

https://dspace.org/

De Montessus de Ballore theorem for Pade approximation.

Chou, Ping

1994
https://hdl.handle.net/10539/26388

Downloaded from a DSpace repository. https://dspace.org/

de Montessus de Ballore Theorem for Padé Approximation

Ping Zhou*

December 2, 1994

Degree awarded with distinction on 20 April 1995.

*A research report submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in partial fulfilment of the degree of Master of Science

I am greatly indebted to my supervisor Professor Doron Lubinsky for his invaluable assistance in promoting my knowledge and ability.

Declaration

I declare that this research report is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

周平 (Zhou Ping)

02 day of Dec., 1994

Abstract

The importance of Padé approximation has been increasingly recognized in recent years. The first convergence result of Padé approximants valid for general meromorphic functions was obtained by de Montessus de Ballore in 1902. He proved that when a function f has precisely n poles in $| z | < R$, then the $(n+1)th$ column in the Padé table of f converges to f in $| z | < R$.

The first part of this report is devoted to the statement and proofs of existence and uniqueness of Padé approximants, as well as to the Padé table and its structure.

de Montessus de Ballore proved his convergence theorem on Padé approximants by using some results of Hadamard (1892) on the location of poles of a function represented by a Taylor series. It was quite lengthy. In the second part, a much shorter and more elegant proof is stated which is due to E.B.Saff (1972), by employing Hermite's contour integral error formula and Hurwitz' theorem.

Now, the de Montessus de Ballore theorem has been extended to many kinds of approximants. In the third part, we will extend the theorem to multipoint Padé approximants and state many other extensions and analogues of the theorem.

Contents

1 Introduction of Padé Approximation 1

1.1 Definition of Padé Approximants 1

1.2 Existence and Uniqueness of Padé Approximants 4

1.3 Padé Table: Its Structure 9

1.4 Padé Table: Its Rows, Columns and Diagonals 15

2 The Classical de Montessus de Ballore Theorem 17

2.1 de Montessus de Ballore Theorem 17

2.2 Results on the Poles 24

3 Extensions of de Montessus de Ballore Theorem 27

3.1 Multipoint Padé Approximants 27

3.2 Walsh Array 38

3.3 Orthogonal Padé Approximants 39

3.4 Smooth Coefficients 42

3.5 Intermediate Rows 45

3.6 Multivariate Padé Approximants 47

Bibliography 54

Chapter 1

Introduction to Padé Approximation

Henri Eugene Padé was born in Abbeville, France on December 17, 1863. In 1886, he obtained the highest teacher's degree in mathematics from the Ecole Normale Superiéure. In 1892, he successfully defended his doctoral thesis on approximation of functions by rational fractions.

In his thesis, Padé arranged the approximants into a table and completely characterized the structure of the table, giving a canonical decomposition of it into different blocks. He also proved convergence of the Padé approximants to $\exp(z)$. These two major achievements justify the approximants being named after him.

Padé's progress in the academic world was quite rapid until his retirement in 1934, at the age of seventy one. Padé died in 1953, at the age of eighty nine.

In this chapter, Padé approximants and the Padé table are defined; the existence and uniqueness of Padé approximants and the structure theorem of the Padé table are proved.

1.1 Definition of Padé Approximants

Before proceeding to a formal definition of Padé approximants, let us recall a few notions

from rational interpolation.

Let $m,n \geq 0$ be integers, if

$$R(x) := \frac{a_0 + a_1 x + a_2 x^2 + ... + a_m x^m}{b_0 + b_1 x + b_2 x^2 + ... + b_n x^n},$$

with at least one $b_j \neq 0, j = 0, 1, ..., n$, where $a_i, b_j \in \mathbb{C}, 0 \leq i \leq m, 0 \leq j \leq n$, then we call $R(x)$ *a rational function of type* (m, n). We let $R_{m,n}$ denote the set of all rational functions of type (m, n).

We know that if $x_j, y_j \in \mathbb{C}$, then there exists a unique polynomial $P(x)$ of degree$\leq n$ such that

$$R(x_j) = \frac{P(x_j)}{Q(x_j)} = y_j, \qquad j = 1, 2, ..., n+1.$$

But if we don't fix the poles of $R \in R_{m,n}, R$ has $(m+1)+(n+1)$ coefficients. Eliminating one for division, we expect to be able to satisfy $m+n+1$ conditions to determine R. Given $x_j, y_j \in \mathbb{C}, j = 1, 2, ..., m+n+1$, all x_j are distinct, we say that the *Corresponding Hermite Interpolation Problem* has a solution if there exists a $R \in R_{m,n}$, such that

$$R(x_j) = y_j, \qquad j = 1, 2, ..., m+n+1.$$

Given $x_j, y_j \in \mathbb{C}, j = 1, 2, ..., m+n+1$, all x_j are distinct, the Corresponding Hermite Problem doesn't always have a solution. If more than max{m,n} y_j are equal, but not all y_j are equal, then there doesn't exist a $R \in R_{m,n}$ such that the Corresponding Hermite Interpolation Problem has a solution R.

To avoid the above problem, we linearize the interpolation conditions: Let $\{z_j\}_{j=1}^{m+n+1} \in \mathbb{C}$ (not necessarily distinct), and let $f : \{z_j\}_{j=1}^{m+n+1} \to \mathbb{C}, R = P/Q \in R_{m,n}$. We say R solves the *Modified Hermite Interpolation Problem associated with* $\{z_j\}_{j=1}^{m+n+1}$ *and* f, if

$$(fQ - P)(z_j) = 0, \qquad j = 1, 2, ..., m+n+1. \tag{1.1}$$

Here if some z_j is repeated, say $z_j = z_{j+1} = ... = z_{j+l+1} \neq z_{j+l+2}$, then we replace (1.1) for $j, j+1, ..., j+l+1$ by

$$(fQ - P)^{(k)}(z_j) = 0, \qquad k = 0, 1, 2, ..., l.$$

In this case, we are assuming $f^{(k)}(z_j)$ exists.

We can prove that the Modified Hermite Interpolation Problem always has a unique solution $R = P/Q$. If in addition $Q(z_j) \neq 0, j = 1, 2, ..., m+n+1$, then R also solves the Corresponding Hermite Interpolation Problem (see Cheney [3]).

A formal power series is an expression

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \qquad a_j \in \mathbb{C}, \qquad j = 0, 1, 2, \cdots.$$

We write for some integer $l \geq 0$,

$$f(z) = O(z^l),$$

if $a_0 = a_1 = a_2 = ... = a_{l-1} = 0$.

We write $f(z) \equiv 0$ if $a_j = 0$ for all $j \geq 0$.

Let F denote the set of all formal power series, then it is clear that

(a) F is closed under "+" and usual power series "×". Furthermore,"×" is commutative.

(b) If

$$f(z) = \sum_{j=o}^{\infty} a_j z^j$$

and $a_0 \neq 0$, then there exists a $g \in F$, such that $g = f^{-1}$, *i.e.* $f \times g = 1$.

(c) If $f(z) = O(z^l)$, for some non-negative integer l, and $g \in F$, then $f \times g = O(z^l)$.

(d) If $k, l \geq 0$ are integers, $k \leq l$ and $f(z) = O(z^l)$, then $f(z) = O(z^k)$.

The Padé approximants are a particular type of rational function approximant to the value of a function. The idea is to match the Taylor series expansion as far as possible.

Definition 1.1. Let $f(z)$ be a formal power series, and m, n be non-negative integers. *The m, n Padé approximant to $f(z)$* is a rational function

$$[m/n] := \frac{P(z)}{Q(z)}$$

of type (m, n), such that

$$(fQ - P)(z) = O(z^{m+n+1}). \tag{1.2}$$

Note that if $f(z)$ is a convergent power series for z near 0, we can rewrite (1.2) as

$$(fQ - P)^{(k)}(0) = 0, \qquad k = 0, 1, 2, ..., m+n$$

So the Padé approximant $[m/n](z)$ is the solution to the Modified Hermite Interpolation Problem for $\{0\}_{j=1}^{m+n+1}$ and f.

1.2 Existence and Uniqueness of Padé Approximants

Theorem 1.2. Let $f(z)$ be a formal power series, then for any integers $m, n \geq 0$, the Padé approximant to $f(z)$, $[m/n](z) = P(z)/Q(z)$ exists and is unique. Further, if after cancelling common factors in P, Q, we obtain $[m/n] = \widehat{P}/\widehat{Q}$, then $\widehat{Q}(0) \neq 0$ and

$$f(z) - [m/n](z) = O(z^{m+n+1-l}), \tag{1.3}$$

where

$$l := \min\{n - \deg(\widehat{Q}), m - \deg(\widehat{P})\}, \tag{1.4}$$

and $\deg(\widehat{Q}), \deg(\widehat{P})$ are the degrees of $\widehat{P}$ and $\widehat{Q}$ respectively.

Proof: Let us write

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

$$P(z) = \sum_{j=0}^{m} p_j z^j,$$

$$Q(z) = \sum_{j=0}^{n} q_j z^j.$$

The condition

$$(fQ - P)(z) = O(z^{m+n+1})$$

becomes

$$\sum_{j=0}^{\infty} \{ \sum_{k=o}^{\min\{j,n\}} a_{j-k} q_k \} z^j - \sum_{j=0}^{m} p_j z^j = O(z^{m+n+1}).$$

Comparing cosfficients of the same powers of z, yields

$$\begin{cases} \sum_{k=0}^{\min\{j,n\}} a_{j-k} q_k - p_j = 0, & j = 0, 1, 2, ..., m, \\ \sum_{k=0}^{\min\{j,n\}} a_{j-k} q_k = 0, & j = m+1, m+2, ..., m+n. \end{cases} \tag{1.5}$$

(1.5) is a system of simultaneous equations of $m+n+1$ homogeneous linear equations in the $m+n+2$ variables $p_0, p_1, ..., p_m, q_0, q_1, ..., q_n$. As there are more variables than equations, (1.5) has a non-trivial solution, i.e. with not all of $p_0, p_1, ..., p_m, q_0, q_1, ..., q_n$ equal to zero. If $Q \equiv 0$ in a solution, i.e. $q_0 = q_1 = ... = q_n = 0$, then the first eqation in (1.5) yields $p_j = 0, j = 0, 1, 2, ..., m$. i.e. $P \equiv 0$, a contradiction. So $Q \neq 0$ and $[m/n](z)$ exists.

Suppose now that $P_1/Q_1 \in R_{m,n}$ can also serve as $[m/n]$. Then recall (1.2):

$$(fQ - P)(z) = O(z^{m+n+1}).$$

Also

$$(fQ_1 - P_1) = O(z^{m+n+1}). \tag{1.6}$$

(1.2)$\times Q_1$ − (1.6) $\times Q$:

$$(-PQ_1 + P_1 Q)(z) = O(z^{m+n+1}).$$

But $-PQ_1 + P_1Q$ is a polynomial of degree $\leq m+n$. We obtain

$$-PQ_1 + P_1Q \equiv 0,$$

or

$$\frac{P}{Q} \equiv \frac{P_1}{Q_1},$$

so [m/n](z) is unique.

