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On first integrals, conservation laws and reduction of classes of Emden and Liénard equations

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Abstract
We present a general method to construct first integrals for some classes of the well known second-order ordinary differential equations, viz., the Emden and Liénard classes of equations. The approach does not require a knowledge of a Lagrangian but, rather, uses the 'multiplier approach' (Anco and Bluman in Eur J Appl Math 13:545–566, 2002; Eur J Appl Math 13:567–585, 2002). It is then shown how a study of the invariance properties and conservation laws are used to 'twice' reduce the equations to solutions. The equations admit five first integrals of which two are independent but the significance of the five are that they correspond to a five-dimensional algebra of Noether symmetries obtained without the need to construct a Lagrangian.

Keywords First integrals · Reduction · Emden and Liénard equations

Mathematics Subject Classification 70H33 · 70S10 · 34A05

1 Introduction

The general version of the modified version of the Emden type equations

$$y'' + \alpha y y' + \beta y^3 = 0 \tag{1.1}$$

has been the subject of many studies; in a detailed study by Mahomed and Leach [3], the linearizability conditions for (1.1) are presented along with the transformations that lead it to the canonical equation $Y'' = 0$, for $\alpha^2 = 9\beta$, that generate the maximal eight-dimensional Lie point symmetry generators of the algebra $sl(3, R)$. As a consequence, first integrals were constructed using the transformations. Nucci and Tamizhmani [4] discussed a procedure to determine Lagrangians using the method of the 'Jacobi last multiplier'. For example, for $\alpha = 3, \beta = 1$, they show that a Lagrangian is $L = \frac{1}{(y^2+\dot{y})^2}$ generating five first integrals. Another important work in this regard is that of Tiwari et al. [5] in which Lie point symmetry

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classifications of similar equations are performed. Lagrangians for differential equations are generally pursued to construct first integrals (conservation laws) via Noether's theorem [6] and, for ordinary differential equations (odes), one can use the variational symmetry and first integral association to perform double reductions of the odes (using the single symmetry) which for second-order odes lead to solutions [7]. For partial differential equations (pdes), a number of works dealing with conservation laws, associated symmetries and double reductions have been done recently, see [8, 9] and very recently, in [10], Anco and Gandarias extended the approach to include the role of 'multipliers'.

The role of multipliers are essential in the construction of conserved flows (conservation laws) of pdes as they surpass the need of a variational principle. Moreover, the existence of higher-order multipliers lead to a larger class of conservation laws and the computation depends on the power of one's software. Much work has been done in this regard, most notably [1, 2, 11]. For odes, however, the multiplier approach to construct first integrals has not been popular largely due to the position of Noether's theorem which relies on the existence and knowledge of a Lagrangian. In what follows, we will show how the multiplier approach can be exploited to construct first integrals of (1.1).

A differential function Q is a **multiplier** that leads to a conservation law of (1.1) if there exists a differential function T that satisfies the total divergence

$$QG = DT, \tag{1.2}$$

where, in our case, $G = y'' + \alpha yy' + \beta y^3$. Then, since the Euler operator annihilates total divergences,

$$\frac{\delta}{\delta y}(QG) = 0, \tag{1.3}$$

where D is the 'total derivative operator' and $\frac{\delta}{\delta y}$ is the 'Euler (-Lagrange) operator'.

In what follows, we study the first integrals and reduction of

$$y'' + 3yy' + y^3 = 0 \tag{1.4}$$

and other classes of similar equations using the multiplier approach. We show how the first integrals are 'associated' with Lie point symmetry generators which provides a 'double reduction' and, in this case, solutions of the ode. The notion of 'association' between symmetries and conservation laws is well known and has been adopted for ordinary Euler (-Lagrange) equations [7] and was generalised to partial differential equations [8]. However, the application to ordinary differential equations and double reduction, irrespective of the variational case, is, to the best of our knowledge, unknown.

