Partition Functions Related To Identities Of Euler Type

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Declaration

This work titled 'Partition functions related to identities of Euler type' was carried out by Beaullah Mugwangwavari under the supervision of Dr. D. Nyirenda of the School of Mathematics, at the University of the Witwatersrand, Johannesburg. It is being submitted for the Degree of Master of Science in Mathematics at the University of Witwatersrand, Johannesburg.

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Abstract

In this research, we shall explore partition functions related to identities of Euler type, mainly focusing on identities due to P. A. MacMahon and M. V. Subbarao. We will give a new bijection for one of the partition theorems due to MacMahon and provide a generalisation of a partition identity due to Subbarao. We will further deduce parity formulas for various related partition functions. Our approach is via generating functions and bijective mappings.

Acknowledgements

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Introduction

1.1 Elementary Aspects of Partitions

The theory of integer partitions involves representing a positive integer as a sum of positive integers. Real development and discovery commenced in the eighteenth century when L. Euler proved important theorems. However, G. E. Andrews in [7] explains that G. W. Leibniz was the first person to consider integer partitions. In addition to Euler's research, partition theory has been further developed through contributions by other mathematicians, namely; S. Ramanujan, L. J. Rogers, I. Schur, MacMahon, and G. E. Andrews, to name a few. Hence, it is not surprising that the theory of partitions has become one of the growing research areas of number theory and combinatorics.

Definition 1.1.1. *A partition of a non-negative integer n is a representation $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_r$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 1$, $\lambda_i \in \mathbb{Z}_{\geq 1}$ for all i and $\sum_{i=1}^{r} \lambda_i = n$. The number n is called the weight of λ which is denoted by $|\lambda|$ and the summands λ_i are called parts.*

Alternatively, a partition can be represented as $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ or $(\mu_1^{m_1}, \mu_2^{m_2}, \mu_3^{m_3}, \ldots, \mu_s^{m_s})$ where m_i is the multiplicity of μ_i and $\mu_i > \mu_{i+1}$ for all $i = 1, 2, \ldots, s-1$. Indeed, $14+14+10+10+7+7+7+1+1+1+1$ can be written as $(14^2, 10^2, 7^3, 1^4)$ or $(14, 14, 10, 10, 7, 7, 7, 1, 1, 1, 1)$.

Visualising a partition as a graph often aids in solving partition problems. The Ferrers graph of a partition $(\lambda_1, \lambda_2, \ldots)$ is a left-justified array of dots with λ_i dots in the i^{th} row. For example, the Ferrers graph of $(4, 2, 1)$ a partition of 7 is given as follows:

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & & & \end{array}$$

Reading the parts as the number of dots in the columns of a Ferrers graph gives a new partition called the *conjugate*. Thus the conjugate of $(4, 2, 1)$ is $(3, 2, 1^2)$. From this transformation, we notice that the number of partitions of n into at most m parts is equal to the number of partitions of n with no part greater than m (see [5]).

Let $n = 7$ and $m = 4$. The following table shows two sets of partitions: In the left column, are partitions of 7 into at most 4 parts, and in the right column, are partitions of 7 with no part greater than 4.

(7)	$\mapsto$	(1^7)
$(6,1)$	$\mapsto$	$(2,1^5)$
$(5,2)$	$\mapsto$	$(2^2,1^3)$
$(5,1^2)$	$\mapsto$	$(3,1^4)$
$(4,3)$	$\mapsto$	$(2^3,1)$
$(4,2,1)$	$\mapsto$	$(3,2,1^2)$
$(4,1^3)$	$\mapsto$	$(4,1^3)$
$(3^2,1)$	$\mapsto$	$(3,2^2)$
$(3,2^2)$	$\mapsto$	$(3^2,1)$
$(3,2,1^2)$	$\mapsto$	$(4,2,1)$
$(2^3,1)$	$\mapsto$	$(4,3)$

The conjugate of a partition can be computed algebraically. The conjugate of a partition λ is denoted by λ'. Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_{r-1}^{m_{r-1}}, \lambda_r^{m_r})$. Then λ' is given by

$$\lambda' = \left(\left(\sum_{i=1}^{r} m_i \right)^{\lambda_r}, \left(\sum_{i=1}^{r-1} m_i \right)^{\lambda_{r-1}-\lambda_r}, \ldots, (m_1)^{\lambda_2-\lambda_1} \right).$$

We further find that conjugation is an involution since it is bijective and $(\lambda')' = \lambda$. If it turns out that a partition is equal to its conjugate, the partition is called *self-conjugate*. The Ferrers graphs of self-conjugate partitions are symmetric about the main diagonal. J. J. Sylvester (see [5]) showed that the number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts.

Apart from conjugation, we also consider some algebraic operations on partitions such as union and sum. The union of two partitions λ and μ is simply the multiset union $\lambda \cup \mu$ where λ and μ are treated as multisets. The sum of two partitions $\lambda + \mu$ is the partition whose i^{th} part is equal to the i^{th} part of λ plus the i^{th} part of μ. We append zeros to the partition with smaller length. For example, let $\lambda = (7^2, 4, 2^2, 1)$ and $\mu = (8^2, 4^2, 2, 1^3)$. Then

$$\lambda \cup \mu = (8^2, 7^2, 4^3, 2^3, 1^4)$$

and

$$\lambda + \mu = (15^2, 8, 6, 4, 2, 1^2).$$

Definition 1.1.2. *The generating function of a sequence* $\{a_n\}_{n=0}^{\infty}$ *is defined as the series*

$$\sum_{n=0}^{\infty} a_n q^n.$$

The generating function for the number of unrestricted partitions of n, denoted $p(n)$, is given by

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \quad \text{where } |q| < 1\,. \tag{1.1.1}$$

Note that

$$\sum_{n=0}^{\infty} p(n)q^n = \left(1 + q^{1.1} + q^{2.1} + \ldots\right)\left(1 + q^{1.2} + q^{2.2} + \ldots\right)$$
$$\times \left(1 + q^{1.3} + q^{2.3} + \ldots\right)\left(1 + q^{1.4} + q^{2.4} + \ldots\right)\ldots.$$

The expansion $(1 + q^j + q^{2j} + q^{3j} + \ldots)$ represents the appearance of a part j in a partition or its non-appearance (represented by 1). Now

$$\sum_{n=0}^{\infty} p(n)q^n = \left(\frac{1}{1-q^1}\right)\left(\frac{1}{1-q^2}\right)\left(\frac{1}{1-q^3}\right)\left(\frac{1}{1-q^4}\right)\cdots$$
$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

The next theorem is one of L. Euler's famous results - a result that is important for this research.

Theorem 1.1.1 (Euler, [4])**.** *Let $d(n)$ denote the number of partitions of n into distinct parts and $o(n)$ denote the number of partitions of n into odd parts. Then $d(n) = o(n)$.*

J. W. L. Glaisher generalized Euler's theorem and gave a beautiful bijective proof which we shall utilize extensively throughout this study.

