CARDINAL SPLINE WAVELETS

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DECLARATION

I declare that this research report is my own, unaided work. It is being submitted for the degree of Master of Science at The University of the Witwatersrand in Johannesburg.

PRBrown

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June 1993

Dedicated to Ria

I am grateful to Professor Doron Lubinsky for his enthusiastic support for my studies in mathematics, and for introducing me to the topic of wavelet analysis.

INTRODUCTION

The subject of wavelet analysis has recently drawn a great deal of attention from mathematical scientists in various disciplines. It is creating a common link among mathematicians, physicists, and electrical engineers.

Although the term *wavelet* was first used in 1984, the concept of a *wavelet basis* originated in the last century with the mathematician Haar , who - in response to the discovery in 1873 by Paul du Bois-Reymond of a *continuous, real-valued* 2π *periodic* function whose Fourier series diverged at a given point - tried to prove the existence of another orthonormal system

$$h_0(x), h_1(x), \ldots, h_n(x), \ldots$$

of functions defined on $[0,1]$ such that for any function $f(x)$ continuous on $[0,1]$, the series

$$< f, h_0 > h_0(x) + < f, h_1 > h_1(x) + \cdots + < f, h_n > h_n(x) + \cdots$$

converges to $f(x)$ uniformly on $[0,1]$. He was successful in discovering the *Haar function* which is defined in this research report.

Throughout the twentieth century, in their work on function spaces, mathematicians defined functions with similar, or part of, the properties of the defenition of our present-day wavelet functions.

Physicists and experts in signal processing were unaware of the developments in mathematics when, faced with specific needs within their disciplines, they "rediscovered" wavelets. This gave impetus to the mathematical development of the subject, and an important advance was made towards the end of the last decade when Ingrid Daubechies constructed a set of *compactly supported orthonormal* basis functions for $L^2(\Re)$ without the strong discontinuities of the Haar basis. These developments in mathematics, in turn, led to exciting developments in signal analysis and numerical analysis.

Recently, many research papers have been published on the topic, and several texts have emerged, including [1], [6], [7], and [8] from which I have compiled this introduction.

Contents

ABSTRACT

This research report deals with the construction of the *semi-orthogonal spline wavelets*, pioneered by Charles Chui and Jian-Zhong Wang, and presented in Charles Chui's book **An introduction to wavelets.** *Orthonormal* wavelets have been studied by Ingrid Daubechies and *orthonormal spline wavelets* have been constructed by Battle and Lemarie. *Biorthogonal spline wavelets* have been constructed by Cohn, Daubechies, and Feauveau. These constructions have different advantages for signal processing. The orthonormal wavelets, being compactly supported, yield finite sequences for decomposition and reconstruction algorithms. These sequences have to be quite large, though, if a certain order of smoothness and regularity is desired. In the case of the semi-orthogonal wavelets constructed in this research report, a principle called the *duality principle* allows us to choose either a finite reconstruction sequence or a finite decomposition sequence, while the other, though not finite, has exponential decay, (see [2]) and so can be truncated so that we can still have an effective algorithm for decomposition or reconstruction. (This is discussed in [1] section 6.5). The B-spline N_m and B-wavelet ψ_m, constructed in this report, have the advantage for filtering applications in signal processing that both have generalised linear phase for *odd* m and linear phase for *even* m (proved in [4]). Furthermore , ψ_m (considered a *band-pass filter*) possesses a remarkable property called *complete oscillation* (see [5]).

In order to contain the length of this research report I have omitted most proofs in the first three chapters, so that I could discuss fully the construction of cardinal spline wavelets in chapter 4.æ

Chapter 1

PRELIMINARIES

1.1 THE FOURIER TRANSFORM

Definition 1.1.1 *The* **fourier transform** $\hat{f}$ *of a function* $f \in L^2(\Re)$ *or* $f \in L^1(\Re)$ *is defined by:*

$$\hat{f}(w) = (\mathcal{F}f)(w) = \int_{-\infty}^{\infty} e^{-ixw} f(x)dx. \tag{1.1}$$

The **inverse fourier** *transform of* $\hat{f}$ *is defined by:*

$$(\mathcal{F}^{-1}\hat{f})(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixw}\hat{f}(w)dw. \tag{1.2}$$

Theorem 1.1.1 *The fourier transform* $\mathcal{F}$ *is a one-to-one map of* $L^2(\Re)$ *onto itself. In other words, to every* $g \in L^2(\Re)$*, there corresponds one and only one* $f \in L^2(\Re)$*, given by:*

$$f(x) := \frac{1}{2\pi}\hat{g^{-}}(x) \tag{1.3}$$

such that $\hat{f} = g$ *.*
i.e. $f(x) := (\mathcal{F}^{-1}g)(x) =: \check{g}(x)$ *is the inverse Fourier transform of* g*.*

Proof : See [1] pg. 35.

Note : We call f^{-} the *reflection* of f and it is defined by:

$$f^{-}(x) := f(-x), \quad \forall x \in \Re. \tag{1.4}$$

We will need the following theorems:

Theorem 1.1.2 *Let $f \in L^1(\Re) \cap L^2(\Re)$. Then the fourier transform $\hat{f}$ of f is in $L^2(\Re)$ and satisfies the parseval identity*

$$< f, f > = \frac{1}{2\pi} < \hat{f}, \hat{f} > . \tag{1.5}$$

Proof:see [1] pg. 32.

$< \cdot, \cdot >$ is the inner product on $L^2(\Re)$ given by

$$< f, g > := \int_{-\infty}^{\infty} f(x)g(x)dx. \tag{1.6}$$

Theorem 1.1.3 *Let $f \in L^1(\Re)$ satisfy the following two conditions:*
1) The series $\sum_{k=-\infty}^{\infty} f(x + 2\pi k)$ converges everywhere to some continuous function, and
2) The fourier series $\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}$ converges everywhere.
Then the following **poisson summation formula** *holds:*

$$\sum_{k=-\infty}^{\infty} f(x + 2\pi k) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \hat{f}(k)e^{ikx}, \quad \forall x \in \Re. \tag{1.7}$$

proof: see [1] pg 45.

1.2 THE WAVELET TRANSFORM

Definition 1.2.1 *If $\psi \in L^2(\Re)$ satisfies the* **admissibility condition**

$$C_\psi := \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{(\omega)} d\omega < \infty \tag{1.8}$$

then ψ is called a *basic wavelet* . Relative to every basic wavelet ψ, the **integral wavelet transform** (IWT) on $L^2(\Re)$ is defined

$$\begin{aligned} (W_\psi f)(b, a) &:= |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t)\overline{\psi\left(\frac{t-b}{a}\right)}dt \\ &= |a|^{-\frac{1}{2}} < f, \psi\left(\frac{\cdot - b}{a}\right) > \end{aligned} \tag{1.9}$$

for $f \in L^2(\Re)$, and $a, b \in \Re$ and $a \neq 0$.
a is called a *dilation* parameter and b is called a *translation* parameter. We can discretise the variables a and b by choosing

$$\begin{aligned} a_j &:= 2^{-j}, \quad j \in Z; \\ b_{j,k} &:= \frac{k}{2^j} b_0, \quad j, k \in Z \end{aligned} \tag{1.10}$$

($b_0 > 0$ is a positive fixed constant).
By introducing the notation

$$\psi_{b_0;j,k}(t) := \psi_{b_{j,k};a_j}(t) = 2^{\frac{j}{2}}\psi(2^j t - kb_0), \tag{1.11}$$

the values of the integral wavelet transform are given by:

$$(W_\psi f)(b_{j,k}, a_j) = < f, \psi_{b_0;j,k} >. \tag{1.12}$$

We are interested in recovering any $f \in L^2(\Re)$ from the values of its IWT in (1.12). We will see that this can be achieved if ψ generates a *Riesz basis* of $L^2(\Re)$.

Definition 1.2.2 *A function $\psi \in L^2(\Re)$ is said to generate a Riesz basis $\{\psi_{b_0;j,k}\}$ with sampling rate b_0 if both the following two properties are satisfied:*
1.)The linear span

$$< \psi_{b_0;j,k} : j, k \in Z > \tag{1.13}$$

is dense in $L^2(\Re)$.
2.)There exists positive constants $0 < A \le B < \infty$ such that

$$A\|\{c_{j,k}\}\|_{l^2}^2 \le \| \sum_{j,k \in Z} c_{j,k}\psi_{b_0;j,k}\|_2^2 \le B\|\{c_{j,k}\}\|_{l^2}^2 \tag{1.14}$$

for all $\{c_{j,k}\} \in l^2(Z^2)$. Here A and B are called **Riesz bounds** *of $\{\psi_{b_0;j,k}\}$.*

If ψ generates a Riesz basis with $b_0 = 1$, then ψ is called an $\mathcal{R}$*-function.* The following theorem will have applications:

Theorem 1.2.1 *For any function $\phi \in L^2(\Re)$ and constants $0 < A \le B < \infty$ the following statements are equivalent:*
1.) $\{\phi(\cdot - k) : k \in Z\}$ satisfies the Riesz condition with Riesz bounds A and B; i.e. for any $\{c_k\} \in l^2$

$$A\|\{c_k\}\|_{l^2}^2 \le \| \sum_{k=-\infty}^{\infty} c_k\phi(\cdot - k)\|_2^2 \le B\|\{c_k\}\|_{l^2}^2. \tag{1.15}$$

2.)The fourier transform $\hat{\phi}$ of ϕ satisfies

$$A \le \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(x + 2\pi k)\right|^2 \le B, \qquad a.e.. \tag{1.16}$$

Proof: see [1] pg 76.

1.3 WAVELET SERIES

In this section we assume $b_0 = 1$ and we use the notation $\psi_{j,k} = \psi_{1;j,k}$. We restrict our attention to *$\mathcal{R}$- functions* ψ, so that $\{\psi_{j,k}\}$ is a Riesz basis of $L^2(\Re)$. An *$\mathcal{R}$- function* $\tilde{\psi}$ is called the *dual* of ψ if it satisfies the duality relationship

$$< \psi_{j,k} \, , \, \tilde{\psi}_{l,m} > = \delta_{j,l}\delta_{k,m}. \tag{1.17}$$

We use the notation $\{\psi^{j,k}\}$ for the dual basis of $\{\psi_{j,k}\}$.
There are R-*functions* that do not have a dual basis $\{\psi^{j,k}\}$ generated by some $\tilde{\psi} \in L^2(\Re)$. This leads to the following definition of wavelets:

Definition 1.3.1 *An $\mathcal{R}$ function $\psi \in L^2(\Re)$ is called an $\mathcal{R}$-wavelet (or wavelet) if it has a dual $\tilde{\psi} \in L^2(\Re)$, in the sense that $\{\psi_{j,k}\}$ and $\{\tilde{\psi}_{j,k}\}$ as defined by* (1.11) *satisfy the duality relationship* (1.17).

Two very important subclasses of *$\mathcal{R}$- wavelets* are the *semi-orthogonal wavelets* and *orthogonal wavelets.* It is easy to characterise the duals for these classes of functions.

Definition 1.3.2 *Let $\psi \in L^2(\Re)$ be an $\mathcal{R}$-function that generates $\{\psi_{j,k}\}$ as in* (1.11). *Then:*
1.) ψ is called an **orthogonal wavelet** *or (***o.n.wavelet***) if $\{\psi_{j,k}\}$ satisfies the* **orthonormality condition**

$$< \psi_{j,k} \, , \, \psi_{l,m} > = \delta_{j,l}\delta_{k,m}, \qquad j,k,l,m \in \mathcal{Z}; \tag{1.18}$$

2.) ψ is called a **semi-orthogonal** *wavelet (or s.o.wavelet), if $\{\psi_{j,k}\}$ satisfies the condition*

$$< \psi_{j,k} \, , \, \psi_{l,m} > = 0 \qquad j \neq l, \qquad j,k,l,m \in \mathcal{Z}. \tag{1.19}$$

An *o.n.wavelet* is self-dual in the sense that $\psi^{j,k} := \tilde{\psi}_{j,k} = \psi_{j,k}, \qquad j,k \in \mathcal{Z}.$

In the following theorem we formulate the dual of a s.o.wavelet:

Theorem 1.3.1 *Let $\psi \in L^2(\Re)$ be an s.o.wavelet and define $\tilde{\psi}$ via its fourier transform*

$$\hat{\tilde{\psi}}(w) := \frac{\hat{\psi}(w)}{\sum_{k=-\infty}^{\infty} |\hat{\psi}(w+2\pi k)|^2}. \tag{1.20}$$

Then $\tilde{\psi}$ is the dual of ψ in the sense that

$$< \psi_{j,k} \, , \tilde{\psi}_{l,m} > = \delta_{j,k}\delta_{k,m}. \tag{1.21}$$

In other words the dual basis $\{\psi^{j,k}\}$ relative to $\{\psi_{j,k}\}$ is given by $\psi^{j,k} = \tilde{\psi}_{j,k}$.

Proof: notice that the bounds in (2) of theorem (1.2.1) guarantee that $\tilde{\psi} \in L^2(\Re)$. For a complete proof see [1] pg 77.