Note. A well known procedure for constructing conservation laws involve Noether's theorem for variational problems. Some cases of (1.1) are variational and the non variational cases require alternative approaches. An approach adopted here requires the determination of **multipliers** followed by certain procedures to then construct the conserved flows/vectors, like the homotopy formula. For a detailed account, see [1, 2, 12], inter alia. We note that the method adopted here recovers all the results where the cases of the equation are variational; Noether's theorem is not utilised here. The reasons are expounded here. Sometimes, construction of Lagrangians are carried out to understand the underlying physics of the model but, usually, a knowledge of the Lagrangian is used to determine first integrals which, in turn, are used to study the integrability of the model. Conservation Laws are now directly obtainable without recourse to a Lagrangian formulation; the computation time and the procedure are far less cumbersome. In the examples below,

2 The Emden equations

The Eq. (1.3) becomes

$$\frac{\delta}{\delta y}[Q(x,y,y')(y''+3yy'+y^3)])=0 \tag{2.1}$$

and after some lengthy calculations and observations, it turns out that the multiplier functions take the form

$$Q=\frac{f(x,y)-y'g(x,y)}{(y^2+y')^3} \tag{2.2}$$

which we enumerate below,

$$\begin{aligned}
Q_1&=\frac{y'}{(y^2+y')^3}\\
Q_2&=\frac{-y^3-yy'}{(y^2+y')^3}\\
Q_3&=\frac{-xy^3+y^2-xyy'}{(y^2+y')^3}\\
Q_4&=\frac{-\frac{1}{2}x^2y^3+xy^2-\frac{2}{3}y-y'(\frac{1}{2}x^2y-\frac{1}{3}x)}{(y^2+y')^3}\\
Q_5&=\frac{-\frac{1}{6}x^3y^3+\frac{1}{2}x^2y^2-\frac{2}{3}xy+\frac{1}{3}-y'(\frac{1}{6}x^3y-\frac{1}{6}x^2)}{(y^2+y')^3}
\end{aligned} \tag{2.3}$$

Each multiplier Q leads to a first integral that satisfy Eq. (2.3). Regarding Q_1, we get the following.

$$D_xI=\frac{y'}{(y^2+y')^3}\left(y''+3yy'+y^3\right), \tag{2.4}$$

where $I=I(x,y,y')$ is the first integral and $D_x=\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+y''\frac{\partial}{\partial y'}+\cdots$ is the total derivative with respect to x. After Expanding and applying the total derivative in Eq. (2.4), we obtain

$$I_x+y'I_y+y''I_{y'}=\frac{y'y''+3y'^2y+y'y^3}{y^6+3y^4y'+3y^2y'^2+y'^3}. \tag{2.5}$$

Splitting on y'' yields

$$\begin{aligned}
y'':I_{y'}&=\frac{y'}{y^6+3y^4y'+3y^2y'^2+y'^3},\\
:I_x+y'I_y&=\frac{3y'^2y+y'y^3}{y^6+3y^4y'+3y^2y'^2+y'^3}.
\end{aligned} \tag{2.6}$$

Integrating Eq. (2.6) with respect to y' gives

$$I(x,y,y')=-\frac{y^2+2y'}{2(y^2+y')^2}+A(x,y), \tag{2.7}$$

where $A(x,y)$ is an arbitrary function of x and y. Substituting the form of $I(x,y,y')$ into Eq. (2.6) leads into the following

$$A_x-\frac{yy'}{(y^2+y')^2}+\frac{2y^3y'}{(y^2+y')^3}+\frac{4yy'^2}{(y^2+y')^3}+y'A_y=\frac{3yy'^2}{(y^2+y')^3}+\frac{y^3y'}{(y^2+y')^3}. \tag{2.8}$$

After simplifying Eq. (2.8), we get that

$$A_x+y'A_y=0. \tag{2.9}$$

Splitting Eq. (2.9) on y' gives

$$\begin{aligned}
y':A_y&=0,\\
:A_x&=0.
\end{aligned} \tag{2.10}$$

Integrating Eq. (2.10) with respect to y yields

$$A(x, y) = B(x), \tag{2.11}$$

where $B(x)$ is an arbitrary function of x. After replacing the value of A into Eq. (2.10) and integrating with respect to x, we obtain $B = C_1$, where C_1 is an arbitrary constant of integration. Therefore, the first integral is given by

$$I(x, y, y') = -\frac{y^2 + 2y'}{2(y^2 + y')^2} + C_1. \tag{2.12}$$

Thus, we may let the first integral corresponding to Q_1 be

$$I_1(x, y, y') = -\frac{y^2 + 2y'}{2(y^2 + y')^2}.$$

It is easy to check that $DI_1 = \frac{y'}{(y^2+y')^3}(y'' + 3yy' + y^3)$.
The first integrals for the rest of the cases in (2.3) are