Now, to prove (1.3), let us write

$$\begin{cases} P(z) = z^r S(z)\widehat{P}(z), \\ Q(z) = z^r S(z)\widehat{Q}(z), \end{cases} \tag{1.7}$$

where $0 \leq r \leq \min\{m,n\}$; $S(z)$ is a polynomial of degree $\leq \min\{m,n\}$ and $S(0) \neq 0$; $\widehat{P}$ and $\widehat{Q}$ have no common factors. By (1.2),

$$z^r S(z)(f\widehat{Q} - \widehat{P})(z) = O(z^{m+n+1}).$$

As $S(0) \neq 0$, we can multiply above equation by the formal power series $1/S(z)$ to deduce that

$$z^r(f\widehat{Q} - \widehat{P})(z) = O(z^{m+n+1})$$

so

$$(f\widehat{Q} - \widehat{P})(z) = O(z^{m+n+1-r}). \tag{1.8}$$

We claim now $\widehat{Q}(0) \neq 0$. For if $\widehat{Q}(0) = 0$, as $r \leq n$, (1.8) shows $\widehat{P}(0)$, the constant coefficient of $\widehat{P}$ is also zero, then $\widehat{P}(z)$ and $\widehat{Q}(z)$ has the common factor z, a contradition. So $\widehat{Q}(0) \neq 0$ and we can multipy (1.8) by the formal power series $1/\widehat{Q}(z)$ to deduce

$$f(z) - \frac{\widehat{P}(z)}{\widehat{Q}(z)} = O(z^{r+n+1-r}),$$

so

$$f(z) - [m/n](z) = f(z) - \frac{z^r S(z)\widehat{P}(z)}{z^r S(z)\widehat{Q}(z)}$$

$$= f(z) - \frac{\widehat{P}(z)}{\widehat{Q}(z)}$$

$$= O(z^{m+n+1-r}). \tag{1.9}$$

Finally from (1.7),

$$m \geq \deg(P) = r + \deg(S) + \deg(\widehat{P}) \geq r + \deg(\widehat{P}),$$

so

$$r \leq m - \deg(\widehat{P}).$$

Similarly

$$r \leq n - \deg(\widehat{Q}),$$

that is

$$r \leq l,$$

where l is defined by (1.4). Then

$$m + n + 1 - r \geq m + n + 1 - l.$$

Hence from (1.9),

$$f(z) - [m/n](z) = O(z^{m+n+1-l}).$$

□

One can prove the following: If $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is a formal power series, then for $m \geq 1$,

(a)

$$[m/0](z) = \sum_{j=0}^{m} a_j z^j. \tag{1.10}$$

(b)

$$[m/1](z) = \sum_{j=0}^{m-1} a_j z^j + \frac{a_m z^m}{1 - a_{m+1} z / a_m}, \qquad a_m \neq 0. \tag{1.11}$$

(c) If $a_0 \neq 0$ and $[m/n](z)$ is the m, n Padé approximant to $f(z)$, then $1/[m,n](z)$ is the n, m Padé approximant to $1/f(z)$, that is, with an obvious notion,

$$[n/m]_{1/f}(z) = 1/[m/n]_f(z). \tag{1.12}$$

(d) It is known from Baker [1] that if $m, n \geq 0$ and if

$$D(m/n) := \det(a_{m-j+k})_{j,k=1}^{n}$$

$$= \det \begin{pmatrix} a_m & a_{m-1} & \cdots & \cdots & a_{m-n+1} \\ a_{m+1} & a_m & \cdots & \cdots & a_{m-n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m+n-1} & a_{m+n-2} & \cdots & \cdots & a_m \end{pmatrix}$$

$$\neq 0,$$

then we can solve the equations in (1.5) with $q_0 = 1$ to obtain

$$[m/n](z) = \frac{\det \begin{pmatrix} \sum_{k=0}^{m} a_k z^k & \sum_{k=1}^{m} a_{k-1} z^k & \cdots & \cdots & \sum_{k=n}^{m} a_{k-n} z^k \\ a_{m+1} & a_m & \cdots & \cdots & a_{m-n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m+n} & a_{m+n-1} & \cdots & \cdots & a_m \end{pmatrix}}{\det \begin{pmatrix} 1 & z & \cdots & \cdots & z^n \\ a_{m+1} & a_m & \cdots & \cdots & a_{m-n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m+n} & a_{m+n-1} & \cdots & \cdots & a_m \end{pmatrix}}. \tag{1.13}$$

This explicit formula can be used, together with other methods, to develop efficient computational algorithms for computing $[m/n](z)$. Padé approximants are widely used by physicists.

1.3 Padé Table: Its structure

We are now ready to state the definition of the Padé table and its structure.

Definition 1.3. The *Padé table* of a formal power series $f(z)$ is the doubly-infinite array

$$\begin{array}{cccccc} [0/0] & [0/1] & [0/2] & [0/3] & \cdots \\ [1/0] & [1/1] & [1/2] & [1/3] & \cdots \\ [2/0] & [2/1] & [2/2] & [2/3] & \cdots \\ [3/0] & [3/1] & [3/2] & [3/3] & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{1.14}$$

The Padé table has a special structure:

Theorem 1.4. (Padé 1892) Let $f(z)$ be a formal power series. The Padé table of $f(z)$ consists of square blocks of size r $(1 \leq r \leq \infty)$ with the follwing properties:

(a) All elements in the square block are identical.

(b) No other entries in the Padé table of $f(z)$ are the same as the elements in this block.

(c) If $[\widehat{m}/\widehat{n}] = \widehat{P}/\widehat{Q} \in R_{\widehat{m},\widehat{n}}$ is the top left hand corner of this block, then

$$\deg(\widehat{P}) = \widehat{m}, \quad \deg(\widehat{Q}) = \widehat{n}, \quad \widehat{Q}(0) \neq 0.$$

Furthermore, if $r < \infty$,

$$f(z) - [\widehat{m}/\widehat{n}](z) = z^{\widehat{m}+\widehat{n}+r}(c_0 + c_1 z + c_2 z^2 + \ldots), \qquad c_0 \neq 0,$$

while if $r = \infty$,

$$f(z) - [\widehat{m}/\widehat{n}](z) \equiv 0$$

Proof: Let us pick any $[m/n]$ in the Padé table of $f(z)$. By Theorem 1.2, after cancelling common factors in numerator and denominator, we can write

$$[m/n] = \widehat{P}/\widehat{Q},$$

where $\widehat{P}$ and $\widehat{Q}$ have no common factors, and

$$\begin{aligned} f(z) - [m/n](z) &= f(z) - \frac{\widehat{P}(z)}{\widehat{Q}(z)} \\ &= O(z^{m+n+1-l}), \end{aligned}$$

where

$$l = \min\{m - \deg(\widehat{P}), n - \deg(\widehat{Q})\}.$$

Write

$$\widehat{m} := \deg(\widehat{P}), \qquad \widehat{n} := \deg(\widehat{Q}).$$

We consider two cases:

Case 1. $f(z) - \widehat{P}(z)/\widehat{Q}(z) \equiv 0.$

Then

$$(f\widehat{Q} - \widehat{P})(z) \equiv 0,$$

so given any integers $j, k \geq 0$, we have that trivially

$$(f\widehat{Q} - \widehat{P})(z) = O(z^{(\widehat{m}+j)+(\widehat{n}+k)+1}),$$

and

$$\frac{\widehat{P}}{\widehat{Q}} \in R_{\widehat{m}+j,\widehat{n}+k},$$

so by uniqueness

$$[\hat{m}+j/\hat{n}+k]=\frac{\hat{P}}{\hat{Q}}, \qquad \forall j,k\geq 0.$$

Thus we have an $\infty \times \infty$ block in the table with top left hand corner $[\hat{m}/\hat{n}]=\hat{P}/\hat{Q}$.

For $[m_1/n_1]$ outside this block, we have

$$m_1<\hat{m}=\deg(\hat{P}) \qquad or \qquad n_1<\hat{n}=\deg(\hat{Q}),$$

then $\hat{P}/\hat{Q}=[\hat{m}/\hat{n}]$ cannot serve as $[m_1/n_1]$.

So we have (a),(b),(c) in this case.

Case 2.

$$f(z)-\hat{P}(z)/\hat{Q}(z)=z^N(c_0+c_1z+c_2z^2+...), \qquad c_0\neq 0, \tag{1.15}$$

for some non-negative integer N.

By (1.3),

$$\begin{aligned} N &\geq m+n+1-l \\ &= m+n+1-\min\{m-\hat{m},n-\hat{n}\} \\ &= m+n+1+\max\{\hat{m}-m,\hat{n}-n\} \\ &= 1+\max\{\hat{m}+n,\hat{n}+m\} \\ &\geq 1+\hat{m}+\hat{n}, \end{aligned}$$

as $m\geq\hat{m}, n\geq\hat{n}$.

Then we can assume that for some integer $r\geq 1$,

$$N=\hat{m}+\hat{n}+r.$$

Let $0\leq j,k\leq r-1$ and $s:=\min\{j,k\}$, j,k are integers, we claim

$$[\hat{m}+j/\hat{n}+k](z)=\frac{z^s\hat{P}(z)}{z^s\hat{Q}(z)}=\frac{\hat{P}(z)}{\hat{Q}(z)}. \tag{1.16}$$

For, $z^s\hat{P}(z)$ has degree $s+\hat{m}\le j+\hat{m}$,

$z^s\hat{Q}(z)$ has degree $s+\hat{n}\le k+\hat{n}$,

and

$$f(z)(z^s\hat{Q}(z))-(z^s\hat{P}(z))=O(z^{N+s}) \qquad (by(1.15))$$
$$=O(z^{\hat{m}+\hat{n}+r+s}),$$

where

$$r+s\ge 1+\max\{j,k\}+\min\{j,k\}=1+j+k,$$

so

$$f(z)(z^s\hat{Q}(z))-(z^s\hat{P}(z))=O(z^{(\hat{m}+j)+(\hat{n}+k)+1}),$$

and by uniqueness, (1.16) follows.

Hence

$$[\hat{m}+j/\hat{n}+k]=[\hat{m}/\hat{n}], \qquad 0\le j,k\le r-1,$$

and

$$f(z)-[\hat{m}/\hat{n}](z)=f(z)-\frac{\hat{P}(z)}{\hat{Q}(z)}=x^{m+n+r}(c_0+c_1z+\ldots), \qquad c_0\neq 0,$$

and

$$\deg(\hat{P})=\hat{m}, \qquad \deg(\hat{Q})=\hat{n}.$$

Then we have (a) and (c) in this case.

Now we prove (b). Firstly as before, we cannot have $[m_1/n_1]=[\hat{m}/\hat{n}]$ if $m_1<\hat{m}$ or $n_1<\hat{n}$.

If

$$m_1\ge\hat{m}, \qquad n_1\ge\hat{n} \qquad and \qquad [m_1/n_1]\equiv[\hat{m}/\hat{n}],$$

then we can write

$$[m_1/n_1](z)=\frac{z^sS(z)\hat{P}(z)}{z^sS(z)\hat{Q}(z)},$$

for some polynomial $S(z)$ with $S(0) \neq 0$, and

$$s + \deg(S) + \hat{m} \leq m_1; \qquad s + \deg(S) + \hat{n} \leq n_1. \tag{1.17}$$

Hence

$$f(z)(z^s S(z)\hat{Q}(z)) - z^s S(z)\hat{P}(z) = O(z^{m_1+n_1+1}).$$

Multiplying it by the formal power series $1/S(z)$ and then cancelling z^s, we obtain

$$f(z)\hat{Q}(z) - \hat{P}(z) = O(z^{m_1+n_1+1-s}).$$

Multiplying it by the formal power series $1/\hat{Q}(z)$ as $\hat{Q}(0) \neq 0$, we have

$$f(z) - \frac{\hat{P}(z)}{\hat{Q}(z)} = O(z^{m_1+n_1+1-s}).$$

By (1.15),

$$m_1 + n_1 + 1 - s = N = \hat{m} + \hat{n} + r,$$

so

$$(m_1 - \hat{m}) + (n_1 - \hat{n}) \leq s + r - 1 \leq \min\{m_1 - \hat{m}, n_1 - \hat{n}\} + r - 1,$$

by (1.17). Then

$$\max\{m_1 - \hat{m}, n_1 - \hat{n}\} + \min\{m_1 - \hat{m}, n_1 - \hat{n}\} \leq \min\{m_1 - \hat{m}, n_1 - \hat{n}\} + r - 1,$$

$$\max\{m_1 - \hat{m}, n_1 - \hat{n}\} \leq r - 1,$$

$$i.e. \qquad m_1 - \hat{m} \leq r - 1, \qquad n_1 - \hat{n} \leq r - 1,$$

then $[m_1/n_1]$ is in the $r \times r$ block already.

□

We have completed the proof of Theorem 1.4. Now, we can deduce

Theorem 1.5. (Kronecker, 1880's) Let $f(z)$ be a formal power series. the following are equivalent:

(a) $f(z)$ is the Maclaurin series of a rational function $R(z) = P(z)/Q(z)$, where $\deg(P) = m$, $\deg(Q) = n, Q(0) \neq 0, P$ and Q have no common factors.

(b) There is an $\infty \times \infty$ block in the Padé table of $f(z)$ with top left hand corner $[m/n](z)$.

Proof: (a)$\Longrightarrow$(b).

From (a) we have

$$f(z) \equiv \frac{P(z)}{Q(z)},$$

then

$$f(z)Q(z) - P(z) \equiv 0.$$

For any integers $j, k \geq 0$,

$$f(z)Q(z) - P(z) = O(z^{(m+j)+(n+k)+1}).$$

By uniqueness,

$$[m+j/n+k] = \frac{P}{Q}, \qquad \forall j, k \geq 0.$$

Since P, Q have no common factors, $[m/n]$ must be the top left hand corner of a block for it cannot serve as $[m'/n']$ with $m' < m$ or $n' < n$.

(b)$\Longrightarrow$(a).

From (b) and the (c) of Theorem (1.4),

$$f(z) - [m/n](z) \equiv 0,$$

so

$$f(z) - \frac{P(z)}{Q(z)} \equiv 0,$$

then

$$R(z) = \frac{P(z)}{Q(z)} \equiv f(z).$$

□

1.4 Padé Table: Its Rows, Columns and Diagonals

A row of the Padé table is a sequence $\{[m/n]\}_{n=0}^{\infty}$ with m a fixed non-negative integer. A column of the Padé table is a sequence $\{[m/n]\}_{m=0}^{\infty}$ with n a fixed non-negative integer; and the diagonal of the Padé table is a sequence $\{[m/m]\}_{m=0}^{\infty}$ lying along the diagonal of the Padé table.

