, then $N(s) \leq N(t)$.
- (iv) The number of events which occur in the interval $(s, t]$ is given by $N(t) - N(s)$ for any s, t with $0 \leq s \leq t$.

A counting process is said to have *independent increments* if the number of events that occur in disjoint time intervals are independent. Hence the number of events that have occurred by time t must be independent of the number of events occurring in the interval $(t, t + \Delta t]$.

A counting process is said to have *stationary increments* if the distribution of the number of events that occur in any time interval depends only on the length of that interval of time. Hence the number of events in the interval $(s + h, t + h]$ has the same distribution as the number of events in the interval $(s, t]$.

Definition 2.1.1

The counting process $\{N(t), t \geq 0\}$ is said to be a *homogeneous Poisson process* with rate λ if:

- (i) $N(0)=0$
- (ii) $\{N(t), t \geq 0\}$ has independent increments
- (iii) Its increments $N(s, t) = N(t) - N(s)$, $0 \leq s \leq t$, have a Poisson distribution with parameter $\lambda(t - s)$:

$$P(N(s, t) = i) = \frac{(\lambda(t - s))^i}{i!} e^{-\lambda(t - s)}; \quad i = 0, 1, \dots,$$

Or, alternatively, introducing the length $\tau = t - s$ of the interval

$$P(N(s, s + \tau) = i) = \frac{(\lambda\tau)^i}{i!} e^{-\lambda\tau}; \quad i = 0, 1, \dots,$$

We can also define the homogeneous Poisson process in terms of the *Landau order*.

Definition 2.1.2

The counting process $\{N(t), t \geq 0\}$ is said to be a *homogeneous Poisson process* with rate λ if:

- (i) $N(0)=0$
- (ii) $\{N(t), t \geq 0\}$ has independent increments
- (iii) $P(N(t + h) - N(t) \geq 2) = o(h)$
- (iv) $P(N(t + h) - N(t) = 1) = \lambda h + o(h)$.

If the rate λ at time t , is allowed to be a function of time i.e. $\lambda(t)$, we obtain the *nonhomogeneous Poisson process*.

Definition 2.1.3

The counting process $\{N(t), t \geq 0\}$ is said to be a *nonhomogeneous Poisson process* with rate function $\lambda(t)$, $t \geq 0$, if:

- (i) $N(0)=0$
- (ii) $\{N(t), t \geq 0\}$ has independent increments
- (iii) $P(N(t+h) - N(t) \geq 2) = o(h)$
- (iv) $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$.

(Before going further, we need to point out that $\lambda(t)$ is also used in section 2.2 below to describe the ageing behaviour of a system. Hence its interpretation will be different. Here $\lambda(t)$ can be modelled as the arrival rate of failures for a system which is repairable).

From definition 2.1.3, if we let

$$\Lambda(t) = \int_0^t \lambda(s) ds, \quad (2.1)$$

which is also known as the integrated failure rate, then for a nonhomogeneous Poisson process, it can be shown that $\forall s, t \geq 0$

$$P(N(t) - N(s) = n) = e^{-\Lambda(t) + \Lambda(s)} \frac{(\Lambda(t) - \Lambda(s))^n}{n!},$$

for $n = 0, 1, \dots$

With the nonhomogeneous Poisson process, we no longer need the assumption of stationary increments which implies that events may be more likely to occur at certain times than at other times.

We considered the following three important aspects of the nonhomogeneous Poisson process:

- (i) the probability distribution of increments $N(s, t)$.

Let $p_n(s, t) = P(N(s, t) = n); 0 \leq s \leq t, n = 0, 1, \dots$

Since $\{N(t), t \geq 0\}$ has independent increments, for $h > 0$,

$$\begin{aligned}
p_0(s, t+h) &= P(N(s, t+h) = 0) \\
&= P\{0 \text{ events in } (s, t) \text{ and } 0 \text{ events in } (t, t+h)\} \\
&= P(N(s, t) = 0, N(t, t+h) = 0) \\
&= P(N(s, t) = 0) \cdot P(N(t, t+h) = 0) \\
&= p_0(s, t)[1 - \lambda(t)h + o(h)]
\end{aligned}$$

Hence,

$$\frac{p_0(s, t+h) - p_0(s, t)}{h} = -\lambda(t)p_0(s, t) + \frac{o(h)}{h}$$

Letting $h \rightarrow 0$ yields

$$p_0'(s, t) = -\lambda(t)p_0(s, t)$$

which is a first order partial differential equation. Using the initial condition $N(0) = 0$ or $p_0(0, 0) = 1$,

$$\log p_0(s, t) = -\int_s^t \lambda(u) du$$

Using (2.1), we obtain

$$p_0(s, t) = e^{-(\Lambda(t) - \Lambda(s))}. \quad (2.2)$$

By induction, for $n \geq 1$ (see proof below),

$$p_n(s, t) = \frac{[\Lambda(t) - \Lambda(s)]^n}{n!} e^{-(\Lambda(t) - \Lambda(s))}; \quad n \geq 1 \quad (2.3)$$

Hence $N(s, t)$ is Poisson distributed with mean $\Lambda(t) - \Lambda(s)$ i.e.

$$\begin{aligned}
E(N(s, t)) &= \Lambda(t) - \Lambda(s) \\
&= \int_s^t \lambda(u) du \quad (2.4)
\end{aligned}$$

Proof of Equation (2.3):

For $n > 0$

$$\begin{aligned} p_n(s, t+h) &= P(N(s, t+h) = n) \\ &= P(N(s, t) = n, N(t, t+h) = 0) \\ &\quad + P(N(s, t) = n-1, N(t, t+h) = 1) \\ &\quad + \sum_{k=2}^n P(N(s, t) = n-k, N(t, t+h) = k) \end{aligned}$$

Using condition (iii) of definition 2.1.3,

$$\sum_{k=2}^n P(N(s, t) = n-k, N(t, t+h) = k) = o(h)$$

Hence by the assumption of independent increments,

$$\begin{aligned} p_n(s, t+h) &= P(N(s, t) = n)P(N(t, t+h) = 0) \\ &\quad + P(N(s, t) = n-1)P(N(t, t+h) = 1) + o(h) \\ &= p_n(s, t)p_0(t, t+h) + p_{n-1}(s, t)p_1(t, t+h) + o(h) \\ &= p_n(s, t)[1 - \lambda(t)h - o(h) - o(h)] \\ &\quad + p_{n-1}(s, t)[\lambda(t)h + o(h)] + o(h) \\ &= p_n(s, t) - p_n(s, t)\lambda(t)h + p_{n-1}(s, t)\lambda(t)h \end{aligned}$$

Therefore

$$\frac{p_n(s, t+h) - p_n(s, t)}{h} = -p_n(s, t)\lambda(t) + p_{n-1}(s, t)\lambda(t)$$

Taking the limit as $h \rightarrow 0$ gives

$$p_n'(s, t) = -p_n(s, t)\lambda(t) + p_{n-1}(s, t)\lambda(t)$$

Multiplying both sides of the above equation by $e^{(\Lambda(t) - \Lambda(s))}$ will give

$$e^{(\Lambda(t) - \Lambda(s))} [p_n'(s, t) + p_n(s, t)\lambda(t)] = e^{(\Lambda(t) - \Lambda(s))} p_{n-1}(s, t)\lambda(t)$$

where the left hand side is simply $\frac{d}{dt} [e^{(\Lambda(t) - \Lambda(s))} p_n(s, t)]$.

For $n = 1$

$$\begin{aligned}\frac{d}{dt}[e^{(\Lambda(t)-\Lambda(s))} p_1(s,t)] &= e^{(\Lambda(t)-\Lambda(s))} p_0(s,t) \lambda(t) \\ &= e^{(\Lambda(t)-\Lambda(s))} e^{-(\Lambda(t)-\Lambda(s))} \lambda(t) \\ &= \lambda(t)\end{aligned}$$

which is a differential equation. Therefore

$$e^{(\Lambda(t)-\Lambda(s))} p_1(s,t) = \int_s^t \lambda(u) du + C$$

Noting that $p_1(0,0) = 0$,

$$\begin{aligned}p_1(s,t) &= \int_s^t \lambda(u) du e^{-(\Lambda(t)-\Lambda(s))} \\ &= [\Lambda(t) - \Lambda(s)] e^{-(\Lambda(t)-\Lambda(s))}\end{aligned}$$

To prove that

$$p_n(s,t) = \frac{[\Lambda(t) - \Lambda(s)]^n}{n!} e^{-(\Lambda(t)-\Lambda(s))}; \quad n \geq 1$$

We assume that it is true for $n = 1$. Then

$$\begin{aligned}\frac{d}{dt}[e^{(\Lambda(t)-\Lambda(s))} p_n(s,t)] &= e^{(\Lambda(t)-\Lambda(s))} e^{-(\Lambda(t)-\Lambda(s))} \frac{[\Lambda(t) - \Lambda(s)]^{n-1}}{(n-1)!} \lambda(t) \\ &= \frac{[\Lambda(t) - \Lambda(s)]^{n-1}}{(n-1)!} \lambda(t)\end{aligned}$$

Therefore solving the above differential equation by the method of substitution (letting $u = [\Lambda(t) - \Lambda(s)]$) and using the fundamental theorem of calculus,

$$e^{(\Lambda(t)-\Lambda(s))} p_n(s,t) = \frac{[\Lambda(t) - \Lambda(s)]^n}{n!} + C$$

Using the condition $p_n(0,0) = 0$

$$p_n(s, t) = \frac{[\Lambda(t) - \Lambda(s)]^n}{n!} e^{-(\Lambda(t) - \Lambda(s))}; \quad n \geq 1$$

which completes the proof.

- (ii) The probability density of the time of occurrence T_n of the n th Poisson event.

Let T_1 be the time of occurrence of the first Poisson event with distribution function $F_{T_1}(t) = P(T_1 \leq t)$ and density function $f_{T_1}(t)$. Then

$$\begin{aligned} p_0(0, t) &= P(T_1 > t) \\ &= 1 - F_{T_1}(t) \end{aligned}$$

From (2.2),

$$p_0(t) = e^{-\Lambda(t)} = e^{-\int_0^t \lambda(u) du}$$

Therefore

$$F_{T_1}(t) = 1 - e^{-\int_0^t \lambda(u) du} \quad \text{and} \quad f_{T_1}(t) = \lambda(t) e^{-\int_0^t \lambda(u) du}, \quad t \geq 0 \quad (2.5)$$

The distribution function of T_n is

$$\begin{aligned} F_{T_n}(t) &= P(T_n \leq t) \\ &= P(N(t) \geq n) \\ &= \sum_{i=n}^{\infty} \frac{[\Lambda(t)]^i}{i!} e^{-\Lambda(t)}, \quad n = 1, 2, \dots \end{aligned} \quad (2.6)$$

And the density is

$$f_{T_n}(t) = \frac{[\Lambda(t)]^{n-1}}{(n-1)!} \lambda(t) e^{-\Lambda(t)}; t \geq 0, n = 1, 2, \dots \quad (2.7)$$

(iii) the joint probability density of $(T_1, T_2, \dots, T_n); n = 1, 2, \dots$

Conditioning on the first Poisson event at time $T_1 = t_1$, the distribution function of T_2 is

$$F_{T_2}(t_2 | t_1) = P(T_2 \leq t_2 | T_1 = t_1)$$

which is the probability that one or more poison events occurs in the interval $(t_1, t_2]$. Using (2.3), we get

$$\begin{aligned} F_{T_2}(t_2 | t_1) &= 1 - p_0(t_1, t_2) \\ &= 1 - e^{-[\Lambda(t_2) - \Lambda(t_1)]} \end{aligned}$$

And the corresponding density is

$$f_{T_2}(t_2 | t_1) = \lambda(t_2) e^{-[\Lambda(t_2) - \Lambda(t_1)]}; \quad 0 \leq t_1 < t_2$$

Recall that the joint probability density of two random variables X and Y is given by

$$f_{X,Y}(x, y) = f_Y(y | x) f_X(x).$$

Hence the joint probability density of (T_1, T_2) is

$$f(t_1, t_2) = \begin{cases} \lambda(t_1) f_{T_1}(t_2) & \text{for } t_1 < t_2 \\ 0 & \text{otherwise} \end{cases}.$$

By induction, the joint density of $(T_1, T_2, \dots, T_n)$ is

$$f(t_1, t_2, \dots, t_n) = \begin{cases} \lambda(t_1) \lambda(t_2) \dots \lambda(t_{n-1}) f_{T_1}(t_n) & \text{for } 0 \leq t_1 < t_2 < \dots < t_n \\ 0 & \text{otherwise} \end{cases} \quad (2.8)$$

Definition 2.1.3

The stochastic process $\{X(t), t \geq 0\}$ is said to be a compound Poisson process if it can be represented by

$$X(t) = \sum_{i=1}^{N(t)} Y_i, t \geq 0 \quad (2.9)$$

where $\{N(t), t \geq 0\}$ is a Poisson process with rate λ , and $\{Y_i, i=1,2,\dots\}$ is a family of independent and identically distributed random variables with a common distribution. The process $\{N(t), t \geq 0\}$ and the sequence $\{Y_i, i \geq 1\}$ are assumed to be independent; in other words the value of the Y_i 's does not depend on the arrival of the events which $N(t)$ is counting. If $N(t) = 0$ in (2.9) then $X(t) = z$ where z is the initial value of $X(t)$ at $t = 0$. The mean value of $X(t)$ is obtained by the method of moment generating function (m.g.f) as follows:

$$\begin{aligned} M_{X(t)}(s) &= E(e^{sX(t)}) \\ &= \sum_{n=0}^{\infty} E(e^{sX(t)} | N(t) = n) P(N(t) = n) \\ &= \sum_{n=0}^{\infty} E(e^{s(Y_1 + \dots + Y_n)} | N(t) = n) e^{-\lambda t} \frac{(\lambda t)^n}{n!} \end{aligned}$$

Following the assumption of independence of $\{N(t), t \geq 0\}$ and the sequence $\{Y_i, i \geq 1\}$, and the independence of the Y_i 's, we get

$$\begin{aligned} M_{X(t)}(s) &= \sum_{n=0}^{\infty} E(e^{s(Y_1 + \dots + Y_n)}) e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\ &= \sum_{n=0}^{\infty} (E(e^{sY_1}))^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} \end{aligned}$$

The m.g.f of the Y_i 's is

$$M_Y(s) = E(e^{sY}).$$

Hence

$$\begin{aligned} M_{X(t)}(s) &= \sum_{n=0}^{\infty} (M_Y(s))^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\ &= e^{-\lambda t} + M_Y(s) e^{-\lambda t} \frac{(\lambda t)}{1!} + (M_Y(s))^2 e^{-\lambda t} \frac{(\lambda t)^2}{2!} + \dots \\ &= e^{-\lambda t} + e^{M_Y(s)} \\ &= e^{\lambda t(M_Y(s)-1)} \end{aligned}$$

Differentiating $M_{X(t)}(s)$ with respect to s yields

$$\frac{d}{ds} M_{X(t)}(s) = \lambda t \left\{ \frac{d}{ds} M_Y(s) \right\} e^{\lambda t(M_Y(s)-1)}.$$

Evaluating at $s = 0$ gives

$$E(X(t)) = \lambda t(E(Y)) \quad (2.10)$$

Alternatively we can make use of Wald identity to obtain the expected value of the compound Poisson process as follows:

$$\begin{aligned} E\left(\sum_{i=1}^{N(t)} Y_i\right) &= E(Y_i)E(N(t)) \\ &= E(Y_i)\lambda t \end{aligned}$$

2.2 Concepts based on the Aging behaviour of a system (Barlow and Proschan [3])

The lifetime of a system is the time span from its starting up time point (birth) to its failure (death), where 'failure' is assumed to be an instantaneous event. In the engineering context, a failure of a system need not be equivalent to the end of its useful life. If X is a lifetime with

distribution function $F(\cdot)$, then $F(x)$ is called failure probability and $\bar{F}(x) = 1 - F(x)$ is called survival probability with regard to the interval $[0, x]$, because $F(x)$ and $\bar{F}(x)$ are the respective probabilities that the system does or does not fail in $[0, x]$.

Residual Lifetime

Let $F_t(x)$ be the distribution function of the residual lifetime X_t of a system, which has already worked for t time units without failing:

$$\begin{aligned}
 F_t(x) &= P(X_t \leq x) \\
 &= P(X - t \leq x \mid X > t) \\
 &= \frac{P(X - t \leq x \cap X > t)}{P(X > t)} \\
 &= \frac{P(t < X \leq t + x)}{P(X > t)} \\
 &= \frac{F(t + x) - F(t)}{\bar{F}(t)} \quad x \geq 0, t \geq 0. \quad (2.11)
 \end{aligned}$$

The corresponding conditional survival probability $\bar{F}_t(x) = 1 - F_t(x)$ is given by

$$\bar{F}_t(x) = \frac{\bar{F}(t + x)}{\bar{F}(t)} \quad x \geq 0, t \geq 0 \quad (2.12)$$

We can also define the mean residual lifetime by

$$m_x(t) = E(X - t \mid X > t) = \int_0^{\infty} \bar{F}_t(x) dx \quad (2.13)$$

Thus $F(x)$ is said to have a decreasing (not strictly) mean residual lifetime (DMRL) if $m_x(t)$ is decreasing in t or if

$$m_x(t_1) \leq m_x(t_2) \quad \text{for } 0 \leq t_1 \leq t_2$$

Definition 2.2.1

A system is aging (rejuvenating) in the interval $[t_1, t_2]$, $t_1 < t_2$, if for an arbitrary but fixed x , the conditional failure probability $F_t(x)$ is increasing (decreasing) for increasing t , $t_1 \leq t \leq t_2$.

Another approach to the aging concept is the failure rate of the system. Consider the conditional failure probability $F_t(\Delta t)$ in $[t, t + \Delta t]$ relative to the length Δt of this interval. This is the conditional failure probability per unit time or the failure probability rate, given by

$$\frac{1}{\Delta t} F_t(\Delta t) = \frac{F(t + \Delta t) - F(t)}{\Delta t} \cdot \frac{1}{\bar{F}(t)}.$$

Hence

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} F_t(\Delta t) = \frac{f(t)}{\bar{F}(t)}$$

This limit is called the failure rate or hazard function and is denoted as $\lambda(t)$:

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)}. \quad (2.14)$$

provided $f(t)$ exists and $\bar{F}(t) > 0$. As mentioned earlier in section 2.1, $\lambda(t)$ is interpreted differently here. It is the rate at which the system is deteriorating because of wear and tear. Hence its tendency to fail. We assumed that the system die at failure.

Integrating both sides of (2.14) yields

$$\int_0^x \lambda(t) dt = -\log \bar{F}(x)$$

Therefore

$$\bar{F}(x) = e^{-\int_0^x \lambda(t) dt} \quad x \geq 0$$

and $F(x) = 1 - e^{-\int_0^x \lambda(t) dt} \quad x \geq 0.$

Using (2.1), we can write $\bar{F}(x)$, $F(x)$, $F_t(x)$, and $\bar{F}_t(x)$ as follows:

$$\begin{aligned} \bar{F}(x) &= e^{-\Lambda(x)} & x \geq 0 \\ F(x) &= 1 - e^{-\Lambda(x)} & x \geq 0 \\ F_t(x) &= 1 - e^{-(\Lambda(t+x) - \Lambda(t))} & x \geq 0, t \geq 0 \\ \bar{F}_t(x) &= e^{-(\Lambda(t+x) - \Lambda(t))} & x \geq 0, t \geq 0 \end{aligned} \quad (2.15)$$

Another useful way of describing the aging property of a system is using (2.15):

A system ages in the interval $[t_1, t_2]$, $t_1 < t_2$, if its failure rate $\lambda(t)$ is increasing in this interval.

