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Dynamical systems in Walrusian general equilibrium

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X be the left invariant vector field[7] on $O(n)$ determined by x. By proposition 2.34 we need only show that $e^{tX} \in O(n-1)$ for all $t \in \mathbb{R}$. For this it is sufficient to prove that the tangent vector to the curve $t \mapsto \tilde{\beta}(e^{tX})$ in S^{n-1} is identically zero, for then $\tilde{\beta}(e^{tX}) \equiv e_n$ which implies that $e^{tX} \in O(n-1)$ for all t. The tangent vector to the curve at t is

$$\begin{aligned} d\tilde{\beta}(X_{\exp tX}) &= d(\eta_{\exp tX} \circ \tilde{\beta} \circ l_{\exp(-tX)})(X_{\exp tX}) \\ &= d\eta_{\exp tX} \circ d\tilde{\beta}(X(e)) = d\eta_{\exp tX} \circ d\tilde{\beta}(x) = 0. \end{aligned}$$

(X_m notation from definition 2.3) thus $d\tilde{\beta}$ is everywhere non-singular and the theorem is proved. □

We have therefore shown that the sphere S^{n-1} is in a natural way diffeomorphic to the homogeneous manifold $O(n)/O(n-1)$.

Remark 2.35. By a similar argument, one can show that there is a diffeomorphism of the sphere S^{n-1} with the homogeneous manifold $SO(n)/SO(n-1)$.

Proposition 2.36. Let H be a subgroup of the Lie group G. If H and G/H are connected, then G is connected.

Proof. Assume that

$$G = U \cup V \tag{2.6}$$

where the U and V are nonempty subsets of G. Then $G/H = \pi(U) \cup \pi(V)$ where $\pi(U)$ and $\pi(V)$ are non-empty open subsets of G/H. Since G/H is connected, there must be a point σH of G/H such that

$$\sigma H \in \pi(U) \cap \pi(V). \tag{2.7}$$

Now, equation (2.6) implies that $\sigma H = (\sigma H \cap U) \cup (\sigma H \cap V)$ where $(\sigma H \cap U)$ and $(\sigma H \cap V)$ are both open subsets of σH (since H has the relative topology). By equation (2.7), both $(\sigma H \cap U)$ and $(\sigma H \cap V)$ are non-empty; thus, since σH is homeomorphic with H and therefore connected, it must be the case that $(\sigma H \cap U) \cap (\sigma H \cap V) \neq \emptyset$ which implies that $U \cap V \neq \emptyset$ which proves that G is connected. □

Theorem 2.37. The Lie group $SO(n)$ is connected for $n \geq 1$ and $O(n)$ has two connected components ($n \geq 1$).

Proof. To prove the connectedness of $SO(n)$ we proceed by induction.

Since $SO(1) = 1 \times 1$ identity matrix, it is connected. Assume now that $SO(n-1)$ is connected. We know that $SO(n)/SO(n-1)$ is diffeomorphic to S^{n-1} which is connected. $SO(n-1)$ is a closed subgroup of the Lie group $SO(n)$. By the previous proposition we are done.

To prove that $O(n)$ has two components, note that every matrix in $O(n)$ has determinant ± 1. The orthogonal group can be written as the following union of two

[7] See the first part of proposition 2.26. Since $O(n)$ is a Lie group if we denote by $\mathfrak{g}$ its set of left invariant fields, there exists a (Lie algebra) isomorphism $\alpha : \mathfrak{g} \to O(n)_e$. Let $X = \alpha^{-1}(x)$

non-empty disjoint connected open subsets : $O(n) = SO(n) \cup \sigma SO(n)$ where

$$\sigma = \begin{bmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Thus $O(n)$ has two components. □

To end this section we prove another theorem. The result of this theorem is of secondary importance to the proof. The particular result we shall use (see proposition 7.16) the fact that any element in $Gl(n)^+ = \{\sigma \in Gl(n) | \det \sigma > 0\}$ can be joined to the identity matrix by a continuous curve.

Theorem 2.38. *$Gl(n)$ has two components.*

Proof. If $Gl(n)^- = \{\sigma \in Gl(n) | \det \sigma < 0\}$, $Gl(n)^+$ and $Gl(n)^-$ are disjoint, open, homeomorphic subsets of $Gl(n)$, so it suffices to prove that $Gl(n)^+$ is connected. This will be done by constructing the continuous curve mentioned previously.

First we show that each element of $Gl(n)$ has a *polar decomposition*, that is, each $\sigma \in Gl(n)$ can be expressed in the form $\sigma = PR$ where P is a positive definite symmetric matrix and $R \in O(n)$ (recall that all the eigenvalues of a symmetric matrix are real and that a symmetric matrix is positive definite if each of its eigenvalues is strictly positive). Let a be an eigenvalue of the symmetric matrix $\sigma\sigma^T$ with eigenvector $v \in \mathbb{R}^n$. Then according to equation (2.4), if $\langle , \rangle$ denotes the standard inner product in $\mathbb{R}^n$,

$$a\langle v, v\rangle = \langle \sigma\sigma^T(v), v\rangle = \langle \sigma^T(v), \sigma^T(v)\rangle.$$