$$\begin{aligned}
&Q_2 : I_2 = \tfrac{y}{y^2+y'} - x \\
&Q_3 : I_3 = y\left(\tfrac{x}{y^2+y'} - \tfrac{1}{2}\tfrac{y}{(y^2+y')^2}\right) - \tfrac{1}{2}x^2 \\
&Q_4 : I_4 = -\tfrac{1}{3}\tfrac{y(xy-1)}{(y^2+y')^2} + \tfrac{1}{6}\tfrac{x(3xy-2)}{y^2+y'} - \tfrac{1}{6}x^3 \\
&Q_5 : I_5 = \left(\tfrac{1}{6}xy - \tfrac{1}{6}\right)\left(\tfrac{x^2}{y^2+y'} - \tfrac{1}{2}\tfrac{2xy-2}{(y^2+y')^2}\right) - \tfrac{1}{24}x^4
\end{aligned} \tag{2.13}$$

For each of the first integrals, I_j above, $I_j(x, y, y') = k$, k constant, is a first-order reduction of (1.4). As an example, the first-order ode (from I_2)

$$y' - \tfrac{1}{x+k}y + y^2 = 0 \tag{2.14}$$

is a reduction of (1.4) and its solution

$$y = \tfrac{2(x+k)}{2kx+x^2+2k_1} \tag{2.15}$$

is a general solution of (1.4).
Note. Whilst the number of independent first integrals is two (which guarantees integrability), the five independent multipliers obtained are equivalent to five 'characteristics' in the Noether case (Noether first integrals) which for second-order ordinary differential equations not only suggests but implies linearizability. This is equivalent to determining the five Noether symmetries forming the $A_{5,40}$ Lie algebra. From above, we note that I_1 and I_2 are independent. Up to a constant multiplier

$$\begin{aligned}
I_3 &= I_2^2, \\
I_4 &= I_2^3 + I_1 I_2, \\
I_5 &= -I_2^4 - 2I_1 I_2^2 - I_1^2.
\end{aligned} \tag{2.16}$$

2.1 The nonlinearizable case

In this section, we consider another class of (1.1) which also generates nontrivial multipliers and, hence, first integrals of the form (2.2), viz.,

$$y'' + \tfrac{7}{2}yy' + \tfrac{3}{2}y^3 = 0. \tag{2.17}$$

Equation (2.17) admits just two such multipliers, i.e.,

$$\begin{aligned} P_1 &= \tfrac{-\frac{2}{3}y^3 - yy'}{(y^2+y')^3} \\ P_2 &= \tfrac{-\frac{2}{3}xy^3 + \frac{5}{9}y^2 - (xy - \frac{8}{9})y'}{(y^2+y')^3}. \end{aligned} \tag{2.18}$$

The first integrals are, respectively,

$$\begin{aligned} J_1 &= \tfrac{1}{3}\, y \left(-\tfrac{1}{2}\, \tfrac{y^2}{(y^2+y')^2} + 3\, \tfrac{1}{y^2+y'} \right) - x \\ J_2 &= -\tfrac{1}{9}\, \tfrac{-9xy+8}{y^2+y'} - \tfrac{1}{6}\, \tfrac{y^2(xy-1)}{(y^2+y')^2} - \tfrac{1}{2}\, x^2. \end{aligned} \tag{2.19}$$

We show the first integrals can be used to perform a double reduction of (2.17). Firstly, (2.17) admit the Lie symmetry generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

A reduced first order ode reduction of (2.17) is given by $J_1 = k$, k constant. This ode, viz.,

$$\tfrac{1}{3}\, y \left(-\tfrac{1}{2}\, \tfrac{y^2}{(y^2+y')^2} + 3\, \tfrac{1}{y^2+y'} \right) - x = k \tag{2.20}$$

inherits X_2 if $k = 0$, since $X_2^{[1]} J_1 =$, where $X_2^{[1]}$ is the first prolongation of X_2. It is, in general, not easy to construct symmetry generators of first-order odes. To reduce (2.20) to a solution using X_2, we need to map X_2 to $X^* = 1\frac{\partial}{\partial t} + 0\frac{\partial}{\partial s}$, where $t = t(x, y)$ and $s = s(x, y)$ are transformed variables obtained by solving

$$x\frac{\partial t}{\partial x} - y\frac{\partial t}{\partial y} = 1, \qquad x\frac{\partial s}{\partial x} - y\frac{\partial s}{\partial y} = 0.$$

