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MASTER'S DISSERTATION

The Bipartite Ramsey Number of Cycles

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Abstract

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Master's of Science in Mathematics

Bipartite Ramsey Number of Cycles

by Ronald J. Maartens, Elizabeth Jonck and Amukelani B. Tivane.

The *Ramsey number* $R(H_1, H_2)$ of two graphs H_1 and H_2 is the smallest positive integer n for which every red-blue coloring of the complete graph K_n of order n results in a subgraph of K_n isomorphic to H_1 all of whose edges are colored red (called a *red* H_1), or a subgraph of K_n isomorphic to H_2 all of whose edges are colored blue (called a *blue* H_2).

The *s-bipartite Ramsey number* $b_s(H_1, H_2)$ of two bipartite graphs H_1 and H_2 is the smallest positive integer t , with $t \geq s$, such that every red-blue coloring of the complete bipartite graph $K_{s,t}$ results in a red H_1 or a blue H_2 . For $s = t$, the *s-bipartite Ramsey number* is known as the *bipartite Ramsey number*, which we denote by $b(H_1, H_2)$.

In this dissertation we investigate $b(C_{2m}, C_{2n})$ and $b_s(C_{2m}, C_{2n})$ for $m \geq 2$ and $n \geq 2$.

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List of Symbols

<u>Symbols</u>	<u>Meaning</u>	<u>Page</u>
$V(G)$	Set of vertices of the graph G	11
$E(G)$	Set of edges of the graph G	11
$N(y)$	Open neighbourhood of the vertex y	11
$\deg_G(y)$	Degree of a vertex y in the graph G	11
$d(y)$	Degree of a vertex y	11
$ V(G) $	Cardinality of the set $V(G)$ /Order of graph G	11
$ E(G) $	Cardinality of the set $E(G)$ /Size of graph G	11
$G < S >$	Vertex set S that induces a subgraph of G	12
ϕ	Bijection indicating an isomorphism between graphs	12
P_n	Path of order n	13
C_n	Cycle of order n	13
$\emptyset$	Empty set	13

G^c	Complement of the graph G	14
K_n	Complete graph of order n	14
$K_{x,y}$	Complete bipartite graph with partite sets of order x and y	14
(β, Δ)	Balanced complete bipartite graph	14
$\chi(G)$	Number of colors required to color the edges of graph G	14
G_R	Subgraph of graph G whose edges are all colored red	15
G_B	Subgraph of graph G whose edges are all colored blue	15
$R(H_1, H_2)$	Ramsey number of the graphs H_1 and H_2	17
$b(H_1, H_2)$	Bipartite Ramsey number of the graphs H_1 and H_2	19
$b(m, n)$	Bipartite Ramsey number of the graphs $K_{m,m}$ and $K_{n,n}$	19
$b_s(H_1, H_2)$	s -Bipartite Ramsey number of the graphs H_1 and H_2	21
$o(1)$	Small o notation	68

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1 Introduction

1.1 Graph Theory

The reader is directed to the textbook *Chromatic Graph Theory* by Chartrand and Zhang [1] for all definitions and notations not mentioned in this chapter.

1.1.1 Introduction to Graph Theory

A *graph* G is a finite nonempty set $V(G)$ of points which are commonly known as *vertices* (also known as nodes, singular vertex or node) together with a set $E(G)$ (possibly empty) of pairs of distinct vertices of G called *edges*. An edge (also known as a link) is represented as a line between two distinct vertices and these vertices are said to be *adjacent*. Edges are adjacent if they are incident to a common vertex. A graph can be represented by $G = (V, E)$ where $V = V(G)$ indicates the set of vertices of G and $E = E(G)$ the set of edges of G . Note that we can omit the “ G ” if the graph under consideration is clear, e.g. V instead of $V(G)$. Adjacent vertices can also be referred to as *neighbours* of each other and so the set of neighbours of the vertex y is called the *neighbourhood* (formally known as a *open neighbourhood*) of y and is denoted by $N(y)$. The *closed neighbourhood* of y is defined as the set $N[y] = N(y) \cup \{y\}$.

The *degree* of a vertex y in the graph G , denoted by $\deg_G(y)$ or just $d(y)$, is the number of vertices in G that are adjacent to the vertex y . Alternatively, $\deg_G(y)$ can be defined as the number of vertices in the neighbourhood of y , that is, $\deg_G(y) = |N(y)|$.

The number of elements in the set V of graph G is called the *order* of G and is denoted by $n = |V(G)|$, while the number of elements in the set E of G is called the *size* of G and is denoted by $m = |E(G)|$.

A graph G' is a subgraph of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. Thus G

contains G' and it is denoted by $G' \subseteq G$. If $V(G') \neq V(G)$, then G' is a proper subgraph of G , while if $V(G') = V(G)$, then G' is a *spanning subgraph* of G .

Suppose that S is a set of vertices such that $S \subset V$ where $G = (V, E)$. We say that $G \langle S \rangle$ is an *induced* subgraph of G if the vertices of S are adjacent in $G \langle S \rangle$ if and only if they are adjacent in G . We say that $G \langle S \rangle$ is a subgraph of G induced by the vertex subset S . We define an *independent set* as a vertex set of a graph that has no two adjacent vertices.

Graphs G and G' are *isomorphic* if a bijection $\phi: V \rightarrow V'$ exists such that two vertices $x, y \in V$ are adjacent in G if and only if $\phi(x)$ and $\phi(y)$ are adjacent in G' . This bijection is called an *isomorphism* and is denoted $G \cong G'$. For example, the bijection $\phi: V \rightarrow V'$ defined by $\phi(x_1) = y_4$, $\phi(x_2) = y_2$, $\phi(x_3) = y_6$, $\phi(x_4) = y_1$, $\phi(x_5) = y_5$, and $\phi(x_6) = y_3$ is illustrated in Figure 1.1. Here we see that G and $\phi(G)$ are isomorphic since the requirements of the definition of isomorphism are met. See for example that $\phi(x_1) = y_4$ and $\phi(x_2) = y_2$ are adjacent in $\phi(G)$ since x_1 and x_2 are adjacent in G .

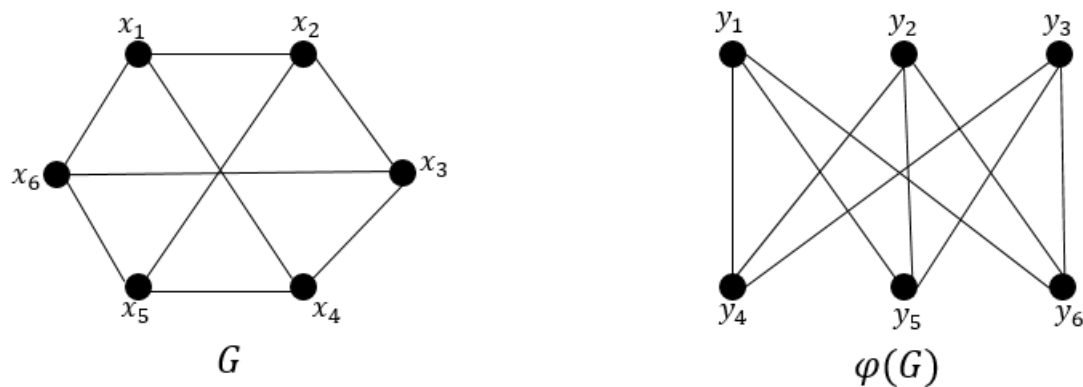


Figure 1.1 Isomorphic graphs.

1.1.2 Paths and Cycles

A *path* on n vertices is a graph, denoted P_n for $n \geq 2$, which has the vertex set $\{x_1, x_2, \dots, x_n\}$ and edge set $\{x_i x_{i+1} | i = 1, 2, \dots, n-1\}$. If path $P_5 = x_1, x_2, \dots, x_5$, then it is said that P_5 is a $x_1 - x_5$ path. Two vertices x_1 and x_2 are *connected* if the graph G contains a $x_1 - x_2$ path, and G is called *connected* if G contains a $x_1 - x_2$ path for every two vertices x_1 and x_2 in G .

A *cycle* is a graph with the vertex sequence $x_1, x_2, \dots, x_n, x_1$ wherein no edge is repeated. A cycle is also defined as a $x_1 - x_5$ path with the edge $x_1 x_5$. The *length* of a cycle is the number of edges in a cycle. A cycle with the length $k \geq 3$ is called a *k-cycle*; consequently a cycle with even (odd) length is called an *even (odd) cycle*. The notation C_n is used to indicate the cycle of length n .

1.1.3 Trees and Forests

A *tree* is a connected graph that does not contain any cycles and a *forest* is any graph without cycles. The order of a tree is n and its size is $m = n - 1$. The notation T_n will be used to represent a tree of order n .

1.1.4 Bipartite Graphs

A graph G is a bipartite graph if the set $V(G)$ can be partitioned into two subsets X and Y such that every edge xy in the graph G joins a vertex x of the set X and a vertex y of the set Y . The vertex set of the bipartite graph can be expressed as $V(G) = X \cup Y$ with $X \cap Y = \emptyset$. Note that a bipartite graph G cannot contain odd cycles, in fact, bipartite graphs are characterized by the following property.

Proposition 1.1 [1]

A graph $G = (V, E)$ is bipartite if and only if G has no odd cycles.

1.1.5 Complete Graphs

A graph G containing n vertices is *complete* if every two vertices of G are adjacent. The complete graph is denoted by K_n and has $\binom{n}{2}$ edges in total. The subgraph K_p , where $p \leq n$, of a graph G is called a *clique*. The graph G^c is called the *complement* of G if $V(G^c) = V(G)$ and $E(G^c) = E(K_n) - E(G)$.

A bipartite graph G with the vertex partite sets X and Y is a *complete bipartite graph* if every vertex of the set X is adjacent to every vertex of the set Y . The complete bipartite graph is denoted by $K_{x,y}$ where x is the order of the set X and y is the order of the set Y . Hence the order of $K_{x,y}$ is $n = x + y$ and the size is $m = xy$. We say that a complete bipartite graph $K_{x,y}$ is a *star* if $x = 1$. A *balanced complete bipartite graph*, denoted (β, Δ) , is a complete bipartite graph $K_{x,y}$ where $x = y$, β is the maximum length between connected vertices in graph G and Δ is the maximum degree of a vertex in graph G .

1.1.6 Edge Colorings

The *edge coloring* of a graph G is an assignment of one color per edge of the graph G from a set of possible colors. Actual colors such as red, blue or yellow are used when a small amount of colors are used, otherwise the color χ_i , where $1 \leq i \leq k$ for some finite integer k , is used. A coloring of a graph G that makes use of k colors will be referred to as *k-edge coloring*. The edge coloring is referred to as *proper edge coloring* if adjacent edges are colored differently. A *red-blue edge coloring* of a graph G is a coloring of the edges of G in which every edge is colored red or blue only. An example of a proper χ_4 -edge coloring is given in Figure 1.2.

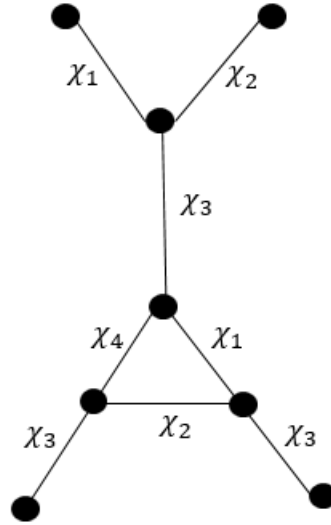


Figure 1.2 χ_4 -edge coloring.

For the purposes of this dissertation we will make use of a red-blue coloring of the edges of graph G whereby we use a dotted line to represent an edge colored red and a solid line to represent an edge colored blue. Note that the red subgraph G_R of G (the graph induced by the edges colored red in G) and the blue subgraph G_B of G (the graph induced by the edges colored blue in G) are complements of each other in G .

1.2 Ramsey Theory

On the social media platform Facebook, there are certain types of relationships. Assuming that there are at least six people on Facebook, then either each Facebook user has at least three people that he/she is friends with or at least three people that he/she is not friends with. In Graph Theory terms, this means that if each edge of a complete graph K_6 is colored red or blue, then every vertex of K_6 has at least three blue edges or at least three red edges incident to it.

However, the Facebook statement above is not true for complete graphs of order less than six. In a group of six people on Facebook, each Facebook user can have at most five friends, resulting in the possible cases.

Case 1: The Facebook user has five friends.

Case 2: The Facebook user has four friends and one person he/she is not friends with.

Case 3: The Facebook user has three friends and two people he/she is not friends with.

Case 4: The Facebook user has two friends and three people he/she is not friends with.

Case 5: The Facebook user has one friend and four people he/she is not friends with.

Case 6: The Facebook user has five people he/she is not friends with.

From Case 1 to Case 3 the Facebook user has more than three people that he/she is friends with and from Case 4 to Case 6 the Facebook user has more than three people that he/she is not friends with. Now let us consider the six Facebook users. Suppose we draw lines joining every pair of users and color them blue if the users are friends and red if the users are not friends. We find that we have colored 15 lines in total and, more interestingly, we find that we always have three lines of the same color drawn from each user. This means that every Facebook user will always have at least three people that he/she is friends with (blue lines) and at least three people that he/she is not friends with (red lines).

The branch of mathematics called Ramsey Theory is named after the mathematician, economist and philosopher Frank Plumpton Ramsey (1903-1930). His first recognized work was the paper *The Foundations of Mathematics* which he completed in the year 1925. In 1928, Ramsey [2] presented his second paper to the London Mathematical Society titled *On a problem of formal logic*. From this study, he observed (in Graph Theory terms) that for all positive integers m and n , there is an integer r such that if each edge of K_r is colored either red or blue, then there will either be a red K_m or a blue K_n in K_r . Note that the number of colors are not necessarily limited to just two. Ramsey's formal theorem in Graph Theory is as follows.

Theorem 1.1 (Ramsey's Theorem)[1]

For positive integers $n_1, n_2, \dots, n_k$, with $k \geq 2$, there exists a positive integer N , such that if the edges of K_N are colored with k colors, then, for some $1 \leq i \leq k$, K_N will have a subgraph K_{n_i} , such that every edge of this K_{n_i} is colored by color i .

Ramsey's theorem gave rise to the study of different versions of his theory done by other mathematicians and researchers, even today.

1.2.1 The Ramsey Number

Let H_1 and H_2 be complete graphs. The *Ramsey number* $R(H_1, H_2) = n$ is the minimum number of vertices of the complete graph $G = K_n$ such that for every red-blue coloring of the edges of G there is a complete subgraph of G that is isomorphic to H_1 , where all of its edges are colored red, or there is a complete subgraph of G that is isomorphic to H_2 , where all of its edges are colored blue. Note that by the natural consequence of symmetry of graphs, we have

$$R(H_1, H_2) = R(H_2, H_1).$$

A popular example in Ramsey Theory is $R(K_3, K_3) = 6$ which is also a solution to the *Friends and Strangers* Facebook example we discussed previously.

We first prove the lower bound $R(K_3, K_3) \geq 6$ as in [2]. Let $V(K_5) = \{x_1, x_2, x_3, x_4, x_5\}$ and define a red-blue coloring of the edges of K_5 by coloring each edge of the 5-cycle induced by the vertex set $\{x_1, x_2, x_3, x_4, x_5\}$ red and the remaining edges blue such that we avoid a red K_3 or blue K_3 ; see Figure 1.3. The upper bound is proved through the observation that in K_6 , any vertex must be adjacent to at least three edges of the same color. See the detailed explanation in the Facebook example above.

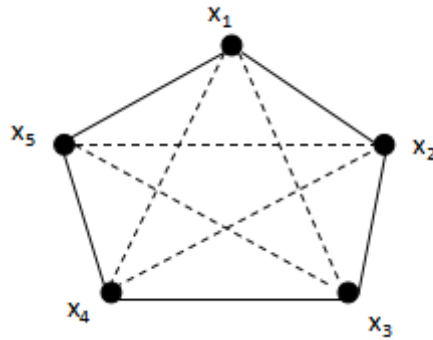


Figure 1.3 Red-blue coloring of the edges of K_5 .

A list of some known classical Ramsey numbers $R(K_s, K_t)$ for $3 \leq s \leq t$ is given below.

- $R(K_3, K_6) = 18$, $R(K_3, K_4) = 9$, $R(K_3, K_5) = 14$ [3].
- $R(K_3, K_9) = 36$ [5], $R(K_3, K_7) = 23$ [4], $R(K_4, K_4) = 18$ [4].
- $R(K_3, K_8) = 28$ [5], $R(K_4, K_5) = 25$ [6].

Although Ramsey's original theorem was in terms of cliques only, the above result can be generalised to any two subgraphs.

1.3 Bipartite Ramsey Theory

Bipartite Ramsey numbers are similarly defined as Ramsey numbers but instead of having a complete graph being colored, we will color a complete bipartite graph. Given bipartite graphs H_1 and H_2 , the bipartite Ramsey number, denoted $b(H_1, H_2)$, is the least positive integer r such that there exists a subgraph G of $K_{r,r}$ such that G is isomorphic to H_1 where all of its edges are colored red or isomorphic to H_2 where all of its edges are colored blue.

The idea behind Bipartite Ramsey numbers dates back to Zarankiewicz [7] who lived from 1902 to 1959 and considered the least positive integer $z = z(s, t, m, n)$ such that any subgraph of $K_{s,t}$ with z edges contains $K_{m,n}$ as a subgraph. The numbers $z(s, t, m, n)$ are referred to as the *Zarankiewicz numbers*.

For positive integers m and n with $m \leq n$, Beineke and Schwenk [8] studied the bipartite Ramsey number $b_2(m, n)$ which they denoted by $R(m, n)$. The bipartite Ramsey number $b(m, n)$ or $b_2(m, n)$ is the least positive integer b such that if the edges of $K_{b,b}$ are colored with red or blue, then a blue $K_{m,m}$ or red $K_{n,n}$ will always exist (Goddard, Henning and Oellermann [9]). Beineke and Schwenk [8] proved that $b(C_4, K_n) \leq 4n - 3$, $b(K_3, K_3) = 17$, $b(K_3, K_n) \leq 8n - 5$ and $b(K_3, K_n) \geq 8n - 7$ while Irving [10] improved the upper bound to $b(K_3, K_n) \leq 8n - 7$. Beineke and Schwenk [8] also conjectured that $b(K_m, K_n) = 2^m(n - 1) + 1$ which Irving [10] disproved by showing that $b(K_4, K_4) \leq 48$. However, Thomason [11] proved part of the conjecture by showing the upper bound $b(K_m, K_n) \leq 2^m(n - 1) + 1$.

It is known that $b(C_6, C_6) = 6$ (Zhang, Sun and Wu [12]) and hence it follows that a complete bipartite graph of order $6 + 6 = 12$ contains either a red C_6 or a blue C_6 . A detailed and more simplified mathematical proof will be given in Chapter 2.

Note that the bipartite Ramsey number $b(H_1, H_2)$ will not exist for non-bipartite graphs H_1 and H_2 because there are no odd cycles in a complete bipartite graph $K_{x,y}$. In mathematical terms, we denote this by $b(H_1, H_2) = \infty$. The existence of a bipartite Ramsey number $b(H_1, H_2)$ where H_1 and H_2 are cycles requires both of them to be even cycles.

Many researchers have studied bipartite Ramsey numbers although the calculation of these numbers is difficult. It is known that $b(K_{2,3}, K_{2,3}) = 9$, $b(K_{2,4}, K_{2,4}) = 13$ and $b(K_{3,3}, K_{3,3}) = 17$. [12]

A popular result is $b(K_{2,2}, K_{2,2}) = 5$ by [8] which we translate into $b(C_4, C_4) = 5$. A popular recreational problem to do with bipartite Ramsey theory is stated by Olejniczak [13] as follows.

Suppose, for some positive integer r , that r girls and r boys are invited to a party where each girl-boy pair is either acquainted or strangers. What is the smallest such number r that guarantees that there exists a group of six people, three girls and three boys, such that either (1) every one of the three girls is acquainted with every one of the three boys or (2) every one of the three girls is a stranger to every one of the three boys?

Solution: $b(K_{3,3}, K_{3,3}) = 17$ [8].

The proof of $b(C_4, C_4) = 5$ and other results of cycles will be given in Chapter 2.

1.4 s -Bipartite Ramsey Theory

We already defined the bipartite Ramsey number $b(m, n)$ in Section 1.3 which is similarly defined as $b_2(m, n)$. This variation of the bipartite Ramsey number will now be generalized. For a positive integer s , the s -bipartite Ramsey number $b_s(H_1, H_2)$ of two bipartite graphs H_1 and H_2 is the smallest positive integer t , with $s \leq t$, such that every red-blue coloring of the edges of the complete bipartite graph $K_{s,t}$ results in a red H_1 or a blue H_2 .

Bi, Chartrand and Zhang studied the s -bipartite Ramsey numbers in their paper called *Party problems and Ramsey numbers* [14]. They were interested in studying the bipartite Ramsey numbers where each subset of vertices X and Y of the complete bipartite graph $V(G) = X \cup Y$ is of a different order.

An example of the above is $b_3(K_{2,2}, K_{2,3}) = 10$ [13] of which the lower bound is illustrated in Figure 1.4 below. Note the lower bound $b_3(K_{2,2}, K_{2,3}) \geq 10$ is achieved by a red-blue coloring of the edges of $K_{3,9}$ that avoids a red subgraph $K_{2,2}$ and a blue subgraph $K_{2,3}$ of $G = K_{3,9}$. To determine this lower bound, we color four edges that are incident to all the vertices from the set $U = \{u_1, u_2, u_3\}$ blue in such a way that only three vertices from the set $W = \{w_1, w_2, w_3, \dots, w_9\}$ are incident to at most two blue edges.

The complete bipartite graph $K_{2,3}$ requires that at least three vertices from W are incident to three blue edges that are also incident to two mutual vertices from the set U , but we have that three vertices of W are incident to two blue edges, hence avoiding a blue $K_{2,3}$. We color the remaining five edges that are incident to every vertex from U red in such a way that the remaining six vertices from W are incident to at most two red edges. We observe that no two vertices of U are adjacent to two

common vertices of W , hence avoiding a red $K_{2,2}$. Note that each solid line represents a red edge and each dashed line represents a blue edge in the red-blue coloring shown in Figure 1.4.

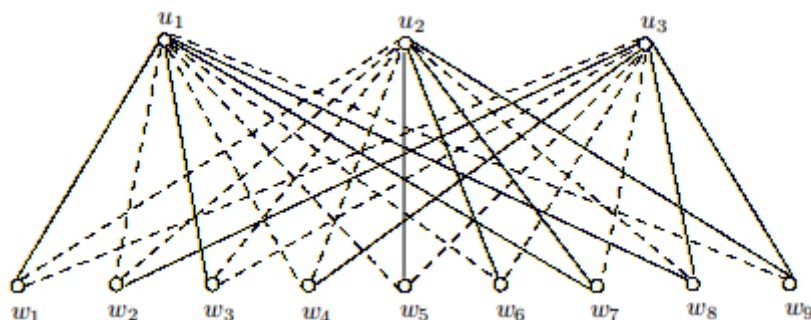


Figure 1.4 Red-blue coloring of the edges of $K_{3,9}$.