It is natural to investigate convergence of sequences of Padé approximants to the function from which they are formed, but it is very complicated. One of the problems is that there is a whole table, not only one sequence. Although from (1.12),

$$[n/m]_{1/f}(z) = \frac{1}{[m/n]_f(z)},$$

we have that every result for columns implies a result for rows, the columns $\{[m/n]\}_{m=0}^{\infty}$, $n = 0, 1, 2, \cdots$,

$$\begin{matrix} [0/n] \\ [1/n] \\ [2/n] \\ \vdots \end{matrix}$$

and the diagonal $\{[m/m\}_{m=0}^{\infty}$,

$$\begin{matrix} [0/0] & & \\ & [1/1] & \\ & & [2/2] \\ & & & \ddots \end{matrix}$$

of the Padé table have radically different convergence properties. In Padé's 1892 thesis, he proved that as $m+n \to \infty$, the (m,n) Padé approximants for e^z, $[m/n](z)$ say, converges to e^z uniformly in compact subset of C. However, there are older results for continued fractions that imply convergence of sequence of Padé approximants to e^z.

From (1.10), we know that the first column of the Padé table is

$$\{[m/0]\}_{m=0}^{\infty} = \left\{\sum_{j=0}^{\infty} a_j z^j\right\}_{m=0}^{\infty},$$

the sequence of partial sums of f, where f is a formal power series or a Taylor series expansion of a function analytic in the disk $D_R := \{z : |z| < R\}\,(0 < R \leq \infty)$. So it converges to f uniformly in the largest circle in which f is analytic. For any n fixed and f analytic in the disk $D_R = \{z : |z| < R\}$ $(0 < R \leq \infty)$ except for exactly n poles, then the $(n+1)th$ column $\{[m/n]\}_{m=0}^{\infty}$ in the Padé table of f converges to f uniformly in the compact subsets of D_R omitting the poles of f. This will be studied in next chapter.

The main results on convergence of the diagonal sequence $\{[m/m]\}_{m=0}^{\infty}$ of the Padé approximants is the Nuttall-Pommerenke theorem.This also applies for more general diagonals, i.e. for the sequences $\{[m_k/n_k]\}_{k=1}^{\infty}$ satisfying

$$\lim_{k\to\infty}(m_k + n_k) = \infty, \qquad and \qquad \frac{1}{\lambda} \leq \frac{m_k}{n_k} \leq \lambda, \qquad all\ k \geq 1,$$

where $\lambda \geq 1$ is independent of k. In 1970, J.Nuttall proved that these sequences converge in planar-Lebesgue measure; In 1973, C.Pommerenke considered more general numerator and denominator degrees, and replaced measure by capacity; In 1978, H.Wallin and A.Gončar replaced Padé approximants by more general rational interpolants. The important results of H.Stahl for functions with branchpoints will be discussed in full in [15].

Chapter 2

The Classical de Montessus de Ballore Theorem

The first convergence result on the columns of the Padé table valid for general meromorphic functions was obtained by de Montessus de Ballore in 1902. He proved that when a function f has precisely n poles in $|z| < R$ $(0 < R \leq \infty)$, then the $(n+1)th$ column in the Padé table of f converges to f in $|z| < R$. He proved this theorem by applying Hadamard's asymtotics for the determinants $\det(a_{m+j-k})_{j,k=1}^{n}$, $m \to \infty$, associated with a meromorphic function. It is quite lengthy. A shorter modern approach, due to E.B.Saff (1972), employs Hermite's contour integral error formula and Hurwitz' theorem.

In this chapter, we will state and prove the de Montessus de Ballore theorem, and discuss some results on the poles.

2.1 de Montessus de Ballore Theorem

Theorem 2.1. (de Montessus de Ballore, 1902,1905) Let f be analytic in the disk $D_R := \{z : |z| < R\}$ $(0 < R \leq \infty)$, except for poles $z_1, z_2, ..., z_n$ (repeated according to multiplicity), $(0 \leq n < \infty)$, none lying at $z = 0$. Then for m large enough, the Padé approximant $[m/n](z)$

to f has poles of total multiplicity exactly n, and satisfies

$$f(z) - [m/n](z) = O(z^{m+n+1}), \qquad as \;\; z \to 0. \tag{2.1}$$

Moreover, as $m \to \infty$, the poles of $[m/n](z)$ converge to the poles of $f(z)$ in D_R according to multiplicity and

$$\lim_{m\to\infty} [m/n](z) = f(z)$$

uniformly in compact subsets of D_R omitting poles of f.

We prove this theorem by using Saff's method. We need two lemmas.

Lemma 2.2. (Hermite's Error Formula for Padé approximants) Let $f(z)$ be analytic in $\overline{D_r} := \{z : |z| \le r\}$, except for poles of total multiplicity $p\,(0 \le p < \infty)$, none lying at 0. Let $S(z)$ be the monic polynomial of degree p such that $(fS)(z)$ is analytic in $\overline{D_r}$. Then for $m \ge 0$ and $n \ge p$, and if $[m/n] := P/Q$,

$$f(z) - [m/n](z) = \frac{1}{2\pi i}\int_{|t|=r} \frac{(fS)(t)}{t-z}\frac{Q(t)}{(SQ)(z)}\left(\frac{z}{t}\right)^{m+n+1} dt, \qquad |z| < r. \tag{2.2}$$

Proof: Note first that as Q and P are polynomials, then $(fQ - P)(z)/z^{m+n+1}$ is analytic in $\overline{D_r}$ except possibly at the poles of f and at $z = 0$. But $f(z)$ is analytic at 0 and by definition of $[m/n](z)$, at $z = 0$,

$$(fQ - P)(z) = z^{m+n+1}(c_0 + c_1 z + c_2 z^2 + ...),$$

where $c_0, c_1, c_2, ...$ are constant in $\mathbb{C}$. So

$$(fQ - P)(z)/z^{m+n+1}$$

is bounded, and so analytic at 0. Then

$$S(z)(fQ - P)(z)/z^{m+n+1}$$

is analytic on $\overline{D_r}$. Cauchy's integral formula yields for $|z| < r$,

$$\frac{S(z)(fQ-P)(z)}{z^{m+n+1}} = \frac{1}{2\pi i}\int_{|t|=r} \frac{S(t)(fQ-P)(t)}{t^{m+n+1}}\frac{dt}{t-z}$$

$$= \frac{1}{2\pi i}\int_{|t|=r} \frac{S(t)(fQ)(t)}{t^{m+n+1}}\frac{dt}{t-z} - \frac{1}{2\pi i}\int_{|t|=r} \frac{(SP)(t)}{t^{m+n+1}}\frac{dt}{t-z}. \tag{2.3}$$

Fix $z \in D_r := \{z : |z| < r\}$. Then $(SP)(t)/(t^{m+n+1}(t-z))$ is a rational function of t with poles only at 0 and z, and so is analytic for $|t| \geq r$.

Since $\deg(SP) \leq p+m \leq n+m$,

and $\deg(t^{m+n+1}(t-z)) = m+n+2$,

there exists a $c > 0$ such that for all $|t|$ large enough,

$$\left|\frac{S(t)P(t)}{t^{m+n+1}(t-z)}\right| \leq c\,|t|^{-2}.$$

Then if $R > r$ is large enough, Cauchy's integral theorem yields

$$\left|\frac{1}{2\pi i}\int_{|t|=r} \frac{S(t)P(t)}{t^{m+n+1}(t-z)}dt\right| = \left|\frac{1}{2\pi i}\int_{|t|=R} \frac{S(t)P(t)}{t^{m+n+1}(t-z)}dt\right|$$

$$\leq \frac{1}{2\pi}\cdot 2\pi R \cdot cR^{-2}$$

$$= \frac{c}{R}$$

$$\to 0, \qquad as\ R \to \infty.$$

So

$$\frac{1}{2\pi i}\int_{|t|=r} \frac{S(t)P(t)}{t^{m+n+1}(t-z)}dt = 0,$$

and (2.3) becomes

$$\frac{S(z)(fQ-P)(z)}{z^{m+n+1}} = \frac{1}{2\pi i}\int_{|t|=r} \frac{S(t)f(t)Q(t)}{t^{m+n+1}(t-z)}dt, \qquad |z| < r. \tag{2.4}$$

By the structure of $S(z)$ we know that $S(0) \neq 0$ and by Theorem 1.2 we can assume that

$Q(0) \neq 0$ then $(SQ)(0) \neq 0$ and we can multiply (2.4) by $z^{m+n+1}/(SQ)(z)$ and deduce

$$f(z) - [m/n](z) = \frac{1}{2\pi i} \int_{|t|=r} \frac{(fS)(t)}{t-z} \frac{Q(t)}{(SQ)(z)} \left(\frac{z}{t}\right)^{m+n+1} dt, \qquad |z| < r.$$

□

Lemma 2.3. (Special case of Hurwitz' Theorem) Let $\{g_n(z)\}_{n=1}^{\infty}$ and $g(z)$ be functions analytic in $|z| < r$ such that

$$\lim_{n\to\infty} g_n(z) = g(z)$$

uniformly in compact subsets of $|z| < r$. If $|z_0| < r$ and z_0 is a zero of multiplicity $\geq s$ of g_n, n large enough, then z_0 is a zero of multiplicity $\geq s$ of g also.

Proof. Firstly

$$g(z_0) = \lim_{n\to\infty} g_n(z_0) = 0.$$

Next, choosing Γ to be a circle centre z_0 contained in $|z| < r$, we have uniformly for $t \in \Gamma$,

$$g(t) = \lim_{n\to\infty} g_n(t),$$

so for each fixed k, $k = 1, 2, 3, ..., s-1$, by Cauchy's integral formula for derivatives,

$$g^{(k)}(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(t)}{(t-z_0)^{k+1}} dt$$

$$= \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\Gamma} \frac{g_n(t)}{(t-z_0)^{k+1}} dt$$

$$= \lim_{n\to\infty} g_n^{(k)}(z_0)$$

$$= 0,$$

i.e. z_0 is a zero of multiplicity $\geq s$ of g.

□

Now, we are raedy to prove de Montessus de Ballore theorem.

Proof of Theorem 2.1 : Write

$$[m/n](z) = \frac{P_m(z)}{Q_m(z)}.$$

Let us normalize Q_m so that $\|Q_m\| = 1, m \geq 1$, i.e.

$$\max_{|t|=R} |Q_m(t)| = 1, \qquad m \geq 1. \tag{2.5}$$

(If $R = \infty$, we replace it by some large positive number R.)

Let $0 < s < r < R$ be such that all poles of $f(z)$ lie inside$\{z : |z| < s\}$ .Then by Lemma 2.2,

$$f(z) - [m/n](z) = \frac{1}{2\pi i} \int_{|t|=r} \frac{(fS)(t)}{t-z} \frac{Q_m(t)}{(SQ_m)(z)} \left(\frac{z}{t}\right)^{m+n+1} dt, \qquad |z| < r,$$

i.e.

$$(fSQ_m)(z) - (SP_m)(z) = \frac{1}{2\pi i} \int_{|t|=r} \frac{(fS)(t)}{t-z} Q_m(t) \left(\frac{z}{t}\right)^{m+n+1} dt, \qquad |z| < r,$$

where $S(z)$ is a monic polynomial of degree n such that $(fS)(z)$ is analytic in D_R.

Then

$$\begin{aligned}
&\max_{|z|\leq s} |(fSQ_m)(z) - (SP_m)(z)| \\
&\leq \frac{1}{2\pi} \cdot 2\pi r \cdot \frac{\max_{|t|=r} |fS|(t)}{r-s} \cdot \max_{|t|=r} |Q_m(t)| \cdot \left(\frac{s}{r}\right)^{m+n+1} \\
&\leq \frac{r}{r-s} \max_{|t|=r} |fS|(t) \left(\frac{s}{r}\right)^{m+n+1},
\end{aligned}$$

by (2.5) and the Maximum-Modulus Principle and $0 < s < r$. Then

$$\max_{|z|\leq s} |(fSQ_m)(z) - (SP_m)(z)| \leq c_1 \left(\frac{s}{r}\right)^{m+n+1},$$

where

$$c_1 = \frac{r}{r-s} \max_{|t|=r} |fS|(t)$$

is independent of m. Since r may be made arbitarily close to R, we deduce that

$$\limsup_{m\to\infty}\left[\max_{|z|\le s}|(fSQ_m)(z)-(SP_m)(z)|\right]^{1/m}\le\frac{s}{R}. \tag{2.6}$$

Now, $\{Q_m\}_{m=1}^{\infty}$ is a sequence of polynomials of degree$\le n$ satisfying (2.5). so it is contained in a closed bounded (so is compact) subset of the set of polynomials of degree$\le n$. It follows that given any infinite subsequence $\aleph$ of the positive integers, it contains another infinite subsequence $\aleph_0$ and there is a polynomial Q of degree$\le n$ with $\|Q\|=1$ such that

$$\lim_{m\to\infty,m\in\aleph_0}\|Q_m-Q\|=0. \tag{2.7}$$

Then we have for $\forall s<R$,

$$\begin{aligned}&\max_{|z|\le s}|(fSQ)(z)-(SP_m)|\\ &\le\max_{|z|\le s}|(fS)(z)(Q(z)-Q_m(z))|+\max_{|z|\le s}|(fSQ_m)(z)-(SP_m)(z)|.\end{aligned}$$

Thus,

$$\lim_{m\to\infty,m\in\aleph_0}\max_{|z|\le s}|(fSQ)(z)-(SP_m)(z)|=0, \tag{2.8}$$

by (2.6) and (2.7).

