Spectral Properties of a Fourth Order Differential Equation with Eigenvalue Dependent Boundary Conditions

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Declaration

I declare that this research report is my own,unaided work. It is been submitted for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any other degree or examination at any other university.

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SPECTRAL PROPERTIES OF A FOURTH ORDER DIFFERENTIAL EQUATION WITH EIGENVALUE DEPENDENT BOUNDARY CONDITIONS

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1. Introduction

This masters dissertation contains a discussion of a variation on the boundary conditions of a problem originally published by M. Möller and V. Pivorvachik [1]. The Birkhoff regularity of the system is discussed. The adjoint of the differential operator and its domain are found. Asymptotic approximations of the eigenvalues of a simplification of the problem are found. Using the methods of [7] eigenvalue asymptotics for the original eigenvalue problems are found.

Small transverse vibrations of a homogeneous beam compressed or stretched by a force g can be described by the partial differential equation

$$\frac{\partial^4}{\partial x^4}u(x,t) - \frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}u(x,t) = -\frac{\partial^2}{\partial t^2}u(x,t).$$

We suppose g to be a sufficiently smooth real-valued function; throughout this dissertation $g \in C^1[0,a]$, $a > 0$, will be assumed. If $g > 0$, then the beam is stretched, if $g < 0$, then it is compressed. Let us impose the following boundary conditions at the left end

$$u(0,t) = 0,$$
$$\left.\frac{\partial^2}{\partial x^2}u(x,t)\right|_{x=0} = 0.$$

In the paper of M. Möller and V. Pivorvachik the boundary conditions corresponding to a hinge connection were

$$u(a,t) = 0,$$
$$\left.\frac{\partial^2}{\partial x^2}u(x,t)\right|_{x=a} = -\alpha\left.\frac{\partial}{\partial t\partial x}u(x,t)\right|_{x=a}.$$

These conditions have now been changed to the following:

$$\frac{\partial^2}{\partial x^2}u(x,t)|_{x=a} = 0,$$
$$\frac{\partial}{\partial x}u(x,t)\bigg|_{x=a} = -\alpha\,\frac{\partial}{\partial t}u(x,t)\bigg|_{x=a}.$$

Substituting $u(x,t) = e^{i\lambda t}y(\lambda,x)$ we obtain the ordinary fourth order differential equation

$$y^{(4)}(\lambda,x) - (gy')'(\lambda,x) = \lambda^2 y(\lambda,x), \tag{1.1}$$

with boundary conditions

$$y(\lambda,0) = 0, \tag{1.2}$$
$$y''(\lambda,0) = 0, \tag{1.3}$$
$$y''(\lambda,a) = 0, \tag{1.4}$$
$$y'(\lambda,a) + i\alpha\lambda y(\lambda,a) = 0. \tag{1.5}$$

The boundary eigenvalue problem converts as per [7, Lemma 6.1.1] to the following first order system,

$$\widetilde{Y}'(x,\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda^2 & g' & g & 0 \end{pmatrix} \widetilde{Y} \text{ where } \widetilde{Y} = \begin{pmatrix} y(x) \\ y'(x) \\ y''(x) \\ y'''(x) \end{pmatrix} \tag{1.6}$$

with the following boundary conditions:

$$W^{(0)}\widetilde{Y}(0,\lambda) + W^{(1)}\widetilde{Y}(a,\lambda) = 0 \tag{1.7}$$

where

$$W^{(0)}(0,\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.8}$$

and

$$W^{(1)}(a,\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\alpha\lambda & 1 & 0 & 0 \end{pmatrix}. \tag{1.9}$$

A change in boundary conditions of the problem of M. Möller and V. Pivorvachik from $y(\lambda,a) = 0$ and $y''(\lambda,a) + i\alpha y'(\lambda,a) = 0$ to

$y''(\lambda, a) = 0$ and $y'(\lambda, a) + i\alpha y(\lambda, a) = 0$ retains the Birkhoff regularity of the problem because all the 4 permutations of the matrix $W_0^{(0)}\Delta + W_0^{(1)}(I - \Delta)$ are still invertible where Δ are block diagonal matrices $\Delta := \operatorname{diag}(\delta_0 I_{n_0}, \dots, \delta_4 I_{n_4})$ and δ_n is either 0 or 1 as per [7, 4.1.21] and [7, 4.1.22] however the whole boundary value problem changes from being self-adjoint to non-self-adjoint. This change results in the loss of a number of eigenvalue properties from [1]; the symmetry of the eigenvalues with respect to the imaginary axis, all the eigenvalues lying in the closed upper half plane for $\alpha \geq 0$ and all the eigenvalues in the negative imaginary axis being semi-simple. We first find the eigenvalues of a simplified problem with $g = 0$ and apply Rouché's theorem's to the characteristic equation. For a general g and for sufficiently large λ we find the asymptotic fundamental system using [7, Theorem 8.2.1]. In order to find the zeros of the characteristic function, methods of estimating exponential sums from [7, Appendix A.2] are used. We then conclude the results of this dissertation with [Theorem 5.2] which states that for $g \in C^1[0, a]$, there is a positive integer k_0 such that the eigenvalues λ_k, $k \in \mathbb{Z}$, of the problem (1.1)– (1.5), counted with multiplicity, can be enumerated in such a way that the eigenvalues λ_k are pure imaginary for $|k| < k_0$, $\lambda_{-k} = -\overline{\lambda_k}$ for $k \geq k_0$, where $\lambda_k = \mu_k^2$ and in particular, there is an odd number of pure imaginary eigenvalues. Finally, we find the asymptotic approximation of the eigenvalues to order three to be $\tau_{k,0} = 0$, $\tau_{k,1} = \frac{i}{2\pi\alpha} + \frac{1}{4\pi}\int_0^a g(x)dx$ and $\tau_{k,2} = -\frac{a}{4\pi^2\alpha^2}$.

2. Birkhoff regularity

Birkhoff regularity is a regularity property which is given in terms of the argument of the nonzero diagonal elements of the leading matrix in the differential system, further by the zero-approximand of the λ-asymptotic fundamental matrix of the differential system and by the limit of the coefficient matrices in the boundary conditions.

Birkhoff regularity for boundary eigenvalue problems for first order $n \times n$ systems of ordinary differential equations, that are asymptotically linear in the eigenvalue parameter λ requires that the leading matrix is supposed to be a diagonal matrix whose nonzero diagonal elements as well as their nonzero difference are assumed to have constant arguments and to be bounded away from zero. The boundary conditions of this boundary eigenvalue problem may depend on λ and have to be asymptotically constant in λ, as λ tends to infinity. Birkhoff regularity for this particular boundary eigenvalue problem is defined in [7, Definition 4.1.2].

Let the boundary eigenvalue problem be defined as follows:

$$y' - (\lambda A_1 + A_0 + \lambda^{-1}A^0(\cdot,\lambda))y = 0, \tag{2.1}$$

$$\sum_{j=0}^{\infty} \widetilde{W}^{(j)}(\lambda)y(a_j) + \int_a^b \widetilde{W}(x,\lambda)y(x)dx = 0, \tag{2.2}$$

where y varies in $(W_p^1(a,b))^n$.

A_0 and A_1 belong to $M_n(L_p(a,b))$, $-\infty < a < b < \infty$, $1 \le p \le \infty$ and $n \in \mathbb{N}\backslash\{0\}$. The boundary eigenvalue problem is considered for a sufficiently large complex number λ, say $|\lambda| \ge \gamma(> 0)$ and $A^0(\cdot,\lambda)$ belongs to $M_n(L_p(a,b))$ for this λ, $A^0(\cdot,\lambda)$ depends holomorphically on it and $A^0(\cdot,\lambda)$ is bounded as $\lambda \to \infty$. More precisely, we suppose A_1 to be a diagonal matrix function,

$$A_1 = \begin{pmatrix} A_0^1 & & & & \\ & A_1^1 & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \cdot & \\ & & & & & A_l^1 \end{pmatrix},$$

where l is a positive integer,

$$A_\nu^1 = r_\nu I_{n_\nu} \ (\nu = 0, \dots, l), \ \sum_{\nu=0}^{l} n_\nu = n,$$

with $n_0 \in \mathbb{N}$ and $n_\nu \in \mathbb{N}\backslash\{0\}$ for $\nu = 1, \dots, l$. According to the block structure of A_1, we write $A_0 = (A_{0,\nu\mu})_{\nu,\mu=0}^{l}$. For the diagonal elements of A_1 we assume: $r_0 = 0$, and for $\nu, \mu = 0, \dots, l$ there are number $\varphi_{\nu\mu} \in [0, 2\pi)$ such that

$$(r_\nu - r_\mu)^{-1} \in L_\infty(a,b) \ if \ \nu \neq \mu, \tag{2.3}$$

$$r_\nu(x) - r_\mu(x) = |r_\nu(x) - r_\mu(x)|e^{i\varphi_{\nu\mu}} \ \text{a.e in } (a,b) \tag{2.4}$$

Note that $\mu = 0$ gives $r_\nu^{-1} \in L_\infty(a,b)$ for $\nu = 1, \dots, l$ and

$$r_\nu(x) = |r_\nu(x)|e^{i\varphi_\nu} \ \text{a.e in } (a,b) \ \ \nu = 1, \dots, l \tag{2.5}$$

where $\varphi_\nu := \varphi_{\nu 0} = \varphi_{0\nu} \pm \pi$ for $\nu = 1, \dots, l$.

If $n_0 = 0$, then we need the conditions (2.3) and (2.4) only for $\nu, \mu \in \{1, \dots, l\}$. On the other hand, the conditions $r_\nu^{-1} \in L_\infty(a,b)$ for $\nu =$

$1, \ldots, l$ and (2.5) are needed in any case. Hence it is no additional assumption if we take $\nu, \mu \in \{1, \ldots, l\}$ in (2.3) and (2.4) also in the case $n_0 = 0$.
For the boundary conditions (2.2) we assume that $a_j \in [a, b]$ for $j \in \mathbb{N}$, that $a_j \neq a_k$ if $j \neq k$, and that $a_0 = a$, $a_1 = b$. We suppose that the matrix function $\widetilde{W}(\cdot, \lambda)$ belongs to $M_n(L_1(a, b))$ for $|\lambda| \geq \gamma$ and that there is $W_0 \in M_n(L_1(a, b))$ such that

$$\widetilde{W}(\cdot, \lambda) - W_0 = O(\lambda^{-1}) \text{ in } M_n(L_1(a, b)) \text{ as } \lambda \to \infty. \tag{2.6}$$

It is assumed that the $\widetilde{W}_j(\lambda)$ are $n \times n$ matrices, defined for $|\lambda| \geq \gamma$, and that there are $n \times n$ matrices $W_0^{(j)}$ such that the estimates

$$\sum_{j=0}^{\infty} |W_0^{(j)}| < \infty \tag{2.7}$$

and

$$\sum_{j=0}^{\infty} |\widetilde{W}^{(j)}(\lambda) - W_0^{(j)}| = O(\lambda^{-1}) \quad \text{as } \lambda \to \infty \tag{2.8}$$

hold. Since

$$\sum_{j=0}^{\infty} |\widetilde{W}^{(j)}(\lambda)| \leq \sum_{j=0}^{\infty} |W_0^{(j)}(\lambda)| + \sum_{j=0}^{\infty} |\widetilde{W}^{(j)}(\lambda) - W_0^{(j)}| < \infty, \tag{2.9}$$

the boundary conditions are well-defined for $|\lambda| \geq \gamma$.
For the definition of Birkhoff regularity we need some further notation: For $\nu = 1, \ldots, l$ let φ_ν be defined as in (2.5) and let $\lambda \in \mathbb{C}\backslash\{0\}$. We set

$$\delta_\nu(\lambda) := \begin{cases} 0 & \text{if } \Re(\lambda e^{i\varphi_\nu}) < 0, \\ 1 & \text{if } \Re(\lambda e^{i\varphi_\nu}) > 0, \\ 0 & \text{if } \Re(\lambda e^{i\varphi_\nu}) = 0 \text{ and } \Im(\lambda e^{i\varphi_\nu}) > 0, \\ 1 & \text{if } \Re(\lambda e^{i\varphi_\nu}) = 0 \text{ and } \Im(\lambda e^{i\varphi_\nu}) < 0 \end{cases} \tag{2.10}$$

For convenience let $\delta_0(\lambda) = \delta_1(\lambda)$. We define the block diagonal matrices

$$\begin{cases} \Delta(\lambda) := \operatorname{diag}(\delta_0(\lambda) I_{n_0}, \ldots, \delta_l(\lambda) I_{n_l}) \\ \Delta_0 := \operatorname{diag}(0 \cdot I_{n_0}, I_{n_1}, \ldots, I_{n_l}) \end{cases} \tag{2.11}$$

which (by definition) reduces to

$$(2.12)\qquad \Delta(\lambda) = \mathrm{diag}(\delta_1(\lambda)I_{n_1}, \dots, \delta_l(\lambda)I_{n_l}), \quad \Delta_0 := I_n,$$

if $n_0 = 0$. Finally we set

$$(2.13)\qquad \widetilde{M_2} := \sum_{j=0}^{\infty} \widetilde{W}_0^{(j)} P^{[0]}(a_j) + \int_a^b W_0(x)P^{[0]}(x)dx.$$

The matrix function $P^{[0]}$ belongs to $M_n(W_p^1(a,b))$ and has block diagonal form according to block structure of A_1, i.e.,

$$(2.14)\qquad P^{[0]} = \mathrm{diag}(P_{00}^{[0]}, P_{11}^{[0]}, \dots, P_{ll}^{[0]}).$$

The diagonal elements $P_{\nu\nu}^{[0]}$ are uniquely given as solutions of the initial value problems

$$(2.15)\qquad \begin{cases} P_{\nu\nu}^{[0]\prime} = A_{0,\nu\nu}P_{\nu\nu}^{[0]}, \\ P_{\nu\nu}^{[0]}(a) = I_{n_\nu}. \end{cases}$$

where the $n_\nu \times n_\nu$ matrix functions $A_{0,\nu\nu}$ are the block diagonal elements of A_0.