The recreational problem stated in [14] that is associated with s -bipartite Ramsey theory is as follows.

There are eight girls at a party. What is the minimum number of boys who must be invited to the party to guarantee that there exists a group of six people, three girls and three boys, such that either (1) every one of the three girls is acquainted with every one of the three boys or (2) every one of the three girls is a stranger to every one of the three boys?.

The answer to the above party recreational problem is the s -bipartite Ramsey number $b_8(K_{3,3}, K_{3,3}) = 29$ [15]. That is, if there are eight girls at a party, then at least 29 boys need to be invited to the party to ensure that there are three girls and three boys where all three girls are acquaintances of all three boys, or all three girls are strangers to all three boys.

Most of the work provided for the s -bipartite Ramsey theory in Chapter 3 is original.

1.5 The Relationship between s -Bipartite Ramsey Numbers and Bipartite Ramsey Numbers

This section investigates the result found in [13] which gives the relationship between s -bipartite Ramsey numbers and bipartite Ramsey numbers.

Proposition 1.2

Let H_1 and H_2 be two bipartite graphs. For every integer s with $2 \leq s \leq b(H_1, H_2)$,

$$b_s(H_1, H_2) \geq b(H_1, H_2).$$

Proof. Let $t = b_s(H_1, H_2)$ for $t \geq s$. If we take a red-blue coloring on $K_{t,t}$, then either there is a red H_1 or a blue H_2 . This implies that $b(H_1, H_2) \leq t = b_s(H_1, H_2)$. ■

2 Bipartite Ramsey Numbers of Cycles

2.1 Background

The bipartite Ramsey number of two even cycles will be studied in this chapter. Refer to Section 1.3 for notation and terminology. In addition, reference will be made to *The Bipartite Ramsey Numbers $b(C_{2m}, C_{2n})$* published in 2013 by Zhang, Sun and Wu [12]. Their work is crucial for this research as they proved the general result of the bipartite Ramsey number of an even cycle with a 6-cycle, that is, $b(C_{2m}, C_6)$, which will be used in the generalisation $b(C_{2m}, C_{2n})$.

2.2 Purpose

The aim is to continue the work of the above-mentioned authors by investigating and finding the bipartite Ramsey number of two cycles.

In Section 2.3 it will be motivated why the bipartite Ramsey number does not exist for odd cycles but exists for even cycles only. Section 2.4 will cover the beginning of the investigation of the bipartite Ramsey number of two even cycles and we will give our own simplified proof for the bipartite Ramsey number of two 4-cycles. In Section 2.5 we will investigate the bipartite Ramsey number of a 4-cycle with a 6-cycle, while in Section 2.6 we will consider the bipartite Ramsey number of a 4-cycle with an 8-cycle.

The bipartite Ramsey number of two 6-cycles is investigated in Section 2.7. In Section 2.8 the bipartite Ramsey number of any even cycle with a 4-cycle will be investigated and determined. In Section 2.9 the bipartite Ramsey number of an even cycle with a 6-cycle will be determined. In Section 2.10 we will investigate the bipartite Ramsey number of two cycles where $m = n$ and lastly in Section 2.11 a summary about the bipartite Ramsey numbers of cycles will be given with some open questions related to the work done in this dissertation.

2.3 The Existence of the Bipartite Ramsey Number of Two Cycles

To prove the existence of the bipartite Ramsey number of two cycles, reference is made to Proposition 1.1 from Section 1.1.

The proposition translates into bipartite Ramsey numbers as follows.

- $b(C_{2k+1}, C_{2k+1})$ does not exist since any complete bipartite graph under consideration will never contain an odd cycle.
- $b(C_{2k}, C_{2k+1})$ does not exist as we can color all the edges of G blue (or red) and so avoid an odd cycle for any complete bipartite graph.
- $b(C_{2k}, C_{2k})$ exists and this is what we ultimately want to find, the bipartite Ramsey number of two even cycles.

2.4 The Bipartite Ramsey Number of Two 4-Cycles

The investigation begins with the bipartite Ramsey number of two 4-cycles.

Theorem 2.1

$$b(C_4, C_4) = 5 \text{ [8]}.$$

Proof. To prove the lower bound, let $G = K_{4,4}$. We partition G into a red C_8 and a blue C_8 as shown in Figure 2.1. Since we only have a red C_8 and a blue C_8 , we can see that there exists a red-blue coloring of the edges of G that avoids a red C_4 and a blue C_4 . Thus $b(C_4, C_4) \geq 5$.

To prove the upper bound, let $X = \{x_1, x_2, \dots, x_5\}$ and $Y = \{y_1, y_2, \dots, y_5\}$ be the 2-partite vertex sets of $V(G) = V(K_{5,5})$. The aim is to show that every red-blue coloring of the edges of G has a red C_4 or a blue C_4 . In a $K_{5,5}$, at least three vertices of any partite set can each be incident to at least three edges of the same color. Suppose

that three vertices $x_1, x_2, x_3 \in X$ are each incident to at least three red edges. Two of those three vertices, say x_1 and x_2 are adjacent to two mutual vertices, which creates a 4-cycle. ■



Figure 2.1 The red-blue coloring of the edges of $K_{4,4}$.

2.5 The Bipartite Ramsey Number of a 4-Cycle and 6-Cycle

We continue our investigation by considering now the bipartite Ramsey number of a 4-cycle and a 6-cycle.

Theorem 2.2

$$b(C_6, C_4) = 5 \text{ [16].}$$

Proof. To prove the lower bound, we let $G = K_{4,4}$ and we want to show that $b(C_6, C_4) > 4$. In Figure 2.1 we had a red C_8 and a blue C_8 in the red-blue coloring of the edges of $K_{4,4}$. Therefore there exists a red-blue coloring of the edges of G that avoids a red C_6 and a blue C_4 , and so $b(C_6, C_4) > 4$.

To prove the upper bound $b(C_6, C_4) \leq 5$, assume to the contrary that $b(C_6, C_4) > 5$. Hence the edges of $G = K_{5,5}$ are red-blue colorable to avoid a red C_6 and a blue C_4 . We define the vertex set of G as $X \cup Y = \{x_1, x_2, \dots, x_5\} \cup \{y_1, y_2, \dots, y_5\}$ with X and Y as the two partite sets and consider a red cycle of order ten induced by the vertices

$$\{x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5\}.$$

From our assumption we can avoid a red C_6 and a blue C_4 , and so we may assume that x_1 is adjacent to any vertex from the set $\{y_2, y_4\}$, say we color the edge x_1y_4 red. However, we color the edge x_3y_4 blue otherwise it gives the red 6-cycle $y_4, x_3, y_2, x_2, y_1, x_1, y_4$. Similarly, we color x_3y_1 blue otherwise it gives the red 6-cycle $x_3, y_1, x_1, y_4, x_4, y_3, x_3$. Since $C_4 \subseteq G_R$, we color the edge x_2y_3 blue to avoid the red 6-cycle $x_2, y_3, x_4, y_4, x_1, y_1, x_2$. Similarly, we color x_2y_5 blue otherwise the red 6-cycle $x_2, y_5, x_5, y_4, x_1, y_1, x_2$ exists.

Edge x_5y_1 is colored red as we may have a red 4-cycle. However, x_4y_2 is colored blue otherwise it gives the red 6-cycle $x_4, y_4, x_1, y_1, x_2, y_2, x_4$ and edge x_4y_5 is colored blue while x_5 is not adjacent to any vertex from the set $\{y_2, y_3\}$. To motivate this, note that the red 6-cycle $x_5, y_3, x_4, y_4, x_1, y_5, x_5$ is formed if x_5y_3 is colored red. It is observed that since $C_6 \not\subseteq G_R$, the edges x_1y_4 and x_5y_1 are colored red. For all other possibilities, it follows that $C_6 \subseteq G_R$ which is a contradiction to our main assumption that $b(C_6, C_4) > 5$.

We now consider G_B as a two partite graph of order ten where we want to avoid a blue 4-cycle. Since $C_6 \subseteq G_B$, we color the edge x_5y_4 blue. Edge x_2y_1 is not colored blue otherwise it gives the blue 4-cycle y_1, x_2, y_4, x_5, y_1 . Similarly, x_3y_2 is colored red to avoid the blue 4-cycle x_3, y_2, x_5, y_4, x_3 . There is at least one edge between the vertices in the set $\{x_1, x_4\}$ and in the set $\{y_3, y_5\}$, say $x_4y_3 \in E(G_B)$, as a red 6-cycle must be avoided.

If $x_4y_3 \in E(G_B)$, then we have that $x_1y_5 \notin E(G_B)$, otherwise it gives the blue 4-cycle y_5, x_1, y_3, x_4, y_5 . Since $C_4 \not\subseteq G_B$ it is observed that x_1 is not adjacent to any

vertex from the set $\{y_1, y_4\}$ and x_2 is not adjacent to any vertex from the set $\{y_2, y_5\}$. By symmetry x_3 is not adjacent to any vertex from the set $\{y_1, y_3\}$, as it gives the blue 4-cycle x_3, y_3, x_2, y_4, x_3 if x_3y_3 is colored blue. Also by symmetry, we find that x_4 is not adjacent to any vertex from the set $\{y_2, y_4\}$, as it gives the blue 4-cycle x_4, y_4, x_3, y_5, x_4 if x_4y_4 is colored blue.

A red 6-cycle can be avoided if any of the edges x_5y_3 or x_5y_5 is colored blue. A red 6-cycle exists if the edges x_5y_4 and x_4y_3 are colored red, see $x_4, y_1, x_5, y_4, x_2, y_3, x_4$. In every possible case, all the adjacencies between vertices of set X and set Y lead to $C_4 \subseteq G_B$ which is a contradiction to the main assumption. Hence, $b(C_6, C_4) > 5$ does not hold. ■

2.6 The Bipartite Ramsey Number of a 4-Cycle and an 8-Cycle

We continue the investigation of the bipartite Ramsey number of a 4-cycle with an even cycle, and in this case consider the even 8-cycle. We first present a lemma that is needed in the proof of Theorem 2.3 that we are going to consider later on.

Lemma 2.1

Let G be a spanning subgraph of $K_{5,5}$ and $C_8 \not\subseteq G$. Let X and Y be two partite subsets of $V(G)$ where $X \cap Y = \emptyset$. If $C_4 \not\subseteq G^c$, then there exists at most one vertex of X or Y whose degree is at most two [12].

Proof. Consider the neighbourhoods $N(x_i)$ and $N(x_j)$ for $1 \leq i, j \leq 5$. If $i \neq j$ in $|N(x_i) \cup N(x_j)| \leq 3$, then there are at least two vertices of set Y that are not adjacent to x_i or x_j and so we have $C_4 \subseteq G^c$. Therefore the following claim can be made. The reader is referred to [12] for the proof of Claim 1.

Claim 1: $|N(x_i) \cup N(x_j)| \geq 4$. Assume to the contrary that there are at least two ver-

tices of set X whose degrees are at most two, say x_1 and x_2 . Since $|N(x_1) \cup N(x_2)| \geq 4$ we may assume that $N(x_1) = \{y_1, y_2\}$ and $N(x_2) = \{y_3, y_5\}$. We now consider the possible cases for $N(y_5)$.

Case 1: Suppose there is at least one vertex from the set $\{x_3, x_4, x_5\}$ that is not adjacent to y_5 , say $x_3y_5 \notin E(G)$.

By Claim 1, $|N(x_1) \cup N(x_3)| = 4$, therefore x_3 is adjacent to both y_3 and y_4 . Similarly, by Claim 1, $|N(x_2) \cup N(x_3)| \geq 4$, so we have that x_3 is adjacent to y_1 or y_2 , and x_4 is adjacent to at least one vertex from the set $\{y_3, y_4\}$, say $x_4y_3 \in E(G)$ as shown in Figure 2.2. Since $C_8 \not\subseteq G$, x_4 is not adjacent to y_1 or y_2 as this results in the red 8-cycle $x_2, y_3, x_4, y_1, x_1, y_2, x_3, y_4, x_2$ if x_4y_1 is colored red. Therefore, we now have that $N(x_2) = \{y_3, y_4\}$ and $N(x_4) \leq 1$. Hence $|N(x_2) \cup N(x_4)| \leq 3$, which is a contradiction to Claim 1. We conclude that at least one vertex from the set $\{x_3, x_4, x_5\}$ is adjacent to y_5 .

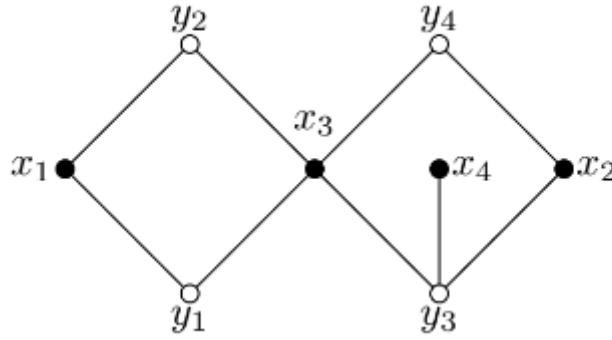


Figure 2.2 The adjacency of x_3 .

Case 2: Suppose that each vertex of $\{x_3, x_4, x_5\}$ is adjacent to y_5 .

We have that $|N(x_1) \cup N(x_3)| \geq 4$ by Claim 1, therefore x_3 is adjacent to at least

one vertex from $\{y_3, y_4\}$, say $x_3y_3 \in E(G)$. Similarly $|N(x_2) \cup N(x_3)| \geq 4$ by Claim 1, and so x_3 is adjacent to at least one vertex from the set $\{y_1, y_2\}$, say $x_3y_1 \in E(G)$.

Suppose there exists at least one edge between the vertices in the set $\{x_4, x_5\}$ and in the set $\{y_2, y_4\}$, say x_4y_2 . Note that x_4 is not adjacent to y_4 , otherwise it gives the 8-cycle $x_4, y_4, x_2, y_3, x_3, y_1, x_1, y_2, x_4$. Since $|N(x_1) \cup N(x_4)| \geq 4$, it follows that x_4 can only be adjacent to y_3 as in Figure 2.3. We observe that x_5 is not adjacent to any vertex from the set $\{y_1, y_2\}$, otherwise it gives the 8-cycle $x_5, y_1, x_1, y_2, x_4, y_3, x_3, y_4, x_5$. We now have that $N(x_2) = \{y_3, y_4\}$ and $N(x_5) \leq 1$, resulting in $|N(x_1) \cup N(x_5)| \leq 3$, which is a contradiction to Claim 1. We conclude that y_5 is not adjacent to any vertex from the set $\{x_3, x_4, x_5\}$.

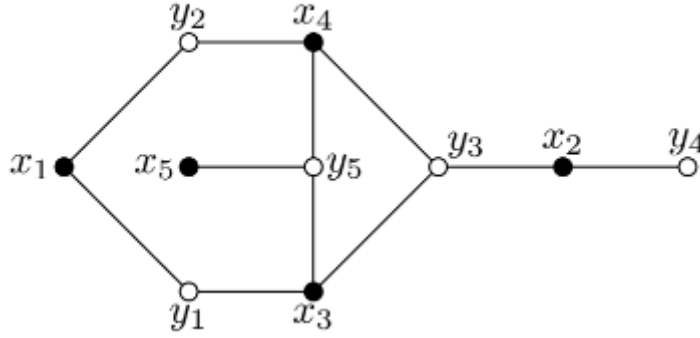


Figure 2.3 The adjacency of y_5 .

By Case 1 and Case 2, Claim 1 does not hold. Therefore we have proven that Lemma 2.1 is true. ■

We are now ready to present the proof of the bipartite Ramsey number of a 4-cycle and an 8-cycle.

Theorem 2.3

$$b(C_8, C_4) \leq 5 \text{ [12].}$$

We refer to the proof of the lower bound of Theorem 2.4 which results in $b(C_8, C_4) \geq 5$ for $m = 4$. We will prove Theorem 2.4 later on. The proof for the upper bound $b(C_8, C_4) \leq 5$ is considered here. Hence the equality $b(C_8, C_4) = 5$ follows from Theorem 2.3 and Theorem 2.4.

Proof. Assume, to the contrary, that $b(C_8, C_4) > 5$. Then, there exists a red-blue coloring of the edges of $K_{5,5}$ that avoids a red 8-cycle and a blue 4-cycle. Since $b(C_6, C_4) \leq 5$, we have that $C_6 \subseteq G$. Consider now a two partite graph G_B of order ten where we want to avoid a blue 4-cycle. The vertex set $V(G) = X \cup Y$ is defined where $X = \{x_1, x_2, \dots, x_5\}$ and $Y = \{y_1, y_2, \dots, y_5\}$.

Note that the longest even cycle in $K_{5,5}$ is of order ten, so assume we color a cycle order ten, induced by the vertices $\{y_1, x_1, y_2, x_2, y_3, x_3, y_4, x_4, y_5, x_5\}$, blue. Since $C_6 \subseteq G$, it follows that x_2 is adjacent to y_4 and x_1 to y_3 . We observe that y_1 is not adjacent to x_4 and y_2 is not adjacent to x_5 , otherwise this produces the 4-cycle x_5, y_2, x_2, y_1, x_5 . Since $C_6 \subseteq G$, y_3 is adjacent to x_1 , or y_5 is adjacent to x_3 , say $x_1y_3 \in E(G_B)$ wlog. Since $C_6 \subseteq G_B$, y_2 is adjacent to x_5 , but this produces the 4-cycle y_2, x_5, y_1, x_2, y_2 , which is a contradiction to our assumption that $b(C_8, C_4) > 5$.

Now consider G_R as a two partite graph of order ten where we want to avoid a red 8-cycle. Note that, since $b(C_6, C_4) \leq 5$, we have $C_6 \subseteq G$. Assume we have an edge set $\{x_1y_1, y_1x_2, x_2y_2, y_2x_3, x_3y_3, y_3x_1\} \in E(G_R)$. Suppose there is at least one edge between the vertices in the set $\{x_4, x_5\}$ and those in the set $\{y_4, y_5\}$, say x_4y_4 .

Assume that x_4 is not adjacent to any vertex from the set $\{y_1, y_2, y_3\}$ and that y_4

is not adjacent to any vertex from the set $\{x_1, x_2, x_3\}$, since said adjacencies may lead to the 8-cycle $y_4, x_1, y_5, x_3, y_2, x_5, y_1, x_4, y_4$ if x_1y_4 is colored red. We have that $d(x_2) \leq 2$ and $d(y_4) \leq 2$. Since $C_4 \subseteq G_R$, x_5 is adjacent to at least two vertices from the set $\{y_1, y_2, y_3\}$, say $x_5y_1, x_5y_2 \in E(G_R)$. Similarly, y_5 is adjacent to at least two vertices from the set $\{x_1, x_2, x_3\}$, say $y_5x_1, y_5x_3 \in E(G_R)$. We have that the vertex set $\{x_2, y_5, x_1, y_3, x_3, y_2, x_5, y_1\}$ induces an 8-cycle if $y_5x_1, y_5x_2 \in E(G_R)$, which is a contradiction to our assumption. Hence, we have $y_5x_1, y_5x_3 \in E(G_R)$ as in Figure 2.4 A where the vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3\}$ induces a cycle of length six with a separate edge x_4y_4 outside the cycle.

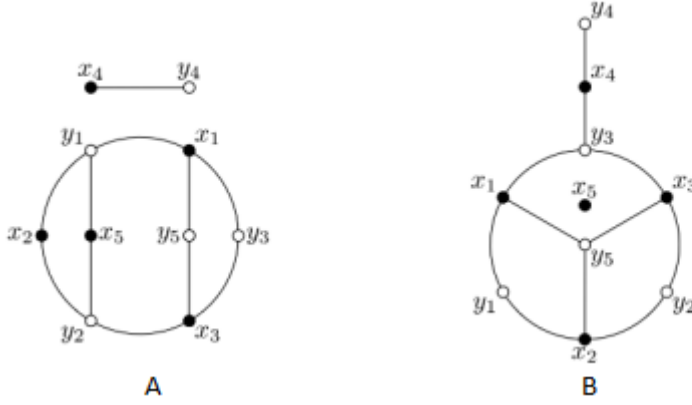


Figure 2.4 A and B.

We now have that x_2 is not adjacent to any vertex from the set $\{y_3, y_5\}$, otherwise this produces the 8-cycle $x_2, y_3, x_3, y_5, x_1, y_1, x_5, y_2, x_2$. So we have $d(x_2) = 2$, as we already had $d(x_4) \leq 2$, which is a contradiction to Lemma 2.1. Hence x_4 is adjacent to at least one vertex from the set $\{y_1, y_2, y_3\}$, contradicting our assumption.

If y_4 is adjacent to any vertex from the set $\{x_1, x_3\}$, then we have the 8-cycle $y_4, x_1, y_5, x_3, y_2, x_5, y_1, x_4, y_4$, which is a contradiction. Therefore, the vertex y_5 is adjacent to any vertex from the set $\{x_1, x_3\}$ since we have $C_4 \subseteq G_R$. Let us now consider x_5 . There are three cases that we observe. Refer to Figure 2.4 B where there

is a cycle of maximum length six with the edge x_4y_4 being incident to the vertex y_3 on the cycle for Case 1.

Case 1: Suppose x_5 is not adjacent to any vertex from the set $\{y_4, y_5\}$.

The vertex y_4 is not adjacent to any vertex from the set $\{x_1, x_3\}$, otherwise it produces the 8-cycle $y_4, x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4$. Hence $d(y_4) \leq 2$. Since Lemma 2.1 says that there is only one vertex of degree at most two, then we have $d(y_5) \geq 3$, and so y_5 has to be adjacent to x_2 as well. Note that y_5 is already adjacent to x_1 and x_3 . We have that x_4 is not adjacent to any vertex from the set $\{y_1, y_2, y_5\}$, otherwise we have the 8-cycle $x_4, y_1, x_1, y_5, x_2, y_2, x_3, y_3, x_4$ if $x_4y_1 \in E(G_R)$. Therefore, $d(x_4) = 2$. By Lemma 2.1 we have $d(x_5) \geq 3$. Hence x_5 is adjacent to any vertex from the set $\{y_1, y_2, y_3\}$. Note that x_5 is already adjacent to y_1 and y_2 . This leaves us with the 8-cycle, $x_5, y_2, x_2, y_5, x_3, y_3, x_1, y_1, x_5$, which is a contradiction to the main assumption. We conclude that x_5 is adjacent to both vertices from the set $\{y_4, y_5\}$.

Case 2: Suppose x_5 is adjacent to just one vertex from the set $\{y_4, y_5\}$.