Now, if z_0 is a zero of S of multiplicity ℓ, then z_0 is a zero of SP_m of multiplicity $\ge\ell$. By (2.8) and lemma 2.3, the same is true for fSQ. But $(fS)(z_0)\neq 0$ (by choice of S), so $Q(z)$ has a zero at z_0 of multiplicity $\ge\ell$. Since this is true for each $z_j, 1\le j\le n$, and $\deg(Q)\le n, \|Q\|=1$, then we deduce that

$$Q=cS,\qquad \textit{for some}\quad c\neq 0,\quad c\in C.$$

Thus, as $m\to\infty, m\in\aleph_0$, the poles of $[m/n](z)$ (the zeros of $Q_m(z)$) approach the poles of $f(z)$ (the zeros of $S(z)$, i.e. the zeros of $Q(z)$) according to multiplicity.

Since $\aleph$ was arbitrary, it follows that for the full sequence of positive integers, the poles of $[m/n](z)$ approach the poles of $f(z)$ as $m\to\infty$ according to multiplicity.

Now, rewriting

$$[m/n](z)=\frac{\hat{P}_m(z)}{\hat{Q}_m(z)},$$

where $\widehat{Q}_m(z)$ is a monic polynomial, we then obtain as $m \to \infty$, the zeros of $\widehat{Q}_m$ approach the zeros of $S(z)$, according to multiplicity. Then

$$\lim_{m\to\infty} \widehat{Q}_m(z) = S(z) \tag{2.9}$$

uniformly in compact subsets of C.

In particular, $\left\{\widehat{Q}_m\right\}_{m=1}^{\infty}$ is uniformly bounded in each compact subset of $\mathbb{C}$. Then (2.6) holds with Q_m, P_m replaced by $\widehat{Q}_m$ *and* $\widehat{P}_m$. That is for $\forall\ 0 < s < R$,

$$\limsup_{m\to\infty} \left[\max_{|z|\le s} \left|\left(fS\widehat{Q}_m\right)(z) - \left(S\widehat{P}_m\right)(z)\right|\right]^{1/m} \le \frac{s}{R}. \tag{2.10}$$

If now K is a compact subset of $|z| < s$ omitting poles of f, then there exists a constant $\delta_k > 0$, depending only on K and $S(z)$, such that

$$\min_{z\in K} |S(z)| \ge 2\delta_k.$$

From (2.9), for m large enough,

$$\min_{z\in K} \left|\widehat{Q}_m(z)\right| \ge \frac{1}{2} \min_{z\in K} |S(z)| \ge \delta_k > 0.$$

Then for m large enough,

$$\min_{z\in K} \left|(S\widehat{Q}_m)(z)\right| \ge 2\delta_k^2.$$

Dividing by $\left|S\widehat{Q}_m(z)\right|$ in (2 10) yields for m large enough and $s < r < R$,

$$\max_{z\in K} |f(z) - [m/n](z)| \le \frac{c_1}{2\delta_k^2}\left(\frac{s}{r}\right)^{m+n+1}$$

$$=: c_2\left(\frac{s}{r}\right)^{m+n+1},$$

where $c_2 := c_1/(2\delta_k^2)$ is independent of m. Hence

$$\lim_{m\to\infty} \max_{z\in K} |f(z) - [m/n](z)| = 0,$$

$$\text{i.e.} \qquad \lim_{m\to\infty}[m/n](z) = f(z)$$

uniformly in compact subsets of D_R omtting poles of f.

Since $\hat{Q}_m(0) \neq 0$ for m large enough, then (2.1) follows.

□

2.2 Results on the poles

If $n = 0$, de Montessus de Ballore theorem tell us that

$$\lim_{m\to\infty}[m/0](z) = \lim_{m\to\infty}\sum_{j=0}^{m} a_j z^j = f(z), \qquad |z| < R,$$

i.e. the Maclaurin series of $f(z)$ converges in the largest circle centre 0 in which $f(z)$ is analytic. So de Montessus de Ballore theorem generalizes the classical conergence theorem for Taylor series. Even more, de Montessus de Ballore theorem shows how to analytically continue a function $f(z)$ from its Maclaurin series into the largest circle centre 0, in which f(z) has n poles.

An essential feature of de Montessus de Ballore theorem is that the number of poles given to the approximants matches exactly the number of the total multiplicity of poles of f. When the approximants are allowed more poles than needed to mimic the behavior of f, these extra poles may wander throughout the domain of meromorphy, preventing pointwise convergence.

Example 2.4. (Perron , 1954) Let

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

be entire. Then $\{[m/0]\}_{m=0}^{\infty}$ converges to $f(z)$ throughout $\mathbb{C}$. But $\{[m/1]\}_{m=0}^{\infty}$ which has only one pole, need not converge uniformly in any open subset of $\mathbb{C}$.

Recall from (1.11), if $a_m \neq 0, a_j \in \mathbb{C}, j = 0, 1, 2, ..., m+1$,

$$[m/1](z) = \sum_{j=o}^{m-1} a_j z^j + \frac{a_m z^m}{1 - a_{m+1}z/a_m}, \qquad m \geq 1,$$

which has a pole at $z = a_m/a_{m+1}$ if $a_{m+1} \neq 0$. Choosing $\{a_m\}_{m=0}^{\infty}$ suitably, we may ensure that every point z_0 in $\mathbb{C}$ is a limit point of poles of $\{[m/1]\}_{m=1}^{\infty}$, so that no matter which $z_0 \in \mathbb{C}$ we choose, $\{[m/1]\}_{m=1}^{\infty}$ is not bounded in any neighborhood of z_0 and cannot converge uniformly in any neighborhood of z_0. To see this, let $\{u_j\}_{j=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$ be dense in $\mathbb{C}$, and let each u_j, $j \geq 1$ be repeated infinitely often in the sequence. We may assume that

$$0 < |u_j| \leq j, \qquad j = 1, 2, 3, \cdots.$$

Define

$$a_{2m} := \frac{u_m}{(2m+1)!}; \qquad a_{2m+1} := \frac{1}{(2m+1)!}, \qquad m = 1, 2, 3, \cdots.$$

Then for $m \geq 1$,

$$|a_{2m}| \leq \frac{m}{(2m+1)!} \leq \frac{1}{(2m+1)!},$$

$$|a_{2m+1}| \leq \frac{1}{(2m+1)!}.$$

So by comparison with

$$e^z = \sum_{m=0}^{\infty} \frac{z^m}{m!},$$

$f(z)$ is entire.

Further, $[2m/1](z)$ has a pole at

$$z = \frac{a_m}{a_{2m+1}} = u_m, \qquad m = 1, 2, 3, \cdots.$$

Thus every point in the plane is a limit point of poles of $\{[m/1]\}_{m=1}^{\infty}$ and the sequence cannot converge at any u_m, $m \geq 1$.

□

However, in 1968, Beardon showed that at least a subsequence of the $\{[m/1]\}_{m=1}^{\infty}$ to an entire function converges locally uniformly throughout $\mathbb{C}$. In 1977, Baker and Graves-Morris conjectured that even when $f(z)$ has $n_0 < n$ poles in $|z| < R$, at least a subsequence of $\{[m/n]\}_{m=1}^{\infty}$ should converge uniformly in compact subsets of $|z| < R$. In 1984, Buslaev, Gončar and Suetin established the conjecture for $R = \infty$, and gave a counter example for $R < \infty$. For

the case $R < \infty$, they showed that the conjecture was still true in some neighborhood of zero. See Lubinsky and Saff [14] for certain related results.

If f has $n_0 > n$ poles in $|z| < R$, according to de Montessus de Ballore theorem, the sequence $\{[m/n]\}_{m=1}^{\infty}$ converges in compact subsets inside the circle containing exactly n poles of f omitting poles of f, but it diverges outside this circle.

Example 2.5 . Let

$$f(z) = (1 - z^2)^{-1}.$$

Then inside $|z| < 1$, $f(z)$ is analytic and the $[m/0]$ Padé approximants converge. Moreover

$$[m/0](z) = \sum_{j=0}^{m} z^{2j}, \qquad |z| < 1.$$

In $|z| \leq R$ with $R > 1$, the $[m/2]$ Padé approximants converge. i.e.

$$\lim_{m \to +\infty} [m/2](z) = f(z)$$

uniformly in compact subsets of $|z| < R$ $(R > 1)$ omitting $z = 1$ and $z = -1$.

But in $|z| \leq R$ with $R > 1$, the $[m/1]$ Padé approximants are Maclaurin polynomials if m is odd, and these diverge for $|z| > 1$.

□

The fact that the poles of $[m/n]$ approach the poles of f geometrically fast, as $m \to \infty$, is useful in numerical analysis for finding poles of functions from their Maclaurin series.

Chapter 3

Extensions of de Montessus de Ballore theorem

In the 1930's, R.Wilson succeeded in obtaining some difficult extensions of de Montessus de Ballore theorem. After him, many extensions were obtained. The extensions to multipoint Padé approximants were given firstly by Saff (1972), and then by Gončar (1975), Warner (1976) and Wallin (1978) amongst others.

In this chapter, some extensions of de Montessus de Ballore theorem will be stated, mostly without proof. We cannot hope in this short research report to discuss all of the extensions. See [15] for others.

3.1 Multipoint Padé Approximants

Let

$$\begin{array}{cccc} x_{11} & & & \\ x_{21} & x_{22} & & \\ x_{31} & x_{32} & x_{33} & \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

denote a triangular system of (not necessarily distinct) interpolation points.

Let

$$\Lambda(\ell) := \{x_{\ell 1}, x_{\ell 2}, ..., x_{\ell \ell}\}, \qquad \ell \geq 1, \tag{3.1}$$

and define the associated polynomials

$$W_\ell(z) := \prod_{j=1}^{\ell}(z - x_{\ell j}), \qquad \ell \geq 1. \tag{3.2}$$

We have

Definition 3.1. If f is analytic in $\Lambda(m+n+1)$, the *multipoint Padé approximant*

$$R_{mn}(f, \Lambda(m+n+1); z) = \frac{P(z)}{Q(z)}$$

is a rational function of type (m,n) such that

$$(f(z)Q(z) - P(z))/W_{m+n+1}(z)$$

is analytic in $\Lambda(m+n+1)$, where $\Lambda(m+n+1)$ and $W_{m+n+1}(z)$ are defined by (3.1) and (3.2).

In the special case that all interpolation points lie at 0, we have

$$R_{mn}(f, \Lambda(m+n+1); z) = [m/n](z).$$

If $f(z)$ is analytic in $\Lambda(m+n+1)$, then there exists a unique multipoint Padé approximant to $f(z)$ on $\Lambda(m+n+1)$.

To describe the extension of de Montessus de Ballore theorem to multipoint Padé approximants, we need some preliminaries.

Let E be a compact set and M=M(E) be the set of all normalized measures on E. If $\Lambda(\ell) \subset$E, $\ell \geq 1$, then *the normalized counting measure* in M(E) is

$$\nu_\ell := \frac{1}{\ell}\sum_{j=1}^{\ell}\delta_{x_{\ell j}}, \qquad \ell \geq 1, \tag{3.3}$$

i.e. ν_ℓ places mass k/ℓ at each interpolation point $x_{\ell j}$ which appears k times in $\Lambda(\ell)$.

Let ν be a non-negative unit Borel measure on a compact subset E of $\mathbb{C}$. Define the *associated potential function*

$$U(z;\nu) := \int_E \log|z-t|^{-1}\, d\nu(t), \qquad z \in \mathbb{C}, \tag{3.4}$$

and the sets

$$E(q) := \left\{ z \in \mathbb{C} : U(z;\nu) > \log\frac{1}{q} \right\}, \qquad q > 0. \tag{3.5}$$

Note that U(z;ν) is continuous in $\mathbb{C}\setminus$E with limit -∞ as $|z| \to \infty$. Furthermore, U(z;ν) is lower semi-continuous in $\mathbb{C}$, that is for all $z \in \mathbb{C}$,

$$\liminf_{\xi \to z} U(\xi;\nu) \geq U(z;\nu).$$

It follows that the sets $E(q)$ $(q > 0)$ are open, and obviously also increase with q.

The *logarithmic energy* or the *energy integral* of ν on E is

$$I[\nu] := \int_E \int_E \log|z-t|^{-1}\, d\nu(t) d\nu(z) = \int_E U(z;\nu) d\nu(z). \tag{3.6}$$

Let

$$V(E) := \inf_{\nu \in M(E)} I[\nu],$$

then there exists a unique $\widehat{\nu} \in M(E)$, called *equilibrium measure*, such that

$$V(E) = I[\widehat{\nu}],$$

and the *corresponding equilibrium potential* is

$$U(z;\widehat{\nu}) = \int_E \log|z-t|^{-1}\, d\widehat{\nu}(t), \qquad z \in \mathbb{C}.$$

The *logarithmic capacity* of E is

$$cap(E) := \exp(-V(E)). \tag{3.7}$$

If V(E)=∞, then cap(E)=0.