Definition 2.1. The boundary eigenvalue problem (2.1) and (2.2) is Birkhoff regular if

$$W_0^{(0)}(I_n - \Delta(\lambda))\Delta_0 + W_0^{(1)}\Delta(\lambda)\Delta_0 + \widetilde{M_2}(I_n - \Delta_0)$$

is invertible for $\lambda \in \mathbb{C}\backslash\{0\}$.

Consider a boundary eigenvalue problem

$$(2.16)\qquad \eta^{(n)} + \sum_{i=0}^{n-1} p_i(\cdot,\lambda)\eta^{(i)} = 0$$

$$(2.17)\qquad \sum_{i=1}^{n} w_{ki}^{(0)}(\lambda)\eta^{i-1}(0) + w_{ki}^{(1)}(\lambda)\eta^{i-1}(a) = 0 \quad (k = 1, \dots, n),$$

where $\lambda \in \mathbb{C}$, and $\eta \in W_p^n(0,a)$. Let

$$(2.18)\qquad p_i(\cdot,\lambda) = \sum_{j=0}^{n-i} \lambda^j \pi_{n-i,j} \quad (i = 0, \dots, n-1),$$

where $\pi_{n-i,j} \in L_p(0,a)$ $(i = 0, \dots, n-1)$, $(j = 0, \dots, n-i)$. We assume $\pi_{n-i,n-i} \neq 0$ for some $i \in \{0, \dots, n-1\}$.

The function π defined by

$$\pi(\cdot,\rho) := \rho^n + \sum_{i=0}^{n-1} \rho^i \pi_{n-i,n-i} \quad (\rho \in \mathbb{C}) \tag{2.19}$$

is called the characteristic function of the differential equation (2.16).

Consider this boundary eigenvalue problem together with its associated first order system defined by the operator

$$T^D(\lambda)y := y' - A(\cdot,\lambda)y \quad (y \in (W_p^1(0,a))^n, \lambda \in \mathbb{C}), \tag{2.20}$$

where

$$A := (\delta_{i,j-1} - \delta_{i,n}p_{j-1})_{i,j=1}^n = \begin{pmatrix} 0 & 1 & & & \\ & & \ddots & 0 & \\ & & & \ddots & \\ & & 0 & 0 & 1 \\ -p_0 & & \cdots & & -p_{n-1} \end{pmatrix}. \tag{2.21}$$

We assume that there are matrix function $C(\cdot,\lambda) \in M_n(W_p^1(0,a))$ depending polynomially on λ and a positive real number γ such that

$$C(\cdot,\lambda) \text{ is invertible in } M_n(W_p^1(0,a)) \text{ if } |\lambda| \geq \gamma \tag{2.22}$$

and such that the equation

$$C^{-1}(\cdot,\lambda)T^D(\lambda)C(\cdot,\lambda)y = y' - \widetilde{A}(\cdot,\lambda)y =: \widetilde{T}^D(\lambda)y \tag{2.23}$$

holds for $|\lambda| \geq \gamma$ and $y \in (W_p^1(0,a))^n$, where

$$\widetilde{A}(\cdot,\lambda) = \lambda A_1 + A_0 + \lambda^{-1}A^0(\cdot,\lambda) \quad (|\lambda| \geq \gamma) \tag{2.24}$$

fulfils the assumptions made above for (2.1).

Consider matrix function

$$W^{(j)}(\lambda) := (w_{ki}^{(j)}(\lambda))_{k,i=1}^n C(a_j,\lambda) \quad (j = 0,1)(a_0 = 0 \text{ and } a_1 = a) \tag{2.25}$$

and set

$$\widehat{T}^R(\lambda)y := W^{(0)}(\lambda)y(0) + W^{(1)}(\lambda)y(a) \quad (y \in (W_p^1(0,a))^n). \tag{2.26}$$

Suppose further that there is a $n \times n$ matrix polynomial $C_2(\lambda)$ whose determinant is not identically zero such that the following property holds:

There are $n \times n$ matrices $W_0^{(0)}$ and $W_0^{(1)}$ such that estimates
(2.27)
$$|C_2^{-1}(\lambda)W^{(0)}(\lambda) - W_0^{(0)}| + |C_2^{-1}(\lambda)W^{(1)}(\lambda) - W_0^{(1)}| = O(\lambda^{-1}) \quad \text{as } \lambda \to \infty$$

holds.

Definition 2.2. The boundary eigenvalue problem (2.16), (2.17) is called Birkhoff regular if $\pi_{nn} \neq 0$ and if there are matrix functions $C(\cdot,\lambda)$ satisfying (2.22) - (2.24) and $C_2(\lambda)$ satisfying (2.27) so that the associated boundary eigenvalue problem

$$\begin{aligned} \widetilde{T}^D(\lambda)y &= 0, \\ C_2(\lambda)^{-1}\widehat{T}^R(\lambda)y &= 0 \end{aligned}$$

is Birkhoff regular in the sense of Definition 2.1.

Theorem 2.3. *Replacing λ with μ^2, the eigenvalue problem* (1.1)-(1.5) *is Birkhoff regular in the sense of Definition 2.2 for all $\alpha > 0$.*

Proof. In the notation of (2.16), $n = 4$, $p_0(\cdot,\mu) = -\mu^4$, $p_1(\cdot,\mu) = -g'$, $p_2(\cdot,\mu) = -g$ and $p_3(\cdot,\mu) = 0$. The characteristic function (2.19) becomes

$$\begin{aligned} \pi(\cdot,\rho) &:= \rho^4 + \sum_{i=1}^{3} \rho^i \pi_{4-i,4-i} \\ &= \rho^4 - 1 \end{aligned}$$

according to [7, 7.1.4], where $\rho \in \mathbb{C}$ and its zeros are $i^{k=1}$, $k = 1, \dots, 4$. The matrix $C(x,\mu) = \mathrm{diag}(1,\mu,\mu^2,\mu^3)(\mathrm{i}^{(k-1)(l-1)})_{k,l=1}^4$ fulfils (2.22) to (2.24). $C(x,\mu)$ is invertible and

$$C^{-1}(\cdot,\lambda)T^D(\lambda)C(\cdot,\lambda)y = y' - \widetilde{A}(\cdot,\lambda)y =: \widetilde{T}^D(\lambda)y.$$

According to [7, Theorem 7.2.4.A], it follows that the boundary matrices defined in [7, (7.3.1)] are given by

$$\widetilde{W}^{(0)}(\mu) = W^{(0)}C(0,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} C(0,\mu)$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{W}^{(1)}(\mu) = W^{(1)}(\mu)C(a,\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\alpha\mu^2 & 1 & 0 & 0 \end{pmatrix} C(a,\mu)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu^2 & -\mu^2 & \mu^2 & -\mu^2 \\ i\alpha\mu^2+\mu & i\alpha\mu^2+i\mu & i\alpha\mu^2-\mu & i\alpha\mu^2-i\mu \end{pmatrix}.$$

The first order system (1.6) fulfils all the assumptions made about (2.1) but the boundary conditions (1.7) are not asymptotically constant in λ. In order for (2.6) to (2.8) hold we require a $n \times n$ matrix polynomial $C_2(\lambda)$ whose determinant is not identically zero such that (2.27) holds. We choose $C_2(\mu) = \operatorname{diag}(1, \mu^2, \mu^2, \mu^2)$ if $\alpha > 0$ because $C_2(\mu)$ is invertible and as $\mu \to \infty$,

$$|C_2(\mu)^{-1}\widetilde{W}^{(j)}(\mu) - W_0^{(j)}| \le M\frac{1}{\mu} \quad \text{i.e.}$$

$$C_2(\mu)^{-1}\widetilde{W}^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$$

meaning that $W_0^{(j)}$ for $j = 0, 1$ are asymptotically constant in μ, where

$$W_0^{(0)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ i\alpha & i\alpha & i\alpha & i\alpha \end{pmatrix}.$$

According to [7, Definition 7.3.1 and Proposition 4.1.7], the matrices Δ of the problem are the four 4×4 diagonal matrices with 2 consecutive ones and 2 consecutive zeros in the diagonal in a cyclic arrangement. By [7, Definition 7.3.1] the problem (1.1)–(1.5) is Birkhoff regular if

$$W_0^{(0)}\Delta + W_0^{(1)}(I - \Delta) \tag{2.28}$$

is invertible for all four choices of Δ. For $\Delta = \operatorname{diag}(1,1,0,0)$

$$W_0^{(0)}\Delta + W_0^{(1)}(I-\Delta) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & i\alpha & i\alpha \end{pmatrix}. \tag{2.29}$$

For $\Delta = \operatorname{diag}(0,1,1,0)$

$$W_0^{(0)}\Delta + W_0^{(1)}(I-\Delta) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ i\alpha & 0 & 0 & i\alpha \end{pmatrix}. \tag{2.30}$$

For $\Delta = \operatorname{diag}(0,0,1,1)$

$$W_0^{(0)}\Delta + W_0^{(1)}(I-\Delta) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ i\alpha & i\alpha & 0 & 0 \end{pmatrix}. \tag{2.31}$$

For $\Delta = \operatorname{diag}(1,0,0,1)$

$$W_0^{(0)}\Delta + W_0^{(1)}(I-\Delta) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & i\alpha & i\alpha & 0 \end{pmatrix}. \tag{2.32}$$

After permutations of columns, the matrices (2.29) to (2.32) are block diagonal matrices consisting of 2×2 blocks taken from two consecutive columns (in the sense of cyclic arrangement) of the first two rows of $W_0^{(0)}$ and the last two rows of $W_0^{(1)}$, respectively. These matrices are invertible. We thus have shown that the problem is Birkhoff regular.

If $\alpha = 0$, then the same conclusion holds with $C_2(\mu) = \operatorname{diag}(1,\mu^2,\mu^2,\mu)$ and

$$W_0^{(0)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$\square$

3. Differential operators and adjoint

Let A, K and M be linear operators acting in $L_2(0,a) \oplus \mathbb{C}$ with domains

$$D(A) = \left\{ Y = \begin{pmatrix} y(x) \\ y(a) \end{pmatrix} : y \in W_2^4(0,a),\ y(0) = y''(0) = y''(a) = 0 \right\},$$
$$D(K) = D(M) = L_2(0,a) \oplus \mathbb{C},$$

given by

$$AY = \begin{pmatrix} y^{(4)} - (gy')' \\ y'(a) \end{pmatrix},$$
$$K = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

We want to find the adjoint A^* of A.

Proposition 3.1. *The adjoint A^* of A is given by*

$$D(A^*) = \left\{ \begin{array}{ll} Z = \begin{pmatrix} z(x) \\ z''(a) - g(a)z(a) \end{pmatrix} : z \in W_2^4, & z(a) = z(0) = 0, \\ & z''(0) = z''(a) = 0, \\ & z''(0) = gz(0) \end{array} \right.$$

$$A^*Z = \begin{pmatrix} z^{(4)} - (gz')' \\ -z^{(3)}(a) + g(a)z'(a)) \end{pmatrix}.$$

In particular, A is not self-adjoint.