Theorem 1.1.2 (Glaisher, [9])**.** *The number of partitions of n where parts occur at most $k-1$ times is equal to the number of partitions of n where parts are not divisible by k.*

Proof. The generating function for the number of partitions of n where parts occur at most $k-1$ times is $\prod_{n=1}^{\infty}\left(1+q^n+q^{2n}+q^{3n}+\ldots+q^{(k-1)n}\right)$. Thus

$$\begin{aligned}\prod_{n=1}^{\infty}\left(1+q^n+q^{2n}+q^{3n}+\ldots+q^{(k-1)n}\right) &= \prod_{n=1}^{\infty}\frac{1-q^{kn}}{1-q^n}\\ &= \prod_{n=1}^{\infty}\prod_{i=0}^{k-1}\frac{1-q^{kn}}{1-q^{kn-i}}\\ &= \prod_{n=1}^{\infty}\prod_{i=1}^{k-1}\frac{1}{1-q^{kn-i}}\end{aligned}$$

which is the generating function for the number of partitions of n where parts are not divisible by k. $\square$

We now describe Glaisher's bijection for Theorem 1.1.2. Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r})$ be a partition of n whose parts are not divisible by k. Note that the notation for λ implies $\lambda_1 > \lambda_2 > \ldots$ are parts with multiplicities $m_1, m_2, \ldots$, respectively. Now, write m_i's in k-ary expansion, i.e.

$$m_i = \sum_{j=0}^{l_i} a_{ij}k^j \quad \text{where} \quad 0 \leq a_{ij} \leq k-1.$$

We map $\lambda_i^{m_i}$ to $\bigcup_{j=0}^{l_i}(k^j\lambda_i)^{a_{ij}}$, where now $k^j\lambda_i$ is a part with multiplicity a_{ij}. The image of λ, which we shall denote by

$\phi(\lambda)$, is given by

$$\bigcup_{i=1}^{r}\bigcup_{j=0}^{l_i}(k^j\lambda_i)^{a_{ij}}.$$

Clearly, this is a partition of n with parts occurring not more than $k-1$ times.

On the other hand, assume that $\mu = (\mu_1^{f_1}, \mu_2^{f_2}, \ldots)$ is a partition of n into parts occurring not more than $k-1$ times. Write $\mu_i = k^{r_i}\cdot a_i$ with $k \nmid a_i$ and then map $\mu_i^{f_i}$ to $(a_i)^{k^{r_i}f_i}$ for each i, where now a_i is a part with multiplicity $k^{r_i}f_i$. The inverse of ϕ is then given by

$$\phi^{-1}(\mu) = \bigcup_{i\geq 1}(a_i)^{k^{r_i}f_i}.$$

In the resulting partition, it is also clear that the parts are not divisible by k. $\square$

Example 1.1.1. *Let* $k = 4$ *and* $\lambda = (14^6, 10^3, 7^4, 5^2, 2^7)$.

Note that λ is a partition whose parts are not divisible by 4, and so we apply ϕ. Representing the multiplicities in 4-ary expansion gives

$$6 = 2.4^0{+}1.4^1,\ 3 = 3.4^0,\ 4 = 1.4^1,\ 2 = 2.4^0,\ 7 = 3.4^0{+}1.4^1.$$

Therefore,

$$14^6 \mapsto \big(14(4^0)\big)^{2.1} \cup \big(14(4^1)\big)^{1} = (56, 14^2),$$

$$10^3 \mapsto \big(10(4^0)\big)^{3.1} = 10^3,$$
$$7^4 \mapsto \big(7(4^1)\big)^{1} = 28,$$
$$5^2 \mapsto \big(5(4^0)\big)^{2.1} = 5^2,$$
$$2^7 \mapsto \big(2(4^0)\big)^{3.1} \cup \big(2(4^1)\big)^{1} = (8, 2^3).$$

Taking the union leads to

$$\phi(\lambda) = (56, 28, 14^2, 10^3, 8, 5^2, 2^3),$$

which is a partition in which parts occur at most 3 times.

To invert the process, we apply ϕ^{-1} to $(56, 28, 14^2, 10^3, 8, 5^2, 2^3)$ in which parts map as follows:

$$56^1 \mapsto 14^4,\ 28^1 \mapsto 7^4,\ 14^2 \mapsto 14^2,\ 10^3 \mapsto 10^3,\ 8^1 \mapsto 2^4,$$
$$5^2 \mapsto 5^2,\ \text{and } 2^3 \mapsto 2^3.$$

Taking the union results in

$$\phi^{-1}((56, 28, 14^2, 10^3, 8, 5^2, 2^3)) = (14^6, 10^3, 7^4, 5^2, 2^7) = \lambda.$$

Euler's theorem (Theorem 1.1.1) is the special case of Glaisher's theorem where $k = 2$. A partition identity that conforms to Euler's theorem in the sense that on one side, parts of partitions have specified multiplicities and on the other side, parts belong to a certain set, is called an identity of Euler type. The following theorems, which are due to MacMahon and Subbarao, are examples of identities of Euler type.

Theorem 1.1.3 (MacMahon, [13]). *The number of partitions of n wherein no part appears with multiplicity one is equal to the number of partitions of n where parts are even or congruent to* $3 \pmod 6$.

Proof. The generating function for the number of partitions of n wherein no part appears with multiplicity one is $\prod\limits_{n=1}^{\infty}\left(1+q^{2n}+q^{3n}+q^{4n}+q^{5n}+q^{6n}+\dots\right)$. Thus

$$
\begin{aligned}
&\prod_{n=1}^{\infty}\left(1+q^{2n}+q^{3n}+q^{4n}+q^{5n}+q^{6n}+\dots\right)\\
&=\prod_{n=1}^{\infty}\left(1+\frac{q^{2n}}{1-q^n}\right)\\
&=\prod_{n=1}^{\infty}\left(\frac{1-q^n+q^{2n}}{1-q^n}\right)\left(\frac{1+q^n}{1+q^n}\right)\\
&=\prod_{n=1}^{\infty}\frac{1}{1-q^{2n}}\times\prod_{n=1}^{\infty}1+q^{3n}\\
&=\prod_{n=1}^{\infty}\frac{1}{1-q^{2n}}\times\prod_{n=1}^{\infty}\frac{1-q^{6n}}{1-q^{3n}}\\
&=\prod_{n=1}^{\infty}\frac{1}{1-q^{2n}}\times\prod_{n=1}^{\infty}\frac{1}{1-q^{6n-3}}
\end{aligned}
$$

which is the generating function for the number of partitions of n where parts are even or congruent to $3 \pmod 6$. □

Theorem 1.1.4 (Subbarao, [16]). *The number of partitions of n in which parts occur with multiplicities $2, 3$ and 5 is equal to the number of partitions of n in which parts are congruent to $\pm 2, \pm 3, 6 \pmod{12}$.*

Proof. The generating function for the number of partitions of n in which parts occur with multiplicities $2, 3$ and 5 is $\prod_{n=1}^{\infty} \left(1 + q^{2n} + q^{3n} + q^{5n}\right)$. Thus

$$\begin{aligned}
\prod_{n=1}^{\infty} \left(1 + q^{2n} + q^{3n} + q^{5n}\right) &= \prod_{n=1}^{\infty} \left(1 + q^{2n}\right)\left(1 + q^{3n}\right) \\
&= \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^{4n-2}}\right)\left(\frac{1}{1 - q^{6n-3}}\right) \\
&= \prod_{n=1}^{\infty} \frac{1}{(1 - q^{12n-2})(1 - q^{12n-3})} \\
&\qquad \times \frac{1}{(1 - q^{12n-6})(1 - q^{12n-9})} \\
&\qquad \times \frac{1}{(1 - q^{12n-10})}
\end{aligned}$$

which is the generating function for the number of partitions of n in which parts are congruent to $\pm 2, \pm 3, 6 \pmod{12}$. $\square$

Euler's theorem has been refined and J. J. Sylvester gave one of its refinements which we shall discuss.

Theorem 1.1.5 (Sylvester, [4]). *For $k \in \mathbb{Z}_{\geq 1}$, let $o_k(n)$ denote the number of partitions of n into k different odd parts and $d_k(n)$ denote the number of partitions of n into k separate sequences of consecutive integers (sequences may be of length one). Then $o_k(n) = d_k(n)$.*

Proof. Firstly, we note that any partition into odd parts can be repositioned into a centrally justified Ferrers graph. This is done by shifting the rows of the original Ferrers graph such that the graph is symmetric about the middle column.