We now consider two subcases here as we mentioned before that there is at least one edge between a vertex from the set $\{x_4, x_5\}$ and a vertex from the set $\{y_4, y_5\}$. We refer to Figure 2.5 where there are vertices of a cycle of maximum length six adjacent to x_4 and y_5 for Case 2.1.

Case 2.1: Suppose $x_5y_4 \in E(G_R)$.

In Figure 2.5, the vertex x_4 is not adjacent to y_5 , otherwise this results in the 8-cycle $y_5, x_4, y_1, x_2, y_2, x_3, y_3, x_1, y_5$. Since $C_4 \subseteq G_R$, y_1 is adjacent to x_4 . Similarly y_2 is adjacent to x_4 . Since y_5 is adjacent to any vertex from the set $\{x_3, x_1\}$, it is not

adjacent to x_2 , otherwise it results in the 8-cycle $y_5, x_2, y_1, x_4, y_2, x_3, y_3, x_1, y_5$. Hence, we have $d(y_5) = 1$. By Lemma 2.1 we have $d(y_4) \geq 3$. Therefore y_4 is adjacent to at least one of the vertices from $\{x_1, x_2, x_3\}$. However, if $y_4x_1 \in E(G_R)$, then we have the 8-cycle, $x_1, y_4, x_4, y_1, x_2, y_2, x_3, y_3, x_1$, which is a contradiction to the main assumption. We conclude that $x_5y_4 \notin E(G_R)$.

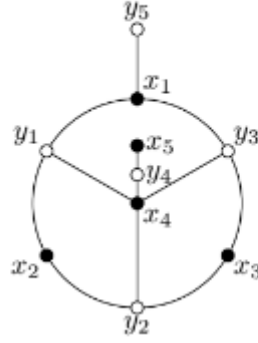


Figure 2.5 The adjacency of x_5 and y_4 .

Case 2.2: Suppose $x_5y_5 \in E(G_R)$.

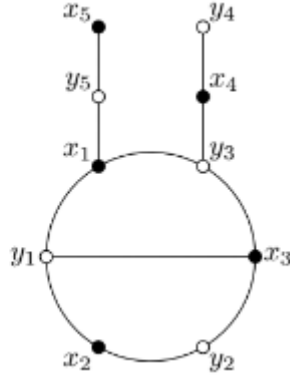


Figure 2.6 The adjacency of x_5 and y_5 .

In Figure 2.6 we have vertices of a cycle of maximum length six incident to the edges $x_3y_1, x_1y_5, y_5x_5, y_3x_4$ and x_4y_4 . The vertex x_5 is not adjacent to y_3 , otherwise it results

in the 8-cycle $x_5, y_3, x_3, y_2, x_2, y_1, x_1, y_5, x_5$. Hence we have $d(x_5) \leq 2$. By Lemma 2.1 it follows that $d(x_3) \geq 3$. The vertex x_3 is not adjacent to y_4 , otherwise the 8-cycle $y_4, x_3, y_2, x_2, y_1, x_1, y_3, x_4, y_4$ is formed. However, x_3 is adjacent to y_1 as we may have a red 4-cycle. By Lemma 2.1 it follows that $d(x_4) \geq 3$. Therefore, x_4 is adjacent to at least one vertex from the set $\{y_1, y_2\}$. The vertex y_5 is now neither adjacent to x_2 nor x_3 , otherwise this produces the 8-cycle $y_5, x_2, y_1, x_3, y_2, x_4, y_3, x_1, y_5$ if x_2y_5 is colored red. We have that $d(y_5) = 2$ but $d(x_5) \leq 2$, which is a contradiction to Lemma 2.1. We conclude that $x_5y_5 \notin E(G_R)$.

Case 3: Suppose x_5 is adjacent to y_4 and y_5 .

We refer to Figure 2.7 where we have two cycles of length six with the mutual edge x_1y_3 . The vertex y_4 is not adjacent to any vertex from the set $\{x_1, x_2, x_3\}$, otherwise, if $y_4x_3 \in E(G_R)$, then we have the 8-cycle $y_4, x_3, y_2, x_2, y_1, x_1, y_5, x_5, y_4$. Hence, $d(y_4) = 2$. By Lemma 2.1 it follows that $d(y_2) \geq 3$. Thus, y_2 is adjacent to at least one vertex from the set $\{x_1, x_4, x_5\}$. We have that $C_8 \subseteq G_R$ in all cases, which is a contradiction to the main assumption. We conclude that x_5 is not adjacent to the vertices y_4 and y_5 .

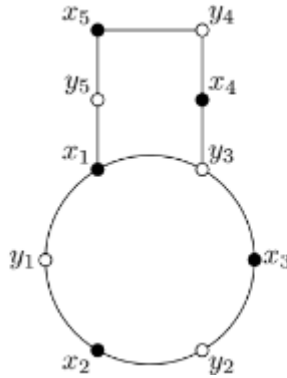


Figure 2.7 x_5 adjacent to y_4 and y_5 .

From Case 1 to Case 3 we have that a red C_8 will exist, which is a contradiction to the main assumption that $b(C_8, C_4) > 5$. Therefore, we have determined the bipartite Ramsey number of a 4-cycle and an 8-cycle, and that is $b(C_4, C_8) = 5$. ■

2.7 The Bipartite Ramsey Number of Two 6-Cycles

We now investigate the bipartite Ramsey number of two 6-cycles. We will make use of the following lemma in the proof of Theorem 2.4.

Lemma 2.2

Let m and n be positive integers. Then $b(C_{2m}, C_{2n}) \geq m + n - 1$.

Proof. We now consider two cases for the values of m and n as follows.

Case 1: $m \neq n$.

Let G_1 and G_2 be the red and the blue subgraphs of $K_{m+n-2, m+n-2}$ respectively. Further let $V(G_1) = X_1 \cup Y$ and $V(G_2) = X_2 \cup Y$, where $Y = \{y_i | 1 \leq i \leq m + n - 2\}$, $X_1 = \{x_i | 1 \leq i \leq m - 1\}$ and $X_2 = \{x_i | m \leq i \leq m + n - 2\}$. Then we see that

$$E(G_1) = \{x_1y_1, x_2y_1, x_2y_2, \dots, x_{m-1}y_{m+n-2}\}$$

and

$$E(G_2) = \{x_my_1, x_{m+1}y_1, x_{m+1}y_2, \dots, x_{m+n-2}y_{m+n-2}\}.$$

Hence, $E(G_1) \cap E(G_2) = \emptyset$ and

$$\begin{aligned} E(G_1) \cup E(G_2) &= \{x_1y_1, x_2y_1, \dots, x_my_1, x_{m+1}y_1, \dots, x_{m+n-2}y_{m+n-2}\} \\ &= E(K_{m+n-2, m+n-2}) \end{aligned}$$

This implies that $C_{2m} \not\subseteq G_1$ and $C_{2n} \not\subseteq G_2$ as the intersection between the two

edge sets is empty. Therefore the edges of $K_{m+n-2, m+n-2}$ are red-blue colorable to avoid a red C_{2m} and a blue C_{2n} . Hence $b(C_{2m}, C_{2n}) \geq m + n - 1$.

Case 2: $m = n$.

Let G_1 and G_2 be the red and blue spanning subgraphs of $K_{m+m-1, m+m-1} = K_{2m-1, 2m-1}$ with $V(G_1) = X_1 \cup Y$ and $V(G_2) = X_2 \cup Y$ where the sets $Y = \{y_i | 1 \leq i \leq 2m - 2\}$, $X_1 = \{x_i | 1 \leq i \leq m - 1\}$, and $X_2 = \{x_i | m \leq i \leq 2m - 2\}$. Further, let

$$E(G_1) = \begin{cases} x_i y_j & \text{if } 1 \leq i, j \leq m - 1, \\ x_i y_j & \text{if } m \leq i, j \leq 2m - 2, \\ x_{2m-1} y_j & \text{if } 1 \leq j \leq 2m - 1, \end{cases} \quad (1)$$

and

$$E(G_2) = \begin{cases} x_i y_j & \text{if } 1 \leq i \leq m - 1, m \leq j \leq 2m - 2, \\ x_i y_j & \text{if } m \leq i \leq 2m - 2, 1 \leq j \leq m - 1, \\ x_i y_{2m-1} & \text{if } 1 \leq i \leq 2m - 2. \end{cases} \quad (2)$$

Similarly to Case 1, we have that $E(G_1) \cap E(G_2) = \emptyset$ and

$$E(G_1) \cup E(G_2) = E(K_{2m-1, 2m-1}).$$

This implies that $C_{2m} \not\subseteq G_1$ and $C_{2m} \not\subseteq G_2$ since the intersection between the edge sets is empty. Therefore, the edges of $K_{2m-1, 2m-1}$ are red-blue colorable to avoid a red G_1 and a blue G_2 . Hence $b(C_{2m}, C_{2n}) \geq 2m$ for $m = n$. ■

Theorem 2.4

$b(C_6, C_6) = 6$ [12].

Proof. The lower bound is obtained using Lemma 2.2, that is $b(C_6, C_6) \geq 6$ for

$m = n = 3$. To prove the upper bound $b(C_6, C_6) \leq 6$, assume, to the contrary, that $b(C_6, C_6) > 6$ and so the edges of $K_{6,6}$ are red-blue colorable to avoid a red C_6 and a blue C_6 . Since $b(C_4, C_6) \geq 5$ (by Theorem 2.2), we have that the edges of $K_{5,5}$ are red-blue colorable in such a way that we cannot avoid a red C_4 or a blue C_6 . Note that an expression for $V(K_{5,5})$ is given by

$$V(K_{5,5}) = V(K_{6,6}) - \{x_6, y_6\}.$$

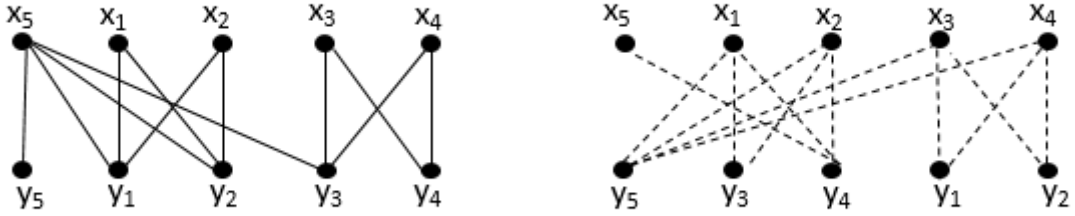


Figure 2.8 Red-blue coloring of the edges of $K_{5,5}$.

We refer to Figure 2.8 where there is a red-blue coloring of the edges of $V(K_{5,5})$ where $V(K_{5,5}) = X \cup Y$ with $X = \{x_1, x_2, \dots, x_5\}$ and $Y = \{y_1, y_2, \dots, y_5\}$. Note that G_1 is the red subgraph while G_2 is the blue subgraph of $K_{5,5}$. By applying Case 1 of Lemma 2.2 we obtain

$$E(G_1) = \begin{cases} x_i y_j & \text{if } 1 \leq i, j \leq 2, \\ x_i y_j & \text{if } 3 \leq i, j \leq 4, \\ x_5 y_j & \text{if } 1 \leq j \leq 5, \end{cases} \quad (3)$$

and

$$E(G_2) = \begin{cases} x_i y_j & \text{if } 1 \leq i \leq 2, 3 \leq j \leq 4, \\ x_i y_j & \text{if } 3 \leq i \leq 4, 1 \leq j \leq 2, \\ x_i y_5 & \text{if } 1 \leq i \leq 4. \end{cases} \quad (4)$$

In $E(G_1)$ we have $\{x_i y_j | 3 \leq i, j \leq 4\}$ and we have $\{x_i y_j | 3 \leq i \leq 4, 1 \leq j \leq 2\}$ in $E(G_2)$ with the two sets containing mutual edges and so we have the intersection $E(G_1) \cap E(G_2) = \{x_i y_j\}$ for all $3 \leq i \leq 4$ and $1 \leq j \leq 2$.

Since $E(G_1) \cap E(G_2) \neq \emptyset$ we have an intersection of edges creating the blue 6-cycle $x_3, y_1, x_1, y_2, x_4, y_3, x_3$, which is a contradiction to the main assumption. We conclude that the edges of $K_{6,6}$ are not red-blue colorable to avoid a red C_6 or a blue C_6 . Therefore $b(C_6, C_6) \leq 6$ and consequently $b(C_6, C_6) = 6$. ■

2.8 The Bipartite Ramsey Number of an Even Cycle with a 4-Cycle

The bipartite Ramsey number of even cycles with a 4-cycle is now considered. To be used in the proof of Theorem 2.5, we state the following lemma and the reader is referred to [16] for the proof.

Lemma 2.3

Let G be a spanning subgraph of $K_{k+1, k+1}$ such that $C_{2k} \subseteq G$ and $x_{k+1}, y_{k+1} \notin V(C_{2k})$. If x_{k+1} and y_{k+1} are adjacent to at least $k - 1$ vertices of $V(C_{2k})$ respectively, then we have $C_{2(k+1)} \subseteq G$.

Theorem 2.5

If $m \geq 4$, then $b(C_{2m}, C_4) = m + 1$ [16].

Proof. We have from Lemma 2.2 with $n = 2$ that $b(C_{2m}, C_4) \geq m + 1$.

We use mathematical induction to prove that $b(C_{2m}, C_4) \leq m + 1$ for $m \geq 4$. Let $P(k)$ be the statement $b(C_{2k}, C_4) \leq k + 1$ where $m = k$ for $k \geq 4$. For $P(4)$ we have that $b(C_8, C_4) \leq 4 + 1 = 5$ which holds by Theorem 2.3. The statement for $P(k + 1)$ will now be proved, in other words, we have to show that $b(C_{2(k+1)}, C_4) \leq k + 2$.

Assume, to the contrary that $b(C_{2(k+1)}, C_4) > k + 2$, which implies that the edges of $K_{k+1, k+1}$ are red-blue colorable to avoid a red cycle of length $2(k + 1)$ and a blue 4-cycle, that is, $C_{2(k+1)} \not\subseteq G_R$ and $C_4 \not\subseteq G_B$. By the inductive hypothesis we have that $b(C_{2k}, C_4) \leq k + 1$ and so, by the definition of a bipartite Ramsey number, there exists a red-blue edge coloring of $K_{k+1, k+1}$ that results in a blue 4-cycle or a $C_{2k} \subseteq G_R$. We define the vertex set $\{x_1, y_1, x_2, y_2, \dots, x_{k+2}, y_{k+2}\}$ where a cycle of length $2k$ is induced by the vertex set $\{x_1, y_1, x_2, y_2, \dots, x_k, y_k\}$ as in Figure 2.9. Now, consider the vertices that do not belong to $V(C_{2k})$, namely x_{k+1} , x_{k+2} , y_{k+1} and y_{k+2} . Since $C_4 \subseteq G_R$ follows from our assumption, we can have at least one edge between any vertex from the set $\{x_{k+1}, x_{k+2}\}$ and any vertex from the set $\{y_{k+1}, y_{k+2}\}$, say $x_{k+1}y_{k+1}$ is the edge. We further assume that there is no edge between x_{k+1} and $V(C_{2k})$.

Since G_R is allowed to have 4-cycles, we have that x_{k+2} is adjacent to at least $k - 1$ vertices from the set $\{y_1, y_2, \dots, y_k\}$ and y_{k+2} to at least $k - 1$ vertices from the set $\{x_1, x_2, \dots, x_k\}$. These vertices belong to $V(C_{2k})$, and so, by Lemma 2.3, it follows that $C_{2(k+1)} \subseteq G_R$, which is a contradiction to the main assumption.

Hence there exists an edge between x_{k+1} and a vertex of $V(C_{2k})$, say $x_{k+1}y_k \in E(G_R)$. We have that y_{k+1} is not adjacent to any vertex from the set $\{x_1, x_k\}$, otherwise this results in the cycle $y_{k+1}, x_1, y_1, \dots, x_{k+1}, y_{k+1}$ of length $2(k + 1)$. In order to avoid a blue 4-cycle, we have that y_{k+2} is adjacent to at least one vertex from the set $\{x_1, x_k\}$,

say $y_{k+2}x_1$ is an edge.

It is observed that x_{k+2} is currently not adjacent to any vertex from the set $\{y_{k+1}, y_{k+2}\}$. Thus, we consider the cases where the vertex x_{k+2} is adjacent or not to the vertices from the set $\{y_{k+1}, y_{k+2}\}$.

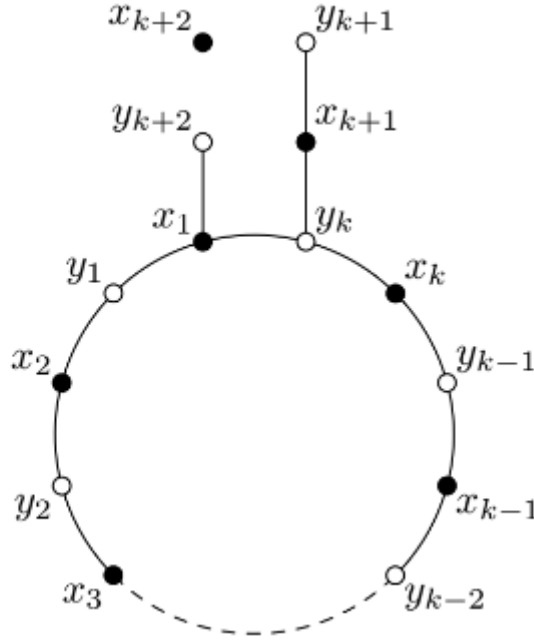


Figure 2.9 Subgraph of G .

Case 1: Suppose there is no edge between x_{k+2} and any vertex from the set $\{y_{k+1}, y_{k+2}\}$.

We then avoid a blue 4-cycle. Therefore, x_k is adjacent to at least one vertex from the set $\{y_{k+1}, y_{k+2}\}$, but then x_k is not adjacent to y_{k+1} since $C_{2(k+1)} \not\subseteq G_R$. We have that y_{k+1} is not adjacent to any vertex from the set $\{x_2, x_{k-1}\}$, otherwise this results in the 4-cycle $x_2, y_{k+1}, x_{k+2}, y_{k+2}, x_2$ in the blue subgraph, which is a contradiction. Hence y_{k+2} is adjacent to both x_2 and x_{k-1} . Since $C_{2(k+1)} \not\subseteq G_R$, the vertex x_{k+1} is not adjacent to any vertex from the set $\{y_1, y_{k-1}, y_{k+2}\}$. See Figure 2.10 for the

vertices that are now adjacent.

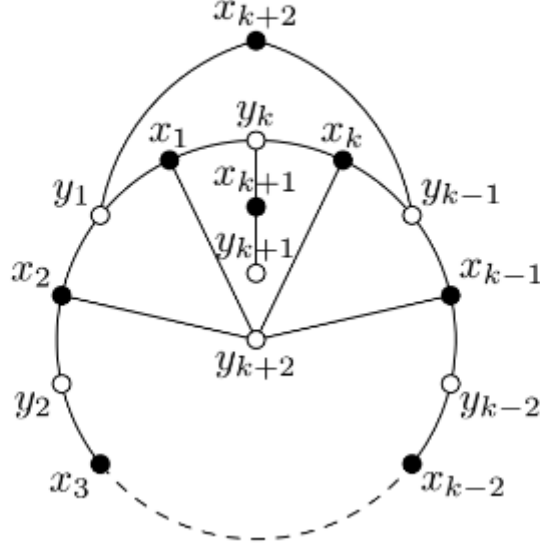


Figure 2.10 Adjacency in the Subgraph of G .

If x_{k+2} is not adjacent to any vertex from the set $\{y_1, y_{k-1}\}$, then the vertices in the set $\{x_{k+1}, x_{k+2}, y_1, y_{k+2}\}$ will induce a 4-cycle in the blue subgraph, which is a contradiction. Therefore x_{k+2} is adjacent to both y_1 and y_{k-1} . Consequently, we have the red cycle $x_1, y_1, x_{k+2}, y_{k-1}, \dots, x_2, y_{k+2}, x_k, y_k, x_1$ of length $2(k+1)$, which is a contradiction to the main assumption.

Case 2: Suppose that there is exactly one edge between x_{k+2} and a vertex from the set $\{y_{k+1}, y_{k+2}\}$. We will now consider two cases.

Case 2.1: Assume $x_{k+2}y_{k+1} \in E(G_R)$.

We define the set $\{x_1, y_1, x_2, \dots, y_{k+1}, x_{k+2}, y_{k+2}\}$ where $V(G) = X \cap Y = \emptyset$ such

that the set $\{x_1, y_1, x_2, y_2, \dots, x_k, y_k\}$ induces a cycle of length $2k$ as in Figure 2.11 A. To avoid a red cycle of length $2(k+1)$, we have that $x_{k+1}y_{k+2}, x_{k+2}y_{k-1} \notin E(G_R)$. If y_{k-1} is not adjacent to x_{k+1} , then the vertices in the set $\{x_{k+1}, x_{k+2}, y_{k-1}, y_{k+2}\}$ induce the blue 4-cycle $x_{k+1}, y_{k-1}, x_{k+2}, y_{k+2}, x_{k+1}$, which is a contradiction. Hence, y_{k-1} is adjacent to x_{k+1} .

Note that x_{k+1} together with $V(C_{2k}) - x_k$ induce a new cycle of length $2k$ which is $x_{k+1}, y_k, x_1, y_1, \dots, x_{k-1}, y_{k-1}, x_{k+1}$. Since $C_{2(k+1)} \not\subseteq G_R$, the vertex y_{k+2} is not adjacent to both vertices from the set $\{x_k, x_{k+2}\}$. However, it remains that the vertices in the set $\{x_k, x_{k+1}, y_{k+1}, y_{k+2}\}$ induce the 4-cycle $x_k, y_{k+2}, x_{k+1}, y_{k+1}, x_k$ in blue the subgraph, which is a contradiction to the main assumption. We conclude that $x_{k+2}y_{k+1} \notin E(G_R)$.

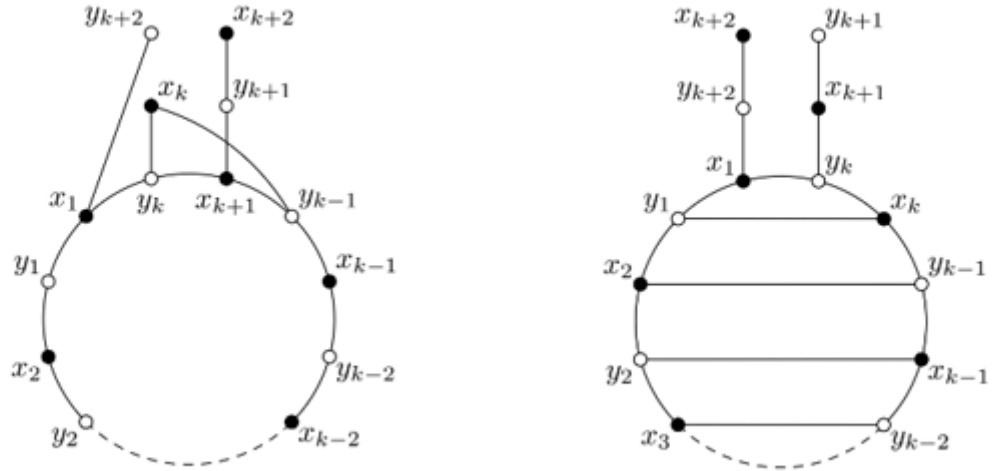


Figure 2.11 A and B.