2.2.1. Classes of Probability Distributions (Barlow and Proschan [3])

IFR (DFR) distribution: $F(x)$ is an increasing (decreasing) failure rate distribution, written as IFR (DFR) if $F_t(x)$ in (2.15) is increasing (decreasing) in $t \geq 0$, for each $x \geq 0$. Similarly if $\bar{F}_t(x)$ in (2.15) is decreasing (increasing) in $t \geq 0$, then $F(x)$ is an increasing (decreasing) failure rate distribution, for each $x \geq 0$.

If the density of $F(x)$ exists, then we can also say that $F(x)$ is an IFR (DFR) if the failure rate $\lambda(t)$ is increasing (decreasing) in $t \geq 0$, for each $x \geq 0$. We can also note that $F(x)$ is IFR iff $\Lambda(t)$ is convex. $\Lambda(t)$ can only be convex if $\lambda(t)$ is increasing.

NBU(NWU) distribution: If the conditional survival probability $\bar{F}_t(x)$ of a system of age t is less(greater) than the corresponding survival probability of a new system $\bar{F}(x)$, we say that $F(x)$ is a New Better Than Used(New Worse Than Used) distribution, written as NBU(NWU) . Alternatively $F(x)$ is a NBU (NWU) distribution if

$$\bar{F}(t+x) \leq (\geq) \bar{F}(t)\bar{F}(x) \quad \forall t \geq 0, x \geq 0.$$

NBUE (NWUE): The mean lifetime of a new system is given by

$$\mu = \int_0^{\infty} \bar{F}(x) dx$$

and the conditional mean residual lifetime of a used system at time t is

$$\mu(t) = \frac{1}{\bar{F}(t)} \int_t^{\infty} \bar{F}(x) dx.$$

Therefore, $F(x)$ is a New Better Than Used in Expectation (New Worst Than Used in Expectation), written as NBUE (NWUE) if

$$\frac{1}{\mu} \int_0^{\infty} \bar{F}(x) dx \leq (\geq) \bar{F}(t) \quad \forall t \geq 0.$$

For further details on this topic see [3], [15].

2.3 Renewal Theory (Cox [16])

Definition 2.3.1 An ordinary renewal process is a sequence of independent, nonnegative, and identically distributed random variables $X_1, X_2, \dots, X_n$ is the time between the $(n-1)th$ and the nth renewal and $X_0 = 0$. The time of the nth renewal is

$$S_n = \sum_{i=1}^n X_i \quad (2.16)$$

As an application to replacement models, consider a repairable system which upon failure, is either repaired or replaced. We assume that maintenance takes negligible amount of time. In this context, the time of failures or repairs are the independent, nonnegative, and identically distributed random variables $X_1, X_2, \dots$. And the time to the nth failure or repair is S_n as defined in (2.16).

When the system is put into operation at time $t = 0$, X_1 is the residual lifetime of a new system. And $X_2, X_3, \dots$ will be residual lifetime of the system after renewal. Hence the distribution of X_1 will be different from the assumed common distribution of $X_2, X_3, \dots$. Hence the following definition follows:

Definition 2.3.2. A delayed renewal process $\{X_1, X_2, \dots\}$ is a sequence of independent, nonnegative random variables such that X_1 is distributed as

$$F_1(t) = P(X_1 \leq t)$$

and $X_2, X_3, \dots$ are commonly distributed with distribution function

$$F(t) = P(X \leq t).$$

The number of events by time t is denoted by $N(t)$, which is the counting process and is called the *renewal process*. More specifically, $N(t)$ is the maximum n for which $S_n \leq t$ provided $N(t)=0$ if $X_1 \geq t$. i.e. $N(t)=\max\{n; S_n \leq t\}$. To obtain the distribution function of $N(t)$, we note that $N(t) \geq n$ iff $S_n \leq t$, hence

$$F_{S_n}(t) = P(S_n \leq t) = P(N(t) \geq n). \quad (2.17)$$

Therefore by independence of the X_i 's and using the method of convolution,

$$F_{S_n}(t) = F_1 * F^{*(n-1)}(t) \text{ for } n=1,2,\dots \quad (2.18)$$

From (2.17) and (2.18) we see that

$$\begin{aligned} P(N(t)=n) &= P(X_1 + X_2 + \dots + X_n \leq t \text{ and } X_1 + X_2 + \dots + X_{n+1} > t) \\ &= F_{S_n}(t) - F_{S_{n+1}}(t) \quad \text{for } n=0,1,\dots \end{aligned} \quad (2.19)$$

2.3.1 The Renewal Function

Definition 2.3.3 The renewal function is the expected number of renewals in $[0, t]$. We denote the renewal function of an ordinary renewal process by $H(t)$, and that of a delayed renewal process by $H_1(t)$. Using (2.17) and (2.18),

$$\begin{aligned} H(t) &= E(N(t)) \\ &= \sum_{n=1}^{\infty} P(N(t) \geq n) \\ &= \sum_{n=1}^{\infty} F^{*(n)}(t). \end{aligned} \quad (2.20)$$

and similarly

$$H_1(t) = \sum_{n=1}^{\infty} F_1 * F^{*(n-1)}(t). \quad (2.21)$$

The integral equations of the renewal functions is obtained by noting that

$$F^{n+1}(t) = \int_0^t F^n(t-x) dF(x) \text{ and } H_1(t) = \sum_{n=0}^{\infty} F_1 * F^{*(n)}(t)$$

Hence

$$\begin{aligned} H_1(t) &= F_1(t) + \sum_{n=1}^{\infty} \int_0^t F_1 * F^{*(n-1)}(t-x) dF(x) \\ &= F_1(t) + \int_0^t \sum_{n=1}^{\infty} F_1 * F^{*(n-1)}(t-x) dF(x) \end{aligned}$$

From (2.21),

$$H_1(t) = F_1(t) + \int_0^t H_1(t-x) dF(x) \quad (2.22)$$

Similar argument leads to

$$H(t) = F(t) + \int_0^t H(t-x) dF(x). \quad (2.23)$$

Equation (2.22) and (2.23) are known as the *Fundamental Renewal equation* of a delayed and ordinary renewal process respectively. Upon differentiating $H_1(t)$ and $H(t)$, the corresponding *renewal densities* $h_1(t)$ and $h(t)$ is obtained provided the densities of X_1 and X exists.

$$\begin{aligned} h_1(t) &= \frac{d}{dt} H_1(t) \\ &= f_1(t) + \int_0^t h_1(t-x) f(x) dx \end{aligned} \quad (2.24)$$

and

$$h(t) = f(t) + \int_0^t h(t-x)f(x)dx \quad (2.25)$$

Alternatively we can differentiate (2.20),

$$h(t) = \frac{d}{dt}H(t) = \sum_{n=1}^{\infty} f^{*(n)}(t)$$

where $h(t)dt$ is the probability of a renewal occurring in $[t, t+dt]$.

One easy way to solve for the above integral equations is to obtain their corresponding Laplace-Stieltjes transform

$$\hat{H}(s) = \int_0^{\infty} e^{-sx} dH(x).$$

Hence taking the transform in (2.22) and (2.23) we get

$$\hat{H}_1(s) = \hat{F}_1(s) + \hat{H}_1(s)\hat{F}_1(s) \text{ and } \hat{H}(s) = \hat{F}(s) + \hat{H}(s)\hat{F}(s)$$

so that

$$\hat{H}_1(s) = \frac{\hat{F}_1(s)}{1 - \hat{F}_1(s)} \quad \text{and} \quad \hat{H}(s) = \frac{\hat{F}(s)}{1 - \hat{F}(s)} \quad (2.26)$$

or equivalently

$$\hat{F}_1(s) = \frac{\hat{H}_1(s)}{1 + \hat{H}_1(s)} \quad \text{and} \quad \hat{F}(s) = \frac{\hat{H}(s)}{1 + \hat{H}(s)} \quad (2.27)$$

Therefore we obtain $H(t)$ or $H_1(t)$ by knowing either $F(t)$ or $F_1(t)$

respectively and vice-versa. We can note that the total number of renewals as $t \rightarrow \infty$ is ∞ , i.e.

$$\lim_{t \rightarrow \infty} N(t) = \infty. \quad (2.28)$$

And also of importance is the rate at which $N(t)$ goes to infinity, which is proved below for an ordinary renewal process. We assume that $E(X) = E(X_1) = \mu$ and is finite.

Theorem 2.1

$$P(\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{\mu}) = 1 \quad (2.29)$$

Proof:

Since

$$S_{N(t)} \leq t \leq S_{N(t)+1}$$

then

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{S_{N(t)+1}}{N(t)}$$

where $\frac{S_{N(t)}}{N(t)}$ is the average of the first $N(t)$ interarrival times of the renewals. Hence by the law of large numbers,

$$\lim_{N(t) \rightarrow \infty} \frac{S_{N(t)}}{N(t)} = \mu.$$

Consequently from (2.28),

$$\lim_{t \rightarrow \infty} \frac{S_{N(t)}}{N(t)} = \mu.$$

We note that $\frac{S_{N(t)+1}}{N(t)}$ can be written as

$$\left(\frac{S_{N(t)+1}}{N(t)+1} \right) \left(\frac{N(t)+1}{N(t)} \right)$$

And then

$$\lim_{t \rightarrow \infty} \frac{S_{N(t)+1}}{N(t)} = \mu$$

Finally by the Sandwich theorem,

$$\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{\mu}$$

Some other important results that we need are given below and the proofs can be found in most textbook on probability theory.

Theorem 2.2(Elementary renewal theorem)

The elementary renewal theorem states that the average renewal rate converge to $\frac{1}{\mu}$ with probability 1, provided $E(X_1)$ is finite i.e.

$$P(\lim_{t \rightarrow \infty} \frac{H_1(t)}{t} = \frac{1}{\mu}) = 1 \quad (2.30)$$

Theorem 2.3(Fundamental (Key) renewal theorem)

If $F(t)$ is not lattice, and if $g(t)$ is Riemann integrable on $[0, \infty]$, then

$$\lim_{t \rightarrow \infty} \int_0^t g(t-x) dH_1(x) = \frac{1}{\mu} \int_0^t g(x) dx \quad (2.31)$$

where $\mu = \int_0^{\infty} \bar{F}(t) dt$

Theorem 2.4(Blackwell's renewal theorem)

If $F(t)$ is not lattice, then

$$\lim_{t \rightarrow \infty} H_1(t+h) - H_1(t) = \frac{h}{\mu} \quad \forall h \geq 0 \quad (2.32)$$

i.e. the expected number of renewals in an interval of length h is

approximately $\frac{h}{\mu}$

2.4 Stochastic Ordering for Non-negative Random Variables.

(Stoyan [44])

Definition 2.4.1

The random variable X , with distribution function F , is said to be stochastically smaller than the random variable Y with distribution function G (written as $X \leq_d Y$) if

$$F(t) \geq G(t) \quad \forall t \in \mathbb{R} \quad (2.33)$$

Or if

$$\bar{F}(t) \leq \bar{G}(t). \quad \forall t \in \mathbb{R} \quad (2.34)$$

We assume that the density functions of the two above variables exist. Please note that the letter d underneath the less than or equal to sign refer to the stochastic ordering of random variables involved. It refers to the absolute sizes of those variables.

Moreover if $X \leq_d Y$ then

$$Ef(X) \leq E(f(Y)) \quad (2.35)$$

for all increasing function f provided the expected values exist.

Stochastic order deal with the absolute sizes of the random variables to be compared. Also of importance is the variability aspect among random variables which leads to the following definition:

Definition 2.4.2

Let X and Y be random variables with finite means. Then we say that X is less than Y in convex order (written as $X \leq_c Y$) if

$$Ef(X) \leq E(f(Y)) \quad (2.36)$$

for all real convex functions f , provided the expected values exist. Similarly if

$$Ef(X) \leq Ef(Y) \quad (2.37)$$

for all increasing convex functions f , we say that X is less than Y in increasing convex order (written as $X \leq_{ic} Y$)

Some other useful results that we need are mentioned below but for more detail explanation and applications see [9], [14], [33], and [44]

(1) Let X and Y be random variables with finite expectations.

(i) If $X \leq_d Y$ then $E(X) \leq E(Y)$.

Proof

$$\begin{aligned} E(X) &= \int_0^{\infty} P(X > t) dt \\ &= \int_0^{\infty} \bar{F}(t) dt \end{aligned}$$

$$\begin{aligned} E(Y) &= \int_0^{\infty} P(Y > t) dt \\ &= \int_0^{\infty} \bar{G}(t) dt \end{aligned}$$

Since $X \leq_d Y$, from (2.34) $\bar{F}(t) \leq \bar{G}(t)$

Therefore

$$\int_0^{\infty} \bar{F}(t) dt \leq \int_0^{\infty} \bar{G}(t) dt$$

Hence

$$E(X) \leq E(Y)$$

(ii) If $X \leq_d Y$ and $E(X) = E(Y)$ then X and Y have the same distribution.

Proof

$$E(X) = \int_0^{\infty} \bar{F}(t) dt \quad E(Y) = \int_0^{\infty} \bar{G}(t) dt$$

Since $E(X) = E(Y)$

$$\Rightarrow \bar{F}(t) = \bar{G}(t) \quad \forall t$$

Hence

$$F(t) = G(t) \quad \forall t$$

(iii) If $X \leq_d Y$ then $f(X) \leq_d f(Y)$ for all increasing functions f .

Proof

If f is increasing, then it is 1-1 and hence have an inverse f^{-1} which is also 1-1.

Let $f(X) = a$ and $f(Y) = b$

Then

$$X = f^{-1}(a) \quad \text{and} \quad Y = f^{-1}(b)$$

Since $X \leq_d Y$

$$\Rightarrow f^{-1}(a) \leq_d f^{-1}(b)$$

Thus

$$a \leq_d b \Rightarrow f(X) \leq_d f(Y)$$

(iv) $X \leq_c Y$ iff $X \leq_{ic} Y$ and $E(X) = E(Y)$

Proof

Let f be an arbitrary convex function and assume that there exist an α such that

$$x \mapsto f(x) + \alpha x \text{ is increasing.}$$

Taking expectation with respect to X and Y we get

$$E(f(X) + \alpha E(X)) \leq E(f(Y) + \alpha E(Y))$$

Since $E(X) = E(Y)$, we obtained

$$Ef(X) \leq Ef(Y)$$

(v) If $X \leq_c Y$ then $F \leq_c G$, where F and G are the distribution function of X and Y respectively.

Proof

Similar to the proof of (iv), let f be an arbitrary convex function and assume that there exist an α such that

$$x \mapsto f(x) + \alpha x.$$

If $X \leq_c Y$, then $Ef(X) \leq Ef(Y)$

Hence

$$E(f(X) + \alpha E(X)) \leq E(f(Y) + \alpha E(Y))$$

provided that $E(X) \leq E(Y)$ for the inequality to holds.

Therefore

$$\int_0^{\infty} F(t) dt \leq \int_0^{\infty} G(t) dt$$

implying that $F \leq_c G$

3. Classical Maintenance Policies

3.1 Policy 1: Age Replacement policy (Barlow and Proschan [2])

Under the *age replacement policy* a unit is replaced upon failure or at a specified time T hours after installation, whichever comes first. T is assumed to be constant i.e. non random and the unit has an increasing failure rate over its age. Hence this policy is suitable when the replacement cost due to failure is higher than the cost of a planned replacement. Let F be the failure distribution function of the unit with density function f and assume that the time taken to replace the unit is negligible. The survival probability of the unit for $nT \leq t < (n+1)T$ is denoted by

$$\begin{aligned} R_A(t, T) &= [P(X > T)]^n [P(X > t - nT)] \\ &= [\bar{F}(T)]^n \bar{F}(t - nT) \end{aligned} \quad (3.1).$$

From (3.1), we can see that if t is before the first planned replacement at time T , $R_A(t, T) = P(X > t)$ and $n = 0$. And if t is between the first and second planned replacement then the probability that the unit lifetime exceeds t is the probability that it survives until the first replacement ($n = 1$) and the interval (T, t) , etc... When F is IFR as define in section 2.2.1, for all $T \geq 0$,

$$R_A(t, T) \geq \bar{F}(t) \quad (3.2)$$

meaning that under age replacement policy, the survival probability is higher. And also for all $T_1 \leq T_2$,

$$R_A(t, T_1) \geq R_A(t, T_2) \quad (3.3).$$

Therefore the survival probability of the unit increases when the planned replacement time is reduced.

The mean time to failure (MTTF) of a unit under no age replacement is given by

$$\mu = \int_0^{\infty} tf(t)dt$$

Integrating by parts we get

$$\begin{aligned}\mu &= [tF(t)]_0^{\infty} - \int_0^{\infty} F(t)dt \\ &= [tF(t) - t]_0^{\infty} + \int_0^{\infty} \bar{F}(t)dt \\ &= \int_0^{\infty} (1 - F(t))dt\end{aligned}\tag{3.4}$$

Under the age replacement policy, the mean time between replacements is given by

$$\begin{aligned}E(X) &= \int_0^T tf(t)dt + \int_T^{\infty} Tf(t)dt \\ &= TF(T) - \int_0^T F(t)dt + T(1 - F(T)) \\ &= \int_0^T (1 - F(t))dt\end{aligned}\tag{3.5}$$

We note that the mean time between replacement is less than T .

And if we let $T \rightarrow \infty$ in (3.5) we obtain the MTTF without replacement.

For the mean time to failure (MTTF) under age replacement we first define

$Y_1, Y_2, \dots$, to be the times between failures (unplanned replacement) and

$$Y_i = N_i T + Z_i \quad \text{for } i = 1, 2, \dots$$

where N_i is a random variable accounting for the number of intervals of length T and Z_i is the time elapsed since the last planned replacement and the current failure. So the mean time to failure under the age replacement policy is given by

$$E(Y_i) = TE(N_i) + E(Z_i)$$

To obtain $E(N_i)$ and $E(Z_i)$ we need the distribution function of N_i and Z_i . So

$$P(N_i = n) = (1 - F(T))^n F(T) \quad n = 0, 1, \dots$$

Hence the expected number of replacement without failures is

$$\begin{aligned} E(N_i) &= \sum_{n=0}^{\infty} nP(N_i = n) \\ &= \sum_{n=0}^{\infty} n[1 - F(T)]^n F(T) \\ &= F(T) + [1 - F(T)]F(T) + 2[1 - F(T)]^2 F(T) + \dots \\ &= \frac{1 - F(T)}{F(T)} \end{aligned} \tag{3.6}$$

And

$$\begin{aligned} P(Z_i \leq t) &= P(X \leq t \mid X \leq T) & 0 \leq t \leq T \\ &= \frac{F(t)}{F(T)} \end{aligned}$$

Hence

$$E(Z_i) = \int_0^T \left(1 - \frac{F(t)}{F(T)}\right) dt$$

Therefore the expected time between failures (unplanned replacement) is

$$\begin{aligned} E(Y_i) &= T \left(\frac{1 - F(T)}{F(T)} \right) + \frac{1}{F(T)} \int_0^T (F(T) - F(t)) dt \\ &= \frac{1}{F(T)} \int_0^T (1 - F(t)) dt \\ &= \frac{\int_0^T (1 - F(t)) dt}{F(T)} \end{aligned} \quad (3.7)$$

Using (3.6), we can see that for $T_1 < T_2$, under the age replacement policy that

$$\frac{\int_0^{T_1} (1 - F(t)) dt}{F(T_1)} \geq \frac{\int_0^{T_2} (1 - F(t)) dt}{F(T_2)} \quad (3.8).$$

implying that the average time to failure increases when the planned replacement time decreases.