If $\alpha > 0$ and $E \subset \mathbb{C}$, the *α-dimensional Hausdorff content (or outer measure)* of E is

$$m_\alpha(E) := \inf \sum_{j=1}^{\infty} (d(B_j))^\alpha, \tag{3.8}$$

where the inf is taken over all covers of E by open balls $\{B_j\}_{j=1}^{\infty}$ with diameters $\{d(B_j)\}_{j=1}^{\infty}$.

Let ν be a non-negative unit Borel measure sequence defined by (3.3), we say that $\{\nu_m\}_{m=1}^{\infty}$ is $(\nu, E)-$ *regular* if

$$\liminf_{m\to\infty} U(z;\nu_m) \geq U(z;\nu), \qquad z \in \mathbb{C}, \tag{3.9}$$

and

$$\lim_{m\to\infty} U(z;\nu_m) = U(z;\nu), \qquad z \in \mathbb{C}\setminus E. \tag{3.10}$$

Now, we state (without proof) some lemmas that will be needed in proof of the extension of de Montessus de Ballore theorem to multipoint Padé approximants, some of the proofs can be found in Hille [7], others in [15].

Lemma 3.2. (Gončar's lemma) Let $\Omega \subset \mathbb{C}$ be open, let $\{f_m\}_{m=1}^{\infty}$ and f be functions from Ω to $\mathbb{C} \cup \{\infty\}$. Assume that $f_m \to f$ in m_1- measure as $m \to \infty$, that is, $\forall \varepsilon > 0$,

$$m_1\{z \in \Omega : |f_m(z) - f(z)| > \varepsilon\} \to 0, \qquad as\ m \to \infty.$$

(a) If $\{f_m\}_{m=1}^{\infty}$ are analytic in Ω, then after redefinition on a set of m_1 measure 0, so is f, and

$$\lim_{m\to\infty} f_m(z) = f(z) \tag{3.11}$$

uniformly in compact subsets of Ω.

(b) If $\{f_m\}_{m=1}^{\infty}$ are meromorphic in Ω, with poles of total multiplicity at most $n < \infty$, then after redefinition on a set of m_1 measure 0, so is f.

(c) If $\{f_m\}_{m=1}^{\infty}$ are meromorphic in Ω, with poles of multiplicity at most $n < \infty$, while f is meromorphic in Ω, with poles of total multiplicity precisely n, then for m large enough, f_m has precisely n poles, counting multiplicity. Further, as $m \to \infty$, the poles of f_m converge to

the poles of f, according to multiplicity, while (3.11) holds uniformly in compact subsets of Ω omitting poles of f.

We remark that Lemma 3.2 (c) may be proved by applying Hurwitz' theorem (Lemma 2.3) much as we did in proving the classical de Montessus de Ballore theorem.

Lemma 3.3. Let $\{\mu_m\}_{m=1}^{\infty}$ and μ be non-negative unit Borel measure on a compact set $E \subset \mathbb{C}$, such that

$$\mu_m \xrightarrow{*} \mu, \qquad m \to \infty.$$

(i.e.

$$\int_E f \, d\mu_m \longrightarrow \int_E f \, d\mu, \qquad m \to \infty,$$

for every continuous function $f : E \to R$.)

(a) Let K be compact and suppose that for $\alpha \in R$ (the set of real numbers),

$$U(z;\mu) > \alpha, \qquad \forall z \in K. \tag{3.12}$$

Then there exists a m_0 such that

$$U(z;\mu_m) > \alpha, \qquad \forall z \in K, \qquad \forall m > m_0. \tag{3.13}$$

(b) Uniformly in compact subsets of $\mathbb{C}$ /E, we have

$$\lim_{m \to \infty} U(z;\mu_m) = U(z;\mu). \tag{3.14}$$

In particular, $\{\mu_m\}_{m=1}^{\infty}$ is $(\mu, E)-$ regular.

Lemma 3.4. Let $\{\mu_m\}_{m=1}^{\infty}$ and μ be non-negative unit Borel measure on a compact set E, such that $\{\mu_m\}_{m=1}^{\infty}$ is $(\mu, E)-$ regular. Then the conclusions (a) and (b) of Lemma 3.3 are valid.

Lemma 3.5. Let $F \subset \mathbb{C}$ be compact and containing the sets of interpolation points

$\Lambda(m), m \geq 1$, let q>0 be such that f is analytic in an open set containing $E(q) \cup E$, except for poles $z_1, z_2, ..., z_n$ in $E(q)$, (repeated according to multiplicity), none of which is an interpolation point, let

$$S(z) := \prod_{j=1}^{n} (z - z_j), \tag{3.15}$$

and let $0 < r < q$ be such that E(r) contains all of $z_1, z_2, ..., z_n$. Let O be an open set containing $\overline{E(r)} \cup E$ in which f is analytic except for $z_1, z_2, ..., z_n$, and let Γ be a closed contour in $O \setminus (\overline{E(r)} \cup E)$ with winding number 1 for each point in $\overline{E(r)} \cup E$, and winding number 0 for each point in $\mathbb{C} \setminus O$. Write

$$R_{mn}(f, \Lambda(m+n+1); z) = \frac{P_{mn}(z)}{Q_{mn}(z)},$$

and let $W_{m+n+1}(z)$ be defined by (3.2). Then for z inside Γ,

$$S(z)f(z)Q_{mn}(z) - S(z)P_{mn}(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{S(t)f(t)Q_{mn}(t)}{t - z} \frac{W_{m+n+1}(z)}{W_{m+n+1}(t)} dt. \tag{3.16}$$

Now, we are ready to state and prove the extension of de Montessus de Ballore theorem to the multipoint Padé approximants due to Wallin (1981). It is also an extension of a result of Warner (1976), and of Saff (1972).

Theorem 3.6. Let $E \subset \mathbb{C}$ be compact, and let

$$\Lambda(m) \subset E, \qquad m = 1, 2, 3, ... \tag{3.17}$$

be the sets of interpolation. Let ν be a non-negative Borel measure on E such that $\{\nu_m\}_{m=1}^{\infty}$ (defined by (3.3)) is $(\nu, E)-$ regular. Let there exist q>0 such that f is analytic in an open set containing $E(q) \cup E$, except for poles $z_1, z_2, ..., z_n$ in E(q) (repeated according to multiplicity), none of which is an interpolation point. Let

$$Q(z) := \prod_{j=1}^{n} (z - z_j),$$

)nd

$$R_{mn}(z) := R_{mn}(f, \Lambda(m+n+1); z), \qquad m \geq 1.$$

(a) Then for m large enough, $R_{mn}(z)$ has exactly n poles, $z_{m1}, z_{m2}, ..., z_{mn}$ (repeated according to multiplicity). Suitably ordered, they satisfy

$$\limsup_{m\to\infty} |z_{mj} - z_j|^{1/m} < 1, \qquad j = 1, 2, ..., n. \tag{3.18}$$

Furthermore, define

$$\hat{Q}_{mn}(z) := \prod_{j=1}^{n} (z - z_{mj}),$$

and

$$\Delta(z) := \limsup_{m\to\infty} \left|\hat{Q}_{mn}(z)\right|^{1/m}, \qquad z \in \mathbb{C}. \tag{3.19}$$

We have for $1 \leq j \leq n$,

$$\Delta(z_j) \leq \frac{s_j}{q}, \tag{3.20}$$

where

$$s_j := \inf\{s : z_j \in E(s)\}. \tag{3.21}$$

(b) If $0 < s < q$ and K is a compact subset of E(s) omitting poles of f, then

$$\limsup_{m\to\infty} \|f - R_{mn}\|^{1/m}_{L_\infty(K)} \leq \frac{s}{q}. \tag{3.22}$$

Note that if none of the poles of f is a limit point of interpolation points, then for m large enough, the poles of R_{mn} cannot coincide with any interpolation point, and so R_{mn} solves the Hermite interpolation problem for f in $\Lambda(m+n+1)$.

Proof of Theorem 3.6 : We split the proof into four steps.

Step 1. Write

$$R_{mn}(z) = R_{mn}(f, \Lambda(m+n+1); z) := \frac{P_{mn}(z)}{Q_{mn}(z)}.$$

Choose R>0 such that the disk $|z| \le R$ contains $\overline{E(q)} \cup E$, and write

$$Q_{mn}(z) = \prod_{|z_{mj}|\le 2R} (z - z_{mj}) \prod_{|z_{mj}|>2R} (1 - \frac{z}{z_{mj}}), \tag{3.23}$$

where $z_{mj}, 1 \le j \le n$ are the poles of R_{mn}. Then $\{Q_{mn}\}_{m=1}^{\infty}$ is uniformly bounded in compact subsets of $\mathbb{C}$.

Choose $0 < s < r < q$ and Γ as in Lemma 3.5. Then (3.16) shows that

$$\max_{z\in\overline{E(s)}} |Q(z)f(z)Q_{mn}(z) - Q(z)P_{mn}(z)| \le c_1 \max_{z\in\overline{E(s)},t\in\Gamma} \left|\frac{W_{m+n+1}(z)}{W_{m+n+1}(t)}\right|, \tag{3.24}$$

where c_1 is independent of m.

Step 2. Let

$$x_j := x_{m+n+1,j}, \qquad j = 1, 2, ..., m+n+1.$$

Since $\{\nu_m\}_{m=1}^{\infty}$ is $(\nu, E)-$ regular, from (3.2) and (3.4), we have

$$\left|\frac{W_{m+n+1}(z)}{W_{m+n+1}(t)}\right| = \left(\prod_{j=1}^{m+n+1} |z - x_j|\right) \div \left(\prod_{j=1}^{m+n+1} |t - x_j|\right)$$

$$= \exp\left(\sum_{j=1}^{m+n+1} \log|t - x_j|^{-1} - \sum_{j=1}^{m+n+1} \log|z - x_j|^{-1}\right)$$

$$= \exp\left[(m+n+1)\left(\int_E \log|t - x_j|^{-1}\, d\nu_{m+n+1}(t) - \int_E \log|z - x_j|^{-1}\, d\nu_{m+n+1}(z)\right)\right]$$

$$= \exp\left[(m+n+1)\left(U(t;\nu_{m+n+1}) - U(z;\nu_{m+n+1})\right)\right],$$

so

$$\left|\frac{W_{m+n+1}(z)}{W_{m+n+1}(t)}\right|^{1/(m+n+1)} = \exp\left(U(t;\nu_{m+n+1}) - U(z;\nu_{m+n+1})\right). \tag{3.25}$$

Now, by (3.5), the definition of E(s),

$$U(z;\nu) \ge \log\frac{1}{s}, \qquad z \in \overline{E(s)},$$

and since $\Gamma \subset \mathbb{C}\setminus E(r)$, so

$$U(t;\nu) \le \log\frac{1}{r}, \qquad t \in \Gamma.$$

Then since also $\Gamma \subset \mathbb{C}\setminus E$, Lemma 3.3 shows that given $0 < \varepsilon < 1$, there exists m_0 such that for $m \ge m_0$,

$$U(z;\nu_{t_m}) > \log\frac{1-\varepsilon}{s}, \qquad z \in \overline{E(s)},$$

and

$$U(t;\nu_m) \le \log\frac{1+\varepsilon}{r}, \qquad t \in \Gamma.$$

Hence for $m \ge m_0$,

$$\max_{z\in\overline{E(s)},t\in\Gamma}\left|\frac{W_{m+n+1}(z)}{W_{m+n+1}(t)}\right| \le \left(\frac{(1+\varepsilon)s}{(1-\varepsilon)r}\right)^{m+n+1}. \tag{3.26}$$

We may assume that ε is so small that

$$\eta := \frac{(1+\varepsilon)s}{(1-\varepsilon)r} < 1. \tag{3.27}$$

Step 3. Let $0 < \delta < 1$, and let B_{mn} be the union of the at most 2n balls of radius $\frac{\delta}{4(n+1)}$ centred on the zeros of QQ_{mn}. Then

$$m_1(B_{mn}) \le \frac{2\delta n}{4(n+1)} \le \frac{\delta n}{2(n+1)} < \delta.$$

Since the choice of R ensures that

$$\left|1-\frac{z}{z_{mj}}\right| \ge \frac{1}{2}, \qquad |z_{mj}| > 2R, \qquad z \in \overline{E(s)},$$

then from (3.23), we have

$$|Q(z)Q_{mn}(z)| \ge \left(\frac{\delta}{8(n+1)}\right)^{2n}, \qquad z \in \overline{E(s)}\setminus B_{mn}. \tag{3.28}$$

Combining (3.24), (3.26), and (3.28) yields for $m \ge m_0$,

$$|f(z) - R_{mn}(z)| \leq c_2 \eta^m, \qquad z \in \overline{E(s)} \backslash B_{mn}, \tag{3.29}$$

where c_2 depends on δ and n but not on m. Thus $R_{mn} \to f$ in m_1-measure in $\overline{E(s)}$ as $m \to \infty$.