Consequently, $a \geq 0$, and since $\sigma\sigma^T$ is non-singular, we must have $a > 0$. Thus $\sigma\sigma^T$ is a positive definite symmetric matrix. Because $\sigma\sigma^T$ is symmetric, there exists an orthogonal matrix $\beta \in O(n)$ such that the matrix

$$\beta\sigma\sigma^T\beta^T \tag{2.8}$$

is diagonal. Because the eigenvalues of $\sigma\sigma^T$ are all positive, the matrix of equation (2.8) has a square root $(\beta\sigma\sigma^T\beta^T)^{1/2}$. Let

$$P = \beta^T(\beta\sigma\sigma^T\beta^T)^{1/2}\beta, \tag{2.9}$$

and

$$R = P^{-1}\sigma.$$

P is a positive definite and symmetric. By equation (2.9) we have that $P^2 = \sigma\sigma^T$ and therefore that

$$\begin{aligned} RR^T &= P^{-1}\sigma\sigma^T(P^{-1})^T = P^{-1}\sigma\sigma^T(P^T)^{-1} \\ &= P^{-1}\sigma\sigma^TP^{-1} = P^{-1}PPP^{-1} = I. \end{aligned}$$

Thus R is orthogonal and $\sigma = PR$ as we asserted.

If $\sigma \in Gl(n)^+$, then σ has polar decomposition where R must now have positive determinant, that is, $R \in SO(n)$. Let $P_t = tI + (1-t)P$ for $t \in [0,1]$. Then P_t is positive definite for each t, so the path $t \mapsto P_tR$ is a continuous curve in $Gl(n)$ joining σ to R. Since $SO(n)$ is connected, and therefore pathwise connected, R can

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be joined to the identity matrix by a continuous curve. Thus $Gl(n)^+$ is pathwise connected, which completes the proof that $Gl(n)$ has two components. □

2.5. **Vector Bundles.** A *vector bundle*, ξ, can be thought of as a family $\{E_x\}_{x\in b}$ of disjoint vector spaces paramatrised by a space B. The union of these vector spaces is a space E and the map $p : E \to B$, $p(E_x) = x$ is continuous. Moreover, p is locally trivial in the sense that locally, with respect to B, E looks like a product with $\mathbb{R}^n$; that is, there are open sets $\{U_i\}$ covering B and homeomorphisms $p^{-1}(U_i) \approx U_i \times \mathbb{R}^n$ mapping each fibre E_x linearly onto $x \times \mathbb{R}^n$.

We view a vector bundle in the same light as a manifold in that it is formed by gluing together elementary objects by maps of a specific kind. For vector bundles these elementary objects are the "trivial" bundles, $U_i \times \mathbb{R}^n$, and the maps are morphisms $U_i \times \mathbb{R}^n \to U_j \times \mathbb{R}^n$ of the form $(x, y) \mapsto (x, g(x)y)$ where $g : U_i \cap U_j \to Gl(n)$, the general linear group.

The main result which we prove here is that the map $SO(n) \to S^{n-1}$ from the special orthogonal group to the $n-1$ sphere which maps a matrix onto its first column, that is, the standard projection in the fibre bundle representation of $SO(n)$, is not homeomorphic to a constant. This result will be used in §7, theorem 7.18.

A *vector bundle chart* on (p, E, B) with domain U and dimension n is a homeomorphism $\phi : p^{-1}(U) \approx U \times \mathbb{R}^n$ where $U \subset B$ is open, such that the following diagram commutes.

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi} & U \times \mathbb{R}^n \\ {\scriptstyle p}\downarrow & \swarrow_{\pi_1} & \\ U & & \end{array}$$

Here, $\pi_1(x, y) = x$. ϕ is called a *trivialisation*. For each $x \in U$, we define the homeomorphism ϕ_x to be the composition

$$\phi_x : p^{-1}(x) \xrightarrow{\phi} x \times \mathbb{R}^n \to \mathbb{R}^n.$$

Thus, if $y \in p^{-1}(x)$, we have the formula

$$\phi(y) = (x, \phi_x(y)).$$

A *vector bundle atlas* Φ on (p, E, B) is a family of vector bundle charts on (p, E, B) with values in the same $\mathbb{R}^n$, whose domains cover B and such that whenever (ϕ, U) and (ψ, V) are in Φ and $x \in U \cap V$, the homeomorphism $\psi_x \phi_x^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is linear. The map $U \cap V \to Gl(n) : x \mapsto \psi_x \phi_x^{-1}$ is required to be continuous. It is called the *transition function* of the pair of charts $(\phi, U), (\psi, V)$. If $\Phi = \{\phi_i, U_i\}_{i\in\Lambda}$ we obtain a family $\{g_{ij}\}$ of transition functions

$$g_{ij} : U_i \cap U_j \to Gl(n).$$

These maps must satisfy the identities

$$g_{ij}(x)g_{jk}(x) = g_{ik}(x)$$
$$g_{ii}(x) = 1_{id} \in Gl(n).$$

A maximal vector bundle atlas Φ is a *vector bundle structure* on (p, E, B).

We call $\xi = (p, E, B, \Phi)$ a vector bundle having *fibre dimension* n, *projection* p, *total space* E and *base space* B. Often Φ is not explicitly mentioned and we may also abuse notation denoting ξ by E or E by ξ where no confusion will arise.

The fibre over $x \in B$ is the space $p^{-1}(x) = \xi_x = E_x$. We give ξ_x vector space structure by making each $\phi_x \xi_x \to \mathbb{R}^n$ an isomorphism, this structure is independent of the choice of $(\phi, U) \in \Phi$. E can therefore be viewed as a "bundle" of vector spaces.