Case 2.2: Assume $x_{k+2}y_{k+2} \in E(G_R)$.

We continue to consider the vertex set $\{x_1, y_1, x_2, \dots, x_{k+2}, y_{k+2}\}$ where the vertices in $\{x_1, y_1, x_2, y_2, \dots, x_k, y_k\}$ induce a cycle of length $2k$; see Figure 2.11 B. We define the edge set as $E(C_{2k}) = \{x_i y_i | 1 \leq i \leq k\} \cup \{y_i x_{i+1} | 1 \leq i \leq k-1\} \cup \{y_k x_1\}$ as shown in Figure 2.11 B. Since we have $C_{2(k+1)} \not\subseteq G_R$, it follows that $x_k y_{k+1}, x_{k+2} y_1 \notin E(G_R)$. Note that since $x_k y_1 \in E(G_R)$, between x_{k+2} and y_{k-1} there is the path $x_{k+2}, y_{k+2}, x_1, y_k, \dots, x_{k-1}, y_{k-1}$ of length $2(k+1)$. Therefore x_{k+2} and y_{k-1} are not adjacent, otherwise the adjacency leads to the red cycle $x_{k+2}, y_{k-1}, x_2, y_1, x_k, \dots, y_{k+2}, x_{k+2}$ of length $2(k+1)$. Similarly, y_{k+1} is not adjacent to x_2 . Since we can have a red 4-cycle in both instances, the vertex x_2 is adjacent to y_{k-1} . Another path of length $2(k+1)$ is observed between x_{k+2} and y_2 , that is $x_{k+2}, y_{k+2}, x_1, y_1, \dots, x_3, y_2$. Hence x_{k+2} and y_2 are not adjacent otherwise the adjacency leads to the red cycle $x_{k+2}, y_2, x_2, y_1, x_k, \dots, y_{k+2}, x_{k+2}$ of length $2(k+1)$. By symmetry, y_{k+1} and x_{k-1} are not adjacent. Therefore, since we may have a red 4-cycle, the vertex y_2 is adjacent to x_{k-1} . We conclude that $x_{k+2}y_{k+2} \notin E(G_R)$.

We now consider the parity of k .

Case 2.2.1: k is even.

If k is even, then we have the sequence,

$$\{x_3 y_{k-2}, y_3 x_{k-2}, x_4 y_{k-3}, y_4 x_{k-3}, \dots, x_{\frac{k}{2}-1} y_{\frac{k}{2}+1}, y_{\frac{k}{2}-1} x_{\frac{k}{2}+1}\}$$

to see which vertices and edges will be incident.

Case 2.2.2: k is odd.

If k is odd, then $k-1$ is even and $k-1$ is divisible by 2. Note that the indices are

increasing along the cycle in an anti-clockwise direction, see Figure 2.11 B.

We have that

$$x_3y_{k-2}, x_{k-2}y_3, x_4y_{k-3}, \dots, x_{\frac{k-1}{2}}y_{\frac{k+3}{2}}, y_{\frac{k-1}{2}}x_{\frac{k+3}{2}} \in E(G_R).$$

Since we can avoid a red cycle of length $2(k+1)$, the vertex x_{k+2} is not adjacent to any vertex from the set $\{y_1, y_2, \dots, y_k\}$. Also, since we may have a red 4-cycle, the vertex x_{k+1} is adjacent to at least $k-1$ vertices from the set $\{y_1, y_2, \dots, y_k\}$. By symmetry y_{k+2} is adjacent to at least $k-1$ vertices from the set $\{x_1, x_2, \dots, x_k\}$ and by Lemma 2.3 it follows that $C_{2(k+1)} \subseteq G_R$, which is a contradiction to the main assumption. Now we know which vertices and edges will be incident.

Case 3: Suppose that x_{k+2} is adjacent to both y_{k+1} and y_{k+2} .

We consider the vertex set $\{x_1, y_1, x_2, \dots, y_{k+1}, x_{k+2}, y_{k+2}\}$ where a cycle of length $2k$ is induced by the vertex set $\{x_1, y_1, x_2, y_2, \dots, x_k, y_k\}$ and where a 6-cycle is induced by the vertex set $\{x_1, y_{k+2}, x_{k+2}, y_{k+1}, x_{k+1}, y_k\}$; see Figure 2.12. Since $C_{2(k+1)} \not\subseteq G_R$, we have that

$$x_{k-1}y_k, x_{k+2}y_1, x_{k+2}y_k \notin E(G_R).$$

If x_{k-1} is not adjacent to y_1 in the red subgraph, then it leads to the blue 4-cycle $x_{k-1}, y_1, x_{k+2}, y_k, x_{k-1}$, which is a contradiction. Therefore, x_{k-1} is adjacent to y_1 in the red subgraph. By symmetry, the vertex y_2 is adjacent to x_k . However, it remains that the red cycle of length $2(k+1)$ exists, which is induced by the vertex set $\{y_{k+1}, x_{k+1}, y_k, x_k, y_2, x_3, \dots, y_{k-2}, x_{k-1}, \dots, y_{k+2}, x_{k+2}\}$, a contradiction to the main assumption. We conclude that x_{k+2} is not adjacent to the vertices y_{k+1} and y_{k+2} .

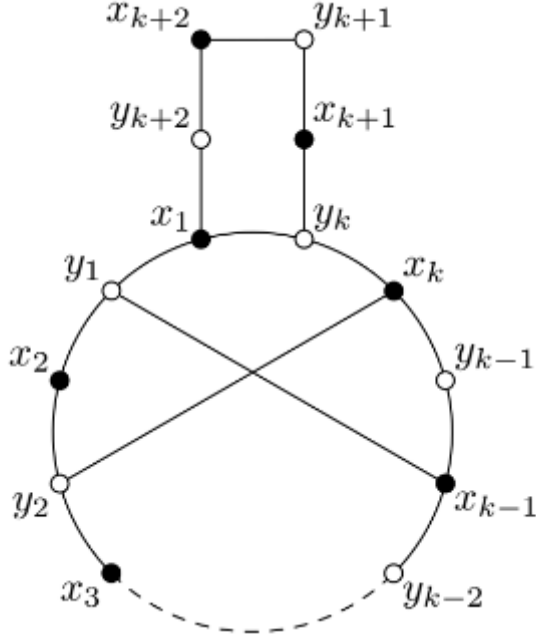


Figure 2.12 x_{k+2} adjacent to y_{k+1} and y_{k+2} .

By Case 1 to Case 3 we see that the assumption $b(C_{2(k+1)}, C_4) > k + 1$ does not hold thus proving the upper bound $b(C_{2k}, C_4) \leq k + 1$. Consequently, we have that $b(C_{2m}, C_4) = m + 1$. ■

2.9 The Bipartite Ramsey Number of an Even Cycle with a 6-Cycle

We now consider the bipartite Ramsey number of an even cycle with a 6-cycle. We introduce the following lemmas that we are going to use in the proof of Theorem 2.6. The reader is referred to [12] for the proof of Lemma 2.4.

Lemma 2.4

Let G be a spanning subgraph of $K_{3,3}$. If $C_6 \not\subseteq G^c$, then $P_3 \subseteq G$.

For the purposes of proving Lemma 2.5 below, we define G to be a spanning subgraph of $K_{k+3,k+3}$ for $k \geq 3$ such that $C_{2(k+1)} \not\subseteq G$ and $C_6 \not\subseteq G^c$ and define our even

cycle as $C_{2k} = \{x_1, y_1, \dots, x_{k-1}, y_{k-1}, x_k, y_k\}$. For two subsets, X and Y of $V(G)$, we define the bipartite graph H_{2k+3} to be the combination of a cycle of length $2k$ and a cycle of length four with a mutual vertex x_i where $i \geq 1$ and the bipartite graph H_{2k+4} to be the combination of a cycle of length $2k$ and a path of length five with a mutual vertex x_i where $i \geq 1$. To prove Lemma 2.5, we introduce the following claims.

Claim 1: $H_{2k+3} \not\subseteq G$.

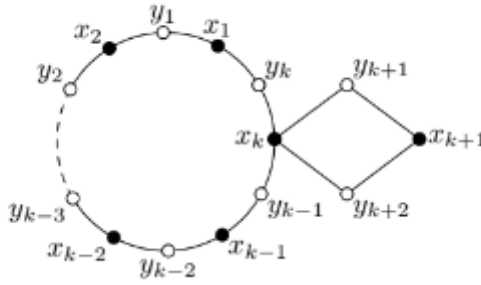


Figure 2.13 The graph H_{2k+3} .

Proof. Assume, to the contrary, that $H_{2k+3} \subseteq G$. Let H_{2k+3} be the combination of a cycle of length $2k$, which is $x_1, y_1, x_2, y_2, \dots, x_k, y_k, x_1$, and a cycle of length four, $x_k, y_{k+1}, x_{k+1}, y_{k+2}, x_k$, with the mutual vertex x_k . Since G is a subgraph of $K_{k+3, k+3}$, we have x_{k+2}, x_{k+3} and y_{k+3} as the vertices not in H_{2k+3} .

Note that we give only one example of a cycle of length $2(k+1)$ because we found that the structure of the contradictory cycles is the same, that is, the cycles are induced by similar sets.

We have assumed that $C_6 \not\subseteq G^c$ and so by Lemma 2.4, we have from the subset of G that $P_3 \subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+3}\} >$. We observe that x_{k+1} is not adjacent to y_{k-1} or y_k , otherwise the vertex set $\{x_{k+1}, y_k, x_1, y_1, \dots, y_{k-1}, x_k, y_{k+2}\}$ may induce a cycle of length $2(k+1)$. Also since $C_{2(k+1)} \not\subseteq G$, we may consider the

following cases. Note that we decided on these cases based on the adjacency in the vertex subset of P_3 .

Case 1: x_{k+2} is adjacent to y_{k-1} and y_{k+3} .

Assume x_{k+2} is adjacent to y_{k-1} and y_{k+3} . Then, we have that each vertex from the set $\{y_{k+1}, y_{k+2}\}$ is not adjacent to any vertex from the set $\{x_1, x_{k-1}, x_{k+2}\}$, otherwise a cycle of length $2(k+1)$ may be induced by the set $\{x_k, y_k, x_1, y_1, \dots, x_{k-1}, y_{k-1}, x_{k+2}, y_{k+2}\}$ if x_{k+2} and y_{k+2} are adjacent. Similarly, y_{k+3} is not adjacent to x_{k-1} , otherwise the vertex set $\{y_{k-1}, x_k, y_k, \dots, x_{k-1}, y_{k+3}, x_{k+2}\}$ may induce a cycle of length $2(k+1)$, a contradiction to our assumption that $C_{2(k+1)} \not\subseteq G$.

Since $C_6 \not\subseteq G^c$, by Lemma 2.4, we have $P_3 \subseteq G < \{x_1, x_{k-1}, x_{k+2}, y_{k+1}, y_{k+2}, y_{k+3}\} >$. Since most of the vertices in the subset of P_3 are not adjacent to each other, we are left with y_{k+3} having to be adjacent to x_1 . We observe that y_k is not adjacent to both vertices from the vertex set $\{x_k, x_{k+2}\}$, otherwise a cycle of length $2(k+1)$ will be induced by the vertex set $\{x_1, y_1, x_2, y_2, \dots, x_k, y_k, x_{k+2}, y_{k+3}\}$. Consequently, we have $P_3 \not\subseteq G < \{x_1, x_{k-1}, x_{k+2}, y_k, y_{k+1}, y_{k+2}\} >$, but by Lemma 2.4, $C_6 \subseteq G^c$ where the cycle of length six is $x_1, y_k, x_{k-1}, y_{k+1}, x_{k+2}, y_{k+2}, x_1$, a contradiction to our assumption that $C_6 \not\subseteq G^c$.

Case 2: y_{k-1} is adjacent to x_{k+2} and x_{k+3} .

Assume y_{k-1} is adjacent to x_{k+2} and x_{k+3} . Then, we have that each vertex from the vertex set $\{y_{k+1}, y_{k+2}\}$ is not adjacent to any vertices from the set $\{x_{k-1}, x_{k+2}, x_{k+3}\}$, otherwise the vertex set $\{x_k, y_k, x_1, y_1, \dots, x_{k-1}, y_{k-1}, x_{k+2}, y_{k+2}\}$ may induce a cycle of length $2(k+1)$ if $x_{k+2}y_{k+2} \in E(G)$. Since $C_6 \not\subseteq G^c$, by Lemma 2.4, we have that $P_3 \subseteq G < \{x_{k-1}, x_{k+2}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\} >$. Hence, from the subset of P_3 we

are left with y_{k+3} having to be adjacent to at least one vertex from the vertex set $\{x_{k+2}, x_{k+3}\}$. We observe that y_{k+3} is not adjacent to x_{k-1} , otherwise the vertex set $\{y_{k-1}, x_k, y_k, \dots, x_{k-1}, y_{k+3}, x_{k+2}\}$ may induce a cycle of length $2(k+1)$. Similarly y_{k+3} is not adjacent to x_k , otherwise the vertex set $\{x_k, y_k, x_1, \dots, y_{k-1}, x_{k+2}, y_{k+3}\}$ may induce a cycle of length $2(k+1)$.

We have that y_k is not adjacent to any vertex from the vertex set $\{x_{k+2}, x_{k+3}\}$, otherwise we may have a cycle of length $2(k+1)$, $y_k, x_1, \dots, y_{k-1}, x_{k+3}, y_{k+3}, x_{k+2}, y_k$. Therefore $P_3 \not\subseteq G < \{x_{k-1}, x_{k+2}, x_{k+3}, y_k, y_{k+1}, y_{k+2}\} >$, but by Lemma 2.4, $C_6 \subseteq G^c$ where the cycle of length six is $x_{k-1}, y_k, x_{k+2}, y_{k+1}, x_{k+3}, y_{k+2}, x_{k-1}$, a contradiction to our assumption that $C_6 \not\subseteq G^c$.

Case 3: x_{k+2} is adjacent to y_{k-1} and y_k .

Assume x_{k+2} is adjacent to y_{k-1} and y_k . Then, each vertex from the vertex set $\{y_{k+1}, y_{k+2}\}$ is not adjacent to any vertex from the set $\{x_1, x_{k-1}, x_{k+2}\}$, otherwise we may have a cycle of length $2(k+1)$, $x_1, y_1, \dots, x_{k-1}, y_{k-1}, x_{k+2}, y_k, x_k, y_{k+1}, x_1$. Since $C_6 \not\subseteq G^c$, by Lemma 2.4, we have $P_3 \subseteq G < \{x_1, x_{k-1}, x_{k+2}, y_{k+1}, y_{k+2}, y_{k+3}\} >$. We are left with y_{k+3} having to be adjacent to at least two vertices from the set $\{x_1, x_{k-1}, x_{k+2}\}$, however y_{k+3} is not adjacent to x_{k+2} , otherwise a cycle of length $2(k+1)$ will be induced by vertex set $\{x_k, y_k, \dots, x_{k-1}, y_{k+3}, x_{k+2}, y_{k-1}\}$. Therefore, y_{k+3} is adjacent to x_1 and x_{k-1} .

Similarly, since $C_6 \not\subseteq G^c$, by Lemma 2.4, we have from the subset P_3 of G that $P_3 \subseteq G < \{x_k, x_{k+1}, x_{k+3}, y_1, y_k, y_{k+3}\} >$. We have that x_k is not adjacent to one vertex from the vertex set $\{y_1, y_{k+3}\}$, otherwise we may have a cycle of length $2(k+1)$, $x_1, y_1, \dots, x_{k-1}, y_{k+3}, x_k, y_{k-1}, x_{k+2}, y_k, x_1$. We also have that x_{k+1} is not adjacent to any vertex from the vertex set $\{y_1, y_k, y_{k+3}\}$, otherwise the vertex set

$\{y_k, x_1, \dots, x_k, y_{k+2}, x_{k+1}\}$ will induce a cycle of length $2(k+1)$.

If x_{k+3} is adjacent to both vertices from the vertex set $\{y_k, y_{k+3}\}$, then the vertex set $\{y_1, x_2, y_2, \dots, x_1, y_{k+3}, x_{k+3}\}$ will induce a cycle of length $2(k+1)$. We are left with x_{k+3} having to be adjacent to y_k . We observe that x_{k+2} is not adjacent to y_1 , otherwise the vertex set $\{y_1, x_2, y_2, \dots, x_{k-1}, y_{k+3}, x_1, y_k, x_k, y_{k-1}, x_{k+2}, y_1\}$ will induce a cycle of length $2(k+1)$. Hence, we have $P_3 \not\subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_1, y_2, y_{k+3}\} >$, but by Lemma 2.4, $C_6 \subseteq G^c$ where the cycle of length six is $x_{k+1}, y_1, x_{k+2}, y_2, x_{k+3}, y_{k+3}, x_{k+1}$, a contradiction to our assumption that $C_6 \not\subseteq G^c$.

Case 4: y_{k+3} is adjacent to x_{k+1} and x_{k+2} .

Assume y_{k+3} is adjacent to x_{k+1} and x_{k+2} . Then, we have that each vertex from the vertex set $\{x_1, x_{k-1}\}$ is not adjacent to any vertex from the set $\{y_{k+1}, y_{k+2}, y_{k+3}\}$, otherwise we may have a cycle of length $2(k+1)$, $x_1, y_1, x_2, y_2, \dots, x_k, y_{k+2}, x_{k+1}, y_{k+1}, x_1$. Since $C_6 \not\subseteq G^c$, by Lemma 2.4, we have $P_3 \subseteq G < \{x_1, x_{k-1}, x_{k+2}, y_{k+1}, y_{k+2}, y_{k+3}\} >$. We are left with x_{k+2} having to be adjacent to at least one vertex from the vertex set $\{y_{k+1}, y_{k+2}\}$, say $x_{k+2}y_{k+1} \in E(G)$. Also, since $C_6 \not\subseteq G^c$, by Lemma 2.4, we have $P_3 \subseteq G < \{x_1, x_{k-1}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\} >$. We are left with x_{k+3} having to be adjacent to at least two vertices from the vertex set $\{y_{k+1}, y_{k+2}, y_{k+3}\}$.

We observe that x_{k+3} is not adjacent to any vertex from the vertex set $\{y_{k-2}, y_{k-1}, y_k\}$, otherwise the vertex set $\{x_k, y_k, x_1, \dots, y_{k-2}, x_{k+3}, y_{k+3}, x_{k+2}, y_{k+1}\}$ will induce a cycle of length $2(k+1)$. We also have that each vertex from the vertex set $\{x_{k+1}, x_{k+2}\}$ is not adjacent to any vertex from the set $\{y_{k-2}, y_{k-1}, y_k\}$, otherwise the vertex set $\{y_k, x_1, y_1, \dots, x_k, y_{k+1}, x_{k+2}\}$ will induce a cycle of length $2(k+1)$. Therefore, we have $P_3 \not\subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-2}, y_{k-1}, y_k\} >$, but by Lemma 2.4, we have $C_6 \subseteq G^c$ where the cycle of length of six is $x_{k+1}, y_{k-2}, x_{k+2}, y_{k-1}, x_{k+3}, y_k, x_{k+1}$, a contradiction

to our assumption that $C_6 \not\subseteq G^c$.

Case 5: y_{k+3} is adjacent to x_{k+2} and x_{k+3} .

Assume y_{k+3} is adjacent to x_{k+2} and x_{k+3} . Then, we have that x_{k+1} is not adjacent to y_{k-1} or y_k , otherwise the vertex set $\{x_k, y_k, x_1, y_1, \dots, x_{k-1}, y_{k-1}, x_{k+1}, y_{k+1}\}$ will induce a cycle of length $2(k+1)$. Since $C_6 \not\subseteq G^c$, by Lemma 2.4, we have $P_3 \subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+1}\} >$. If y_k and y_{k-1} are adjacent to both vertices from the set $\{x_{k+2}, x_{k+3}\}$, then we may have a cycle of length $2(k+1)$, $y_k, x_1, \dots, y_{k-1}, x_{k+2}, y_{k+3}, x_{k+3}, y_k$. Therefore we are left with y_{k+1} having to be adjacent to at least one vertex from the vertex set $\{x_{k+2}, x_{k+3}\}$, say $x_{k+2}y_{k+1} \in E(G)$.

We observe that y_1 is not adjacent to x_{k+2} or x_{k+3} , otherwise a cycle of length $2(k+1)$ will be induced by the vertex set $\{y_1, x_2, y_2, \dots, x_k, y_{k+2}, x_{k+1}, y_{k+1}, x_{k+2}\}$. Hence $P_3 \not\subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_1, y_{k-1}, y_k\} >$ but, by Lemma 2.4, we have $C_6 \subseteq G^c$ where the cycle of length six is $x_{k+1}, y_1, x_{k+2}, y_{k-1}, x_{k+3}, y_k, x_{k+1}$, a contradiction to our assumption that $C_6 \not\subseteq G^c$. By Case 1 to Case 5 we have $H_{2k+1} \not\subseteq G$. ■

Claim 2: $H_{2k+4} \not\subseteq G$.

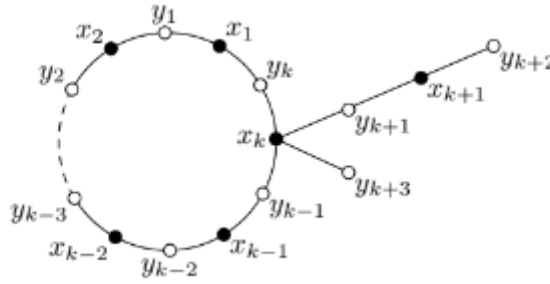


Figure 2.14 The graph H_{2k+4} .

Proof. Assume, to the contrary, that $H_{2k+4} \subseteq G$. Let H_{2k+4} be the combination of a cycle of length $2k$, which is $x_1, y_1, x_2, y_2, \dots, x_k, y_k, x_1$, and a path of length five, $y_{k+3}, x_k, y_{k+1}, x_{k+1}, y_{k+2}$, with a mutual x_k . Since G is a subgraph of $K_{k+3, k+3}$, we have x_{k+2} and x_{k+3} as the vertices not in H_{2k+4} .

Note that we give only one example of a cycle of length $2(k+1)$ because we found that the structure of the contradictory cycles is the same, that is, the cycles are induced by similar sets.

