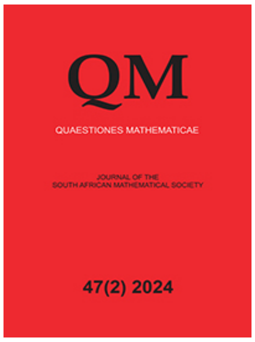

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FORM OF THE SOLUTIONS OF DIFFERENCE EQUATIONS VIA LIE SYMMETRY ANALYSIS AND FIBONACCI NUMBERS

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Abstract. In this paper, we give the form of the solutions of the following rational difference equation

$$x_{n+1} = \frac{x_{n-3}x_{n-1}}{a_n x_{n-3} + b_n x_{n-1}}, \tag{1}$$

where a_n and b_n are sequences of real numbers, by using Lie symmetry analysis and associated with Fibonacci numbers, respectively. We find an interesting relation between the exact solution of Equation (1) and the classical Fibonacci sequence.

Mathematics Subject Classification (2020): 39A10, 39A99, 39A13.

Key words: Difference equation, form of the solutions, reduction, Lie symmetry analysis, Fibonacci numbers.

1. Introduction. Recently, there has been a lot of interest in studying the dynamics of rational difference equations. There are many papers in the field of difference equations, see, [2, 3, 5, 10, 11, 22] and the related references therein. One of the reasons is that difference equations have been applied in several mathematical models in biology, economics, genetics, psychology, ecology, physics, engineering, population dynamics, etc. Obviously, the form of the solutions of rational difference equations have also been widely studied by many authors with various methods, for example, [1, 2, 3, 5, 6, 7, 8, 9, 10].

Moreover, the studies dealt with the relation between closed form of the solutions of nonlinear difference equations and linear sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Padovan, Perrin numbers have received great attention by researchers. Because some of the solution forms of these equations are even expressible in terms of well-known integer sequences (see, e.g., [12, 14, 13 15, 16, 21, 19, 20, 24, 23, 25]).

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In this paper, motivated by these interesting results, we find a fascinating relation between the exact solution of

$$x_{n+1} = \frac{x_{n-3}x_{n-1}}{a_n x_{n-3} + b_n x_{n-1}}$$

and the classical Fibonacci sequence. Then we give the form of the solutions of Equation (1) by using Lie symmetry analysis and associated with Fibonacci numbers, respectively.

2. Preliminaries.

We begin by introducing some basic definitions and some theorems needed in the sequel. For details, see [18].

Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. A difference equation of order $k+1$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \qquad n = 0, 1, \cdots. \tag{2}$$

A solution of Equation (2) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Equation (2) for all $n \geq -k$.

LEMMA 1. *For every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation (2) has a unique solution $\{x_n\}_{n=-k}^{\infty}$.*

DEFINITION 2. A point $\overline{x} \in I$ is called an *equilibrium point* of Equation (2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is,

$$x_n = \overline{x} \text{ for } n \geq 0$$

is a solution of Equation (2) or equivalently, $\overline{x}$ is a fixed point of f .

DEFINITION 3. (Stability)

(a) The equilibrium point $\overline{x}$ of Equation (2) is locally stable if for every $\varepsilon > 0$; there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \varepsilon, \quad \text{for all } n \geq -k.$$

(b) The equilibrium point $\overline{x}$ of Equation (2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Equation (2)and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(c) The equilibrium point $\overline{x}$ of Equation (2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(d) The equilibrium point $\overline{x}$ of Equation (2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Equation (2).

(e) The equilibrium point $\overline{x}$ of Equation (2) is unstable if $\overline{x}$ is not locally stable.

Suppose that the function f is continuously differentiable in some open neighborhood of an equilibrium point $\overline{x}$. We use

$$q_i = \frac{\partial f}{\partial u_i}(\overline{x}, \overline{x}, ..., \overline{x}), \quad \text{for } i = 0, 1, ..., k$$

to denote the partial derivative of $f(u_0, u_1, ..., u_k)$ with respect to u_i evaluated at the equilibrium point $\overline{x}$ of Equation (2).

DEFINITION 4. The equation

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + ... + q_k y_{n-k} = \sum_{i=0}^{k} \frac{\partial f}{\partial u_i}(\overline{x}, \overline{x}, ..., \overline{x}) y_{n-i}, \quad n = 0, 1, ... \tag{3}$$

is called the linearized equation of Equation (2) about the equilibrium point $\overline{x}$, and the equation

$$\lambda^{k+1} - q_0 \lambda^k - ... - q_{k-1}\lambda - q_k = 0 \tag{4}$$

is called the characteristic equation of Equation (3) about $\overline{x}$.