The mean number of replacement in an interval $(0, t)$ denoted by $E(N_A(t))$ is simply t divided by the mean time between replacement which is

$$E(N_A(t)) = \frac{t}{\int_0^T (1 - F(t)) dt} \quad (3.9)$$

The obvious question is the financial implication of such policy. Let c_1 be the cost of a preventive maintenance at time T and c_2 be the cost incurred at

failure for a perfect repair, where $c_1 < c_2$. Also denote $N_A^1(t)$ and $N_A^2(t)$ be the number of preventive maintenances and number of perfect repairs in $(0, t)$ respectively. Hence the expected cost during $(0, t)$ is given by

$$C(T) = c_1 E(N_A^1(t)) + c_2 E(N_A^2(t)) \quad (3.10)$$

We seek to minimise the average cost per unit time which is

$$\lim_{t \rightarrow \infty} \frac{C(T)}{t} = \lim_{t \rightarrow \infty} \left[c_1 \frac{E(N_A^1(t))}{t} + c_2 \frac{E(N_A^2(t))}{t} \right]$$

Using (3.6) and (3.9), we get

$$\lim_{t \rightarrow \infty} \frac{C(T)}{t} = \frac{c_1 \bar{F}(T) + c_2 F(T)}{\int_0^T (1 - F(t)) dt} \quad (3.11)$$

To obtain the optimum replacement interval, we need the value $T = T_0$ that minimise (3.11). By differentiating (3.11) with respect to T and setting the equation equal to zero we get

$$\frac{f(T)}{\bar{F}(T)} \int_0^T \bar{F}(t) dt - F(T) = \frac{c_1}{c_2 - c_1} \quad (3.12)$$

From (2.14) we obtain

$$\lambda(T) \int_0^T \bar{F}(t) dt - F(T) = \frac{c_1}{c_2 - c_1} \quad (3.13)$$

Using the assumption of IFR, there exists a unique solution $T = T_0$ provided that the failure rate is strictly increasing to infinity and is continuous. And the optimum cost rate is

$$C(T_0) = (c_2 - c_1) \lambda(T_0) \quad (3.14)$$

If the right hand side of equation (3.11) tends to infinity as T tends to zero, then it is decreasing and hence $T_0 = \infty$. Therefore the optimum age

replacement policy will be to replace at failure only. Berg [10] proves that the policy itself is the optimal decision rule among other policies.

3.2 Policy 2: Block Replacement policy (Barlow and Proschan[2])

Under the block replacement policy, the unit undergoes preventive maintenance (planned replacement) at times $kT (k = 1, 2, \dots)$ irrespective of its age. If the unit failed, then it is correctively maintained (unplanned replacement). Similar to the age replacement policy, we need to obtain the expected number of failures in $(0, t)$ denoted as $N_B(t)$, where the subscript stands for block.

Theorem 3.2.1

$$P(\lim_{t \rightarrow \infty} \frac{N_B(t)}{t} = \frac{E(N_B(T))}{T}) = 1 \quad (3.15)$$

Proof

Consider an interval of length T and let the number of failures in that particular interval $((i-1)T, iT)$ for $i = 1, 2, \dots$, be denoted by $N_{Bi}(T)$. The counting process $N_{Bi}(T)$ is independent and identically distributed random variables according to the poisson process. Using theorem (2.1), for $i = 1, 2, \dots$ and $k = 0, 1, \dots$

$$\sum_{i=1}^k \frac{N_{Bi}(T)}{kT(\frac{k+1}{k})} \leq \frac{N_B(t)}{t} \leq \sum_{i=1}^{k+1} \frac{N_{Bi}(T)}{(k+1)T(\frac{k}{k+1})} \quad (3.16)$$

When $t \rightarrow \infty$, means that the number of intervals of length T goes to infinity as well i.e. $k \rightarrow \infty$. By the strong law of large numbers and the sandwich theorem,

$$\begin{aligned}\lim_{k \rightarrow \infty} \sum_{i=1}^k \frac{N_{Bi}(T)}{kT(\frac{k+1}{k})} &= \frac{N_{B1}(T) + N_{B2}(T) + \dots}{T} \\ &= \frac{E(N_B(T))}{T}\end{aligned}$$

Hence

$$\lim_{t \rightarrow \infty} \frac{N_B(t)}{t} = \frac{E(N_B(T))}{T} \quad (3.17)$$

We can also note that over an infinite time span, that the number of failures per unit time is simply the expected number of failures per unit interval.

Or

$$\lim_{t \rightarrow \infty} \frac{E(N_B(t))}{t} = \frac{E(N_B(T))}{T} \quad (3.18)$$

Barlow and Proschan ([2]) show that the number of replacements under the block replacement policy is stochastically larger than the number of replacements under the age replacement policy i.e.

$$P(N_B^r(t) \geq n) \geq P(N_A^r(t) \geq n) \text{ for } n = 0, 1, \dots \quad (3.19)$$

where the superscript stands for replacement. On the other hand, with IFR, the number of failures under the block policy is less than the number of failures under the age policy. Therefore by theorem 3.2.1,

$$\lim_{t \rightarrow \infty} \frac{E(N_B^r(t))}{t} \geq \lim_{t \rightarrow \infty} \frac{E(N_A^r(t))}{t} \quad (3.20)$$

For the block replacement policy,

$$\lim_{t \rightarrow \infty} \frac{E(N_B^r(t))}{t} = \frac{E(N_B(t))}{T} + \frac{1}{T} \quad (3.21)$$

And for the age replacement policy, by the elementary renewal theorem (2.2)

$$\lim_{t \rightarrow \infty} \frac{E(N_B^r(t))}{t} = \frac{1}{\int_0^T (1-F(t))dt}$$

Hence from (3.20)

$$\begin{aligned} \frac{E(N_B(t))}{T} + \frac{1}{T} &\geq \frac{1}{\int_0^T (1-F(t))dt} \\ \Rightarrow E(N_B(T)) &\geq \frac{T}{\int_0^T (1-F(t))dt} - 1 \end{aligned} \quad (3.22)$$

Equation (3.22) provides a lower bound for the expected number of failures under the block replacement policy. To obtain an upper bound, Barlow and Proshan [2] showed that for a system with IFR, the number of failures under the block replacement policy is stochastically less than the number of failures under the age replacement policy i.e.

$$P(N_B(t) \geq n) \leq P(N_A(t) \geq n) \text{ for } n = 0, 1, \dots \quad (3.23)$$

Thus for an infinite time span,

$$\lim_{t \rightarrow \infty} \frac{E(N_B(t))}{t} \leq \lim_{t \rightarrow \infty} \frac{E(N_A(t))}{t} \quad (3.24)$$

From (3.7)

$$\lim_{t \rightarrow \infty} \frac{E(N_A(t))}{t} = \frac{F(T)}{\int_0^T (1-F(t))dt}$$

Therefore from (3.18) and the above equation,

$$\frac{E(N_B(T))}{T} \leq \frac{F(T)}{\int_0^T (1-F(t))dt} \quad (3.25)$$

Hence

$$E(N_B(T)) \leq \frac{TF(T)}{\int_0^T (1-F(t))dt} \quad (3.26)$$

Combining (3.22) and (3.26) we get bounds for the expected number of failures in $(0, T)$:

$$\frac{T}{\int_0^T (1-F(t))dt} - 1 \leq E(N_B(T)) \leq \frac{TF(T)}{\int_0^T (1-F(t))dt} \quad (3.27).$$

Similar to the age replacement policy cost factors are important. Again let c_1 and c_2 be the cost of a preventive and a corrective maintenance respectively.

We know from (3.17) that over a long period of time, the expected number of failures with planned maintenance scheduled at time $kT (k=1, 2, \dots)$ is

$\frac{E(N_B(T))}{T}$. Hence the expected cost per unit time for a block replacement

policy is given by

$$C(T) = \frac{c_1 + c_2 E(N_B(T))}{T} \quad (3.28)$$

The optimum interval $T = T_0$ that minimise (3.28) is obtained by differentiating (3.28) with respect to T and setting the equation equal to zero giving

$$T_0 = \frac{c_1 + c_2 E(N_B(T))}{c_2 \frac{d}{dT} E(N_B(T))} \quad (3.29)$$

Substituting (3.29) into (3.28) we get the minimum expected cost per unit time which is

$$C_{\min}(T_0) = c_2 \frac{d}{dT_0} E(N_B(T_0)) \quad (3.30)$$

More comparison between the age and block replacement policies is done by Berg [11].

3.2.1 Policy 3: Periodic replacement with minimal repair at failure.

This policy is similar to the block replacement policy such that preventive maintenance is carried out at specified times $kT (k = 1, 2, \dots)$. But when the system failed, a minimal repair is carried out. Under such repair, the system carries on functioning with the same failure rate value it had prior to failure. And the system is said to be “as bad as old”. This policy was introduced by Barlow and Hunter ([1]) and they refer to it as “Policy II”. As before the expected cost during $(0, t)$ is given by

$$C(t) = c_1 E(N_1^r(t)) + c_3 E(N_3(t))$$

where c_1 and c_3 are the cost of a preventive maintenance (replacement) and minimal repair respectively. And $N_1^r(t)$ and $N_3(t)$ is the number of

planned replacement and minimal repairs in $(0, t)$ respectively. The average cost per unit time is

$$\lim_{t \rightarrow \infty} \frac{C(t)}{t} = \lim_{t \rightarrow \infty} \left[c_1 \frac{E(N_1'(t))}{t} + c_3 \frac{E(N_3(t))}{t} \right] \quad (3.31)$$

Using (3.21),

$$\lim_{t \rightarrow \infty} \left[c_1 \frac{E(N_1'(t))}{t} \right] = \frac{E(N_1(t))}{T} + \frac{1}{T}$$

To obtain $E(N_3(t))$, we note firstly that the counting process $N_3(t)$ follows a nonhomogeneous poisson process. In section 2.1, we obtained the probability distribution of the increments of such a random variable (see equation 2.3). And we also obtained its expected value in terms of the integrated failure rate (see equation (2.4)).

Hence using (2.4) we obtained

$$\begin{aligned} E(N_3(t)) &= \int_0^t \lambda(s) ds \\ &= \Lambda(t) \end{aligned}$$

Therefore from (3.17),

$$\lim_{t \rightarrow \infty} \frac{E(N_3(t))}{t} = \frac{\Lambda(t)}{T}$$

Thus the average cost per unit interval of length T for such policy is then

$$C(T) = \frac{c_1 + c_3 \Lambda(T)}{T} \quad (3.32)$$

Differentiation of (3.32), and setting the resulting equation to be zero will give the optimum interval T_0 that will minimise $C(T)$ i.e. the following inequality should be satisfied

$$T_0\lambda(T_0) - \Lambda(T_0) = \frac{c_1}{c_3} \quad (3.33)$$

and substituting c_1 from (3.33) into (3.32) gives the minimum average cost per unit interval of length T_0 , which is

$$C_{\min}(T_0) = c_3\lambda(T_0) \quad (3.34)$$

3.3 Policy 4: Repair cost limit policy

The repair cost limit policy introduced by Drinkwater and Hastings ([19]) was solely based on the age of the system. They defined the repair cost limit as the amount of money that can be spent on the repair of a unit. If the repair needed costs above the estimated repair cost limit, the unit is replaced, else the necessary repair is undertaken.

We define the initial cost of the unit by c_0 . The total cost of the unit at time t is denoted by $C(t)$. $C(t)$ consists of the initial cost c_0 and the cost of repair(s) up to time t . Thus the average cost per unit time is

$$C_{\text{average}}(t) = \frac{C(t)}{t} \quad (3.35)$$

To obtain the repair cost limit equation at time t , we define the following quantities:

$c(t)$ = Cost of repair at time t

$c_f(t)$ = Expected total cost of future repair(s)

$R(t)$ = expected residual lifetime of the unit.

L = Total average cost per unit time of the unit.

Upon failure, the unit is repaired if

$$\frac{c(t) + c_f(t)}{R(t)} < L \quad (3.36)$$

where the left hand side of (3.36) represents the expected future total cost per unit time. If the unit is replaced at failure, then the expected future total cost per unit time is L . Therefore the repair cost limit equation is obtained when

$$\frac{c(t) + c_f(t)}{R(t)} = L \quad (3.37)$$

Hence the repair cost limit at time t is

$$c(t) = LR(t) - c_f(t) \quad (3.38)$$

In this original policy, two methods were used to obtain the optimum repair cost limits. The first one was a simulation done in the form of successive approximation to obtain a set of repair limits by using equation (3.38). And the second one was by mathematical derivation of the average cost equation, which was then minimised to obtain the minimal repair limit. The second method was done by observing the frequency of the number of repairs in a given time interval and also the cost of repairs. It was found that the number of repairs follows a poison distribution and that the cost of repairs follows a log-normal distribution closely, or might even be modelled by the negative exponential distribution. Hence the derivation was done. After studying the various extensions to the age replacement and block replacement policies, we will be in a better position to consider a detailed analysis of this policy because of the different types of failures that can occur to the unit. Some

failures are minor whereas others can cause complete system breakdown. Consequently different types of repairs can be done.

4. Extensions to the Age replacement Policy

4.1. Policy 5: The unit is replaced at fixed time T or when the number of failures reaches $N \geq 1$, whichever comes first, and any failure(s) between planned replacements is fixed by minimal repair.

We used same notation as in section 2.3 on renewal theory. The counting process $\{N(t): t \geq 0\}$ represents the number of failures in $(0, t)$. Since failures are removed by minimal repairs, the value of the failure rate remains the same and the unit is said to be “as bad as old”. We assume that the unit wears out due to its age (continuous deteriorations) and due to failures (because it has been used a number of times say N). Hence this policy is a combination of continuous and discrete replacements. Replacement can occur either at age T , which implies that the number of failures did not reach N and $\{N(T) = n; n = 0, 1, 2, \dots, N - 1\}$, or before T meaning that the unit has already failed N times. When failures occurs due to age, the integrated failure rate is

$$\Lambda(t) = \int_0^t \lambda(s) ds$$

And when failure occurs due to the number of times the unit has been used, the cumulative failure rate up to the n th breakdown is given by

$$\Lambda_n = \sum_{j=1}^n \lambda_j. \quad (4.1)$$

$\Lambda(t)$ represents the expected number of failures due to age in the interval $(0, t]$, whereas Λ_n is the expected number of failures due to the number of times the unit has been used until the n th time. Using the method of

convolution and renewal theory (see equation 2.17, 2.18, 2.19), the probability that the number of failures is equal to n is

$$P(N(t) = n) = F^{*n}(t) - F^{*(n+1)}(t) \quad n = 0, 1, \dots \quad (4.2)$$

where $F(t)$ is the distribution function of the time between the $(n-1)st$ and nth failure of the unit due to the number of uses.

With possible replacement at time S_N or T , the expected time to replacement is simply

$$T\bar{F}^{*N}(t) + \int_0^T t f^{*N}(t) dt$$

Integrating the above by parts, we get

$$\int_0^T (1 - F^{*N}(t)) dt \quad (4.3)$$

Equation 4.3 and 3.5 are basically the same.

If replacement takes place at S_N , the expected number of failures is the sum of the expected number of failures due to age and due to the number of uses.

Using equation (2.23)

$$E(N(t)) = \int_0^T \Lambda(t) dF^{*N}(t) + \int_0^T \Lambda_N dF^{*N}(t) \quad (4.4)$$

And if replacement takes place at scheduled time T , then $N(t) \leq N-1$.

Using (2.20) and (4.2),

$$E(N(T)) = \sum_{n=0}^{N-1} [\Lambda(T) + \Lambda_n] (F^{*n}(T) - F^{*(n+1)}(T)) \quad (4.5)$$

Hence the expected number of failures before replacement is

$$\begin{aligned}
& \int_0^T \Lambda(t) dF^{*N}(t) + \int_0^T \Lambda_N dF^{*N}(t) \\
& + \sum_{n=0}^{N-1} [\Lambda(T) + \Lambda_n] (F^{*n}(T) - F^{*n+1}(T))
\end{aligned} \tag{4.6}$$

Note that in the above equation

$$\begin{aligned}
\sum_{n=0}^{N-1} (F^{*n}(T) - F^{*n+1}(T)) &= 1 - F^{*N}(T) \\
&= \bar{F}^{*N}(T)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T \Lambda_N dF^{*N}(t) &= \Lambda_N F^{*N}(T) \\
&= \sum_{n=1}^N \lambda_n F^{*n}(T)
\end{aligned}$$

Integrating by parts, equation (4.6) becomes

$$\int_0^T \lambda(t) (1 - F^{*N}(t)) dt + \sum_{n=1}^N \lambda_n F^{*n}(T) \tag{4.7}$$

Let the cost for a planned preventive maintenance be c_1 at time T or when the number of failures reaches N ($N \geq 1$) and the cost of a minimal repair be c_2 . Then the expected cost per unit time is given by

$$C(T) = \frac{c_1 + c_2 \left\{ \int_0^T \lambda(t) (1 - F^{*N}(t)) dt + \sum_{n=1}^N \lambda_n F^{*n}(T) \right\}}{\int_0^T (1 - F^{*N}(t)) dt} \tag{4.8}$$

To obtain the optimum expected cost per unit time we need that particular value of T and N that will minimise equation 4.8. Partial differentiation of

(4.8) with respect to T and setting the derivative to be zero will give the optimal interval that minimise (4.8). Let

$$U(T) = c_1 + c_2 \left\{ \int_0^T \lambda(t)(1 - F^{*N}(t))dt + \sum_{n=1}^N \lambda_n F^{*n}(T) \right\}$$

and $V(T) = \int_0^T (1 - F^{*N}(t))dt$

Then

$$\frac{\partial}{\partial T} U(T) = c_2 \frac{\partial}{\partial T} \left\{ \int_0^T \lambda(t)(1 - F^{*N}(t))dt \right\} + c_2 \sum_{n=1}^N \lambda_n \frac{\partial}{\partial T} F^{*n}(T)$$

and by the fundamental theorem of calculus

$$\frac{\partial}{\partial T} V(T) = 1 - F^{*N}(T)$$

Using the quotient rule,

$$\frac{\partial}{\partial T} C(T) = \frac{\left(\int_0^T (1 - F^{*N}(t))dt \{ c_2 \lambda(T) + c_2 \int_0^T (1 - F^{*N}(t))dt + c_2 \sum_{n=1}^N \lambda_n f^{*n}(T) \} - (c_1 + c_2 \{ \int_0^T \lambda(t)(1 - F^{*N}(t))dt + \sum_{n=1}^N \lambda_n F^{*n}(T) \}) (1 - F^{*N}(T)) \right)}{\left(\int_0^T (1 - \bar{F}^{*N}(t))dt \right)^2}$$

Setting the above equation to zero, we get

$$\begin{aligned}
\frac{\partial}{\partial T} C(T) &= \int_0^T (1 - F^{*N}(t)) dt c_2 \lambda(T) + \frac{c_2 \int_0^T (1 - F^{*N}(t)) dt \sum_{n=1}^N \lambda_n f^{*n}(T)}{(1 - F^{*N}(T))} - c_1 \\
&\quad - c_2 \int_0^T \lambda(t) (1 - F^{*N}(t)) dt - c_2 \sum_{n=1}^N \lambda_n F^{*n}(T) \\
&= 0 \\
\Rightarrow c_2 \left\{ \begin{aligned} &\int_0^T (1 - F^{*N}(t)) dt \lambda(T) + \frac{\int_0^T (1 - F^{*N}(t)) dt \sum_{n=1}^N \lambda_n f^{*n}(T)}{(1 - F^{*N}(T))} \\ &- \int_0^T \lambda(t) (1 - F^{*N}(t)) dt - \sum_{n=1}^N \lambda_n F^{*n}(T) \end{aligned} \right\} = c_1 \\
\Rightarrow \left\{ \begin{aligned} &\int_0^T (1 - F^{*N}(t)) dt \lambda(T) + \frac{\int_0^T (1 - F^{*N}(t)) dt \sum_{n=1}^N \lambda_n f^{*n}(T)}{(1 - F^{*N}(T))} \\ &- \int_0^T \lambda(t) (1 - F^{*N}(t)) dt - \sum_{n=1}^N \lambda_n F^{*n}(T) \end{aligned} \right\} = \frac{c_1}{c_2} \quad (4.9)
\end{aligned}$$

Thus the value of $T = T_0$ that satisfies (4.9), gives the optimal expected cost per unit time given that $N \geq 0$ is fixed.