Assuming, as we can, that $\{z_j\}_{j=1}^n \subset E(s)$, Lemma 3.2 implies that for m large enough, R_{mn} has exactly n poles, counting multiplicity, and these poles converge to the poles of f as $m \to \infty$, counting multiplicity.

If K is a compact subset of E(s) omitting poles of f, then for m large enough,

$$K \cap B_{mn} = \phi,$$

and so (3.29) holds uniformly for $z \in K$. As we may choose r arbitrarily close to q, (3.29) and (3.27) give

$$|f(z) - R_{mn}(z)|^{1/m} \leq c_2^{1/m} \eta = c_2^{1/m} \frac{(1+\varepsilon)s}{(1-\varepsilon)q}, \qquad z \in K.$$

Since $\varepsilon > 0$ is arbitrary and c_2 is independent of m, then

$$\limsup_{m \to \infty} \|f - R_{mn}\|_{L_\infty(K)}^{1/m} \leq \frac{s}{q}.$$

This completes the proof of (b).

Step 4. We proceed to the proof of (3.20), which immediately implies (3.18).

Note first that for m large enough, all zeros of Q_{mn} (poles of R_{mn}) lie in E(q), but $|z| \leq R$ contains $\overline{E(q)} \cup E$ (by choice of R), then all zeros of Q_{mn} lie in $|z| \leq 2R$. So our normalisation (3.23) ensure that for m large enough,

$$Q_{mn}(z) = \prod_{j=1}^{n} (z - z_{mj}) = \hat{Q}_{mn}(z).$$

Fix $1 \leq j \leq n$ and choose $0 < s < r < q$ with $z_j \in E(s)$, then $(fQ)(z) \neq 0$, for the order of pole of f at z_j is exactly matched by the multiplicity of the zero of Q there. Setting $z = z_j$ in (3.24) and using (3.26) leads to the estimate

$$|(fQ)(z_j) Q_{mn}(z_j)| \leq c_2 \eta^m,$$

where c_2 is independent of m and η is given by (3.27). Then

$$|Q_{mn}(z_j)| \leq \frac{c_2}{|fQ|(z_j)}\eta^m =: c_3\eta^m,$$

where c_3 is independent of m.

As we may choose s arbitrarily close to s_j, which is given by (3.21), r arbitrarily close to q, and ε arbitrarily small, we obtain

$$\begin{aligned}\Delta(z_j) &= \limsup_{m\to\infty} \left|\widehat{Q}_{mn}(z_j)\right|^{1/m} \\ &= \limsup_{m\to\infty} |Q_{mn}(z_j)|^{1/m} \\ &\leq \limsup_{m\to\infty} c_3^{1/m}\eta \\ &\leq \frac{s_j}{q}.\end{aligned}$$

Hence we have (3.20), then (3.18) follows.

□

As a special case of Theorem 3.6, we can deduce the first de Montessus de Ballore theorem for multipoint Padé approximants, due to Saff (1972).

Corollary 3.7. Let $E \subset \mathbb{C}$ be compact with $cap(E) > 0$. Assume further that $\overline{\mathbb{C}}\setminus E$ is connected and possesses a classical Green's function $G(z)$ with a pole at infinity, i.e.

$$G(z) = V(E) - U(z;\widehat{\nu}),$$

where $U(z;\widehat{\nu})$ is the corresponding equilibrium potential and $V(E) := \inf_{\nu\in M(E)} I[\nu]$.

Let the interpolation points satisfy (3.17) and the polynomials $\{W_m\}_{m=1}^{\infty}$ of (3.2) satisfy

$$\lim_{m\to\infty} |W_m(z)|^{1/m} = cap(E)\exp(G(z)), \qquad z \in \mathbb{C}\setminus E. \tag{3.30}$$

Let

$$\widehat{E}(s) := \begin{cases} E \cup \{z \in \mathbb{C}\setminus E : G(z) < \log s\}, & s > 1, \\ E, & 0 < s \leq 1. \end{cases} \tag{3.31}$$

Suppose that there exists q>1 such that f is analytic in $\widehat{E}(q)$, except for poles $z_1, z_2, ..., z_n$ (repeated according to multiplicity), none of which is an interpolation point. Let

$$R_{mn}(z) := R_{mn}(f, \Lambda(m+n+1); z), \qquad m \geq 1.$$

Then the conclusions (a) and (b) of Theorem 3.6 remain valid if we replace E(s) by $\widehat{E}(s)$ there.

3.2 Walsh Array

Now, let us recall best uniform rational approximation. Let $E \subset \mathbb{C}$ be a compact set, $f(z)$ be analytic on E, m,n be non-negative integers. Then for each pair (m,n), there exists a rational function $W_{m,n}(z) \in R_{m,n}$, which is of best uniform approximation to $f(z)$ on E in the sense that for all rational function $r_{m,n}(z)$ of type (m,n), we have

$$\|f(z) - W_{m,n}(z)\|_{L_\infty(E)} \leq \|f(z) - r_{m,n}(z)\|_{L_\infty(E)}.$$

Definition 3.8. The *Walsh array* of a function $f(z)$ analytic on a compact set $E \subset \mathbb{C}$ is the table

$$\begin{array}{cccc} W_{0,0}(z) & W_{0,1}(z) & W_{0,2,}(z) & \cdots \\ W_{1,0}(z) & W_{1,1}(z) & W_{1,2}(z) & \cdots \\ W_{2,0}(z) & W_{2,1}(z) & W_{2,2}(z) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \tag{3.32}$$

J.L.Walsh extended the de Montessus de Ballore theorem to the Walsh array in 1965. It is a close relative of Corollary 3.7 but for the Walsh array replacing multipoint Padé approximants.

Theorem 3.9. Let $E \subset \mathbb{C}$ be compact with cap(E)>0. Assume further that $\overline{\mathbb{C}} \setminus E$ is connected and possesses a Green's function G(z) with a pole at infinity. Let $\widehat{E}(s)$ be defined

by (3.31) for s>1, i.e.

$$\widehat{E}(s) := E \cup \{z \in \mathbb{C} \setminus E : G(z) < \log s\}, \qquad s > 1.$$

Suppose that $f : E \to \mathbb{C}$ is analytic in E and for some q>1, is meromorphic in $\widehat{E}(q)$, with poles $z_1, z_2, \ldots, z_n$ (repeated according to multiplicity). Let $\{r_{m,n}(z)\}_{m=0}^{\infty}$ be a sequence of rational functions of respective types (m,n) which satisfy

$$\limsup_{m\to\infty} \|f(z) - r_{m,n}(z)\|_{L_\infty(E)}^{1/m} \le \frac{1}{q}. \tag{3.33}$$

(In particular, this holds if $\{r_{m,n}\}_{m=0}^{\infty}$ is the (n+1)th row in the Walsh array for f on E.)

Then the conclusions (a) and (b) of Theorem 3.6 remain valid if we replace E(s) by $\widehat{E}(s)$ there.

3.3 Orthogonal Padé Approximants

Throughout this section, we assume that $\alpha(x)$ is a nondecreasing function on $\Delta = [-1, 1]$, such that $\alpha'(x) > 0$ almost everywhere on Δ. We denote by $\{\phi_j(x)\}_{j=0}^{\infty}$ a sequence of orthonormal polynomials, that is, for $j \ge 0, \phi_j$ is a polynomial of degree exactly j and normalized to have positive leading coefficient and

$$\int_{-1}^{1} \phi_i(x)\phi_j(x)d\alpha(x) = \delta_{ij}, \qquad i, j = 0, 1, 2, \cdots. \tag{3.34}$$

For any $\rho > 1$, we denote by D_ρ the interior of the ellipse with foci at the points ± 1 and the sum of the semiaxes equal to ρ. That is, D_ρ is the interior of $\Gamma_\rho := \{z : |\varphi(z)| = \rho\}$, where $\varphi(z) := z + \sqrt{z^2 - 1}$, and the branch of the square root is chosen so that $|\varphi(z)| > 1$ outside Δ. The domain $D_\rho, \rho \in (1, +\infty)$, is said to be canonical (with respect to Δ).

Let $f(z)$ be an analytic function on Δ (hence f is also analytic in a neighborhood of Δ). For an arbitrary $n \ge 0$, we denote by $D_n(f)$ the maximal canonical domain in which f can be continued to a meromorphic function with at most n poles (counted according to multiplicity). Then $D_0(f)$ is the maximal canonical domain in which f can be continued as an analytic

function and f can be expanded in a Fourier series with respect to the system $\{\phi_j\}_{j=0}^{\infty}$:

$$f(z) = \sum_{j=0}^{\infty} a_j \phi_j(z), \tag{3.35}$$

where

$$a_j = \langle f, \phi_j \rangle = \int_{-1}^{1} f(x)\phi_j(x)d\alpha(x), \qquad j = 0, 1, 2, \cdots. \tag{3.36}$$

and the series (3.35) converges uniformly to f on compact subsets of $D_0(f)$.

Definition 3.10. Let m, n be non-negative integers and $f(x)$ be a real function on Δ. The *linear orthogonal Padé approximant of type (m,n) to f* is a rational function

$$R_{mn}^{\ell} := \frac{P_{mn}^{\ell}}{Q_{mn}^{\ell}} \in R_{m,n},$$

where $Q_{mn}^{\ell}(x) \neq 0$ on Δ, and for $x \in \Delta$,

$$\left(Q_{mn}^{\ell} f - P_{mn}^{\ell}\right)(x) = \sum_{k=m+n+1}^{\infty} b_k \phi_k(x), \qquad b_k \in \mathbb{C}. \tag{3.37}$$

The *non-linear orthogonal Padé approximant of type (m,n) to f* is a rational function

$$R_{mn}^{n} := \frac{P_{mn}^{n}}{Q_{mn}^{n}} \in R_{m,n},$$

where $Q_{mn}^{n}(x) \neq 0$ on Δ, and for $x \in \Delta$,

$$(f - R_{mn}^{n})(x) = \sum_{k=m+n+1}^{\infty} c_k \phi_k(x), \qquad c_k \in \mathbb{C}. \tag{3.38}$$

Since (3.37) implies

$$< Q_{mn}^{\ell} f, \phi_k >= 0, \qquad k = n+1, n+2, ..., n+m, \tag{3.39}$$

i.e.

$$\int_{-1}^{1} \left(Q_{mn}^{\ell} f \phi_k\right)(x)d\alpha(x) = 0, \qquad k = n+1, n+2, ..., n+m,$$

we have obtained a linear homogeneous system of m equations in m+1 unknowns for the coefficients of Q^{ℓ}_{mn}, it always has a non-trivial solution, and P^{ℓ}_{mn} is uniquely determined by Q^{ℓ}_{mn} :

$$P^{\ell}_{mn}(x) = \sum_{k=0}^{n} < Q^{\ell}_{mn} f, \phi_k > \phi_k(x). \tag{3.40}$$

So linear orthogonal Padé approximants to $f(z)$ exist, but need not be unique.

From (3.38), we know that the non-linear orthogonal Padé approximants to $f(z)$ need not exist, but if it exists, it is unique.

We have analogues of the de Montessus de Ballore theorem to series of linear and non-linear orthogonal Padé approximants (Suetin (1978,1980)).

Theorem 3.11. Let f have n $(0 \leq n < \infty)$ poles in the maximal canonical domain $D_n(f)$, and let $\left\{R^{\ell}_{mn}\right\}_{m=0}^{\infty}$ be the sequence of linear orthogonal Padé approximants to f. Then the following assertions are true:

(a) For sufficiently large m, the rational functions R^{ℓ}_{mn} have n finite poles (i.e. Q^{ℓ}_{mn} has degree n), the finite poles of R^{ℓ}_{mn} (the zeros of Q^{ℓ}_{mn}) tend to the poles of f in $D_n(f)$ as $m \to \infty$; and each pole of f "attracts" a number of poles of R^{ℓ}_{mn} equal to its multiplicity.

(b) The sequence $\left\{R^{\ell}_{mn}\right\}_{m=0}^{\infty}$ converges to f uniformly in compact subsets of $D_n(f)$ omitting poles of f.

Theorem 3.12. Let f have n $(0 \leq n < \infty)$ poles in the maximal canonical domain $D_n(f)$. Then for each sufficiently large m, there exists a non-linear orthogonal Padé approximant R^{n}_{mn} of type (m,n) to f. In addition

(a) The rational function R^{n}_{mn} has n finite poles; the finite poles of R^{n}_{mn} tend to the poles of f in $D_n(f)$ as $m \to \infty$, and each poles of f "attracts" as many poles of R^{n}_{mn} as its multiplicity.