Proof. Let $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in L_2(0,a) \oplus \mathbb{C}$. Then by definition of A^*, $Z \in D(A^*)$ if and only if there is a $W \in L_2(0,a) \oplus \mathbb{C}$ such that $(AY, Z) = (Y, W)$ for all $Y \in D(A)$; for such Z, $A^*Z = W$.

Firstly, choose an arbitrary $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in D(A^*)$ for $y \in W_2^4(0,a)$ with $y^{(k)}(a) = y^{(k)}(0) = 0$ for $k = 0, ..., 3$, $Y = \begin{pmatrix} y \\ y^{(k)}(a) \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$. Given the bilinear form $(,)$ and integrating by parts we get;

$$\begin{aligned}(AY,Z) &= (y^{(4)} - (gy')', z)_0 \\ &= -(y^{(3)} - gy', z')_1 \\ &= (y'', z'')_2 + (y', gz')_1 \\ &= -(y', z^{(3)})_3 - (y, (gz')')_2 \\ &= (y, z^{(4)} - (gz')')_4\end{aligned}$$

with $(,)_j$ being the sesquilinear form in the dual pair of spaces W_2^j and W_2^{-j}. But $y^{(k)}(a) = y^{(k)}(0) = 0$ and therefore $Y \in D(A)$. Writing $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and $Y = \begin{pmatrix} y \\ 0 \end{pmatrix}$, it follows that for an arbitrary z there is a $w_1 \in L_2(0,a)$ (which is a set of functionals in the dual space of $L_2(0,a)$ which is self-dual) such that

$$(AY,Z) = (Y,W) = (y, w_1) = (y, w_1)_4$$

whence

$$z^{(4)} - (gz')' = w_1 \in L_2(0,a).$$

Therefore, $z \in W_2^4(0,a)$ for each $Z = \begin{pmatrix} z \\ d \end{pmatrix} \in D(A^*)$. Thus, taking $Z = \begin{pmatrix} z \\ d \end{pmatrix}$ with $z \in W_2^4(0,a)$ we can write

$$\begin{aligned}(AY,Z) &= \int_0^a (y^{(4)} - (gy')')\overline{z}\, dx + y'(a)\overline{d} \\ &= \int_0^a (\overline{z}^{(4)} - (g\overline{z}')')y\, dx \\ &\quad + y^{(3)}(a)\overline{z}(a) - y^{(3)}(0)\overline{z}(0) \\ &\quad - y''(a)\overline{z}'(a) + y''(0)\overline{z}'(0) \\ &\quad + y'(a)\overline{z}''(a) - y'(0)\overline{z}''(0) \\ &\quad - y(a)\overline{z}^{(3)}(a) + y(0)\overline{z}^{(3)}(0) \\ &\quad - y'(a)(g\overline{z})(a) + y'(0)(g\overline{z})(0) \\ &\quad + y(a)(g\overline{z}')(a) - y(0)(g\overline{z}')(0) \\ &\quad + y'(a)\overline{d}.\end{aligned}$$

A comparison of (AY, Z) and (Y, W) to the definition of an adjoint which states that $Z \in D(A^*)$ i.e $(AY, Z) = (Y, A^*Z)$, shows that $Z \in D(A^*)$ if and only if there is a $w_2 \in \mathbb{C}$ such that, for all $y \in D(A)$

$$\begin{aligned} y(a)\overline{w_2} = y^{(3)}(a)\overline{z}(a) - y^{(3)}(0)\overline{z}(0) \\ - y''(a)\overline{z}'(a) + y''(0)\overline{z}'(0) \\ + y'(a)\overline{z}''(a) - y'(0)\overline{z}''(0) \\ + y(0)\overline{z}^{(3)}(0) - y(a)\overline{z}^{(3)}(a) \\ - y'(a)(g\overline{z})(a) + y'(0)(g\overline{z})(0) \\ + y(a)(g\overline{z}')(a) - y(0)(g\overline{z}')(0) \\ + y'(a)\overline{d}. \end{aligned}$$

Since $Y = \begin{pmatrix} y \\ y(a) \end{pmatrix} \in D(A)$ satisfies $y(0) = y''(0) = y''(a) = 0$

$$\begin{aligned} y(a)\overline{w_2} &= y^{(3)}(a)\overline{z}(a) - y^{(3)}(0)\overline{z}(0) \\ &- y'(0)(\overline{z}''(0) - g\overline{z}(0)) \\ &- y'(a)(-\overline{z}''(a) + g\overline{z}(a) + \overline{d} \cdot \\ &- y(a)(\overline{z}^{(3)}(a) - g\overline{z}'(a)). \end{aligned} \tag{3.1}$$

We can choose a particular $y \in D(A)$ such that $y(a) = 1$ with $y^{(3)}(a) = y^{(3)}(0) = y'(0) = y'(a) = 0$. Then (3.1) becomes

$$w_2 = -z^{(3)}(a) + g(a)z(a).$$

Similarly choose $y \in D(A)$ such that $y'(a) = 1$ with $y^{(3)}(a) = y^{(3)}(0) = y'(0) = y(a) = 0$. Then

$$\bar{d} = z''(a) - g(a)z(a).$$

To find the boundary conditions on z. We need to find z such the $(AY, Z) = (Y, W)$. Using (3.1) we can substitute for d and w_2 as given above and since y and its derivatives are linearly independent then;

$$\begin{aligned} 0 = y^{(3)}(a)\overline{z}(a) - y^{(3)}(0)\overline{z}(0) \\ - y'(0)(\overline{z}''(0) - g\overline{z}(0)) \end{aligned}$$

implies that $z(a) = z(0) = 0, z''(0) = gz(0)$ giving the boundary conditions on z.

We now show that

$$B = \left\{ Z = \begin{pmatrix} z(x) \\ z''(a) - g(a)z(a) \end{pmatrix} : z \in W_2^4, z(a) = z(0) = 0, z''(0) = gz(0) \right\}$$

satisfies $B \subset D(A^*)$.

Let

$$Z = \begin{pmatrix} z(x) \\ z''(a) - g(a)z(a) \end{pmatrix} \in B$$

and

$$W = \begin{pmatrix} z^{(4)} - (gz')' \\ -z^{(3)}(a) + g(a)z'(a) \end{pmatrix}.$$

$$\begin{aligned}(AY, Z) &= \int_0^a (y^{(4)} - (gy')')\overline{z}\, dx + y'(a)\overline{(z''(a) - g(a)z(a))} \\ &= \int_0^a (\overline{z}^{(4)} - (g\overline{z}')')y\, dx \\ &\quad + y^{(3)}(a)\overline{z}(a) - y^{(3)}(0)\overline{z}(0) \\ &\quad - y''(a)\overline{z}'(a) + y''(0)\overline{z}'(0) \\ &\quad + y'(a)\overline{z}''(a) - y'(0)\overline{z}''(0) \\ &\quad + y(0)\overline{z}^{(3)}(0) - y(a)\overline{z}^{(3)}(a) \\ &\quad - y'(a)g\overline{z}(a) + y'(0)g\overline{z}(0) \\ &\quad + y(a)g\overline{z}'(a) - y(0)g\overline{z}'(0) \\ &\quad + y'(a)(\overline{z}''(a) - g(a)\overline{z}(a)).\end{aligned}$$

Substituting for $Y \in D(A)$ i.e $y(0) = y''(0) = y''(a) = 0$ and $Z \in B$ i.e. $z(a) = z(0) = 0, z''(0) = gz(0)$ we get

$$\begin{aligned}(AY, Z) &= \int_0^a (\overline{z}^{(4)} - (g\overline{z}')')y\, dx + (-\overline{z}^{(3)}(a) + g(a)\overline{z}(a))y(a) \\ &\quad + 2y'(a)\overline{z}''(a).\end{aligned}$$

Showing that if $Z \in B$ then $Z \in D(A^*)$. In order for $(AY, Z) = (Y, W)$ we require that $z''(a) = 0$, thus $C = B \cap \{z''(a) = 0\}$ then $D(A^*) = C$

and $A^*Z = W$.

Showing that A is not self-adjoint since $A \neq A^*$. $\square$

4. Asymptotics of eigenvalues for $g = 0$

In this section we consider the eigenvalue problem (1.1)–(1.5) with $g = 0$;

$$y^{(4)}(\lambda, x) = \lambda^2 y(\lambda, x), \tag{4.1}$$

As in the paper of M. Möller and V. Pivorvachik a formula for the asymptotic distribution of eigenvalues is proved. However the current boundary eigenvalue problem is not self-adjoint and hence [1, Lemma 3.1], [1, Lemma 3.2], [1, Lemma 3.3] and [1, Lemma 3.4] do not hold. Take the canonical fundamental system y_j, $j = 1, \dots, 4$, with $y_j^{(m)}(0) = \delta_{j,m+1}$ for $m = 0, \dots, 3$, which is analytic on $\mathbb{C}$ with respect to λ. Because of the boundary conditions $y(0) = y''(0) = 0$ we only need y_2 and y_4. We put $\mu = \sqrt{\lambda}$, $\lambda \neq 0$. It is easy to see that

$$\begin{aligned} y_2(x, \lambda) &= \frac{1}{2\mu}\sin(\mu x) + \frac{1}{2\mu}\sinh(\mu x)\text{and} \\ y_4(x, \lambda) &= -\frac{1}{2\mu^3}\sin(\mu x) + \frac{1}{2\mu^3}\sinh(\mu x), \end{aligned}$$

interpreting $\lambda = 0$ as a limit and applying l'Hospital's rule becomes

$$\begin{aligned} y_2(x, 0) &= \lim_{\mu \to 0} y_2(x, \lambda) \\ &= \lim_{\mu \to 0} \frac{(\sin(\mu x) + \sinh(\mu x))}{2\mu} \\ &= \lim_{\mu \to 0} x \frac{(\cos(\mu x) + \cosh(\mu x))}{2} \\ &= x \\ &\text{and} \\ y_4(x, 0) &= \lim_{\mu \to 0} y_4(x, \lambda) \\ &= \lim_{\mu \to 0} \frac{(-\sin(\mu x) + \sinh(\mu x))}{2\mu^3} \\ &= \lim_{\mu \to 0} x^3 \frac{(\cos(\mu x) + \cosh(\mu x))}{12} \\ &= \frac{x^3}{6}. \end{aligned}$$

Representing the boundary conditions (1.5), (1.4) by functionals B_1, B_2 and $y(0) = y''(0) = 0$, the (reduced) characteristic matrix of the boundary value problem, $T(\lambda) := \begin{pmatrix} T^D(\lambda) \\ T^R(\lambda) \end{pmatrix} : (W_2^1(0,a))^4 \to (L_2(0,a))^4 \times \mathbb{C}^4$ as represented by

$$T(\lambda)Y = \begin{pmatrix} y^{(4)} - (gy')' \\ y(0) \\ y''(0) \\ y''(a) \\ y'(a) + i\alpha\lambda y(a) \end{pmatrix}$$

becomes

$$M = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \begin{pmatrix} y_2 & y_4 \end{pmatrix}$$

where for $y \in W_2^4(0,a)$

$$\begin{aligned} B_1 y &= y''(a) \\ B_2 y &= y'(a) + i\alpha\mu^2 y(a) \end{aligned}$$

since

$$\begin{pmatrix} y_2(x) & y_4(x) \end{pmatrix} = \begin{pmatrix} \sin(\mu x) & \sinh(\mu x) \end{pmatrix} \begin{pmatrix} \frac{1}{2\mu} & -\frac{1}{2\mu^3} \\ \frac{1}{2\mu} & \frac{1}{2\mu^3} \end{pmatrix}.$$

The characteristic equation is given by $\det M = 0$, where

$$\det M = \det \left[\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \begin{pmatrix} \sin(\mu x) & \sinh(\mu x) \end{pmatrix} \right] * \det \begin{pmatrix} \frac{1}{2\mu} & -\frac{1}{2\mu^3} \\ \frac{1}{2\mu} & \frac{1}{2\mu^3} \end{pmatrix}$$

with the two determinants given by

$$\begin{aligned} &\det \left[\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \begin{pmatrix} \sin(\mu x) & \sinh(\mu x) \end{pmatrix} \right] \\ &= \det \begin{pmatrix} -\mu^2 \sin(\mu a) & \mu^2 \sinh(\mu a) \\ \mu \cos(\mu a) + i\alpha\mu^2 \sin(\mu a) & \mu \cosh(\mu a) + i\alpha\mu^2 \sinh(\mu a) \end{pmatrix} \\ &= -\mu^3 \sin(\mu a) \cosh - \mu^3 \cos(\mu a) \sinh(\mu a) \\ &\quad - 2i\alpha\mu^4 \sin(\mu a) \sinh(\mu a) \end{aligned}$$

and

$$\det\begin{pmatrix} \frac{1}{2\mu} & -\frac{1}{2\mu^3} \\ \frac{1}{2\mu} & \frac{1}{2\mu^3} \end{pmatrix} = \frac{1}{2\mu^4}.$$