The dual $\tilde{\psi}$ of a wavelet ψ is itself a wavelet, i.e. we always consider *pairs* of wavelets. From the density condition (1) in the definition of a Riesz basis, it is possible to expand every $f \in L^2(\Re)$ as

$$f(x) = \sum_{j,k \in Z} c_{j,k}\psi_{j,k}(x) = \sum_{j,k \in Z} d_{j,k}\tilde{\psi}_{j,k}(x). \tag{1.22}$$

These two doubly infinite series are called *wavelet series* and convergence is in $L^2(\Re)$. By the duality relationship (1.17) it follows that

$$\begin{aligned} c_{j,k} &= < f \, , \, \tilde{\psi}_{j,k} >; \\ d_{j,k} &= < f \, , \, \psi_{j,k} > . \end{aligned} \tag{1.23}$$

In summary we have the following scheme for reconstructing any $f \in L^2(\Re)$ from discrete samples of its integral wavelet transform:

Theorem 1.3.2 *Let ψ be a wavelet with dual $\tilde{\psi}$. For any $f \in L^2(\Re)$, consider its IWT, using both ψ and $\tilde{\psi}$ as basic wavelets, evaluated at $(b,a) = (\frac{k}{2^j}, \frac{1}{2^j} : j, k \in Z)$, namely*

$$\begin{aligned} d_{j,k} &= < f \, , \, \psi_{j,k} > = (W_\psi f)\left(\frac{k}{2^j}, \frac{1}{2^j}\right); \\ c_{j,k} &= < f \, , \, \tilde{\psi}_{j,k} > = (W_{\tilde{\psi}} f)\left(\frac{k}{2^j}, \frac{1}{2^j}\right), \end{aligned} \tag{1.24}$$

then f can be reconstructed from either $\{d_{j,k}\}$ or $\{c_{j,k}\}$ by using either of two wavelet series in (1.22). Furthermore, the inner product of any two $L^2(\Re)$ functions can also be recovered from the analogous discrete samples of their IWT, by using the formula

$$< f \, , g > = \sum_{j,k \in Z} < f \, , \, \psi_{j,k} >< \tilde{\psi}_{j,k} \, , \, g > . \tag{1.25}$$

Chapter 2

SCALING FUCTIONS AND WAVELETS

2.1 MULTIRESOLUTION ANALYSIS

The idea behind Wavelet analysis is the construction of a sequence of sets $\{V_j\}$ in $L^2(\Re)$, which, if it satisfies the properties below, is called a multiresolution approximation (MRA) of $L^2(\Re)$.
1.) $\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$;
2.) $clos_{L^2(\Re)} \left(\bigcup_{j \in Z} V_j\right) = L^2(\Re)$;
3.) $f(x) \in V_j \rightarrow f(2x) \in V_{j+1} \quad j \in Z$;
4.) $f(x) \in V_j \rightarrow f(x + \frac{1}{2^j}) \in V_j \quad j \in Z$.
The method we use to construct such a sequence of sets is to find a *scaling function* $\phi \in L^2(\Re)$ that *generates* such a sequence. We have the following definition:

Definition 2.1.1 *A function $\phi \in L^2(\Re)$ is called a* **scaling function** *if the subspaces V_j of $L^2(\Re)$ defined by:*

$$V_j := clos_{L^2(\Re)} < \phi_{j,k} \; : \; k \in Z >, \quad j \in Z, \tag{2.1}$$

[where $\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi(2^j x - k)$]
satisfies the four properties above and if $\{\phi(\cdot - k) \; : k \in Z\}$ is a Riesz basis of V_0. We say that the scaling function ϕ generates a multiresolution approximaion of $L^2(\Re)$.

If $\{\phi(\cdot - k) \; : k \in Z\}$ is a Riesz basis of V_0 then it follows easily that the family $\{\phi_{j,k} \; : k \in Z\}$ is also a Riesz basis of V_j. Then since $\phi \in V_0 \subset V_1$, and $\{\phi_{j,k} \; : \; k \in Z\}$

is a Riesz basis of V_1, there exists a unique l^2-sequence $\{p_n\}$ that describes the *two scale relation*

$$\phi(x) = \sum_{k=-\infty}^{\infty} p_k \phi(2x - k) \tag{2.2}$$

of the scaing fuction ϕ. The sequence $\{p_k\}$ is called a *two scale sequence* of ϕ. Corresponding to this sequence we can introduce the *two scale symbol* P defined by

$$P(z) = P_\phi(z) := \frac{1}{2} \sum_{k=-\infty}^{\infty} p_k z^k. \tag{2.3}$$

We can formulate (2.2) by taking its fourier transform:

$$\begin{aligned} \hat{\phi}(w) &= P(e^{-i\frac{w}{2}})\hat{\phi}(\frac{w}{2}) \\ &= P(z)\hat{\phi}(\frac{w}{2}) \end{aligned} \tag{2.4}$$

where $z = e^{-i\frac{w}{2}}$.

This equation can also be the defining equation of the two scale symbol P_ϕ of a given scaling function ϕ. If $\{p_k\}$ is a finite sequence then ϕ has compact support, and in fact, if

$$\phi(x) = \sum_{k=0}^{N_\phi} p_k \phi(2x - k) \tag{2.5}$$

for some positive integer N_ϕ then it can be shown that $\text{supp}(\phi) = [0, N\phi]$ (see [1] pg. 128-131).

In general we will assume that the sequence $\{p_k\}$ is in l^1. In this case the corresponding two-scale symbol belongs to the so called *Wiener Class*.

Definition 2.1.2 *A laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$ is said to belong to the Wiener class $\mathcal{W}$ if its coefficient sequence $\{a_n\}$ is in l^1.*

$\mathcal{W}$ is an algebra in the sense that if $G(z) = \sum_n g_n z^n$ and $H(z) = \sum_n h_n z^n$ for l^1 sequences $\{g_n\}$ and $\{h_n\}$, then if $Q(z) = G(z)H(z) = \sum_n q_n z^n$ then $\{q_n\}$ is also an l^1 sequence. (**proof**: see [1] pg. 140).

Furthermore, we have the following theorem due to N. Wiener:

Theorem 2.1.1 *let $f \in \mathcal{W}$ and suppose $f(z) \neq 0$ $\forall z$ on the unit circle $|z| = 1$. Then $\frac{1}{f} \in \mathcal{W}$ also.*

Proof: see [1] pg. 140.

Let ϕ be a scaling function with *two-scale symbol* $P_\phi(z) = \frac{1}{2}\sum_k p_k z^k$. Now consider any other l^1 sequence $\{q_k\}$ and its symbol

$$Q(z) = \frac{1}{2} \sum_{k=-\infty}^{\infty} q_k z^k. \tag{2.6}$$

Then Q is also in $\mathcal{W}$ and defines a function

$$\psi(x) := \sum_k q_k \phi(2x-k) \tag{2.7}$$

in V_1. This function also generates a closed subspace W_0 in the same manner as ϕ generates V_0, namely

$$W_0 := clos_{L^2(\Re)} < \psi(\cdot - k) : k \in \mathcal{Z} > . \tag{2.8}$$

We are interested in the relationship between V_0 and W_0, both subsets of V_1. Obviously, this depends on the relationship between P and Q. We would like (for reasons stated later) V_0 and W_0 to be complementary spaces of V_1, i.e.

$$V_1 = V_0 \dot{+} W_0 \tag{2.9}$$

where the notation $\dot{+}$ means *direct sum* ($V_0 \cap W_0 = \{0\}$ and $V_1 = V_0 \cup W_0$). In our analysis of the relationship between P and Q it is useful to construct the matrix

$$M_{P,Q}(z) := \begin{bmatrix} P(z) & Q(z) \\ P(-z) & Q(-z) \end{bmatrix}$$

and the determinant

$$\Delta_{P,Q}(z) := det M_{P,Q}. \tag{2.10}$$

Since P and Q are in $\mathcal{W}$, $\Delta_{P,Q}$ is in $\mathcal{W}$ and if $\Delta_{P,Q}(z) \neq 0$ on $|z| = 1$ then $\frac{1}{\Delta_{P,Q}} \in \mathcal{W}$ from theorem 2.1.1 .

We can construct the following functions G and H which are also in $\mathcal{W}$.

$$G(z) := \frac{Q(-z)}{\Delta_{P,Q}(z)} = \frac{1}{2} \sum_{n=-\infty}^{\infty} g_n z^n \tag{2.11}$$

$$H(z) := \frac{-P(-z)}{\Delta_{P,Q}(z)} = \frac{1}{2} \sum_{n=-\infty}^{\infty} h_n z^n .$$

where $\{g_n\}$ and $\{h_n\}$ are in l^1. The reason for constructing G and H is apparent in the following theorem:

Theorem 2.1.2 *A necessary and sufficient condition for the direct-sum decomposition (2.9) to hold is that the continuous function $\Delta_{P,Q}$ never vansishes on the unit circle $|z| = 1$. Furthermore, if $\Delta_{P,Q}(z) \neq 0$ for all $|z| = 1$ then the family $\{\psi(\cdot - k) : k \in \mathcal{Z}\}$ governed by $Q(z)$ as in (2.6) and (2.7) is a Riesz basis of W_0, and the decomposition relation*

$$\phi(2x - l) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \{g_{2k-l}\phi(x-k) + h_{2k-l}\psi(x-k)\} \tag{2.12}$$

holds $\forall x \in \Re$. $\{g_n\}$ and $\{h_n\}$ are the sequences mentioned above.

proof: see [1] pg. 142-145.
Let us define as in (2.8)

$$W_j := clos_{L^2(\Re)} \{\psi(2^j \cdot - k) : k \in \mathbb{Z}\}, \quad j \in \mathbb{Z}. \tag{2.13}$$

Suppose now that two-scale symbol P and Q satisfy the conditions stated in theorem 2.1.2. Then the following argument shows that we can extend (2.9) to the more general case

$$V_{j+1} = V_j \dot{+} W_j, \quad j \in \mathbb{Z}. \tag{2.14}$$

Let

$$f_{j+1} \in V_{j+1}$$

i.e.

$$f_{j+1}(x) = \sum_k \alpha_k \phi(2^{j+1}x - k)$$

for some sequence $\{\alpha_j\}$.
Then

$$\begin{aligned} f_{j+1}(2^{-j}x) &= \sum_k \alpha_k \phi(2^{j+1}(2^{-j}x) - k) \\ &= \sum_k \alpha_k \phi(2x - k) \quad (\in V_1) \\ &= \sum_k \alpha_k \phi(x - k) + \sum_k \delta_k \psi(x - k) \\ &= \sum_k \alpha_k \phi(2^j(2^{-j}x) - k) + \sum_k \delta_k \psi(2^j(2^{-j}x) - k) \end{aligned}$$

where we have applied (2.9) in step 3; the first and second terms on the r.h.s are in V_0 and W_0 respectively.
Thus

$$f_{j+1}(x) = \sum_k \alpha_k \phi(2^j x - k) + \sum_k \delta_k \psi(2^j x - k),$$

where terms on the r.h.s. are in V_j and W_j respectively. i.e.

$$V_{j+1} = V_j + W_j. \tag{2.15}$$

Now suppose that $f_j \in V_j$ and $g_j \in W_j$, and $f_j = g_j$. Then

$$f_j(x) = \sum_k \alpha_k \phi(2^j x - k) = \sum_k \beta_k \psi(2^j x - k) = g_j(x)$$

for some sequence $\{\beta_j\}$, and

$$f_j(2^{-j}x) = \sum_k \alpha_k \phi(x - k) = \sum_k \beta_k \psi(x - k) = g_j(2^{-j}x).$$

Therefore

$$f_j(2^{-j}x) = g_j(2^{-j}x)$$

(terms in V_0 and W_0 respectively).
Therefore

$$f_j(x) = g_j(x) \;=\; 0$$

(since $V_0 \cap W_0 = \{0\}$),
and

$$V_j \cap W_j \;=\; \{0\}. \tag{2.16}$$

(2.15) and (2.16) give (2.14).
Now we can establish the following about the sets V_j and W_j:

$$\begin{aligned} V_{j+1} &= V_j \dot{+} W_j \\ &= W_j \dot{+} (V_{j-1} \dot{+} W_{j-1}) \\ &= W_j \dot{+} W_{j-1} \dot{+} W_{j-2} \dot{+} V_{j-2} \\ & \quad etc. \end{aligned}$$

From property (2) of a multiresolution approximation we can deduce that

$$L^2(\Re) \;=\; \cdots \dot{+} W_{-1} \dot{+} W_0 \dot{+} W_1 \dot{+} \cdots . \tag{2.17}$$

If $\{\psi(\cdot - k) \;: k \in \mathcal{Z}\}$ is a Riesz basis of W_0 as determined in theorem 2.1.2 it is easy to show that $\{\psi(2^j \cdot -k) \;: k \in \mathcal{Z}\}$ is also a Riesz basis of V_j , but we need to do more analysis to establish if ψ generates a Riesz basis of $L^2(\Re)$ and even then we have to investigate the existence of a dual $\tilde{\psi} \in L^2(\Re)$ so that ψ is a *wavelet function*. This is the purpose of the following discussion.

We change the emphasis slightly and begin by considering *two-scale symbols* P_ϕ and $G^*_{\tilde{\phi}}$ corresponding to *scaling functions* ϕ and $\tilde{\phi}$ where

$$G^*(z) \;=\; \overline{G(z)} \;=\; \overline{G}(\frac{1}{z}) \;=\; \frac{1}{2}\sum_n \overline{g}_{-n} z^n \quad |z| = 1 \tag{2.18}$$

($\{g_n\}$ is the *two-scale sequence* of the *two-scale symbol* G).
We assume that ϕ generates a MRA $\{V_j\}$ and $\tilde{\phi}$ generates a MRA $\{\tilde{V}_j\}$ of $L^2(\Re)$. If we select Laurent series Q and H that satisfy

$$\Delta_{P,Q}(z) \neq 0, \quad \Delta_{G,H}(z) \neq 0, \quad |z| = 1, \tag{2.19}$$

then according to theorem 2.1.2 and (2.17) we will obtain unrelated direct sum decompositions of $L^2(\Re)$:
$L^2(\Re) \;=\; \cdots \dot{+} W_{-1} \dot{+} W_0 \dot{+} W_1 \cdot + \cdots ;$

$L^2(\Re) = \cdots \dotplus \tilde{W}_{-1} \dotplus \tilde{W}_0 \dotplus \tilde{W}_1 \dotplus \cdots$

(where W_j and $\tilde{W}_j$ are spaces corresponding to functions ψ and $\tilde{\psi}$ determined from Q and H respectively).

To make a connection between these two decompositions we start with the following definition:

Definition 2.1.3 *Scaling functions ϕ and $\tilde{\phi}$ generating possibly different MRA's $\{V_j\}$ and $\{\tilde{V}_j\}$ respectively, of $L^2(\Re)$, are said to be* **dual scaling functions** *if they satisfy the condition*

$$< \phi(\cdot - j), \tilde{\phi}(\cdot - k) > = \int_{-\infty}^{\infty} \phi(x-j)\overline{\tilde{\phi}(x-k)}dx = \delta_{j,k}, \quad j,k \in Z, \tag{2.20}$$

and the following theorem:

Theorem 2.1.3 *Suppose $P = P_\phi$ and $G^* = G^*_{\tilde{\phi}}$ are two-scale symbols with dual scaling functions ϕ and $\tilde{\phi}$ respectively. Then, if $Q, H \in \mathcal{W}$ are chosen from the class*

$$\begin{aligned} Q(z) &= z^{-1}G(-z)k(z^2); \\ H(z) &= zP(-z)k^{-1}(z^2), \end{aligned} \tag{2.21}$$

where $k \in \mathcal{W}$ and $k(z) \neq 0$ for $|z| = 1$, then the functions ψ and $\tilde{\psi}$ defined

$$\begin{aligned} \psi(z) &= \sum_k q_k z^k \\ \tilde{\psi}(z) &= \sum_k \overline{h}_{-k} z^k \end{aligned} \tag{2.22}$$

satisfy

$$< \psi_{j,k}, \tilde{\psi}_{l,m} > = \delta_{j,l}\delta_{k,m}, \quad j,k,l,m \in Z. \tag{2.23}$$

Furthermore $\psi \in W_0$ and $\tilde{\psi} \in \tilde{W}_0$ are dual wavelet functions.

proof: see [1] pg 142-143.