(b) The sequence $\{R^{n}_{mn}\}_{m=0}^{\infty}$ converges to f unifomly in compact subsets of $D_n(f)$ omitting poles of f.

3.4 Smooth Coefficients

It is known by Padé approximators that when the Maclaurin series coefficients $\{a_j\}_{j=0}^{\infty}$ of

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

are "smooth", the Padé approximants behave well. What constitutes smoothness? In Baker[2], some examples of smoothness conditions were treated. In Lubinsky [13], smoothness was quantified by using the ratio $a_{j-1}a_{j+1}/a_j^2$. That is, if

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

is a formal power series with $a_j \neq 0$, j large enough, let

$$q_j := a_{j-1}a_{j+1}/a_j^2, \qquad j \ large \ enough. \tag{3.41}$$

We say that the coefficients $\{a_j\}_{j=0}^{\infty}$ of $f(z)$ are *smooth* if there exists some $q \in \mathbb{C}\setminus\{0\}$, such that

$$\lim_{j\to\infty} q_j = \lim_{j\to\infty} \left(a_{j-1}a_{j+1}/a_j^2\right) = q. \tag{3.42}$$

Let $\theta \in (0, 2\pi)$ be such that $\theta/(2\pi)$ is irrational, and let $q := e^{i\theta}$. Define *the partial theta function*

$$h_q(z) := \sum_{j=0}^{\infty} q^{j(j-1)/2} z^j, \qquad |z| < 1. \tag{3.43}$$

Then for $h_q(z)$,

$$a_{j-1}a_{j+1}/a_j^2 = q,$$

and we see that the partial theta function h_q has smooth coefficients.

Definition 3.13. Given $q \in \mathbb{C}$, define the *nth Rogers-Szegö polynomial*

$$G_n(z) = G_n(z;q) := \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} z^j, \tag{3.44}$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix} := \begin{cases} \frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-j+1})}{(1-q)(1-q^2)\ldots(1-q^j)}, & 1 \le j \le n, \\ 1, & j = 0, \end{cases} \tag{3.45}$$

Note that $\begin{bmatrix} n \\ j \end{bmatrix}$ is a polynomial in q, so is well defined even if q is a root of unity. In particular, if $q = 1$, $\begin{bmatrix} n \\ j \end{bmatrix} = \binom{n}{j}$.

For $n \ge 1$ and $q \in C$, we have (see Lubinsky [13])

$$G_n(z) = G_{n-1}(z) + zq^{n-1}G_{n-1}(z/q), \tag{3.46}$$

and $G_n(0) = 1$.

Theorem 3.14. Let n be a positive integer. Let

$$f(z) := \sum_{j=0}^{\infty} a_j z^j$$

be a formal power series with smooth coefficients, i.e. for some $q \in \mathbb{C} \setminus \{0\}$

$$\lim_{j\to\infty} \left(a_{j-1}a_{j+1}/a_j^2 \right) = q.$$

Assume further that either the following conditions (A) or (B) is satisfied.

(A) The number q is not a jth root of unity for $j < n$, that is

$$q^j \ne 1, \qquad j = 1, 2, 3, \ldots, n-1.$$

(B) The number q is a jth root of unity for $j < n$, but if ℓ denote the smallest such j, for some positive integer N with $n \le \ell N + 1$ and $\{c_j\}_{j=1}^{N} \subset \mathbb{C} \setminus \{0\}$,

$$q_m := a_{m-1}a_{m+1}/a_m^2 = q[1 + \sum_{j=1}^{N} c_j m^{-j} + o(m^{-N})]. \tag{3.47}$$

Let $G_n(z) = G_n(z;q)$ be the Rogers-Szegö polynomial defined by (3.44), and for $m \geq 1$, let $Q_{mn}(z)$ denote the denominator of $[m/n](z)$ normalized (if possible) by

$$Q_{mn}(0) = 1$$

Then, locally uniformly in $\mathbb{C}$,

$$\lim_{m\to\infty} Q_{mn}(ua_m/a_{m+1}) = G_n(-u). \tag{3.48}$$

Furthermore, if $u_1, u_2, \ldots, u_n$ denote the zeros of $G_n(-u)$, and for n large enough, $z_{m1}, z_{m2}, \ldots, z_{mn}$ denote the zeros of $Q_{mn}(z)$, suitably ordered, we have

$$\lim_{m\to\infty} \frac{z_{mj}}{a_m/a_{m+1}} = u_j, \qquad 1 \leq j \leq n. \tag{3.49}$$

Finally, if $\sigma := \min_j \{|u_j|\}$, and

$$R := \liminf_{m\to\infty} |a_m/a_{m+1}| > 0, \tag{3.50}$$

then

$$\lim_{m\to\infty} [m/n](z) = f(z), \qquad |z| < \sigma R. \tag{3.51}$$

From this theorem, we have that for series with "smooth" coefficients, the asymptotic behavior of the Padé rows may be completely determined. In particular, for large classes of entire functions of zero, finite,and infinite order, all the rows of the Padé table converge, and the poles of $[m/n](z)$, n=1,2,3,..., approach ∞ with rate a_m/a_{m+1} as $m \to \infty$.

For the partial theta function h_q defined by (3.43), if $m > n-1 \geq 0$ and $q^j \neq 1$, some $1 \leq j \leq n$, then the Padé denominator $Q_{mn}(z)$ normalized by $Q_{mn}(0) = 1$ satisfies

$$Q_{mn}(-zq^{-m}) = G_n(z). \tag{3.52}$$

(See Lubinsky and Saff [14].)

In this case, (3.48) is valid without the limit sign. So h_q is really the model function for smoothness in the sense (3.42).

3.5 Intermediate Rows

Let $f(z)$ be a meromorphic function in $\mathbb{C}$. Assume that:

$f(z)$ has 1 pole in $|z| < 1$,

$f(z)$ has 1 pole on $|z| = 1$,

$f(z)$ has 1 pole of order 3 on $|z| = 2$,

$f(z)$ has 2 poles in $2 < |z| < 3$.

Then by de Montessus's theorem,

$$\lim_{m\to\infty} [m/1] = f$$

uniformly in compact subsets of $|z| < 1$ omitting the pole of f;

$$\lim_{m\to\infty} [m/2] = f$$

uniformly in compact subsets of $|z| < 2$ omitting the 2 poles of f;

$$\lim_{m\to\infty} [m/5] = f$$

uniformly in compact subsets of $|z| < 3$ omitting the poles of f.

What can we say about the convergence of the sequences $\{[m/3]\}_{m=1}^{\infty}$ and $\{[m/4]\}_{m=1}^{\infty}$? Here we call $\{[m/3]\}_{m=1}^{\infty}$ and the 5th column $\{[m/4]\}_{m=1}^{\infty}$ of the Padé table for this function) *intermediate rows (columns)*.

In general, if $f(z)$ has μ poles in E(q),q>0 (defined by (3.5)) and is meromorphic with poles of total order ν ($\nu > 1$) on the boundary of E(q), there are some known convergence res) extensions of de Montessus de Ballore theorem) for the intermediate rows $\left\{R_{m(\mu+1)}(f, \Lambda(m+\mu+2); z)\right\}_{m=1}^{\infty}$ through $\left\{R_{m(\mu+\nu)}(f, \Lambda(m+\mu+\nu+1); z)\right\}_{m=1}^{\infty}$ for multi-points Padé approximants. (See Liu [10], chapter 3).

Let $E \subset \mathbb{C}$ be a compact set and $K = \overline{\mathbb{C}} \backslash E$ be connected and possess a classical Green's function $G_n(z)$ having a pole at infinity. Set

$$\Gamma_q := \{z:\ G(z) = \log q\}, \qquad q > 1.$$

$$E(q) := \textit{interior}\ (\Gamma_q). \tag{3.53}$$

$$H(z) := -\frac{\partial G(z)}{\partial x} + i\frac{\partial G(z)}{\partial y}, \quad (z = x + iy). \tag{3.54}$$

$$\Delta := cap(E), \tag{3.55}$$

where cap(E) is defined by (3.7).

Theorem 3.15. Let E be assumed as above. Let $f(z)$ be analytic on E, meromorphic with precisely μ ($\mu \geq 0$) poles in E(q) and analytic on Γ_q except for a pole at α of order τ, $H(\alpha) \neq 0$. Let the interpolation schemes be described as in (3.1) and $\Lambda(\ell) \subset E$, $\ell = 1, 2, 3, \ldots$, and the associated polynomials $W_\ell(z)$ defined by (3.2) satisfy

$$|W_\ell(z)| \leq M\Delta^\ell, \quad \textit{for } z \in E,$$

$$\textit{and} \tag{3.56}$$

$$|G(z) + \log\Delta - \ell^{-1}\log|W_\ell(z)|| \leq M\ell^{-1},$$

for z on each compact subset of K. Here M is a constant independent of ℓ and of the compact set.

Suppose α is not a critical point of $G(z)$ and $0 \leq \nu \leq \tau$. Then

(a) for m large enough, there exists a unique multipoint Padé approximant

$$R_{m(\mu+\nu)}(f, \Lambda(m+\mu+\nu+1); z) =: R_{m(\mu+\nu)}$$

of type $(m, \mu+\nu)$, which interpolates to $f(z)$ on the set $\Lambda(m+\mu+\nu+1)$. Each $R_{m(\mu+\nu)}$ has precisely $\mu+\nu$ finite poles and, as $m \to \infty$, μ of which approach the μ poles of $f(z)$ respectively in E(q), the other ν poles tend to α (counting multiplicity).

(b)

$$\lim_{m\to\infty} R_{m(\mu+\nu)}(f, \Lambda(m+\mu+\nu+1); z) = f(z) \tag{3.57}$$

uniformly in compact subsets of E(q) omitting poles of $f(z)$.

Now, applying this theorem to the question stated in the beginning of this section, we have following results:

(1) For m large enough, the Padé approximants [m/3] and [m/4] to f are unique. [m/3] (respectively [m/4]) has precisely 3 (resp. 4) poles and, as $m \to \infty$, 2 of the poles approach the 2 poles of f in $|z| < 2$ respectively, and the other 1 pole (resp. 2 poles) tends to the pole of f on $|z| = 2$.

(2)

$$\lim_{m\to\infty} [m/3] = f$$

and

$$\lim_{m\to\infty} [m/4] = f$$

uniformly in compact subsets of $|z| < 2$ omitting the 2 poles of f in $|z| < 2$.

In the case that $f(z)$ has more than one pole on the boundary of E(q), the situation is much more complicated than the case of one pole and I refer to Liu [10] for details.

3.6 Multivariate Padé Approximants

There are two types of multivariate Padé approximants, the homogeneous and non-homogeneous Padé approximants. In this section, definitions are given in the sense of Cuyt [4],[5],[6].

Let $p \geq 2$, and let $\underline{r} := (r_1, r_2, ..., r_p)$ be a p-tuple of positive numbers. Define the *polydisc*

$$B(0;\underline{r}) := \{(z_1, z_2, ..., z_p) \in \mathbb{C}^p : |z_j| < r_j,\ 1 \leq j \leq p\}. \tag{3.58}$$

If $\underline{k} := (k_1, k_2, ..., k_p)$ is a p-tuple of non-negative integers, then for $\underline{z} := (z_1, z_2, ..., z_p) \in \mathbb{C}^p$, we define

$$\underline{z}^{\underline{k}} := z_1^{k_1} z_2^{k_2} ... z_p^{k_p},$$

and say that the term $\underline{z}^k$ has degree $k := k_1 + k_2 + ... + k_p$.

Given an integer k, we denote by

$$C_k\underline{z}^k \quad or \quad A_k\underline{z}^k \quad or \quad B_k\underline{z}^k$$

the sum of terms

$$C_k\underline{z}^k := \sum_{k_1+k_2+...+k_p=k} c_{k_1k_2...k_p} z_1^{k_1} z_2^{k_2} ... z_p^{k_p},$$

$$A_k\underline{z}^k := \sum_{k_1+k_2+...+k_p=k} a_{k_1k_2...k_p} z_1^{k_1} z_2^{k_2} ... z_p^{k_p},$$

$$B_k\underline{z}^k := \sum_{k_1+k_2+...+k_p=k} b_{k_1k_2...k_p} z_1^{k_1} z_2^{k_2} ... z_p^{k_p}.$$

The *formal power series* $f(\underline{z})$ is

$$f(\underline{z}) = \sum_{i=0}^{\infty} C_k\underline{z}^k.$$

The *degree of a polynomial* $P(\underline{z})$ is the highest degree amongst its non-zero terms. The *order of a formal power series* $f(\underline{z})$, denote by $\partial_0 f(\underline{z})$ is the lowest degree amongst its non-zero terms.