Let $\gamma = (\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_r)$ denote a partition of n into k different odd parts. Consider the centrally justified Ferrers graph of γ. Let a_i denote the number of dots in the i^{th} column of the graph, and a_1 be the middle column. The columns are alternatively counted from left to right such that the a_{2i}'s are on the left. Similarly, let b_i be the number of dots in the i^{th} row of the graph alternatively counted from right to left starting from a_1 (see Figure 1.1). Thus we have the set $\{a_1, a_2, a_3, \ldots, a_r, b_1, b_2, b_3, \ldots, b_r\}$. Now construct

$$\upsilon = (b_1 + a_1 - 1, b_2 + a_2 - 1, \ldots, b_r + a_r - 1)\,. \qquad (1.1.2)$$

It can be shown that υ is a partition of n into k separate sequences of consecutive integers (see [4]).

Now to invert, we find a_i's and b_i's from the equation

$$a_i + b_i = v_i + 1 \tag{1.1.3}$$

used together with the following:
If $r \equiv 0 \pmod 2$, then

$$b_r = v_r.$$

Otherwise,

$$a_r = v_r.$$

If $i \equiv 1 \pmod 2$ and $i < r$, then

$$b_i = b_{i+1} + 1.$$

If $i \equiv 0 \pmod 2$ and $i < r$, then

$$a_i = a_{i+1} + 1.$$

Having obtained the sets $\{a_1, a_2, a_3, \ldots, a_r\}$ and $\{b_1, b_2, b_3, \ldots, b_r\}$, we now construct a centrally justified Ferrers graph (see Figure 1.1), from which a partition γ is obtained. Actually γ contains k different odd parts.

$\square$

For instance, let $\gamma = \left(21, 17^2, 13, 9^3, 5, 3\right)$. The partition γ is a partition of 103 into 6 different odd parts.

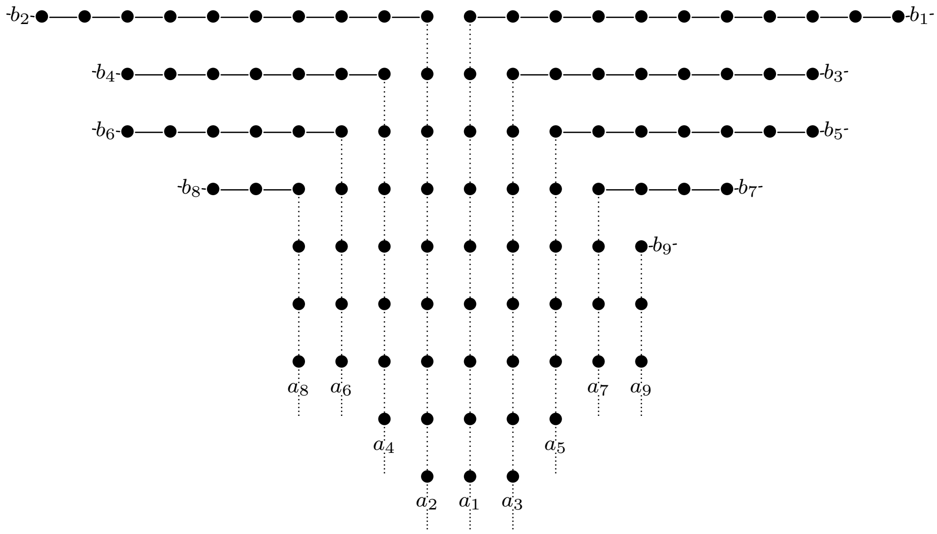


Figure 1.1: Centrally justified Ferrers graph of γ

From the centrally justified Ferrers graph, we obtain

$a_1 = 9,\ \ a_2 = 9,\ \ a_3 = 8,\ \ a_4 = 7,\ \ a_5 = 6,\ \ a_6 = 5,\ \ a_7 = 4,$

$a_8 = 4,\ \ a_9 = 3,\ \ b_1 = 11,\ \ b_2 = 10,\ \ b_3 = 8,\ \ b_4 = 7,\ \ b_5 = 7,$

$b_6 = 6,\ \ b_7 = 4,\ \ b_8 = 3$ and $b_9 = 1$.

The parts of υ are computed as follows:

$\upsilon_1 = 11 + 9 - 1 = 19,\quad \upsilon_2 = 10 + 9 - 1 = 18,$
$\upsilon_3 = 8 + 8 - 1 = 15,\quad \upsilon_4 = 7 + 7 - 1 = 13,$
$\upsilon_5 = 7 + 6 - 1 = 12,\quad \upsilon_6 = 6 + 5 - 1 = 10,$
$\upsilon_7 = 4 + 4 - 1 = 7,\quad \upsilon_8 = 3 + 4 - 1 = 6,$
$\upsilon_9 = 1 + 3 - 1 = 3.$

Therefore,

$$\begin{aligned}\upsilon &= (19, 18, 15, 13, 12, 10, 7, 6, 3)\\ &= ((19, 18), (15), (13, 12), (10), (7, 6), (3))\end{aligned}$$

which is a partition of 103 into 6 separate sequences of consecutive integers.

To invert, let $\upsilon = (19, 18, 15, 13, 12, 10, 7, 6, 3)$. We have $r = 9$, which is odd. As such $a_r = 3$. Thus

$$3 = b_9 + 3 - 1 \Rightarrow b_9 = 1,$$

$$a_8 = 3 + 1 = 4,$$

$$6 = b_8 + 4 - 1 \Rightarrow b_8 = 3,$$

$$b_7 = 3 + 1 = 4,$$

$$7 = 4 + a_7 - 1 \Rightarrow a_7 = 4,$$

$$a_6 = 4 + 1 = 5,$$

$$10 = b_6 + 5 - 1 \Rightarrow b_6 = 6,$$

$$b_5 = 6 + 1 = 7,$$

$$12 = 7 + a_5 - 1 \Rightarrow a_5 = 6,$$

$$a_4 = 6 + 1 = 7,$$

$$13 = b_4 + 7 - 1 \Rightarrow b_4 = 7,$$

$$b_3 = 7 + 1 = 8,$$

$$15 = 8 + a_3 - 1 \Rightarrow a_3 = 8,$$

$$a_2 = 8 + 1 = 9,$$

$$18 = b_2 + 9 - 1 \Rightarrow b_2 = 10,$$

$$b_1 = 10 + 1 = 11,$$

$$19 = 11 + a_1 - 1 \Rightarrow a_1 = 9.$$

Therefore,

$$\{a_1, a_2, a_3, \ldots, a_r\} = (9, 9, 8, 7, 6, 5, 4, 4, 3),$$

and

$$\{b_1, b_2, b_3, \ldots, b_r\} = (11, 10, 8, 7, 7, 6, 4, 3, 1).$$

Having constructed the centrally justified graph, we obtain

$$\gamma = \left(21, 17^2, 13, 9^3, 5, 3\right).$$

1.2 q-Series

We use the following notation. For $a, q \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}(a;q)_n &= (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1})\\ &= \prod_{i=0}^{n-1}(1-aq^i)\end{aligned}$$

where $(a;q)_0 = 1$. Also

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n,$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{1.2.1}$$

Using the q-series notation, the generating function for $p(n)$ is given as

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

The following results were obtained by A. Cauchy, Euler and J. Jacobi and shall be useful in our work. The first theorem is due to Cauchy, followed by two special cases from Euler.