We get

$$t = \ln x, \qquad s = xy.$$

Therefore,

$$\begin{aligned} y' = \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}s} \Big/ \frac{\mathrm{d}x}{\mathrm{d}s} \\ &= \frac{1 - s\frac{\mathrm{d}t}{\mathrm{d}s}}{x^2 \frac{\mathrm{d}t}{\mathrm{d}s}} \end{aligned}$$

so that

$$\tfrac{1}{3}\, y \left(-\tfrac{1}{2}\, \tfrac{y^2}{(y^2+y')^2} + 3\, \tfrac{1}{y^2+y'} \right) - x = k \tag{2.21}$$

becomes the variable separable ode

$$6\,s^4 t'^2 - 17\,s^3 t'^2 + 12\,s^2 t'^2 + 12\,s^2 t' - 18\,s t' + 6 = 0, \tag{2.22}$$

where $t = t(s)$. We get, for the special case, $k = 0$

$$t' = \frac{-6\,s + 9 + \sqrt{-6\,s + 9}}{\left(6\,s^2 - 17\,s + 12\right) s}$$

so that

$$t = -\tfrac{1}{4}\ln\left(\sqrt{-6s+9}+3\right) + \tfrac{3}{4}\ln\left(\sqrt{-6s+9}+1\right) - \tfrac{3}{4}\ln\left(\sqrt{-6s+9}-1\right) + \tfrac{1}{4}\ln\left(\sqrt{-6s+9}-3\right) + \tfrac{3}{4}\ln(s) - \tfrac{3}{4}\ln(3s-4) + k_1.$$

Note. The above procedure of dealing with reduction of first-order equations is known ([7]) but the transformations can often be cumbersome. Here, however, we were able to induce elegant transformations, via the symmetries, with the associated first integrals to a solution which would otherwise be problematic.

3 Classes of the quadratic Liénard equations

In [5], a detailed Lie point symmetry analysis of the Liénard equations, $y'' + f(y)y'^2 + g(y) = 0$, is done. In particular, they showed two classes are linearizable as they admit an 8-dimensional Lie algebra of point symmetries. The complete set of five independent first integrals are constructed below. The detailed calculations are omitted as they follow the procedure above.

3.1 $f = \alpha$ constant

We first consider the class

$$y'' + \alpha y'^2 + \tfrac{1}{\alpha}\beta^2\left(1 - e^{-\alpha y}\right) = 0 \tag{3.1}$$

which admits the $sl(3, R)$ algebra of Lie point symmetry generators with basis

$$\begin{aligned}
&\partial_x, && (1-e^{-\alpha y})\partial_y,\\
&-\tfrac{1}{\alpha}\cos(\beta x)e^{-\alpha y}\partial_y, && -\tfrac{1}{\alpha}\sin(\beta x)e^{-\alpha y}\partial_y,\\
&\tfrac{1}{2\beta}\sin(2\beta x)\partial_x + \tfrac{1}{2\alpha}\cos(2\beta x)(1-e^{-\alpha y})\partial_y, &&\\
&-\tfrac{1}{2\beta}\cos(2\beta x)\partial_x + \tfrac{1}{2\alpha}\sin(2\beta x)(1-e^{-\alpha y})\partial_y, &&\\
&-\sin(\beta x)(1-e^{\alpha y})\partial_x - \tfrac{\beta}{\alpha}(2-e^{\alpha y})\cos(\beta x)\partial_y, &&\\
&-\cos(\beta x)(1-e^{\alpha y})\partial_x + \tfrac{\beta}{\alpha}(2-e^{\alpha y})\sin(\beta x)\partial_y. &&
\end{aligned} \tag{3.2}$$