To obtain the value of N that minimise (4.8), Nakagawa [37], uses the fact that for a fix value of T ,

the average cost per unit time when the number of failures is $N + 1 >$

the average cost per unit time when the number of failures is N

and

the average cost per unit time when the number of failures is $N \leq$

the average cost per unit time when the number of failures is $N - 1$

Using the above inequalities and (4.8) he showed that

$$\frac{\int_0^T (1 - F^{*N}(t)) dt}{\int_0^T (F^{*N}(t) - F^{*N+1}(t)) dt} \left\{ \int_0^T \lambda(t) [F^{*N}(t) - F^{*N+1}(t)] dt + \lambda_{N+1} F^{*N+1}(T) \right\}$$

$$-\int_0^T \lambda(t)(1 - F^{*N}(t)) dt - \sum_{n=1}^N \lambda_n F^{*n}(T) > \frac{c_1}{c_2} \quad (4.10)$$

and

$$\frac{\int_0^T (1 - F^{*N}(t)) dt}{\int_0^T (F^{*N}(t) - F^{*N+1}(t)) dt} \left\{ \int_0^T \lambda(t) [F^{*N}(t) - F^{*N+1}(t)] dt + \lambda_{N+1} F^{*N+1}(T) \right\}$$

$$-\int_0^T \lambda(t)(1 - F^{*N}(t)) dt - \sum_{n=1}^N \lambda_n F^{*n}(T) \leq \frac{c_1}{c_2} \quad (4.11)$$

Thus the value of $N = N_0$ that satisfies (4.10) and (4.11) will give the optimal expected cost per unit time given that T is fixed.

4.2 Policy 6: The unit can incur one of two types of failures at time t : Type I failure which occurs with probability $1 - p(t)$ and is fixed by a minimal repair and Type II failure which occurs with probability $p(t)$ and is fixed by a replacement. The unit is replaced at either the n th Type I failure or the first Type II failure or at planned age T whichever occur first.

(Note that most of the theory needed to describe policy 6 comes from Beichelt ([9], page 136-140 and [4])

For this policy, we are dealing with a generalised nonhomogeneous poisson process $\{N(t); t \geq 0\}$ with failure rate $\lambda(t)$ and two types of events, Type I failure, and Type II failure which occurs independently of each other with respective probabilities of $1 - p(t)$ and $p(t)$. Thus $N(t)$ can be represented as

$$N(t) = N_1(t) + N_2(t) \quad (4.12)$$

where $\{N_1(t); t \geq 0\}$ and $\{N_2(t); t \geq 0\}$ are independent nonhomogeneous poisson process with respective failure rate $(1 - p(t))\lambda(t)$ and $p(t)\lambda(t)$, and counts the number of Type I and Type II failure respectively. Let Y be the time to the first occurrence of a Type II failure with distribution function $G(t) = P(Y_1 \leq t)$ and survival function $\bar{G}(t)$. Beichelt [9] showed that

$$\bar{G}(t) = P(N_2(t) = 0) = e^{-\int_0^t p(u)\lambda(u)du}, \quad t \geq 0 \quad (4.13)$$

Conditioning on the number of Type II failures, we can obtain the probability that the number of Type I failures is exactly k i.e.

$$P(N_1(t) = k \mid Y = t) = \frac{e^{-\int_0^t (1-p(u))\lambda(u)du} \left(\int_0^t (1-p(u))\lambda(u)du\right)^k}{k!} \quad (4.14)$$

for $k \geq 0$

If we take $T = \infty$, let X_1 be the time to the first replacement due to either a Type II failure or we reach the n th Type I failure. Then the survival distribution function of X_1 is

$$\bar{F}_{X_1}(t) = P(X_1 > t)$$

$$\begin{aligned}
&= \sum_{k=0}^{n-1} P(N_2(t)=0, N_1(t)=k) \\
&= \sum_{k=0}^{n-1} \bar{G}(t) P(N_1(t)=k)
\end{aligned} \tag{4.15}$$

Let L be the random length of a replacement cycle i.e. the time between any two successive replacements.

For this policy, with planned replacement at age T ,

$$L = \begin{cases} X_1 & \text{when } X_1 \leq T \\ T & \text{when } X_1 > T \end{cases} \tag{4.16}$$

Therefore the expected value of L is

$$\begin{aligned}
E(L) &= \int_0^T t dF_{X_1}(t) + T \bar{F}_{X_1}(T) \\
&= \int_0^T \bar{F}_{X_1}(t) dt \\
&= \sum_{k=0}^{n-1} \int_0^T \bar{G}(t) P(N_1(t)=k) dt \quad (\text{from 4.15})
\end{aligned} \tag{4.17}$$

The next step is to find the expected number of minimal repairs in a replacement cycle i.e. $E(N_1(T))$. Let $S_1, S_2, \dots$, be arrival time of the first, second, minimal repairs respectively.

$$E(N_1(T)) = E\left[\sum_{k=1}^{n-1} P(Y > S_k) I_{[0,T]}(S_k)\right] \tag{4.18}$$

where

$$I_{[0,T]}(S_k) = \begin{cases} 1 & \text{if } S_k \in [0, T] \\ 0 & \text{otherwise} \end{cases}$$

Equation (4.18) makes sense because the Type II failure must occur after the k th minimal repair to obtain the expected value of $N_1(T)$.

And $E(I_{[0,T]}(S_k)) = P(S_k \in [0, T])$

Thus

$$E(N_1(T)) = \sum_{k=1}^{n-1} \int_0^T P(Y > t) P(S_k \in [0, T])$$

We note that

$$P(S_k \leq T) = P(N_1(T) \geq k)$$

Using (4.14), and after some algebra,

$$E(N_1(T)) = \sum_{k=1}^{n-1} \int_0^T \bar{G}(t) P(N_1(t) = k-1) (1-p(t)) \lambda(t) dt \quad (4.19)$$

Let c_1, c_2 , and c_3 be the cost of, a planned replacement at time T , replacement at the n th failure and minimal repair respectively.

Therefore the total expected cost per unit time is

$$C(T) = \frac{c_3 \sum_{k=1}^{n-1} \int_0^T \bar{G}(t) P(N_1(t) = k-1) (1-p(t)) \lambda(t) dt + c_1 \bar{F}_{X_1}(T) + c_2 F_{X_1}(T)}{\sum_{k=0}^{n-1} \int_0^T \bar{G}(t) P(N_1(t) = k) dt} \quad (4.20)$$

If $n = \infty$ in (4.20), we obtain the policy considered by Beichelt [4].

And if $n = \infty$, $p(t) = 1$, and $c_3 = 0$ then we get the original age replacement policy.

And finally if $n = \infty$, $p(t) = 0$, we get Policy 3.

What we need next is the value of T and n that will minimise (4.20).

In the case that $n = \infty$, the expected cost per unit time becomes

$$C(T) = \frac{c_3 \int_0^T \bar{G}(t)(1-p(t))\lambda(t)dt + c_1 \bar{F}_{X_1}(T) + c_2 F_{X_1}(T)}{\int_0^T \bar{G}(t)dt} \quad (4.21)$$

Letting $p(t) = p$ at time t , Beichelt [5], showed that there is an optimal interval $T = T_0$, that will minimise (4.21) and it is a solution to the equation

$$p\lambda(T) \int_0^T \bar{G}(t)dt - G(T) = \frac{pc_1}{(c_2 - c_1 - c_3)p + c_3} \quad (4.22)$$

Beichelt [4] also proved that there exists a unique T_0 if $\lambda(t)$ is strictly increasing to infinity and $c_2 - c_1 > c_3(1+p)/p$.

Sheu et al [40], gives detailed mathematical literature on how to minimise (4.20) with respect to both n and T under various assumptions and showed that a unique n_0 and T_0 exists. They assumed that the cost of a minimal repair at failure is a function of a random part and a deterministic part.

4.3 Policy 7: The unit undergoes $(k-1)th$ imperfect repairs. After that, two types of failures are possible at time t . Type I failure occurs with probability $1-p(t)$ and is removed by a minimal repair, and a Type II failure occurs with probability $p(t)$ and is fixed by a replacement. After the $(k-1)th$ imperfect repairs, the unit is replaced at planned age T or at the first occurrence of Type II failure, whichever occur first. This policy is due to Wang and Phang [46] and they called it the “mixed age pm model”.

We need to point out that the unit is new at time $t = 0$. And hence we don't expect any major breakdown at the beginning of its life. Thus the repairs done are called imperfect in the sense that the operating time of the unit decreases by a fraction after each of those repairs. And the value of the failure rate is somewhere between the one when it is new and when it undergoes a minimal repair. Therefore the first $(k - 1)$ repairs are imperfect. Any repairs done afterwards are either minimal (due to Type I failure) or perfect (due to Type II failure). After a replacement, the time is set to zero and once again $(k - 1)$ imperfect repairs are done.

Policy 7 involves a geometric process and hence we need to define such process as given by Yeh Lam [48]

Definition 4.3.1

Given a sequence of non-negative random variables $\{X_n; n = 1, 2, \dots\}$. If they are independent and the distribution function of X_n is given by $F(a^{n-1}t)$ for $n = 1, 2, \dots$, where a is a positive constant, then $\{X_n; n = 1, 2, \dots\}$ is called a geometric process.

If $a > 1$, then we have a decreasing geometric process, i.e. $X_{n+1} \leq_d X_n$. If

$0 < a < 1$ then we have an increasing geometric process, i.e. $X_n \leq_d X_{n+1}$ and

if $a = 1$ it's simply a renewal process. Suppose that $E(X_1) = \zeta$ and $Var(X_1) = \sigma^2$, then for $n = 2, 3, \dots$

$$E(X_n) = \frac{\zeta}{a^{n-1}} \quad \text{and} \quad Var(X_n) = \frac{\sigma^2}{a^{2(n-1)}}$$

Under this policy, the imperfect repair is in between a perfect repair and a minimal repair. After such repair, the unit is said to be between "as good as

new” and “as bad as old” (Barlow and Hunter [1]). We assume that minimal repair takes negligible amount of time but now the replacement time is taken to be a random variable R with $E(R) = r$. The expected lifetime of a new unit is μ_1 and the expected time of the first repair of a new unit is μ_2 . Let X_k be the time between the $(k-1)$ st and the k th repairs with distribution function $F_{X_k}(t)$ and survival function $\bar{F}_{X_k}(t)$, assuming that the corresponding densities exists and the failure rate being $\lambda_k(t)$. After an imperfect repair, the lifetime decreases by a factor of a ($0 < a < 1$), thus giving us the following renewal process: $\{a^{k-1}X_k, k = 1, 2, \dots\}$. Then $\{X_k; k = 1, 2, \dots\}$ is an increasing geometric process. Imperfect maintenance times are independent random variables $\{Z_k; k = 1, 2, \dots\}$ and increases by a factor of b ($b > 1$) after each maintenance. Thus $\{b^{k-1}Z_k, k = 1, 2, \dots\}$ is a decreasing geometric process. We assume that the renewal processes $\{a^{k-1}X_k; k = 1, 2, \dots\}$ and $\{b^{k-1}Z_k; k = 1, 2, \dots\}$ are independent.

Both processes form a geometric progression:

$$\text{For the lifetimes: } \mu_1 \quad a\mu_1 \quad a^2\mu_1 \quad \dots a^{k-2}\mu_1$$

$$\text{For the imperfect maintenance times: } \mu_2 \quad b\mu_2 \quad b^2\mu_2 \quad \dots b^{k-2}\mu_2$$

Note that

$$F_{X_k}(t) = F_{X_1}\left(\frac{t}{a^{k-1}}\right), \quad \lambda_k(t) = \frac{1}{a^{k-1}} \lambda_1\left(\frac{t}{a^{k-1}}\right), \quad E(X_k) = \frac{E(X_1)}{a^{k-1}}$$

$$E(Z_1) = \mu_2, \quad E(X_1) = \mu_1, \quad E(Z_k) = \frac{E(Z_1)}{b^{k-1}}$$

For convenience, we will write $\frac{1}{a^{k-1}}$ as a^{1-k} and $\frac{1}{b^{k-1}}$ as b^{1-k}

Of obvious interest is the counting process $\{N_m(t); t \geq 0\}$ for the number of minimal repairs in $(0, t)$. This counting process is studied extensively in policy 6 and Beichelt ([4], [9]). Hence most of the required theory here comes from the above-mentioned references. As in policy 6, Y denotes the time of the first Type II failure with survival distribution given by

$$\begin{aligned}\bar{G}(t) &= e^{-\int_0^t p(u)\lambda_k(u)du} \\ &= e^{-a^{1-k}\int_0^t p(u)\lambda_1(a^{1-k}u)du}\end{aligned}\quad (4.23)$$

The expected length of a replacement cycle is

$$E(L) = \frac{\mu_1(1-a^{k-1})}{1-a} + \frac{\mu_2(1-b^{k-1})}{1-b} + \int_0^T \bar{G}(t)dt + r \quad (4.24)$$

The first term on the right of (4.24) is the expected lifetimes of the unit until the $((k-1)th)$ imperfect repairs. The second term is the expected imperfect maintenance times. The third term is the expected time of a first Type II failure since the last imperfect maintenance. And finally the last term is the expected duration of a replacement. The first two terms are obtained by taking expectation of the sum of the first $(k-1)th$ terms of the two respective geometric progressions above.

To obtain the expected number of minimal repairs in the interval $(0, \min(t, Y))$, Beichelt ([9], Pg 140, Eq 3.69-3.72) showed that

$$\begin{aligned}E(N_m(t) | Y < t) &= \frac{1}{G(t)} \int_0^t \int_0^y p(u)\lambda_k(u)dudG(y) \\ &= \frac{a^{1-k}}{G(t)} \int_0^t \int_0^y p(u)\lambda_1(a^{1-k}u)dudG(y)\end{aligned}\quad (4.25)$$

and

$$\begin{aligned}
 E(N_m(t) | Y \geq t) &= \int_0^t p(u) \lambda_k(u) du \\
 &= a^{1-k} \int_0^t p(u) \lambda_1(a^{1-k}u) du
 \end{aligned} \tag{4.26}$$

Therefore the unconditional mean value of $N_m(t)$ is

$$\begin{aligned}
 E(N_m(t)) &= E(N_m(t) | Y < t)G(t) + E(N_m(t) | Y \geq t)\bar{G}(t) \\
 &= a^{1-k} \int_0^t \bar{G}(u) \lambda_1(a^{1-k}u) du - G(t)
 \end{aligned} \tag{4.27}$$

Let c_1, c_2, c_3, c_4 , and c_5 be the following cost respectively: planned replacement, replacement due to Type II failure, minimal repair, fixed cost involved for an imperfect repair, and the variable costs at the n th imperfect repair which is $\frac{n(n-1)}{2}c_5$ ($n = 1, 2, \dots, k-1$).

Therefore the total expected cost per unit time is given by

$$\begin{aligned}
 C(T) &= \\
 &\frac{\left(c_4(k-1) + c_5 \left(\frac{k(k-1)}{2} \right) + c_1 \bar{G}(T) + c_2 G(T) + \right. \\
 &\quad \left. c_3 \left\{ a^{1-k} \int_0^T \bar{G}(u) \lambda_1(a^{1-k}u) du - G(T) \right\} \right)}{\frac{\mu_1(1-a^{k-1})}{1-a} + \frac{\mu_2(1-b^{k-1})}{1-b} + \int_0^T \bar{G}(t) dt + r}
 \end{aligned} \tag{4.28}$$

The optimal value of $T = T_0$ and $k = k_0$ that minimise (4.28) is obtained by approximation method.

5. Extensions to the Block Replacement policy

5.1 Policy 8: A unit is replaced at planned maintenance times. Within each renewal cycle of length T , if the unit fail before reaching a certain age, $T - s$ ($0 \leq s \leq T$), it is replaced by a new one. Else if it fails after age $T - s$, it is replaced by a used one.

This policy is due to Tango [45].

With regard to the original block replacement policy, where a failed unit is always replaced, policy 8 considers using either a new unit or a used one at failure. And this aspect makes it more economical. In such policy the renewal cycle is of length T and it is split into two time interval: the first one of length $T - s$ ($0 \leq s \leq T$) and the second one of length s . The idea is that if a unit fail in the first interval i.e. in $[(k-1)T, kT - s]$, it is replaced by a new one and if it fails in the second interval $[kT - s, kT]$, it is replaced by a used one. Since we are dealing with used units, we assume that they come from previous planned replacement and that they did not fail in a renewal cycle of length T . Thus having different distribution compare to new ones. Therefore let $F_T(x)(\bar{F}_T(x))$ represents the conditional failure (survival) probability of a used unit. And for a new one the failure (survival) probability is $F(x)(\bar{F}(x))$. In this policy, of importance to the cost function are: the number of replacements (new unit) due to failure in the interval $[(k-1)T, kT - s]$, the number of replacement (used unit) due to failure in the interval $[kT - s, kT]$, and the number of replacement at schedule times kT . And we denote them by $N_1(t)$, $N_2(t)$, and $N_3(t)$ respectively for the interval $(0, t)$.

The total expected cost in $(0, t)$ is given by

$$C(t) = c_1 E(N_1(t)) + c_2 E(N_2(t)) + c_3 E(N_3(t)) \quad (5.1)$$

where c_1, c_2 , and c_3 are: cost of replacement with a new unit at failure, cost of replacement with a used unit at failure, cost of planned replacement respectively.

And it is known that from the renewal reward process (Ross [39], Theorem 3.6.1, Pg 133) and equation (3.18) in section 3, that the expected cost per unit time is

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{C(t)}{t} &= c_1 \lim_{t \rightarrow \infty} \frac{E(N_1(t))}{t} + \lim_{t \rightarrow \infty} c_2 \frac{E(N_2(t))}{t} + c_3 \lim_{t \rightarrow \infty} \frac{E(N_3(t))}{t} \\ &= \frac{\text{Expected cost over the renewal cycle of length } T}{T} \end{aligned} \quad (5.2)$$

The number of planned replacement made in $(0, t)$ is exactly $\frac{t}{T}$. Therefore

$$\lim_{t \rightarrow \infty} \frac{E(N_3(t))}{t} = \frac{1}{T}$$

We note that replacement at failure can only take place in an interval of length $T - s$, which means that $\lim_{t \rightarrow \infty} \frac{E(N_1(t))}{t}$ is the same as: (expected number of replacement in an interval of length $T - s$) / T

$$\lim_{t \rightarrow \infty} \frac{E(N_1(t))}{t} = \frac{E(N_1(T - s))}{T}$$

Replacement at failure with a used unit takes place in the interval $[T - s, T]$.

Suppose that the first failure in this interval occurs at age x , then the conditional expected number of replacement by used unit in that interval is $1 + E(N_2(T - x))$.