Let $\varphi = \det M$. Then, φ becomes

$$\varphi(\mu) = -\varphi_0(\mu) - i\alpha\varphi_1(\mu), \tag{4.2}$$

where

$$\varphi_0(\mu) = \frac{1}{2\mu}\sin(\mu a)\cosh(\mu a) + \frac{1}{2\mu}\cos(\mu a)\sinh(\mu a),$$
$$\varphi_1(\mu) = \sin(\mu a)\sinh(\mu a).$$

In order to find zeros of $\varphi(\mu)$, we first find the zeros of $\varphi_0(\mu)$.
As in the paper of M. Möller and V. Pivorvachik, zero is a zero of double multiplicity. The nonzero zeros of $\varphi_0(\mu)$ are given by those $\mu \neq 0$ for which $\sin(\mu a)\sinh(\mu a) = 0$. The non-zero zeros $\sin(\mu a)$ are simple and are given by:

$$\tilde{\mu}_k^+ = k\frac{\pi}{a}, \quad \tilde{\mu}_k^- = -k\frac{\pi}{a}, \quad k = 1, 2, \ldots$$

As $\sin(i\gamma a) = i\sinh(\gamma a)$ the non-zero zeros of $\sinh(\mu a)$ are simple and given by

$$\tilde{\mu}_{-k}^+ = ik\frac{\pi}{a}, \quad \tilde{\mu}_{-k}^- = -ik\frac{\pi}{a}, \quad k = 1, 2, \ldots,$$

which shows that zeros of φ_0, counted with multiplicity, are $\tilde{\mu}_k^\pm$ and $\tilde{\mu}^\pm_{-k}$, $k \in \mathbb{Z}$.
Let $\varphi_2(\mu) = \tanh(\mu a) + \tan(\mu a)$. Since

$$\tan\left(\left(\frac{j\pi}{a} + i\gamma\right)a\right) = \tan(i\gamma a) = -i\tanh(\gamma a) \in i\mathbb{R} \cup \{\infty\}$$

for $j \in \mathbb{Z}$, $\gamma \in \mathbb{R}$ we have

$$|\tan(\mu a)| \leq 1 \quad \text{for} \quad \mu = \frac{j\pi}{a} + i\gamma, \quad \gamma \in \mathbb{R}.$$

For $\mu a = x + iy$ we conclude

$$\tanh(\mu a) = \frac{e^{x+iy} - e^{-x-iy}}{e^{x+iy} + e^{-x-iy}} \to \pm 1$$

uniformly in y as $x \to \pm\infty$. Hence there is a $j_0 \in \mathbb{N}$ such that $|j| \geq j_0$,

$$\left|\tanh\left(\left(\frac{j\pi}{a} + i\gamma\right)a\right) - (1)^{sgn(j)}\right| < \frac{1}{2}$$

for all $j \in \mathbb{Z}$ with $|j| \geq j_0$ and $\gamma \in \mathbb{R}$. These two estimate lead to,

$$\begin{aligned}\left|\tanh((\frac{j\pi}{a}+i\gamma)a)+\tan(\mu a)\right|-|\pm 1| &\leq \left|\tanh((\frac{j\pi}{a}+i\gamma)a)+\tan(\mu a)\pm 1\right| \\ &\leq |\tan(\mu a)|+\left|\tanh((\frac{j\pi}{a}+i\gamma)a)\pm 1\right| \\ &< \frac{1}{2}+1\end{aligned}$$

$$|\varphi_2(\mu)| < \frac{5}{2} \quad \text{for} \quad \mu = \frac{j\pi}{a}+i\gamma, \quad j \in \mathbb{Z}, \quad |j| \geq j_0, \quad \gamma \in \mathbb{R}.$$

By interchanging tanh and tan we obtain the same estimate for $\mu = \gamma + i\frac{j\pi}{a}$, $j \in \mathbb{Z}$, $\gamma \in \mathbb{R}$. Hence, for μ on the square with vertices $\pm j\frac{\pi}{a} \pm ij\frac{\pi}{a}$, where $j \in \mathbb{N}$,

$$\begin{aligned}\alpha|\varphi_0(\mu)| &= \frac{\alpha}{2\mu}|\varphi_2(\mu)||\sin(\mu a)\sinh(\mu a)| \\ &> \frac{5\alpha}{4|\mu|}|\sin(\mu a)\sinh(\mu a)| \\ &= \frac{5\alpha}{4|\mu|}|\varphi_1(\mu))| \\ &> \frac{\alpha}{|\mu|}|\varphi_1(\mu)|.\end{aligned}$$

Since both φ_1 and φ_0 are analytic functions in $\mathbb{C}$ with squares given by vertices $\pm j\frac{\pi}{a} \pm ij\frac{\pi}{a}$, then according to Rouché's theorem, if $|\varphi_0| > |\varphi_1|$ inside these squares, then φ has the same number of zeros inside these squares as φ_0.

Putting

$$\begin{aligned}\varphi_{00} &= \sin(\mu a), \\ \varphi_{01} &= \frac{-i}{2\alpha\mu}\left(\cos(\mu a)+\sin(\mu a)\tanh(\mu a)\right),\end{aligned}$$

we get

$$\varphi_{02} := \frac{\varphi(\mu)}{i\alpha\sinh(\mu a)} = -\varphi_{00} - \varphi_{01}.$$

Note that

$$\mu^{(0)} = k\frac{\pi}{a}, \qquad k \in \mathbb{Z},$$

are the zeros of φ_{00}. Let C_k^ρ be the circle of radius $\rho < \frac{\pi}{2a}$ with center at μ_k^{00}. Due to $\rho < \frac{\pi}{2a}$ these circles do not intersect. Since $|\varphi_{00}(\mu)|$ is periodic with period $\frac{\pi}{a}$, there is constant $p(\rho) > 0$ such that $|\varphi_{00}(\mu)| > p(\rho)$ for all $\mu \in C_k^\rho$ and all $k \in \mathbb{Z}$. We estimate φ_{01} on these circles for sufficiently large positive k:

$$|\cos(\mu a)| < C_2, \quad |\sin(\mu a)| < C_2, \quad |\tanh(\mu a)| < C_3$$

$\mu \in C_k^\rho$, where the constants C_j are independent of ρ and k is large enough.
Thus we obtain

$$|\varphi_{01}(\mu a)| < \frac{1}{|\mu|} C_2(1 + C_3)$$

for $\mu \in C_k^\rho$ and $k \geq k_0(\rho)$ large enough. Since the right hand side tends to 0 as $|\mu| \to \infty$, it follows that

$$|\varphi_{01}(\mu)| < |\varphi_{00}(\mu)|, \qquad (\mu \in C_k^\rho, \quad k > k_0(\rho)).$$

Applying Rouché's theorem we obtain that each of these circles (for k large enough) contains exactly one zero of φ_{02}, and thus exactly one zero of φ.

Inside sufficiently large squares φ and φ_0 have the same number of zeros, and there are zeros which have the same first order approximations as the zeros of φ_0.

We summarize the results of this section as follows:

Lemma 4.1. *For $g = 0$ there is a positive integer k_0 such that the eigenvalues $\tilde{\lambda}_k$, $k \in \mathbb{Z}$, of the problem (1.1)–(1.5), counted with multiplicity, can be enumerated in such a way that the eigenvalues λ_k are pure imaginary for $|k| < k_0$, $\lambda_{-k} = -\overline{\lambda_k}$ for $k \geq k_0$, where $\lambda_k = \mu_k^2$ with*

$$\mu_k = k\frac{\pi}{a} + o(k)$$

$k \in \mathbb{Z}$, as $k \to \infty$. In particular, there is an odd number of pure imaginary eigenvalues.

Proof. Consider the characteristic function $\varphi(\mu)$ above, then

$$\begin{aligned}
\varphi(i\overline{\mu}) &= -\varphi_0(i\overline{\mu}) - i\alpha\varphi_1(i\overline{\mu}) \\
&= -\left(\frac{1}{2i\overline{\mu}}\sin(i\overline{\mu}a)\cosh(i\overline{\mu}a) + \frac{1}{2i\overline{\mu}}\cos(i\overline{\mu}a)\sinh(i\overline{\mu}a)\right) \\
&\quad - i\alpha\sin(i\overline{\mu})\sinh(i\overline{\mu}a) \\
&= -\left(\frac{1}{2i\overline{\mu}}i\sinh(\overline{\mu}a)\cos(\overline{\mu}a) + \frac{1}{2i\overline{\mu}}\cosh(\overline{\mu}a)i\sin(\overline{\mu}a)\right) \\
&\quad - i\alpha i\sinh(\overline{\mu})i\sin(\overline{\mu}a) \\
&= -\varphi_0(\overline{\mu}) + i\alpha\varphi_1(\overline{\mu}) = \overline{\varphi(\mu)}.
\end{aligned}$$

Hence, if $\lambda = \mu^2$ is an eigenvalue of (1.1) - (1.5) for $g = 0$, then also $(i\overline{\mu})^2 = -\overline{\mu}^2 = -\overline{\lambda}$ is an eigenvalue of the same multiplicity. Thus $\lambda_{-k} = \overline{\lambda_k}$.

□

5. Asymptotics of eigenvalues

We will use the results of [7, Chapter VIII] for asymptotic fundamental systems of differential equations discussed in this paper. [7, Chapter VIII] is concerned with regular two-point boundary eigenvalue problems for the n-th order λ-linear differential equation of the type $\mathbf{K}\eta = \lambda\mathbf{H}\eta$, where $\mathbf{K}$ and $\mathbf{H}$ are differential operators such that $\mathbf{K}$ is of higher order than $\mathbf{H}$ and $L^D := \mathbf{K} - \lambda\mathbf{H}$, where L^D is as defined previously. [7, Theorem 8.2.1], gives properties of the fundamental system of $L^D\eta$ in terms of that of $\mathbf{H}\eta$ where λ is replaced with μ^2 and establishes the existence of functions φ_r. [7, Theorem 8.2.1] is quoted below.

Theorem 5.1. *Suppose that $h_{n_0} = 1$, set $l = n - n_0$, and let $k \in \mathbb{N}$. Suppose that $k \geq max\{l, n_0 - 1\}$ if $n_0 > 0$. Suppose that*