Definition 3.16. Let $f(\underline{z})$ be a formal power series, and $m, n \geq 0$ be integers, *The (m,n) multivariate homogeneous Padé approximant to* f is the rational function

$$[m/n](\underline{z}) := \frac{P(\underline{z})}{Q(\underline{z})} \tag{3.59}$$

with

$$P(\underline{z}) := \sum_{j=mn}^{mn+m} A_j\underline{z}^j,$$

and

$$Q(\underline{z}) := \sum_{j=mn}^{mn+n} B_j\underline{z}^j,$$

where $Q(\underline{z})$ is not identically zero, such that

$$\delta_0(fQ - P)(\underline{z}) \geq mn + m + n + 1. \tag{3.60}$$

Cuyt has shown that $[m/n](\underline{z})$ exists and is unique and also proved the following analogue of de Montessus de Ballore theorem for multivariate Padé approximants (see Cuyt [4]):

Theorem 3.17. Suppose that f is analytic in the origin and is meromorphic in the polydisc $B(0,\underline{r})$ with pole set

$$P := \{\underline{z} \in B(0,\underline{r}) : S(\underline{z}) = 0\},$$

where S is a polynomial of degree n. Let I denote the subsequence of positive integers m, such that the irreducible form of $[m/n](\underline{z})$, say $P_m(\underline{z})/Q_m(\underline{z})$ has $Q_m(\underline{0}) \neq 0$. Assume that I is infinite. Then

$$\lim_{m\to\infty, m\in I} [m/n](\underline{z}) = f(\underline{z}) \tag{3.61}$$

uniformly in compact subsets of $B(0,\underline{r})\backslash P$, and after a suitable normalization, we have

$$\lim_{m\to\infty, m\in I} Q_m(\underline{z}) = S(\underline{z}) \tag{3.62}$$

uniformly in compact subsets of $B(0,\underline{r})$.

Now, we discuss the non-homogeneous type of multivariate Padé approximants. In order to avoid notational difficulties, we will restrict to the case of bivariate functions. The generalization to more than two variables is strightforward.

Definition 3.18. Let $f(x,y)$ be a formal power series, i.e.

$$f(x,y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j, \tag{3.63}$$

and let M,N be index sets in $\mathbb{N} \times \mathbb{N} =: \mathbb{N}^2$. Let us define m and n by

$$\#M = m+1, \quad \#N = n+1, \qquad m,n > 0. \tag{3.64}$$

The (M,N) multivariate non-homogeneous Padé approximant to $f(x,y)$ is a rational function

$$[M/N]_E(x,y) := \frac{P(x,y)}{Q(x,y)}$$

with polynomials$\mathbb{N}$

$$P(x,y) = \sum_{(i,j)\in M} a_{ij}x^i y^j,$$

$$Q(x,y) = \sum_{(i,j)\in N} b_{ij}x^i y^j,$$

and an interpolation set E such that

$$(fQ-P)(x,y) = \sum_{(i,j)\in \mathbb{N}^2\setminus E} d_{ij}x^i y^j, \tag{3.65}$$

with

$$M \subset E, \tag{3.66}$$

$$\#(E\setminus M) = \#N - 1 = n, \tag{3.67}$$

and E satisfies the inclusion property:

$$(i,j)\in E \Longrightarrow (k,l)\in E \ \ for\ \ 0\le k\le i \ \ and\ \ 0\le l\le j. \tag{3.68}$$

Clearly the equation (3.65) can be rewritten as

$$\frac{\partial^{(i+j)}(fQ-P)}{\partial x^i \partial y^j}\Big|_{(0,0)} = 0, \qquad for\ (i,j)\in E, \tag{3.69}$$

and this clarifies the terminology of interpolation set.

Condition (3.66) enable us to split the system as equations

$$d_{ij} = 0, \quad (i,j) \in E$$

in an inhomogeneous part defining the numerator coefficients

$$\sum_{\mu=0}^{i} \sum_{\nu=0}^{j} c_{\mu\nu} b_{i-\mu,j-\nu} = a_{ij}, \quad (i,j) \in M, \tag{3.70}$$

and a homogeneous part defining the denominator coefficients

$$\sum_{\mu=0}^{i} \sum_{\nu=0}^{j} c_{\mu\nu} b_{i-\mu,j-\nu} = 0, \quad (i,j) \in E \backslash M. \tag{3.71}$$

By convention,

$$b_{i,j} = 0 \quad if \ \ (i,j) \notin N. \tag{3.72}$$

Then condition (3.67) guarantees the existence of a nontrivial denominator $Q(x,y)$ because the homogeneous system has one equation less than the number of unknowns and so one unknown coefficient can be chosen freely.

Condition(3.68) finally takes care of the Padé approximation property, namely if $Q(0,0) \neq 0$, then

$$(f - \frac{P}{Q})(x,y) = \sum_{(i,j) \in \mathbb{N}^2 \backslash E} e_{ij} x^i y^j. \tag{3.73}$$

Cuyt proved a multivariate non-homogeneous type analogue of the univariate de Montessus de Ballore theorem in 1990 (Cuyt [6]), for the case of simple poles. Before stating the theorem, we introduce some notation. By the set $M * N$ we denote the index set that results from the multiplication of a polynomial indexed by M with a polynomial indexed by N,

$$M * N = \{(i+k, j+l) : (i,j) \in M, (k,l) \in N\}. \tag{3.74}$$

Since the set E satisfies the inclusion property, we can inscribe isosceles triangles in E, with top at (0,0) and base along the antidiagonals (see graphs below). Let τ be the largest of these inscribed triangles and let r_τ be the "length" of the two equal sides. We call r_τ the *range of the*

triangle τ. On the other hand, because $M * N$ is a finite subset of $\mathbb{N} \times \mathbb{N}$, we can circumscribe it with such triangles. Let T be the smallest of these circumscribing triangles and let r_T be the "length" of the two equal sides. We call r_T the *range of the triangle T.*

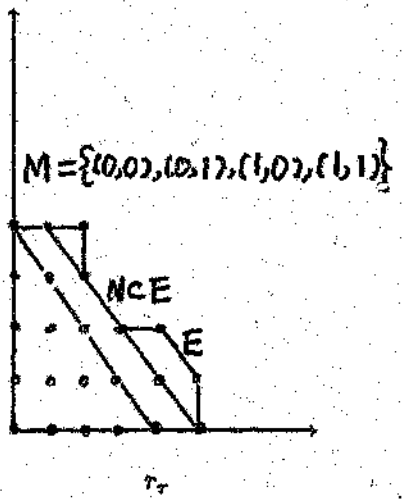

Fig. 1

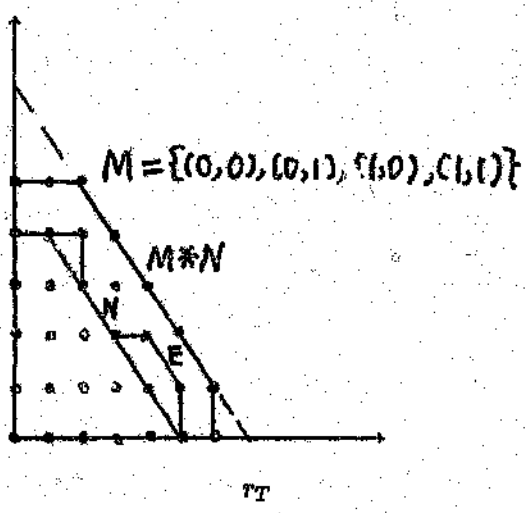

Fig. 2

Theorem 3.19. Let $f(x,y)$ be a function which is meromorphic in the polydisc

$$B(0;R_1,R_2) := \{(x,y) : |x| < R_1, |y| < R_2\},$$

meaning that there exists a polynomial

$$R_N(x,y) = \sum_{(d,e)\in N\subseteq \mathbb{N}^2} r_{de}x^d y^e = \sum_{i=0}^{n} r_{d_i e_i} x^{d_i} y^{e_i}, \tag{3.75}$$

such that $(fR_N)(x,y)$ is analytic in the polydisc above. Further, we assume that $R_N(0,0) \neq 0$ so that necessarily $(0,0)\in N$. Let there also exist n zeros $(x_h, y_h) \in B(0;R_1,R_2)$, $(1 \leq h \leq n)$ of $R_N(x,y)$ satisfying

$$(fR_N)(x_h,y_h) \neq 0, \quad h = 1,2,\ldots,n,$$

and

$$\det \begin{pmatrix} x_1^{d_1} y_1^{e_1} & \dots & x_1^{d_n} y_1^{e_n} \\ \vdots & \vdots & \vdots \\ x_n^{d_1} y_n^{e_1} & \dots & x_n^{d_n} y_n^{e_n} \end{pmatrix} \neq 0, \tag{3.76}$$

Then the Padé approximants $[M/N]_E(x,y) = (P/Q)(x,y)$ with N fixed as given above, and M and E growing, converges to $f(x,y)$ uniformly on compact subsets of

$$\{(x,y) : |x| < R_1, |y| < R_2, R_N(x,y) \neq 0\},$$

and its denominator

$$Q(x,y) = \sum_{i=0}^{n} b_{d_i e_i} x^{d_i} y^{e_i}$$

converges to $R_N(x,y)$ under the following conditions for M and E: the range of the largest inscribed triangle r_τ in E and the range of the smallest triangle circumscribing $M * N$, r_T should both tend to infinity as $n \to \infty$.

Bibliography

[1] G.A. Baker, Jr., *Essentials of Padé Approximants*, Academic Press, New York, 1975.

[2] G.A. Baker, Jr. and P. Graves-Morris, *Padé Approximants. Part 1: Basic Theory*, Encyclopedia of Mathematics and its Applications, Volume 13, Addision-Wesley, Reading, 1981.

[3] E.W.Cheney, *Introduction to Approximation Theory*, McGraw-Hill, New York, 1966.

[4] A.Cuyt, *A Montessus de Ballore Theory for Multivariate Padé Approximants*, Journal of Approximation Theory, Vol. 43, No.1, 1985, pp.43—52.

[5] A.Cuyt, *A review of Multivariate Padé Approximation Theory*, Journal of Computation... and Applied Mathematics, 12 & 13 (1985), 221—232.

[6] A.Cuyt, *A Multivariate Convergence Theorem of the "de Montessus de Ballore" Type*, Journal of Computational and Applied Mathematics, 32 (1990) 47—57.

[7] E. Hille, *Analytic Function Theory*, Vol. 2, Ginn and Company, Boston, 1962.

[8] J. Karlsson and H. Wallin, *Rational Approximation by An Interpolation Procedure in Several Variables*, (in) Padé and Rational Approximation (eds. P.R. Graves-Morris, E.B.Saff, R.S. Varga), Academic Press, New York, 1977, pp.83—100.

[9] A.L.Levin and D.S.Lubinsky, *Rows and Diagonals of the Walsh Array for Entire Functions with Smooth Maclaurin Series Coefficients*, Constr. Approx. (1990) 6: 257—286.

[10] X.Liu, *Generalization of de Montessus de Ballore's Theorem on the Row Convergence of Rational Approximants*, Ph.D. Thesis, University of South Florida, 1992.

[11] D.S.Lubinsky, *Exceptional Sets of Padé Approximants*, Ph.D. Thesis, Witwatersrand University, Johannesburg, 1980.

[12] D.S.Lubinsky and A.Sidi, *Convergence of Linear and Nonlinear Padé Approximants from Series of Orthogonal Polynomials*, Transactions of The American Mathematics Society. Vol. 278, No. 1, 1983.

[13] D.S.Lubinsky, *Uniform Convergence of Rows of the Padé Table for Functions with Smooth Maclaurin Series Coefficients*, Constr. Approx. (1987) 3: 307--330.

[14] D.S.Lubinsky and E.B.Saff, *Convergence of Padé Approximants of Partial Theta Functions and the Roger-Szegö Polynomials*, Constr. Approx. (1987) 3: 331—361.

[15] D.S.Lubinsky, E.B.Saff and H.Stahl, Book, in preparation.

[16] H.Stahl, *Existence and Uniqueness of Rational Interpolants with Free and Prescribed Poles*, Springer Lecture Notes, Vol. 1287, Springer, Berlin, 1987, pp.180—208.

[17] S.P.Suetin, *On the Convergence of Rational Approximations to Polynomial Expansions in Domains of Meromophy of A Given Function*, Math. USSR Sbornik, Vol. 34 (1978), No.3.

[18] S.P.Suetin, *On de Montessus de Ballore's Theorem for Nonlinear Padé Approximants of Orthogonal Expansions and Faber Series*, Soviet Math. Dokl. Vol. 22 (1980), No. 1.



Author:Zhou P
Name of thesis:de Montessus de Ballore Theorem for Pade Approximation

PUBLISHER:
University of the Witwatersrand, Johannesburg
©2015

LEGALNOTICES:

Copyright Notice: All materials on the University of the Witwatersrand, Johannesburg Library website are protected by South African copyright law and may not be distributed, transmitted, displayed or otherwise published in any format, without the prior written permission of the copyright owner.

Disclaimer and Terms of Use: Provided that you maintain all copyright and other notices contained therein, you may download material (one machine readable copy and one print copy per page)for your personal and/or educational non-commercial use only.

The University of the Witwatersrand, Johannesburg, is not responsible for any errors or omissions and excludes any and all liability for any errors in or omissions from the information on the Library website.