We observe that x_{k+1} is not adjacent to any vertex from the set $\{y_{k-1}, y_k\}$, otherwise this may create a cycle of length $2(k+1)$, $y_k, x_1, y_1, \dots, x_k, y_{k+1}, x_{k+1}, y_k$. We also have that x_{k+1} is not adjacent to y_{k+3} , otherwise we may have the combination of a cycle of length $2k$, $x_1, y_1, x_2, y_2, \dots, x_k, y_k, x_1$, and that of length four, $x_k, y_{k+3}, x_{k+1}, y_{k+1}, x_k$, with a mutual x_k , which will be $H_{2k+3} \subseteq G$, a contradiction to Claim 1.

Since $C_6 \not\subseteq G^c$, by Lemma 2.4, $P_3 \subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+3}\} >$. We have that one vertex from the set $\{x_{k+2}, x_{k+3}\}$ is not adjacent to both vertices from the set $\{y_{k-1}, y_{k+3}\}$, otherwise the vertex set $\{x_k, y_k, \dots, x_{k-1}, y_{k-1}, y_{k+3}\}$ will induce a cycle of length $2(k+1)$. Since $C_{2(k+1)} \not\subseteq G$, we consider the following three cases. Note that we decided on these cases based on the adjacency in the vertex subset of P_3 .

Case 1: x_{k+2} is adjacent to y_{k-1} and y_k .

Assume x_{k+2} is adjacent to y_{k-1} and y_k . Then, each vertex from the set $\{y_{k+1}, y_{k+3}\}$ is not adjacent to any vertex from the set $\{x_1, x_{k-1}, x_{k+2}\}$, otherwise the vertex set $\{x_k, y_k, \dots, x_{k-1}, y_{k-1}, x_{k+2}, y_{k+3}\}$ will induce a cycle of length $2(k+1)$. We also have that y_{k+2} is not adjacent to any vertex from the set $\{x_1, x_{k-1}\}$, otherwise the vertex set $\{x_1, y_1, x_2, \dots, y_{k-1}, x_k, y_{k+1}, x_{k+1}, y_{k+2}\}$ will induce a cycle of length $2(k+1)$. Hence

$P_3 \not\subseteq G < \{x_1, x_{k-1}, x_{k+2}, y_{k+1}, y_{k+2}, y_{k+3}\} >$ but, by Lemma 2.4, we have $C_6 \subseteq G^c$ where the cycle of length six is $x_1, y_{k+1}, x_{k-1}, y_{k+2}, x_{k+2}, y_{k+3}, x_1$, a contradiction to our assumption that $C_6 \not\subseteq G^c$.

Case 2: y_{k-1} is adjacent to x_{k+2} and x_{k+3} .

Assume y_{k-1} is adjacent to x_{k+2} and x_{k+3} . Then, each vertex from the set $\{y_{k+1}, y_{k+3}\}$ is not adjacent to any vertex from the set $\{x_{k+2}, x_{k+3}\}$, otherwise the vertex set $\{x_k, y_k, x_1, \dots, y_{k-1}, x_{k+2}, y_{k+3}\}$ will induce a cycle of length $2(k+1)$. Also, y_k is not adjacent to x_{k+1} , otherwise the vertex set $\{y_k, x_1, \dots, y_{k-1}, x_k, y_{k+1}, x_{k+1}\}$ will induce a cycle of length $2(k+1)$. If y_k is adjacent to one vertex from $\{x_{k+2}, x_{k+3}\}$, then we get a graph similar to that in Case 1 which was proved to be a contradiction above. We have that x_{k+1} is not adjacent to y_{k+3} , otherwise we have the combination of a cycle of length $2k$, $x_1, y_1, x_2, y_2, \dots, x_k, y_k, x_1$ and that of length four, $x_k, y_{k+3}, x_{k+1}, y_{k+1}, x_k$, with a mutual x_k , which will be $H_{2k+3} \subseteq G$, a contradiction to Claim 1. Hence $P_3 \not\subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_k, y_{k+1}, y_{k+3}\} >$ but, by Lemma 2.4, we have $C_6 \subseteq G^c$ where the cycle of length six is $x_{k+1}, y_k, x_{k+2}, y_{k+1}, x_{k+3}, y_{k+3}, x_{k+1}$, a contradiction to our assumption that $C_6 \not\subseteq G^c$.

Case 3: y_{k+3} is adjacent to x_{k+2} and x_{k+3} .

Assume y_{k+3} is adjacent to x_{k+2} and x_{k+3} . Then, we have that each vertex from the set $\{y_{k-1}, y_k\}$ is not adjacent to any vertex from the set $\{x_{k+1}, x_{k+2}, x_{k+3}\}$, otherwise we may have a cycle of length $2(k+1)$, $x_k, y_k, x_1, \dots, y_{k-1}, x_{k+3}, y_{k+3}, x_k$. Since $C_6 \not\subseteq G^c$, by Lemma 2.4, we have $P_3 \subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+1}\} >$. We are left with y_{k+1} having to be adjacent to at least one vertex from the set $\{x_{k+2}, x_{k+3}\}$, but this creates the combination of a cycle of length $2k$, $x_1, y_1, x_2, y_2, \dots, x_k, y_k, x_1$, and that of length four, $x_k, y_{k+3}, x_{k+2}, y_{k+1}, x_k$, with a mutual x_k , which will be $H_{2k+3} \subseteq G$,

a contradiction to Claim 1.

By Case 1 to Case 3, we have $H_{2k+4} \not\subseteq G$. ■

Claim 3: $(C_{2k} \cup C_4) \not\subseteq G$.

Proof. Assume, to the contrary, that $C_{2k} \cup C_4 \subseteq G$. Let $x_k, y_k, x_1, \dots, y_{k-2}, x_{k-1}, x_{k-1}, x_k$ be a cycle of length $2k$ and $x_{k+1}, y_{k+1}, x_{k+2}, y_{k+2}, x_{k+1}$ be a cycle of length four. Since G is a subgraph of $K_{k+3, k+3}$, we have x_{k+3} and y_{k+3} as the vertices not in $C_{2k} \cup C_4$.

Since $C_6 \not\subseteq G^c$, by Lemma 2.4, we have $P_3 \subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+2}\} >$. We have that vertices from the set $\{x_{k+1}, x_{k+2}\}$ are not adjacent to both vertices from the set $\{y_{k-1}, y_k\}$, otherwise we may have a cycle of length $2(k+1)$ which is $y_k, x_1, y_1, \dots, x_{k-1}, y_{k-1}, x_{k+2}, y_{k+1}, x_{k+1}, y_k$. We also have that the vertices from the set $\{y_{k+1}, y_{k+2}\}$ are not adjacent to both vertices from the set $\{x_{k-1}, x_k\}$, otherwise the vertex set $\{x_k, y_k, \dots, x_{k-1}, y_{k+1}, x_{k+1}, y_{k+2}\}$ will induce a cycle of length $2(k+1)$, a contradiction to our assumption that $C_{2(k+1)} \not\subseteq G$.

We have that x_k is not adjacent to both vertices from the set $\{y_{k+1}, y_{k+2}\}$, otherwise we may have the combination of a cycle of length $2k$, $x_k, y_k, x_1, \dots, y_{k-2}, x_{k-1}, x_{k-1}, x_k$, and that of length four, $x_k, y_{k+1}, x_{k+1}, y_{k+2}, x_k$, with a mutual x_k , which will be $H_{2k+3} \subseteq G$, a contradiction to Claim 1. Since $C_6 \not\subseteq G^c$, by Lemma 2.4, we have from the subset P_3 that $P_3 \subseteq G < \{x_{k-1}, x_k, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\} >$. Hence x_k remains to be adjacent to y_{k+1} and y_{k+3} , but this creates the combination of a cycle of length $2k$, $x_k, y_k, x_1, \dots, y_{k-2}, x_{k-1}, x_{k-1}, x_k$, and a path of length five, $y_{k+3}, x_k, y_{k+1}, x_{k+1}, y_{k+2}$, with a mutual x_k , which will be $H_{2k+4} \subseteq G$, a contradiction to Claim 2. Since $C_{2(k+1)} \not\subseteq G$, we consider the following two cases. Note that we decided on these cases based on the adjacency in the vertex subset of P_3 .

Case 1: x_{k+3} is adjacent to y_{k+2} and y_{k+3} .

Assume x_{k+3} is adjacent to y_{k+2} and y_{k+3} . Since $C_{2(k+1)} \not\subseteq G$, y_{k+3} is adjacent to x_{k+1} . We have that each vertex from the set $\{x_{k-1}, x_k\}$ is not adjacent to at least two vertices from the set $\{y_{k+1}, y_{k+2}, y_{k+3}\}$ and that each vertex from the set $\{y_{k-1}, y_k\}$ is not adjacent to at least two vertices from the set $\{x_{k+1}, x_{k+2}, x_{k+3}\}$, otherwise we may have a cycle of length $2(k+1)$, $x_k, y_k, x_1, \dots, y_{k-1}, x_{k+2}, y_{k+1}, x_k$ if $x_k y_{k+1}, y_{k-1} x_{k+2} \in E(G)$.

This means that any vertices from the cycle of length $2k$ are not adjacent to at least two vertices from the set $\{x_{k+1}, x_{k+2}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\}$, otherwise $C_{2(k+1)} \subseteq G$, a contradiction to our assumption that $C_{2(k+1)} \not\subseteq G$. We observe that x_k is not adjacent to two vertices from the set $\{y_{k+1}, y_{k+2}, y_{k+3}\}$, otherwise we may have the combination of a cycle of length $2k$, $x_k, y_k, x_1, \dots, y_{k-2}, x_{k-1}, x_{k-1}, x_k$, and that of length four, $x_k, y_{k+1}, x_{k+1}, y_{k+2}, x_k$, with a mutual x_k , which will be $H_{2k+3} \subseteq G$, a contradiction to Claim 1. Hence $P_3 \not\subseteq G < \{x_1, x_{k-1}, x_k, y_{k+1}, y_{k+2}, y_{k+3}\} >$ but, by Lemma 2.4, we have $C_6 \subseteq G^c$ where the cycle of length six is $x_1, y_{k+1}, x_{k-1}, y_{k+2}, x_k, y_{k+3}, x_1$, a contradiction to our assumption that $C_6 \not\subseteq G^c$.

Case 2: y_{k+1} is adjacent to x_k and x_{k+3} .

Assume y_{k+1} is adjacent to x_k and x_{k+3} . Then, any vertex from set $\{x_{k+1}, x_{k+2}, x_{k+3}\}$ is not adjacent to any vertex from the set $\{y_{k-1}, y_k\}$, otherwise a cycle of length $2(k+1)$ will be induced by the vertex set $\{x_k, y_k, \dots, x_{k-1}, y_{k-1}, x_{k+3}, y_{k+1}\}$. We observe that y_{k+1} is not adjacent to any vertex from the set $\{x_1, x_{k-1}\}$, otherwise we have the combination of a cycle of length $2k$, $y_{k+1}, x_1, y_1, \dots, x_{k-1}, y_{k+1}$, and that of length four, $y_{k+1}, x_{k+1}, y_{k+2}, x_{k+2}, y_{k+1}$, with a mutual y_{k+1} , which will be $H_{2k+3} \subseteq G$,

a contradiction to Claim 1. If y_{k+3} is adjacent to both x_{k+2} and x_{k+3} , then this gives us a graph similar to that in Case 1, which was proved to be a contradiction above. Since $C_6 \not\subseteq G^c$, by Lemma 2.4, we have $P_3 \subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+3}\} >$. Hence x_{k+1} is adjacent to y_{k+3} .

We have that any vertex from the set $\{x_1, x_{k-1}\}$ is not adjacent to any vertex from the set $\{y_{k+2}, y_{k+3}\}$, otherwise we may have a cycle of length $2(k+1)$, for example, $x_1, y_1, \dots, x_k, y_{k+1}, x_{k+1}, y_{k+3}, x_1$ if $x_1 y_{k+3} \in E(G)$. Hence, by Lemma 2.4, we have $P_3 \not\subseteq G < \{x_1, x_2, x_{k-1}, y_{k+1}, y_{k+2}, y_{k+3}\} >$, since $C_6 \subseteq G^c$ where the cycle of length six is $x_1, y_{k+1}, x_2, y_{k+2}, x_{k-1}, y_{k+3}, x_1$, a contradiction to our assumption that $C_6 \not\subseteq G^c$.

By Case 1 to Case 2, we have $C_{2k} \cup C_4 \not\subseteq G$. ■

Claim 4: $(C_{2k} \cup P_5) \not\subseteq G$.

Proof. Assume, to the contrary, that $C_{2k} \cup P_5 \subseteq G$. Let $x_1, y_1, \dots, x_k, y_k, x_1$ be a cycle of length $2k$ and $y_{k+3}, x_{k+1}, y_{k+1}, x_{k+2}, y_{k+2}$ be a path of length five. Since G is a subgraph of $K_{k+3, k+3}$, we have x_{k+3} as the vertex not in $C_{2k} \cup P_5$.

Note that we give only one example of a cycle of length $2(k+1)$ because we found that the structure of the contradictory cycles is the same, that is, the cycles are induced by similar sets.

We observe that at least two vertices from the set $\{x_1, y_1, x_2, y_2, \dots, x_k, y_k\}$ are not adjacent to at least two vertices from the set of vertices that are not contained in the cycle of length $2k$, which is $\{x_{k+1}, y_{k+1}, x_{k+2}, y_{k+2}, y_{k+3}\}$, otherwise we may have a cycle of length $2(k+1)$, $x_k, y_k, x_1, \dots, y_{k-1}, x_{k+1}, y_{k+1}, x_k$ if $x_k y_{k+1}, x_{k+1} y_{k-1} \in E(G)$.

We also have that x_k is not adjacent to any two vertices from the set $\{y_{k+1}, y_{k+2}, y_{k+3}\}$, otherwise we may have the combination of a cycle of length $2k$, $x_1, y_1, \dots, x_k, y_k, x_1$, and that of length four, $x_k, y_{k+1}, x_{k+1}, y_{k+2}, x_k$, with a mutual x_k , which will be $H_{2k+3} \subseteq G$, a contradiction to Claim 1. Also, x_{k+1} is not adjacent to both y_{k-1} and y_k , otherwise we may have the combination of a cycle of length $2k$, $x_{k+1}, y_k, x_1, \dots, y_{k-1}, x_{k+1}$, and a path of length five, $y_{k+3}, x_{k+1}, y_{k+1}, x_{k+2}, y_{k+2}$, with a mutual x_{k+1} , which will be $H_{2k+4} \subseteq G$, a contradiction to Claim 2.

Since $C_6 \not\subseteq G^c$, by Lemma 2.4, we have $P_3 \subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\} >$. Therefore, x_{k+3} is adjacent to at least two vertices from the set $\{y_{k+1}, y_{k+2}, y_{k+3}\}$. Also, since $C_6 \not\subseteq G^c$, by Lemma 2.4, $P_3 \subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+3}\} >$. Hence x_{k+3} is adjacent to y_{k-1} and y_k but this creates the combination of a cycle of length $2k$, $x_{k+3}, y_k, x_1, \dots, y_{k-1}, x_{k+3}$, and that of length four, $x_{k+3}, y_{k+1}, x_{k+2}, y_{k+2}, x_{k+3}$, with a mutual x_{k+3} , which will be $H_{2k+3} \subseteq G$, a contradiction to Claim 1.

Hence $P_3 \not\subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_1, y_{k-1}, y_k\} >$, but by Lemma 2.4, $C_6 \subseteq G^c$ where the cycle of length six is $x_{k+1}, y_1, x_{k+2}, y_{k-1}, x_{k+3}, y_k, x_{k+1}$, a contradiction to our assumption that $C_6 \not\subseteq G^c$. Note that we chose y_1 from the set $\{x_1, y_1, x_2, y_2, \dots, x_k, y_k\}$, we could have chosen any other vertex but the structure of the cycle will still be same. This concludes our proof. ■

We now prove Lemma 2.5 as follows.

Lemma 2.5

Let G be a spanning subgraph of $K_{k+3, k+3}$ for $k \geq 3$. If $C_{2k} \subseteq G$ and $C_6 \not\subseteq G^c$, then $C_{2(k+1)} \subseteq G$ [12].

Proof. Assume, to the contrary, that $C_{2(k+1)} \not\subseteq G$ and suppose that

$$E(C_{2k}) = \{x_1y_1, y_1x_2, x_2y_2, \dots, x_ky_k, y_kx_1\}.$$

Since $C_6 \not\subseteq G^c$, by Lemma 2.4 we have that

$$P_3 \subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\} >, \text{ say } x_{k+1}y_{k+1}, x_{k+1}y_{k+2} \in E(G).$$

Also since $C_6 \not\subseteq G^c$, by Lemma 2.4 we have $P_3 \subseteq G < \{x_k, x_{k+2}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\} >$.

The following contradictions are observed.

Contradiction to Claim 1.

$H_{2k+3} \subseteq G$, if x_k is adjacent to both vertices from $\{y_{k+1}, y_{k+2}\}$.

Contradiction to Claim 2.

$H_{2k+4} \subseteq G$, if x_k is adjacent to both vertices from the set $\{y_{k+1}, y_{k+2}\}$ or to both vertices from the set $\{y_{k+2}, y_{k+3}\}$.

Contradiction to Claim 3.

$(C_{2k} \cup C_4) \subseteq G$, if exactly one vertex from the set $\{x_{k+2}, x_{k+3}\}$ is adjacent to both vertices from the set $\{y_{k+1}, y_{k+2}\}$.

Contradiction to Claim 4.

$(C_{2k} \cup P_5) \subseteq G$, if exactly one vertex from the set $\{x_{k+2}, x_{k+3}\}$ is adjacent to both vertices from the set $\{y_{k+1}, y_{k+3}\}$ or both vertices from the set $\{y_{k+2}, y_{k+3}\}$.

Note that contradictions to the claims will lead to $C_{2(k+1)} \subseteq G$, a contradiction to our main assumption. From the above subset of P_3 , we have the following four cases.

Case 1: Suppose that y_{k+1} is adjacent to x_k and x_{k+2} .

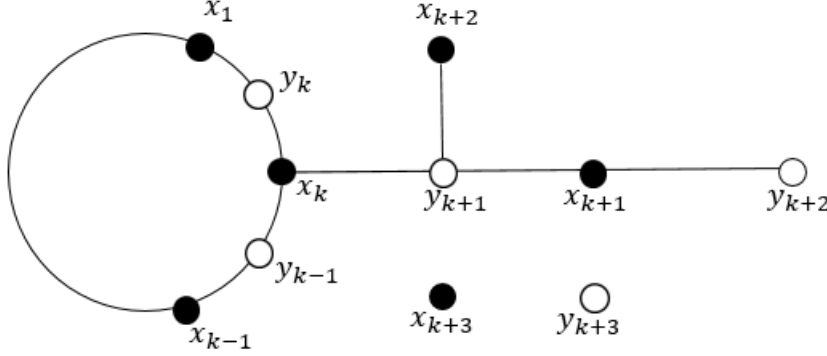


Figure 2.15 y_{k+1} adjacent to x_k and x_{k+2} .

We consider the vertex set $\{x_1, y_1, x_2, \dots, y_{k+2}, x_{k+3}, y_{k+3}\}$ where we have a cycle of length $2(k+1)$ as in Figure 2.15. Since $C_6 \not\subseteq G^c$, by Lemma 2.4, we have from the subset P'_3 of G that $P'_3 \subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+2}\} >$. Each vertex from the set $\{x_{k+1}, x_{k+2}\}$ is not adjacent to any vertex from the set $\{y_{k-1}, y_k\}$, otherwise the vertex set $\{x_{k+2}, y_k, x_1, \dots, y_{k+1}\}$ will induce a cycle of length $2(k+1)$. If the vertices x_{k+2} and y_{k+2} are adjacent to each other, then $(C_{2k} \cup C_4) \subseteq G$, which is a contradiction to Claim 3. If the vertices x_{k+3} and y_{k+2} or the vertices x_{k+2} and y_{k+3} are adjacent to each other, then $(C_{2k} \cup P_5) \subseteq G$, which is a contradiction to Claim 4. Therefore from $P'_3 \subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_{k-1}, y_k, y_{k+2}\} >$, we observe that since we have shown which vertices in P'_3 cannot be adjacent to each other, we only have x_{k+3} remaining to be adjacent to both vertices from the set $\{y_{k-1}, y_k\}$ as illustrated in Figure 2.16 below.



The diagram shows a graph with nodes labeled $x_1, x_{k-1}, x_k, x_{k+1}, x_{k+2}, y_k, y_{k+1}, y_{k+2}, y_{k+3}$. Nodes $x_1, x_{k-1}, x_k, x_{k+1}, x_{k+2}$ are black circles, while $y_k, y_{k+1}, y_{k+2}, y_{k+3}$ are white circles. A cycle is formed by nodes $x_1, y_k, x_k, y_{k-1}, x_{k-1}, y_{k+3}, y_{k+1}, y_k$. A path connects y_{k+1} to x_{k+2} and y_{k+2} . The nodes x_{k+1} and y_{k+1} are connected by an edge, as are x_{k+2} and y_{k+1} , and x_{k+1} and y_{k+2} .

Figure 2.17 The adjacency of x_{k+3} and y_{k+3} .

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we conclude for Case 1 that y_{k+3} is adjacent to both vertices from the set $\{x_1, x_{k-1}\}$, that is, $y_{k+3}x_{k-1}, y_{k+3}x_1 \in E(G)$. See Figure 2.17 for vertices that are adjacent to each other.

Case 2: Suppose that y_{k+1} is adjacent to x_{k+2} and x_{k+3} .

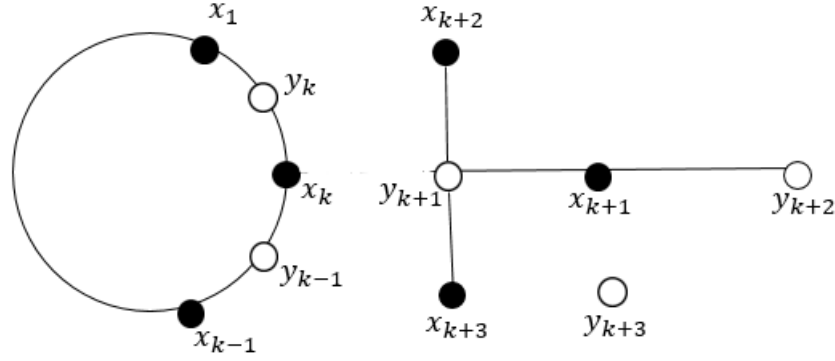


Figure 2.18 y_{k+1} adjacent to x_{k+2} and x_{k+3} .