Theorem 1.2.1 (q-binomial Theorem, [4]). *If* $|q|, |t| < 1$, *then*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n t^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1-atq^n}{1-tq^n}. \tag{1.2.2}$$

Proof. Let

$$F(t) = \prod_{n=0}^{\infty} \frac{1-atq^n}{1-tq^n}.$$

It turns out that $F(t)$ has a Maclaurin series expansion, i.e.

$$F(t) = \sum_{n=0}^{\infty} A_n t^n \tag{1.2.3}$$

where A_n depends on q (see [3]). Thus

$$\begin{aligned} F(t) &= \prod_{n=0}^{\infty} \frac{1-atq^n}{1-tq^n} \\ (1-t)F(t) &= (1-at)\prod_{n=1}^{\infty} \frac{1-atq^n}{1-tq^n} \\ &= (1-at)\prod_{n=0}^{\infty} \frac{1-atq^{n+1}}{1-tq^{n+1}} \\ &= (1-at)F(tq). \end{aligned}$$

We have

$$(1-t)F(t) = (1-at)F(tq). \tag{1.2.4}$$

Substituting (1.2.3) into (1.2.4) leads to

$$\sum_{n=0}^{\infty} A_n t^n - \sum_{n=0}^{\infty} A_n t^{n+1} = \sum_{n=0}^{\infty} A_n t^n q^n - \sum_{n=0}^{\infty} a A_n t^{n+1} q^n. \tag{1.2.5}$$

The constant term of the Maclaurin series is A_0. Note that $A_0 = F(0) = 1$. Comparing coefficients of t^n in (1.2.5) yields,

$$\begin{aligned} A_n - A_{n-1} &= q^n A_n - a q^{n-1} A_{n-1} \\ A_n - q^n A_n &= A_{n-1} - a q^{n-1} A_{n-1} \\ A_n &= \frac{(1 - a q^{n-1})}{(1 - q^n)} A_{n-1}. \end{aligned}$$

Continuing through with the iteration, A_n can be expressed in terms of q. We have

$$\begin{aligned} A_n &= \frac{(1 - a q^{n-1})(1 - a q^{n-2}) \ldots (1 - a) A_0}{(1 - q^n)(1 - q^{n-1}) \ldots (1 - q)} \\ &= \frac{(a; q)_n}{(q; q)_n}. \end{aligned}$$

Hence

$$F(t) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} t^n. \tag{1.2.6}$$

$\square$

The proof for the next corollary mirrors that of Theorem 1.2.1. As such, by making essential replacements, the proof is easily obtained.

Corollary 1.2.1 (Euler)**.** *For* $|t|, |q| < 1$,

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{1-tq^n}, \tag{1.2.7}$$

$$\sum_{n=0}^{\infty} \frac{t^n q^{\frac{n(n-1)}{2}}}{(q;q)_n} = \prod_{n=0}^{\infty} (1+tq^n). \tag{1.2.8}$$

Proof. We begin with (1.2.7). Set $a = 0$ in Theorem 1.2.1 to obtain

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{1-tq^n}.$$

Now for (1.2.8), set $a = \frac{a}{c}$ and $t = cz$ in Theorem 1.2.1,

$$\sum_{n=0}^{\infty} \frac{(1-\frac{a}{c})(1-\frac{a}{c}q)\ldots(1-\frac{a}{c}q^{n-1})(cz)^n}{(1-q)(1-q^2)\ldots(1-q^n)} = \prod_{n=0}^{\infty} \frac{(1-\left(\frac{a}{c}\right)(cz)q^n)}{(1-(cz)q^n)},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{(c-a)(c-aq)\ldots(c-aq^{n-1})}{c^n}\right)(cz)^n}{(1-q)(1-q^2)\ldots(1-q^n)} = \prod_{n=0}^{\infty} \frac{(1-azq^n)}{(1-czq^n)},$$

$$\sum_{n=0}^{\infty} \frac{(c-a)(c-aq)\ldots(c-aq^{n-1})z^n}{(1-q)(1-q^2)\ldots(1-q^n)} = \prod_{n=0}^{\infty} \frac{(1-azq^n)}{(1-czq^n)}.$$

Setting $c = 0$ and $a = -1$ gives

$$\sum_{n=0}^{\infty} \frac{z^n q^{\sum_{i=1}^{n-1} i}}{(1-q)(1-q^2)\ldots(1-q^n)} = \prod_{n=0}^{\infty} (1+zq^n),$$

$$\sum_{n=0}^{\infty} \frac{z^n q^{\frac{n(n-1)}{2}}}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{n=0}^{\infty}(1+zq^n).$$

$\square$

Theorem 1.2.2 (Jacobi's Triple Product, [4]). *For* $z \neq 0$ *and* $|q| < 1$,

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=0}^{\infty} \left(1-q^{2n+2}\right)\left(1+zq^{2n+1}\right)\left(1+z^{-1}q^{2n+1}\right).$$

Proof. For $|q| < |z| < \frac{1}{|q|}$, $|q| < 1$,

$$\begin{aligned}
\prod_{n=0}^{\infty}(1+zq^{2n+1}) &= \prod_{n=0}^{\infty}\left(1+zq(q^2)^n\right) \\
&= \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n} \quad \text{(by (1.2.8))} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} z^n q^{n^2} (q^{2n+2};q^2)_\infty \quad \text{(by (1.2.1))} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} (q^{2n+2};q^2)_\infty.
\end{aligned}$$

When n is negative, $(q^{2n+2};q^2)_\infty = 0$. Thus the range of the sum can be extended because essentially no new terms have been added. From (1.2.8), replacing t with $-t$, it is clear that

$$(t;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n t^n q^{\frac{n(n-1)}{2}}}{(q;q)_n}. \tag{1.2.9}$$

Using (1.2.9), we have

$$(q^{2n+2};q^2)_\infty = \sum_{r=0}^{\infty} \frac{(-1)^r q^{r^2+2nr+r}}{(q^2;q^2)_r}. \tag{1.2.10}$$

Then

$$\begin{aligned}
\prod_{n=0}^{\infty}(1+zq^{2n+1}) &= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} \sum_{r=0}^{\infty} \frac{(-1)^r q^{r^2+2nr+r}}{(q^2;q^2)_r} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(-1)^r z^{-r} q^r}{(q^2;q^2)_r} \sum_{n=-\infty}^{\infty} z^{n+r} q^{(n+r)^2} \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{r=0}^{\infty} \frac{(\frac{-q}{z})^r}{(q^2;q^2)_r} \sum_{t=-\infty}^{\infty} z^t q^{t^2} \\
&= \frac{1}{(q^2;q^2)_\infty(\frac{-q}{z};q^2)_\infty} \sum_{t=-\infty}^{\infty} z^t q^{t^2} \quad \text{(by (1.2.7))}.
\end{aligned}$$

For all $z \neq 0$ outside the condition $|q| < |z| < \frac{1}{|q|}$, the result still holds (see [3]).

$\square$

Corollary 1.2.2. *For* $|q| < 1$,

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}.$$

Proof. Replacing z by z^2q in Theorem 1.2.2,

$$\sum_{n=-\infty}^{\infty} z^{2n}q^{n^2+n} = \prod_{n=0}^{\infty}\left(1+z^2q^{2n+2}\right)\left(1+z^{-2}q^{2n}\right)\left(1-q^{2n+2}\right)$$
$$= (-z^2q^2;q^2)_\infty \left(-\frac{1}{z^2};q^2\right)_\infty (q^2;q^2)_\infty .$$

Dividing both sides by $1+\frac{1}{z^2}$ yields,

$$\frac{\sum_{n=-\infty}^{\infty} z^{2n}q^{n^2+n}}{1+\frac{1}{z^2}} = \frac{(-z^2q^2;q^2)_\infty \left(-\frac{1}{z^2};q^2\right)_\infty (q^2;q^2)_\infty}{1+\frac{1}{z^2}}$$
$$= \frac{(-z^2q^2;q^2)_\infty \left(1+\frac{1}{z^2}\right)\left(-\frac{q^2}{z^2};q^2\right)_\infty (q^2;q^2)_\infty}{1+\frac{1}{z^2}}$$
$$= (-z^2q^2;q^2)_\infty \left(-\frac{q^2}{z^2};q^2\right)_\infty (q^2;q^2)_\infty .$$