The following shows that the choice of functions G and H in theorem 2.1.3 is consistent with the equations defining G and H in (2.11) that are needed for the decomposition relation in (2.12).

Given $P(z)$ and $G(z)$, we chose $Q(z) = z^{-1}G(-z)k(z^2)$ so that

$$\begin{aligned} \Delta_{P,Q}(z) &= P(z)Q(-z) - P(-z)Q(z) \\ &= P(z)(-z)^{-1}G(z)k(z^2) - z^{-1}G(-z)k(z^2)P(-z) \\ &= \frac{k(z^2)}{z}[-P(z)G(z) - P(-z)G(-z)] \\ &= -\frac{k(z^2)}{z}[P(z)G(z) + P(-z)G(-z)] \end{aligned}$$

In [1] pg. 151 it is proved that if ϕ and $\tilde{\phi}$ are dual scaling functions as in theorem 2.1.3 then their two-scale symbols P and G^* of satisfy the identity $P(z)G(z) + P(-z)G(-z) = 1$. Therefore the above expression becomes

$$\Delta_{P,Q}(z) = -\frac{k(z^2)}{z}$$

Now the expression for $G(z)$ in (2.11) is consistent with our choice of Q, for

$$\frac{Q(-z)}{\Delta_{P,Q}(z)} = (-z)^{-1}G(z)k(z^2)\frac{-z}{k(z^2)} = G(z) \quad .$$

Also, according to (2.11)

$$\begin{aligned} H(z) = \frac{-P(-z)}{\Delta_{P,Q}(z)} &= -P(-z)\frac{-z}{k(z^2)} \\ &= zP(-z)k^{-1}(z^2) \end{aligned}$$

which is the same as the definition of H(z) we had in (2.21).

In chapter 1 we showed that if ψ and $\tilde{\psi} \in L^2(\Re)$ are dual *wavelet functions*, then any $f \in L^2(\Re)$ can be represented in two unique wavelet series:

$$f(x) = \sum_{j,k} < f, \tilde{\psi}_{j,k} > \psi_{j,k}(x); \tag{2.24}$$

$$f(x) = \sum_{j,k} < f, \psi_{j,k} > \tilde{\psi}_{j,k}(x), \tag{2.25}$$

where the coefficients are the values of the IWT of f relative to the *basic wavelets* $\tilde{\psi}$ and ψ respectively; evaluated at the time-scale positions

$$b, a = \left(\frac{k}{2^j}, \frac{1}{2^j}\right) .$$

See section 1.2.

We want algorithms for finding the IWT values of f, and for reconstructing f from its IWT values. It turns out that the *two-scale sequences* $\{\overline{g}_{-n}\}$ and $\{\overline{h}_{-n}\}$ from *two-scale symbols* $G^* = G^*_{\tilde{\phi}}$ and H^* as given in (2.11) and (2.18) can be used for obtaining the IWT values $< f, \tilde{\psi}_{j,k} >$ .This computational scheme, called the *decomposition algorithm* is a consequence of the *decomposition relation* (2.12) in theorem 2.1.2. On the other hand, the two-scale sequences $\{p_n\}$ and $\{q_n\}$ (from *two-scale symbols* P and Q as given in 2.3 and 2.6) can be used for reconstruction of f from its IWT values. This computational scheme called the *reconstruction algorithm* is a consequence of *two-scale relations* (2.2) and (2.7) If we want to use ψ instead of $\tilde{\psi}$ as the *basic wavelet* then the *two-scale sequences* $\{p_n\}$ and $\{q_n\}$ are used in the *decomposition algorithm* and $\{\overline{g}_{-n}\}$ and $\{\overline{h}_{-n}\}$ are used in the *reconstruction algorithm*. In other words, the role of the pairs ($\{\overline{g}_{-n}\}$,$\{\overline{h}_{-n}\}$)

and ($\{p_n\}$,$\{q_n\}$) for *decomposition* and *reconstruction* purposes are interchanged if the information

$$\left\{W_{\tilde{\psi}} f\left(\tfrac{k}{2^j}, \tfrac{1}{2^j}\right); \quad j,k \in Z\right\} \tag{2.26}$$

is replaced by the IWT information

$$\left\{W_{\psi} f\left(\tfrac{k}{2^j}, \tfrac{1}{2^j}\right); \quad j,k \in Z\right\}. \tag{2.27}$$

This is called the *duality principle in wavelet decomposition-reconstruction.*

In what follows we only discuss the IWT in (2.26) using $\tilde{\psi}$ as a basic wavelet. For any $f \in L^2(\Re)$ let f_n be a projection of f onto V_n.
Since

$$\begin{aligned} V_{N+1} &= W_N \dot{+} V_N \\ &= (W_{N-1} \dot{+} V_{N-1}) \dot{+} W_N \\ &\vdots \\ &= W_{N-1} \dot{+} \cdots \dot{+} W_{N-M} \dot{+} V_{N-M} \end{aligned} \tag{2.28}$$

for any positive integer M, f_N has a unique decomposition

$$f_N = g_{N-1} + g_{N-2} + \cdots + g_{N-M} + f_{N-M} \tag{2.29}$$

where

$$\begin{aligned} g_j(x) &\in W_j \quad j = N-M, \cdots, N-1 \\ f_{N-M} &\in V_{N-M} \end{aligned} \tag{2.30}$$

Let us write

$$f_j(x) = \sum_k c_k^j \phi(2^j x - k) \tag{2.31}$$

where $\mathbf{c}^j := \{c_k^j\} \quad k \in Z$,
and

$$g_j(x) = \sum_k d_k^j \psi(2^j x - k) \tag{2.32}$$

where $\mathbf{d}^j := \{d_k^j\} \quad k \in Z$;
then the decomposition in (2.29) is uniquely determined by the sequences $\mathbf{c}^j := \{c_k^j\}$ and $\mathbf{d}^j := \{d_k^j\}$ in (2.31) and (2.32). Again, note that

$$d_k^j = \left\{W_{\tilde{\psi}} f\left(\tfrac{k}{2^j}, \tfrac{1}{2^j}\right); \quad j,k \in Z\right\}. \tag{2.33}$$

We introduce the notation

$$\begin{aligned} a_n &:= \frac{1}{2} g_{-n}; \\ b_n &:= \frac{1}{2} h_{-n} \end{aligned} \tag{2.34}$$

(where $\{\overline{g}_{-n}\}$ and $\{\overline{h}_{-n}\}$ are the *two-scale sequences* corresponding to G_ϕ^* and H^* respectively). Hence the *decomposition relation* (2.12) becomes

$$\phi(2x-l) = \sum_{-\infty}^{\infty} a_{l-2k}\phi(x-k) + b_{l-2k}\psi(x-k), \quad l \in \mathbb{Z}. \tag{2.35}$$

We can now derive the *decomposition* and *reconstruction* algorithms:

1) Decomposition algorithm

$$\begin{aligned} c_k^{j-1} &= \sum_l a_{l-2k} c_l^j; \\ d_k^{j-1} &= \sum_l b_{l-2k} c_l^j. \end{aligned} \tag{2.36}$$

This can be represented diagramatically as follows:

$$\begin{array}{ccccccccc} & & d^{N-1} & & d^{N-2} & & \cdots & & d^{N-M} \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\ c^N & \rightarrow & c^{N-1} & \rightarrow & c^{N-2} & \rightarrow & \cdots & \rightarrow & c^{N-M}, \end{array}$$

2) Reconstruction algorithm

$$c_k^j = \sum_l p_{k-2l} c_l^{j-1} + q_{k-2l} d_l^{j-1}, \tag{2.37}$$

which is represented diagramatically:

$$\begin{array}{ccccccccc} d^{N-M} & & d^{N-M+1} & & \cdots & & d^{N-1} & & \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ c^{N-M} & \rightarrow & c^{N-M+1} & \rightarrow & \cdots & \rightarrow & c^{N-1} & \rightarrow & c^N. \end{array}$$

proof of 1) *By applying* (2.35) *we have*

$$\begin{aligned} f_j(x) &= \sum_l c_l^j \phi(2^j x - l) \\ &= \sum_l c_l^j \left[\sum_k \left\{a_{l-2k}\phi(2^{j-1}x-k) + b_{l-2k}\psi(2^{j-1}x-k)\right\}\right] \\ &= \sum_k \left\{\sum_l a_{l-2k} c_l^j\right\} \phi(2^{j-1}x-k) + \sum_k \left\{\sum_l b_{l-2k} c_l^j\right\} \psi(2^{j-1}x-k). \end{aligned}$$

Hence, from the decomposition relation $f_j(x) = f_{j-1}(x) + g_{j-1}(x)$, *where* $f_{j-1}(x)$ *and* $g_{j-1}(x)$ *are given as in* (2.31) *and* (2.32) *with* j *replaced by* $j-1$, *it follows that*

$$\sum_k \left\{\sum_l a_{l-2k} c_l^j - c_k^{j-1}\right\} \phi(2^{j-1}x-k) + \sum_k \left\{\sum_l b_{l-2k} c_l^j - d_k^{j-1}\right\} \psi(2^{j-1}x-k) = 0$$

so that (2.37) *follows from the* l^2 *linear independence of* $\{\phi_{j-1,k} : k \in \mathbb{Z}\}$ *and* $\{\psi_{j-1,k} : k \in \mathbb{Z}\}$ *and the fact that* $V_{j-1} \cap W_{j-1} = \{0\}$.

proof of 2) *By applying the two-scale relations* (2.2) *and* (2.7) *we have*

$$\begin{aligned} f_j(x) &= f_{j-1}(x) + g_{j-1}(x) \\ &= \sum_l \left[c_l^{j-1}\phi(2^{j-1}x - l) + d_l^{j-1}\psi(2^{j-1}x - l) \right] \\ &= \sum_l \left[c_l^{j-1} \sum_k p_k \phi(2^j x - 2l - k) + d_l^{j-1} \sum_k q_k \phi(2^j x - 2l - k) \right] \\ &= \sum_l \sum_k \left(c_l^{j-1} p_{k-2l} + d_l^{j-1} q_{k-2l} \right) \phi(2^j x - k) \\ &= \sum_k \left\{ \sum_l \left[p_{k-2l} c_l^{j-1} + q_{k-2l} d_l^{j-1} \right] \right\} \phi(2^j x - k), \end{aligned}$$

and comparing with (2.31) *we obtain* (2.37).

Chapter 3

CARDINAL SPLINE ANALYSIS

3.1 THE CARDINAL B-SPLINE

In this chapter we give an example of a *multiresolution approximation* of $L^2(\Re)$ and a *scaling function* that generates it.

For a non-negative integer n let π_n denote the collection of all *algebraic polynomials* of degree at most n, and $C^n = C^n(\Re)$ the collection of all functions f such that $f', f'', \ldots, f^{(n)}$ are continuous everywhere.

Definition 3.1.1 *For each positive integer m, the space S_m of* **cardinal splines** *of order m and with knot sequence $\mathcal{Z}$ is the collection of all functions $f \in C^{m-2}$ such that the restriction of f to any interval $[k, k+1)$, $k \in \mathcal{Z}$, is in π_{m-1}. i.e.*

$$f|_{[k,k+1)} \in \pi_{m-1}, \qquad k \in \mathcal{Z}\,. \tag{3.1}$$

We can extend the definition to cardinal splines with knot sequence 2^{-j}, $j \in \mathcal{Z}$ and denote by S_m^j the space of *cardinal splines* with this knot sequence. In this case, the above definition applies to the restriction $f|_{[2^{-j}k, 2^{-j}(k+1))} \in \pi_{m-1}$, $\quad k \in \mathcal{Z}$. If $j_1, j_2, \in \mathcal{Z}$ with $j_2 > j_1$, then $2^{-j_2}\mathcal{Z} \supset 2^{-j_1}\mathcal{Z}$, so if $f \in S_m^{j_1}$ then $f \in S_m^{j_2}$, i.e. if f is a *spline function* with knot sequence $2^{-j_1}\mathcal{Z}$, then f is also a *spline function* with knot sequence $2^{-j_2}\mathcal{Z}$. We can deduce that $S_m^{j_1} \subset S_m^{j_2}$, or, more generally, we have a nested sequence

$$\cdots \subset S_m^{-1} \subset S_m^0 \subset S_m^1 \subset \cdots \tag{3.2}$$

of cardinal splines (with $S_m^0 := S_m$).
To obtain a multiresolution approximation of $L^2(\Re)$ we form the intersections of the spaces

S_m^j with $L^2(\Re)$, i.e. define

$$V_j^m = clos_{L^2(\Re)}\left(S_m^j \cap L^2(\Re)\right). \tag{3.3}$$

This gives us a nested sequence

$$\cdots \subset V_{-1}^m \subset V_0^m \subset V_1^m \subset \cdots \tag{3.4}$$

of *cardinal spline spaces* of $L^2(\Re)$.
It is clear that

$$clos_{L^2(\Re)}\left(\bigcup_{j \in \mathbb{Z}} V_j^m\right) = L^2(\Re) \tag{3.5}$$

for even in the trivial case when $m = 1$, the piecewise constant functions on intervals $[2^{-j}k, 2^{-j}(k+1))$ can approximate a function in $L^2(\Re)$ arbitrarily closely as $j \to \infty$.