α) $k_j \in L_{p'}(a,b)$ for $j = 0, \dots, n-1-k$ and $k_{n-1-j} \in W_{p'}^{k-j}(a,b)$ for $j = 0, \dots, mink-1, n-1$ if $n_0 = 0$,
β) $h_0, \dots, h_{n_0-1} \in W_{p'}^{k}(a,b)$, $k_0, \dots, k_{n_0-1} \in W_{p'}^{k-l}(a,b)$, and $k_{n-1-j} \in W_{p'}^{k-j}(a,b)$ for $j = 0, \dots, l-1$ if $n_0 > 0$. Let $\{\pi_1, \dots, \pi_{n_0}\} \subset W_{p'}^{k+n_0}(a,b)$ be a fundamental system of $\boldsymbol{H}\eta = 0$.
For sufficiently large λ the differential equation $\boldsymbol{K}\eta = \lambda^l\boldsymbol{H}\eta$ has the

fundamental system $\{\eta_1(\cdot,\lambda),\dots,\eta_n(\cdot,\lambda)\}$ *with the following properties:*

i) There are functions $\pi_{\nu r} \in W_{p'}^{k+n_0-lr}(a,b)$ $(1 \le \nu \le n_0, 1 \le r \le [\frac{k}{l}])$ *such that*

$$\begin{aligned}
(5.1)\ \eta_\nu^{(\mu)}(\cdot,\lambda) &= \pi_\nu^{(\mu)} + \sum_{r=1}^{[\frac{k}{l}]} \lambda^{-lr}\pi_{\nu r}^{(\mu)} + \{o(\lambda^{-k})\}_\infty \\
&\quad (\nu = 1,\dots,n_0; \mu = 0,\dots,n_0-1) \\
(5.2)\ \eta_\nu^{(\mu)}(\cdot,\lambda) &= \pi_\nu^{(\mu)} + \sum_{r=1}^{[\frac{k-\mu+n_0-1}{l}]} \lambda^{-lr}\pi_{\nu r}^{(\mu)} + \{o(\lambda^{-k+\mu-n_0+1})\}_\infty \\
&\quad (\nu = 1,\dots,n_0; \mu = n_0,\dots,n-1),
\end{aligned}$$

ii) Set $\tilde{k} := min\{k, k+1-n_0\}$. *Let* $\omega_j = \exp\{\frac{2\pi i(j-1)}{j}\}$ $(j = 1,\dots,l)$. *There are functions* $\varphi_r \in W_{p'}^{k+1-r}(a,b)$, $r = 0,\dots,\tilde{k}$ *such that* φ_0 *is the solution of the initial value problem*

$$(5.3) \qquad \varphi_0' - \frac{1}{l}(h_{n_0-1} - k_{n-1})\varphi_0 = 0, \quad \pi_0(a) = 1,$$

and

$$\begin{aligned}
(5.4)\ \eta_\nu^{(\mu)}(x,\lambda) &= \left[\frac{d^\mu}{dx^\mu}\right]\{\sum_{r=0}^{\tilde{k}}(\lambda\omega_{\nu-n_0})^{-r}\varphi_r(x)e^{\lambda\omega_{\nu-n_0}(x-a)}\} \\
&\quad +\{o(\lambda^{-\tilde{k}+\mu})\}_\infty e^{\lambda\omega_{\nu-n_0}(x-a)} \\
&\qquad (\nu = n_0+1,\dots,n; \mu = n_0,\dots,n-1),
\end{aligned}$$

where $\left[\frac{d^\mu}{dx^\mu}\right]$ *means that we omit those terms of the Leibniz expansion which contain a function* $\varphi_r^{(j)}$ *with* $j > \tilde{k} - r$.

In order to apply the theorem above we need to replace λ with μ^2 and (1.1) has an asymptotic fundamental system $\{\eta_1,\eta_2,\eta_3,\eta_4\}$ of the form (we write a formula which also gives the derivatives, because we need them for the characteristic determinant)

$$(5.5) \qquad \eta_\nu^{(j)}(x,\mu) = \delta_{\nu,j}(x,\mu)e^{\mu i^{\nu-1}x},$$

where

(5.6)

$$\delta_{\nu,j}(x,\mu) = \frac{d^j}{dx^j}\left\{\sum_{r=0}^{k}(\mu i^{\nu-1})^{-r}\varphi_r(x)e^{\mu i^{\nu-1}x}\right\}e^{-\mu i^{\nu-1}x} + o(\mu^{-k+j}),$$

$k = 0, 1, \ldots,$ and k can be chosen to be sufficiently large if g is sufficiently regular. Since the coefficient of $y^{(3)}$ in (1.1) is zero, [7, (8.2.3)] immediately gives $\varphi_0(x) = 1$. Further functions $\varphi_1, \varphi_2, \cdots \in W_2^{k+1-r}(0, a)$ are determined below. We then recall that

$$W_0^{(0)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ i\alpha & i\alpha & i\alpha & i\alpha \end{pmatrix}.$$

Where it now follows that the characteristic function $D(\mu)$ of (1.1)–(1.5) is the determinant of the associated characteristic matrix given by

$$M(\lambda) = W_0^{(0)}(\mu)Y(0, \mu) + W_0^{(1)}(\mu)Y(a, \mu)$$

$$= \begin{pmatrix} \delta_{1,0}(0,\mu) & \delta_{2,0}(0,\mu) & \delta_{3,0}(0,\mu) & \delta_{4,0}(0,\mu) \\ \delta_{1,2}(0,\mu) & \delta_{2,2}(0,\mu) & \delta_{3,2}(0,\mu) & \delta_{4,2}(0,\mu) \\ \delta_{1,2}(a,\mu)e^{\mu a} & \delta_{2,2}(a,\mu)e^{i\mu a} & \delta_{3,2}(a,\mu)e^{-\mu a} & \delta_{4,2}(a,\mu)e^{-i\mu a} \\ [\delta_{1,1}(a,\mu) + i\alpha\mu^2\delta_{1,0}(a,\mu)]e^{\mu a} & [\delta_{2,1}(a,\mu) + i\alpha\mu^2\delta_{2,0}(a,\mu)]e^{i\mu a} & [\delta_{3,1}(a,\mu) + i\alpha\mu^2\delta_{3,0}(a,\mu)]e^{-\mu a} & [\delta_{4,1}(a,\mu) + i\alpha\mu^2\delta_{4,0}(a,\mu)]e^{-i\mu a} \end{pmatrix}$$

where $Y(x, \mu) = (\eta_j^{(i-1)}(x, \mu))_{i,j=1}^4$ is the fundamental matrix associated with the fundamental system $\{\eta_1, \eta_2, \eta_3, \eta_4\}$.

For the case $g = 0$ we already know the asymptotic distribution of the eigenvalues, see Lemma 4.1. Denote the corresponding characteristic function D by D_0. Due to the Birkhoff regularity, g only influences lower order terms in D, and therefore it follows from the estimates in [7, Appendix A.2] that, away from small disks around the zeros of D_0, $|D(\lambda) - D_0(\lambda)| < |D_0(\lambda)|$ if $|\lambda|$ is sufficiently large. The function $D(\lambda)$ is not analytic, but this estimate extends to the analytic equivalents with, e. g., a fundamental system y_j, $j = 1, \ldots, 4$, with $y_j^{(m)}(0) = \delta_{j,m+1}$ for $m = 0, \ldots, 3$, since changing initial conditions results in multiplying the fundamental matrix with the same λ dependent matrices so that both D and D_0 are multiplied by the same matrix. Hence applying Rouché's theorem both to large circles centered at zero avoiding the small disks and to the boundaries of the small discs which are sufficiently far away from 0, it follows that the eigenvalue problem for general g have the same asymptotic distribution as for $g = 0$. Hence Lemma 4.1 leads to

Lemma 5.2. *For $g \in C^1[0,a]$ there is a positive integer k_0 such that the eigenvalues λ_k, $k \in \mathbb{Z}$, of the problem (1.1)–(1.5), counted with multiplicity, can be enumerated in such a way that the eigenvalues λ_k are pure imaginary for $|k| < k_0$, $\lambda_{-k} = -\overline{\lambda_k} + o(1)$ for $k \geq k_0$, where $\lambda_k = \mu_k^2$ with*

$$\mu_k = k\frac{\pi}{a} + o(k)$$

as $k \to \infty$. In particular, there is an odd number of pure imaginary eigenvalues.

In [7, Appendix A] an estimate below of $D(\mu)$ has been given. With the methods used there one can also obtain more precise estimates of the location of the zeros of D. To this end we first observe that D has the form

$$D(\mu) = \sum_{m=1}^{5} \psi_m(\mu) e^{\omega_m \mu a},$$

where the $\psi_j(\mu)$ are polynomials in $\delta_{\nu,k}(0,\mu)$ and $\delta_{\nu,k}(a,\mu)$ and ω_m is the sum of the powers of the exponential in $\eta_\nu^{(j)}(a,\mu) = \delta_{\nu,j}(a,\mu)e^{\mu i^{\nu-1}a}$ when computing the characteristic function $D(\lambda)$. For $m = 1, ..., 5$ we have $\omega_1 = 1 + i$, $\omega_2 = -1 + i$, $\omega_3 = -1 - i$, $\omega_4 = 1 - i$, $\omega_5 = 0$. For example, we can write

$$D_1(\mu) := D(\mu)e^{-\omega_1 \mu a} = \psi_1(\mu) + \sum_{m=2}^{5} \psi_m(\mu) e^{(\omega_m - \omega_1)\mu a}.$$

It now follows from $\omega_2 - \omega_1 = -2$, $\omega_3 - \omega_1 = -2-2i$, $\omega_4 - \omega_1 = -2i$, $\omega_5 - \omega_1 = -1 - i$, that for $\arg\mu \in (-\frac{3\pi}{8}, \frac{\pi}{8})$ we have $|e^{(\omega_m - \omega_1)\mu a}| \leq e^{-\sin\frac{\pi}{8}|\mu|a}$ for $m = 2, 3, 5$, and these terms therefore can be absorbed by $\psi_1(\mu)$ as they are of the form $o(\mu^{-s})$ for any integer s. This estimate holds since $arg(\omega_m - \omega_1) \in \{-\frac{3}{4}\pi, \pi\}$. Hence, in the sector $\arg\mu \in (-\frac{3\pi}{8}, \frac{\pi}{8})$,

$$\begin{aligned} D_1(\mu) &= \psi_1(\mu) + \psi_4(\mu)e^{(\omega_4 - \omega_1)\mu a} \\ &= \psi_1(\mu) + \psi_4(\mu)e^{-2i\mu a}. \end{aligned} \tag{5.7}$$

This will be used to find the eigenvalue asymptotics along the positive real axis. Our assumption has to be that ψ_1 and ψ_2 have nonzero terms

apart from the o-terms. But this condition is satisfied as we already know that the problem is Birkhoff regular, see Theorem 2.3.
Due to the symmetry of the problem, the asymptotics on the other half-axes are not explicitly needed. We now want to find the asymptotics

$$\mu_k = k\frac{\pi}{a} + \tau_k, \quad \tau_k = \sum_{m=0}^{n} \tau_{k,m} k^{-m} + o(k^{-n}), \quad k = 1, 2, \ldots . \tag{5.8}$$

where will find $\tau_{k,0}$ and $\tau_{k,1}$ from the expansion of $D_1(\mu)$.
Next we find the terms $\tau_{k,0}$, $\tau_{k,1}$ and $\tau_{k,2}$ in the expansion of τ_k in (5.8).
We calculate

$$\begin{aligned}
\psi_1(\mu) &= \begin{vmatrix} \delta_{3,0}(0,\mu) & \delta_{4,0}(0,\mu) \\ \delta_{3,2}(0,\mu) & \delta_{4,2}(0,\mu) \end{vmatrix} \cdot \\
&\quad \begin{vmatrix} \delta_{1,2}(a,\mu) & \delta_{2,2}(a,\mu) \\ \delta_{1,1}(a,\mu) + i\alpha\mu^2\delta_{1,0}(a,\mu) & \delta_{2,1}(a,\mu) + i\alpha\mu^2\delta_{2,0}(a,\mu) \end{vmatrix} \\
&= [\delta_{3,0}(0,\mu)\delta_{4,2}(0,\mu) - \delta_{4,0}(0,\mu)\delta_{3,2}(0,\mu)] \\
&\quad \times \ [\delta_{1,2}(a,\mu)[\delta_{2,1}(a,\mu) + i\alpha\mu^2\delta_{2,0}(a,\mu)] \\
&\quad - \delta_{2,2}(a,\mu)[\delta_{1,1}(a,\mu) + i\alpha\mu^2\delta_{1,0}(a,\mu)]], \\
\psi_4(\mu) &= \begin{vmatrix} \delta_{2,0}(0,\mu) & \delta_{3,0}(0,\mu) \\ \delta_{2,2}(0,\mu) & \delta_{3,2}(0,\mu) \end{vmatrix} \cdot \\
&\quad \begin{vmatrix} \delta_{1,2}(a,\mu) & \delta_{4,2}(a,\mu) \\ \delta_{1,1}(a,\mu) + i\alpha\mu^2\delta_{1,0}(a,\mu) & \delta_{4,1}(a,\mu) + i\alpha\mu^2\delta_{4,0}(a,\mu) \end{vmatrix} \\
&= [\delta_{2,0}(0,\mu)\delta_{3,2}(0,\mu) - \delta_{3,0}(0,\mu)\delta_{2,2}(0,\mu)] \\
&\quad \times \ [\delta_{1,2}(a,\mu)[\delta_{4,1}(a,\mu) + i\alpha\mu^2\delta_{4,0}(a,\mu)] \\
&\quad - \delta_{4,2}(a,\mu)[\delta_{1,1}(a,\mu) + i\alpha\mu^2\delta_{1,0}(a,\mu)]].
\end{aligned}$$