Let $\xi = (p, E, B, \Phi)$ and $\xi' = (p', E', B', \Phi')$ be vector bundles. A *fibre map* $F : \xi \to \xi'$ is a map $F : E \to E'$ which covers a map $f : B \to B'$, that is, there is a commuting diagram :

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}$$

Thus if $x \in B$ and $f(x) = y$, F maps the fibre over x into the fibre over y by the map $F_x : \xi_x \to \xi'_y$. If each F_x is linear we call F a morphism of vector bundles. If F_x is bijective F is called a *vector bundle map*. So, in particular we point out the following : If $U'_j \in \Phi'$ and $U_k \in \Phi$ are such that $f^{-1}(U'_j) \cap U_k \neq \emptyset$ and if $x \in f^{-1}(U'_j) \cap U_k$, the transition function between the different charts defined by

$$\bar{g}_{jk}(x) = \phi'_{j,f(x)} \circ F_x \circ \phi^{-1}_{k,x}$$

which maps $\mathbb{R}^n$ into $\mathbb{R}^n$ coincides with an element of $Gl(n)$.

Fix a differentiability class, C^r, $1 \leq r \leq \omega$. The above definition makes sense if all spaces involved are required to be C^r manifolds and maps are required to be C^r maps. In this way we obtain C^r vector bundles, morphisms and bundle maps.

Example 2.39 (Cosets). Consider the Lie group $SO(n)$ acting as a transitive group of transformations on the manifold S^{n-1}. A projection is defined by selecting the point $e_n \in S^{n-1}$ and defining $p(l) = le_n$. Since $SO(n-1)$ is the subgroup of $SO(n)$ which leaves e_n fixed, the fibres are just the left cosets of $SO(n-1)$ in $SO(n)$. There are many natural correspondences $SO(n-1) \to SO(n-1)_x$, any $l \in SO(n-1)_x$ defines one by $y \to l \cdot y$. However, any two such $y \to l \cdot y$, $y \to l' \cdot y$ differ by the left translation of $SO(n-1)$ corresponding to $l^{-1}l'$. Thus the transition functions map overlapping co-ordinates into $SO(n-1) \subset Gl(n)$ and act on $SO(n-1)$ by left translation.

Definition 2.40. Let $\xi' = (p', E', B', \Phi')$. A continuous map $f : B \to B'$ induces a bundle $\xi = (p, E, B, \Phi)$, denoted $f^{-1}\xi'$, as follows :

Form the product space $B \times E'$ and let $p : B \times E' \to B$ and $F : B \times E' \to E'$ be the natural projections. Define E to be the subspace of $B \times E'$ of those pairs (x, ξ'_y) such that $f(x) = p'(\xi'_y)$. We therefore have commutativity of the following diagram

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}$$

To complete the definition of ξ, we need to define the induced vector bundle atlas. If $\Phi' = \{\phi'_i, U'_i\}_{i\in\Lambda}$, define $U_i = f^{-1}(U'_i)$ and if $y \in p^{-1}(x)$, set

$$\phi_{i,x}(y) = (x, \phi_{i,f(x)}(y)). \tag{2.10}$$

It can be shown that equation (2.10) provides a bundle structure to ξ, but again since we shall not deal with this bundle structure directly the details are omitted.

Definition 2.41. Let $I = [0,1]$ and $\xi = (p, E, B, \Phi)$. The bundle $\xi \times I$ is defined by

$$\xi \times I = (q, E \times I, B \times I, \Psi)$$

where $q(\xi_x, t) = (p(\xi_x), t)$ and $(\psi_i, V_i) \in \Psi$ is defined by $(\phi_i, U_i) \in \Phi$ where $V_i = U_i \times I$ and $\psi_{i,x}(y,t) = (\phi_{i,x}(y), t)$. Then the co-ordinate transformations of $\xi \times I$ satisfy $g_{ij}(x,t) = g_{ij}(x)$.

Definition 2.42. Let ξ, ξ' be two bundles having the same fibre and let F_0 and F_1 be two bundle maps of ξ into ξ'. We shall say that F_0 and F_1 are *homotopic (as bundle maps)*, written $F_0 \simeq F_1$, if there exists a bundle map $F : \xi \times I \to \xi'$ such that $F(x,0) = F_0(x)$ and $F(x,1) = F_1(x)$. The map F is called a *homotopy*.

One of the most important results in the study of bundles is the following homotopy extension theorem also known as :

Theorem 2.43 (First covering homotopy theorem). *Let ξ, ξ' be two bundles having the same fibre and such that the transition functions map into the same group $G \subset Gl(n)$. Let B be normal, locally compact and such that any open cover of B is reducible to an countable subcover (such a space is called a C_σ space). Let $F_0 : \xi \to \xi'$ be a bundle map which induces a map $f_0 : B \to B'$. Let $f : B \times I \to B'$ be a homotopy of f_0. Then there exits a homotopy $F : \xi \times I \to \xi'$ whose induced map is f.*

The homotopy F is said to *cover* f. The geometric idea behind the proof is to decompose the homotopy F into a succession of homotopies each of which is small in the sense that all motion which takes place does so inside a single co-ordinate neighbourhood U'_j of Φ'. Since the part of ξ' over U'_j is a product space and the theorem is trivial for product spaces, the complete theorem is built up in pieces.

The proof of this theorem is lengthy and since it will not be used in our applications in the generality stated here, a proof for the special case where B is compact is given. This is the case that we shall require later and it is also the first step to prove the theorem in the generality stated here.