We consider the vertex set $\{x_1, y_1, x_2, \dots, y_{k+2}, x_{k+3}, y_{k+3}\}$, where $V(G) = X \cup Y$ and the set $\{x_1, y_1, x_2, y_2, \dots, x_k, y_k\}$ induces a cycle of length $2k$ as in Figure 2.18. Since $C_6 \not\subseteq G^c$, by Lemma 2.4 we have $P_3''' \subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_k\} >$. If y_k is adjacent to both vertices from the set $\{x_{k+1}, x_{k+2}\}$, then $H_{2k+3} \subseteq G$ which is a contradiction to Claim 1. Also, if y_k is adjacent to both vertices from the set $\{x_{k+2}, x_{k+3}\}$, then $H_{2k+3} \subseteq G$ which is a contradiction to Claim 1.

If any vertex from the set $\{x_{k+2}, x_{k+3}\}$ is adjacent to y_{k+2} or if both vertices from the set $\{x_{k+2}, x_{k+3}\}$ are adjacent to y_{k+2} , then we have $(C_{2k} \cup C_4) \subseteq G$, which is a contradiction to Claim 3. If any vertex from the set $\{x_{k+2}, x_{k+3}\}$ is adjacent to y_{k+3} , then we have $(C_{2k} \cup P_5) \subseteq G$, which is a contradiction to Claim 4. Therefore, from the vertices in $P_3''' \subseteq G < \{x_{k+1}, x_{k+2}, x_{k+3}, y_k\} >$, we are left with y_{k+3} having to be adjacent to x_{k+1} .

Similarly, since $C_6 \not\subseteq G^c$, we have $P_3^{(4)} \subseteq G < \{x_1, x_k, x_{k+2}, y_{k+1}, y_{k+2}, y_{k+3}\} >$ by Lemma 2.4, say $x_1 y_{k+3}, x_k y_{k+3} \in E(G)$ as illustrated in Figure 2.19 where there is a graph $V(G) = X \cup Y$ that contains a cycle of length $2k$.

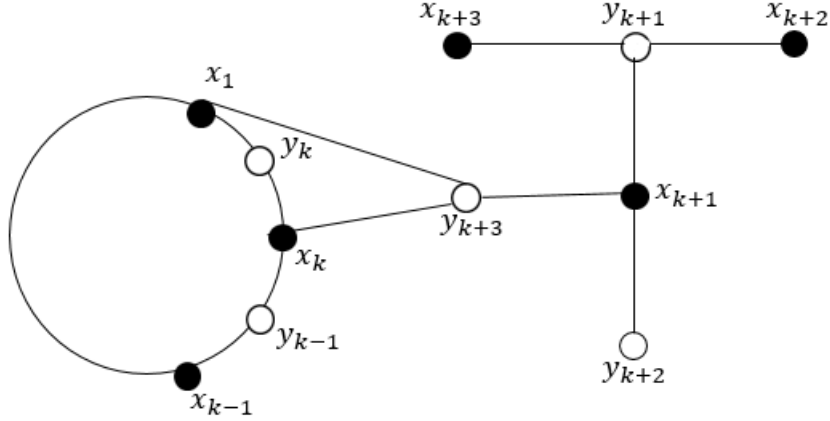


Figure 2.19 The adjacency of y_{k+3} to vertices from set X .

In order to avoid a $C_{2(k+1)}$, y_{k+1} is not adjacent to any vertex from the set $\{x_1, x_k\}$. If any vertex from the set $\{x_1, x_k\}$ is adjacent to both vertices from the set $\{y_{k+2}, y_{k+3}\}$, then we have $H_{2k+3} \subseteq G$ which is a contradiction to Claim 1. If any vertex from the set $\{x_{k+2}, x_{k+3}\}$ is adjacent to y_{k+2} or y_{k+3} , then we have $(C_{2k} \cup C_4) \subseteq G$ which is a contradiction to Claim 3. If each vertex from the set $\{x_1, x_k\}$ is adjacent to each vertex from the set $\{y_{k+2}, y_{k+3}\}$, that is, $x_1 y_{k+2}, x_k y_{k+3} \in E(G)$, then the vertex set $\{x_1, y_{k+3}, x_{k+1}, y_{k+2}, x_k, y_{k-1}, \dots, x_2, y_1\}$ will induce a cycle of length $2(k+1)$, which is a contradiction to the main assumption. We conclude that no two vertices are adjacent in $P_3^{(4)} \subseteq G < \{x_1, x_k, x_{k+2}, y_{k+1}, y_{k+2}, y_{k+3}\} >$. Hence, y_{k+1} is not adjacent to x_{k+2} and x_{k+3} .

Case 3: Suppose that y_{k+3} is adjacent to x_k and x_{k+2} .

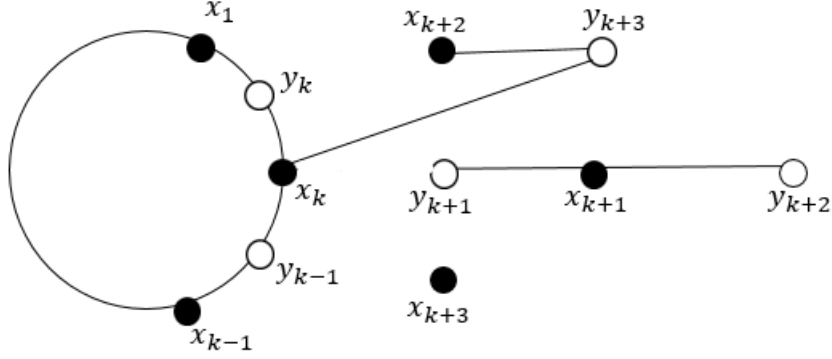


Figure 2.20 y_{k+3} adjacent to x_k and x_{k+2} .

We consider the vertex set $\{x_1, y_1, x_2, \dots, y_{k+2}, x_{k+3}, y_{k+3}\}$ where $\{x_1, y_1, x_2, y_2, \dots, x_k, y_k\}$ induces a cycle of length $2k$ as in Figure 2.20. Since $C_6 \not\subseteq G^c$, by Lemma 2.4 we have $P_3^{(5)} \subseteq G < \{x_k, x_{k+2}, x_{k+3}, y_{k-1}, y_{k+1}, y_{k+2}\} >$. If x_k is adjacent to any vertex from the set $\{y_{k+1}, y_{k+2}\}$, then we have a subgraph isomorphic to H_{2k+4} , that is, $H_{2k+4} \subseteq G$ which is a contradiction to Claim 2. If x_{k+2} is adjacent to any vertex from the set $\{y_{k+1}, y_{k+2}\}$, then we have $(C_{2k} \cup P_5) \subseteq G$ which is a contradiction to Claim 4. If x_{k+3} is adjacent to both vertices from the set $\{y_{k+1}, y_{k+2}\}$, then we have $(C_{2k} \cup C_4) \subseteq G$ which is a contradiction to Claim 3. The vertices y_{k-1} and x_{k+2} are not adjacent, otherwise we may have $y_{k-1}, x_{k+2}, y_{k+3}, x_k, \dots, y_{k-2}, x_{k-1}, y_{k-1}$, a cycle of length $2(k+1)$. Hence from the vertices in $P_3^{(5)} \subseteq G < \{x_k, x_{k+2}, x_{k+3}, y_{k-1}, y_{k+1}, y_{k+2}\} >$, we are left with x_{k+3} having to be adjacent to y_{k-1} , that is, $x_{k+3}y_{k-1} \in E(G)$, as illustrated in Figure 2.21.

By symmetry, if x_{k+3} is not adjacent to y_k , then we have for the path $P_3^{(6)}$ that $P_3^{(6)} \not\subseteq G < \{x_k, x_{k+2}, x_{k+3}, y_k, y_{k+1}, y_{k+2}\} >$ and by Lemma 2.4 we have $C_6 \subseteq G^c$ which is a contradiction. Hence we have an edge $x_{k+3}y_k$ as illustrated in Figure 2.21.

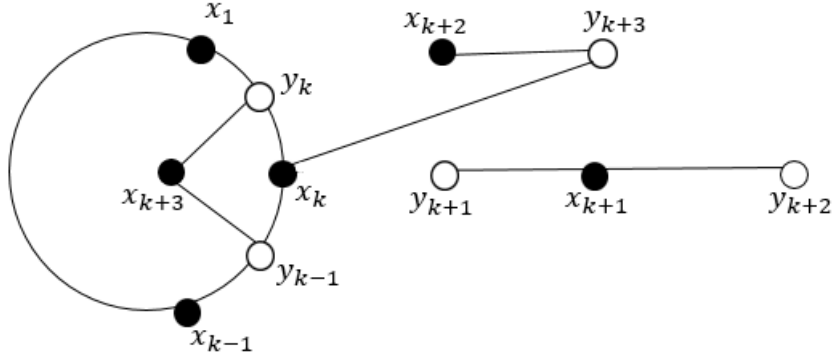


Figure 2.21 x_{k+3} adjacent to y_{k-1} and y_k .

Similarly, since $C_6 \not\subseteq G^c$, we have $P_3^{(7)} \subseteq G < \{x_1, x_{k+2}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\} >$ by Lemma 2.4. If any vertex from the set $\{x_1, x_{k+3}\}$ is adjacent to both vertices from the set $\{y_{k+1}, y_{k+2}\}$, we then have a subgraph isomorphic to H_{2k+3} , that is, $H_{2k+3} \subseteq G$ which is a contradiction to Claim 1. Just as in $P_3^{(6)}$, if x_{k+2} is adjacent to any vertex from the set $\{y_{k+1}, y_{k+2}\}$, we then have $(C_{2k} \cup P_5) \subseteq G$ which is a contradiction to Claim 4. Since $C_{2(k+1)} \not\subseteq G$, the vertex y_{k+3} is not adjacent to any vertex from the set $\{x_1, x_{k+3}\}$ and any vertex from the set $\{y_{k+1}, y_{k+2}\}$ is not adjacent to both vertices from the set $\{x_1, x_{k+3}\}$. We conclude that this results in no two vertices that are adjacent in $P_3^{(7)} \subseteq G < \{x_1, x_{k+2}, x_{k+3}, y_{k+1}, y_{k+2}, y_{k+3}\} >$. Therefore, y_{k+3} is not adjacent to x_k and x_{k+2} .

Case 4: Suppose that y_{k+3} is adjacent to x_{k+2} and x_{k+3} .

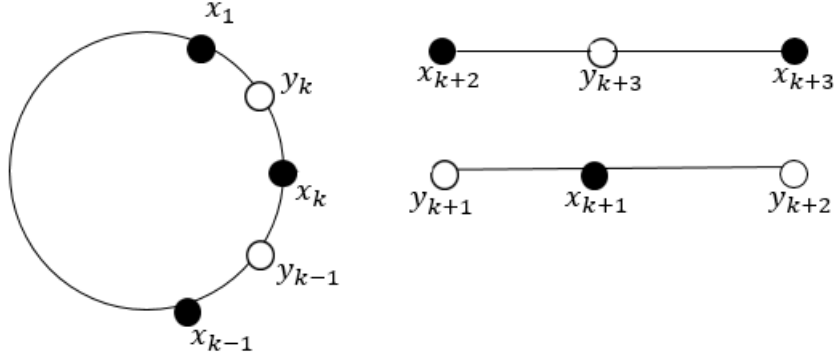


Figure 2.22 y_{k+3} adjacent to x_{k+2} and x_{k+3} .

We consider the vertex set $\{x_1, y_1, x_2, \dots, y_{k+2}, x_{k+3}, y_{k+3}\}$ where $\{x_1, y_1, x_2, y_2, \dots, x_k, y_k\}$ induces a cycle of length $2k$ as in Figure 2.22. Since $C_6 \not\subseteq G^c$, by Lemma 2.4 we have $P_3^{(8)} \subseteq G < \{x_k, x_{k+2}, x_{k+3}, y_k, y_{k+1}, y_{k+2}\} >$. If y_k is adjacent to both vertices from the set $\{x_{k+2}, x_{k+3}\}$ and if x_k is adjacent to both vertices from the set $\{y_{k+1}, y_{k+2}\}$, we then have a subgraph isomorphic to H_{2k+3} , that is, $H_{2k+3} \subseteq G$ which is a contradiction to Claim 1. If any vertex from the set $\{x_{k+2}, x_{k+3}\}$ is adjacent to any vertex from the set $\{y_{k+1}, y_{k+2}\}$, we then have $(C_{2k} \cup P_5) \subseteq G$ which is a contradiction to Claim 4.

Alternatively, if any vertex from the set $\{x_{k+2}, x_{k+3}\}$ is adjacent to both vertices from the set $\{y_{k+1}, y_{k+2}\}$, we then have $(C_{2k} \cup C_4) \subseteq G$ which is a contradiction to Claim 3. We thus conclude, for the path $P_3^{(8)}$, that we have no two vertices adjacent in $P_3^{(8)} \subseteq G < \{x_k, x_{k+2}, x_{k+3}, y_k, y_{k+1}, y_{k+2}\} >$.

In Case 1 to Case 4 we have $C_{2(k+1)} \subseteq G$, which is a contradiction to the main assumption. Thus Lemma 2.5 holds. ■

If we substitute $k = 4$ into Lemma 2.5, then we have the following corollary.

Corollary 2.1

$$b(C_8, C_6) \leq 6 \text{ [12].}$$

Finally, from the last three results we proved, we are now able to prove Theorem 2.6 which is the main result for this section.

Theorem 2.6

If $m \geq 4$, then $b(C_{2m}, C_6) = m + 2$ [12].

Proof. To prove the lower bound, we refer to Case 1 of Theorem 2.4 which we gave as $b(C_{2m}, C_{2n}) \geq m + n - 1$. Letting $n = 3$, we get $b(C_{2m}, C_6) \geq m + 2$.

To prove the upper bound $b(C_{2m}, C_6) \leq m + 2$, we use the principle of mathematical induction and define our inductive statement by $P(k)$: $b(C_{2k}, C_6) \leq k + 2$ for $k \geq 4$. For $P(4)$ we have that $b(C_8, C_6) \leq 6$ which is true by Corollary 2.1. We now show that the statement for $P(k + 1)$ is also true, that is $b(C_{2(k+1)}, C_6) \leq k + 3$ for $k \geq 3$.

Assume, to the contrary, that $b(C_{2(k+1)}, C_6) > k + 3$ for $k \geq 3$. Therefore the edges of $K_{k+3, k+3}$ are red-blue colorable to avoid a red $C_{2(k+1)}$ and a blue C_6 . However, by the inductive statement we see that there exists a red-blue coloring of the edges of $K_{k+3, k+3}$ that cannot avoid a red C_{2k} and blue C_6 , that is, $C_{2k} \subseteq G$ and $C_6 \not\subseteq G^c$. By Lemma 2.5 we have $C_{2(k+1)} \subseteq G$ which is a contradiction to the main assumption.

Since the assumption does not hold, we conclude that the statement $P(k + 1)$: $b(C_{2(k+1)}, C_6) \leq k + 3$ is true for $k \geq 3$. ■

2.10 The Bipartite Ramsey Number of Two m -Cycles

We consider the bipartite Ramsey number of two even cycles of the same size ($m = n$) in general. Theorem 2.7 below requires statistical analysis, hence we only state it. The reader is referred to [17] for the proof thereof. However, Corollary 2.2 [17] that results from Theorem 2.7 is proved here.

Theorem 2.7

For each fixed $0 < \gamma < 1$ and every natural number $\Delta \geq 1$, there exists a constant $\beta = \beta(\gamma, \Delta) > 0$ and a natural number n_0 such that for every bipartite balanced (β, Δ) -graph H on $n \geq n_0$ vertices we have $b(H, H) \leq (1 + \gamma)n$.

Corollary 2.2

$b(C_{2n}, C_{2n}) = (2 + o(1))n$ where $n \geq 2$.

Proof. Let G be a complete bipartite graph with the partite sets X and Y where $X = \{x_1, x_2, \dots, x_{2n-2}\}$ and $Y = \{y_1, y_2, \dots, y_{2n-2}\}$. We want to show that there exists a red-blue coloring of the edges of G that avoids a red C_{2n} and a blue C_{2n} .

We partition G into a red subgraph G_R by coloring the edge $x_i y_j$ red if it satisfies the conditions $x_i \in X$ and $y_j \in Y$ for $i \geq 1$ and $1 \leq j \leq n-1$. Note that to have a red cycle of length $2n$ in a bipartite graph, we need G_R to be balanced, but since $1 \leq j \leq n-1$, it follows that $|X| \neq |Y|$. Therefore we can avoid a red C_{2n} .

Consequently, we also partition G into a blue subgraph G_B by coloring the edge $x_i y_j$ blue that must satisfy the conditions $x_i \in X$ and $y_j \in Y$ for $i \geq 1$ $j \geq n$. Hence the condition for set Y leads to $|X| \neq |Y|$ as it does not include the vertices $\{y_j | 1 \leq j \leq n\}$ which are included in set X and so we can avoid a blue C_{2n} . Thus $b(C_{2n}, C_{2n}) \geq 2n - 1$.

Theorem 2.7 implies the upper bound $b(C_{2n}, C_{2n}) \leq (2 + o(1))n$ for sufficiently large n . Since we have proved the lower bound and the upper bound we may conclude that Corollary 2.2 holds. ■

2.11 In Conclusion

After conducting the investigation on the bipartite Ramsey number of even cycles, we find that the following theorem summarizes all of our findings and results in Chapter 2.

Theorem 2.8

$$b(C_{2m}, C_4) = \begin{cases} 5 & \text{if } m = 2 \text{ or } 3, \\ m + 1 & \text{if } m \geq 4. \end{cases} \quad (5)$$

$$b(C_{2m}, C_6) = \begin{cases} 5 & \text{if } m = 2, \\ 6 & \text{if } m = 3, \\ m + 2 & \text{if } m \geq 4. \end{cases} \quad (6)$$

$$b(C_{2m}, C_8) = \begin{cases} 5 & \text{if } m = 2, \\ 6 & \text{if } m = 3. \end{cases} \quad (7)$$

$$b(C_{2m}, C_{2n}) = \begin{cases} (2 + o(1))n & \text{if } m = n. \end{cases} \quad (8)$$

By looking at the results combined, we see that $b(C_{2m}, C_8)$ for $m \geq 4$ in (7) and $b(C_{2m}, C_{2n})$ for $m \neq n$ in (8) still remain as open questions. We hope to continue with the topic in future studies.

The following conjecture appeared in [12] which we already know is true for $n = 2$ and $n = 3$ by Theorem 2.5 and Theorem 2.6 respectively. Through trial-and-error looking at some specific examples, we believe this conjecture is true.

Conjecture 2.1

$b(C_{2m}, C_{2n}) = m + n - 1$ if $m > n$.

3 The s -Bipartite Ramsey Number of Two Cycles

3.1 Background

The s -bipartite Ramsey number of two even cycles is studied in this chapter. Given two bipartite graphs H_1 and H_2 , the s -bipartite Ramsey Number $b_s(H_1, H_2)$ is the least positive integer t such that any red-blue coloring of edges of the complete graph $K_{s,t}$, where $s \leq t$, has a subgraph H_1 where all of its edges are colored red or a subgraph H_2 where all of its edges are colored blue.

In general terms, if $b(H_1, H_2) = r$ for two bipartite graphs H_1 and H_2 , then there exists a red-blue coloring of the edges of $K_{r,r}$ which results in a red H_1 or a blue H_2 , while on the other hand, there is a red-blue coloring of the edges of $K_{r-1, r-1}$ where a red H_1 and a blue H_2 can be avoided. The latter made room for considering a red-blue coloring of the edges of the complete graph $K_{r-1, r}$ which led to a concept called 2-Ramsey, another type of Ramsey number.

For two bipartite graphs H_1 and H_2 , the 2-Ramsey number, denoted $b_2(H_1, H_2)$, is the least positive integer r such that every red-blue coloring of the edges of the complete bipartite graph $K_{r-1, r}$, results in a red H_1 or blue H_2 .

Previously it was mentioned that Beineke and Schwenk [8] showed that $b(C_4, C_4) = 5$. If we consider the red-blue coloring of the edges of the graph $K_{r-1, r}$, then there is a link between the 2-Ramsey number and bipartite Ramsey number, which is either $b_2(H_1, H_2) = 2b(H_1, H_2)$ or $b_2(H_1, H_2) = 2b(H_1, H_2) - 1$ [15].

For example, we will show that $b_2(C_4, C_4) = 10$ [18]. Since $b(C_4, C_4) = 5$ [8], it

then follows from the definitions of different Ramsey numbers that

$$b_2(C_4, C_4) \leq 2b(C_4, C_4) = 2 \times 5 = 10.$$

Further, $b_2(C_4, C_4) = 9$ [18] which, using a bipartite Ramsey number, is given by

$$b_2(C_4, C_4) = 2b(C_4, C_4) - 1 = 10 - 1 = 9.$$

We let $r - 1 \leq s$ and $r = t$ in the complete graph $K_{r-1,r}$. It is known that the s -bipartite Ramsey number $b_s(C_4, C_4) = 7$ if $s = 3, 4$ [15]. Reference will also be made to the dissertation *Colorings of Graphs and Their Applications* by Johnston [19] where the relationship between s -bipartite Ramsey numbers and bipartite Ramsey numbers is studied. We will study $b_s(C_{2m}, C_{2n})$ in this chapter for certain positive integers $m, n \geq 2$ and s .

3.2 Purpose

In Section 3.3 the non-existence of some s -bipartite Ramsey numbers of two cycles is investigated. In Section 3.4 the s -bipartite Ramsey number of two 4-cycles is determined while in Section 3.5 the s -bipartite Ramsey number of a 4-cycle and 6-cycle is determined. In Section 3.6 the s -bipartite Ramsey number of two 6-cycles is determined for some values of s . Lastly, in Section 3.7, we summarize all of our findings. Note that Section 3.3, Section 3.5 and Section 3.6 contain original work.

3.3 The Non-Existence of some s -Bipartite Ramsey Numbers of Two Cycles

We begin our study of the s -bipartite Ramsey number of two cycles by first considering when this number will not exist. Recall that in Section 2.1 we stated that odd cycles cannot exist in any bipartite graph. Hence the s -bipartite Ramsey number of

any two odd cycles will not exist, that is, $b_s(C_m, C_n) = \infty$ if $m \geq 3$ and $n \geq 3$ are odd numbers for any positive integer s . Note that the s -bipartite Ramsey number of any two even cycles does not exist when $s = 1$, that is, $b_1(C_m, C_n) = \infty$ for $m, n \geq 4$. This is due to the fact that a star $K_{1,t}$ does not have any cycles as subgraphs for any positive integer t .

If $s = 2$, then consider $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \dots, y_t\}$ to be 2-partite sets of vertices of G where all edges from $x_1 y_j$ are colored red and all edges from $x_2 y_j$ are colored blue for $1 \leq j \leq t$ and any positive integer t . In this red-blue coloring of the edges of $K_{2,t}$, a vertex from set X and all the vertices from set Y induce a red/blue star in the two colored subgraphs, which does not have any cycles as subgraphs. Therefore, $b_2(C_m, C_n) = \infty$ for $m, n \geq 4$.