Taking the limit as $z \to i$ of $\dfrac{\sum_{n=-\infty}^{\infty} z^{2n+1}q^{n^2+n}}{z+\frac{1}{z}}$, gives the indeterminate form $\frac{0}{0}$. To see this observe that

$$\lim_{z\to i} z+\frac{1}{z} = i+\frac{1}{i} = \frac{1+i^2}{i} = 0$$

and

$$
\begin{aligned}
&\lim_{z\to i} \sum_{n=-\infty}^{\infty} z^{2n+1} q^{n^2+n} \\
&= \sum_{n=-\infty}^{\infty} i^{2n+1} q^{n^2+n} \\
&= i \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+n} \\
&= i \left(\sum_{n=-\infty}^{-1} (-1)^n q^{n^2+n} + 1 + \sum_{n=1}^{\infty} (-1)^n q^{n^2+n} \right) \\
&= i \left(\sum_{n=1}^{\infty} (-1)^n q^{n^2-n} + 1 + \sum_{n=1}^{\infty} (-1)^n q^{n^2+n} \right) \\
&= i \left(-\sum_{n=0}^{\infty} (-1)^n q^{n^2+n} + 1 + \sum_{n=1}^{\infty} (-1)^n q^{n^2+n} \right) \\
&= i \left(-1 - \sum_{n=1}^{\infty} (-1)^n q^{n^2+n} + 1 + \sum_{n=1}^{\infty} (-1)^n q^{n^2+n} \right) \\
&= 0.
\end{aligned}
$$

Applying $L'H\hat{o}spital's$ rule (differentiation with respect to z) leads to the following results:

$$
\begin{aligned}
&\lim_{z\to i}\sum_{n=-\infty}^{\infty}\frac{z^{2n+1}q^{n^2+n}}{z+\frac{1}{z}}\\
&=\lim_{z\to i}\sum_{n=-\infty}^{\infty}\frac{(2n+1)z^{2n}q^{n^2+n}}{1-\frac{1}{z^2}}\\
&=\sum_{n=-\infty}^{\infty}\frac{(2n+1)i^{2n}q^{n^2+n}}{1-\frac{1}{i^2}}\\
&=\frac{1}{2}\left(\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n^2+n}\right)\\
&=\frac{1}{2}\left(\sum_{n=-\infty}^{-1}(-1)^n(2n+1)q^{n^2+n}+\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}\right)\\
&=\frac{1}{2}\left(\sum_{n=1}^{\infty}(-1)^n(1-2n)q^{n^2-n}+\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}\right)\\
&=\frac{1}{2}\left(\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}+\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}\right)\\
&=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}.
\end{aligned}
$$

Also,

$$\begin{aligned}
&\lim_{z\to i}(-z^2q^2;q^2)_\infty(-\frac{q^2}{z^2};q^2)_\infty(q^2;q^2)_\infty \\
&= (-(i)^2q^2;q^2)_\infty(-\frac{q^2}{i^2};q^2)_\infty(q^2;q^2)_\infty \\
&= (q^2;q^2)^3_\infty.
\end{aligned}$$

Now setting $q = q^{\frac{1}{2}}$, we obtain

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n^2+n}{2}} = (q;q)^3_\infty.$$

$\square$

Euler observed that a pattern emerged while multiplying out expressions of the following form:

$(1-q)(1-q^2)(1-q^3)\ldots = 1-q-q^2+q^5+q^7-q^{12}\ldots.$

Notably, the series is the reciprocal of $\prod_{n=1}^{\infty}\frac{1}{1-q^n}$. His observation gave rise to the following theorem.

Theorem 1.2.3 (Euler's Pentagonal Number Theorem, [3])**.** *For* $|q|<1$,

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = \prod_{n=1}^{\infty}(1-q^n). \tag{1.2.11}$$

Proof. Note that

$$\begin{aligned}
\prod_{n=1}^{\infty}(1-q^n) &= \prod_{n=1}^{\infty}(1-q^{3n})(1-q^{3n-1})(1-q^{3n-2}) \\
&= \prod_{n=0}^{\infty}(1-q^{3(n+1)})(1-q^{3(n+1)-1})(1-q^{3(n+1)-2}) \\
&= \prod_{n=0}^{\infty}(1-q^{3n+3})(1-q^{3n+2})(1-q^{3n+1}).
\end{aligned}$$

Now setting $z = -q^{-\frac{1}{2}}$ and $q = q^{\frac{3}{2}}$ in Theorem 1.2.2 yields

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = \prod_{n=0}^{\infty}\left(1-q^{3n+3}\right)\left(1-q^{3n+2}\right)\left(1-q^{3n+1}\right).$$

$\square$

Numbers of the form $\frac{n(3n-1)}{2}$ where $n \in \mathbb{Z}$ are called generalised pentagonal numbers. This is because they can be represented in a pentagonal pattern of dots (see Figure 1.2).

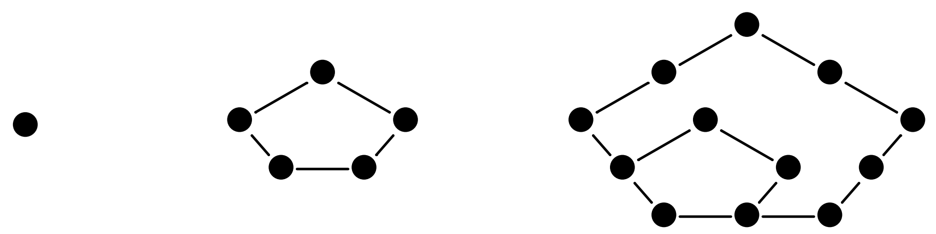


Figure 1.2: Patterns for $n = 1, 2$ and 3

Let $d_e(n)$ and $d_o(n)$ be the number of partitions of n into distinct parts, with an even and odd number of parts, respectively. A. M. Legendre made the following observation.

Theorem 1.2.4 (Legendre, [4])**.**

$$d_e(n) - d_o(n) = \begin{cases} (-1)^j, & \textit{if } n = \frac{j(3j\pm1)}{2} \;\; j \geq 0; \\ 0, & \textit{otherwise}. \end{cases}$$

For example, partitions of 7 into distinct parts are 7, $(6, 1)$, $(5, 3)$, $(4, 3)$, $(4, 2, 1)$. Thus $d(7) = 5$. Observe that $d_e(7) = 3$ and $d_o(7) = 2$ so that $d_e(7) - d_o(7) = 1$. If we use the theorem, note that $n = 7$ is a pentagonal number with $j = 2$, hence $d_e(7) - d_o(7) = (-1)^2 = 1$.

In the following chapter, we discuss bijective proofs for Theorem 1.1.3.

Chapter 2

Bijections of Theorem 1.1.3

G. E. Andrews, S. Fu and J. Sellers proved Theorem 1.1.3 bijectively. In this chapter, we discuss their bijections. The bijection by Fu and Sellers shall be referred to as Fu-Sellers bijection.

2.1 Andrews' Bijection

We describe Andrews' bijection for Theorem 1.1.3. This mapping is given in [6].

Let C_n denote the set of partitions of n wherein no part appears with multiplicity one and D_n the set of partitions of n wherein parts are even or congruent to $3 \pmod 6$. Andrews' bijection goes as follows.

Let $\lambda = (l^{h_l}, (l-1)^{h_{l-1}}, \ldots, 3^{h_3}, 2^{h_2}, 1^{h_1})$ be a partition in C_n where h_i is the multiplicity of i. Since no part appears exactly once, it follows that $h_i \in \{0, 2, 3, 4, \ldots\}$. Furthermore, h_i can be written uniquely as $h_i = k_i + g_i$ where $k_i = 3\left(\frac{1-(-1)^{h_i}}{2}\right)$ and $g_i = h_i - 3\left(\frac{1-(-1)^{h_i}}{2}\right)$. Note that

$k_i \in \{0, 3\}$ and $g_i \in \{0, 2, 4, 6, 8, \ldots\}$. Define d_j as follows:

$$\begin{aligned}
d_{6t+1} &= 0, \qquad\qquad (t = 0, 1, 2, 3, \ldots) \\
d_{6t+5} &= 0, \\
d_{6t+2} &= \frac{1}{2} g_{3t+1}, \\
d_{6t+4} &= \frac{1}{2} g_{3t+2}, \\
d_{6t+3} &= \frac{1}{3} k_{2t+1} + g_{6t+3}, \\
d_{6t+6} &= \frac{1}{3} k_{2t+2} + g_{6t+6}.
\end{aligned}$$

Since $d_{6t+1} = d_{6t+5} = 0$, it is clear that the partition $(f^{d_f}, (f-1)^{d_{f-1}}, ..., 2^{d_2}, 1^{d_1})$ is in D_n.