Proof. Since $B \times I$ is compact, the covering $\{f^{-1}(U'_j)\}$ has a refinement of the form $\{U_\lambda \times I_\nu\}$ where $\{U_\lambda\}$ is an open covering of B and $I_1, \ldots, I_r$ is a finite sequence of open intervals covering I such that I_ν meets only $I_{\nu-1}$ and $I_{\nu+1}$ for each $\nu = 2, \ldots, r-1$. Choose numbers

$$0 = t_0 < t_1 < \ldots < t_r = 1$$

so that $t_\nu \in I_\nu \cap I_{\nu+1}$, $\nu = 1, \ldots, r-1$. We shall suppose, inductively, that $F(\xi_x, t)$ has been defined over $\xi \times [0, t_\nu]$ so as to satisfy the conditions of the theorem over this part and proceed to extend F over $\xi \times [t_\nu, t_{\nu+1}]$.

For each $x \in B$ there is a pair of neighbourhoods (W, W') such that $x \in W$, $\overline{W} \subset W'$, and $\overline{W}'$ lies in some U_λ. Choose a finite number of such pairs (W_α, W'_α), $\alpha = 1, \ldots, s$ such that the $\{W_\alpha\}$ cover B. According to Urysohn's lemma, there is a map $u_\alpha : B \to [t_\nu, t_{\nu+1}]$ such that $u_\alpha(\overline{W}_\alpha) = t_{\nu+1}$ and $u_\alpha(B \backslash \overline{W}'_\alpha) = t_\nu$. Define $\tau_0(x) = t_\nu$ for $x \in B$ and

$$\tau_\alpha(x) = \max[u_1(x), \ldots, u_\alpha(x)], \qquad x \in B, \alpha = 1, \ldots, s.$$

Then

$$t_\nu = \tau_0(x) \leq \tau_1(x) \leq \ldots \leq \tau_s(x) = t_{\nu+1}.$$

Define B_α to be the set of pairs (x, t) such that $t_\nu \leq t \leq \tau_\alpha(x)$. Let ξ_α be the part of $\xi \times I$ over B_α. Then

$$\xi \times t_\nu = \xi_0 \subset \xi_1 \subset \ldots \subset \xi_s = \xi \times [t_\nu,\ t_{\nu+1}].$$

We suppose inductively that F has been properly extended over $\xi_{\alpha-1}$ and shall proceed to define its extension over ξ_α.

By definition of the τ's, the set $B_\alpha \backslash B_{\alpha-1}$ is contained in $W'_\alpha \times [t_\nu, t_{\nu+1}]$ and by definition of the W's, $\overline{W}'_\alpha \times [t_\nu, t_{\nu+1}] \subset U_\lambda \times I_\nu$ and is mapped by f into some U'_j. With j fixed in this manner, define

$$F(\xi_x, t) = \left(\phi'_{j,f(x,t)}\right)^{-1} \left(\phi'_{j,f(x,\tau)} F(\xi_x, \tau)\right), \text{ for } (x,t) \in B_\alpha \backslash B_{\alpha-1}, \tag{2.11}$$

where $x = p(\xi_x)$ and $\tau = \tau_{\alpha-1}(x)$. Notice that the right hand side is defined and continuous in the closure of $B_\alpha \backslash B_{\alpha-1}$, and, for $t = \tau_{\alpha-1}(x)$, it reduces to $F(b, \tau_{\alpha-1}(x))$. Hence F on $\xi_{\alpha-1}$ has been continuously extended over ξ_α. Clearly, $p'F(b,t) = f(x,t)$, so F maps fibres into fibres and covers f.

To show that F is a bundle map of ξ_α, it suffices to prove this on the closure of $\xi_\alpha \backslash \xi_{\alpha-1}$. Abbreviate $\tau_{\alpha-1}(x)$ by τ, then

$$\begin{aligned}
\bar{g}_{jk}(x,t) \cdot y &= \phi'_{j,f(x,t)} F_{(x,t)} \psi^{-1}_{k,x}(y,t) = \phi'_{j,f(x,t)} F(\phi^{-1}_{k,x}(y), t) \\
&= \phi'_{j,f(x,t)} \left(\phi'_{j,f(x,t)}\right)^{-1} \left(\phi'_{j,f(x,\tau)} F(\phi^{-1}_{k,x}(y), \tau)\right) \qquad \text{by 2.11} \\
&= \phi'_{j,f(x,\tau)} F(\phi^{-1}_{k,x}(y), \tau) \\
&= \phi'_{j,f(x,\tau)} F_{(x,\tau)} \psi^{-1}_{k,x}(y\ \tau) \\
&= \bar{g}_{jk}(x, \tau_{\alpha-1}(x)) \cdot y.
\end{aligned}$$

The hypothesis that F is a bundle map of $\xi_{\alpha-1}$ in ξ' means that $\bar{g}_{jk}(x,\tau)$ yields a continuous map into G of the intersections of $V_k \times I$ with the boundary of $B_\alpha \backslash B_{\alpha-1}$. It follows that that $\bar{g}_{jk}(x,\tau)$ yields a continuous map into G of the intersections of $V_k \times I$ with the closure of $B_\alpha \backslash B_{\alpha-1}$. Therefore, F is a bundle map of ξ_α into ξ'. This completes the general step of the two-fold induction and proves the theorem for a compact space B.

The remainder of the proof for the general case can be found in [35] page 50. □

Theorem 2.44 (Second covering homotopy theorem). Let $\xi' = (p', E', B', \Phi')$ and B be a C_σ-space, and $f_0 : B \to E'$ be a map and let $\bar{f} : B \times I \to B'$ be a homotopy of $p'f_0 = \bar{f}_0$. Then there exists a homotopy $f : B \times I \to E'$ covering $\bar{f}$ (that is $p'f = \bar{f}$).