Recall that for all random variables X and Y (Ross [53], pg 20-21),

$$\begin{aligned} E(X) &= E[E(X | Y)] \\ &= \int E[X | Y = y] dF(y) \\ &= \int \left[\int x f(x | y) dx \right] dF(y) \\ &= \int \left[\int x dF(x | y) \right] dF(y) \end{aligned}$$

Therefore the unconditional expected number of replacements by used items in the interval of length T is

$$\begin{aligned} E(N_2(T)) &= \int_{T-s}^T (1 + E(N_2(T-x))) dF(x) + \\ &\quad \int_0^{T-s} \int_{T-s}^T (1 + E(N_2(T-x))) dF(x-u) dE(N_1(u)) \end{aligned} \quad (5.3)$$

Using renewal theory from section 2.4, and the distribution of the lifetimes of new and used units, then

$$H_1(t) = E(N_1(t)) = \sum_{n=1}^{\infty} F^{*n}(x)$$

and

$$H_2(t) = E(N_2(t)) = \sum_{n=1}^{\infty} F_T^{*n}(x)$$

Hence equation (5.3) can be written as

$$\begin{aligned} H_2(T) &= \int_{T-s}^T (1 + H_2(T-x)) dF(x) + \\ &\quad \int_0^{T-s} \int_{T-s}^T (1 + H_2(T-x)) dF(x-u) dH_1(y) \end{aligned} \quad (5.4)$$

Therefore the expected cost per unit time is

$$C(T) = \frac{c_1 H_1(T) + c_2 H_2(T) + c_3}{T} \quad (5.5)$$

An alternative way to obtain the expected number of replacement by used units was also considered by Tango [45]. He suggested viewing the arrivals of the replacement as a delayed renewal process. At time of first failure in the interval $[T-s, T]$, at age x , the unit had distribution function $F(x)$, because it was a new unit that either started operation in the first interval or one that did not fail since $t = 0$. After age x , replacement is carried out with used units; hence their lifetime distributions ($F_T(x)$) will be different. For more details see Tango [45].

To obtain the optimal value of T and s that will minimise $C(T)$ it is necessary to use appropriate computer software. But Tango [45], gave upper bound for $C(T)$.

5.2 Policy 9: Preventive maintenances are carried out at scheduled times kT ($k = 1, 2, \dots, N$). The unit is replaced with a new one at the N th preventive maintenance. Any failure is removed by a minimal repair.

(This policy is due to Nakagawa [38])

Policy 9 is a modification of policy 3 (periodic replacement with minimal repair at failure) where replacement takes place at the time NT instead at times $T, 2T, \dots$, as in policy 3. In each of the intervals $[(k-1)T, kT]$, for $k = 1, 2, \dots, N$ the failure rate is $\lambda_k(t)$ ($(k-1)T \leq t \leq kT$). In section 3.2.1, we obtained the expected number of minimal repairs in $(0, t)$, in terms of both the failure rate and the integrated failure rate. In our case here, we take

t to be in the interval $(0, NT)$. And also the length of a renewal/replacement cycle is NT . We denote the number of minimal repairs in $(0, t)$ by $N_3(t)$.

For $0 < t \leq T$,

$$E[N_3(t)] = \int_0^T \lambda_1(t) dt$$

If $T < t < 2T$, then

$$E[N_3(t)] = \int_0^T \lambda_1(t) dt + \int_T^t \lambda_2(s) ds$$

Hence the expected number of minimal repairs in a replacement cycle i.e. in the time interval $(0, NT)$ is simply

$$E[N_3(NT)] = \int_0^T \lambda_1(t) dt + \int_T^{2T} \lambda_2(t) dt + \dots + \int_{(N-1)T}^{NT} \lambda_N(t) dt \quad (5.6)$$

$$= \sum_{k=1}^N \int_0^T \lambda_k(t) dt \quad (5.7)$$

We also know from the original block replacement policy that the number of replacement $(N_1(t))$ in $(0, t)$ here is exactly $\frac{t}{NT}$. Therefore using the renewal reward process, the expected cost per unit time $(C(T))$ is simply the expected cost in a replacement cycle divided by the length of the replacement cycle. As before the cost of: a preventive maintenance, replacement and minimal repair are c_1, c_2 , and c_3 respectively. In a replacement cycle we only have one replacement and $(N-1)$ preventive maintenances. Thus we get

$$C(T) = \frac{c_1(N-1) + c_3 \sum_{k=1}^N \int_0^T \lambda_k(t) dt + c_2}{NT} \quad (5.8)$$

Or in terms of the integrated failure rate, $\Lambda_k(T) = \int_0^T \lambda_k(t) dt$ equation (5.8)

becomes

$$C(T) = \frac{c_1(N-1) + c_3 \sum_{k=1}^N \Lambda_k(T) + c_2}{NT} \quad (5.9)$$

The optimum value of $T = T_0$ and $N = N_0$ that minimise (5.8) or (5.9) is easily derived by firstly setting the derivative of $C(T)$ with respect to T equal to zero. Using the quotient rule and the fundamental theorem of calculus we get

$$c_3 \left[T \sum_{k=1}^N \lambda_k(T) - \sum_{k=1}^N \Lambda_k(T) \right] = c_1(N-1) + c_2 \quad (5.10)$$

The optimum T_0 should satisfy

$$T_0 \sum_{k=1}^N \lambda_k(T_0) - \sum_{k=1}^N \Lambda_k(T_0) = \frac{c_1(N-1) + c_2}{c_3} \quad (5.11)$$

Nakagawa [36] proved that T_0 is unique. And he also showed that a unique N_0 exists by assuming that the failure rate increases after each preventive maintenance i.e. $\lambda_{k+1}(t) > \lambda_k(t)$ for $k = 1, 2, \dots, N$. He also gave the equations that should be satisfied by N_0 for the expected cost per unit time to be minimal. For more details, see [38].

5.3 Policy 10: The unit undergoes $(k-1)$ th imperfect repairs. After the $(k-1)$ th imperfect repairs, perfect preventive maintenance is carried out at times $kT (k = 1, 2, \dots)$, and two types of failures can occur

at time t : Type I failure which is fixed by a minimal repair having probability of occurrence $p(t)$; and Type II failure which is fixed by a replacement having probability of occurrence $1 - p(t)$. This policy is due to Wang and Phang[46] and they called it the “Periodic pm model 2”.

This policy is very similar to policy 7 except that after the $(k-1)th$ imperfect repairs, the original block replacement policy starts operating with two possible types of failures during a renewal cycle L (which is the time between two successive replacements at times kT). The following quantities used in policy 7 have the same meaning in our present policy: $\mu_1, \mu_2, a, b, X_k, Z_k, F_{X_k}(t), \bar{F}_{X_k}(t), \lambda_k(t), \{N_m(t), t \geq 0\}, Y, \bar{G}(t), c_1, c_2, c_3, c_4$, and c_5 .

Therefore

$$\begin{aligned} E(N_m(t) | Y < t) &= \frac{1}{G(t)} \int_0^t \int_0^y p(u) \lambda_k(u) du dG(y) \\ &= \frac{a^{1-k}}{G(t)} \int_0^t \int_0^y p(u) \lambda_1(a^{1-k}u) du dG(y) \end{aligned} \quad (5.12)$$

and

$$\begin{aligned} E(N_m(t) | Y \geq t) &= \int_0^t p(u) \lambda_k(u) du \\ &= a^{1-k} \int_0^t p(u) \lambda_1(a^{1-k}u) du \end{aligned} \quad (5.13)$$

When t is within a renewal cycle ($t \leq T$), the unconditional mean value of $N_m(t)$ is

$$E(N_m(t)) = E(N_m(t) | Y < t)G(t) + E(N_m(t) | Y \geq t)\bar{G}(t) + \int_0^t N_m(t-u)dG(u) \quad (5.14)$$

Using equation (4.25)-(4.27) we get

$$E(N_m(t)) = a^{1-k} \int_0^t \bar{G}(u) \lambda_1(a^{1-k}u) du - G(t) + \int_0^t N_m(t-u)dG(u) \quad (5.15)$$

The expected number of replacements due to a Type II failure ($E(N_r(t))$) is obtained from Yeh Lam [48], [49].

We assume that minimal repairs takes negligible amount of time but replacement at planned times kT is a random variable T_p with $E(T_p) = t_p$.

The expected length of a renewal cycle is

$$E(L) = \frac{\mu_1(1-a^{k-1})}{1-a} + \frac{\mu_2(1-b^{k-1})}{1-b} + T + t_p \quad (5.16)$$

Therefore the expected cost per unit time is

$$C(T) = \frac{\left(c_4(k-1) + c_5\left(\frac{k(k-1)}{2}\right) + c_2E(N_r(T)) + c_1 \right.}{\frac{\mu_1(1-a^{k-1})}{1-a} + \frac{\mu_2(1-b^{k-1})}{1-b} + T + t_p} \left. c_3 \left\{ a^{1-k} \int_0^T \bar{G}(u) \lambda_1(a^{1-k}u) du - G(T) + \int_0^T N_m(t-u)dG(u) \right\} \right) \quad (5.17)$$

To obtain the optimal value of $T = T_0$ and $k = k_0$ (for the number of imperfect repairs), Wang and Pham [46] suggests the maximisation of the unit's availability by numerical methods which will give the value T_0 and k_0 that will minimise (5.17).

6. Various Repair cost limit policies.

Repair cost limit policies are policies base on either the cost of a repair or the cost rate. Once a unit failed, the cost of repair is determined and if this figure exceeds a predetermined level, known as the repair cost limit, the unit is replaced else it repaired. As mentioned in the introductory section, there are policies that do not depend solely on the cost of a single repair as done by Drinkwater and Hastings [19], [24], and in policy 4. Instead, Beichelt [5], introduced policies where the decision to replace a unit depends on the whole history of the repair process. And he used the repair cost rate to decide on a replacement. All policies below dealt with an infinite time span. They involve either the repair cost rate or simply the repair cost as decision variable.

6.1 Policy 11: A unit is replaced when the repair cost rate is at least a predetermined level C_0 . The repair cost rate function is assumed to be linear

As we have seen in the previous policies, the expected cost per unit time in $(0, t)$ is given by the total expected cost in $(0, t)$ divided by t , i.e. $\frac{C(t)}{t}$.

And this can be obtained by using the renewal reward process, $\lim_{t \rightarrow \infty} \frac{C(t)}{t}$.

For our policy, let $C(t)$ be total cost of repair in $(0, t)$, and thus giving a repair cost rate $R(t) = \frac{C(t)}{t}$ and that for $0 < t_1 < t_2 < t$, $R(t_1) < R(t_2)$; meaning that over time, as the unit ages, the cost of repair increases, hence the repair cost rate. The time between the $(i-1)th$ and the ith replacement

is denoted by X_i , where the X_i 's are i.i.d random variables. Obviously we will have replacement cycles of length X_i . And a replacement takes place when $R(X_i) = C_0$.

At the n th replacement, when the time is $S_n = \sum_{i=1}^n X_i$, the total cost per unit time is simply: the total cost in $(0, S_n)$ divided by S_n which is $\frac{C(S_n)}{S_n}$. If the

cost of a replacement is c_1 , then the total cost in $(0, S_n)$ is just the cost of the n replacements plus the cost of operating the unit during each replacement cycle of length X_i . The total cost is thus

$$nc_1 + [C(X_1) + \dots + C(X_n)]$$

where

$$\begin{aligned} C(X_i) &= \text{Cost of operating the unit during } (0, X_i) \\ &= X_i \times \text{cost rate during } (0, X_i) \\ &= X_i \times C_0 \end{aligned}$$

assuming that the cost rate is constant in the interval $(0, X_i)$. Therefore the cost rate in $(0, S_n)$ is

$$R(S_n) = \frac{C(S_n)}{S_n} = \frac{nc_1 + \sum_{i=1}^n X_i C_0}{S_n} \quad (6.1)$$

Using the renewal reward process, the long run expected cost per unit time is given by

$$R(S_\infty) = \lim_{n \rightarrow \infty} \frac{C(S_n)}{S_n}$$

$$= \frac{\text{expected cost in a replacement cycle}}{\text{expected length of the replacement cycle}}$$

Let the L = length of a replacement cycle

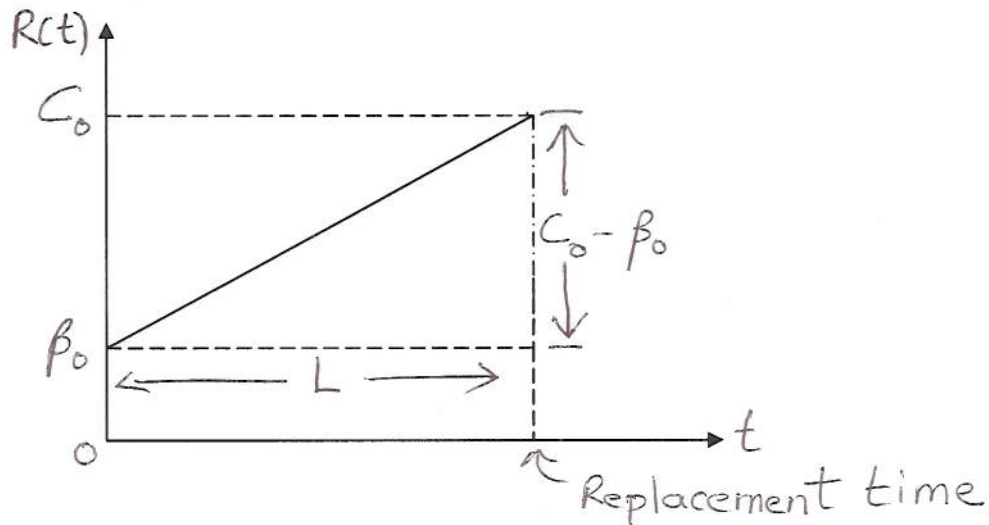
Then

$$\begin{aligned} R(S_{\infty}) &= \frac{c_1 + LC_0}{E(L)} \\ &= \frac{c_1}{E(L)} + C_0 \end{aligned} \quad (6.2)$$

Beichelt [5] assumes that the repair cost rate is a linear function of time with linear equation

$$R(t) = \beta_0 + \beta_1 t \quad (6.3)$$

where $0 \leq \beta_0 \leq C_0$ (which is the intercept) and β_1 , being the positive slope, is the rate of change of $R(t)$ and is random.



Note from the graph above that $R(L) = C_0$. The slope of the line is

$$\beta_1 = \frac{C_0 - \beta_0}{L}$$

Hence $L = \frac{C_0 - \beta_0}{\beta_1}$. And $E(L) = \frac{C_0 - \beta_0}{E(\beta_1)}$. Substituting $E(L)$ into (6.2),

we get

$$\begin{aligned} R(S_\infty) &= \frac{c_1}{\frac{C_0 - \beta_0}{E(\beta_1)}} + C_0 \\ &= \frac{c_1 E(\beta_1) + C_0 (C_0 - \beta_0)}{C_0 - \beta_0} \end{aligned} \quad (6.4)$$

Therefore the value of C_0 that minimize the expected cost per unit time is obtained by setting derivative of (6.4) with respect to C_0 equal to zero, giving

$$C_0 = \sqrt{c_1 E(\beta_1)} + \beta_0 \quad (6.5)$$

When equality holds in (6.4), a replacement should take place. Thus the minimum expected cost per unit time is

$$\begin{aligned} R_{\min}(S_\infty) &= \frac{c_1 E(\beta_1) + [\sqrt{c_1 E(\beta_1)} + \beta_0][\sqrt{c_1 E(\beta_1)}]}{\sqrt{c_1 E(\beta_1)}} \\ &= 2\sqrt{c_1 E(\beta_1)} + \beta_0 \end{aligned} \quad (6.6)$$

In this policy one key assumption was that the repair cost rate function is linear. But this doesn't necessarily have to be the case as shown in policy 12 below, which was introduced by Beichelt [6].

The same notation used here applied to policy 12.

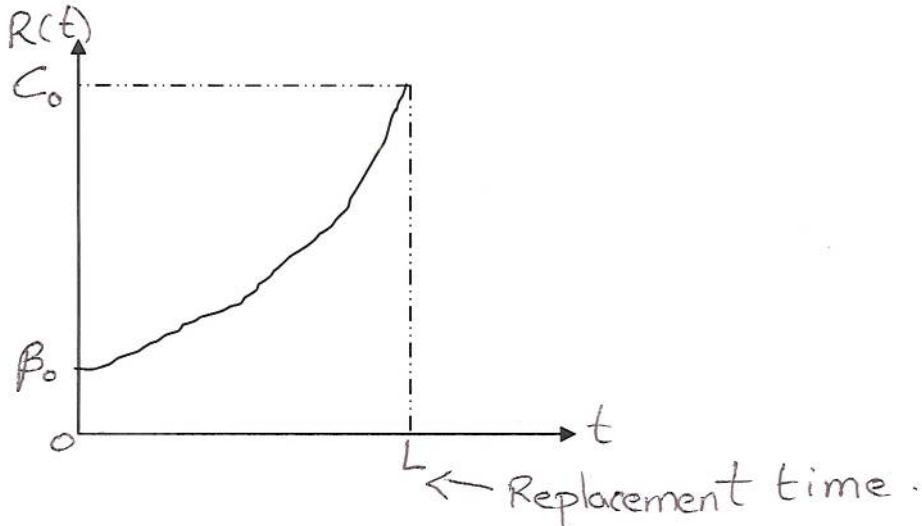
6.2 Policy 12: A unit is replaced when the repair cost rate is at least a predetermined level C_0 with a general repair cost rate function. At failure the unit is replaced.

We saw in policy 11 that $R(S_\infty) = \frac{c_1}{E(L)} + C_0$ and that $R(t) = \beta_0 + \beta_1 t$. But

now we assume that

$$R(t) = \beta_0 + \beta_1 f(t) \quad (6.7)$$

with $E(\beta_1)$ being finite. And also β_0 and β_1 doesn't represent the intercept or slope of a line respectively. And for $0 < t_1 < t_2 < t$, $f(t_1) < f(t_2)$, and $f(0) = 0$. The graph below depicts what the repair cost rate function might look like.



From the graph, $R(L) = C_0$ and

$$L = f^{-1}\left(\frac{C_0 - \beta_0}{\beta_1}\right) \quad (6.8)$$

Using (6.2),

$$R(S_\infty) = \frac{c_1}{f^{-1}\left(\frac{C_0 - \beta_0}{E(\beta_1)}\right)} + C_0 \quad (6.9)$$

If the function $f(t) = t$, then we get policy 11. Beichelt [6] proved that the decision to replace the unit base on the repair cost rate is more economical than using its age.

6.3 Policy 13: A unit is replaced if the repair cost at failure time t is greater than $c_0(t)$. Else the unit is minimally repaired.

Under this policy, two types of failure can occur. Type I (Type II) failure which is removed by minimal repair (perfect repair) having probability of occurrence $1 - p(t)$ ($p(t)$) respectively. The length L , between two successive replacements (due to a Type II failure) is the replacement cycle. In policy 6, Beichelt ([4], [9]) showed that the survival function of L was

$$\bar{G}(t) = e^{-\int_0^t p(u)\lambda(u)du}, \quad t \geq 0$$

The repair cost C_t at time t is a random variable with distribution function given by

$$R_t(x) = P(C_t \leq x)$$

What needs to be done now is to associate the events Type I and Type II failures with the repair cost limit $c_0(t)$. A Type II failure means replacement of the unit, which is the same as the repair cost being greater than $c_0(t)$.

Thus

$$\begin{aligned} P(\text{Type II failure}) &= p(t) \\ &= P(C_t > c_0(t)) \\ &= 1 - R_t(c_0(t)) = \bar{R}_t(c_0(t)) \end{aligned} \quad (6.10)$$

Similarly,

$$P(\text{Type I failure}) = 1 - p(t) = R_t(c_0(t)) \quad (6.11)$$

Therefore the repair cost limit $c_0(t)$ can be embedded into the survival function of L giving

$$\bar{G}(t) = e^{-\int_0^t \bar{R}_u(c_0(u))\lambda(u)du}, \quad t \geq 0 \quad (6.12)$$

The expected length of a replacement cycle is obtained by substituting $T = \infty$ in equation (4.17) of policy 6. Thus

$$\begin{aligned} E(L) &= \int_0^\infty \bar{G}(t) dt \\ &= \int_0^\infty e^{-\int_0^t R_u(c_0(u))\lambda(u)du} dt \end{aligned} \quad (6.13)$$

Hence the expected cost per unit time is either obtained by either using the renewal reward process, since we have $E(L)$ or simply substituting $T = \infty$ and $n = \infty$ into equation (4.21) and get

$$\begin{aligned} C(\infty) &= \frac{c_3 \int_0^\infty \bar{G}(t)(1 - p(t))\lambda(t)dt + c_1}{\int_0^\infty \bar{G}(t)dt} \\ &= \frac{c_3 \int_0^\infty e^{-\int_0^t R_u(c_0(u))\lambda(u)du} R_t(c_0(t))\lambda(t)dt + c_1}{\int_0^\infty e^{-\int_0^t R_u(c_0(u))\lambda(u)du} dt} \end{aligned} \quad (6.14)$$

The best approach to obtain the optimal value of $c_0(t)$ that will minimise (6.14), is to consider a unit with a specific lifetime distribution and the same

for the repair cost variable. Typical lifetime distribution is the Weibull and for the repair cost variable a normal or negative exponential distribution will do.