To verify property (3) of a MRA (see first paragraph of chapter 2) we need to show that $f(x) \in V_j^m \Rightarrow f(2x) \in V_{j+1}^m$.
Therefore, suppose that

$$f(x) \in V_j^m$$

for some $k \in \mathbb{Z}$ and fix

$$x \in [2^{-(j+1)}k, 2^{-(j+1)}(k+1)).$$

Then

$$2x \in [2^{-j}k, 2^{-j}(k+1)).$$

Since

$$f|_{[2^{-j}k, 2^{-j}(k+1))} \in \pi_{m-1}$$

we have

$$f((2x)) = p((2x))$$

for some polynomial p of degree $\leq m-1$.
If we define

$$p_0(x) = p(2x)$$

then

$$p_0 \in \pi_{m-1}$$

also, and

$$f(2x) = p_0(x).$$

Since x was chosen arbitrarily in $[2^{-(j+1)}k, 2^{-(j+1)}(k+1))$ we have

$$f(2\cdot)|_{[2^{-(j+1)}k, 2^{-(j+1)}(k+1))} \in \pi_{m-1}.$$

i.e.

$$f(2x) \in V_{j+1}^m$$

since k was chosen arbitrarily.

Lastly, to verify (4) of a MRA we use a similar argument to show that

$$f(x) \in V_j^m \;\Rightarrow\; f\left(x+\frac{1}{2^j}\right) \in V_j^m. \tag{3.6}$$

We have found a multiresolution approximation of $L^2(\Re)$. To find a *generating scaling function* we define, using a *convolution operation*, the *B-spline functions*.

Definition 3.1.2 *Let $f, g \in L^1(\Re)$. Then the* **convolution** *h of f and g is also in $L^1(\Re)$ and is defined by*

$$h(x) \;=\; (f*g)(x) \;:=\; \int_{-\infty}^{\infty} f(x-y)g(y)dy. \tag{3.7}$$

It is easy to show that $h \in L^1(\Re)$, and that the convolution operation is commutative $(f*g = g*f)$ and associative $(\{f*g\}*u = f*\{g*u\})$ for $L^1(\Re)$ functions f, g, u.

Definition 3.1.3 *Let $N_1(x)$ be the characteristic function on the unit interval $[0,1)$ and define*

$$\begin{aligned} N_m(x) \;:=\; (N_{m-1}*N_1)(x) \;&=\; \int_{-\infty}^{\infty} N_{m-1}(x-t)N_1(t)dt \\ &=\; \int_0^1 N_{m-1}(x-t)dt. \end{aligned} \tag{3.8}$$

We call N_m the **m$^{\text{th}}$ order cardinal B-spline** *N_m.*

Clearly, $N_m(x)$ has compact support which also implies that $N_m(x) \in L^1(\Re)$.

N_m has several important properties listed in the following theorem.

Theorem 3.1.1 *The m^{th} order cardinal B-spline N_m satisfies the following properties:*
1) for every $f \in C = C^0$ (continuous functions)

$$\int_{-\infty}^{\infty} f(x)N_m(x)dx \;=\; \int_0^1 \cdots \int_0^1 f(x_1+\ldots+x_n)dx_1\ldots dx_m, \tag{3.9}$$

2) for every $g \in C^m$

$$\int_{-\infty}^{\infty} g^{(m)}(x)N_m(x)dx \;=\; \textstyle\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}g(k), \tag{3.10}$$

3) $SuppN_m = [0, m]$;
4) $N_m(x) > 0, \quad 0 < x < m$;
5) $\sum_{-\infty}^{\infty} N_m(x-k) = 1, \quad \forall x \in \Re$;
6) $N_m'(x) = (\Delta N_{m-1})(x) := N_{m-1}(x) - N_{m-1}(x-1)$;
7) Cardinal B-splines N_m and N_{m-1} are related by the identity

$$N_m(x) = \frac{x}{m-1}N_{m-1}(x) + \frac{m-x}{m-1}N_{m-1}(x-1); \tag{3.11}$$

8) N_m is symmetric with respect to the centre of its support, ie

$$N_m\left(\frac{m}{2}+x\right) = N_m\left(\frac{m}{2}-x\right). \tag{3.12}$$

proof: see [1] section 4.1 pg 81-87.
It can be shown [see [1] pg 88-89] that if we form the set of integer translates of the B-spline N_m

$$B := \{N_m(x-k) : k \in Z\} \tag{3.13}$$

then B is a Riesz basis of V_0^m.
Similarly the family

$$B_j := \{2^{\frac{j}{2}} N_m(2^j x - k) : k \in Z\} \tag{3.14}$$

is a Riesz basis of V_m^j with the same Riesz bounds. By the density property of a Riesz basis [see 1.13] we have

$$V_j^m = clos_{L^2(\Re)}\{2^{\frac{j}{2}} N_m(2^j x - k) : j, k \in Z\}. \tag{3.15}$$

This means that the B-spline N_m is a scaling function that generates the MRA in (3.4) . [see defn 2.1.1].

Our next step is to determine the two-scale symbol P_{N_m} of N_m [see (2.3)].
First note the following property of the convolution operation.

$$(f*g)^\wedge(\omega) = \hat{f}(\omega)\hat{g}(\omega) \tag{3.16}$$

as can be seen from the following computation

$$\begin{aligned}(f*g)^\wedge(\omega) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x-y)g(y)dy\right)e^{-i\omega x}dx\\ &= \int_{-\infty}^{\infty} g(y)\left(\int_{-\infty}^{\infty} e^{-i\omega x} f(x-y)dx\right)dy\\ &= \int_{-\infty}^{\infty} g(y)\left(\int_{-\infty}^{\infty} e^{-i\omega(t+y)} f(t)dt\right)dy\\ &= \int_{-\infty}^{\infty} g(y)e^{-i\omega y}\left(\int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt\right)dy\\ &= \hat{f}(\omega)\hat{g}(\omega)\end{aligned}$$

where we have substituted $t = x - y$ in the third step, and applied Fubini's theorem[1] in the second step.

We know from (2.4) that we can write

$$\hat{N}_m(\omega) = P_{N_m}\left(e^{-i\omega/2}\right)\hat{N}_m\left(\frac{\omega}{2}\right) \tag{3.17}$$

where P_{N_m} is the two-scale symbol of the scaling function N_m. We can now state a property of the fourier transform of N_m.

Since

$$N_m(x) = (N_{m-1} * N_1)(x) \tag{3.18}$$

(see (3.8)), we have

$$\begin{aligned} \hat{N}_m(w) &= \hat{N}_{m-1}(\omega)\hat{N}_1(\omega) \\ &= \hat{N}_{m-2}(\omega)(\hat{N}_1(\omega))^2 \\ &\vdots \\ &= (\hat{N}_1(\omega))^m. \end{aligned} \tag{3.19}$$

We know from defn 3.1.3 that $N_1(x)$ is the characteristic function of the interval $[0,1)$. Therefore

$$\begin{aligned} \hat{N}_1(\omega) &= \int_0^1 e^{-i\omega x}dx \\ &= \frac{1}{-i\omega}e^{-i\omega x}|_0^1 \\ &= \frac{1-e^{-i\omega}}{i\omega} \end{aligned} \tag{3.20}$$

Therefore

$$\hat{N}_m(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^m \tag{3.21}$$

(by substitution into (3.19)).

[1] Fubini's theorem states that if

$$\int[\int |f(x,y)|dy]dx < \infty$$

then

$$\int[\int f(x,y)dy]dx = \int[\int f(x,y)dx]dy$$

0

Subsitution of (3.21) into (3.17) gives

$$P_{N_m}\left(e^{-i\omega/2}\right) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^m \left(\frac{i\omega/2}{1-e^{-i\omega/2}}\right)^m \tag{3.22}$$
$$= \left(\frac{1+e^{-i\omega/2}}{2}\right)^m$$
$$= 2^{-m}\sum_{k=0}^{m}\binom{m}{k}e^{-ik\omega/2}.$$

Note that for $z = e^{-i\omega/2}$ we have

$$P_{N_m}(z) = \left(\frac{1+z}{2}\right)^m. \tag{3.23}$$

If the two-scale sequence of N_m is denoted by $\{ p_{m,k} \}$ then we can write

$$\frac{1}{2}\sum_{-\infty}^{\infty} p_{m,k}e^{-ik\omega/2} = 2^{-m}\sum_{k=0}^{m}\binom{m}{k}e^{-ik\omega/2} \tag{3.24}$$

where

$$p_{m,k} = \left\{ \begin{array}{cl} 2^{-m+1}\binom{m}{k} & ; \; 0 \le k \le m \\ 0 & ; \; otherwise \end{array} \right\}. \tag{3.25}$$

Therefore the two-scale relation for cardinal B-splines of order m is given by

$$N_m(x) = 2^{-m+1}\sum_{k=0}^{m}\binom{m}{k}N_m(2x-k). \tag{3.26}$$

In wavelet analysis we want to project a given function $f \in L^2(\Re)$ onto some space V_j, a subspace of a multiresolution approximation as defined in chapter 2. If we know the values of a function $f \in L^2(\Re)$ at all the integer points, say

$$f_k := f(k), \quad \forall k \in Z \tag{3.27}$$

then we could project f onto a cardinal spline space V_j^m by means of interpolation. We do this using the *centred* cardinal B-spline $N_m(x+\frac{m}{2})$, i.e. we determine the solution $\{c_k\}$ in the equation

$$\sum_{j=-\infty}^{\infty} c_j N_m(x+\frac{m}{2}-j)|_{x=k} = f_k \tag{3.28}$$

or more simply, we determine the *fundamental cardinal spline* function given by the sequence $\{c_k^m\}$

$$L_m(x) = \sum_{k=-\infty}^{\infty} c_k^m N_m(x+\frac{m}{2}-k) \tag{3.29}$$

which satisfies the property

$$L_m(j) = \delta_{j,0} \tag{3.30}$$

so that we can solve (3.28) by forming the linear combination

$$\sum_{k=-\infty}^{\infty} L_m(x-k)f_k. \tag{3.31}$$

Chapter 4

CARDINAL SPLINE WAVELETS

This chapter uses the theory developed in chapter 2 to find the wavelet functions corresponding to the scaling functions N_m defined in chapter 3.

The first section introduces a class of functions known as the Euler-Frobenius polynomials, which are useful in the construction of spline-wavelets.

4.1 EULER-FROBENIUS POLYNOMIALS

For a positive integer n the **Euler-Frobenius polynomial** $E_{n-1}(x)$ is defined:

$$E_{n-1}(z) := (n-1)! z^{[(n-1)/2]} \sum_{k=-\infty}^{\infty} N_n\left(\frac{n}{2}+k\right) z^k \tag{4.1}$$

where $[x]$ denotes the largest integer not exceeding x and N_n is the *B-spline* of order n defined in the previous chapter.
If n is *even* then $n = 2m$ for some positive integer m and (4.1) becomes

$$\begin{aligned} E_{2m-1}(z) &:= (2m-1)! z^{m-1} \sum_{k=-\infty}^{\infty} N_{2m}(m+k) z^k \\ &= (2m-1)! z^{m-1} \sum_{k=-m+1}^{m-1} N_{2m}(m+k) z^k \end{aligned} \tag{4.2}$$

and if n is *odd* then $n = 2m+1$ for a postive integer m and (4.1) becomes

$$\begin{aligned} E_{2m}(z) &:= (2m)! z^m \sum_{k=-\infty}^{\infty} N_{2m+1}\left(\frac{2m+1}{2}+k\right) z^k \\ &= (2m)! z^m \sum_{k=-m}^{m} N_{2m+1}\left(\frac{2m+1}{2}+k\right) z^k. \end{aligned} \tag{4.3}$$

If we define

$$E_{N_m}(z) := \sum_{k=-m+1}^{m-1} N_{2m}(m+k)z^k \tag{4.4}$$

and

$$\Pi_m(z) = z^{m-1}E_{N_m}(z) \tag{4.5}$$

then

$$\begin{aligned} E_{2m-1}(z) &= (2m-1)!\Pi_m(z) \\ &= (2m-1)!z^{m-1}E_{N_m}(z) \end{aligned} \tag{4.6}$$

and we can prove some important identities for $E_{N_m}(z)$ and $\Pi_m(z)$.

Theorem 4.1.1 *The following set of identities relate $E_{N_m}(z)$, $\Pi_m(z)$ and $P_{N_m}(z)$:*
1) Both $E_{N_m}(z)$ and $\Pi_m(z)$ never vanish on $|z|=1$;
2)

$$E_{N_m}(e^{-i\omega}) = \sum_{k=-\infty}^{\infty}|\hat{N}_m(\omega+2\pi k)|^2; \tag{4.7}$$

3)

$$|P_{N_m}(e^{-i\omega/2})|^2E_{N_m}(e^{-i\omega/2}) + |P_{N_m}(-e^{-i\omega/2})|^2E_{N_m}(-e^{-i\omega/2}) = E_{N_m}(e^{-i\omega}), \quad \forall\omega\in\Re; \tag{4.8}$$

4)

$$(P_{N_m}(z))^2\Pi_m(z)-(P_{N_m}(-z))^2\Pi_m(-z) = z\Pi_m(z^2), \qquad \forall z\in C. \tag{4.9}$$

proof
The identity in (4.7) *is an application of the poisson summation formula* (1.7) *[see [1] section 2.5]. From theorem 1.2.1. we have (1) since $\{N_m(\cdot-k)\}$ is a Riesz basis of V_0^m. To derive* (4.8) *we start with* (4.7) *and apply* (3.17)

$$\begin{aligned}
E_{N_m}(e^{-i\omega}) &= \sum_{k=-\infty}^{\infty}|\hat{N}_m(\omega+2\pi k)|^2 \\
&= \sum_{k=-\infty}^{\infty}|P_{N_m}\left(e^{-i\frac{\omega+2\pi k}{2}}\right)\hat{N}_m\left(\frac{\omega+2\pi k}{2}\right)|^2 \\
&= \sum_{k=-\infty}^{\infty}|P_{N_m}\left(e^{-i\omega/2}e^{-i\pi k}\right)\hat{N}_m\left(\frac{\omega+2\pi k}{2}\right)|^2 \\
&= |P_{N_m}(e^{-i\omega/2})|^2\sum_{k\,even}|\hat{N}_m\left(\frac{\omega+2\pi k}{2}\right)|^2+|P_{N_m}(-e^{-i\omega/2})|^2\sum_{k\,odd}|\hat{N}_m\left(\frac{\omega+2\pi k}{2}\right)|^2 \\
&= |P_{N_m}(e^{-i\omega/2})|^2\sum_{k=-\infty}^{\infty}|\hat{N}_m\left(\frac{\omega+4\pi k}{2}\right)|^2 \\
&+ |P_{N_m}(-e^{-i\omega/2})|^2\sum_{k=-\infty}^{\infty}|\hat{N}_m\left(\frac{\omega+2\pi(2k+1)}{2}\right)|^2 \\
&= |P_{N_m}(e^{-i\omega/2})|^2\sum_{k=-\infty}^{\infty}|\hat{N}_m\left(\frac{\omega}{2}+2\pi k\right)|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \quad |P_{N_m}(-e^{-i\omega/2})|^2\sum_{k=-\infty}^{\infty}|\hat{N}_m \quad \frac{\omega+2\pi}{2}+2\pi k \quad |^2 \\
&= \quad |P_{N_m}(e^{-i\omega/2})|^2 E_{N_m}(e^{-i\omega/2}) + |P_{N_m}(-e^{-i\omega/2})|^2 E_{N_m}(-e^{-i\omega/2}).
\end{aligned}
$$