A straightforward but lengthy calculation see appendix 1, leads to $\psi_1(\mu) = -4i\alpha\mu^6 + O(\mu^5)$ and $\psi_4(\mu) = 4i\alpha\mu^6 + O(\mu^5)$ as $\mu \to \infty$.
Hence we can write

$$\mu^{-6}\psi_1(\mu) = -4i\alpha + \psi_{1,1}\mu^{-1} + \psi_{1,2}\mu^{-2} + o(\mu^{-2}) \tag{5.9}$$

$$\mu^{-6}\psi_4(\mu) = 4i\alpha + \psi_{4,1}\mu^{-1} + \psi_{4,2}\mu^{-2} + o(\mu^{-2}). \tag{5.10}$$

$$
\begin{aligned}
\frac{1}{\mu_k} &= \left(k\frac{\pi}{a} + \sum_{m=0}^{n} \tau_{k,m} k^{-m} + o(k^{-n})\right)^{-1} \\
&= \frac{a}{\pi k}\left(1 + \frac{a}{\pi k}\tau_{k,0} + \frac{a}{\pi k^2}\tau_{k,1} + \frac{a}{\pi k^3}\tau_{k,2}\right. \\
&\quad \left. + \frac{a}{\pi k}\left(\sum_{m=3}^{n} \tau_{k,m} k^{-m} + o(k^{-n})\right)\right)^{-1} \\
&= \frac{a}{\pi k}\left(1 - \left(\frac{a}{\pi k}\tau_{k,0} + \frac{a}{\pi k^2}\tau_{k,1} + \frac{a}{\pi k^3}\tau_{k,2}\right.\right. \\
&\quad \left.\left. + \frac{a}{\pi k}\left(\sum_{m=3}^{n} \tau_{k,m} k^{-m} + o(k^{-n})\right)\right) + \dots\right) \\
&= \frac{a}{\pi k}\left(1 - \left(\frac{a}{\pi k}\tau_{k,0} + \frac{a}{\pi k^2}\tau_{k,1} + \frac{a}{\pi k^3}\tau_{k,2}\right.\right. \\
&\quad \left. + \frac{a}{\pi k}\left(\sum_{m=3}^{n} \tau_{k,m} k^{-m} + o(k^{-n})\right)\right) \\
&\quad \left. + \left(\frac{a}{\pi k}\tau_{k,0} + \dots\right)^2 + \dots\right) \\
&= \frac{a}{\pi k}\left(1 - \frac{a}{\pi k}\tau_{k,0} - \left(\frac{a}{\pi}\tau_{k,1} - (\frac{a}{\pi})^2 \tau_{k,0}^2\right)\frac{1}{k^2} + o(k^{-2})\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\mu_k^2} &= \left(\frac{a}{\pi k}\right)^2 \Big(1 - \Big(\frac{a}{\pi k}\tau_{k,0} + \frac{a}{\pi k^2}\tau_{k,1} + \frac{a}{\pi k^3}\tau_{k,2} \\
&\quad + \frac{a}{\pi k}\left(\sum_{m=3}^{n} \tau_{k,m} k^{-m} + o(k^{-n})\right)\Big) \\
&\quad + \Big(\frac{a}{\pi k}\tau_{k,0} + \ldots\Big)^2 + \ldots\Big)^2 \\
&= \left(\frac{a}{\pi k}\right)^2 \Big(1 - 2\Big(\frac{a}{\pi k}\tau_{k,0} + \frac{a}{\pi k^2}\tau_{k,1} + \frac{a}{\pi k^3}\tau_{k,2} \\
&\quad + \frac{a}{\pi k}\left(\sum_{m=3}^{n} \tau_{k,m} k^{-m} + o(k^{-n})\right)\Big) \\
&\qquad + \Big(\frac{a}{\pi k}\tau_{k,0} + \frac{a}{\pi k^2}\tau_{k,1} + \frac{a}{\pi k^3}\tau_{k,2} \\
&\quad + \frac{a}{\pi k}\left(\sum_{m=3}^{n} \tau_{k,m} k^{-m} + o(k^{-n})\right)\Big)^2 \\
&\qquad + 2\Big(\frac{a}{\pi k}\tau_{k,0} + \frac{a}{\pi k^2}\tau_{k,1} + \frac{a}{\pi k^3}\tau_{k,2} \\
&\quad + \frac{a}{\pi k}\left(\sum_{m=3}^{n} \tau_{k,m} k^{-m} + o(k^{-n})\right)\Big)^2 + \ldots\Big) \\
&= \left(\frac{a}{\pi k}\right)^2 + o(k^{-2})
\end{aligned}
$$

and

$$
\begin{aligned}
e^{-2i\tau_k a} &= \exp\left(-2ia\left(\tau_{k,0} + \frac{\tau_{k,1}}{k} + \frac{\tau_{k,2}}{k^2} + \frac{\tau_{k,3}}{k^3} + o(k^{-3})\right)\right) \\
&= e^{-2ia\tau_{k,0}} \exp\left(-2ia\frac{\tau_{k,1}}{k}\right) \exp\left(-2ia\frac{\tau_{k,2}}{k^2}\right) \\
&\qquad \exp\left(-2ia\frac{\tau_{k,3}}{k^3} + o(k^{-3})\right) \\
&= e^{-2ia\tau_{k,0}} \left(1 - 2ia\frac{\tau_{k,1}}{k} - 2a^2\frac{\tau_{k,1}^2}{k^2} + o(k^{-2})\right) \\
&\quad \left(1 - 2ia\frac{\tau_{k,2}}{k^2} + o(k^{-2})\right)\left(1 - 2ia\frac{\tau_{k,3}}{k^3} + o(k^{-3})\right) \\
&= e^{-2ia\tau_{k,0}} \left(1 - 2ia\frac{\tau_{k,1}}{k} + (-2a^2\tau_{k,1}^2 - 2ia\tau_{k,2})\frac{1}{k^2} + o(k^{-2})\right)
\end{aligned}
$$

implies

$$\begin{aligned}
\mu_k^{-6}\psi_1(\mu_k) &= -4i\alpha + \psi_{1,1}\mu_k{}^{-1} + \psi_{1,2}\mu^{-2} + o(\mu^{-2}) \\
&= -4i\alpha + \psi_{1,1}(\frac{a}{\pi k} - (\frac{a}{\pi k})^2\tau_{k,0} + \ldots) \\
&\quad + \psi_{1,2}((\frac{a}{\pi k})^2 + \ldots) + o(k^{-2}) \\
\mu_k^{-6}\psi_4(\mu_k) &= 4i\alpha + \psi_{4,1}\mu_k{}^{-1} + \psi_{4,2}\mu^{-2} + o(\mu^{-2}) \\
&= 4i\alpha + \psi_{4,1}(\frac{a}{\pi k} - (\frac{a}{\pi k})^2\tau_{k,0} + \ldots) \\
&\quad + \psi_{4,2}((\frac{a}{\pi k})^2 + \ldots) + o(k^{-2}).
\end{aligned}$$

Since $D_1(\mu_k) = 0$ can be written as

$$\begin{aligned}
0 &= \mu_k^{-6}\psi_1(\mu_k) + \mu_k^{-6}\psi_4(\mu_k)e^{-2i\tau_k a} \\
&= -4i\alpha + \psi_{1,1}(\frac{a}{\pi k} - (\frac{a}{\pi k})^2\tau_{k,0} + \ldots) + \psi_{1,2}((\frac{a}{\pi k})^2 + \ldots) + o(k^{-2}) \\
&\quad + \left(4i\alpha + \psi_{4,1}(\frac{a}{\pi k} - (\frac{a}{\pi k})^2\tau_{k,0} + \ldots) + \psi_{4,2}((\frac{a}{\pi k})^2 + \ldots) + o(k^{-2})\right) \\
&\quad\quad e^{-2ia\tau_{k,0}}\left(1 - 2ia\frac{\tau_{k,1}}{k} + (-2a^2\tau_{k,1}^2 - 2ia\tau_{k,2})\frac{1}{k^2} + o(k^{-2})\right),
\end{aligned}$$

where after a lengthy but straightforward calculation of

(5.11)
$$\psi_{1,1}(\mu_k) = -2(1+i) + 4\alpha(1+i)(\varphi_1(0) - \varphi_1(a))$$

(5.12)
$$\psi_{4,1}(\mu_k) = 2(1-i) - 4\alpha(1-i)(\varphi_1(a) - \varphi_1(0)),$$

(5.13)
$$\psi_{1,2}(\mu_k) = -4(\varphi_1(a) - \varphi_1(0)) + 8\alpha\varphi_1(0)\varphi_1(a) - 4\alpha(\varphi_1^2(0) + \varphi_1^2(a)) \quad \text{and}$$

(5.14)
$$\psi_{4,2}(\mu_k) = 4(\varphi_1(a) - \varphi_1(0)) + 8\alpha\varphi_1(0)\varphi_1(a) - 4\alpha(\varphi_1^2(a) + \varphi_1^2(0)).$$

A comparison of coefficients of k^0, k^{-1} and k^{-2} in $D_1(\mu_k) = 0$ leads to

$$
\begin{aligned}
k^0 : 0 &= -4i\alpha + 4i\alpha e^{-2ia\tau_{k,0}} \\
1 &= e^{-2ia\tau_{k,0}}
\end{aligned}
$$

$$
k^{-1} : 0 = \psi_{1,1}\frac{a}{\pi} - 4i\alpha(2ia\tau_{k,1})e^{-2ia\tau_{k,0}} + \psi_{4,1}\frac{a}{\pi}e^{-2ia\tau_{k,0}}
$$

$$
\begin{aligned}
k^{-2} : 0 &= (\frac{a}{\pi})^2(\psi_{1,2} - \psi_{1,1}\tau_{k,0}) + 4i\alpha(-2a^2\tau_{k,1}^2 - 2ia\tau_{k,2})e^{-2ia\tau_{k,0}} \\
&\quad - 2ia(\frac{a}{\pi})\psi_{4,1}\tau_{k,1}e^{-2ia\tau_{k,0}} - (\frac{a}{\pi})^2(\psi_{4,1}\tau_{k,0} - \psi_{4,2})e^{-2ia\tau_{k,0}} \\
&= (\frac{a}{\pi})^2((\psi_{1,2} - \psi_{1,1}\tau_{k,0}) - (\psi_{4,1}\tau_{k,0} - \psi_{4,2})e^{-2ia\tau_{k,0}}) \\
&\quad - 2i(4\alpha a^2\tau_{k,1}^2 + \frac{a^2}{\pi}\psi_{4,1}\tau_{k,1})e^{-2ia\tau_{k,0}} + 8\alpha a\tau_{k,2}e^{-2ia\tau_{k,0}},
\end{aligned}
$$

thus

$$
\tau_{k,0} = 0,
$$

$$
\begin{aligned}
\tau_{k,1} &= -\frac{1}{8\pi\alpha}(\psi_{1,1} + \psi_{4,1}) \\
&= -\frac{1}{8\pi\alpha}(-4i + 8\alpha(\varphi_1(0) - \varphi_1(a))) \\
&= \frac{i}{2\pi\alpha} + \frac{1}{\pi}\varphi_1(a) \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
\tau_{k,2} &= \frac{1}{8a\alpha}\left((\frac{a}{\pi})^2((\psi_{4,1}\tau_{k,0} - \psi_{4,2}) - (\psi_{1,2} - \psi_{1,1}\tau_{k,0}))\right) \\
&\quad + \frac{1}{8a\alpha}\left(2i(4\alpha a^2\tau_{k,1}^2 + \frac{a^2}{\pi}\psi_{4,1}\tau_{k,1})\right) \\
&= \frac{1}{8a\alpha}\left(-\frac{a^2}{\pi^2}(\psi_{4,2} + \psi_{1,2})\right) \\
&\quad + \frac{i}{4a\alpha}\left(4a^2\alpha\tau_{k,1}^2 + \frac{a^2}{\pi}\psi_{4,1}\tau_{k,1}\right) \\
&= \frac{1}{8a\alpha}\left(-\frac{a^2}{\pi^2}(16\alpha\varphi_1(0)\varphi_1(a) - 8\alpha(\varphi_1^2(0) + \varphi_1^2(a)))\right) \\
&\quad ia\tau_{k,1}^2 + \frac{ia}{4\pi\alpha}\left(2(1-i) - 4\alpha(1-i)(\varphi_1(a) - \varphi_1(0))\right)\tau_{k,1}.
\end{aligned}
$$