The following result, known as the *Linearized Stability Theorem*, is very useful in determining the local stability character of the equilibrium point $\overline{x}$ of Equation (2).

THEOREM 5. (The linearized stability theorem)
Assume that the function f is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\overline{x}$. Then the following statements are true:

(a) *When all the roots of Equation (4) have absolute value less than one, then the equilibrium point $\overline{x}$ of Equation (2) is locally asymptotically stable.*

(b) *If at least one root of Equation (4) has absolute value greater than one, then the equilibrium point $\overline{x}$ of Equation (2) is unstable.*

THEOREM 6. (Clark theorem) *Assume that $q_0, q_1, ..., q_k$ are real numbers such that*

$$\sum_{i=1}^{k} |q_i| = |q_0| + |q_1| + ... + |q_k| < 1.$$

Then all roots of Equation (4) lie inside the unit disk.

Using the Clark Theorem 6, we have the following result.

THEOREM 7. *Assume that* $q_i \in \mathbb{R}$, $i = 1, 2, ...$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$\sum_{i=1}^{k} |q_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation,

$$x_{n+k} + q_1 x_{n+k-1} + ... + q_k x_n = 0, n = 0, 1, \ldots.$$

THEOREM 8. *Let* $g : [a, b]^{k+1} \rightarrow [a, b]$ *be a continuous function, where* k *is a positive integer, and where* $[a, b]$ *is an interval of real numbers. Consider the difference equation*

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}), \qquad n = 0, 1, \ldots. \tag{5}$$

Suppose that g *satisfies the following conditions:*

(a) *For each integer* i *with* $1 \leq i \leq k+1$, *the function* $g(z_1, z_2, ..., z_{k+1})$ *is weakly monotonic in* z_i *for fixed* $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

(b) *If* (m, M) *is a solution of the system*

$$m = g(m_1, m_2, ..., m_{k+1}) \text{ and } M = g(M_1, M_2, ..., M_{k+1})$$

then $m = M$, *where for each* $i = 1, 2, ..., k+1$, *we set*

$$m_i = \left\{ \begin{array}{l} m, \text{ if } g \text{ is non-decreasing in } z_i \\ M, \text{ if } g \text{ is non-increasing in } z_i \end{array} \right\}$$

and

$$M_i = \left\{ \begin{array}{l} M, \text{ if } g \text{ is non-decreasing in } z_i \\ m, \text{ if } g \text{ is non-increasing in } z_i \end{array} \right\}.$$

Then there exists exactly one equilibrium $\bar{x}$ *of Equation* (5), *and every solution of Equation* (5) *converges to* $\bar{x}$.

DEFINITION 9. Let G be a local group of transformations acting on a manifold M. A subset $S \subset M$ is called G-invariant, and G is called symmetry group of S, if whenever $x \in S$, and $g \in G$ is such that $g.x$ is defined, then $g.x \in S$.

DEFINITION 10. Let G be a connected group of transformations acting on a manifold M. A smooth real-valued function $V : M \rightarrow \mathbb{R}\mathbb{R}$ is an invariant function for G if and only if

$$X(V) = 0 \quad \textit{for all} \quad x \in M, \tag{6}$$

and every infinitesimal generator X of G.

DEFINITION 11. A parameterized set of point transformations,

$$\Gamma_\varepsilon : x \rightarrow \hat{x}(x; \varepsilon), \tag{7}$$

where $x = x_i$, $i = 1, ..., p$ are continuous variables, is a one-parameter local Lie group of transformations as long as the following conditions are met:

a. Γ_0 is the identity map if $\hat{x} = x$ when $\varepsilon = 0$.

b. $\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0.

c. Each $\hat{x}_i$ can be represented as a Taylor series (in a neighborhood of $\varepsilon = 0$ that is determined by x), and therefore

$$\hat{x}_i\left(x;\varepsilon\right) = x_i + \varepsilon \xi_i\left(x\right) + O(\varepsilon^2), i:1,...,p.$$