Proof. Let $\xi = \bar{f}_0^{-1}\xi'$ and let $F_0 : \xi \to \xi'$ be the induced map. Let $F : \xi \times I \to \xi'$ be a covering homotopy (see theorem 2.43) of $\bar{f}$. Define the cross section σ of ξ as $\sigma(x) = (x, f_0(x))$. Then $F_0\alpha = f_0$. Define $f(x,t) = (\alpha(x), t)$. It follows that f is the desired homotopy. □

A map $f : X \to Y$ is called *reducible* if it is homotopic to a map of X into a proper subset of Y, otherwise it is called *irreducible*. It is called *inessential* if it is homotopic to a constant map; otherwise it is called *essential*. A space is called reducible (irreducible) if its identity map is reducible (irreducible).

Theorem 2.45. Let ξ be a bundle in which E is a C_σ-space and $p : E \to B$ is reducible. Then E is reducible.

Proof. We have only to apply theorem 2.44 to the identity map of E. □

Corollary 2.46. If E is irreducible, so also are p and B. In particular, if B has more than one point, then p is essential.

Proof. Since b reducible implies p reducible, the corollary is immediate. □

Remark 2.47. It is easy to see (see remark 2.55) that a compact n-manifold is irreducible. As a consequence, this corollary applies to the bundles constructed out of coset spaces of compact Lie groups.

We have already seen that $SO(n)/SO(n-1) = S^{n-1}$ and $SO(n)$ is a bundle over S^{n-1} with fibre and group $SO(n-1)$. Corollary 2.46 shows that the projection of $SO(n)$ into its S^{n-1} is not homotopic to a constant.

2.6. **Oriented Vector Bundles.** Let V be a (real) finite dimensional vector space of dimension $n \geq 0$. An *orientation* is an equivalence class of bases where the coordinate change has positive Jacobian. If $\dim V > 0$ there are two orientations, denoted ω and $-\omega$.

An *oriented vector space* is a pair (V, ω) where ω is an orientation of V. Given (V, ω) and (W, ω'), an isomorphism $L : V \to W$ is called orientation preserving if $L(\omega) = \omega'$; otherwise it is called *orientation reversing.*

The standard orientation ρ^n of $\mathbb{R}^n$, $n > 0$ is $[e_1, \ldots e_n]$ where e_i is the i-th unit vector. Any n dimensional submanifold of $\mathbb{R}^n$ is also given this standard orientation.

Let $\xi = (p, E, B)$ be a vector bundle. An *orientation for* ξ is a family $\omega = \{\omega_x\}_{x \in B}$ where ω_x is an orientation of the fibre E_x such that ξ has an atlas Φ with the property that if $\phi : \xi|U \to \mathbb{R}^n$ is in Φ then

$$\phi_x : (E_x, \omega_x) \to (\mathbb{R}^n, \rho^n)$$

is orientation preserving. We call ω a *coherent family of orientations of the fibres.*

If ξ has an orientation ω, then ξ is called orientable and the pair (ξ, ω) is an *oriented bundle.*

Let M be a manifold. M is called *orientable* if TM is an orientable vector bundle. An *orientation* of M means an orientation of TM. An *oriented manifold* is a pair

(M, ω) where ω is an orientation of M. Define $-\omega$ to be the orientation of M such that $(-\omega)_x = -\omega_x$, as orientations of M_x for all $x \in M$

If (M, ω) and (N, θ) are oriented n-manifolds, the product orientation of $\omega \times \theta$ for $M \times N$ assigns to $(x, y) \in M \times N$ the orientation $\omega_x \oplus \theta_y$ of $M_x \oplus N_y$.

Each orientation ω of M determines an orientation $\partial\omega$ for ∂M as follows : For $x \in \partial M$, choose a positively oriented basis $(e_1, \ldots, e_n)$ for M in such a way that $e_2, \ldots, e_n$ are tangent to the boundary (assuming $n \geq 2$) and that e_1 is an "inward" vector. Then $(e_2, \ldots, e_n)$ determines the required orientation of ∂M at x. If $n = 1$ then each boundary point x is assigned an orientation -1 or +1 accordingly as positively oriented vectors at x point inward or outward.

Let θ be an orientation of M. We usually give $M \times I$ the product orientation $\omega = \theta \times \rho^1$ where, ρ^1 is the standard orientation of I. Since the induced orientation faces inward along the boundary,

$$\partial\omega | M \times 0 = \theta \quad \text{and} \quad \partial\omega | M \times 1 = -\theta.$$

We shall frequently speak of "the oriented manifold M" not naming the orientation explicitly. In this case, ∂M and $M \times I$ are also oriented manifolds, as is any submanifold of M of the same dimension. If $-M$ denotes the manifold M with opposite orientation, then

$$\partial(M \times I) = (M \times 0) \cup (-M \times 1)$$

as oriented manifolds.

Let $S \subset M$ be a C^{r+1} submanifold. The *algebraic normal bundle* of S in M is the C^r quotient bundle $T_S M / TS$; it is canonically C^r isomorphic to TS.

Theorem 2.48. The algebraic normal bundle ν of ∂M in M is trivial and therefore orientable.

2.7. **The Degree of a Map.** One of the most important tools in differential topology is the degree of a map $f : M \to N$, where M and N are compact n-manifolds, N is connected and $\partial M = \partial N = \emptyset$. This degree is an integer if M and N are oriented.