6.4. Policy 14: An age dependent repair cost limit policy

For this policy, we used the same notation as in policy 13. At failure time t , the cost of repair is estimated and if $C_t \leq c_0(t)$, the unit is minimally repaired else it is replaced. We assume that the optimal repair cost is less than or equal to a replacement cost and that the repair cost is stochastically increasing.

If the unit is repaired at time t then the conditional “survival” probability of the replacement cycle is

$$\begin{aligned} & P(L > t + u \mid L > t) \\ &= \frac{P(L > t + u)}{P(L > t)} \\ &= \frac{\bar{G}(t + u)}{\bar{G}(t)} \end{aligned}$$

Thus the mean remaining length of the replacement cycle is

$$\begin{aligned} E(L - t \mid L > t) &= \int_0^{\infty} \frac{\bar{G}(t + u)}{\bar{G}(t)} du \\ &= \int_t^{\infty} \frac{\bar{G}(u)}{\bar{G}(t)} du \end{aligned} \tag{6.15}$$

The probability of a failure occurring in the remaining length of the replacement cycle is

$$\int_t^{\infty} \frac{\bar{G}(u)}{\bar{G}(t)} \lambda(u) du \tag{6.16}$$

The expected repair cost in the remaining length of the replacement cycle due to a failure is

$$\begin{aligned} & \int_0^{c_0(u)} P(C_u \leq s) ds \\ &= \int_0^{c_0(u)} R_u(s) ds \end{aligned} \quad (6.17)$$

The expected remaining cost is then

$$C_t + \int_t^{\infty} \frac{\bar{G}(u)}{\bar{G}(t)} \lambda(u) du \int_0^{c_0(u)} R_u(s) ds$$

From the renewal reward process, the expected cost per unit time is then given by

$$\frac{C_t + \int_t^{\infty} \frac{\bar{G}(u)}{\bar{G}(t)} \lambda(u) du \int_0^{c_0(u)} R_u(s) ds}{\int_t^{\infty} \frac{\bar{G}(u)}{\bar{G}(t)} du} \quad (6.18)$$

In this case a repair takes place if

$$\frac{C_t + \int_t^{\infty} \frac{\bar{G}(u)}{\bar{G}(t)} \lambda(u) du \int_0^{c_0(u)} R_u(s) ds}{\int_t^{\infty} \frac{\bar{G}(u)}{\bar{G}(t)} du} < c^* \quad (6.19)$$

where c^* is the minimum expected cost per unit time. Hence the equilibrium optimal repair cost limits equation at time t (when $C_t = c_0(t)$) should satisfy

$$c_0(t) = c^* \int_t^{\infty} \frac{\bar{G}(u)}{\bar{G}(t)} \lambda(u) du - \int_t^{\infty} \frac{\bar{G}(u)}{\bar{G}(t)} du \int_0^{c_0(u)} R_u(s) ds \quad (6.20)$$

Substituting $t = 0$ in (6.20) will give the value of c^* .

Differentiation of (6.20) with respect to t leads to the optimal solution, and it should satisfy

$$\frac{d}{dt}c_0(t) = \lambda(t) \int_0^{c_0(t)} \bar{R}_t(s) ds - c^* \quad (6.21)$$

6.5. Policy 15: The unit undergoes $(k-1)th$ imperfect repairs. After that, two types of failures are possible at time t . Type I failure occurs with probability $1-p(t)$ and is removed by a minimal repair, and a Type II failure occurs with probability $p(t)$ and is fixed by a replacement. After the $(k-1)th$ imperfect repairs, the unit is replaced at the first occurrence of Type II failure, or when the repair cost $C_t > c_0(t)$, whichever occur first.

This policy is a combination of Policies 7 and 13 except that the unit is not replaced based on its age but rather on cost or complete system breakdown.

The meaning of imperfect repair in this policy is exactly the same as in Policy 7. The same notations are used here as in the above-mentioned policies. Recall that

$$\bar{G}(t) = e^{-\int_0^t \bar{R}_u(c_0(u)) \lambda(u) du}, \quad t \geq 0$$

Hence the expected length of the replacement cycle in terms of the cost function is

$$\begin{aligned} E(L) &= \mu_1 \frac{(1-a^{k-1})}{1-a} + \mu_2 \frac{(1-b^{k-1})}{1-b} + \int_0^\infty \bar{G}(t) dt + r \\ &= \mu_1 \frac{(1-a^{k-1})}{1-a} + \mu_2 \frac{(1-b^{k-1})}{1-b} + \int_0^\infty e^{-\int_0^t \bar{R}_u(c_0(u)) \lambda(u) du} dt + r \end{aligned} \quad (6.22)$$

And the expected cost in a replacement cycle is

$$c_4(k-1) + c_5\left(\frac{k(k-1)}{2}\right) + c_2G(t) + c_3\left\{a^{1-k} \int_0^t e^{-\int_0^u \bar{R}_s(c_0(s))\lambda(s)ds} \lambda_1(a^{1-k}u)du - (1 - e^{-\int_0^t \bar{R}_u(c_0(u))\lambda(u)du})\right\} \quad (6.23)$$

Therefore the expected cost per unit time is

$$\frac{\mu_1 \frac{(1-a^{k-1})}{1-a} + \mu_2 \frac{(1-b^{k-1})}{1-b} + \int_0^\infty e^{-\int_0^t \bar{R}_u(c_0(u))\lambda(u)du} dt + r}{\left(c_4(k-1) + c_5\left(\frac{k(k-1)}{2}\right) + c_2G(t) + c_3\left\{a^{1-k} \int_0^t e^{-\int_0^u \bar{R}_s(c_0(s))\lambda(s)ds} \lambda_1(a^{1-k}u)du - (1 - e^{-\int_0^t \bar{R}_u(c_0(u))\lambda(u)du})\right\} \right)} \quad (6.24)$$

Therefore the optimal value of $c_0(t)$ that will minimize (6.24) can be obtained by appropriate computer software.

7. Maintenance Policies under General repairs, using the virtual age of the system.

The Virtual Age concept was first introduced by Kijima et al [32]. Guo et al [23] give the following description of what it stands for: “...*Three cars of the same make which started operation at the same time, say, on January 1, 1998, by 31st December, end of year 2000, two car’s mileage meter both indicated 100 000 km, and the third car only recorded 7500 km. It is obvious three cars have the same chronological age, 2 years. But the third car with 7500 km mileage record will be more reliable than the first two cars with 100 000 km mileage record. Even for cars with same records, different driving behavior and road conditions may cause the same mileage cars to have different performance in the future. In order to recognize such an intrinsic character of a system, just like the bio clock of human beings, researchers named it a **virtual age**...*” In Kijima Model I, the maintenance measure fixes the damages that the system incurred during one operating cycle. In his second model (named Model II), the maintenance measure fixes the damages incurred prior to the previous failure and during the operating period leading to the current one. Before describing the models, the explanation below follows closely to the one given by Kijima [31].

On failure a general repair Θ is carried out instantaneously and the amount of time between the $(n-1)$ th failure and the n th failure is X_n . X_n depends on the virtual age V_{n-1} after the $(n-1)$ th repair. Let $\Theta = (\theta_1, \theta_2, \dots)$ be a sequence of repairs. The degree of repair upon the

n th failure is denoted by θ_n , where θ_n is a random variable and $0 \leq \theta_n \leq 1$.

Using equation (2.11) and (2.12) the distribution of X_n is given by

$$F_y(x) = P(X_n \leq x | V_{n-1} = y) = \frac{F(x+y) - F(y)}{1 - F(y)} \quad (7.1)$$

$$\text{with survival function denoted by } \bar{F}_y(x) = \frac{\bar{F}(y+x)}{\bar{F}(y)}. \quad (7.2)$$

Using the results from section 2.2, equations (7.1) and (7.2) can also be written as

$$F_y(x) = 1 - e^{-(\Lambda(y+x) - \Lambda(y))}, \quad y \geq 0, x \geq 0. \quad (7.3)$$

$$\bar{F}_y(x) = e^{-(\Lambda(y+x) - \Lambda(y))}, \quad y \geq 0, x \geq 0. \quad (7.4)$$

Hence $F(x)$ is said to be IFR (Increasing Failure Rate) if $\bar{F}_y(x)$ is decreasing in $y \geq 0$ for any $x \geq 0$, or if $F_y(x)$ is increasing in $y \geq 0$ for any $x \geq 0$. This can easily be seen from the fact that if $\Lambda(y+x) - \Lambda(y)$ is increasing then $F_y(x)$ is increasing too. Hence $\bar{F}_y(x)$ is decreasing. Thus $F(x)$ is IFR iff $\Lambda(t)$ is convex. $\Lambda(t)$ can only be convex if $\lambda(t)$ is increasing.

Let Z_y ($y \geq 0$) be the life time of a system whose virtual age is y . It can be shown that the survival function of Z_y is $\bar{F}_y(x)$ and that $X_n \equiv Z_{V_{n-1}}$. In other words the probability that the system fails after the n th repair is equal to the probability that the system fails given that the virtual age is V_{n-1} . Also of importance is the time to the n th failure, $S_n = \sum_{k=1}^n X_k$ ($n \geq 0$). So far we have two processes: $\{X_n; n \geq 1\}$ which is the failure process and

$\{S_n; n \geq 0\}$ which represents the real age process. The other relevant one is $N(t) = \max\{n \geq 0; S_n \leq t\}$, which is the repair counting process.

Section 7.1 gives detail descriptions of Model I and as for Model II, see Kijima [31] for the results of the monotonicity properties. They are derived in similar manner as the first Model.

7.1 Kijima Model I (Kijima [31])

The Virtual age after the n th repair is:

$$V_n = V_{n-1} + \theta_n X_n. \quad (7.5)$$

Upon failure, the virtual age changes which determines the updated failure rate. Before the n th repair, the virtual age is

$$V_n^- = V_{n-1} + X_n. \quad (7.6)$$

The n th repair can reset the failure rate as far back as the virtual age at the start of the last operating cycle and it decreases its accumulated age. Therefore the “degree of repair” applied at the n th repair only fixes the damages that the system incurred during X_n . If $\theta_n = 0 \forall n \geq 1$, we get $V_n = 0$ which is a replacement and if $\theta_n = 1 \forall n \geq 1$, we get $V_n = V_{n-1} + X_n$, which is a minimal repair. If $\theta_n < 1$ then the system is rejuvenated by the repair since $V_n < S_n$. Let the degree of repair at the $(n+1)$ th failure be $\theta_{n+1} = \theta$ ($0 \leq \theta \leq 1$). Then using (7.1) and (7.5), we can obtain the survival function of the virtual age process $\{V_n; n \geq 0\}$ for the system after the $(n+1)$ th repair as follows:

$$\bar{V}_{n+1}(x) = P(V_{n+1} > x)$$

$$\begin{aligned}
&= P(V_n + \theta X_{n+1} > x) \\
&= P(V_n + \theta X_{n+1} > x | V_n > x)P(V_n > x) + \\
&\quad P(V_n + \theta X_{n+1} > x | V_n \leq x)P(V_n \leq x) \\
&= P(V_n > x) + P(V_n + \theta X_{n+1} > x \cap V_n \leq x) \\
&= P(V_n > x) + \int_0^x P(V_n + \theta X_{n+1} > x \cap V_n \leq x | V_n = y) dV_n(y) \\
&= \bar{V}_n(x) + \int_0^x P(y + \theta X_{n+1} > x) dV_n(y) \\
&= \bar{V}_n(x) + \int_0^x P(X_{n+1} > \frac{x-y}{\theta}) dV_n(y) \tag{7.7}
\end{aligned}$$

Hence

$$\bar{V}_{n+1}(x) = \bar{V}_n(x) + \int_0^x \frac{\bar{F}(y + \frac{x-y}{\theta})}{\bar{F}(y)} dV_n(y), \quad x \geq 0. \tag{7.8}$$

For all $n \geq 1$, the density function of $V_n(x)$ exist and let $v_n(x) = \frac{d}{dx} V_n(x)$.

$$V_{n+1}(x) = \int_0^x \frac{F(y + \frac{x-y}{\theta}) - F(y)}{\bar{F}(y)} dV_n(y) \tag{7.9}$$

Therefore differentiating (7.9) gives the probability density function of the virtual age process after the $(n+1)th$ repair

$$\begin{aligned}
v_{n+1}(x) &= \frac{d}{dx} V_{n+1}(x) \\
&= \frac{d}{dx} \int_0^x \frac{F(y + \frac{x-y}{\theta}) - F(y)}{\bar{F}(y)} v_n(y) dy
\end{aligned}$$

$$\begin{aligned}
&= \int_0^x \frac{\frac{d}{dx} F(y + \frac{x-y}{\theta})}{\bar{F}(y)} v_n(y) dy \\
&= \frac{1}{\theta} \int_0^x \frac{f(y + \frac{x-y}{\theta})}{\bar{F}(y)} v_n(y) dy, \quad n \geq 1. \quad (7.10)
\end{aligned}$$

At the $(n+1)th$ failure the degree of repair applied is θ given that the virtual age of the system after the nth repair is y . Hence we can define the conditional survival probability of the system after the $(n+1)th$ repair in the interval $(0, \frac{x-y}{\theta})$ by

$$g_x^\theta(y) = \begin{cases} \bar{F}_y(\frac{x-y}{\theta}), & y \leq x \\ 1 & , \quad y > x \\ 0 & , \quad \theta = 0, y \leq x \end{cases} \quad (7.11)$$

Using (7.7) and (7.11), we can see that for $y > x$

$$P(X_{n+1} > \frac{x-y}{\theta} \mid y > x) = 1$$

Thus $\bar{V}_{n+1}(x)$ can be written as

$$\begin{aligned}
\bar{V}_{n+1}(x) &= \int_0^\infty P(X_{n+1} > \frac{x-y}{\theta}) dV_n(y) \\
&= \int_0^\infty g_x^\theta(y) dV_n(y), \quad x \geq 0. \quad (7.12)
\end{aligned}$$

Equation (7.12) represents the probability that the $(n+1)th$ operating period is greater than $\frac{x-y}{\theta}$, with the degree of the $(n+1)th$ repair being θ , given that the virtual age of the system after the nth repair is in the interval $(y, y+dy)$. Or it's simply the survival probability of the system after the nth repair in the interval $(0, \frac{x-y}{\theta})$.

For any repair Θ , the virtual age process $\{V_n^\Theta, n \geq 0\}$ is a Markov process and using (7.9), (7.11) and (7.12) the transition probability is given by

$$\begin{aligned} P(V_{n+1}^\Theta \leq x | V_n^\Theta = y) &= \int_0^x (1 - g_x^\theta(y)) dV_n(y) \\ &= \int_0^1 (1 - g_x^\theta(y)) d\Theta_{n+1}(\theta). \end{aligned} \quad (7.13)$$

Equation (7.13) is the conditional probability that the system fails in the interval $(0, \frac{x-y}{\theta})$ after the nth repair, with the degree of the $(n+1)th$ repair $\Theta_{n+1} \in (0, 1)$.

Define the expected value of the virtual age of the system after the $(n+1)th$ repair in the interval (x, ∞) given that the virtual age after the nth repair is y as

$$\begin{aligned} G_x^\theta(y) &= \int_x^\infty P(V_{n+1} > u) du \\ &= \int_x^\infty P(X_{n+1} > \frac{u-y}{\theta}) du \\ &= \int_x^\infty \bar{F}_y(\frac{u-y}{\theta}) du \end{aligned}$$

$$= \int_x^\infty g_u^\theta(y) du \quad (7.14)$$

$G_x^\theta(y)$ is also the expected value of X_{n+1} in the interval $(\frac{u-y}{\theta}, \infty)$ given that the system doesn't fail in the interval $(0, \frac{u-y}{\theta})$.

Then from (7.12),

$$\begin{aligned} \int_x^\infty \bar{V}_{n+1}(u) du &= \int_x^\infty \int_0^\infty g_x^\theta(y) dV_n(y) du \\ &= \int_0^\infty G_x^\theta(y) dV_n(y), \quad x \geq 0 \end{aligned} \quad (7.15)$$

$G_x^\theta(y)$ can be simplified by using (2.13), (7.11) and the change of variable technique as follows

$$\begin{aligned} G_x^\theta(y) &= \int_x^\infty g_u^\theta(y) du \quad y \leq x \\ &= \int_x^\infty \bar{F}_y\left(\frac{u-y}{\theta}\right) du \quad y \leq x \end{aligned} \quad (7.16)$$

$$= \theta \int_{(x-y)/\theta}^\infty \bar{F}_y(u) du \quad y \leq x \quad (7.17)$$

For $y > x$

$$\begin{aligned} G_x^\theta(y) &= \int_x^y \bar{F}_y\left(\frac{u-y}{\theta}\right) du + \int_y^\infty \bar{F}_y\left(\frac{u-y}{\theta}\right) du \\ &= \int_x^y du + \theta \int_0^\infty \bar{F}_y(x) dx \\ &= y - x + \theta m_x(y) \quad y > x \end{aligned} \quad (7.18)$$

Thus

$$G_x^a(y) = \begin{cases} \theta \int_{(x-y)/\theta}^{\infty} \bar{F}_y(u) du, & y \leq x \\ y - x + \theta m_x(y), & y > x \end{cases} \quad (7.19)$$

Therefore for $y > x$, the expected value of the Virtual age of the system after the $(n+1)th$ repair is simply $y - x + \theta(E(X_{n+1}) \text{ in the } (0, \infty))$ according to (7.5).

The process $\{S_n; n \geq 0\}$ is a nonhomogeneous Poisson process but doesn't possess the markovian property, which makes the computation of the $E(S_n^\Theta)$ for the cost function difficult. Hence bounds are obtained for the expected values of S_n and $N(t) = \max\{n \geq 0; S_n \leq t\}$, which is the repair counting process. Therefore monotonicity properties of the different stochastic processes needed to compute the cost function are obtained in the following sections as described by Kijima [31].

7.1.1. Monotonicity Properties of $\{V_n^\Theta\}$ with respect to the repair Θ .

Lemma 1

- (i) $g_x^\theta(y)$ is increasing in both $y \geq 0$ and $0 \leq \theta \leq 1$
- (ii) If $m_x(y)$ is convex in y , then $G_x^\theta(y)$ is increasing and convex in $y \geq 0$ and in $0 \leq \theta \leq 1$. ($m_x(y)$ is defined as in equation 2.13 except here we conditioned on the virtual age y).

Proof.

For part (i): We just need to show that the first derivative of $g_x^\theta(y)$ w.r.t $y \geq 0$

$$\begin{aligned}
\frac{d}{dy}(g_x^\theta(y)) &= \frac{d}{dy} \left\{ \frac{1 - F(y + \frac{x-y}{\theta})}{\bar{F}(y)} \right\} \\
&= \frac{\bar{F}(y) \{ -(1 - 1/\theta) f(y + \frac{x-y}{\theta}) \} + \{ \bar{F}(y + \frac{x-y}{\theta}) f(y) \}}{(\bar{F}(y))^2} \\
&= \lambda(y) \bar{F}_y(\frac{x-y}{\theta}) + \frac{\{ -(\frac{\theta-1}{\theta}) f(y + \frac{x-y}{\theta}) \}}{\bar{F}(y)}
\end{aligned} \tag{7.20}$$

where $\lambda(y) = \frac{f(y)}{\bar{F}(y)}$.