Consider the pth-order difference equation

$$u_{n+p} = \Omega\left(u_n, ..., u_{n+p-1}\right), \tag{8}$$

for some smooth function Ω. Assume the point transformations are of the form

$$\hat{n} = n; \qquad \hat{u}_n = u_n + \varepsilon Q\left(n, u_n\right) O(\varepsilon^2) \tag{9}$$

with the corresponding infinitesimal symmetry generator

$$X = Q\left(n, u_n\right)\frac{\partial}{\partial u_n} + S^{(1)}Q\left(n, u_n\right)\frac{\partial}{\partial u_{n+1}} +,,, + S^{(p-1)}Q\left(n, u_n\right)\frac{\partial}{\partial u_{n+p-1}},$$

where $S^{(j)}$ is the shift operator, i.e., $S^{(j)}: n \to n+j$. The symmetry condition is defined as

$$\hat{u}_{n+p} = \Omega\left(n, \hat{u}_n, \hat{u}_{n+1}, ..., \hat{u}_{n+p-1}\right), \tag{10}$$

whenever (8) is true. Substituting the Lie point symmetries (9) into the symmetry condition (10) leads to the linearized symmetry condition

$$S^{(p)}Q - X\Omega = 0|_{u_{n+p}=\Omega(u_n,...,u_{n+p-1})}. \tag{11}$$

3. Exact solutions of Equation (1).

3.1. Exact solutions by using Lie symmetry analysis. Consider the higher order difference equation

$$x_{n+1} = \frac{x_{n-3}x_{n-1}}{a_n x_{n-3} + b_n x_{n-1}}, \tag{12}$$

where a_n and b_n are random sequences of real numbers. The corresponding forward shift is given by

$$u_{n+4} = \frac{u_n u_{n+2}}{A_n u_n + B_n u_{n+2}}, \tag{13}$$

where A_n and A_n are random sequences.

3.1.1. Symmetries and reductions. In order to obtain the appropriate change of variables, we apply the criterion of invariance

$$Q(n+4, u_{n+4}) - \mathbf{X}(u_{n+4}) = 0 \tag{14}$$

whenever (13) holds.

The procedure for solving this type of functional equation is now well known [6, 7, 8]. The steps are as follows:

- Apply the differential operator $\frac{\partial}{\partial u_n} - \frac{B_n u_{n+2}^2}{A_n u_n^2} \frac{\partial}{\partial u_{n+2}}$ on (14) and multiply the resulting equation by $(A_n u_n + B_n u_{n+2})^2$. This gives:

$$\begin{aligned} &B_n u_{n+2}^2 Q'(n+2, u_{n+2}) - 2B_n u_{n+2} Q(n+2, u_{n+2}) - B_n u_{n+2}^2 Q'(n, u_n) \\ &+ \frac{2B_n u_{n+2}^2}{u_n} Q(n, u_n) = 0. \end{aligned} \tag{15}$$

- Then, differentiate with respect to u_n twice and divide by $-B_n u_{n+2}^2$. This gives:

$$Q^{(3)}(n, u_n) - \frac{2}{u_n} Q''(n, u_n) + \frac{4}{u_n^2} Q'(n, u_n) - \frac{4}{u_n^3} Q(n, u_n) = 0. \tag{16}$$

- Solve for Q and substitute in (14) to get rid of any dependency between the two arbitrary constants.

This set of calculations leads to six independent characteristics

$$\begin{aligned} &Q_1(n, u_n) = \alpha_1(n) u_n^2; \quad Q_2 n, u_n) = \alpha_2(n) u_n^2; \\ &Q_3(n, u_n) = \alpha_3(n) u_n^2; \quad Q_4(n, u_n) = \alpha_4(n) u_n^2; \\ &Q_5(n, u_n) = u_n; \quad Q_6(n, u_n) = (-1)^n u_n; \end{aligned} \tag{17}$$

where the α_i's are solutions of

$$\tilde{\alpha}_{n+4} - A_n \tilde{\alpha}_{n+2} - B_n \tilde{\alpha}_n = 0.$$

So, the symmetry generators are given by

$$X_1 = \alpha_1(n) u_n^2 \partial\, u_n, \quad X_2 = \alpha_2(n) u_n^2 \partial\, u_n \tag{18}$$

$$X_3 = \alpha_3(n) u_n^2 \partial\, u_n, \quad X_4 = \alpha_4(n) u_n^2 \partial\, u_n \tag{19}$$

$$X_5 = u_n \partial\, u_n, \quad X_6 = (-1)^n u_n \partial\, u_n. \tag{20}$$

The reduction will be done using any of the generator X_i, $i = 1, \ldots, 4$. We start by introducing the canonical coordinate

$$\begin{aligned} s_n &= \int \frac{du_n}{\tilde{\alpha}_n {u_n}^2} \\ &= -\frac{1}{\tilde{\alpha}_n u_n}. \end{aligned} \tag{21}$$