To prove (4.9) we need an expression for $\overline{P_{N_m}(z)}$ in terms of $P_{N_m}(z)$:

$$
\begin{aligned}
\overline{P_{N_m}(z)} &= 2^{-m}\sum_{k=0}^{m}\binom{m}{k}z^{-k} \qquad |z| = e^{-i\omega/2} \\
&= 2^{-m}(z^m z^{-m})\sum_{k=0}^{m}\binom{m}{k}z^{-k} \\
&= 2^{-m}z^{-m}\sum_{k=0}^{m}\binom{m}{k}z^{m-k} \\
&= z^{-m}2^{-m}\sum_{j=0}^{m}\binom{m}{m-j}z^{j} \\
&= z^{-m}2^{-m}\sum_{j=0}^{m}\binom{m}{j}z^{j} \\
&= z^{-m}P_{N_m}(z).
\end{aligned}
$$

We have used the fact that $\binom{m}{m-j} = \binom{m}{j} \quad 0 \le j \le m$. For the first step see (3.22). In the fourth step we have used the substitution $j = m-k$.
Thus we have

$$
\overline{P_{N_m}(z)} = z^{-m}P_{N_m}(z). \tag{4.10}
$$

If we rewrite (4.8) in terms of z for $z = e^{-i\omega/2}$ we obtain

$$
P_{N_m}(z)\overline{P_{N_m}(z)}E_{N_m}(z) + P_{N_m}(-z)\overline{P_{N_m}(-z)}E_{N_m}(-z) = E_{N_m}(z^2), \qquad |z| = 1. \tag{4.11}
$$

If we substitute (4.10) and $E_{N_m}(z) = z^{-(m-1)}\Pi_m(z)$ (from (4.6)) into (4.11) we get

$$
z^{-m}(P_{N_m}(z))^2 z^{1-m}\Pi_m(z) + (-z)^{-m}(P_{N_m}(-z))^2(-z)^{-(m-1)}\Pi_m(-z) = z^{-2(m-1)}\Pi_m(z^2) \tag{4.12}
$$

which simplifies to

$$
z^{1-2m}(P_{N_m}(z))^2\Pi_m(z) - (z)^{1-2m}(P_{N_m}(-z))^2\Pi_m(-z) = z^{-2m+2}\Pi_m(z^2) \tag{4.13}
$$

so that

$$
(P_{N_m}(z))^2\Pi_m(z) - (P_{N_m}(-z))^2\Pi_m(-z) = z\Pi_m(z^2). \tag{4.14}
$$

Since both sides of (4.9) are entire functions (being algebraic polynomials in z), and (4.14) holds for $|z| = 1$ they must be identical for all z.

An important theorem about the roots of the Euler-Frobenius polynomial E_{2m-1} is proved in [1] section 6.4, pg 195-199. The statement is the following:

Theorem 4.1.2 *let m be any positive integer. Then the Euler-Frobenius polynomial E_{2m-1} of order $2m-1$ (or degree $2m-2$) can be written as*

$$E_{2m-1}(z) = \Pi_{j=1}^{2m-2}(z-\lambda_{m,j}) \tag{4.15}$$

where

$$\lambda_{m,2m-2} < \lambda_{m,2m-3} < \cdots < \lambda_{m,m} < 1 < \lambda_{m,m-1} < \cdots < \lambda_{m,1} < 0 \tag{4.16}$$

and

$$\lambda_{m,1}\lambda_{m,2m-2} = \cdots = \lambda_{m,m-1}\lambda_{m,m} = 1. \tag{4.17}$$

Equations (4.2) and (4.3) can be formulated slightly differently:

$$\begin{aligned} E_{2m-1}(z) &= (2m-1)!z^{m-1}\sum_{k=-m+1}^{m-1} N_{2m}(m+k)z^k \\ &= (2m-1)!\sum_{k=-m+1}^{m-1} N_{2m}(m+k)z^{k+m-1} \\ &= (2m-1)!\sum_{j=0}^{2m-1} N_{2m}(j+1)z^j \end{aligned}$$

with the substitution $j = k+m-1$ in the last step.
We now have

$$E_{2m-1}(z) = (2m-1)!\sum_{k=0}^{2m-1} N_{2m}(k+1)z^k. \tag{4.18}$$

Similarly it can be shown that

$$E_{2m}(z) = (2m)!\sum_{k=0}^{2m} N_{2m+1}\left(k+\frac{1}{2}\right)z^k. \tag{4.19}$$

We now have the "*machinery*" needed to construct the compactly supported spline-wavelets.

4.2 COMPACTLY SUPPORTED SPLINE WAVELETS

The method used to construct cardinal spline wavelets is an application of theorem 2.1.3, but first we need:

Theorem 4.2.1 *Suppose that we have, in the general case, a scaling function ϕ generating the sets V_j constituting a MRA of $L^2(\Re)$ with a dual scaling function $\tilde{\phi}$ (see defn 2.1.3) generating the same MRA of $L^2(\Re)$, ie $\tilde{V}_j = V_j \quad j \in \mathcal{Z}$ (where $\tilde{V}_j$ are the sets generated by $\tilde{\phi}$). Then $\tilde{\phi}$ must be determined via the following expression:*

$$\hat{\tilde{\phi}}(\omega) = \frac{\hat{\phi}(\omega)}{\sum_{k=-\infty}^{\infty}|\hat{\phi}(\omega + 2\pi k)|^2}. \tag{4.20}$$

Proof *Since ϕ and $\tilde{\phi}$ are dual scaling functions, for each $n \in \mathcal{Z}$ we have*

$$\begin{aligned}
\delta_{n,0} = <\phi, \tilde{\phi}(\cdot - n)> &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\phi}(\omega)\overline{\hat{\tilde{\phi}}(\omega)}e^{-in\omega}dw \\
&= \sum_{k=-\infty}^{\infty}\frac{1}{2\pi}\int_{2\pi k}^{2\pi(k+1)}\hat{\phi}(\omega)\overline{\hat{\tilde{\phi}}(\omega)}e^{-in\omega}dw \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=-\infty}^{\infty}\hat{\phi}(\omega+2\pi k)\overline{\hat{\tilde{\phi}}(\omega+2\pi k)}\right)e^{-in\omega}dw
\end{aligned}$$

(using the parseval identity (1.5)*) in the first step).*
It follows that

$$\sum_{k=-\infty}^{\infty}\hat{\phi}(\omega + 2\pi k)\overline{\hat{\tilde{\phi}}(\omega + 2\pi k)} \equiv 1. \tag{4.21}$$

We know that $\phi \in V_0$ and $\tilde{\phi} \in \tilde{V}_0$ but we have assumed $V_j = \tilde{V}_j, \quad \forall j \in \mathcal{Z}$. Therefore $V_0 = \tilde{V}_0$. So $\phi \in \tilde{V}_0$ and we can express ϕ in terms of $\tilde{\phi}$ using some sequence $\{\alpha_k\}$ in the following:

$$\phi(x) = \sum_{k=-\infty}^{\infty}\alpha_k\tilde{\phi}(x-k). \tag{4.22}$$

Taking Fourier transforms of both sides gives

$$\hat{\phi}(\omega) = \hat{\tilde{\phi}}(\omega)\sum_{j=-\infty}^{\infty}\alpha_j e^{-i\omega j}. \tag{4.23}$$

Therefore

$$\begin{aligned}
\hat{\phi}(\omega + 2\pi k) &= \hat{\tilde{\phi}}(\omega + 2\pi k)\sum_{j=-\infty}^{\infty}\alpha_j e^{-i(\omega+2\pi k)j} \qquad (4.24)\\
&= \hat{\tilde{\phi}}(\omega + 2\pi k)\sum_{j=-\infty}^{\infty}\alpha_j e^{-i\omega}.
\end{aligned}$$

Substituting (4.24) *in* (4.21) *gives*

$$\sum_{k=-\infty}^{\infty}\hat{\phi}(\omega + 2\pi k)\overline{\frac{\hat{\phi}(\omega + 2\pi k)}{\sum_{j=-\infty}^{\infty}\alpha_j e^{-i\omega j}}} \equiv 1. \tag{4.25}$$

Therefore

$$\begin{aligned} \sum_{k=-\infty}^{\infty}|\hat{\phi}(\omega+2\pi k)|^2 &= \overline{\sum_{j=-\infty}^{\infty}\alpha_j e^{-i\omega j}} \\ &= \sum_{j=-\infty}^{\infty}\alpha_j e^{-i\omega j} \\ &= \frac{\hat{\phi}(\omega)}{\hat{\tilde{\phi}}(\omega)}. \end{aligned}$$

The second step follows from the fact that the LHS is positive and the last step from (4.23).
Therefore

$$\hat{\tilde{\phi}}(\omega) = \frac{\hat{\phi}(\omega)}{\sum_{k=-\infty}^{\infty}|\hat{\phi}(\omega+2\pi k)|^2}. \tag{4.26}$$

Once a MRA of $L^2(\Re)$ (consisting if sets V_j determined by the scaling function ϕ) has been determined it is always possible to introduce the sets W_j satisfying $V_{j+1} = V_j \dot{+} W_j$ [see (2.14)]. Theorem 2.1.2 states under what conditions a function ψ generates the sets W_j. In theorem 4.2.1 we have a dual scaling function $\tilde{\phi}$ generating *the same* MRA with sets $\tilde{V}_j = V_j, \quad \forall j \in \mathcal{Z}$. Clearly then the sets $\tilde{W}_j$ determined by the $\tilde{V}_j$ are the same as the sets W_j determined by the V_j, i.e. $W_j = \tilde{W}_j, \quad \forall j \in \mathcal{Z}$. Theorem 2.1.3 states that it is possible here to choose *dual* wavelets $\psi \in W_0$ and $\tilde{\psi} \in \tilde{W}_0$ so that the sets W_j are generated by ψ and the sets $\tilde{W}_j$ are generated by $\tilde{\psi}$. However, as a result of the equivalence of the sets W_j and $\tilde{W}_j$, the sets W_j ($\tilde{W}_j$) are orthogonal *(i.e. if $f \in W_j$ and $g \in W_k$ then $< f, g >= \delta_{j,k}$)* and the wavelets ψ and $\tilde{\psi}$ are semi-orthogonal wavelets. (See definition 1.3.2 (1.19)). Consider the following argument:
Choose $\psi_{j,k} \in W_j$ and $\psi_{l,m} \in W_l$ for some $j, k, l, m \in \mathcal{Z}$
Since $W_l = \tilde{W}_l$, $\psi_{l,m} \in \tilde{W}_l$, we can write

$$\psi_{l,m}(x) = \sum_{t=-\infty}^{\infty}\alpha_t\tilde{\psi}(2^l x - t) \tag{4.27}$$

for some $\{\alpha_t\}$.
Therefore

$$\begin{aligned} < \psi_{j,k}, \psi_{l,m} > &= < \psi_{j,k}, \sum_{t=-\infty}^{\infty}\alpha_t\tilde{\psi}(2^l x - t) > \\ &= \sum_{t=-\infty}^{\infty}\alpha_t < \psi_{j,k}, \tilde{\psi}_{l,t} > \\ &= \sum_{t=-\infty}^{\infty}\alpha_t\delta_{j,l}\delta_{k,t} \\ &= \left\{ \begin{array}{lll} \alpha_k & if & j = l \\ 0 & if & j \neq l \end{array} \right\}. \end{aligned}$$

Instead of (2.17) we now have

$$L^2(\Re) = \cdots \oplus W_{-1} \oplus W_0 \oplus \cdots \tag{4.28}$$

(an orthogonal sum), and instead of (2.14) the orthogonal sum

$$V_{j+1} = V_j \oplus W_j. \tag{4.29}$$

Consider now the scaling function N_m with finite two-scale sequence derived in (3.25) and two-scale symbol in (3.22). For $z = e^{-i\omega/2}$ (4.7) becomes

$$E_{N_m}(z) = \sum_{k=-\infty}^{\infty} |\hat{N}_m\left(\frac{\omega}{2} + 2\pi k\right)|^2, \qquad z = e^{-i\omega/2}. \tag{4.30}$$

The dual scaling function $\tilde{N}_m$ of N_m generating the same MRA as N_m can be defined using (4.20) in theorem 4.2.1 and identity (4.7):

$$\hat{\tilde{N}}_m = \frac{\hat{N}_m(\omega)}{E_{N_m}(z^2)}, \qquad z = e^{-i\omega/2}. \tag{4.31}$$

Note that (4.7) shows that $E_{N_m}(z) > 0$ on $|z| = 1$.
The two-scale symbol $G^* = G^*_{\tilde{N}_m}$ can be determined from

$$\begin{aligned} \hat{\tilde{N}}_m(\omega) &= \frac{1}{E_{N_m}(z^2)}\hat{N}_m(\omega) \\ &= \frac{1}{E_{N_m}(z^2)} P_{N_m}(z)\hat{N}_m\left(\frac{\omega}{2}\right) \\ &= \frac{E_{N_m}(z)}{E_{N_m}(z^2)} P_{N_m}(z)\hat{\tilde{N}}_m(\omega/2). \end{aligned}$$

As mentioned in section 2.1, this can be used as the defining equation for G^*. Hence

$$G^*(z) = \frac{E_{N_m}(z)}{E_{N_m}(z^2)} P_{N_m}(z). \tag{4.32}$$

In theorem 2.1.3 we chose the two-scale symbol Q for a wavelet ψ relative to a scaling function ϕ from the class

$$Q(z) = z^{-1} G(-z) k(z^2) \tag{4.33}$$

where $k \in \mathcal{W}$ and $k(z) \neq 0$ for $|z| = 1$.