Example 2.1.1. *Let* $n = 12$*. Then* $C_{12} = \{1^{12}, (2^2, 1^8), (2^3, 1^6), (2^4, 1^4), (2^5, 1^2), 2^6, (3^2, 1^6), (3^2, 2^2, 1^2), (3^2, 2^3), (3^3, 1^3), 3^4, (4^2, 2^2), (4^2, 1^4), 4^3, (5^2, 1^2), 6^2\}$.

In the partition 2^6, the multiplicity of 2 is 6. Thus $h_2 = k_2 + g_2 = 0 + 6$. This gives $d_4 = \frac{1}{2}(6) = 3$ so that

$$2^6 \mapsto 4^3.$$

Similarly, for partitions 4^3 and 6^2, we have

$$3 = h_4 = k_4 + g_4 = 3 + 0,$$

$$d_{12} = \frac{1}{3}(3) + 0 = 1,$$

and

$$2 = h_6 = k_6 + g_6 = 0 + 2,$$
$$d_6 = \frac{1}{3}(0) + 2 = 2.$$

Thus

$$4^3 \mapsto 12,$$
$$6^2 \mapsto 6^2.$$

Applying the mapping to the remaining partitions, we get the following:

C_{12}		D_{12}
1^{12}	$\mapsto$	2^6
$(2^2, 1^8)$	$\mapsto$	$(4, 2^4)$
$(2^3, 1^6)$	$\mapsto$	$(6, 2^3)$
$(2^4, 1^4)$	$\mapsto$	$(4^2, 2^2)$
$(2^5, 1^2)$	$\mapsto$	$(6, 4, 2)$
2^6	$\mapsto$	4^3
$(3^2, 1^6)$	$\mapsto$	$(3^2, 2^3)$
$(3^2, 2^2, 1^2)$	$\mapsto$	$(4, 3^2, 2)$
$(3^2, 2^3)$	$\mapsto$	$(6, 3^2)$
$(3^3, 1^3)$	$\mapsto$	$(9, 3)$
3^4	$\mapsto$	3^4
$(4^2, 2^2)$	$\mapsto$	$(8, 4)$
$(4^2, 1^4)$	$\mapsto$	$(8, 2^2)$
4^3	$\mapsto$	12
$(5^2, 1^2)$	$\mapsto$	$(10, 2)$
6^2	$\mapsto$	6^2

2.2 Fu-Sellers Bijection

We describe Fu-Sellers bijection for Theorem 1.1.3. This mapping is given in [8]. Fu-Sellers bijection goes as follows. Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_t^{m_t})$ be a partition wherein no part appears with multiplicity one. Depending on the parity of m_i, the following cases are considered.
Case 1 : m_i is even

$$\lambda_i^{m_i} \mapsto (2\lambda_i)^{\frac{m_i}{2}}.$$

These new parts are even.

Case 2 : m_i is odd $(m_i \geq 3)$
We begin by peeling off three copies of each λ_i and so the remaining parts have even multiplicity. On the parts with even multiplicity, Case 1 is applied. This now leaves parts with multiplicity exactly 3. From these parts, we form a distinct-part sub-partition by taking one copy of each part. Now, any map that converts distinct-part partitions into odd-part partitions, for example Glaisher or Sylvester, is applied. Lastly, multiply each part in this odd-part sub-partition by 3. Each of these parts will then be congruent to $3 \pmod 6$.

To invert, let $\mu = (\mu_1^{m_1}, \mu_2^{m_2}, \ldots, \mu_s^{m_s})$ be a partition wherein parts are even or congruent to $3 \pmod 6$. We now consider two cases.
Case 1 : λ_i is even

$$\lambda_i^{m_i} \mapsto \left(\frac{\lambda_i}{2}\right)^{2m_i}.$$

These new parts have even multiplicities.

Case 2 : λ_i is congruent to $3 \pmod 6$
Firstly,

$$\lambda_i^{m_i} \mapsto \left(\frac{\lambda_i}{3}\right)^{m_i}.$$

These new parts are odd. Thus we apply any map that converts odd-part partitions into distinct-part partitions. Finally, multiply each part in the distinct-part sub-partition by 3. $\square$

Example 2.2.1. *Let $n = 12$. The set $S = \{1^{12}, (2^2, 1^8), (2^3, 1^6), (2^4, 1^4), (2^5, 1^2), 2^6, (3^2, 1^6), (3^2, 2^2, 1^2), (3^2, 2^3), (3^3, 1^3), 3^4, (4^2, 2^2), (4^2, 1^4), 4^3, (5^2, 1^2), 6^2\}$ is precisely the set of partitions of 12 wherein no part appears with multiplicity one.*

The part 2 in the partition 2^6 has an even multiplicity. Thus

$$2^6 \mapsto 4^3.$$

Similarly,

$$6^2 \mapsto 12.$$

The part 4 in partition 4^3 has an odd multiplicity, exactly 3. We now apply Glaisher map ϕ ($k = 2$) which gives

$$4 \mapsto 1^4.$$

Multiplying each part by 3, we get

$$4^3 \mapsto 3^4.$$

Applying the mapping to the remaining partitions, we get the following:

$$\boxed{\begin{array}{ccc}
1^{12} & \mapsto & 2^6 \\
(2^2, 1^8) & \mapsto & (4, 2^4) \\
(2^3, 1^6) & \mapsto & (3^2, 2^3) \\
(2^4, 1^4) & \mapsto & (4^2, 2^2) \\
(2^5, 1^2) & \mapsto & (4, 3^2, 2) \\
2^6 & \mapsto & 4^3 \\
(3^2, 1^6) & \mapsto & (6, 2^3) \\
(3^2, 2^2, 1^2) & \mapsto & (6, 4, 2) \\
(3^2, 2^3) & \mapsto & (6, 3^2) \\
(3^3, 1^3) & \mapsto & (9, 3) \\
3^4 & \mapsto & 6^2 \\
(4^2, 2^2) & \mapsto & (8, 4) \\
(4^2, 1^4) & \mapsto & (8, 2^2) \\
4^3 & \mapsto & 3^4 \\
(5^2, 1^2) & \mapsto & (10, 2) \\
6^2 & \mapsto & 12
\end{array}}$$

In the following chapter, we discuss Andrews' generalisation for Theorem 1.1.3.

Chapter 3

A generalisation of Theorem 1.1.3

We introduce Andrews' generalisation for Theorem 1.1.3 followed by Subbarao's theorem which finitizes Andrews' generalisation. We then discuss bijections due to Fu and Sellers, H. Gupta, M. Kanna and B. Dharmendra for Subbarao's theorem. The bijection by M. Kanna and B. Dharmendra shall be referred to as Kanna-Dharmendra bijection. Furthermore, the bijection by Fu and Sellers shall be referred to as Fu-Sellers bijection.