The multipliers that lead to nontrivial and independent first integrals are

$$\begin{aligned}
Q_1 &= -e^{2\alpha y}y',\\
Q_2 &= \mathrm{e}^{\alpha y}\cos(\beta x),\\
Q_3 &= \mathrm{e}^{\alpha y}\sin(\beta x),\\
Q_4 &= -2\,(\mathrm{e}^{\alpha y})^2\sin(\beta x)\cos(\beta x)\,y' + \tfrac{2(\mathrm{e}^{\alpha y})^2(\cos(\beta x))^2\beta - 2\,\mathrm{e}^{\alpha y}(\cos(\beta x))^2\beta - (\mathrm{e}^{\alpha y})^2\beta + \mathrm{e}^{\alpha y}\beta}{\alpha},\\
Q_5 &= -\left(2\,(\mathrm{e}^{\alpha y})^2(\cos(\beta x))^2 - (\mathrm{e}^{\alpha u})^2\right)y' + \tfrac{-2(\mathrm{e}^{\alpha y})^2\sin(\beta x)\cos(\beta x)\beta + 2\,\mathrm{e}^{\alpha y}\sin(\beta x)\cos(\beta x)\beta}{\alpha}
\end{aligned} \tag{3.3}$$

with corresponding first integrals

$$
\begin{aligned}
I_1 &= -\tfrac{1}{\alpha^2}[\tfrac{1}{2}\,\alpha^2 \mathrm{e}^{2\alpha y} y'^2 + \tfrac{1}{2}\,\mathrm{e}^{2\alpha y}\beta^2 - \mathrm{e}^{\alpha y}\beta^2 + \tfrac{1}{2}\,\beta^2],\\
I_2 &= -\tfrac{1}{\alpha}[-\mathrm{e}^{\alpha y}\cos(\beta x)\,\alpha\, y' - \mathrm{e}^{\alpha y}\sin(\beta x)\,\beta + \sin(\beta x)\,\beta],\\
I_3 &= -\tfrac{1}{\alpha}[-\mathrm{e}^{\alpha y}\sin(\beta x)\,\alpha\, y' + \mathrm{e}^{\alpha y}\cos(\beta x)\,\beta - \cos(\beta x)\,\beta],\\
I_4 &= -\tfrac{1}{\alpha^2}[\sin(\beta x)\cos(\beta x)\,\mathrm{e}^{2\alpha y}\alpha^2 y'^2 - 2\,(\cos(\beta x))^2\,\mathrm{e}^{2\alpha y}\alpha\,\beta\, y'\\
&\quad - \sin(\beta x)\cos(\beta x)\,\mathrm{e}^{2\alpha y}\beta^2 + 2\,\mathrm{e}^{\alpha y}\,(\cos(\beta x))^2\,\beta\, y'\alpha + 2\,\mathrm{e}^{\alpha y}\cos(\beta x)\,\beta^2\sin(\beta x)\\
&\quad + \mathrm{e}^{2\alpha y}\alpha\,\beta\, y' - \cos(\beta x)\,\beta^2\sin(\beta x) - \mathrm{e}^{\alpha y}\alpha\,\beta\, y'],\\
I_5 &= -\tfrac{1}{\alpha^2}[(\cos(\beta x))^2\,\mathrm{e}^{2\alpha y}\alpha^2 y'^2 + 2\cos(\beta x)\sin(\beta x)\,\mathrm{e}^{2\alpha y}\alpha\,\beta\, y'\\
&\quad - (\cos(\beta x))^2\,\mathrm{e}^{2\alpha y}\beta^2 - 2\,\mathrm{e}^{\alpha y}\cos(\beta x)\,\alpha\, y'\sin(\beta x)\,\beta - \tfrac{1}{2}\mathrm{e}^{2\alpha y}\alpha^2 y'^2\\
&\quad + 2\,\mathrm{e}^{\alpha y}\,(\cos(\beta x))^2\,\beta^2 - (\cos(\beta x))^2\,\beta^2 + \tfrac{1}{2}\mathrm{e}^{2\alpha y}\beta^2 - \mathrm{e}^{\alpha y}\beta^2 + \tfrac{1}{2}\beta^2]
\end{aligned}
\tag{3.4}
$$

The double reduction to attain solutions by quadrature will require a test for association between a symmetry and first integral pair followed by the procedure presented above.