If $Q(z)$ determines a wavelet ψ relative to the scaling function N_m then we can express it in terms of the Euler-Frobenius polynomial $E_{2m-1}(z)$ using equation (4.32).

$$\begin{aligned} G(z) &= \overline{\frac{E_{N_m}(z)}{E_{N_m}(z^2)} P_{N_m}(z)}, \qquad |z| = 1 \\ &= \frac{E_{N_m}(z)}{E_{N_m}(z^2)} \overline{P_{N_m}(z)}, \quad . \quad |z| = 1. \end{aligned}$$

Therefore, recalling (3.23) and (4.6) we have

$$G(-z) = \frac{E_{N_m}(-z)}{E_{N_m}(z^2)} \overline{P(-z)} \tag{4.34}$$

$$
\begin{aligned}
&= \frac{E_{N_m}(-z)}{E_{N_m}(z^2)}\left(\frac{1-z}{2}\right)^m \\
&= \frac{E_{2m-1}(-z)}{E_{2m-1}(z^2)}\frac{(-z)^{1-m}}{z^{3-2m}}\left(\frac{z-1}{2z}\right)^m \\
&= -\frac{1}{z}\frac{E_{2m-1}(-z)}{E_{2m-1}(z^2)}\left(\frac{1-z}{2}\right)^m.
\end{aligned}
$$

Substituting (4.34) into (4.33) gives:

$$
Q(z) = -z^2\left(\frac{1-z}{2}\right)^m E_{2m-1}(-z)\frac{k(z^2)}{E_{2m-1}(z^2)} \tag{4.35}
$$

where $k \in \mathcal{W}, \quad k(z) \neq 1$ on $|z|^2 = 1$.
For the wavelet ψ determined by $Q(z)$ to have compact support the sequence $\{q_k\}$ determined by Q must be finite, i.e. $k(z^2)$ must be chosen so that $Q(z)$ is a polynomial [see section 2.1.1].

Suppose α is a zero of $E_{2m-1}(z^2)$. Then $(-\alpha)$ is also a zero (because $E_{2m-1}(\alpha^2) = E_{2m-1}((-\alpha)^2)$), i.e. we could write

$$
E_{2m-1}(z) = (z-\alpha)(z+\alpha)R_1(z^2) \tag{4.36}
$$

for some remainder polynomial $R_1(z)$. Notice from theorem 4.1.2 that the zeros of $E_{2m-1}(z)$ are all negative, which implies that the zeros of $E_{2m-1}(-z)$ are all positive, and so only one of the terms $(z-\alpha)$, $(z+\alpha)$ may be a common factor with $E_{2m-1}(-z)$. If we were to cancel this term, the remaining term $(z-\alpha)R_1(z^2)$ or $(z+\alpha)R_1(z^2)$ would not be a polynomial consisting of powers of z^2 (the highest order term is odd) and so could not cancel with $k(z^2)$. Therefore, to choose $k(z^2)$ so that $Q(z)$ is a polynomial it is necessary and sufficient that $k(z^2)$ contains a factor $E_{2m-1}(z^2)$ i.e.

$$
k(z) = c_0 z^{n_0} E_{2m-1}(z) \tag{4.37}
$$

where c_0 is a non-zero constant and n_0 is any positive integer.

If we choose $c_0 = \frac{-1}{(2m-1)!}$ and $n_0 = 1$ then (4.35) simplifies to

$$
Q_m(z) := \frac{1}{2}\textstyle\sum_{n=-\infty}^{\infty} q_n z^n := \left(\frac{1-z}{2}\right)^m \displaystyle\sum_{k=0}^{2m-2} N_{2m}(k+1)(-z)^k \tag{4.38}
$$

(where we have used (4.18)) and we can define

$$
\psi_m(x) := \textstyle\sum_{n=-\infty}^{\infty} q_n N_m(2x-n) \tag{4.39}
$$

using the sequence $\{q_n\}$ in (4.38). ψ_m is called the m^{th} *order B-wavelet*.

The sequence $\{q_n\}$ can be determined explicitly:

$$\begin{aligned}
\textstyle\sum_{n=-\infty}^{\infty} q_n z^n &:= 2\left(\frac{1-z}{2}\right)^m \sum_{k=0}^{2m-2} N_{2m}(k+1)(-z)^k \\
&= \frac{1}{2^{m-1}} \sum_{l=0}^{m} \binom{m}{l} (-z)^l \sum_{k=0}^{2m-2} N_{2m}(k+1)(-z)^k \\
&= \frac{1}{2^{m-1}} \sum_{l=0}^{m} \sum_{k=0}^{2m-2} \binom{m}{l} N_{2m}(k+1)(-z)^{k+l} \\
&= \textstyle\sum_{j=-\infty}^{\infty} (-1)^j \frac{1}{2^{m-1}} \left(\sum_{l=0}^{m} \binom{m}{l} N_{2m}(j-l+1) \right) z^j
\end{aligned}$$

where we had the substitution $j = k + l$ in the last step.
Therefore

$$q_n = \frac{(-1)^n}{2^{m-1}} \sum_{l=0}^{m} \binom{m}{l} N_{2m}(n+1-l). \tag{4.40}$$

The support of ψ_m can also be determined:
Since

$$suppN_{2m} = [0, 2m]$$

by (3) of theorem 3.1.1,
$q_n \neq 0$ for

$$0 < n+1 < 2m \tag{4.41}$$

for $l = 0$ in (4.40) or

$$0 < n+1-m < 2m \tag{4.42}$$

for $l = m$ in (4.40).
(we only need to consider the smallest and largest values of l) [see (4) of theorem 3.1.1].
i.e. $q_n \neq 0$ for $n = 0, 1, \cdots, 3m-2$.
Therefore

$$\psi_m(x) = \sum_{n=0}^{3m-2} q_n N_m(2x-n). \tag{4.43}$$

$\psi_m(x) \neq 0$ when $N_m(2x-n) \neq 0$ for some $n = 0, \cdots, 3m-2$.
$suppN_m = [0, m]$, again by (3) of theorem 3.1.1 .
Therefore

$$\psi_m \neq 0$$

when $0 < 2x - n < m \quad n = 0, \cdots, 3m-2$.
Consider extreme cases
(1) $n = 0$ and
(2) $n = 3m-2$.
Then
1) $0 < 2x - (0) < m$

i.e. $0 < x < \frac{m}{2}$;

2) $0 < 2x - (3m-2) < m$

i.e. $\frac{3m}{2} + 1 < x < 2m - 1$.

Therefore $\psi_m(x) \neq 0$ if $0 < x < 2m - 1$ (combining (1) and (2)).

i.e.

$$supp\psi_m = [0, 2m-1]. \tag{4.44}$$

The results in this section can be summarised in the following theorem:

Theorem 4.2.2 *Let m be any positive integer. Also let N_m be the m^{th} order cardinal B-spline and ψ_m the B-wavelet defined in* (4.39) *with coefficient sequence given by* (4.40). *Then $\{\psi_m(\cdot - k) : k \in \mathcal{Z}\}$ is a Riesz basis of W_0 (the wavelet space generated by ψ_m). Furthermore ψ_m has compact support given by $supp\psi_m = [0, 2m-1]$.*

In the next section the dual wavelets $\tilde{\psi}_m$ are defined in terms of *interpolation splines*.

4.3 CONSTRUCTION OF A DUAL WAVELET

The scaling function N_m generates a MRA of $L^2(\Re)$ with sets V_j. In the previous section the wavelet functions ψ_m with compact support that generate the orthogonal spaces W_j satisfying

$$V_{j+1} = V_j \oplus W_j, \qquad \forall j \in \mathcal{Z} \tag{4.45}$$

were determined. In this section the spline functions $\psi_{I,m}$ that do not have compact support (for $m \geq 2$) that also generate the sets W_j are determined as well as the relationship of these functions with the dual $\tilde{\psi}_m$ of ψ_m.

If equation (4.40) is applied for $m = 1$, then

$$q_n = (-1)^n [\binom{1}{0} N_2(n+1) + \binom{1}{1} N_2(n)] \tag{4.46}$$

Therefore

$$\begin{aligned} q_0 &= N_2(1) + N_2(0) \\ q_1 &= -(N_2(1) + N_2(2)) \end{aligned} \tag{4.47}$$

An application of definition 3.1.3 for $m = 2$ shows that

$$N_2(x) = \left\{ \begin{array}{lcl} 2-x & : & 1 \leq x < 2 \\ x & : & 0 \leq x < 1 \end{array} \right\}. \tag{4.48}$$

Therefore

$$q_0 = 1, \qquad q_1 = -1.$$

So, by (4.43)

$$\begin{aligned} \psi_1(x) &= q_0 N_1(2x) + q_1 N_1(2x-1) \\ &= N_1(2x) - N_1(2x-1) \\ &= \left\{ \begin{array}{rcl} 1 & : & 0 \leq x < \frac{1}{2} \\ -1 & : & \frac{1}{2} \leq x < 1 \end{array} \right\}. \end{aligned} \tag{4.49}$$

$\psi_H := \psi_1$ is called the *Haar wavelet.*
It follows from (6) in theorem 3.1.1 that ψ_H is related to the derivative of the second order cardinal B-spline N_2, namely

$$\psi_H(x) = N_2'(2x) = N_1(2x) - N_1(2x-1). \tag{4.50}$$

The B-spline N_2 can be viewed as a fundamental cardinal spline [see (3.29)]. In fact, the second order cardinal spline L_2 is defined by:

$$L_2(x) = N_2(x-1). \tag{4.51}$$

Furthermore, with an application again of (6) in theorem 3.1.1

$$\begin{aligned} L_2'(x) &= \frac{d}{dx}N_2(x-1) \\ &= \triangle N_1(x+1) \\ &= N_1(x+1) - N_1(x). \end{aligned}$$

So

$$L_2'(2x-1) = N_1(2x) - N_1(2x-1), \tag{4.52}$$

which gives an equivalence between $\psi_H(x)$ and $L_2'(2x-1)$:

$$\psi_H(x) = L_2'(2x-1). \tag{4.53}$$

This statement can be generalised in the following theorem:

Theorem 4.3.1 *Let $\{V_j^m\}$ be the MRA of $L^2(\Re)$ generated by the m^{th} order cardinal B-spline and let $\{W_j^m : j \in \mathbb{Z}\}$ denote the orthogonal complementary wavelet spaces such that*

$$V_{j+1}^m = V_j^m \oplus W_j^m \quad j \in \mathbb{Z}.$$

Define

$$\psi_{I,m}(x) := L_{2m}^{(m)}(2x-1), \tag{4.54}$$

where L_{2m} is the $(2m)^{th}$ order fundamental cardinal spline. Then $\psi_{I,m}$ generates the wavelet spaces $\{W_j^m : j \in \mathbb{Z}\}$, i.e.

$$W_j^m = clos_{L^2(\Re)} < 2^{j/2}\psi_{I,m}(2^jx-k) : k \in \mathbb{Z} >. \tag{4.55}$$

proof

We first verify that $\psi_{I,m} \in W_0^m$.

If (6) of theorem 3.1.1 is succesfully applied m times to determine the m^{th} derivative of the m^{th} order cardinal B-spline then we get a finite linear combination of integer translates of the delta distribution:

$$N_m^{(m)}(x) = \sum_{k=0}^{m} \frac{1}{2m}(-1)^{m-k}\binom{m}{k}\delta(x-k).$$

Hence, for every $n \in \mathbb{Z}$

$$\begin{aligned} < N_m(\cdot - n), \psi_{I,m} > &= \int_{-\infty}^{\infty} N_m(x-n)L_{2m}^{(m)}(2x-1)dx \\ &= \frac{(-1)^m}{2^m}\int_{-\infty}^{\infty} N_m^{(m)}(x-n)L_{2m}(2x-1)dx \\ &= \sum_{k=0}^{m}\frac{(-1)^{m-k}}{2^m}\binom{m}{k}\int_{-\infty}^{\infty}\delta(x-n-k)L_{2m}(2x-1)dx \\ &= \sum_{k=0}^{m}\frac{(-1)^{m-k}}{2^m}\binom{m}{k} L_{2m}(2n+2k-1)dx \\ &= 0, \end{aligned}$$

since $L_{2m}(l) = \delta_{l,0}, \quad l \in Z$ *. Use integration by parts to get the second step.*

We investigate the two-scale relation of $\psi_{I,m}$ *with respect to* $N_m(2x-k), \quad k \in Z$ *, i.e, we are interested in the* l^2 *sequence* $\{q_k\}$ *for which*

$$\psi_{I,m} \;=\; L_{2m}^{(m)}(2x-1) \;=\; \sum_{k=-\infty}^{\infty} q_k N_m(2x-k). \tag{4.56}$$

If we can detetmine such a sequence then we would have established that $\psi_{I,m} \in V_1$ *and the above result would ensure that* $\psi_{I,m} \in W_0$*.*

Keeping the same notation as in (3.29)

$$L_{2m}(x) \;=\; \sum_{k=-\infty}^{\infty} c_k^{(2m)} N_{2m}(x+m-k). \tag{4.57}$$

By repeatedly applying (6) of theorem 3.1.1 it follows that

$$\begin{aligned} N_{2m}^{(m)}(x) &= (\triangle N_{2m-1}^{(m-1)})(x) \\ &\;\vdots \\ &= (\triangle^m N_m)(x) \\ &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} N_m(x-k). \end{aligned} \tag{4.58}$$

So if we differentiate (4.57) m *times we have*

$$\begin{aligned} \psi_{I,m}(x) = L_{2m}^{(m)}(2x-1) &= \sum_{k=-\infty}^{\infty} c_k^{(2m)} N_{2m}^{(m)}(2x-1+m-k) \\ &= \sum_{k=-\infty}^{\infty} c_k^{(2m)} \sum_{l=0}^{m} (-1)^l \binom{m}{k} N_m^{(m)}(2x-1+m-k-l) \\ &= \sum_{n=-\infty}^{\infty} q_n^\dagger N_m(2x-n) \end{aligned}$$

with $q_n^\dagger = \sum_{l=0}^{m} (-1)^l \binom{m}{k} c_{m+n-1-l}^{(2m)}$*.*

We can determine the two-scale symbol $Q^\dagger$ *corresponding to the two-scale sequence* $\{q_k^\dagger\}$*:*

$$\begin{aligned} Q^\dagger(z) &= \frac{1}{2}\sum_{n=-\infty}^{\infty} q_n^\dagger z^n \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \left(\sum_{l=0}^{m} (-1)^l \binom{m}{l} c_{m+n-1-l}^{(2m)} \right) z^n \\ &= \frac{1}{2}\sum_{j=-\infty}^{\infty} \left(\sum_{l=0}^{m} (-1)^l \binom{m}{l} c_j^{(2m)} z^{j-m+l+1} \right) \\ &= \frac{1}{2} z^{-m+1} \sum_{j=-\infty}^{\infty} \left(\sum_{l=0}^{m} (-1)^l z^l \binom{m}{l} \right) c_j^{(2m)} z^j \\ &= \frac{1}{2} z^{-m+1} (1-z)^m \sum_{j=-\infty}^{\infty} c_j^{(2m)} z^j \end{aligned} \tag{4.59}$$