3.1 Andrews' Generalisation

Theorem 3.1.1 (Andrews, [2]). *Let $A_r(n)$ denote the number of partitions of n wherein parts occurring an odd number of times, occur at least $2r+1$ times. Let $B_r(n)$ be the number of partitions of n wherein parts are either even or else congruent to $2r+1 \pmod{4r+2}$. Then $A_r(n) = B_r(n)$.*

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} A_r(n)q^n &= \prod_{n=1}^{\infty}\left(1+q^{2n}+\ldots+q^{2rn}+q^{(2r+1)n}+\ldots\right) \\
&= \prod_{n=1}^{\infty}\Big(1+q^{2n}+q^{4n}\ldots+q^{(2r-2)n}+q^{2rn} \\
&\qquad\qquad + q^{(2r+1)n}(1+q^n+q^{2n}+\ldots)\Big) \\
&= \prod_{n=1}^{\infty}\left(\frac{1-q^{(2r+2)n}}{1-q^{2n}}+\frac{q^{(2r+1)n}}{1-q^n}\right) \\
&= \prod_{n=1}^{\infty}\left(\frac{1+q^{(2r+1)n}}{1-q^{2n}}\right) \\
&= \prod_{n=1}^{\infty}\frac{1}{1-q^{2n}} \times \prod_{n=1}^{\infty} 1+q^{(2r+1)n} \\
&= \prod_{n=1}^{\infty}\frac{1}{1-q^{2n}} \times \prod_{n=1}^{\infty}\frac{1-q^{2(2r+1)n}}{1-q^{(2r+1)n}} \\
&= \prod_{n=1}^{\infty}\frac{1}{1-q^{2n}} \times \prod_{n=1}^{\infty}\frac{1}{1-q^{(4r+2)n-2r-1}} \\
&= \sum_{n=0}^{\infty} B_r(n)q^n.
\end{aligned}$$

□

The case where $r = 0$ describes unrestricted partitions, and the case where $r = 1$ results in Theorem 1.1.3.

3.2 Subbarao's Finitization

The following theorem finitizes Theorem 3.1.1.

Theorem 3.2.1 (Subbarao, [16]). *Let $m > 1, r \geq 0$ be integers, and let $C_{m,r}(n)$ denote the number of partitions of n such that all even multiplicities of the parts are less than $2m$, and all odd multiplicities are at least $2r+1$ and at most $2(m+r)-1$. Let $D_{m,r}(n)$ be the number of partitions of n in which parts are either odd and congruent to $2r+1 \pmod{4r+2}$ or even and not congruent to $0 \pmod{2m}$. Then $C_{m,r}(n) = D_{m,r}(n)$.*

Proof.

$$
\begin{aligned}
&\sum_{n=0}^{\infty} C_{m,r}(n)q^n \\
&= \prod_{n=1}^{\infty} \Big(1 + q^{2n} + q^{4n} + \ldots + q^{(2m-2)n} + q^{(2r+1)n} \\
&\qquad\qquad + q^{(2r+3)n} + \ldots + q^{(2(m+r)-1)n}\Big) \\
&= \prod_{n=1}^{\infty} \Big(1 + q^{2n} + q^{4n} + \ldots + q^{(2m-2)n} \\
&\qquad\qquad + q^{(2r+1)n}(1 + q^{2n} + \ldots + q^{2(m-1)})\Big) \\
&= \prod_{n=1}^{\infty} \left(\frac{1-q^{2mn}}{1-q^{2n}} + \frac{q^{(2r+1)n}(1-q^{2mn})}{1-q^{2n}}\right)
\end{aligned}
$$

$$= \prod_{n=1}^{\infty} \frac{\left(1+q^{(2r+1)n}\right)\left(1-q^{2mn}\right)}{1-q^{2n}}$$

$$= \prod_{\substack{n=1 \\ n \text{ even} \\ n\not\equiv 0 \pmod{2m}}}^{\infty} \frac{1}{(1-q^n)} \prod_{n=1}^{\infty} \frac{1}{\left(1-q^{(4r+2)n-2r-1}\right)}$$

$$= \sum_{n=0}^{\infty} D_{m,r}(n) q^n.$$

□

A special case of Theorem 3.2.1 is Theorem 1.1.4 obtained by setting $m = 2, r = 1$.

3.3 Bijections for Theorem 3.2.1

The following bijection is due to H. Gupta (see [10]).

3.3.1 Gupta's Bijection

Let $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_l^{a_l})$ be a partition enumerated by $C_{m,r}(n)$. As such

$$a_i = \begin{cases} 2v_i, & 0 < v_i < m; \\ 2(w_i + r) + 1, & 0 \leq w_i < m. \end{cases} \tag{3.3.1}$$

Without loss of generality, we can assume that in λ the first j_0 of the λ_i's have an even multiplicity, the next j_1 have an odd multiplicity exceeding $2r + 1$ and the rest have an odd multiplicity equal to $2r+1$. Then we have the following decomposition:

$$\lambda = \left(\bigcup_{i=1}^{j_0} \lambda_i^{2v_i}, \bigcup_{i=j_0+1}^{j_0+j_1} \lambda_i^{(2(w_i+r)+3)}, \bigcup_{i=j_0+j_1+1}^{j} \lambda_i^{2r+1} \right)$$

which can be rearranged to

$$\lambda = \left(\bigcup_{i=1}^{j_0} \lambda_i^{2v_i}, \bigcup_{i=j_0+1}^{j_0+j_1} \lambda_i^{2(w_i+1)}, \bigcup_{i=j_0+1}^{j} \lambda_i^{2r+1} \right).$$

Note that $\bigcup\limits_{i=1}^{j_0} \lambda_i^{2v_i}$ and $\bigcup\limits_{i=j_0+1}^{j_0+j_1} \lambda_i^{2(w_i+1)}$ are sub-partitions wherein parts have even multiplicities. Taking the union of these sub-partitions, λ rearranges to

$$\lambda = \left(\bigcup_{i=1}^{j_0+j_1} \lambda_i^{2u_i}, \bigcup_{i=j_0+1}^{j} \lambda_i^{(2r+1)} \right)$$

where $u_i = v_i$ or $u_i = w_i + 1$. Let $S = \bigcup\limits_{i=1}^{j_0+j_1} \lambda_i^{2u_i}$ and $T = \bigcup\limits_{i=j_0+1}^{j} \lambda_i^{(2r+1)}$. Then $\lambda = (S,\ T)$.

The sub-partition T is a partition with parts appearing exactly $2r+1$ times. Taking one copy of each part, we obtain a distinct sub-partition of T. We then apply Sylvester's map to the distinct sub-partition of T to get $\gamma = (\gamma_1^{c_1}, \gamma_2^{c_2}, \ldots, \gamma_o^{c_o})$, which is a partition into odd parts. Lastly, we multiply each part γ_i by $2r+1$ to obtain $\{(2r+1)\gamma_1\}^{c_1}, \ldots, \{(2r+1)\gamma_o\}^{c_o}$.

The sub-partition S is a partition with parts appearing with even multiplicities. We perform the following transformation on parts in S;

$$\lambda_i^{2u_i} \mapsto (2\xi_i)^{b_i}$$

with

$$\xi_i = \frac{\lambda_i}{m^t} \; ; \; b_i = m^t u_i$$

where t the largest non-negative integer such that m^t divides λ_i. Hence,

$$\left((2\xi_1)^{b_1}, \ldots, (2\xi_h)^{b_h}, \{(2r+1)\gamma_1\}^{c_1}, \ldots, \{(2r+1)\gamma_o\}^{c_o}\right) \tag{3.3.2}$$

is a partition of n enumerated by $D_{m,r}(n)$.

The inverse of this map can easily be constructed. We will not provide the details here. The reader is encouraged to consult [10].

We illustrate the bijection with $\lambda = \left(15^4, 14^4, 11^6, 8^9, 5^9, 3^{12}, 2^{13}\right)$. Note that λ is a partition enumerated by $C_{7,4}(361)$. By going through the decomposition of λ and necessary rearrangements, we get:

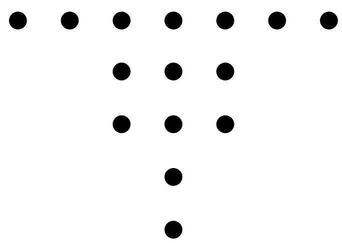
$$S = \left(15^4, 14^4, 11^6, 3^{12}, 2^4\right)$$

$$T = \left(8^9, 5^9, 2^9\right).$$

Firstly, we apply Sylvester's map to the sub-partition $(8, 5, 2)$. The length of the sub-partition is 3, which is odd and thus $v_3 = a_3 = 3$. Solving the remaining equations recursively, we obtain

$$a_1 = 5, \quad a_2 = 3, \quad a_3 = 2, \quad b_1 = 4, \quad b_2 = 3, \quad b_3 = 1$$

and construct the following centrally justified Ferrers graph.