Let

$$r_n = s_{n+2} - s_n \tag{22a}$$
$$= \frac{1}{\tilde{\alpha}_n u_n} - \frac{1}{\tilde{\alpha}_{n+2} u_{n+2}}. \tag{22b}$$

Applying the forward shift operator on (22) twice and replacing u_{n+4} by its expression given in (13), we get

$$r_{n+2} = -\frac{\tilde{\alpha}_n B_n}{\tilde{\alpha}_{n+4}}\, r_n. \tag{23}$$

Iterating (23), we get

$$r_{2n+j} = r_j \prod_{k=0}^{n-1} \left(-\frac{\tilde{\alpha}_{2k+j} B_{2k+j}}{\tilde{\alpha}_{2k+j+4}} \right), \quad j = 0, 1.$$

Iterating (22a), we have

$$s_{2n+j} = s_j + \sum_{l=0}^{n-1} r_{2l+j}, \quad j = 0, 1.$$

Invoking (21), we have

$$\begin{aligned} u_{2n+j} &= -\frac{1}{\tilde{\alpha}_{2n+j} s_{2n+j}} \\ &= -\frac{1}{\tilde{\alpha}_{2n+j}\left(s_j + \sum\limits_{l=0}^{n-1} r_{2l+j} \right)} \\ &= -\frac{1}{\tilde{\alpha}_{2n+j}\left(s_j + \sum\limits_{l=0}^{n-1} \left[r_j \prod\limits_{k=0}^{l-1} \left(-\frac{\tilde{\alpha}_{2k+j} B_{2k+j}}{\tilde{\alpha}_{2k+j+4}} \right) \right] \right)} \\ &= -\frac{1}{\tilde{\alpha}_{2n+j}\left\{ -\frac{1}{\tilde{\alpha}_j u_j} + \sum\limits_{l=0}^{n-1} \left[\left(\frac{1}{\tilde{\alpha}_j u_j} - \frac{1}{\tilde{\alpha}_{j+2} u_{j+2}} \right) \prod\limits_{k=0}^{l-1} \left(-\frac{\tilde{\alpha}_{2k+j} B_{2k+j}}{\tilde{\alpha}_{2k+j+4}} \right) \right] \right\}} \\ &= \frac{\tilde{\alpha}_j u_j}{\tilde{\alpha}_{2n+j}\left\{ 1 - \left(1 - \frac{\tilde{\alpha}_j u_j}{\tilde{\alpha}_{j+2} u_{j+2}} \right) \sum\limits_{l=0}^{n-1} \left[\prod\limits_{k=0}^{l-1} \left(-\frac{\tilde{\alpha}_{2k+j} B_{2k+j}}{\tilde{\alpha}_{2k+j+4}} \right) \right] \right\}}, \quad j = 0, 1. \end{aligned} \tag{24}$$

Note that $\tilde{\alpha}_n$ satisfies

$$\tilde{\alpha}_{n+4} - A_n \tilde{\alpha}_{n+2} - B_n \tilde{\alpha}_n = 0,$$

that is,

$$\begin{bmatrix} \alpha_{n+2} \\ \alpha_{n+4} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ B_n & A_n \end{bmatrix} \begin{bmatrix} \alpha_n \\ \alpha_{n+2} \end{bmatrix}.$$

To derive the solutions of (12) from those of (13), we back shift (24) four times and we replace A_n, B_n by a_n, b_n, respectively. We obtain

$$\begin{aligned} & x_{2n+j-3} \\ & = \frac{\alpha_j x_{j-3}}{\alpha_{2n+j}\left\{1-\left(1-\frac{\alpha_j x_{j-3}}{\alpha_{j+2}x_{j-1}}\right)\sum\limits_{l=0}^{n-1}\left[\prod\limits_{k=0}^{l-1}\left(-\frac{\alpha_{2k+j}b_{2k+j}}{\alpha_{2k+j+4}}\right)\right]\right\}}, \quad j=0,1, \qquad (25)\\ & = \frac{\alpha_j x_{j-3}}{\alpha_{2n+j}\left\{1-\left(1-\frac{\alpha_j x_{j-3}}{\alpha_{j+2}x_{j-1}}\right)\sum\limits_{l=0}^{n-1}\left[\left(\prod\limits_{k=0}^{l-1}(-b_{2k+j})\right)\left(\frac{\alpha_j\alpha_{2+j}}{\alpha_{2l+j}\alpha_{2l+j+2}}\right)\right]\right\}}, \quad j=0,1, \qquad (26)\end{aligned}$$