The L.H.S of equation (7.20) is greater than or equal to zero implying that $g_x^\theta(y)$ is increasing in $y \geq 0$.

Similarly, the first derivative of $g_x^\theta(y)$ w.r.t θ is

$$\begin{aligned}
\frac{d}{d\theta}(g_x^\theta(y)) &= \frac{\bar{F}(y) \frac{d}{d\theta} \bar{F}(y + \frac{x-y}{\theta})}{(\bar{F}(y))^2} \\
&= \frac{\bar{F}(y) f(y + \frac{x-y}{\theta}) (\frac{x-y}{\theta^2})}{(\bar{F}(y))^2}
\end{aligned}$$

For $0 \leq \theta \leq 1$, the above equation is greater than or equal to zero.

For the proof of the part (ii), we first check whether $G_x^\theta(y)$ is increasing in $y \geq 0$ and $0 \leq \theta \leq 1$ and then check for convexity.

By definition $G_x^\theta(y) = \int_x^\infty g_u^\theta(y) du$

$$\frac{d}{dy}(G_x^\theta(y)) = \int_x^\infty \frac{d}{dy}(g_u^\theta(y))du \geq 0 \text{ (from part (i))}$$

Hence $G_x^a(y)$ is increasing in $y \geq 0$ and $0 \leq a \leq 1$.

To check for convexity, we need to show separately that the second derivative of $G_x^\theta(y)$ w.r.t y and w.r.t a is greater than or equal to zero respectively.

So for $y \leq x$, from (7.16) and (7.17),

$$\begin{aligned} \frac{d}{dy}(G_x^\theta(y)) &= \int_x^\infty \frac{d}{dy} \left\{ \bar{F}_y \left(\frac{u-y}{\theta} \right) \right\} du \\ &= \int_x^\infty \frac{d}{dy} \left\{ \frac{\bar{F}(y + \frac{u-y}{\theta})}{\bar{F}(y)} \right\} du \\ &= \int_x^\infty \left\{ \frac{\bar{F}(y) \{ -(1-1/\theta) f(y + \frac{u-y}{\theta}) \}}{(\bar{F}(y))^2} + \frac{\{ \bar{F}(y + \frac{u-y}{\theta}) f(y) \}}{(\bar{F}(y))^2} \right\} du \\ &= \int_x^\infty \frac{\{ -(1-1/\theta) f(y + \frac{u-y}{\theta}) \}}{\bar{F}(y)} du + \int_x^\infty \frac{\{ \bar{F}(y + \frac{u-y}{\theta}) \lambda(y) \}}{\bar{F}(y)} du \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dy}(G_x^\theta(y)) &= \frac{-(1-1/\theta)}{\bar{F}(y)} \int_x^\infty f(y + \frac{u-y}{\theta}) du + \lambda(y) \int_x^\infty \frac{\bar{F}(y + \frac{u-y}{\theta})}{\bar{F}(y)} du \\ &= \frac{(1-\theta)}{\theta} \left\{ \frac{\theta \bar{F}(y + \frac{x-y}{\theta})}{\bar{F}(y)} \right\} + \lambda(y) \int_x^\infty \frac{\bar{F}(y + \frac{u-y}{\theta})}{\bar{F}(y)} du \end{aligned}$$

$$\begin{aligned}
&= (1-\theta)\bar{F}_y\left(\frac{x-y}{\theta}\right) + \lambda(y)G_x^\theta(y) \\
&= (1-\theta)g_x^\theta(y) + \lambda(y)G_x^\theta(y) \quad y \leq x \quad (7.21)
\end{aligned}$$

Using similar reasoning, for $y > x$

$$\frac{d}{dy}(G_x^\theta(y)) = 1 - \theta + \theta\lambda(y)m_x(y). \quad (7.22)$$

If we differentiate (7.21) and (7.22) once more, we will see that

$$\frac{d^2}{dy^2}(G_x^\theta(y)) \geq 0 \text{ for } y \leq x \text{ and } y > x.$$

Hence $G_x^\theta(y)$ is convex in y .

To prove the convexity of $G_x^\theta(y)$ w.r.t. θ , we need to show that the second derivative w.r.t $\theta \geq 0$ (for $y \leq x$).

$$\begin{aligned}
\frac{d}{d\theta}(G_x^\theta(y)) &= \int_x^\infty \frac{d}{d\theta} \left\{ \frac{\bar{F}(y + \frac{u-y}{\theta})}{\bar{F}(y)} \right\} du \\
&= \frac{1}{\bar{F}(y)} \int_x^\infty -\frac{(u-y)}{\theta^2} \frac{d}{d\theta} \bar{F}(y + \frac{u-y}{\theta}) du \\
&= \frac{1}{\theta \bar{F}(y)} \int_x^\infty \left(\frac{u-y}{\theta} \right) f(y + \frac{u-y}{\theta}) du \\
&= \frac{1}{\bar{F}(y)} \int_{\frac{x-y}{\theta}}^\infty z f(y+z) dz \quad \left(\text{letting } z = \frac{u-y}{\theta} \right)
\end{aligned}$$

Using integration by parts and the change of variable techniques,

$$\begin{aligned}
\frac{d}{d\theta}(G_x^\theta(y)) &= \frac{1}{\bar{F}(y)} \left(-\left(\frac{x-y}{\theta}\right) \bar{F}\left(y + \frac{x-y}{\theta}\right) - \int_{\frac{x-y}{\theta}}^{\infty} \bar{F}(y+z) dz \right) \\
&= \left(-\left(\frac{x-y}{\theta}\right) \frac{\bar{F}\left(y + \frac{x-y}{\theta}\right)}{\bar{F}(y)} - \int_{\frac{x-y}{\theta}}^{\infty} \frac{\bar{F}(y+u)}{\bar{F}(y)} du \right)
\end{aligned}$$

Using (7.12) and (7.16), we obtain

$$\begin{aligned}
\frac{d}{d\theta}(G_x^\theta(y)) &= \left(-\left(\frac{x-y}{\theta}\right) \bar{F}_y\left(\frac{x-y}{\theta}\right) - \int_{\frac{x-y}{\theta}}^{\infty} \bar{F}_y(u) du \right) \\
&= -\frac{1}{\theta} \left((x-y) g_x^\theta(y) + G_x^\theta(y) \right) \quad (y \leq x). \tag{7.23}
\end{aligned}$$

$$\frac{d^2}{d\theta^2}(G_x^\theta(y)) = \frac{x-y}{\theta} \frac{d}{d\theta}(g_x^\theta(y)) \geq 0$$

For $y > x$, from (7.19)

$$\frac{d}{d\theta}(G_x^\theta(y)) = m_x(y) \text{ and } \frac{d^2}{d\theta^2} = 0$$

Hence $G_x^\theta(y)$ is convex in θ which completes the proof.

Lemma 2

If $F \in \text{DMRL} \Rightarrow F \in \text{IFR}$.

Proof.

Note that $F \in \text{DMRL}$ implies that $m_x(y)$ is decreasing in y or

$$\begin{aligned}
& \frac{d}{dy}(m_x(y)) \leq 0. \\
\text{i.e. } & \int_0^\infty \frac{d}{dy} \left\{ \frac{\bar{F}(y+x)}{\bar{F}(y)} \right\} dx \leq 0 \\
\Rightarrow & \int_0^\infty \frac{\bar{F}(y) \frac{d}{dy} \bar{F}(y+x) - \bar{F}(y+x) \frac{d}{dy} \bar{F}(y)}{(\bar{F}(y))^2} \leq 0 \\
\Rightarrow & \int_0^\infty \frac{-\bar{F}(y) f(y+x) + \bar{F}(y+x) f(y)}{(\bar{F}(y))^2} \leq 0 \\
\Rightarrow & -\int_0^\infty \frac{f(y+x)}{\bar{F}(y)} dx + \lambda(y) \int_0^\infty \frac{\bar{F}(y+x)}{\bar{F}(y)} dx \leq 0 \\
\Rightarrow & \lambda(y) m_x(y) - 1 \leq 0 \tag{7.24} \\
\Rightarrow & \lambda(y) \leq \frac{1}{m_x(y)}
\end{aligned}$$

In Lemma 1(ii), for $m_x(y)$ to be convex,

$$\frac{d^2}{dy^2}(m_x(y)) \geq 0$$

Differentiating the L.H.S of the inequality in (7.24) by the product rule and substituting $\frac{d}{dy}(m_x(y)) = \lambda(y)m_x(y) - 1$ in the resulting equation, will give us

$$\frac{d^2}{dy^2}(m_x(y)) = \frac{m_x(y) \frac{d}{dy} \lambda(y) + \lambda(y)[\lambda(y)m_x(y) - 1]}{(m_x(y))^2}$$

which should be greater than or equal zero. Hence $\lambda(y)$ must be increasing in y , which implies that $F \in \text{IFR}$.

The following theorems is from Kijima [31] and is based on two sequences of repairs $A = (A_1, A_2, \dots)$ and $B = (B_1, B_2, \dots)$

Theorem 7.1.

$$(i) \quad A \underset{d}{<} B \Rightarrow V_n^A \underset{d}{<} V_n^B \quad \forall n \geq 1.$$

$$(ii) \quad \text{If } m_x(y) \text{ is convex then } A \underset{c}{<} B \Rightarrow V_n^A \underset{c}{<} V_n^B \quad \forall n \geq 1.$$

Proof.

Part (i):

$$\text{For } n=1, V_1^A = A_1 X_1 \text{ and } V_1^B = B_1 X_1$$

$$\text{If } A_1 < B_1 \Rightarrow V_1^A < V_1^B$$

$$\text{For } n=2, V_2^A = A_1 X_1 + A_2 X_2 \text{ and } V_2^B = B_1 X_1 + B_2 X_2.$$

$$\text{If } A_i < B_i \text{ for } i=1, 2 \Rightarrow V_2^A < V_2^B$$

Assume that $V_k^A < V_k^B$ for $k = n+1$

$$\text{Then } V_{n+1}^A = V_n^A + A_{n+1}^A X_{n+1}^A \text{ and } V_{n+1}^B = V_n^B + A_{n+1}^B X_{n+1}^B$$

$$\text{But } A_{n+1}^A X_{n+1}^A < A_{n+1}^B X_{n+1}^B \text{ since } A \underset{d}{<} B$$

$$\text{Hence } V_n^A \underset{d}{<} V_n^B \quad \forall n \geq 1$$

For part (ii):

From (7.13),

$$\bar{V}_{n+1}^A(x) = \int_0^1 \int_0^x g_x^a(y) dV_n(y) dA_{n+1}(a) \quad \text{and}$$

$$\bar{V}_{n+1}^B(x) = \int_0^1 \int_0^x g_x^b(y) dV_n(y) dB_{n+1}(b)$$

From Lemma 1, since $G_x^a(y)$ and $G_x^b(y)$ are convex implies that

$g_x^a(y)$ and $g_x^b(y)$ are convex too. Hence for $A <_c B$

$$\bar{V}_{n+1}^A(x) > \bar{V}_{n+1}^B(x) \quad \forall n \geq 1$$

Theorem 7.2.

- (i) If $A <_d B$ and $F \in \text{DMRL}$ then $E(S_n^A) \geq E(S_n^B) \quad \forall n \geq 1$.
- (ii) If theorem 7.1(i) holds and $E(A_n) = E(B_n) = p$ then

$$E(S_n^A) \leq E(S_n^B) \quad \forall n \geq 1.$$

Proof.

Part (i):

$$\begin{aligned} E(X_n) &= \int_0^\infty E(X_n | V_{n-1} = y) dV_{n-1}(y) \\ &= \int_0^\infty \int_0^\infty \bar{F}_y(x) dx dV_{n-1}(y) \\ &= \int_0^\infty m_x(y) dV_{n-1}(y) \end{aligned}$$

If $F \in \text{DMRL}$, then $\frac{d}{dy} m_x(y) \leq 0$ i.e $m_x(y)$ is decreasing in y . And hence

from theorem 1, $V_{n-1}^A <_d V_{n-1}^B \quad \forall n \geq 1$ for $A <_d B$.

$$\text{Thus} \quad V_{n-1}^A(y) \geq V_{n-1}^B(y) . \quad (7.25)$$

$$E(X_n^A) = \int_0^\infty m_x(y) dV_{n-1}^A(y) \text{ and}$$

$$E(X_n^B) = \int_0^\infty m_x(y) dV_{n-1}^B(y).$$

Consequently from (7.25),

$$E(X_n^A) \geq E(X_n^B) \quad \forall n \geq 1$$

Hence

$$E(S_n^A) = E(X_1^A) + \dots + E(X_n^A) \geq E(S_n^B) = E(X_1^B) + \dots + E(X_n^B).$$

For part (ii):

$$\begin{aligned} E(S_n^A) &= E(X_1^A) + \dots + E(X_n^A) \\ \Rightarrow pE(S_n^A) &= p \sum_{k=1}^n E(X_k^A) \\ &= p(E(X_1^A) + \dots + E(X_n^A)) \\ &= E(V_n^A) \end{aligned}$$

Since $A \underset{c}{<} B \Rightarrow V_n^A \underset{c}{<} V_n^B$, then

$$\begin{aligned} E(V_n^A) &\leq E(V_n^B) \\ \Rightarrow pE(S_n^A) &\leq pE(S_n^B) \\ \Rightarrow E(S_n^A) &\leq E(S_n^B) \end{aligned}$$

Theorem 7.3.

Let $A_n = A$ and $B_n = B$ with $A \underset{d}{<} B \quad \forall n \geq 1$. If $F \in \text{IFR}$ then

$$N^A(t) < N^B(t).$$

Proof .

Recall that $N^A(t) = \max\{n \geq 1; S_n^A \leq t\}$.

Therefore the proof is similar to show that $S_n^A > S_n^B \quad \forall n \geq 1$.

For $n=1$, $S_1^A = S_1^B = X_1$

Assume that $S_n^A > S_n^B$. For $A_n = A$, $V_n^A = AS_n^A$, then

$$\begin{aligned}
 S_{n+1}^A(x) &= P(S_{n+1}^A \leq x) \\
 &= P\left(\frac{V_{n+1}^A}{A} \leq x\right) \\
 &= \int_0^x P\left(\frac{V_{n+1}^A}{A} \leq x \mid V_n^A = ay\right) dV_n^A(y) \\
 &= \int_0^x P\left(\frac{V_n^A}{A} + X_{n+1}^A \leq x \mid V_n^A = ay\right) dV_n^A(y)
 \end{aligned}$$

Here $A = a$ and $S_n^A = y$, hence $V_n^A = ay$

So

$$\begin{aligned}
 S_{n+1}^A(x) &= \int_0^x P(X_{n+1}^A \leq x - y \mid V_n^A = ay) dV_n^A(y) \\
 &= \int_0^1 \int_0^x P(X_{n+1}^A \leq x - y \mid V_n^A = ay) dS_n^A(y) dA(a)
 \end{aligned} \tag{7.26}$$

Using (7.1) we can see that

$$P(X_{n+1}^A \leq x - y \mid V_n^A = ay) = \frac{F(x - y + ay) - F(ay)}{\bar{F}(ay)}$$

Therefore

$$\begin{aligned}
 P(X_{n+1}^A > x - y \mid V_n^A = ay) &= \frac{\bar{F}(x - y + ay)}{\bar{F}(ay)} \\
 &= g_{ax}^a(ay)
 \end{aligned}$$

Hence using (7.26)

$$\bar{S}_{n+1}^A(x) = \int_0^1 \int_0^\infty g_{ax}^a(ay) dS_n^A(y) dA(a) \quad (7.27)$$

From Theorem 7.1, $g_{ax}^a(ay)$ is increasing in y for $0 \leq a \leq 1$ but decreasing in a since $F \in \text{IFR}$. As $A <_d B \forall n \geq 1$, implies that $\bar{S}_{n+1}^A(x) \geq \bar{S}_{n+1}^B(x)$.

7.1.2 An approximation for $E(S_n^\Theta)$

In most maintenance policies, it is important to know the expected number of failures in a given time interval in order to minimise cost over long period of time. Hence for a general degree of repair Θ , we need to find $E(S_n^\Theta)$ or $E(N^\Theta(t))$. The minimal repair $\Theta^M = (1, 1, \dots)$ provides a lower bound for $E(S_n^\Theta)$ and the imperfect repair $\Theta^I = (I_1, I_2, \dots)$ gives the corresponding upper bound. For any repair Θ , the survival function of X_n is given by

$$\bar{X}_n^\Theta(x) = \int_0^\infty \bar{F}_y(x) dV_{n-1}^\Theta(y)$$

and we showed that from the proof of theorem 7.2(i) that

$$E(X_n^\Theta) = \int_0^\infty m_x(y) dV_{n-1}^\Theta(y). \quad (7.28)$$

We can either use bounds to approximate the value of $E(S_n^\Theta)$ for various repair Θ . Or alternatively the density function of V_n^Θ can be obtained recursively from (7.11). Then the expected value of X_n^Θ is obtained from (7.28) such that $E(S_n^\Theta) = \sum_{k=1}^n E(X_k^\Theta)$ which is more difficult. Hence Kijima derived a recursion formula for the upper bound as follows:

Define the following indicator function

$$I_{N(t)=n-1} = \begin{cases} 1 & \text{if the } (n-1)\text{th failure occurs in } [0, t] \\ 0 & \text{otherwise} \end{cases}$$

where $N(t)$ is the nonhomogeneous Poisson process with failure rate $\lambda(t)$.

Under a minimal repair, $\Theta^M = (1, 1, \dots)$ and

$$\begin{aligned} \mu_1 &= E(X_1^{\Theta^M}) \\ &= E\left(\int_0^\infty I_{N^{\Theta^M}(t)=0} dt\right) \\ &= \int_0^\infty E(I_{N^{\Theta^M}(t)=0}) dt \\ &= \int_0^\infty P(N^{\Theta^M}(t) = 0) dt \end{aligned}$$

Similarly

$$\mu_2 = \int_0^\infty P(N^{\Theta^M}(t) = 1) dt$$

And

$$\begin{aligned} \mu_n &= E(X_n^{\Theta^M}) \\ &= \int_0^\infty P(N^{\Theta^M}(t) = n-1) dt \quad \forall n \geq 1 \end{aligned} \tag{7.29}$$

Recall from definition (2.1.2) that

$$P(N(t+s) - N(t) = n) = e^{-(\Lambda(t+s) - \Lambda(t))} \frac{(\Lambda(t+s) - \Lambda(t))^n}{n!}, \quad \forall n \geq 0$$

Hence

$$\mu_n = \frac{1}{(n-1)!} \int_0^\infty e^{-(\Lambda(t))} (\Lambda(t))^{n-1} dt \quad \forall n \geq 1 \quad (7.30)$$

Under an imperfect repair, $\Theta^I = (I_1, I_2, \dots)$ and at failure the system can either undergoes a minimal repair with probability p where $I_n = 1$ or is replaced with probability $1 - p$ where $I_n = 0$. Assuming that the system was already in use before failure, and went through $(m-1)$ minimal repairs (say), then the expected real age at the n th repair (depending on whether it receives a minimal repair or not) is defined as

$$\begin{aligned} E(S_n^{\Theta^I}) &= K^{\Theta^I}(m, n) \\ &= \mu_m + (1-p)K^{\Theta^I}(m, n-1) + pK^{\Theta^I}(m+1, n-1) \end{aligned} \quad (7.31)$$

and μ_m is the expected lifetime of the system.