Lemma 2.49. Let (W, ω) be an oriented manifold with boundary. Suppose $K \subset U$ is an embedded arc which is transverse to ∂M at its endpoints $u, v \in \partial W$. Let κ be an orientation of K and consider the orientation ω/κ of the algebraic normal bundle of κ. Then

$$\omega_u / \kappa_u = (\partial\omega)_u \Leftrightarrow \omega_v / \kappa_v = -(\partial\omega)_v.$$

Proof. Let X_u and X_v be tangent vectors to K at u, v respectively. Then X_u is inward if and only if X_v is outward; this is equivalent to the lemma. □

Let (M, ω) and (N, θ) be compact oriented manifolds of the same dimension without boundaries. Assume N is connected. Let $f : M \to N$ be a C^1 map $x \in M$ a regular point of f with $y = f(x)$. We say x has positive type if the isomorphism $T_x f : M_x \to N_y$ preserves orientation, that is, it sends ω_x to θ_y. In this case, we write $\deg_x f = 1$. It $T_x f$ reverses orientation, then x has negative type and we write $\deg_x f = -1$. We call $\deg_x f$ the *degree* of f at x.

Suppose $y \in N$ is any regular value for f. Define the degree of f over y to be

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \deg_x f.$$

If $f^{-1}(y)$ is empty, then $\deg(f, y) = 0$. To stress the orientations, we can write $\deg(f, y; \omega, \theta)$.

Lemma 2.50. Let W be a compact oriented manifold of dimension $n+1$, N a compact oriented n-manifold without boundary and $h : W \to N$ a C^∞ map. Let $y \in N$ be a regular value for both h and $h|\partial W$. Then $\deg(h, y) = 0$.

Proof. Let ω and θ be orientations of W and N respectively. Let $M_1, \ldots M_m$ be the components of ∂W.

Since y is a regular value, $h^{-1}(y)$ is a compact 1-dimensional submanifold of W whose boundary is $(h|\partial W)^{-1}(y)$. Let $u \in h^{-1}(y)$. Then there is a unique $v \in h^{-1}(y), v \neq u$ and a component arc $K \subset h^{-1}(y)$ such that $\partial K = \{u, v\}$. It suffices to show that u and v are of opposite type for $h|\partial W$

Let $\nu = TW/TK$, the algebraic normal bundle of K in W. Since y is a regular value, Tf induces a bimorphism $\Phi : \nu \to N_y$. There are natural identifications $\nu_u = (\partial W)_u$, $\nu_v = (\partial W)_v$.

Since K is an arc there is a unique orientation κ of ν such that $\kappa_u = (\partial\omega)_u$. By lemma 2.49, $\kappa_v = -(\partial\omega)_v$.

Suppose u is of positive type (for $h|\partial W$). Then $\Phi_x(\kappa_x) = \theta_y$ for all $x \in K$. It follows that

$$\begin{aligned} T(h|\partial W)(\partial\omega)_u &= \Phi_u \kappa_u \\ &= \theta_y \end{aligned}$$

and

$$\begin{aligned} T(h|\partial W)(-\partial\omega)_v &= \Phi_v \kappa_v \\ &= \theta_y. \end{aligned}$$

Therefore, v is of negative type. This shows that $h|\partial W$ has equal numbers of points of positive and negative type in $h^{-1}(y)$. □

Corollary 2.51. Let (M, ω) and (N, θ) be compact oriented N-manifolds without boundary. Let N be connected and $f, g : M \to N$ homotopic C^∞ maps having common regular value $y \in N$. Then $\deg(f, y) = \deg(g, y)$.

Proof. There is a homotopy $h : M \times I \to N$ from f to g and one can make h C^∞ and transverse to y. As oriented manifolds

$$\partial(M \times I) = (M \times 0, \omega) \cup (M \times 1, -\omega).$$

By lemma 2.50 we have

$$\begin{aligned} 0 = &\deg\left(h|\partial(M \times I)\right) \\ &= \deg(f, y; \omega, \theta) + \deg(g, y; -\omega, \theta) \\ &= \deg(f, y; \omega, \theta) - \deg(g, y; \omega, \theta). \end{aligned}$$

□

Lemma 2.52. Let (M, ω) and (N, θ) be compact oriented N-manifolds without boundary. Let N be connected and $y, z \in N$ be regular values for a C^∞ map $f : M \to N$. Then $\deg(f, y) = \deg(f, z)$.

Proof. If there is a diffeomorphism $h : N \to N$, homotopic to the identity, such that $h(y) = z$ then by corollary 2.51 $\deg(h, z) = \deg_y h = 1$. It is easy to see that this implies

$$\deg(f, y) = \deg(hf, z).$$

But hf is homotopic to f, hence

$$\deg(hf, z) = \deg(f, z).$$

Thus if we can construct such an h, we are done. If y and z are in the same co-ordinate, we construct such an h using the local co-ordinates. The relation between y and z that such an h exists is an equivalence relation on N whose equivalence classes are thus disjoint open sets. Since N is connected, an two points are equivalent. □

Finally, note the following lemma which is not proved here

Lemma 2.53. Let M, N manifolds and $f : M \to N$ a continuous map. Then f can be approximated by C^∞ maps homotopic to f.

Let M, N be oriented compact n-manifolds, $n \geq 1$, without boundary and N connected. The previous results allow us to define the *degree* of a continuous map $f : M \to N$ as $\deg(g, z)$ where $g : M \to N$ is a C^∞ map homotopic to f and $z \in N$ is a regular value for g. They also yield :

Theorem 2.54 (Hirsch [12])**.** Let M and N be compact oriented manifolds without boundary, N connected. Then homotopic maps $M \to N$ have the same degree.