We can use the argument above for $s = 1$ and $s = 2$ to generalize that the s -bipartite Ramsey number will not exist for any two even cycles of length four or more. Results given below show that the s -bipartite Ramsey number of a 6-cycle and other even cycles also does not exist if $s = 3$ or $s = 4$.

Theorem 3.1

If $s = 3$, then $b_s(C_m, C_n) = \infty$ for even numbers $m \geq 4$ and $n \geq 6$.

Proof. We have to prove that there exists a red-blue coloring of the edges of $G = K_{3,t}$, where we will be able to avoid both a red C_m and a blue C_n in G for $m \geq 4$ and $n \geq 6$.

Let $G = K_{3,t}$ be the complete bipartite graph with the partite sets X and Y where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, \dots, y_t\}$ for any positive integer t .

We begin by coloring all the edges of the vertex x_3 red. This gives us a star $K_{1,t}$

and so we can avoid a red C_m for any even number $m \geq 4$. We then color the remaining edges incident to vertices x_1 and x_2 blue which gives a complete bipartite graph $K_{2,t}$. Note that for a cycle with length longer than six to exist, we will need at least three vertices from the set X to be adjacent to at least three vertices from the set Y . Since we have a blue $K_{2,t}$ we are able to avoid a blue C_n for $n \geq 6$. This concludes our proof. ■

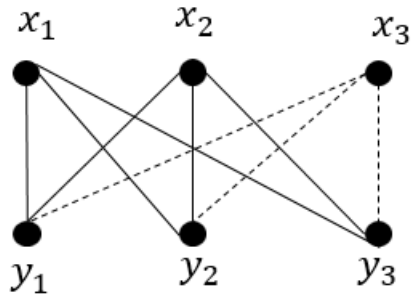


Figure 3.1 Red-blue coloring of the edges of $K_{3,3}$.

An example of Theorem 3.1 is given in Figure 3.1 for $m = 4$ and $t = 3$. We observe from Figure 3.1 a red-blue coloring of the edges of $G = K_{3,3}$ since we can avoid a red C_4 in G as there can never exist a cycle in star $K_{1,3}$ and we can always avoid a blue C_6 in G as there can never exist a C_6 in a complete bipartite graph $K_{2,3}$.

Theorem 3.2

If $s = 4$, then $b_s(C_m, C_n) = \infty$ for even numbers $m, n \geq 6$.

Proof. We have to prove that there exists a red-blue coloring of the edges of $G = K_{4,t}$, where we will be able to avoid both a red C_m and a blue C_n in G for $m, n \geq 6$.

Let $G = K_{4,t}$ be the complete bipartite graph with the partite set X and Y where

$X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, \dots, y_t\}$ for any positive integer t .

We begin by coloring all the edges incident to the vertices x_1 and x_2 red. This gives us a complete bipartite graph $K_{2,t}$. All cycles with length longer than four require at least three vertices from the set X . We are able to avoid a red C_m in the graph $K_{2,t}$ for any even number $m \geq 6$. We then color all the remaining edges incident to the vertices x_3 and x_4 blue which results in a complete bipartite graph $K_{2,t}$. Again, we avoid a blue C_n for all even numbers $n \geq 6$. This concludes our proof. ■

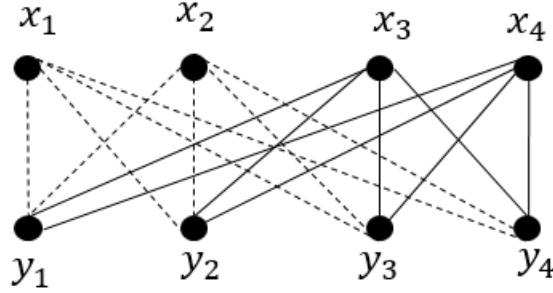


Figure 3.2 Red-blue coloring of the edges of $K_{4,4}$.

An example of Theorem 3.2 is given in Figure 3.2 where $m = 6$ and $n = 8$ for $t = 4$. We observe from Figure 3.2 a red-blue coloring of the edges of $G = K_{4,4}$ since we can avoid a red C_6 and a blue C_8 as there can never exist a cycle with length longer than four in $K_{2,4}$.

Now that we have discussed the cases where the s -bipartite Ramsey number of some two cycles does not exist, the next step is to investigate the actual value of the s -bipartite Ramsey number for some cycles where it exists.

3.4 The s -Bipartite Ramsey Number of Two 4-Cycles

In this section, we investigate the s -bipartite Ramsey number of two 4-cycles for $s \geq 3$. Note that when proving the upper bound t in $b_s(C_m, C_n) \leq t$, we consider the complete bipartite graph $G = K_{s,t}$ and redraw it such that any vertex that does not lie on the cycle is not enclosed by the vertices of the cycle. We define the *interior* chords of a cycle to be the chords between the vertices on the inside of a cycle and *exterior* edges to be the ones between a vertex on the cycle and a vertex on the outside of the cycle, or between two vertices on the outside of the cycle.

Theorem 3.3

$$b_3(C_4, C_4) = 7 \text{ [15].}$$

Proof. First the lower bound $b_3(C_4, C_4) \geq 7$ is proved by showing that there exists a red-blue coloring of the edges of $G = K_{3,6}$ that avoids both a red C_4 and a blue C_4 .

In this regard let X and Y be the partite sets of G where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, \dots, y_6\}$. Suppose that we color three edges incident to every vertex of set X red and the remaining edges incident to every vertex of set X blue as in Figure 3.3. Then there is neither a blue C_4 nor a red C_4 . Thus, we found a red-blue coloring of the edges of $K_{3,6}$ where a red C_4 and a blue C_4 will not appear. Therefore, $b_3(C_4, C_4) > 6$.

We now give a proof for the upper bound $b_3(C_4, C_4) \leq 7$. Assume, to the contrary, that $b_3(C_4, C_4) > 7$, which means there exists a red-blue coloring of the edges of $G = K_{3,7}$ that avoids both a red C_4 and a blue C_4 . G has 21 edges which are partitioned into a red subgraph G_R and a blue subgraph G_B .

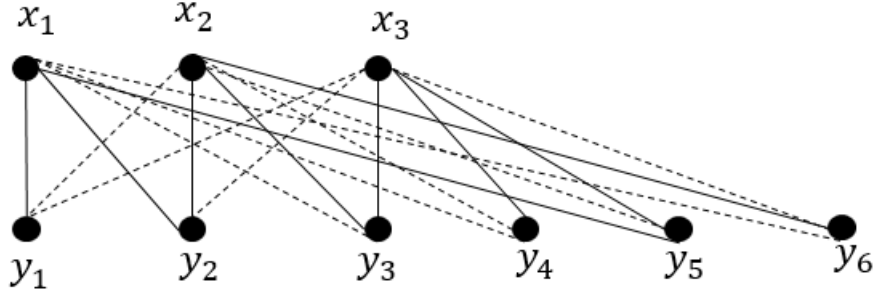


Figure 3.3 Red-blue coloring of the edges of $K_{3,6}$.

Assume, without loss of generality, that the size of G_R is at least 11, i.e., $|E(G_R)| \geq 11$. We know that G_R is not a tree nor a forest otherwise it would be of maximum size $m \leq 9$. Therefore, we may assume that G_R is not a tree and let C_k be the longest cycle in G_R for an even number $k > 4$. Note that we cannot have $k = 4$ from our assumption that G contains no red C_4 's. Since $|X| = 3$, we see that the longest possible cycle in G_R can only have length six. Consider a vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3\}$ that induces a C_6 .

In the exterior of the 6-cycle, we have four vertices $\{y_j | 4 \leq j \leq 7\}$ from the set Y . Note that for a C_4 to exist, we need at least two vertices from the set X to be both adjacent to the same two vertices from the set Y . Therefore, we can have at most four edges from the edge set $\{x_i y_j | 1 \leq i \leq 3, 4 \leq j \leq 7\}$, otherwise, if we have five or more edges, then an example is that y_4 and y_2 will both be adjacent to x_2 and x_3 which leads to a red C_4 .

In the interior of the 6-cycle we have three remaining chords: $x_1 y_2$, $x_2 y_3$ and $x_3 y_1$. If we color any of these edges red, then it will lead to a red C_4 . Hence we cannot add any more edges. This leaves us with a total number of six edges from the 6-cycle plus the four edges in the exterior of the 6-cycle, making a total of 10 edges colored red,

a contradiction to our assumption that $|E(G_R)| \geq 11$. This concludes our proof. ■

Theorem 3.4

$$b_4(C_4, C_4) = 7 \text{ [15].}$$

Proof. To prove the lower bound $b_4(C_4, C_4) \geq 7$, we let $G = K_{3,7}$ be the graph with the partite sets X and Y where $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, \dots, y_6\}$. Suppose that we color three edges incident to every vertex of set X red and the remaining three edges incident to every vertex of set X blue as in Figure 3.4. Note that there is neither a red C_4 nor a blue C_4 in Figure 3.4. We can find a red-blue coloring of the edges of $K_{4,6}$ where neither a red C_4 nor a blue C_4 will appear. Therefore, $b_4(C_4, C_4) \geq 7$.

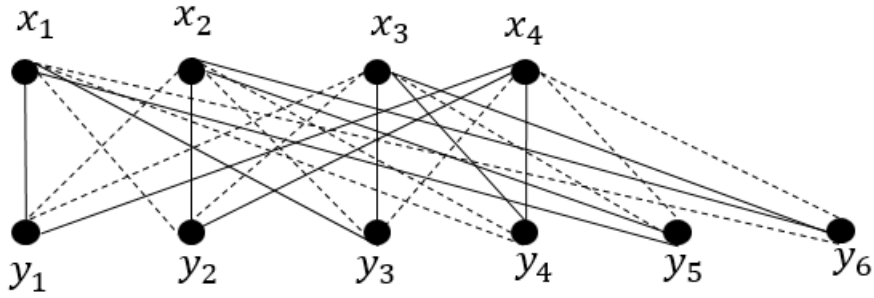


Figure 3.4 Red-blue coloring of the edges of $K_{4,6}$.

We now give a proof for the upper bound $b_4(C_4, C_4) \leq 7$. Assume, to the contrary, that $b_4(C_4, C_4) > 7$, which means there exists a red-blue coloring of the edges of $G = K_{4,7}$ that avoids both a red C_4 and a blue C_4 . G has 28 edges which are partitioned into a red subgraph G_R and a blue subgraph G_B .

Assume, without loss of generality, that the size of G_R is at least 14, i.e., $|E(G_R)| \geq 14$. We know that G_R is not a tree nor a forest otherwise it would be of maximum size

$m \leq 10$. Therefore, we may assume that G_R is not a tree and let C_k be the longest cycle in G_R for an even number $k > 4$. Note that we cannot have $k = 4$ from our assumption that G contains no red C_4 's. Since $|X| = 4$, we have the following cases.

Case 1: The longest cycle in G_R has length six.

Consider a vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3\}$ which induces a C_6 in G_R . It can be drawn so that the vertex $x_4 \in X$ and four vertices $\{y_j | 4 \leq j \leq 7\}$ from the set Y are on the exterior of this C_6 , meaning that the cycle does not enclose these vertices. We observe that we can color at most four edges incident to vertex x_4 red without producing any red even cycle, therefore we only consider the following subcases.

Subcase 1: x_4 is adjacent in G_R to all four vertices $\{y_j | 4 \leq j \leq 7\}$.

We can have only one edge from the edge set $\{x_4 y_j | 1 \leq j \leq 3\}$, otherwise if we have two or more edges, then x_4 and x_2 will both be adjacent to y_1 and y_2 , which leads to a red C_4 .

Subcase 2: x_4 is not adjacent in G_R to any vertex $\{y_j | 4 \leq j \leq 7\}$.

In the exterior of the 6-cycle we may have at most four edges from the edge set $\{x_i y_j | 1 \leq i \leq 3, 4 \leq j \leq 7\}$ otherwise, if we have five or more edges, then y_4 and y_1 will both be adjacent to x_1 and x_2 , which leads to a red C_4 . We may have one edge from $\{x_4 y_j | 1 \leq j \leq 3\}$, otherwise, if we have two or more edges, then it may lead to a red C_4 as at least two vertices from the set X will be both adjacent to the same two vertices from the set Y . For example, if x_4 is adjacent to y_1 and y_2 , then the vertex set $\{x_4, y_1, x_2, y_2\}$ will induce a red C_4 .

Note that we only aim here to determine if we have at least 14 edges colored red in total. Therefore, we can exclude the subcases for vertex x_4 being adjacent to one, two or three vertices of Y in Case 1. Note that these possibilities are covered by Subcase 1 by vertex x_4 being adjacent to all four vertices from the set $\{y_j | 4 \leq j \leq 7\}$.

In the interior of the 6-cycle we have three chords x_1y_2, x_2y_3 and x_3y_1 such that if we color anyone of them red, then this will lead to a C_4 . Hence, we do not have any chords in the interior of a 6-cycle. In any subcase above we have edges from the 6-cycle, four edges added from the exterior of the 6-cycle plus one more edge making a total of 11 edges colored red in G_R , a contradiction to our main assumption that $|E(G_R)| \geq 14$.

Case 2: The longest cycle in G_R has length eight.

Consider a vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4\}$ which induces a C_8 in G_R . In the exterior of the 8-cycle, we have three vertices $y_j \in Y$ for $5 \leq j \leq 7$, so we can have in G_R at most three edges from the edge set $\{x_iy_j | 1 \leq i \leq 4, 5 \leq j \leq 7\}$, otherwise, if we have four or more edges, then two vertices from the set Y will be adjacent to the same two vertices from the set X . For example, y_5 and y_1 will both be adjacent to x_1 and x_2 which leads to a red C_4 .

In the interior of the 8-cycle we have the eight remaining chords: $x_1y_2, x_1y_3, x_2y_4, x_3y_1, x_3y_4, x_4y_1$ and x_4y_2 . If we color any of these edges red, then we may have a red C_4 . This is mainly because if we have a 6-cycle existing in the interior of an 8-cycle, then there will exist a red C_4 as one of the subgraphs of C_8 . Therefore, we cannot color any more edges red, leaving us with a total of eight edges colored red from the cycle plus the three edges in the exterior of the 8-cycle making it 11 edges colored red, a contradiction to our assumption that $|E(G_R)| \geq 14$. This concludes our proof. ■

We now prove the general case for $s \geq 5$.

Theorem 3.5

$b_s(C_4, C_4) = 5$ for $s \geq 5$.

Proof. We consider Theorem 3.5 for when $s = 5$ and when $s > 5$.

Case A: $s = 5$.

If $s = 5$, then we have that $b_5(C_4, C_4) = 5$ [15].

First, the lower bound $b_5(C_4, C_4) \geq 5$ has to be proved which means we need to show that there exists a red-blue coloring of the edges of $K_{4,5}$ that avoids both a red C_4 and a blue C_4 . Figure 3.5 below illustrates a red-blue coloring of the edges of $G = K_{4,5}$ where the longest cycle in the blue subgraph of G is a C_8 and where the longest cycle in the red subgraph of G is also a C_8 , while the shortest cycles are a red C_6 and a blue C_6 respectively. We observe that there is no C_4 in any red or blue subgraph of G . Thus we have proved the lower bound of $G = K_{4,5}$ and so $b_4(C_4, C_4) \geq 5$.

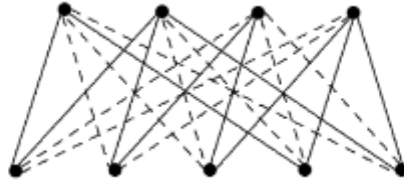


Figure 3.5 The red-blue coloring of the edges of $K_{4,5}$.

We now give the proof for the upper bound $b_5(C_4, C_4) \leq 5$. Assume, to the contrary, that $b_5(C_4, C_4) > 5$, which means there exists a red-blue coloring of the edges of

$G = K_{5,5}$ that avoids both a red C_4 and a blue C_4 . Note that G has 25 edges which can be partitioned into a red subgraph G_R and a blue subgraph G_B .

Assume, without loss of generality, that the size of G_R is at least 13, i.e., $|E(G_R)| \geq 13$. We know that G_R is not a tree nor a forest as it would then be of maximum size $m = 9$. Therefore, we may assume that G_R is not a tree and let C_k be the longest cycle in G_R for an even number $k > 4$. Note that we cannot have $k = 4$ from our assumption that G does not contain any red C_4 's. Since $|X| = 5$, we have the following cases.

Case 1: The longest cycle in G_R has length six.

Consider a vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3\}$ which induces a C_6 . We observe that any vertex from the set $\{x_4, x_5\}$ can be adjacent to at most two vertices from the set $\{y_4, y_5\}$ without producing a red even cycle. We also observe that the vertices x_4 and x_5 cannot be adjacent to both the vertices y_4 and y_5 as otherwise the adjacency may lead to a red C_4 . Therefore, we only consider the following subcases.

Subcase 1: One vertex from the set $\{x_4, x_5\}$, say x_4 , is adjacent in G_R to y_4 and y_5 .

In the exterior of the 6-cycle we observe that we may only have one edge in G_R from the edge set $\{x_i y_j | 1 \leq i \leq 3, 4 \leq j \leq 5\}$, otherwise, if we have two or more edges from the set, then at least two vertices from the set X may be adjacent to the same two vertices from the set Y , resulting in a red C_4 . We may have only one edge from $\{x_5 y_j | 4 \leq j \leq 5\}$ in G_R , otherwise the vertex set $\{x_5, y_4, x_4, y_5\}$ will induce a red C_4 . We can also have the two edges $x_4 y_3$ and $x_2 y_4$ in order to avoid a red C_4 .

In the interior of a 6-cycle we cannot color any chords red as this and the inci-

dency of the edges with vertices from the set $\{x_1, y_1, x_3, y_3\}$ may lead to a red C_4 . Therefore we have a total of 12 edges colored red in at most, a contradiction to our main assumption that is $|E(G_R)| \geq 13$.

Subcase 2: In G_R , the vertices from the set $\{x_4, x_5\}$ are not adjacent to any vertex from the set $\{y_4, y_5\}$.

In the exterior of the 6-cycle we observe that we can have two edges from the edge set $\{x_i y_j | 1 \leq i \leq 3, 4 \leq j \leq 5\}$ and two edges from the set $\{x_i y_j | 4 \leq i \leq 5, 1 \leq j \leq 3\}$. If we have more than two edges from any of the two edge sets colored red, then we will have two vertices from the set X being adjacent to the same two vertices from the set Y or *vice versa*, which leads to a red C_4 .

In the interior of a 6-cycle we cannot color any chord red as the vertex set $\{x_2, y_1, x_3, y_2\}$ will lead to a red C_4 . Thus we have a total of at most 10 edges in G_R , a contradiction to our main assumption which is $|E(G_R)| \geq 13$.

Case 2: The longest cycle in G_R has length eight.

Consider a vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4\}$ which induces a C_8 in G_R . In the exterior of the 8-cycle we only have the two vertices x_5 and y_5 . We will consider the following two subcases.

Note that if we color any chord in the interior of the 8-cycle red, then we will have a red C_6 but this also means that a red C_4 will exist as one of the subgraphs of C_8 , for example x_1, y_1, x_4, y_4, x_1 . Therefore we cannot color any chord inside the 8-cycle red.

Subcase 1: x_5 and y_5 are adjacent in G_R .

We may have at most two edges adjacent to the edge x_5y_5 if and only if the edge is not of the form $\{x_iy_j|i = j\}$, otherwise this may create a C_{10} which contradicts our assumption that the longest cycle has length eight. By Subcase 1 we have a total of at most 11 edges in G_R , a contradiction to our assumption that $|E(G_R)| \geq 13$.

Subcase 2: x_5 and y_5 are not adjacent in G_R .

We may have at most two edges from the set $\{x_5y_j|1 \leq j \leq 5\}$, otherwise if we have three or more edges colored red, then x_5 and x_2 will both be adjacent to y_1 and y_2 , which leads to a red C_4 . We may also have at most two edges from the set $\{x_iy_5|1 \leq i \leq 5\}$ otherwise the vertex set $\{x_3, y_3, x_4, y_5\}$ will induce a red C_4 .

By Subcase 2 we have a total of at most 12 edges in G_R , contradicting our main assumption which is $|E(G_R)| \geq 13$.

Case 3: The longest cycle in G_R has length ten.

Consider a vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5\}$ which induces a C_{10} in G_R .

In a C_{10} , we will have a C_6 by coloring any chord red which satisfies the condition $\{x_iy_j|i \geq 1, j = i + 2\}$. We observe that we may color only two chords red, otherwise, if we color more than two chords red, then the vertex set $\{x_3, y_5, x_1, y_3\}$ will induce a red C_4 . However, we cannot have a C_8 as coloring the chord inside of a C_{10} red to form a C_8 will also produce a red C_4 , for example x_1, y_1, x_5, y_5, x_1 . Hence, in G_R we have a total of at most 10 edges in G_R , a contradiction to our assumption that $|E(G_R)| \geq 13$.

Case B: $s > 5$.

Since $K_{5,5} \subset K_{5,6} \subset \dots \subset K_{5,s}$ for $s \geq 5$, it can be concluded that by symmetry, the results will be the same for $s \geq 5$. That is, since we cannot avoid a red C_4 or a blue C_4 in a $K_{5,5}$, we will also not be able to avoid a red C_4 or a blue C_4 in a $K_{5,s}$ as $K_{5,5}$ is a subgraph of $K_{5,s}$. Therefore $b_s(C_4, C_4) = 5$ for all $s \geq 5$. ■

In their published work, Bi, Chartrand and Zhang [15] stated that $b_s(C_4, C_4) = s$ for all $s \geq 5$ without proof. Their previous article [14] which contains the proof is not published yet. We question their result as the following observation shows why.

Suppose that $s = 5$ and so $b_5(C_4, C_4) = 5$ according to [14] and [15]. It follows that the red-blue coloring of the edges of $K_{5,5}$ cannot avoid a red C_4 or a blue C_4 . Since $K_{5,5} \subset K_{5,6}$, we also cannot avoid a red C_4 or a blue C_4 in the red-blue coloring of the edges of $K_{5,6}$. Now let $s = 6$, then $b_6(C_4, C_4) = 6$ [15] but the lower bound of this result will be $b_6(C_4, C_4) \geq 5$ where we can avoid a red C_4 and a blue C_4 in the red-blue edge coloring of the edges of $K_{5,6}$. However, we have a contradiction as $K_{5,6}$ will always contain a red C_4 or a blue C_4 from our argument above. Hence, $b_s(C_4, C_4) \neq s$ for all $s \geq 5$ and, from Theorem 3.5, we have proved that $b_s(C_4, C_4) = 5$.

Table 1 illustrates our findings for $b_s(C_4, C_4) = t$ where “Y” means we can avoid a red C_4 and a blue C_4 , and “N” means we cannot avoid a red or a blue C_4 . Since $b_3(C_4, C_4) = 7$ from Theorem 3.3, we see in the table that if $s = 3$ and $t = 4$, then we will be able to avoid a red C_4 and a blue C_4 . Also, if $s = 3$ and $t = 8$, then we will not be able to avoid a red C_4 or a blue C_4 .