Finally, for all $\tau_{k,1}$ and $\tau_{k,2}$, we need to find the function φ_1 and others as needed. According to [7, (8.2.45)],

$$\varphi_{\nu r} = \varepsilon^{\mathsf{T}} V Q_{22}^{[r]} \varepsilon_{\nu - n_0}$$

where $\varepsilon^{\mathsf{T}} = (1,1,1,1)$, and the 4×4 matrix function $Q^{[r]}$ is a solution of (see [7, (8.2.33) and (8.2.34)] where $Q^{[0]} = I_4$ by [7, (8.2.18)])

$$\Omega_4 Q_{22}^{[r]} - Q_{22}^{[r]} \Omega_4 = Q_{22}^{[r-1]\prime} + \frac{1}{4} \sum_{j=1}^{4} k_{n-1-j} \Omega_4 \epsilon \epsilon^{\mathsf{T}} \Omega_4^{-1-j} Q_{22}^{[r-1-j]}, \tag{5.15}$$

$$Q_{22}^{[k]\prime} + \frac{1}{4} \sum_{j=1}^{4} k_{n-1-j} \Omega_4 \epsilon \epsilon^{\mathsf{T}} \Omega_4^{-1-j} Q_{22}^{[k-j]}. \tag{5.16}$$

Here $\Omega_4 = \operatorname{diag}(\omega_1, \omega_2, \omega_3, \omega_4) = \operatorname{diag}(1, i, -1, -i)$.Let $Q_{22}^{[r]} = q_{ij}^{[r]}$, whence

$$q_{ij}^{[1]} = \begin{cases} \frac{1}{4}\omega_i^{-1} \int_0^x g(t)\,dt & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$q_{ij}^{[2]} = \begin{cases} \frac{1}{2}(\frac{1}{4} \int_0^x g(t)\,dt)^2 + \frac{1}{4} g(x) \omega_1^{-2} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$q_{ij}^{[3]} = \begin{cases} \frac{1}{6}(\frac{1}{4} \int_0^x g(t)\,dt)^3 + \frac{1}{16} g(x) \int_0^x g(t)\,dt & \text{if } i = j = 1 \\ 0 & \text{if } i \neq j \end{cases}$$

and thus

$$\varphi_1(x) = \frac{1}{4} \int_0^x g(t)\;dt, \tag{5.17}$$

$$\varphi_2(x) = \frac{1}{2} \left(\frac{1}{4} \int_0^x g(t)\;dt \right)^2 + \frac{1}{4} g(x), \tag{5.18}$$

$$\varphi_3(x) = \frac{1}{6} \left(\frac{1}{4} \int_0^x g(t)\;dt \right)^3 + \frac{1}{16} g(x) \int_0^x g(t)\;dt \tag{5.19}$$

Altogether, we obtain

(5.20)

$$\tau_{k,1} = \frac{i}{2\pi\alpha} + \frac{1}{4\pi}\int_0^a g(x)\,dx$$

(5.21)

$$\begin{aligned}
\tau_{k,2} &= -\frac{a}{8\pi^2\alpha}(-8\alpha\varphi_1^2(a)) + ia\tau_{k,1}^2 \\
&\quad + \frac{a}{2\pi\alpha}(1+i)\tau_{k,1} - \frac{a}{\pi}(1+i)\varphi_1(a)\tau_{k,1} \\
&= \frac{a}{16\pi^2}\left(\int_0^a g(x)\,dx\right)^2 + ia\left(-\frac{1}{2i\pi\alpha} + \frac{1}{4\pi}\int_0^a g(x)\,dx\right)^2 \\
&\quad + \frac{a}{2\pi\alpha}(1+i)\left(-\frac{1}{2i\pi\alpha} + \frac{1}{4\pi}\int_0^a g(x)\,dx\right) \\
&\quad - \frac{a}{\pi}(1+i)\left(\frac{1}{4}\int_0^a g(x)\,dx\right)\left(-\frac{1}{2i\pi\alpha} + \frac{1}{4\pi}\int_0^a g(x)\,dx\right) \\
&= \frac{a}{16\pi^2}\left(\int_0^a g(x)\,dx\right)^2 + \frac{ia}{16\pi^2}\left(\int_0^a g(x)\,dx\right)^2 \\
&\quad - \frac{a}{16\pi^2}(1+i)\left(\int_0^a g(x)\,dx\right)^2 - \frac{2a}{8\pi^2\alpha}\left(\int_0^a g(x)\,dx\right) \\
&\quad + \frac{a}{8\pi^2\alpha}(1+i)\left(\int_0^a g(x)\,dx\right) + \frac{a}{8\pi^2\alpha}(1-i)\left(\int_0^a g(x)\,dx\right) \\
&\quad - \frac{ia}{4\pi^2\alpha^2} - \frac{a}{4\pi^2\alpha^2}(1-i) \\
&= -\frac{a}{4\pi^2\alpha^2}.
\end{aligned}$$

Theorem 5.3. *For $g \in C^1[0,a]$, there is a positive integer k_0 such that the eigenvalues λ_k, $k \in \mathbb{Z}$, of the problem (1.1)–(1.5), counted with multiplicity, can be enumerated in such a way that the eigenvalues λ_k are pure imaginary for $|k| < k_0$, $\lambda_{-k} = -\overline{\lambda_k} + o(1)$ for $k \geq k_0$, where $\lambda_k = \mu_k^2$ with*

$$\begin{aligned}
\mu_k = k\frac{\pi}{a} &+ \frac{1}{k}\left(\frac{i}{2\pi\alpha} + \frac{1}{4\pi}\int_0^a g(x)dx\right) \\
&+ \frac{1}{k^2} - \frac{a}{4\pi^2\alpha^2} + O(k^{-2})
\end{aligned}$$

as $k \to \infty$. In particular, there is an odd number of pure imaginary eigenvalues.

Appendix 1

$$\delta_{\nu,j}(x,\mu) = \frac{d^j}{dx^j}\left\{\sum_{r=0}^{k}(\mu i^{\nu-1})^{-r}\varphi_r(x)e^{\mu i^{\nu-1}x}\right\}e^{-\mu i^{\nu-1}x}+o(\mu^{-k+j}),\ k=0,1,\ldots.$$

$$\delta_{1,0}(x,\mu) = \left\{\sum_{r=0}^{k}\mu^{-r}\varphi_r(x)\right\} + o(\mu^{-k})$$

$$\begin{aligned}
\delta_{1,1}(x,\mu) &= \frac{d}{dx}\left\{\sum_{r=0}^{k}\mu^{-r}\varphi_r(x)e^{\mu x}\right\}e^{-\mu x} + o(\mu^{-k+1})\\
&= \sum_{r=0}^{k}\mu^{-r}\left(\mu\varphi_r(x)+\varphi_r'(x)\right) + o(\mu^{-k+1})\\
\delta_{1,2}(x,\mu) &= \frac{d^2}{dx^2}\left\{\sum_{r=0}^{k}\mu^{-r}\varphi_r(x)e^{\mu x}\right\}e^{-\mu x} + o(\mu^{-k+2})\\
&= \sum_{r=0}^{k}\mu^{-r}(\mu^2\varphi_r(x)+2\mu\varphi_r'(x)+\varphi_r''(x)) + o(\mu^{-k+2})\\
\delta_{2,0}(x,\mu) &= \left\{\sum_{r=0}^{k}(\mu i)^{-r}\varphi_r(x)\right\} + o(\mu^{-k})\\
\delta_{2,1}(x,\mu) &= \frac{d}{dx}\left\{\sum_{r=0}^{k}(\mu i)^{-r}\varphi_r(x)e^{\mu ix}\right\}e^{-\mu ix} + o(\mu^{-k+1})\\
&= \sum_{r=0}^{k}(\mu i)^{-r}((\mu i)\varphi_r(x)+\varphi_r'(x)) + o(\mu^{-k+1})\\
\delta_{2,2}(x,\mu) &= \left\{\sum_{r=0}^{k}(\mu i)^{-r}((\mu i)^2\varphi_r(x)+2(\mu i)\varphi_r'(x)+\varphi_r''(x))\right\} + o(\mu^{-k+2})\\
\delta_{3,0}(x,\mu) &= \left\{\sum_{r=0}^{k}(-\mu)^{-r}\varphi_r(x)\right\} + o(\mu^{-k})\\
\delta_{3,1}(x,\mu) &= \left\{\sum_{r=0}^{k}(-\mu)^{-r}((-\mu)\varphi_r(x)+\varphi_r'(x))\right\} + o(\mu^{-k+1})
\end{aligned}$$

$$\delta_{3,2}(x,\mu)=\left\{\sum_{r=0}^{k}(-\mu)^{-r}((-\mu)^2\varphi_r(x)+2(-\mu)\varphi_r'(x)+\varphi_r''(x))\right\}+o(\mu^{-k+2})$$
$$\delta_{4,0}(x,\mu)=\left\{\sum_{r=0}^{k}(-\mu i)^{-r}\varphi_r(x)\right\}+o(\mu^{-k})$$
$$\delta_{4,1}(x,\mu)=\left\{\sum_{r=0}^{k}(-\mu i)^{-r}((-\mu i)\varphi_r(x)+\varphi_r'(x))\right\}+o(\mu^{-k+1})$$
$$\delta_{4,2}(x,\mu)=\left\{\sum_{r=0}^{k}(-\mu i)^{-r}((-\mu i)^2\varphi_r(x)+2(-\mu i)\varphi_r'(x)+\varphi_r''(x))\right\}+o(\mu^{-k+2})$$

$$
\begin{aligned}
\psi_1(\mu) &= [\delta_{3,0}(0,\mu)\delta_{4,2}(0,\mu) - \delta_{4,0}(0,\mu)\delta_{3,2}(0,\mu)] \\
&\quad \times\ [\delta_{1,2}(a,\mu)[\delta_{2,1}(a,\mu) + i\alpha\mu^2\delta_{2,0}(a,\mu)] \\
&\quad - \delta_{2,2}(a,\mu)[\delta_{1,1}(a,\mu) + i\alpha\mu^2\delta_{1,0}(a,\mu)]] \\
&= \delta_{3,0}(0,\mu)\delta_{4,2}(0,\mu)\delta_{1,2}(a,\mu)[\delta_{2,1}(a,\mu) + i\alpha\mu^2\delta_{2,0}(a,\mu)] \\
&\quad - \delta_{3,0}(0,\mu)\delta_{4,2}(0,\mu)\delta_{2,2}(a,\mu)[\delta_{1,1}(a,\mu) + i\alpha\mu^2\delta_{1,0}(a,\mu)] \\
&\quad - \delta_{4,0}(0,\mu)\delta_{3,2}(0,\mu)\delta_{1,2}(a,\mu)[\delta_{2,1}(a,\mu) + i\alpha\mu^2\delta_{2,0}(a,\mu)] \\
&\quad + \delta_{4,0}(0,\mu)\delta_{3,2}(0,\mu)\delta_{2,2}(a,\mu)[\delta_{1,1}(a,\mu) + i\alpha\mu^2\delta_{1,0}(a,\mu)] \\
&= (1 + (-\mu)^{-1}\varphi_1(0) + (-\mu)^{-2}\varphi_2(0) + \ldots) \\
&\quad ((-\mu i)^2 + (-\mu i)\varphi_1(0) + 2\varphi_1'(0) + \varphi_2(0) + \ldots) \\
&\quad \times (\mu^2 + \mu\varphi_1(a) + 2\varphi_1'(a) + \varphi_2(a) + \ldots) \\
&\quad \times \left[((\mu i) + \varphi_1(a) + \ldots) + i\alpha\mu^2(1 + (\mu i)^{-1}\varphi_1(a) + (\mu i)^{-2}\varphi_2(a) + \ldots)\right] \\
&\quad - (1 + (-\mu)^{-1}\varphi_1(0) + (-\mu)^{-2}\varphi_2(0) + \ldots) \\
&\quad ((-\mu i)^2 + (-\mu)\varphi_1(0) + 2\varphi_1'(0) + \varphi_2(0) + \ldots) \\
&\quad \times ((\mu i)^2 + (\mu i)\varphi_1(a) + 2\varphi_1'(a) + \varphi_2(a) + \ldots) \\
&\quad \times \left[(\mu(a) + \varphi_1(a) + \ldots) + i\alpha\mu^2(1 + \mu^{-1}\varphi_1(a) + \mu^{-2}\varphi_2(a) + \ldots)\right] \\
&\quad - (1 + (-\mu i)^{-1}\varphi_1(0) + (-\mu i)^{-2}\varphi_2(0) + \ldots) \\
&\quad ((-\mu)^2 + (-\mu)\varphi_1(0) + 2\varphi_1'(0) + \varphi_2(0) + \ldots) \\
&\quad \times (\mu^2 + \mu\varphi_1(a) + 2\varphi_1'(a) + \varphi_2(a) + \ldots) \\
&\quad \times \left[((\mu i) + \varphi_1(a) + \ldots) + i\alpha\mu^2(1 + (\mu i)^{-1}\varphi_1(a) + (\mu i)^{-2}\varphi_2(a) + \ldots)\right] \\
&\quad + (1 + (-\mu i)^{-1}\varphi_1(0) + (-\mu i)^{-2}\varphi_2(0) + \ldots) \\
&\quad ((-\mu)^2 + (-\mu)\varphi_1(0) + 2\varphi_1'(0) + \varphi_2(0) + \ldots) \\
&\quad \times (\mu^2 + 2\mu\varphi_1(a) + 2\varphi_1'(a) + \varphi_2(a) + \ldots) \\
&\quad \times ((\mu i)^2 + (\mu i)\varphi_1(a) + 2\varphi_1'(a) + \varphi_2(a) + \ldots) \\
&\quad \times \left[(\mu(a) + \varphi_1(a) + \ldots) + i\alpha\mu^2(1 + (\mu i)^{-1}\varphi_1(a) + (\mu i)^{-2}\varphi_2(a) + \ldots)\right] \\
&= -4i\alpha\mu^6 + (-2(1+i) + 4\alpha(1+i)(\varphi_1(0) - \varphi_1(a)))\mu^5 \\
&\quad + (4(\varphi_1(0) - \varphi_1(a)) + 8\alpha\varphi_1(0)\varphi_1(a) - 4\alpha(\varphi_1^2(0) + \varphi_1^2(a)))\mu^4 + O(\mu^3)
\end{aligned}
$$