In the third step we used the substitution $j = m+n-1-l$*.*

Form the interpolatory property $L_{2m}(l) = \delta_{l,0}$ *it follows from* (4.57) *that*

$$\begin{aligned} \sum_{k=-\infty}^{\infty} c_k^{(2m)} N_{2m}(l+m-k) &= 0 \qquad l \neq 0 \\ \sum_{k=-\infty}^{\infty} c_k^{(2m)} N_{2m}(m-k) &= 1. \end{aligned} \tag{4.60}$$

We can establish the following identity

$$\begin{aligned}\left(\sum_{-m+1}^{m-1} N_{2m}(m+k)z^k\right)\left(\sum_{n=-\infty}^{\infty} c_n^{(2m)} z^n\right) &= \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} N_{2m}(m+k)c_n^{(2m)} z^{n+k} \\ &= \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} N_{2m}(m+l-n)c_n^{(2m)} z^l \\ &= \sum_{n=-\infty}^{\infty} N_{2m}(m+l-n)c_n^{(2m)} z^l|_{l=0} \\ &= \sum_{n=-\infty}^{\infty} N_{2m}(m-n)c_n^{(2m)} \\ &= 1.\end{aligned}$$

In the second step we have substituted $l = n+k$ and in the third and final steps used equation (4.60).
Therefore

$$\begin{aligned}\sum_{n=-\infty}^{\infty} c_n^{(2m)} z^n &= \frac{1}{\sum_{-m+1}^{m-1} N_{2m}(m+k)z^k} \\ &= \frac{1}{E_{N_m}(z)}.\end{aligned} \tag{4.61}$$

Applying this to (4.59) *gives*

$$Q^\dagger(z) = \frac{z^{-m+1}}{2}(1-z)^m \frac{1}{E_{N_m}(z)}. \tag{4.62}$$

Let us construct the set:

$$W_0^* := clos_{L^2(\Re)}\{\psi_{I,m}(x-k) : k \in Z\}.$$

Then we know from the first part of this proof that $W_0^* \subset W_0$. However, if we can establish that

$$V_1 = V_0 + W_0^*$$

then, in fact, we have $W_0^* = W_0$.
This is where we need theorem 2.1.2.

Since the two-scale symbol of N_m is given by (3.23)*, we can compute the determinant $\Delta_{P,Q^\dagger}$ defined in* (2.10)*:*

$$\begin{aligned}\Delta_{P,Q^\dagger}(z) &= det\begin{bmatrix} P(z) & Q^\dagger(z) \\ P(-z) & Q^\dagger(-z)\end{bmatrix} \\ &= \frac{(-z)^{-m+1}(1+z)^{2m}}{2^{m+1}E_{N_m}(-z)} - \frac{(z)^{-m+1}(1-z)^{2m}}{2^{m+1}E_{N_m}(z)}.\end{aligned}$$

Recall from section 4.1 that $\Pi_m(z) := z^{m-1}E_{N_m}(z)$. Thus

$$\Delta_{P,Q^\dagger}(z) = 2^{m-1}\left(\frac{(P_{N_m}(z))^2}{\Pi_m(-z)} - \frac{(P(-z))^2}{\Pi_m(z)}\right)$$

$$
\begin{aligned}
&= 2^{m-1}\ \frac{\Pi_m(z)(P_{N_m}(z))^2 - \Pi_m(-z)(P_{N_m}(-z))^2}{\Pi_m(z)\Pi_m(-z)} \\
&= 2^{m-1}\frac{z\Pi_m(z^2)}{\Pi_m(z)\Pi_m(-z)} \\
&= 2^{m-1}\frac{zE_{N_m}(z^2)}{E_{N_m}(-z)E_{N_m}(z)}
\end{aligned}
$$

where the third step is an application of identity (4.9) in theorem 4.1.1.
(1) of theorem 4.1.1 states that $E(z)$ never vanishes on the unit circle. Therefore it follows that $\Delta_{P,Q}(z) \neq 0 \quad on |z| = 1$.
The result in theorem 2.1.2 sh ws that we have completed the proof of this theorem.

We now establish that the family $\{\psi_{I,m}(2^j x - k) \; : k \in \mathbb{Z}\}$ is a Riesz basis of $W_j, \quad j \in \mathbb{Z}$. We begin by defining the space

$$V_1^{2m,0} := \{s \in V_1^{2m} \; : s(k) = 0, \quad k \in \mathbb{Z}\} \tag{4.63}$$

of cardinal splines of order $2m$ with knot sequence $\frac{1}{2}\mathbb{Z}$ and vanishing at all the integers. It is clear that the function

$$\Psi_{2m}(x) := \frac{1}{2^m} L_{2m}(2x-1) \tag{4.64}$$

is in $V_1^{2m,0}$ and so are all its integer translates. In fact, we have the next result:

Theorem 4.3.2 *For each m, the family*

$$\{\Psi_{2m}(\cdot - k) \; : k \in \mathbb{Z}\} \tag{4.65}$$

is a Riesz basis of $V_1^{2m,0}$.

proof *To show that the linear span of the family in (4.65) is dense in $V_1^{2m,0}$ we let $G \in V_1^{2m,0}$ be chosen arbitrarily. Then by applying theorem 3.1.1 (2), we have*

$$\int_{-\infty}^{\infty} G^{(m)}(x) N_m(x-l)dx = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}G(k+l) = 0, \qquad l \in \mathbb{Z}. \tag{4.66}$$

Consequently, since it is clear from theorem 3.1.1 (6) that $G^{(m)}$ is in V_1^m, the derivation in (4.66) shows that $G^{(m)}$ is orthogonal to the subspace V_0^m of V_1^m and hence lies in W_0^m. By theorem (4.3.1) we have

$$G^{(m)}(x) = \sum_{n=-\infty}^{\infty} a_n \psi_{I,m}(x-n) \tag{4.67}$$

for some sequence $\{a_n\} \in l^2$.
Also, observe from the definition (4.64) of Ψ_{2m} we have

$$\Psi_{2m}^{(m)}(x) = \psi_{I,m}(x) \tag{4.68}$$

so that

$$D^m\{G(x) - \textstyle\sum_{n=-\infty}^{\infty} a_n \Psi_{2m}(x-n)\} \;=\; 0, \qquad x \in \Re, \tag{4.69}$$

where D^m denotes the m^{th} order differential operator. Since $V_1^{2m,0}$ consists only of functions in $L^2(\Re)$ that vanish at Z, we have

$$G(x) \;=\; \textstyle\sum_{n=-\infty}^{\infty} a_n \Psi_{2m}(x-n), \qquad \{a_k\} \in l^2. \tag{4.70}$$

To show that the basis in (4.65) is a Riesz basis we note that

$$\begin{aligned} \hat{\Psi}_{2m}(w) &= \frac{1}{2^m}\hat{L}_{2m}(2\cdot -1) \\ &= \frac{1}{2^m}\textstyle\int_{-\infty}^{\infty} e^{-i\omega x} L_{2m}(2x-1)dx \\ &= \frac{1}{2^m}\frac{1}{2}\textstyle\int_{-\infty}^{\infty} e^{-i\omega(y+1)/2} L_{2m}(y)dy \\ &= \frac{e^{-i\omega/2}}{2^{m+1}}\hat{L}_{2m}\left(\frac{\omega}{2}\right). \end{aligned} \tag{4.71}$$

Furthermore

$$\begin{aligned} \hat{L}_{2m}(\omega) &= \textstyle\int_{-\infty}^{\infty} e^{-i\omega x} L_{2m}(x) \\ &= \textstyle\int_{-\infty}^{\infty} e^{-i\omega x}\sum_{k=-\infty}^{\infty} c_k^{(2m)} N_{2m}(x+m-k)dx \\ &= \textstyle\sum_{k=-\infty}^{\infty} c_k^{(2m)} \int_{-\infty}^{\infty} e^{-i\omega(y-m+k)} N_{2m}(y)dy \\ &= e^{-i\omega m}\hat{N}_{2m}(\omega)\textstyle\sum_{k=-\infty}^{\infty} c_k^{(2m)} e^{-i\omega k}. \end{aligned}$$

Therefore

$$\hat{L}_{2m}\left(\frac{\omega}{2}\right) \;=\; e^{i(\omega/2)m}\hat{N}_{2m}\left(\frac{\omega}{2}\right)\textstyle\sum_{k=-\infty}^{\infty} c_k^{(2m)} e^{-i(\omega/2)k}. \tag{4.72}$$

Upon substitution of (4.72) into (4.71) we have

$$\hat{\Psi}_{2m}(\omega) \;=\; 2^{-m-1} e^{i(m-1)\omega/2}\left(\textstyle\sum_{k=-\infty}^{\infty} c_k^{(2m)} e^{-ik\omega/2}\right)\hat{N}_{2m}\left(\frac{\omega}{2}\right) \tag{4.73}$$

from which it follows that

$$\begin{aligned} &\textstyle\sum_{k=-\infty}^{\infty} |\hat{\Psi}_{2m}(\omega+2\pi k)|^2 \\ &= \textstyle\sum_{k=-\infty}^{\infty} (\hat{\Psi}_{2m}(\omega+2\pi k))\overline{(\hat{\Psi}_{2m}(\omega+2\pi k))} \\ &= 2^{-2m-2}\textstyle\sum_{k=-\infty}^{\infty}(\sum_{j=-\infty}^{\infty} c_j^{(2m)} e^{-ij\frac{\omega+2\pi k}{2}})\overline{(\sum_{j=-\infty}^{\infty} c_j^{(2m)} e^{-ij\frac{\omega+2\pi k}{2}})}|\hat{N}_{2m}\left(\frac{\omega+2\pi k}{2}\right)|^2 \\ &= 2^{-2m-2}\left\{\begin{array}{c} \sum_{k \text{ even}}\left[\left(\sum_{j=-\infty}^{\infty} c_j^{(2m)}(e^{-i\omega/2})^j\right)\overline{\left(\sum_{j=-\infty}^{\infty} c_j^{(2m)}(e^{-i\omega/2})^j\right)}|\hat{N}_{2m}\left(\frac{\omega+2\pi k}{2}\right)|^2\right] \\ \sum_{k \text{ odd}}\left[\left(\sum_{j=-\infty}^{\infty} c_j^{(2m)}(-e^{-i\omega/2})^j\right)\overline{\left(\sum_{j=-\infty}^{\infty} c_j^{(2m)}(-e^{-i\omega/2})^j\right)}|\hat{N}_{2m}\left(\frac{\omega+2\pi k}{2}\right)|^2\right] \end{array}\right\} \\ &= 2^{-2m+2}|\hat{N}_{2m}(\omega/2+2\pi k)|^2\textstyle\sum_{k=-\infty}^{\infty}\frac{1}{(E_{N_m}(z))^2} + |\hat{N}_{2m}(\omega/2+2\pi k+\pi)|^2\sum_{k=-\infty}^{\infty}\frac{1}{(E_{N_m}(-z))^2}. \end{aligned}$$

To obtain the last step we have applied the identity in (4.6.2), using $z = e^{-i\omega/2}$ and the positivity of $E(z)$ on $|z| = 1$.
Then since

$$E_{N_{2m}}(z) \;=\; \textstyle\sum_{k=-\infty}^{\infty} |\hat{N}_{2m}(\omega/2 + 2\pi k)|^2 \tag{4.74}$$

and

$$E_{N_{2m}}(-z) \;=\; \textstyle\sum_{k=-\infty}^{\infty} |\hat{N}_{2m}(\omega/2 + 2\pi k + \pi)|^2 \tag{4.75}$$

(4.7) *for $z = e^{-i\omega/2}$ we get*

$$\textstyle\sum_{k=-\infty}^{\infty} |\hat{\Psi}_{2m}(\omega + 2\pi k)|^2 \;=\; 2^{-2m-2}\left\{\frac{E_{N_{2m}}(z)}{(E_{N_m}(z))^2} + \frac{E_{N_{2m}}(-z)}{(E_{N_m}(-z))^2}\right\} \tag{4.76}$$

for $z = e^{-i\omega/2}$.
By the positivity of $E_{N_m}(z)$ on the unit circle $|z| = 1$ (see theorem 4.1.1 (1)) the result in theorem 1.2.1 tells us that the set in (4.65) is a Riesz basis of $V_1^{2m,0}$.

As a consequence of theorems 4.3.1 and theorem 4.3.2 we have the following result:

Theorem 4.3.3 *For each positive integer m, the m^{th} order differential operator D^m maps the spline space $V_1^{2m,0}$ one-to-one onto the wavelet space W_0^m. In addition, the Riesz basis $\{\Psi_{2m}(\cdot - k) \;:\; k \in \mathcal{Z}\}$ of $V_1^{2m,0}$ corresponds to the Riesz basis $\{\psi_{I,m}(\cdot - k) \;:\; k \in \mathcal{Z}\}$ of W_0^m via the relation $\psi_{I,m} = D^m \Psi_{2m}$.*

We are now able to determine the dual wavelet $\tilde{\psi}_m$ of the B-wavelet ψ_m defined in the previous section.