$s \backslash t$	1	2	3	4	5	6	7	8
1	Y	Y	Y	Y	Y	Y	Y	Y
2	Y	Y	Y	Y	Y	Y	Y	Y
3	Y	Y	Y	Y	Y	Y	N	N
4	Y	Y	Y	Y	Y	Y	N	N
5	Y	Y	Y	Y	N	N	N	N
6	Y	Y	Y	Y	N	N	N	N
7	Y	Y	N	N	N	N	N	N
8	Y	Y	N	N	N	N	N	N

Table 1: Our results for $b_s(C_4, C_4)$.

3.5 The s -Bipartite Ramsey Number of a 4-Cycle and a 6-Cycle

In this section we investigate the s -bipartite Ramsey number of a 4-cycle and 6-cycle for $s \geq 4$ as the part for $s \leq 3$ has been covered in Section 3.3. As explained previously, note that when proving the upper bounds, we consider the graph G and redraw it such that there are no vertices inside an even cycle and that all the remaining vertices are on the outside and not enclosed by the vertices of the cycle. We define the *interior* chords of a cycle to be the chords between the vertices on the inside of a cycle and the *exterior* edges to be the ones between a vertex on the cycle and a vertex on the outside of the cycle, or between two vertices on the outside of the cycle.

Theorem 3.6

$$b_4(C_4, C_6) = 5.$$

Proof. To prove the lower bound $b_4(C_4, C_6) \geq 5$, let $G = K_{4,4}$ be a complete bipartite graph with the partite sets X and Y of $V(G)$ where $X = \{x_1, x_2, x_3, x_4\}$ and

$Y = \{y_1, y_2, y_3, y_4\}$. Suppose that we color two edges incident to every vertex of X red and then color the remaining two edges incident to every vertex of X blue as in Figure 3.6.

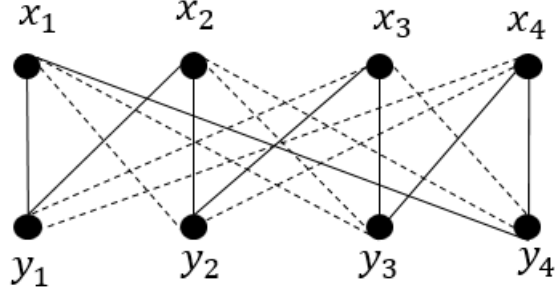


Figure 3.6 Red-blue coloring of the edges of $K_{4,4}$.

We find that the only cycle in G_R and in G_B is a C_8 . Therefore neither a blue C_6 nor a red C_4 exists in Figure 3.6. Hence a red-blue coloring of the edges of $K_{4,4}$ exists where we can avoid a red C_4 and blue C_6 . Therefore, $b_4(C_4, C_6) \geq 5$.

We now give a proof for the upper bound $b_4(C_4, C_6) \leq 5$. Assume, to the contrary, that $b_4(C_4, C_6) > 5$, which means there exists a red-blue coloring of the edges of $G = K_{4,5}$ that avoids both a red C_4 and a blue C_6 . We know that G is not a tree nor a forest as it would then be of maximum size $m \leq 7$ which is not true because $|E(G)| = 16$. Since we want to avoid a red C_4 , we assume the following cases in the red subgraph of G .

Case 1: The longest cycle in G_R has length six.

Consider a vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3\}$ that induces a red 6-cycle in G_R . This cycle can be drawn so that the vertex x_4 is on the exterior of this 6-cycle and so the

cycle will not enclose x_4 . We observe that x_4 can be adjacent to three vertices from the set $\{y_2, y_4, y_5\}$. Note that any vertex from the set $\{y_4, y_5\}$ cannot be adjacent to any vertex from the set $\{x_2, x_3\}$ otherwise this may lead to a red C_4 , for example x_4, y_5, x_2, y_2, x_4 . Therefore we color all the edges between vertices from the vertex set $\{y_4, y_5\}$ and the vertex set $\{x_2, x_3\}$ blue, and then we color one of the edges between x_1 and the vertex set $\{y_4, y_5\}$ red. If we color both edges red, then the vertex set $\{x_1, y_4, x_4, y_5\}$ will induce a red C_4 .

In the interior of the 6-cycle we have the chords x_1y_2 , x_2y_3 and x_3y_1 which may lead to a red C_4 if we color any of them red, for example x_1, y_1, x_3, y_3, x_1 . Therefore we color all three of them blue. From the edges that cannot be colored red in the red subgraph, we observe that the vertices of the set $\{y_1, x_4, y_3, x_2, y_4, x_3\}$ induce a blue 6-cycle; a contradiction to our assumption that $b_4(C_4, C_6) > 5$. Thus we have that there is at least one edge that cannot be colored red or blue to avoid a red C_4 and a blue C_6 , for example, the edge x_3y_1 . Therefore the longest cycle in G_R cannot be of length six.

Case 2: The longest cycle in G_R has length eight.

Consider a vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4\}$ that induces a red 8-cycle in G_R . In the exterior of an 8-cycle we have that the vertex y_5 can be adjacent to at most two vertices from the set Y since a red C_6 may exist if y_5 is adjacent to two vertices from the set X whose indices differ by two. For example, we have the 6-cycle $x_1, y_1, x_2, y_2, x_3, y_5, x_1$ if x_1y_5 and x_3y_5 are colored red. Also, y_5 cannot be adjacent to more than two vertices from set X as this may lead to a red C_4 since two vertices from set X will both be adjacent to y_5 and another vertex from set Y ; see for example x_4, y_3, x_3, y_5, x_4 . This also means y_5 will be adjacent to at most two vertices whose indices differ by two in the blue subgraph, for example, x_2 and x_4 .

In the interior of a C_8 we cannot have any chord colored red otherwise this may lead to a red C_4 ; for example x_1, y_1, x_4, y_4, x_1 if the chord x_4y_1 is colored red. Therefore we color all the chords $x_1y_2, x_1y_3, x_2y_3, x_2y_4, x_3y_1, x_3y_4, x_4y_1$ and x_4y_2 blue which form a blue C_8 . However, since y_5 is incident to two blue edges, for example y_5x_2 and y_5x_4 in the blue subgraph, we observe that the vertices in the set $\{x_2, y_5, x_4, y_1, x_3, y_4\}$ will induce a blue C_6 . Therefore at least one edge cannot be colored red or blue to avoid a red C_4 and a blue C_6 , e.g., the edge x_2y_5 . Hence, the longest cycle in G_R cannot be of length eight. This concludes our proof. ■

We now prove the case for $s \geq 5$ as follows.

Theorem 3.7

$$b_s(C_4, C_6) = 5.$$

Proof. We prove Theorem 3.7 for when $s = 5$ and when $s > 5$. Hence we have the following two cases.

Case 1: $s = 5$.

If $s = 5$, then $b_5(C_4, C_6) = 5$. To prove the lower bound, we refer to the proof of Case 1 for Theorem 3.5. We have that we can avoid a red C_4 and a blue C_4 in a red-blue coloring of the edges of $G = K_{4,5}$. We observe that there exists a red C_6 and a blue C_8 in Figure 3.5 while avoiding a red C_4 and a blue C_6 . Thus $b_5(C_4, C_6) > 4$.

To prove the upper bound $b_5(C_4, C_6) \leq 5$, assume, to the contrary, that $b_5(C_4, C_6) > 5$ which means there exists a red-blue coloring of the edges of $G = K_{5,5}$ that avoids both a red C_4 and a blue C_6 . Since we want to avoid a red C_4 , we can have the

following subcases for the longest cycle in the red subgraph of G .

Subcase 1: The longest cycle has length six.

Consider the vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3\}$ that induces a red 6-cycle in G_R . In the exterior of the 6-cycle we can have that x_4 is adjacent to three vertices from the set $\{y_2, y_4, y_5\}$. If it is adjacent to four or more edges, then the vertex set $\{x_4, y_1, x_2, y_2\}$ will induce a red C_4 . We also have that x_5 is adjacent to at most two vertices of set Y whose indices do not differ by two, e.g., x_5y_1 and x_5y_5 , as we only have three vertices whose indices differ by two in set Y . Similarly, y_5 is adjacent to at most two vertices of set X whose indices do not differ by two. If x_5 is adjacent to any two vertices of set Y whose indices differ by two, e.g. x_5y_2 , and x_5y_4 , then this will lead to a red C_4 , for example x_5, y_2, x_4, y_4, x_5 . Hence x_5 and y_5 are each incident to at most three blue edges.

In the interior of the 6-cycle we cannot color any chord red as this may lead to a red C_4 . For example, x_1, y_1, x_3, y_3, x_1 if the chord x_3y_1 is colored red. Therefore, we color the chords x_1y_2 , x_2y_3 , and x_3y_1 blue. We observe that the vertex set $\{x_1, y_2, x_5, y_3, x_2, y_5\}$ induces a blue C_6 , a contradiction to our assumption that $b_5(C_4, C_6) > 5$. We have that at least one of the edges on the blue C_6 cannot be colored red or blue, for example the chord x_1y_2 . Therefore the longest cycle in G_R cannot be of length six.

Subcase 2: The longest cycle in G_R has length eight.

Consider a vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4\}$ that induces a red 8-cycle. In the exterior of the 8-cycle we only have one edge between x_5 and any vertex from set Y . If we have two or more edges, then the vertex set $\{x_5, y_1, x_2, y_2\}$ will induce a red C_4 . We also have one edge between y_5 and any vertex from set X , otherwise, if we have

two or more edges, then the vertex set $\{x_3, y_3, x_4, y_5\}$ will induce a red C_4 . Therefore x_5 and y_5 are incident to at most three blue edges.

In the interior of an 8-cycle we refer to Case 2 of Theorem 3.6 where the edges that cannot be colored red form a blue C_8 in the blue subgraph of G . However, since x_5 and y_5 are incident to at most three blue edges, we have that the edges of a blue C_8 together with the edges incident to x_5 and y_5 create a blue C_6 induced by the vertex set $\{x_5, y_4, x_3, y_1, x_4, y_5\}$, a contradiction to our assumption that $b_5(C_4, C_6) > 5$. Therefore the longest cycle in G_R cannot be of length eight.

Subcase 3: The longest cycle in G_R has length ten.

Consider a vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5\}$ that induces a red C_{10} . We recall from Theorem 3.5 that in the interior of a C_{10} we can color at most two chords red which creates a red C_6 , thus avoiding a red C_4 . We cannot have a red C_8 in a red C_{10} as the chord x_5y_1 on the C_8 makes way for the existence of a red C_4 , namely x_1, y_1, x_5, y_5, x_1 . Hence we have all the remaining chords colored in blue: $x_2y_3, x_2y_5, x_1y_3, x_1y_4, x_3y_1, x_4y_1, x_4y_2, x_5y_1, x_5y_2$ and x_5y_3 . We observe that the edges x_4y_1, x_4y_2, x_5y_1 and x_5y_2 induce a blue C_4 . Note that if two vertices incident to an edge x_iy_j are adjacent to two vertices of the vertex set that induces a C_4 , then this will lead to a C_6 . We observe that y_3 and x_1 incident to the blue edge x_1y_3 are adjacent to x_5 from x_5y_3 and y_2 from x_1y_2 . This adjacency leads to a blue C_6 , $x_1, y_2, x_4, y_1, x_5, y_3, x_1$, which is a contradiction to our assumption $b_5(C_4, C_6) > 5$. Therefore, since an edge such as x_1y_3 of the blue C_6 cannot be colored red or blue, it follows that we cannot avoid a red C_4 or a blue C_6 in the graph G . Hence, we cannot have the longest cycle in G_R to be of length ten. This concludes the proof for case 1.

Case 2: $s > 5$.

Since $K_{5,4} \subset K_{5,5} \subset \dots \subset K_{5,s}$ for $s \geq 5$, it can be concluded by symmetry that $b_s(C_4, C_6) = 5$ for $s \geq 5$. That is, since we cannot avoid a red C_4 or a blue C_6 in a $K_{5,5}$, we will also not be able to avoid a red C_4 or a blue C_6 in a $K_{5,s}$ as $K_{5,5}$ is a subgraph of $K_{5,s}$. Therefore $b_s(C_4, C_6) = 5$ for all $s \geq 5$. ■

In Table 2 our findings for $b_s(C_4, C_6) = t$ are presented, where “Y” means we can avoid a red C_4 and a blue C_6 , and “N” means we cannot avoid a red C_4 or a blue C_6 . Since $b_4(C_4, C_6) = 5$ from Theorem 3.6, we see in the table that if $s = 3$ and $t = 5$, then we will be able to avoid a red C_4 and a blue C_6 . Also, if $s = 4$ and $t = 6$, then we will not be able to avoid a red C_4 or a blue C_6 .

$s \backslash t$	1	2	3	4	5	6	7	8
1	Y	Y	Y	Y	Y	Y	Y	Y
2	Y	Y	Y	Y	Y	Y	Y	Y
3	Y	Y	Y	Y	Y	Y	Y	Y
4	Y	Y	Y	Y	N	N	N	N
5	Y	Y	Y	N	N	N	N	N
6	Y	Y	Y	N	N	N	N	N
7	Y	Y	Y	N	N	N	N	N
8	Y	Y	Y	N	N	N	N	N

Table 2: Our results for $b_s(C_4, C_6)$.

3.6 The s -Bipartite Ramsey Number of Two 6-Cycles

We now consider the s -bipartite Ramsey number of two 6-cycles for $s \geq 5$. As explained previously, note that when proving the upper bounds, we consider the graph

G and redraw it such that there are no vertices inside an even cycle and that all the remaining vertices are on the outside and not enclosed by the vertices of the cycle. We define the *interior* chords of a cycle to be the chords between the vertices on the inside of a cycle and *exterior* edges to be the ones between a vertex on the cycle and a vertex on the outside of the cycle, or between two vertices on the outside of the cycle. If $s = 5$, then we have the following theorem.

Theorem 3.8

$$b_5(C_6, C_6) = 5.$$

Proof. To prove the lower bound $b_5(C_6, C_6) \geq 5$, let $G = K_{5,4}$ with X and Y as the partite sets of $V(G)$ where $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, \dots, y_5\}$. We recall the red-blue coloring of the edges of $K_{4,4}$ in Figure 3.2 where we have a red subgraph $K_{2,4}$ and a blue subgraph $K_{2,4}$. We may construct the graph $K_{4,5}$ by adding two blue edges between y_5 and any two vertices of the red $K_{2,4}$, say x_1 and x_2 . We also add two red edges between y_5 and any two vertices of the blue $K_{2,4}$, say x_3 and x_4 . It follows that in the red subgraph of $K_{4,5}$, we have a red $K_{2,4}$ and two edges y_5x_3 and y_5x_4 while avoiding a red C_6 . In the blue subgraph of $K_{4,5}$ we have a blue $K_{2,4}$ and two edges y_5x_1 and y_5x_2 while avoiding a blue C_6 . Therefore, $b_s(C_6, C_6) \geq 5$.

We now give a proof for the upper bound $b_5(C_6, C_6) \leq 5$. Assume, to the contrary, that $b_5(C_6, C_6) > 5$. Hence, there exists a red-blue coloring of the edges of $G = K_{5,5}$ that avoids a red C_6 and a blue C_6 . Note that G has 25 edges which can be partitioned into a red subgraph G_R and a blue subgraph G_B .

Assume, without loss of generality, that the size of G_R is at least 13, i.e, $|E(G_R)| \geq 13$. We know that G_R is not a tree nor a forest as it would then be of maximum size $m \leq 10$. We may assume that G_R is not a tree and let C_k be the longest cycle in G_R

for an even number $k \geq 4$. Note that we cannot have $k = 6$ from our assumption that G_R does not contain any red C_6 's. Since $|X| = 5$, we consider the following cases.

Case 1: The longest cycle in G_R has length four.

Consider the vertex set $\{x_1, y_1, x_2, y_2\}$ that induces a red C_4 in G_R . In the exterior of the 4-cycle we have three vertices from set X and three vertices from set Y . We cannot have all three vertices from set X being adjacent to all three vertices from set Y , otherwise this will lead to a red C_6 , for example $x_3, y_3, x_4, y_4, x_5, y_5, x_3$. Therefore, we only consider the following subcases.

Subcase 1: Both vertices from the set $\{x_3, x_4\}$ are adjacent in G_R to all three vertices from the set $\{y_3, y_4, y_5\}$.

We may have at most two edges from the edge set $\{x_i y_j | 1 \leq i \leq 2, 3 \leq j \leq 5\}$, otherwise, if we have three or more edges, then the vertex set $\{x_1, y_3, x_3, y_4, x_2, y_1\}$ will induce a red C_6 . We have that x_5 can only be adjacent to one vertex from the set $\{y_1, y_2\}$, otherwise, if it is adjacent to both of them, then we will have a red C_6 , for example $x_5, y_1, x_2, y_3, x_1, y_2, x_5$. We may not have any chords in the interior of a C_4 since no two vertices in any partite set of a bipartite graph can be adjacent. This gives us a total of 13 edges colored red in G_R , proving our main assumption that $|E(G_R)| \geq 13$. However, we observe that the edges that cannot be colored red in G_R produce a blue C_6 in the blue subgraph G_B , for example, the edge set $\{x_1 y_3, y_3 x_5, x_5 y_5, y_5 x_2, x_2 y_4, y_4 x_1\}$ which contradicts our assumption that G contains no red C_6 's or blue C_6 's. Therefore, both vertices from the set $\{x_3, x_4\}$ are not adjacent to all the vertices from the set y_3, y_4, y_5 and so we cannot have the longest cycle in G_R to be of length four.

Subcase 2: x_5 is adjacent to all three vertices from the set $\{y_3, y_4, y_5\}$.

We may have three edges from the edge set $\{x_i y_j | 1 \leq i \leq 3, 3 \leq j \leq 5\}$ where x_i may be adjacent to all three vertices from set Y , otherwise, if we have four or more edges, then the vertex set $\{x_1, y_3, x_5, y_4, x_2, y_2\}$ will induce a red C_6 . We may also have the two edges $x_3 y_5$ and $x_4 y_4$. Since we cannot have any chords inside a C_4 it follows that we have a total of at most 12 edges colored red in G_R , a contradiction to our main assumption that $|E(G_R)| \geq 13$. Therefore, x_5 is not adjacent to all three vertices from the set y_3, y_4, y_5 and so we cannot have the longest cycle in G_R to be of length four.

Case 2: The longest cycle in G_R has length eight.

Consider the vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4\}$ that induces a red C_8 in G_R . In the exterior of the 8-cycle we may have at most two edges between x_5 and any vertex from set Y , otherwise, if we have three or more edges, then the vertex set $\{x_5, y_4, x_1, y_1, x_2, y_2\}$ will induce a red C_6 . Similarly, we may have at most two edges between y_5 and any vertex from set X . We have that x_5 is not adjacent to y_5 , otherwise this may lead to a cycle longer than that of length eight, namely a C_{10} , contradicting our assumption for Case 2.

If we color any chord inside a C_8 , then we will have a red C_6 . For example, if we color the chord $x_4 y_1$ red, then we will have $x_4, y_1, x_2, y_2, x_3, y_3, x_4$ as a red C_6 . Therefore, we may not color any chord red. This gives us a total of at most 12 edges colored red in G_R , a contradiction to our main assumption that $|E(G_R)| \geq 13$. Therefore, we cannot have the longest cycle in G_R to be of length eight.

Case 3: The longest cycle in G_R has length ten.

Consider the vertex set $\{x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5\}$ that induces a red C_{10} in G_R . To avoid a red C_6 in a C_{10} , we may color the chord x_5y_1 red. This will lead to a red C_4 , x_1, y_1, x_5, y_5, x_1 , and a red C_8 , $x_5, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5$. We cannot color any more chords red as it will lead to a red C_6 , for example, $x_2, y_2, x_3, y_3, x_4, y_4, x_2$ if we color the chord x_2y_4 red. Hence, we have a total of at most 11 edges colored red in G_R , a contradiction to our main assumption that $|E(G_R)| \geq 13$. Therefore, we cannot have the longest cycle in G_R to be of length ten.

Since $K_{5,5} \subset K_{5,6} \subset \dots \subset K_{5,s}$ for $s \geq 5$, we can conclude that by symmetry, $b_s(C_6, C_6) = 5$ for $s \geq 5$. That is, since we cannot avoid a red C_6 or a blue C_6 in a $K_{5,5}$, we will also not be able to avoid a red C_6 or a blue C_6 in a $K_{5,s}$ as $K_{5,5}$ is a subgraph of $K_{5,s}$. Therefore, $b_s(C_6, C_6) \geq 5$ for all $s \geq 5$. ■

Our findings for $b_s(C_6, C_6) = t$ are presented in Table 3, where “Y” indicates that we can avoid a red C_6 or a blue C_6 and “N” indicates that we cannot avoid a red C_6 or a blue C_6 . Since $b_5(C_6, C_6) = 5$ from Theorem 3.8, we see in the table that if $s = 5$ and $t = 4$, then we will be able to avoid a red C_6 or a blue C_6 . Also, if $s = 5$ and $t = 6$, then we will not be able to avoid a red C_6 or a blue C_6 .

$\begin{smallmatrix} t \\ s \end{smallmatrix}$	1	2	3	4	5	6	7	8
1	Y	Y	Y	Y	Y	Y	Y	Y
2	Y	Y	Y	Y	Y	Y	Y	Y
3	Y	Y	Y	Y	Y	Y	Y	Y
4	Y	Y	Y	Y	Y	Y	Y	Y
5	Y	Y	Y	Y	N	N	N	N
6	Y	Y	Y	Y	N	N	N	N
7	Y	Y	Y	Y	N	N	N	N
8	Y	Y	Y	Y	N	N	N	N

Table 3: Our results for $b_s(C_6, C_6)$.

3.7 In Conclusion

We have the following theorem that summarizes all of our findings for the s -bipartite Ramsey numbers of two cycles.

Theorem 3.12

Let $m \geq 4$. Then,

$$b_s(C_m, C_4) = \begin{cases} \infty & \text{if } 1 \leq s \leq 2, \\ 7 & \text{if } 3 \leq s \leq 4 \text{ and } m = 4, \\ 5 & \text{if } s \geq 5, \end{cases} \quad (9)$$

and

$$b_s(C_m, C_6) = \begin{cases} \infty & \text{if } s \leq 4 \text{ and } m \geq 6, \\ 5 & \text{if } s = 4 \text{ and } m = 4, \\ 5 & \text{if } s \geq 5 \text{ and } m \geq 6. \end{cases} \quad (10)$$

We plan to continue to investigate the following question in future studies:

What is the general s -bipartite Ramsey number of two even cycles?

That is, $b_s(C_{2m}, C_{2n})$ for positive integers $m, n \geq 2$.

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