$$\begin{aligned}
\psi_4(\mu) &= [\delta_{2,0}(0,\mu)\delta_{3,2}(0,\mu) - \delta_{3,0}(0,\mu)\delta_{2,2}(0,\mu)] \\
&\quad \times\ [\delta_{1,2}(a,\mu)[\delta_{4,1}(a,\mu) + i\alpha\mu^2\delta_{4,0}(a,\mu)] \\
&\quad - \delta_{4,2}(a,\mu)[\delta_{1,1}(a,\mu) + i\alpha\mu^2\delta_{1,0}(a,\mu)]] \\
&= \delta_{2,0}(0,\mu)\delta_{3,2}(0,\mu)\delta_{1,2}(a,\mu)[\delta_{4,1}(a,\mu) + i\alpha\mu^2\delta_{4,0}(a,\mu)] \\
&\quad - \delta_{2,0}(0,\mu)\delta_{3,2}(0,\mu)\delta_{4,2}(a,\mu)[\delta_{1,1}(a,\mu) + i\alpha\mu^2\delta_{1,0}(a,\mu)] \\
&\quad - \delta_{3,0}(0,\mu)\delta_{2,2}(0,\mu)\delta_{1,2}(a,\mu)[\delta_{4,1}(a,\mu) + i\alpha\mu^2\delta_{4,0}(a,\mu)] \\
&\quad + \delta_{3,0}(0,\mu)\delta_{2,2}(0,\mu)\delta_{4,2}(a,\mu)[\delta_{1,1}(a,\mu) + i\alpha\mu^2\delta_{1,0}(a,\mu)] \\
&= (1 + (\mu i)^{-1}\varphi_1(0) + (\mu i)^{-2}\varphi_2(0) + \ldots) \\
&\quad ((-\mu)^2 + (-\mu)\varphi_1(0) + 2\varphi_1'(0) + \varphi_2(0) + \ldots) \\
&\quad \times (\mu^2 + \mu\varphi_1(a) + 2\varphi_1'(a) + \varphi_2(a) + \ldots) \\
&\quad \times \left[((-\mu i) + \varphi_1(a) + \ldots) + i\alpha\mu^2(1 + (-\mu i)^{-1}\varphi_1(a) + (-\mu i)^{-2}\varphi_2(a) + \ldots)\right] \\
&\quad - (1 + (\mu i)^{-1}\varphi_1(0) + (\mu i)^{-2}\varphi_2(0) + \ldots) \\
&\quad ((-\mu)^2 + (-\mu)\varphi_1(0) + 2\varphi_1'(0) + \varphi_2(0) + \ldots) \\
&\quad \times ((-\mu i)^2 + (-\mu i)\varphi_1(a) + 2\varphi_1'(a) + \varphi_2(a) + \ldots) \\
&\quad \times \left[(\mu + \varphi_1(a) + \ldots) + i\alpha\mu^2(1 + \mu^{-1}\varphi_1(a) + \mu^{-2}\varphi_2(a) + \ldots)\right] \\
&\quad - (1 + (-\mu)^{-1}\varphi_1(0) + (-\mu)^{-2}\varphi_2(0) + \ldots) \\
&\quad ((\mu i)^2 + (\mu i)\varphi_1(0) + 2\varphi_1'(0) + \varphi_2(0) + \ldots) \\
&\quad \times (\mu^2 + \mu\varphi_1(a) + 2\varphi_1'(a) + \varphi_2(a) + \ldots) \\
&\quad \times \left[((-\mu i) + \varphi_1(a) + \ldots) + i\alpha\mu^2(1 + (-\mu i)^{-1}\varphi_1(a) + (-\mu i)^{-2}\varphi_2(a) + \ldots)\right] \\
&\quad + (1 + (-\mu)^{-1}\varphi_1(0) + (-\mu)^{-2}\varphi_2(0) + \ldots) \\
&\quad ((\mu i)^2 + (\mu i)\varphi_1(0) + 2\varphi_1'(0) + \varphi_2(0) + \ldots) \\
&\quad \times ((-\mu i)^2 + (-\mu i)\varphi_1(a) + 2\varphi_1'(a) + \varphi_2(a) + \ldots) \\
&\quad \times \left[(\mu + \varphi_1(a) + \ldots) + i\alpha\mu^2(1 + \mu^{-1}\varphi_1(a) + \mu^{-2}\varphi_2(a) + \ldots)\right] \\
&= 4i\alpha\mu^6 + (2(1-i) - 4\alpha(1-i)(\varphi_1(a) - \varphi_1(0)))\mu^5 \\
&\quad + (4(\varphi_1(a) - \varphi_1(0)) + 8\alpha\varphi_1(0)\varphi_1(a) - 4\alpha(\varphi_1^2(a) + \varphi_1^2(0)))\mu^4 + O(\mu^3)
\end{aligned}$$

From (5.5)

$$\eta_\nu^{(j)}(x,\mu) = \delta_{\nu,j}(x,\mu)e^{\mu i^{\nu-1}x},$$

Substituting into (4.1) for $j = 0$ and λ replaced with μ^2 we get

$$\frac{d^4}{dx^4}\left\{\delta_\nu(x,\mu)e^{\mu i^{\nu-1}x}(\mu,x)\right\} = \mu^4\delta_\nu(x,\mu)e^{\mu i^{\nu-1}x}(\mu,x),$$
$$\Rightarrow \ \delta_\nu(x,\mu) \text{ is a non-zero constant in } x,\ \delta_\nu(\mu).$$

Thus $\{\eta_1,\eta_2,\eta_3,\eta_4\}$ is given by $\{\delta_1(\mu)e^{\mu x},\delta_2(\mu)e^{\mu ix},\delta_3(\mu)e^{-\mu x},\delta_4(\mu)e^{-\mu ix}\}$. Choose all $\delta_\nu = 1$ and $\{\eta_\nu^{(j)}\}$ for $j=1,2$. becomes $\{\mu e^{\mu x}, i\mu e^{\mu ix}, -\mu e^{-\mu x}, -i\mu e^{-\mu ix}\}$ and $\{\mu^2 e^{\mu x}, -\mu^2 e^{\mu ix}, \mu^2 e^{-\mu x}, -\mu^2 e^{-\mu ix}\}$ respectively. A comparison of the the two systems to (5.5), we have $\{\delta_{\nu,j}\}$ equal to $\{\mu, i\mu, -\mu, -i\mu\}$ and $\{\mu^2,-\mu^2,\mu^2,-\mu^2\}$ for $j=1$ and 2.

$$\begin{aligned}
\psi_1(\mu) &= \begin{vmatrix}\delta_{3,0}(0,\mu) & \delta_{4,0}(0,\mu)\\ \delta_{3,2}(0,\mu) & \delta_{4,2}(0,\mu)\end{vmatrix}\cdot\\
&\quad \begin{vmatrix}\delta_{1,2}(a,\mu) & \delta_{2,2}(a,\mu)\\ \delta_{1,1}(a,\mu)+i\alpha\mu^2\delta_{1,0}(a,\mu) & \delta_{2,1}(a,\mu)+i\alpha\mu^2\delta_{2,0}(a,\mu)\end{vmatrix}\\
&= [\delta_{3,0}(0,\mu)\delta_{4,2}(0,\mu)-\delta_{4,0}(0,\mu)\delta_{3,2}(0,\mu)]\\
&\quad\times\ [\delta_{1,2}(a,\mu)[\delta_{2,1}(a,\mu)+i\alpha\mu^2\delta_{2,0}(a,\mu)]\\
&\quad-\delta_{2,2}(a,\mu)[\delta_{1,1}(a,\mu)+i\alpha\mu^2\delta_{1,0}(a,\mu)]],\\
&= \delta_{3,0}(0,\mu)\delta_{4,2}(0,\mu)\delta_{1,2}(a,\mu)[\delta_{2,1}(a,\mu)+i\alpha\mu^2\delta_{2,0}(a,\mu)]\\
&\quad-\delta_{3,0}(0,\mu)\delta_{4,2}(0,\mu)\delta_{2,2}(a,\mu)[\delta_{1,1}(a,\mu)+i\alpha\mu^2\delta_{1,0}(a,\mu)]\\
&\quad-\delta_{4,0}(0,\mu)\delta_{3,2}(0,\mu)\delta_{1,2}(a,\mu)[\delta_{2,1}(a,\mu)+i\alpha\mu^2\delta_{2,0}(a,\mu)]\\
&\quad+\delta_{4,0}(0,\mu)\delta_{3,2}(0,\mu)\delta_{2,2}(a,\mu)[\delta_{1,1}(a,\mu)+i\alpha\mu^2\delta_{1,0}(a,\mu)]\\
&= -4i\alpha\mu^6-2(1+i)\mu^5,\\
\psi_4(\mu) &= \begin{vmatrix}\delta_{2,0}(0,\mu) & \delta_{3,0}(0,\mu)\\ \delta_{2,2}(0,\mu) & \delta_{3,2}(0,\mu)\end{vmatrix}\cdot\\
&\quad \begin{vmatrix}\delta_{1,2}(a,\mu) & \delta_{4,2}(a,\mu)\\ \delta_{1,1}(a,\mu)+i\alpha\mu^2\delta_{1,0}(a,\mu) & \delta_{4,1}(a,\mu)+i\alpha\mu^2\delta_{4,0}(a,\mu)\end{vmatrix}\\
&= [\delta_{2,0}(0,\mu)\delta_{3,2}(0,\mu)-\delta_{3,0}(0,\mu)\delta_{2,2}(0,\mu)]\\
&\quad\times\ [\delta_{1,2}(a,\mu)[\delta_{4,1}(a,\mu)+i\alpha\mu^2\delta_{4,0}(a,\mu)]\\
&\quad-\delta_{4,2}(a,\mu)[\delta_{1,1}(a,\mu)+i\alpha\mu^2\delta_{1,0}(a,\mu)]]\\
&= \delta_{2,0}(0,\mu)\delta_{3,2}(0,\mu)\delta_{1,2}(a,\mu)[\delta_{4,1}(a,\mu)+i\alpha\mu^2\delta_{4,0}(a,\mu)]\\
&\quad-\delta_{2,0}(0,\mu)\delta_{3,2}(0,\mu)\delta_{4,2}(a,\mu)[\delta_{1,1}(a,\mu)+i\alpha\mu^2\delta_{1,0}(a,\mu)]\\
&\quad-\delta_{3,0}(0,\mu)\delta_{2,2}(0,\mu)\delta_{1,2}(a,\mu)[\delta_{4,1}(a,\mu)+i\alpha\mu^2\delta_{4,0}(a,\mu)]\\
&\quad+\delta_{3,0}(0,\mu)\delta_{2,2}(0,\mu)\delta_{4,2}(a,\mu)[\delta_{1,1}(a,\mu)+i\alpha\mu^2\delta_{1,0}(a,\mu)]\\
&= 4i\alpha\mu^6+2(1-i)\mu^5.
\end{aligned}$$

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