3.2 $f = -\frac{2\alpha}{1+\alpha y}$

The class

$$
y'' - \tfrac{2\alpha}{1+\alpha y}y'^2 + \beta^2\left(y + \alpha y^2\right) = 0 \tag{3.5}
$$

also admits the maximal eight-dimensional Lie algebra of symmetries isomorphic to $sl(3, R)$. The multipliers that lead to nontrivial and independent first integrals are

$$
\begin{aligned}
Q_1 &= \tfrac{\cos(\beta x)}{\alpha^2 y^2 + 2\alpha\, y + 1},\\
Q_2 &= \tfrac{\sin(\beta x)}{\alpha^2 y^2 + 2\alpha\, y + 1},\\
Q_3 &= -\tfrac{y'}{(\alpha y + 1)^4},\\
Q_4 &= \tfrac{2\,(\cos(\beta x))^2\beta\, y - \beta\, y}{\alpha^3 y^3 + 3\alpha^2 y^2 + 3\alpha\, y + 1} - y'\left(\tfrac{2\sin(\beta x)\cos(\beta x)}{\alpha^4 y^4 + 4\alpha^3 y^3 + 6\alpha^2 y^2 + 4\alpha\, y + 1}\right),\\
Q_5 &= -\tfrac{-2\beta y\sin(\beta x)\cos(\beta x)}{\alpha^3 y^3 + 3\alpha^2 y^2 + 3\alpha\, y + 1} - y'\left(\tfrac{(2\,(\cos(\beta x))^2 - 1}{\alpha^4 y^4 + 4\alpha^3 y^3 + 6\alpha^2 y^2 + 4\alpha\, y + 1}\right)
\end{aligned}
\tag{3.6}
$$

with corresponding first integrals

$$
\begin{aligned}
I_1 &= \tfrac{\sin(\beta x)\alpha\,\beta\, y^2 + \sin(\beta x)\beta\, y + \cos(\beta x)y'}{(\alpha\, y + 1)^2},\\
I_2 &= \tfrac{\sin(\beta x)y'}{\alpha^2 y^2 + 2\alpha\, y + 1} + \tfrac{\cos(\beta x)\beta}{\alpha\,(\alpha\, y + 1)} - \tfrac{\cos(\beta x)\beta}{\alpha},\\
I_3 &= -\tfrac{1}{2}\,\tfrac{y'^2}{(\alpha y + 1)^4} - \beta^2\left(-\tfrac{1}{\alpha^2(\alpha\, y + 1)} + \tfrac{1}{2}\,\tfrac{1}{\alpha^2(\alpha\, y + 1)^2}\right),\\
I_4 &= -\tfrac{1}{2}\,\tfrac{\sin(2\beta x)y'^2}{(\alpha\, y + 1)^4} + \tfrac{\cos(2\beta x)\beta\, yy'}{(\alpha\, y + 1)^3} + \sin(2\beta x)\,\beta^2\left(\tfrac{1}{2}\,\tfrac{1}{\alpha^2(\alpha\, y + 1)^2} - \tfrac{1}{\alpha^2(\alpha\, y + 1)}\right)\\
&\quad + \tfrac{1}{2}\,\tfrac{\sin(2\beta x)\beta^2}{\alpha^2},\\
I_5 &= -\tfrac{1}{2}\,\tfrac{\cos(2\beta x)y'^2}{(\alpha\, y + 1)^4} - \tfrac{\sin(2\beta x)\beta\, yy'}{(\alpha\, y + 1)^3} + \cos(2\beta x)\,\beta^2\left(\tfrac{1}{2}\,\tfrac{1}{\alpha^2(\alpha\, y + 1)^2} - \tfrac{1}{\alpha^2(\alpha\, y + 1)}\right)\\
&\quad + \tfrac{1}{2}\,\tfrac{\cos(2\beta x)\beta^2}{\alpha^2}.
\end{aligned}
\tag{3.7}
$$

Note. In all of the cases above, the differential equations are linearizable as they generate five first integrals (two independent ones) which will correspond to a five-dimensional algebra of variational symmetries which, in the case of (3.1), would be a subalgebra of the $sl(3, R)$ (3.2).

Conclusions

We presented a general method to construct first integrals for some classes of some well known second-order ordinary differential equations, viz., the Emden and Liénard classes of equations. The approach does not require a knowledge of a Lagrangian and, instead, utilises the 'multiplier approach'. Then, we showed how a study of the invariance properties and conservation laws are used to 'twice' reduce the equations to solutions. In each case, the equations admitted five first integrals of which two are independent but the significance of the five are that they correspond to a five-dimensional algebra of Noether symmetries obtained without the need to construct a Lagrangian.
In short, there is an emphasis in the paper to show that, inter alia, one could still conclude the linearizability of the odes based on the knowledge (cardinality) of the multipliers. This is important as the algebraic nature of the variational symmetries used to be the indicator of the 'optimal' Lagrangian.

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