7.2 Kijima Model II (Kijima [31])

The virtual age after the n th repair is

$$V_n = \theta_n(V_{n-1} + X_n) \quad (7.32)$$

The “degree of repair” applied fixes the damages that the system incurred before the $(n-1)$ th repair and during X_n . If $\theta_n = 0 \forall n \geq 1$, we get $V_n = 0$ which is a replacement and if $\theta_n = 1 \forall n \geq 1$, we get $V_n = V_{n-1} + X_n$, which is a minimal repair. Before the n th repair the virtual age is $V_n^- = V_{n-1} + X_n$. As mentioned earlier, detail description of model II can be found in Kijima [38]. For this second model, bounds are also used to estimate $E(S_n^\Theta)$ and he showed that the following formula is used for the upper bound

$$E(S_n^{\Theta^I}) - E(S_{n-1}^{\Theta^I}) = (1-p) \sum_{k=1}^{n-1} p^{k-1} \mu_k + p^{n-1} \mu_n \quad (7.33)$$

When the system fails, a general repair $\Theta = (\theta_1, \theta_2, \dots)$ of Kijima type I or type II is applied. We assumed that $E(\theta_n) = p \quad \forall n \geq 1$. Hence $E(S_n^{\Theta^M})$ and $E(S_n^{\Theta^I})$ are used to estimate the lower and upper bounds for $E(\Theta_n)$ respectively. If the difference between the lower and upper bounds is small, a minimal repair is more appropriate to the system. For numerical examples see Kijima [31].

7.3 Applications of Kijima Models for General Repair

Example 1:

The following policy is due to Jiang, Makis and Jardine [26]. The type of repair that the unit undergoes at failure is a Kijima Type I general repair. The core part of the policy is mentioned since its detailed description used a measure theoretical approach and is quite advanced. The policy is given in terms of a theorem by Jiang *et al* [26].

Firstly let us define the notation used just like in [26].

c_0 = Optimal average cost.

c_1 = Cost of planned replacement.

c_2 = Cost due to failure alone.

$C(v)$ = Random repair cost in terms of the virtual age with distribution function $G_v(c)$ and that $\forall v_1 < v_2, C(v_1) \leq_d C(v_2)$.

$\theta[v, C(v)]$ = Random degree of repair which is a function, $0 \leq \theta[v, C(v)] \leq 1$.

And $\forall v_1 < v_2, 0 \leq p \leq 1$,

$$\theta[v_1, G_{v_1}^{-1}(p)] \leq_d \theta[v_2, G_{v_2}^{-1}(p)].$$

(v, x) = State of the unit where v the virtual age after last repair is, and x is the length the unit has been working since last repair. Therefore at the n th failure, the state is $(v, x, C(V_n^-))$ where $(v, x) = (V_{n-1}, X_n)$.

$R(v) = R_0(v, 0)$ = Residual value of the repaired unit in state $(v, 0)$.

$R_0(v, x)$ = Residual value of the unit in state (v, x) .

$R_1^-(v, x, C(V_n^-))$ = Residual value of the failed unit before repair, given that the repair cost is $C(V_n^-)$.

$R_1^+(v, x, C(V_n^-))$ = Residual value of the unit after repair.

$R_2(v, x)$ = Mean residual value of the failed unit in state (v, x)

Policy 16(Jiang et al, [26])

“The optimal policy exists, and is a combination of a generalized repair cost limit policy for failure replacement and an age based policy for preventive replacement, i.e. the optimal replacement time t_0 is determined by

$$t_0 = \begin{cases} S_n & \text{if } C(V_n^-) \geq R_1^+(V_{n-1}, X_n, C(V_n^-)) \\ S_{n-1} + (T(V_{n-1}) - V_{n-1}) & \text{if no replacement took place at} \\ & \text{or before } S_{n-1} \text{ and} \\ & S_{n-1} + (T(V_{n-1}) - V_{n-1}) < S_n \end{cases} \quad (7.34)$$

The repair cost limit is given by

$$R_1^+(v, x, C(V_n^{-1})) = E(R(v + \theta(v + x, C(V_n^{-1})x))$$

where

$$R(t) = \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(u) \{c_0 - \lambda(u)(c_2 - R_2(t, u - t))\}^+ du$$

and

$$R_2(v, x) = E(R_1^-(v, x, C(V_n^{-1})))$$

$$R_1^- = E(R(v + \theta(v + x, C(V_n^{-1})x) - C(V_n^{-1}))^+)$$

Also

$$\begin{aligned} T(v) &= v + \min\{x : R_0(v, x) = 0\} \\ &= \min\{t \geq v : c_0 \leq \lambda(t)(c_2 - R_2(v, t - v))\} \end{aligned}$$

where

$$R_0(v, x) = \frac{1}{\bar{F}(v)} \int_{v+x}^\infty \bar{F}(u) [c_0 - \lambda(u)(c_2 - R_2(v, u - v))]^+ du$$

And

$$R(0) = c_1, \quad R(\bar{T}) = 0 \quad \text{when } \bar{T} = \min\{t : c_0 \leq \lambda(t)c_2\} \quad "$$

Thus according to policy 16,

Replace the unit at virtual age $T(v)$ when $R_0(v, T(v) - v) = 0$.

At failure, replace the unit when $C(V_n^-) \geq R_1^+(V_{n-1}, X_n, C(V_n^-))$.

Jiang et al [26] showed various modifications to this policy based on the different general degree of repairs.

Example 2

Kahle and Love [27] used data from the British Columbia Hydro-Electric Power Generation System for one turbine unit to find a suitable maintenance

policy that best fit the collected information. Based on their observation, corrective maintenance is of three forms: a minimal repair, minor repair or a major repair. Preventive maintenance is either a minor or a major repair. The minor or major repairs are of the Kijima type I or II models and affect the failure rate which is dependent on the virtual age of the unit. Thus if a repair at the n th failure is of Kijima type I, then the failure rate immediately after, will have the same value as the failure rate immediately after the $(n-1)$ th repair i.e. when the virtual age is V_{n-1} . And if it is of Kijima type II, then the failure rate immediately after is directly proportional to the virtual age. They suggested two degrees of repair for the three possible maintenance actions (corrective or preventive) as follows:

At the n th failure the degree of repair is $\theta_{1n}\theta_{2n}$.

Thus if it's a minimal repair then $\theta_{1n} = 1$ and $\theta_{2n} = 1$

For a minor repair, $\theta_{1n} = \theta_1$ and $\theta_{2n} = 1$ ($0 \leq \theta_1 \leq 1$)

For a major repair, $\theta_{1n} = \theta_1$ and $\theta_{2n} = \theta_2$ ($0 \leq \theta_2 \leq 1$)

Thus starting at $t = 0$, $V_0 = 0$ and the failure rate during the first operating cycle is given by $\lambda_{X_1}(t) = \lambda(t)$. Then the unit stops operating due to failure or preventive maintenance and undergoes a repair which consequently affects the failure rate and the virtual age. Hence after the first repair the failure rate during the second operating cycle changes to

$$\lambda_{X_2}(t) = \lambda(t - S_1 + V_1)$$

where

$$V_1 = \theta_{11}\theta_{21}X_1 \text{ if the repair was of Kijima type I or II}$$

After the second repair we have the following

$$\lambda_{X_3}(t) = \lambda(t - S_2 + V_2)$$

and

$$V_2 = V_1 + \theta_{12}\theta_{22}X_2 \quad \text{if the repair was of Kijima type I}$$

$$\text{or } V_2 = \theta_{12}\theta_{22}(V_1 + X_2) \quad \text{if the repair was of Kijima type II.}$$

Hence after the n th repair, the failure rate and the virtual age is given by

$$\lambda_{X_n}(t) = \lambda(t - S_{n-1} + V_{n-1}) \quad (7.35)$$

$$V_n = V_{n-1} + \theta_{1n}\theta_{2n}X_n \quad \text{if the repair was of Kijima type I} \quad (7.36)$$

$$V_n = \theta_{1n}\theta_{2n}(V_{n-1} + X_n) \quad \text{if the repair was of Kijima type II} \quad (7.37)$$

A Weibull distribution was used to model the lifetime distribution of the turbine unit. And the the method of maximum likelihood was used to estimate the 2 parameters of the distribution and the two degrees of repair θ_1 and θ_2 . Kahle and love [27] tried 7 different maintenance policies and obtained estimates (for each policy) for the 2 parameters of the Weibull distribution and for θ_1 and θ_2 . Consequently they tried to find the policy that best fit the data collected. When the time to carry out repairs is taken into account, it was found that the failure rate follows one close to an exponential distribution. They also assumed that each repair (minor or major) has its own degree of repair given by

$$\theta_n = 1 - \Phi(\log(r_n) - 2.4) \quad (7.38)$$

where

θ_n = degree of repair at the n th corrective/preventive maintenance.

r_n = time taken to repair the unit after the n th operating cycle.

Φ = distribution function of the standard normal distribution

and the value 2.4 is the mean value of the log repair times.

The models used to fit their data showed that the failure rate is constant. Hence they concluded that the turbine unit undergoes either a minor or a major repair base on its current state such that the failure process fits an exponential model.

Kahle [28] further analyse the data collected from the British Columbia Hydro-Electric Power Generation System and derived an optimal maintenance policy that leads to a constant failure rate. The policy is optimal if repairs during preventive maintenance lead to constant failure intensity.

8. Conclusions.

Many more maintenance policies have been studied over the years and we have presented some of those that fall under the age, block and cost limit categories for a single unit system. Other categories of maintenance policies are: sequential policies, failure limit policies, repair time limit policies, just to mention a few. Also there are policies that dealt with multi-units system which are more complicated. As we have seen in this survey, the main aim for any policy is to minimise cost in the long run. The age replacement policies considered here involves either minimal repairs, imperfect repairs or both and we made the assumptions that certain type of repairs(preventive or corrective) takes negligible amount of time and that the unit has an IFR lifetime distribution. But in reality, some of the assumptions we made here might not be appropriate. Jhang [25] present another policy in which many more realistic factors are taken into account like delivery times of a unit, inspection time, cost due to ordering, shortages etc.... Another variation of the age replacement policy is by Block [12], [13], where the assumption of IFR distribution is relaxed and weaker assumptions are made. See Barlow [3] for detail literature of the weaker assumptions. The original block replacement policy is known to be wasteful and hence the modifications made to it are more economical as we saw in policy 8 where used units were kept and used at failure if required. Minimal repair at failure strategies has been used extensively in this study. For more on minimal repair policies see Beichelt [8], Feldman [20], and Nakagawa [35], [36]. Policies involving geometric processes can be look upon as more realistic than others, because we would normally expect the lifetime of a unit to decrease after a repair and maintenance costs to increase as the unit ages. For more on those types of

policies refer to Stanley [43], Yeh Lam [47]-[50] and Zang [51]. The repair cost limit policies considered are very flexible in the sense that repair cost limit can be “mixed” with the other two classical categories of policies. We did not use the markov decision process approach because of its complexity and the high number of states involve. Kapur [29] modified the repair cost limit policy by taking into consideration the number of repairs and the cost of a repair as the two main variables to decide on the type of repair. In Kapur [30], they presented a repair cost limit policies base on age, type of failures, cost of minimal repairs, and they use the markov decision process method. Dagpunar [18] presented a new approach to repair cost limit policy similar to the ones by Beichelt. Maintenance policies with a general degree of repair are very common nowadays. We analysed the Kijima models which involves the virtual age process of the unit. Various modifications have been made to the two models. Stadje and Zuckerman [42] suggested a repair action that is based on Model I and does two things: Firstly it restores the system into operation with the same virtual age prior to failure and secondly it reduces the virtual age by d (say) units. Further work in the generalisation of Model I was done by Makis and Jardine [34], where the repair process was formulated as a semi-Markov decision process in a two dimensional state space involving the number of failures and the real age of the system.

The extensions/generalisations of Model II have also been modelled. Dagpunar [17] suggested the idea of a deterministic repair function for the degree of repair. Apart from system restoration, the virtual age is not restricted to the most recent operating cycle. Policies involving general repair is a whole research topic on its own. Guo [23] gave a survey on the

different categories of general repairs and also mentioned the limitation of the policies base on virtual age.

REFERENCES:

- 1) BARLOW, R.E AND HUNTER, L.C, Optimum Preventive Maintenance Policies, Operation Research, **8**, (1960).
- 2) BARLOW, R.E AND PROSCHAN, F.: Mathematical Theory of Reliability. Wiley, New York, 1965
- 3) BARLOW, R.E AND PROSCHAN, F.: Statistical Theory of Reliability and Life Testing, Probability Models, 1975
- 4) BEICHELT, F., A General Preventive Maintenance Policy, Mathematische Operationsforschung und statistic Series, Statistics **7** (1976), pg (927-932)
- 5) BEICHELT, F., Replacement policies based on system age and maintenance cost limits, Mathematische Operationsforschung und statistic Series, Statistics **12** 4, 1981b
- 6) BEICHELT, F., A replacement policy based on limits for the repair cost rate, IEEE Transactions on Reliability, **31** 4, 1982
- 7) BEICHELT, F., A General Maintenance Model and its Application to Repair Limit Replacement Policies, Microelectronics Reliability, **32** 8, 1992.
- 8) BEICHELT, F., A Unifying Treatment of Replacement Policies with Minimal Repair, Naval Research Logistics, **40**, (51-67), 1993.
- 9) BEICHELT, F., Stochastic Processes in Science, Engineering and Finance. Chapman & Hall/CRC (2006)
- 10) BERG M, A proof of optimality for age replacement policies, Journal of Applied Probability, **13**, (751-759), 1976
- 11) BERG M AND EPSTEIN B, Comparison of age, block, and failure replacement policies, IEEE Transactions on Reliability, **27** 1, 1978
- 12) BLOCK HM, BORGES WS, SAVITS TH, A general age replacement model with minimal repair, Naval Research Logistics, **35** 5, 1988

- 13) BLOCK HM, LANGBERG NA, SAVITS TH, Repair replacement policies, *Journal Of Applied Probability*, **30** 1, 1993
- 14) BROWN, M, Bounds, Inequalities, and Monotonicity Properties for Some Specialized Renewal Process, *The Annals of Probability*, **8** 2, 1980
- 15) CHEN Y, Classes of life distributions and Renewal Counting Process, *Journal of Applied Probability*, **31**, (1110-1115), 1994
- 16) COX D R, *Renewal Theory*, Methuen, London, 1962
- 17) DAGPUNAR, J.S, Some Properties and Computational Results for a General Repair Process, *Naval Research Logistics*, **45**, 391-405(1998).
- 18) DAGPUNAR, J.S, A new approach for solving repair limit problems, *European Journal of Operational Research*, **113**, (137-146,1999
- 19) DRINKWATER R.W AND HASTINGS N.V.J, An economic replacement model. *Operational Research Quarterly*, **18**, (121-138), 1967
- 20) FELDMAN R M AND CHEN M, Optimal Replacement Policies with Minimal Repair and Age dependent costs, *European Journal of Operational Research*, **98**, (75-84),1997
- 21) FELLER W, *An introduction to Probability Theory and its Application*, Vol. II, 2nd Edition, Wiley, New York, 1971
- 22) GARDENT P AND NONANT L, Entretien et renouvellement d'un parc de machines, *Revue Francaise de Recherche Operationelle*, **7**, 5-19(1963).
- 23) GUO, R; ASCHER, H; LOVE, E: Generalized Models of Repairable Systems: A Survey via Stochastic Process Formalism, *ORION* **16**, 87-128.
- 24) HASTINGS N.A.J, The Repair limit method, *Operational Research Quarterly*, **20**, (337-349), 1969
- 25) JHANG J P, A Generalized Age Replacement policy with random delivery time and inspection, *International Journal of Systems Science*, **32**, 3, 321-329, 2001

- 26) JIANG XIAOYUE, MAKIS VILIAM and JARDINE K.S.ANDREW, Optimal Repair/Replacement Policy for a General Repair, *Advances in Applied Probability*, **33**, 206-222(2001)
- 27) KAHLE W, LOVE C E, Modeling the Influence of Maintenance Actions, *Mathematical and Statistical Methods in Reliability*, Eds. B.H.Lindquist, K.A.Doksum, World Scientific Publishing Co., Series on Quality, Reliability and Engineering Statistics, **387-400(2003)**
- 28) KAHLE W, Optimal Maintenance Policies in Incomplete Repair Models,
- 29) KAPUR P.K, GARG R.B, Optimal Number of Minimal Repairs before Replacement with Repair Cost Limit, *Reliability Engineering and System Safety*, **26**, (35-46), 1989
- 30) KAPUR P.K, GARG R.B, BUTANI N.L, Some replacement policies with minimal repairs and repair cost limit, *International Journal of System Science*, **20** 2, 1989
- 31) KIJIMA, M., Some results for Repairable Systems with General Repair, *Journal of Applied Probability*, **26**, 89-102(1989).
- 32) KIJIMA M, MORIMURA H AND SUZUKI Y, Periodical replacement problem without assuming minimal repair, *European Journal of Operational Research*, **37**, (194-203), 1988.
- 33) LAST, G. and SZEKLI, R., Asymptotic and Monotonicity Properties of Some Repairable Systems, *Advances in Applied Probability*, **30**, 1089-1110(1998).
- 34) MAKIS V and JARDINE K.S.A, A note on Optimal Replacement Policy under General Repair, *European Journal of Operational Research*, **69**, 75-82(1993)

- 35) NAKAGAWA T, A summary of periodic replacement with minimal repair at failure, Journal of the Operations Research Society of Japan, **24**, (213-228), 1981a
- 36) NAKAGAWA T, Modified periodic replacement with minimal repair at failure, IEEE Transactions on Reliability, **30**, (165-168), 1981b
- 37) NAKAGAWA T, Optimal policy of continuous and discrete replacement with minimal repair at failure, Naval Research Logistics Quarterly, **31** 4, 1984
- 38) NAKAGAWA T, Periodic and Sequential preventive maintenance policies, Journal of Applied Probability, **23** 2, 1986
- 39) ROSS, S.M, Stochastic Processes. Second Edition. Wiley, New York, 1996
- 40) SHEU S H, GRIFFITH W S, AND NAKAGAWA T, Extended Optimal Replacement Model with Random Minimal Repair Costs, European journal of Operational Research, **85**, (636-649), 1995
- 41) SHIZUO S, A Probabilistic Approach to Preventive Maintenance, Journal of Operational Research Society of Japan, **49** 1, 1957
- 42) STADJE W., and ZUCKERMAN D., Optimal Maintenance Strategies for Repairable Systems with General Degree of Repair, Journal of Applied Probability, **28**, 384-396 (1991).
- 43) STANLEY J A D, On Geometric Processes and repair Replacement Problems, Microelectronics Reliability, Vol **33**, No 4, pg 489-491, 1993
- 44) STOYAN D AND MULLER A.: Comparisons Methods for Stochastic Models and Risks. Wiley, New York, 2002
- 45) TANGO T, Extended Block Replacement policy with Used Items, Journal of Applied Probability, **15**, 560-572 (1978)

- 46) WANG H AND PHAM H, Some Maintenance Models and Availability with Imperfect Maintenance in Production Systems, *Annals of Operations Research*, **91**, (305-318), 1999.
- 47) YEH LAM, A note on the Optimal Replacement Problem, *Advances in Applied Probability*, **20**, 479-482, 1988.
- 48) YEH LAM, Geometric **Process** and Replacement Problem, *Actae Mathematicae Applicatae Sinica*, **4**, pg 479-482, 1988
- 49) YEH LAM, Optimal Geometric Process Replacement Model, *Actae Mathematicae Applicatae Sinica*, **8**, pg 73-81, 1992
- 50) YEH LAM AND CHAN S K, Statistical Inference for geometric Process with Lognormal Distribution, *Computational Statistics and Data Analysis*, **27**, 99-112, 1998.
- 51) ZANG Y L, A Geometric Process Repair Model with Good as New Preventive Repair, *IEEE Transactions on Reliability*, **51**, 2, 2002