where α_n satisfies

$$\alpha_{n+4}-a_n\alpha_{n+2}-b_n\alpha_n=0. \qquad (27)$$

3.1.2. The case when a_n and b_n are periodic with period two. Let $a_n = (a_0, a_1, a_0, a_1, \dots)$ and $b_n = (b_0, b_1, b_0, b_1, \dots)$. Then (27) can be written as

$$\begin{bmatrix}\alpha_{n+2}\\ \alpha_{n+4}\end{bmatrix}=\begin{bmatrix}0 & 1\\ b_n & a_n\end{bmatrix}\begin{bmatrix}\alpha_n\\ \alpha_{n+2}\end{bmatrix}. \qquad (28)$$

- The case $b_j \neq -a_j^2/4$

Here, (28) takes the form

$$\begin{aligned}\begin{bmatrix}\alpha_{2n+j}\\ \alpha_{2n+2+j}\end{bmatrix} &= \begin{bmatrix}0 & 1\\ b_j & a_j\end{bmatrix}^n\begin{bmatrix}\alpha_j\\ \alpha_{j+2}\end{bmatrix} \qquad (29)\\ &= \frac{1}{\lambda_{2j}-\lambda_{1j}}\begin{bmatrix}\lambda_{1j}^n\lambda_{2j}-\lambda_{2j}^n\lambda_{1j} & \lambda_{2j}^n-\lambda_{1j}^n\\ \lambda_{1j}^{n+1}\lambda_{2j}-\lambda_{2j}^{n+1}\lambda_{1j} & \lambda_{2j}^{n+1}-\lambda_{1j}^{n+1}\end{bmatrix}\begin{bmatrix}\alpha_j\\ \alpha_{j+2}\end{bmatrix}, j=0,1 \qquad (30)\end{aligned}$$

where $\lambda_{1j} = \frac{a_j-\sqrt{a_j^2+4b_j}}{2}$ and $\lambda_{2j} = \frac{a_j+\sqrt{a_j^2+4b_j}}{2}$, $j = 0, 1$. One merely applies the method of diagonalization to find (30). Hence

$$\alpha_{2n+j}=\left(\sum_{s=0}^{n-1}\lambda_{2j}^{n-1-s}\lambda_{1j}^{s}\right)\alpha_{j+2}-\left(\sum_{s=0}^{n-2}\lambda_{2j}^{n-1-s}\lambda_{1j}^{s+1}\right)\alpha_j \qquad (31)$$

and, using (25), we get

$$x_{2n+j-3}=\frac{\alpha_j x_{j-3}}{\alpha_{2n+j}\left\{1-\left(1-\frac{\alpha_j x_{j-3}}{\alpha_{j+2}x_{j-1}}\right)\sum\limits_{l=0}^{n-1}\left[(-b_j)^l\left(\frac{\alpha_j\alpha_{2+j}}{\alpha_{2l+j}\alpha_{2(l+1)+j}}\right)\right]\right\}}, \qquad (32)$$

where $j = 0, 1$. The expressions of α_{2l+j}'s are given in (31)

- The case $b_j = -a_j^2/4$

Here, (28) takes the form

$$\alpha_{n+4} - a_j\alpha_{n+2} + \frac{a_j^2}{4}\alpha_n = 0. \tag{33}$$

Since the coefficients are independent of n, one merely applies the method for solving linear difference equations to get

$$\alpha_n = (c_1 + nc_2)\left(\frac{\sqrt{a_j}}{2}\right)^n + (c_3 + nc_4)\left(\frac{-\sqrt{a_j}}{2}\right)^n, j = 0, 1, \tag{34}$$

for some constants $c_i = 1, \ldots 4$.