Because the degree of a map is easy to calculate, this theorem provides a convenient method for distinguishing homotopy classes. It is the only homotopy invariant for maps into S^n.

Remark 2.55. As referred to in remark 2.47, it is now easy to see that a compact n-manifold is irreducible. Since the identity map is surjective its degree $\neq 0$ and any map f from the manifold M to itself which is not surjective has degree $= 0$ (let $y \in M \backslash f(M)$ then $f^{-1}(y)$ is empty and by definition $\deg y = 0$).

2.8. **Homology Theory.** In this section, some elementary notions in homology theory are introduced.

Definition 2.56. For $k, N \in \mathbb{N}, v_0, \ldots, v_k \in \mathbb{R}^N$ such that $\{v_i - v_0\}_{i=1}^k$ is linearly independent, the convex hull σ of $\{v_0, \ldots, v_k\}$ is called the (k-dimensional) *simplex* spanned by $\{v_0, \ldots, v_k\}$ and the v_i are called the *vertices* of the simplex. The simplices spanned by a subset of $\{v_0, \ldots, v_k\}$ are called the *faces* of σ. A *simplical complex* S is a finite collection of simplices such that any two simplices intersect in a common face. The union of the simplices of S is denoted $|S|$. A *triangulation of a manifold* M is a pair (S, h) consisting of a simplical complex S and a homeomorphism $h : S \to M$. The images of simplices, faces and vertices under h are also

called simplices, faces and vertices. Identifying two orderings of the set $\{v_0, \ldots, v_k\}$ of vertices of a simplex in M if they differ by an even permutation, we call a simplex with a choice of ordering of the vertices (modulo even permutations) an *oriented simplex*. The chosen orientation is called the *positive orientation* and the other one the *negative orientation*. If σ is an oriented simplex then denote by $-\sigma$ the same simplex with opposite orientation.

Note that an orientation of a simplex induces an orientation on each face since the vertices of a face form a subset of the vertices of the simplex.

Now fix a triangulation (S, h) of M and consider formal sums $\sum n_i\sigma_i$ where the σ_i are oriented k-simplices and $n_i \in \mathbb{Z}$. For $n_i < 0$ we define $n_i\sigma_i := (-n_i)(-\sigma_i)$. The set of such formal sums with the obvious additive structure is the free abelian group generated by the k-simplices and subject to the relations $\sigma + (-\sigma) = 0$ and $\sigma_i + \sigma_j = \sigma_j + \sigma_i$, that is, a finitely generated free abelian group.

Definition 2.57. A formal sum $\sum n_j\sigma_j$ of oriented k-simplices is called a *k-chain*. Denote by C_k the set of k-chains. The *boundary operator* $\partial : C_k \to C_{k-1}$ is defined by setting

$$\partial\sigma := \sum_{i=0}^{k} \sigma_i$$

for an oriented k-simplex σ, where the σ_i are the $(k-1)$-dimensional faces of σ with induced orientation, and extending linearly to C_k.

An important combinatorial fact is that the boundary of a boundary is zero.

Theorem 2.58 (Poincaré Lemma). $\partial^2 = 0$.

Since $\partial : C_k \to C_{k-1}$ is by definition a homeomorphism with respect to the additive structure, the Poincaré Lemma shows that the set $B_k := \partial(C_{k+1}) \subset C_k$ of k-dimensional *boundaries* is a subgroup of the group $Z_k := \ker\partial = \{c \in C_k | \partial c = 0\}$ of k dimensional *cycles*. Since C_k is abelian, B_k is normal in C_k.

Definition 2.59. $H_k(M, \mathbb{Z}) := H_k := Z_k/B_k$ is called the *k-th homology group* of M over the integers. H_k is a finitely generated abelian group.

The k-th homology group can also be defined over groups other than the integers. For our purposes we need to define this homology over $\mathbb{Z}_2$, the integers modulo 2. This is done by restricting the coefficients of the k-simplices in the k-chains to be elements of $\mathbb{Z}_2$. We shall denote this homology group by $H_k(M, \mathbb{Z}_2)$.

Proposition 2.60. Let M be a connected orietable manifold without boundary, then

$$H_m(M, \mathbb{Z}_2) = \mathbb{Z}_2$$

where $m = \dim M$.

Proof. Note that $B_m = 0$ and thus $H_m = Z_m$. Suppose that a simplex σ occurs i times $i = 0, 1$ in an n-cycle c. Since the boundary is zero, the $(n-1)$-faces of σ have to be cancelled by other terms in ∂c. Thus each neighbouring simplex has to appear

i times with appropriate orientation. Thus c is the i fold sum of all n simplices of the triangulation, all oriented compatibly. There is exactly one such chain for each i, so $H_m = Z_m = \mathbb{Z}_2$. □

Remark 2.61. All homology groups are homotopy invariants.

3. Smale's Model

3.1. **Introduction.** The main results in this section prove, in a very general setting, the existence of finitely many, "stable" equilibria for "most" economies. We shall follow Smale [25].

A traditional approach to the dynamic study of equilibrium theory, as outlined in the introduction, involves the study of a vector field generated by the excess demand function on the price simplex. The approach offered in this section will differ from the classical approach in that existence of excess demand functions will not be assumed. Instead, we work directly with the more fundamental utility functions. This allows for added realism since there is no evidence to support the fact that individual decisions are uniquely determined by prices.

This generality should not, however, be interpreted as the desire of a student of mathematics to offer the most general result on purely aesthetic grounds[8] . Rather, it yields all our required results while being easily extended to more sophisticated economies, that is, economies with production[9].