We have an equation in theorem 2.1.3 to determine the two-scale symbol H^* of $\tilde{\psi}$ relative to $\tilde{N}_m$:

$$H(z) \;=\; zP(z)k^{-1}(z^2) \tag{4.77}$$

where $H(z) = \overline{H^*(z)}$ for $|z| = 1$.
In view of the dual relationship (4.31) for $\tilde{N}_m$ we have for $z = e^{-i\omega/2}$

$$\begin{aligned} \hat{\tilde{\psi}}_m(\omega) &= H^*(z)\hat{\tilde{N}}_m(\frac{\omega}{2}) \\ &= H^*(z)\frac{1}{E_{N_m}(z)}\hat{N}_m(\frac{\omega}{2}). \end{aligned} \tag{4.78}$$

The two-scale symbol $Q^\dagger(z)$ in (4.62) determines the relationship between $\hat{N}_m$ and $\hat{\psi}_{I,m}$:

$$\begin{aligned} \hat{\psi}_{I,m}(\omega) &= Q^\dagger(z)\hat{\tilde{N}}_m(\frac{\omega}{2}), \qquad z = e^{-i\omega/2} \\ &= \frac{z^{-m+1}}{2}(1-z)^m \frac{1}{E_{N_m}(z)}\hat{N}_m(\frac{\omega}{2}). \end{aligned}$$

Therefore

$$\hat{N}_m(\frac{\omega}{2}) = \hat{\psi}_{I,m}(\omega)\frac{2z^{m-1}}{(1-z)^m}E_{N_m}(z). \tag{4.79}$$

We can substitute (4.79) into (4.78) to establish a relationship between the dual wavelet $\tilde{\psi}_m$ and $\psi_{I,m}$.

$$\begin{aligned}
\hat{\tilde{\psi}}_m(\omega) &= H^*(z)\frac{1}{E_{N_m}(z)}\left(E_{N_m}\frac{2}{z^{-m+1}(1-z)^m}\right)\hat{\psi}_{I,m}(\omega)\\
&= H^*(z)\left(\frac{2}{z^{-m+1}(1-z)^m}\right)\hat{\psi}_{I,m}(\omega)\\
&= \overline{zP(-z)k^{-1}(z^2)}\left(\frac{2z^{m-1}}{(1-z)^m}\right)\hat{\psi}_{I,m}(\omega)\\
&= \frac{z^{-1}(1-1/z)^m}{2^m k(z^2)}\left(\frac{2z^{m-1}}{(1-z)^m}\right)\hat{\psi}_{I,m}(\omega)\\
&= (-1)^m 2^{-m+1}z^{-2}\frac{1}{k(z^2)}\hat{\psi}_{I,m}(\omega), \qquad z = e^{-i\omega/2}.
\end{aligned}$$

Recall that $k(z)$ was defined for ψ_m in (4.37) and we chose $c_0 = \frac{-1}{(2m-1)!}$ and $n_0 = 1$. Using this expression for $k(z^2)$ in the above gives

$$\begin{aligned}
\hat{\tilde{\psi}}_m(\omega) &= (-1)^{m+1}2^{-m+1}(2m-1)!z^{-4}\frac{1}{E_{2m-1}(z^2)}\hat{\psi}_{I,m}(\omega)\\
&= \frac{(-1)^{m+1}}{2^{m-1}}z^{-2(m+1)}\frac{1}{E_{N_m}(z^2)}\hat{\psi}_{I,m}(\omega)\\
&= \frac{(-1)^{m+1}}{2^{m-1}}e^{i\omega(m+1)}\textstyle\sum_{n=-\infty}^{\infty}c_n^{(2m)}e^{-i\omega n}\hat{\psi}_{I,m}(\omega)
\end{aligned}$$

(substituting $z = e^{-i\omega/2}$ and using identity (4.61)).
Now we use the inverse Fourier transform defined in (1.2).

$$\begin{aligned}
\tilde{\psi}_m(x) &= \frac{1}{2\pi}\textstyle\int_{-\infty}^{\infty}e^{i\omega x}\hat{\tilde{\psi}}_m(\omega)d\omega \qquad (4.80)\\
&= \frac{1}{2\pi}\frac{(-1)^{m+1}}{2^{m-1}}\textstyle\int_{-\infty}^{\infty}e^{i\omega x}e^{i\omega(m+1)}(\sum_{n=-\infty}^{\infty}c_n^{(2m)}e^{-i\omega n})\hat{\psi}_{I,m}(\omega)d\omega\\
&= \frac{(-1)^{m+1}}{2^{m-1}}\left(\textstyle\sum_{n=-\infty}^{\infty}c_n^{(2m)}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega(x+m+1-n)}\right)\hat{\psi}_{I,m}(\omega)d\omega\\
&= \frac{(-1)^{m+1}}{2^{m-1}}\left(\textstyle\sum_{n=-\infty}^{\infty}c_n^{(2m)}\psi_{I,m}(x+m+1-n)\right).
\end{aligned}$$

This is the required formulation of $\tilde{\psi}$ in terms of $\psi_{I,m}$.

So far, we have determined the two-scale symbol P_{N_m} of N_m (in (3.23)) and the two-scale symbol Q_m of ψ_m in (4.38). Actually, we see from (4.18) that

$$\begin{aligned}
Q_m(z) &= \left(\frac{1-z}{2}\right)^m\sum_{k=0}^{2m-2}N_{2m}(k+1)(-z)^k \qquad (4.81)\\
&= \left(\frac{1-z}{2}\right)^m\frac{E_{2m-1}(-z)}{(2m-1)!}.
\end{aligned}$$

The two-scale symbol G of $\tilde{N}_m$ was given in (4.34). We have

$$G(z) = \frac{1}{z}\left(\frac{1+z}{2}\right)^m \frac{E_{2m-1}(-z)}{E_{2m-1}(z^2)} \tag{4.82}$$

and the two-scale symbol H of $\tilde{\psi}_m$ was given in (4.7). In terms of our defenition of $k(z)$ chosen in (4.37) we get (for our previous choice of constants c_0 and n_0)

$$\begin{aligned} H(z) &= z\left(\frac{1-z}{2}\right)^m \frac{-(2m-1)!}{z^2 E_{2m-1}(z^2)} \\ &= -z^{-1}\left(\frac{1-z}{2}\right)^m \frac{(2m-1)!}{E_{2m-1}(z^2)}. \end{aligned} \tag{4.83}$$

We now have the pairs of functions $(N_m \psi_m)$ and $(\tilde{N}_m \tilde{\psi}_m)$, with two-scale symbols (P,Q) and (G,H) respectively, that are needed to apply the decomposition and reconstruction algorithms discussed in chapter 2 (pg. 11-14) to cardinal splines.

Chapter 5

A BRIEF REVIEW OF PAPERS BY CHUI AND WANG

5.1 A GENERAL APPROACH TO MULTIRESOLUTION ANALYSIS

I will discuss briefly certain aspects of [2], [3], and [4], relating to my research report.

In [2] and [3] the authors verify that the interpolatory spline wavelet

$$\psi_{I,m} := L_{2m}^{(m)}(2x-1) \tag{5.1}$$

and the B-spline

$$\psi_m(x) := \frac{1}{2^{m-1}} \sum_{j=0}^{2m-2} (-1)^j N_{2m}(j+1) N_{2m}^{(m)}(2x-j) \tag{5.2}$$

do generate the spaces W_j determined by the MRA $\{V_j\}$ generated by the B-spline N_m. The authors first show that $\psi_{I,m}, \psi_m \in W_0$ and establish that each is a basis of W_0 by using a similar argument to the proof of theorem 2.1.2 in this research report. In [3] the authors continue to discuss the duality principle and verify that the dual function $\tilde{\psi}$ can be expressed in terms of $\psi_{I,m}$.

In [4] the author shows that the B-wavelet ψ_m and interpolatory wavelet $\psi_{I,m}$ are particular examples of a more general class of wavelet functions. The author begins by defining the class $\Phi = \{\phi\}$ of all functions generating a particular multiresolution analysis of $L^2(\Re)$ with *finite* two-scale sequence, i.e. for each such ϕ there is some positive value N_ϕ such that

$$\phi(x) \;=\; \sum_{n=0}^{N_\phi} p_n^\phi \phi(2x-n), \qquad x \in \Re, \tag{5.3}$$

for a finite sequence $\{p_n^\phi\}$. We may assume without loss of generality that $p_0^\phi, p_N^\phi \neq 0$, so that ϕ has compact support $[0, N_\phi]$.

With respect to some $\phi \in \Phi$ we have the following definitions
1)

$$\gamma_\phi(x) \;=\; \int_{-\infty}^{\infty} \phi(x+y)\overline{\phi(y)}dy. \tag{5.4}$$

2) For γ_ϕ as in (1) we define

$$B_\phi(z) \;=\; \sum_{n=-\infty}^{\infty} \gamma_\phi(n) z^n. \tag{5.5}$$

It is clear that $\gamma_\phi(-x) = \overline{\gamma_\phi(x)}$ and $supp\gamma_\phi \subset [-N_\phi, N_\phi]$ so that $B_\phi(z)$ is a finite Laurent series.
3) Let k_ϕ be the non-negative integer such that $\gamma_\phi(k_\phi) \neq 0$ but $\gamma_\phi(n) = 0$ for all $n > k_\phi$.
4) Let $\Pi_\phi(z) = z^{k_\phi} B_\phi(z)$. (Hence Π_ϕ is a polynomial with degree exactly equal to $2k_\phi$).
5) For any $P \in \pi$ let $\check{P}$ denote its reciprocal polynomial. Hence

$$\check{\Pi}_\phi(z) \;=\; z^{2k_\phi}\overline{\Pi_\phi(\frac{1}{z})}. \tag{5.6}$$

(Here π is the collection of all algebraic polynomials with complex coefficients).
6) z_0 is called a symmetric root or $(z^2 - z_0^2)$ a symmetric factor of $p \in \pi$ if $z_0 \neq 0$ and $p(z_0) = p(-z_0)$.

In general if $\phi \in \Phi$ we call ϕ a (generalised) *spline* and the spaces $\{V_k\}$ $k \in Z$ *spline spaces* even though ϕ may not be a piecewise polynomial. In particular, if $\phi \in \Phi$ is a ϕ with minimum support, we call ϕ a (generalised) *B-spline*. In this paper the author shows that for a function to be a *B-spline* it is necessary and sufficient that the two-scale symbol P_ϕ has no symmetric roots. Thus it turns out that the scaling function N_m is a *B-spline* of the MRA it generates. In fact it is also shown in this paper that it is the only generator of its MRA with a finite two-scale sequence.

In what follows we let $\nu \in \Phi$ denote the (generalised) B-spline of a particular MRA $\{V_k\}$.

In this paper the author uses the following definition of a wavelet:

Definition 5.1.1 *An $L^2(\Re)$ function η is called a wavelet for a given MRA provided that*
1) $\{\eta(\cdot - n) \;:\; n \in Z\}$ is a Riesz basis of W_0;
2) The symbol function

$$Q_\eta(z) \;:=\; \frac{1}{2}\sum_{n=-\infty}^{\infty} q_n^\eta z^n \tag{5.7}$$

of the two-scale sequence $\{q_n^\eta\}$ is analytic in a neighbourhood of $|z| = 1$.

The collection of all wavelets will be denoted by Ψ.

Remark The analyticity condition of Q_η on $|z| = 1$ is equivalent to the condition that both sequences $\{q_n^\eta\}$ and $\{q_{-n}^\eta\}$ have exponential decay as $n \to \infty$. This is important for practical purposes such as applications to time-frequency analysis.

To give a characterisation of all wavelets $\eta \in \Psi$ we define the polynomial

$$\beta_\nu(z) := z^{N_\phi - k_\phi - 1}\Pi_\nu(z)\tilde{P}_\nu(z). \tag{5.8}$$

β_ν may be factorised as follows:

$$\beta_\nu(z) = \mu_\nu(z)\lambda_\nu(z^2) \tag{5.9}$$

where $\mu_\nu, \lambda_\nu \in \pi$ such that $\lambda_\nu(1) = 1$, z^2 does not divide $\mu_\nu(z)$ and μ_ν has no symmetric roots.

In stating our characterisation theorem we need the following notation:

Definition 5.1.2 *Let $\mathcal{A}$ denote the class of all functions f analytic in a neighbourhood $|z| = 1$ such that $f(z) \neq 0$ on $|z| = 1$.*

Remark $f \in \mathcal{A}$ if and only if $1/f \in \mathcal{A}$. Furthermore $fg \in \mathcal{A}$ if both f and g are in $\mathcal{A}$.

The main result is the following characterization theorem:

Theorem 5.1.1 *Let μ_ν be as defined in* (5.9). *Then $\eta \in \Psi$ if and only if*

$$Q_\eta(z) = \mu_\nu(-z)\omega_\eta(z^2) \tag{5.10}$$

where $\omega_\eta \in \mathcal{A}$.

For a proof of this theorem see [4]. If $\omega_\eta \in \pi$ then the wavelet η has compact support and if ω_η is chosen so that $\omega_\eta(z) = 1$ then η has *minimum* support. Under the normalisation condition $\lambda_\nu(1) = 1$, the minimally supported η is unique. This minimally supported wavelet is called the *B-wavelet* of the multiresolution analysis.

We consider again the polynomial spline setting where ν is the m^{th} order B-spline N_m. Then in the factorisation (5.9) we have $\lambda_{N_m}(z) = 1$ so that $\mu_{N_m} = \beta_{N_m}$. In fact, it can be shown that

$$\mu_{N_m}(z) = \frac{1}{(2m-1)!}\left(\frac{1+z}{2}\right)^m \Pi_{2m-1}(z) \tag{5.11}$$

where Π_{2m-1} is the Euler-Frobenius polynomial E_{2m-1}. Hence, by choosing

$$\omega_{\eta_m}(z^2) := \frac{2^m((2m-1)!)^2}{\Pi_{2m-1}(z)\Pi_{2m-1}(-z)} \tag{5.12}$$

we have

$$Q_{\eta_m}(z) = (2m-1)!\frac{(1-z)^m}{\Pi_{2m-1}(z)} \tag{5.13}$$

(note that this is the same as $Q^\dagger$ in (4.62) if we recall identity (4.6)). This gives us the wavelet

$$\eta_{N_m}(x) = L_{2m}^{(m)}(2x-1) \tag{5.14}$$

we had previously. In addition by choosing $\omega_{\psi_m} = 1$, we have

$$Q_{\psi_m}(z) = \frac{1}{(2m-1)!}\left(\frac{1-z}{2}\right)^m \Pi_{2m-1}(-z) \tag{5.15}$$

which is the definition given in (4.81) of the two-scale symbol Q_m. Thus we have determined the compactly supported wavelet

$$\psi_m(x) = \frac{1}{2^{m-1}}\sum_{j=0}^{2m-2}(-1)^j N_{2m}(j+1)N_{2m}^{(m)}(2x-j). \tag{5.16}$$

As stated previously this is the minimally supported B-wavelet of the polynomial spline spaces.

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