3.1.3. The case when a_n and b_n are constant. Let $a_n = (a, a, a, a, \ldots)$, $b_n = (b, b, b, b, \ldots)$. Hence, from (32),

$$x_{2n+j-3} = \frac{\alpha_j x_{j-3}}{\alpha_{2n+j}\left\{1 - \left(1 - \frac{\alpha_j x_{j-3}}{\alpha_{j+2}x_{j-1}}\right)\sum_{l=0}^{n-1}\left[(-b)^l\left(\frac{\alpha_j\alpha_{2+j}}{\alpha_{2l+j}\alpha_{2(l+1)+j}}\right)\right]\right\}}, \tag{35}$$

where $j = 0, 1$. When $b \neq -a^2/4$, the expressions of α_{2l+j}'s are given by

$$\alpha_{2n+j} = \left(\sum_{s=0}^{n-1}\lambda_2^{n-1-s}\lambda_1^s\right)\alpha_{j+2} - \left(\sum_{s=0}^{n-2}\lambda_2^{n-1-s}\lambda_1^{s+1}\right)\alpha_j$$

where $\lambda_1 = \frac{a-\sqrt{a^2+4b}}{2}$ and $\lambda_2 = \frac{a+\sqrt{a^2+4b}}{2}$.

3.2. Exact solutions associated with Fibonacci numbers. When we take $a_n = b_n = 1$ in the Equation (1), we obtain the following equation

$$x_{n+1} = \frac{x_{n-3}x_{n-1}}{x_{n-3} + x_{n-1}} \tag{36}$$

In this part, our aim is to determine the general solution of Equation (36) in explicit form and determine the asymptotic behavior of solutions of Equation (36). Here we should recall that the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ has been defined by the recursive equation

$$F_{n+2} = F_{n+1} + F_n,\ n \in \mathbb{N}_0 \tag{37}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. The solution of equation (37) is given by following formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

which is called Binet formula of Fibonacci numbers, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Also, the backward Fibonacci sequence is given as

$$F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1}F_n.$$

Furthermore, it follows that

$$\lim_{n\to\infty}\frac{F_{n+r}}{F_n}=\alpha^r,\ r\in\mathbb{Z}.$$

Now, we can give the form of the solutions of Equation (36) associated with Fibonacci numbers.

THEOREM 12. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Equation (36). Then, for $n=0,1,2,...$, the forms of solutions $\{x_n\}_{n=-3}^{\infty}$ are given by*

$$x_{2n-1}=\frac{x_{-3}x_{-1}}{F_{n+1}x_{-3}+F_nx_{-1}} \tag{38}$$

$$x_{2n}=\frac{x_{-2}x_0}{F_{n+1}x_{-2}+F_nx_0} \tag{39}$$

where the initial conditions $x_{-3},x_{-2,}x_{-1},x_0\in\mathbb{RR}-F$, with F being the forbidden set of Equation (36) given by

$$F=\bigcup_{n=1}^{\infty}\left\{(x_{-3},x_{-2,}x_{-1},x_0):F_{n+1}x_{-3}+F_nx_{-1}=0 \text{ and } F_{n+1}.x_{-2}+F_n.x_0=0\right\}.$$

Proof. For $n=0$ the result holds. Assume that $n>0$ and that our assumption holds for $n-1,n-2$. That is,

$$\begin{aligned} x_{2n-3} &= \frac{x_{-3}x_{-1}}{F_nx_{-3}+F_{n-1}x_{-1}}, \qquad x_{2n-5}=\frac{x_{-3}x_{-1}}{F_{n-1}x_{-3}+F_{n-2}x_{-1}} \\ x_{2n-2} &= \frac{x_{-2}x_0}{F_nx_{-2}+F_{n-1}x_0}, \qquad x_{2n-4}=\frac{x_{-2}x_0}{F_{n-1}x_{-2}+F_{n-2}x_0}. \end{aligned}$$

From the above equations and Equation (36), it follows that

$$\begin{aligned} x_{2n-1} &= \frac{x_{2n-3}x_{2n-5}}{x_{2n-3}+x_{2n-5}} \\ &= \frac{\frac{x_{-3}x_{-1}}{F_nx_{-3}+F_{n-1}x_{-1}}\frac{x_{-3}x_{-1}}{F_{n-1}x_{-3}+F_{n-2}x_{-1}}}{\frac{x_{-3}x_{-1}}{F_nx_{-3}+F_{n-1}x_{-1}}+\frac{x_{-3}x_{-1}}{F_{n-1}x_{-3}+F_{n-2}x_{-1}}} \\ &= \frac{x_{-3}x_{-1}}{(F_{n-1}+F_n)x_{-3}+(F_{n-2}+F_{n-1})x_{-1}} \\ &= \frac{x_{-3}x_{-1}}{F_{n+1}x_{-3}+F_nx_{-1}}. \end{aligned}$$