Consequently, the fixed or equilibrium points studied are not the classical ones obtained when using excess demand functions. However, the *extended equilibria* maintain all of the desirable qualities of their classical counterparts and in the special case where utility curves define excess demand functions, the extended equilibria coincide with the classical ones (Theorem A).

3.2. **Utility Functions.** Given an m consumer, l commodity economy, define the *commodity space* as $\bar{P} = \{x = (x^1, \ldots, x^l) \in \mathbb{R}^l \mid x^i \geq 0 \text{ for each } i\}$ (we shall assume in this section that a commodity bundle may have zero components since it seems natural for an agent to hold a zero amount of some good). A *utility function* will be a C^2 function $u : \bar{P} \to \mathbb{R}$ and the space of all such functions will be denoted $C^2(\bar{P}, \mathbb{R})$.

The first derivative $Du(x)$ at $x \in \bar{P}$ is a linear map from $\mathbb{R}^l$ to $\mathbb{R}$ and, identifying $\mathbb{R}^l$ with its dual space, may be considered as a vector in $\mathbb{R}^l$. The second derivative $D^2u(x)$ of u at $x \in \bar{P}$ is a symmetric bilinear form.

Remark 3.1. At this stage we note the fact that no desirability conditions have been imposed on the utility functions, that is, there is, no restriction on $Du(x)$.

The space $C^2(\bar{P}, \mathbb{R})$ is now examined with the C^2 strong topology, which by linearity is defined here in terms of neighbourhoods of zero. Since $\bar{P}$ is paracompact and $\mathbb{R}$ is a metric space, such a neighbourhood N_ϵ is defined by each strictly positive continuous function $\epsilon : \bar{P} \to \mathbb{R}$ (see §2.1).

---

[8] Indeed, the methods presented here are not the most general - Smale [26] shows how these results can be extended to include commodity vectors of the price equilibria which may lie on the boundary $\partial\bar{P}$ of the commodity space. Also in [38] it has been shown that assumption (3.1) is not necessary. In that paper, where normalised gradients are no longer available, Lagrange multipliers are used explicitly in order to relate $Du_i(x)$ and p.

[9] Many famous examples of economies with production show that production plans are not always uniquely determined by equilibrium prices. To examine such economies would require a theory in which agent's decisions were not determined by price, that is, the approach taken here.

We work with an open set $U \subset C^2(\bar{P}, \mathbb{R})$ of utility functions defined by

$$U = \{u \in C^2(\bar{P}, \mathbb{R}) | Du(x) \neq 0, \forall x \in \bar{P}\}. \tag{3.1}$$

Such utilities will exclude "unstable" preferences.

An *economy* is a point $(r, u) \in (\bar{P})^m \times (U)^m = \mathcal{E}$ where $(r, u) = (r_1, \ldots, r_m, u_1, \ldots, u_m)$ with r_i, u_i respectively the initial resources and utility function for the i-th agent.

The *state*, $\mathcal{S} = (\bar{P})^m \times \bar{S}$, $\bar{S} = \{x \in \bar{P} | \|x\| = 1\}$, of an economy consists of a set of commodities for each agent together with a price system (in this section, a price system *is* allowed to have zero components).

3.3. **Price Equilibria.** A *(classical) price equilibrium* of an economy (r, u) is a state (x, p) which satisfies

A. $\sum x_i = \sum r_i$.
B. $p(x_i) = p(r_i)$ for $i = 1, \ldots, m-1$ (note (A) and (B) imply $p(x_m) = p(r_m)$)
C. For each i, x_i is a maximum for u_i on the budget set $B_{r_i} = \{x \in \bar{P} \mid p(x) \leq p(r_i)\}$.

Define, for some initial allocation $r \in (\bar{P})^m, r = (r_1, \ldots, r_m)$

$$\Sigma_r = \left\{(x, p) \in \mathcal{S} \middle| \sum x_i = \sum r_i,\ p(x_i) = p(r_i);\ i = 1, \ldots, m-1\right\}$$

Thus Σ_r represents the set of *attainable states* relative to a budget constraint r. For an economy (r, u), Σ_r contains the classical price equilibria and the "extended price equilibria" which we shall now define.

Since no assumptions regarding the convexity of the utility functions are made, there is no guarantee that these x_i of condition (C) are well defined. Thus, in this generalised setting, a new set of equilibria, called the *extended equilibria* and denoted $E_{ex}(r, u)$, is needed.

It is easy to see that condition (C) and equation (3.1) imply

C': For each i, there is a number $\mu_i \geq 0$ such that $\mu_i Du_i(x_i) = p$.

(Since p is normal to the budget plane, condition (C') can also be interpreted as saying : x_i is a critical point of the restriction $u_i|B_{i,p}$.) We say that (x, p) is an extended price equilibria or $(x, p) \in E_{ex}(r, u)$ if it satisfies conditions (A), (B) and (C').

We make the following remarks.

1. Although some of the x_i may be on $\partial\bar{P}$, the definition still makes sense.
2. It is easlily verified that $E_{ex}(r, u) \supset E_{cl} \cap [(P)^m \times \bar{S}]$. That is, a classical price equilibrium where x_i are not in $\partial\bar{P}$, is also an extended price equilibrium. Thus if the number of extended price equilibriua are finite, the same will be true for the classical ones defined on P.
3. There are examples of extended price equilibria that are not classical equilibria, see Smale [25] example 4 section 4 (here a two person economy has an equilibrium at which one agent's utility is *minimised* at an extended price equilibria !).

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