From the definition of F_n, we can easliy see that $F_{n-1}+F_n=F_{n+1}$ and $F_{n-2}+F_{n-1}=F_n$. Hence similarly, we obtain

$$\begin{aligned} x_{2n} &= \frac{x_{2n-2}x_{2n-4}}{x_{2n-2}+x_{2n-4}} \\ &= \frac{\frac{x_{-2}x_0}{F_nx_{-2}+F_{n-1}x_0}\frac{x_{-2}x_0}{F_{n-1}x_{-2}+F_{n-2}x_0}}{\frac{x_{-2}x_0}{F_nx_{-2}+F_{n-1}x_0}+\frac{x_{-2}x_0}{F_{n-1}x_{-2}+F_{n-2}x_0}} \\ &= \frac{x_{-2}x_0}{(F_{n-1}+F_n)x_{-2}+(F_{n-2}+F_{n-1})x_0} \\ &= \frac{x_{-2}x_0}{F_{n+1}x_{-2}+F_nx_0}. \end{aligned}$$

Hence the proof is complete.

Some qualitative behavior of these difference equations such that local stability and global attractor of the equilibrium point was studied in [1], [2], [3], [4], [5]. □

4. Numerical examples. In this section we give some numerical examples to support our theoretical results which was mentioned in the previous sections.

Example 13. Consider the equation $x_{n+1} = \dfrac{x_{n-3}x_{n-1}}{x_{n-3} + x_{n-1}}$ with initial conditions $x_{-3} = 5$, $x_{-2} = 3$, $x_{-1} = 0.2$, $x_0 = -0.1$ to verify our theoretical results. (See Figure 1)

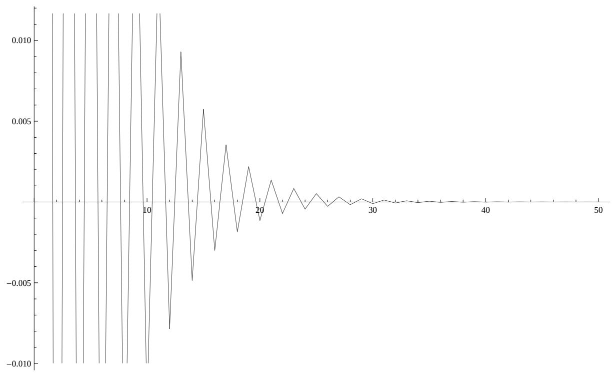

Figure 1: Plot of the difference equation $x_{n+1} = \dfrac{x_{n-3}x_{n-1}}{x_{n-3} + x_{n-1}}$.

Example 14. Consider the equation $x_{n+1} = \dfrac{x_{n-3}x_{n-1}}{(-1)^n x_{n-3} + (-1)^{n+1} x_{n-1}}$ with initial conditions $x_{-3} = 0.5$, $x_{-2} = 3$, $x_{-1} = 0.2$, $x_0 = -0.1$ to verify our theoretical results. (See Figure 2)

5. Conclusion. We have obtained the solutions of the difference equation (1) using Lie symmetry analysis and specific cases were exhibited. Furthermore, it was seen that the form of solutions obtained for these difference equations in some cases are associated with Fibonacci numbers.

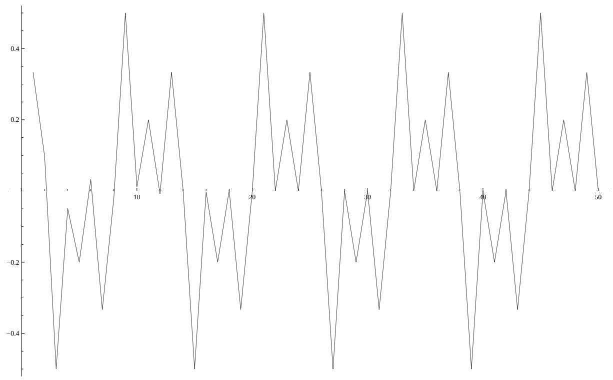

Figure 2: Plot of the difference equation $x_{n+1} = \dfrac{x_{n-3}x_{n-1}}{(-1)^n x_{n-3} + (-1)^{n+1} x_{n-1}}$.

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