Therefore, we get the partition $\left(7, 3^2, 1^2\right)$. Multiplying each part by 9 gives $\left(63, 27^2, 9^2\right)$. We now transform $(15^4, 14^4, 11^6, 3^{12}, 2^4)$ to get

$\xi_1 = 15, \quad \xi_2 = 2, \quad \xi_3 = 11, \quad \xi_4 = 3, \quad \xi_5 = 2,$
and
$b_1 = 2, \quad b_2 = 14, \quad b_3 = 3, \quad b_4 = 6, \quad b_5 = 2\ .$

Thus we get the sub-partition $\left(30^2, 22^3, 6^6, 4^{16}\right)$. Lastly, taking the union of the two sub-partitions gives

$$\left(63, 30^2, 27^2, 22^3, 9^2, 6^6, 4^{16}\right)$$

which is enumerated by $D_{7,4}(361)$.

3.3.2 Kanna-Dharmendra Bijection

Let $\lambda = (\lambda_1^{s_1}, \lambda_2^{s_2}, \ldots, \lambda_l^{s_l})$ be a partition enumerated by $C_{m,r}(n)$. Depending on the parity of s_i for each i, the following two cases are considered.

Case 1: Suppose $s_i = 2t$ with $1 \leq t \leq m-1$.

$$\lambda_i^{2t} \mapsto \begin{cases} (2\lambda_i)^t, & \text{if } m \nmid \lambda_i; \\ \left(\frac{2\lambda_i}{m^x}\right)^{t.m^x}, & \text{otherwise,} \end{cases}$$

where $\frac{2\lambda_i}{m^x} \not\equiv 0 \pmod{2m}$ and x is the largest non-zero integer that satisfies the equation.

Case 2: Suppose $s_i = 2t-1$ with $r+1 \leq t \leq m+r$.

$$\lambda_i^{2t-1} \mapsto \begin{cases} (2r+1)\lambda_i,\ \lambda_i^{2(t-r-1)}, & \text{if } \lambda_i \text{ is odd ;} \\ \left(\frac{(2r+1)\lambda_i}{2^y}\right)^{2^y}, \lambda_i^{2(t-r-1)}, & \text{if } \lambda_i \text{ is even,} \end{cases}$$

where $\frac{(2r+1)\lambda_i}{2^y} \equiv 2r+1 \pmod{4r+2}$ and y is the largest non-negative integer that satisfies the equation. Note that the multiplicity in $\lambda_i^{2(t-r-1)}$ is even as such we apply Case 1.
The image partition is taken as the union of all image parts.

To invert, let $\mu = (\mu_1^{\omega_1}, \mu_2^{\omega_2}, \ldots, \mu_t^{\omega_t})$ be a partition enumerated by $D_{m,r}(n)$. We consider two cases depending on the parity of μ_i.

Case 1: Suppose μ_i is even.

We write ω_i as

$$\omega_i = m^{\rho_1} + m^{\rho_2} + m^{\rho_3} + \ldots + m^{\rho_l} + \eta \tag{3.3.3}$$

where $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_l > 0$ and $0 \leq \eta < m$. The representation of ω_i in (3.3.3) is unique. This uniqueness comes from the m-ary expansion of ω_i. Construct a partition

$$x_i = \left(m^{\rho_1} \times \frac{\mu_i}{2}\right)^2 \cup \left(m^{\rho_2} \times \frac{\mu_i}{2}\right)^2 \cup \ldots \cup \left(m^{\rho_l} \times \frac{\mu_i}{2}\right)^2 .$$

Thus

$$\mu_i^{\omega_i} \mapsto \begin{cases} x_i, & \text{if } \eta = 0; \\ x_i \cup \left(\frac{\mu_i}{2}\right)^{2\eta}, & \text{if } 0 < \eta < m . \end{cases}$$

Case 2: Suppose μ_i is odd. We write ω_i as $\omega_i = 2^{\rho_1} + 2^{\rho_2} + 2^{\rho_3} + \ldots + 2^{\rho_l} + \zeta$ where $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_l > 0$ and $0 \leq \zeta < 2$ (this representation is unique). Construct a partition

$$y_i = \left(2^{\rho_1} \times \frac{\mu_i}{2r+1}\right)^{(2r+1)} \cup \left(2^{\rho_2} \times \frac{\mu_i}{2r+1}\right)^{(2r+1)}$$

$$\cup \left(2^{\rho_3} \times \frac{\mu_i}{2r+1}\right)^{(2r+1)} \cup \ldots \cup \left(2^{\rho_l} \times \frac{\mu_i}{2r+1}\right)^{(2r+1)} .$$

Thus

$$\mu_i^{\omega_i} \mapsto \begin{cases} y_i, & \text{if } \zeta = 0; \\ y_i \cup \left(\frac{\mu_i}{2r+1}\right)^{2r+1}, & \text{if } \zeta = 1 . \end{cases}$$

The image of μ is now given by taking the union of all image parts. $\square$

Example 3.3.1. *Let* $m = 7, r = 4$. *Then* $\lambda = (15^4, 14^4, 11^6, 8^9, 5^9, 3^{12}, 2^{13})$ *is a partition enumerated by* $C_{7,4}(361)$.

Applying Kanna-Dharmendra map, we obtain the following:

$$15^4 \mapsto 30^2, \; (s_i = 2(2);\; t = 2);$$
$$14^4 \mapsto 4^{14}, \; (s_i = 2(2);\; 14 = 2(7);\; x = 1);$$
$$11^6 \mapsto 22^3, \; (s_i = 2(3);\; t = 3);$$
$$8^9 \mapsto 9^8, \; (s_i = 2(5) - 1;\; y = 3;\; 2(t - r - 1) = 0);$$
$$5^9 \mapsto 45, \; (s_i = 2(5) - 1;\; 2(t - r - 1) = 0);$$
$$3^{12} \mapsto 6^6, \; (s_i = 2(6);\; t = 6);$$
$$2^{13} \mapsto (9^2, 4^2), \; (s_i = 2(7) - 1;\; y = 1;\; 2(t - r - 1) = 4).$$

Taking the union we obtain

$$(45, 30^2, 22^3, 9^{10}, 6^6, 4^{16})$$

which is a partition enumerated by $D_{7,4}(361)$.

3.3.3 Fu-Sellers Bijection

Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_t^{m_t})$ be a partition enumerated by $C_{m,r}(n)$. We now consider two cases, depending on the parity of m_i for each i, in order to define the bijection.

Case 1: m_i is even

$$\lambda_i^{m_i} \mapsto \left(\frac{2\lambda_i}{m^{\sigma_i}}\right)^{\frac{m_i m^{\sigma_i}}{2}}$$

where $\sigma_i = \mathrm{ord}_m(\lambda_i) := \max\{\sigma \in \mathbb{Z}_{\geq 0} : m^\sigma \mid \lambda_i\}$.

Case 2: m_i is odd ($m_i \geq 2r+1$)
Peel off $2r+1$ copies of the part λ_i and apply the previous algorithm to the remaining parts. From the $2r+1$ copies of each λ_i that were peeled off, take a copy of each part formulating a sub-partition into distinct parts. Then apply any map that converts distinct-part partitions into odd-part partitions. Finally, multiply each of the parts in this odd-part sub-partition by $2r+1$. Each of these parts will then be odd and congruent to $2r+1 \pmod{4r+2}$.
The image of λ is the union of all image parts.

$\square$

For example, consider a partition $\lambda = \left(13^8, 10^{13}, 9^4, 7^7, 6^{11}, 3^6\right)$ enumerated by $C_{5,3}(403)$. We separate the partition according to the parity of multiplicities. The separation is $\left(13^8, 9^4, 3^6\right) \cup \left(10^{13}, 7^7, 6^{11}\right)$. From the sub-partition in which parts have odd multiplicities $\left(10^{13}, 7^7, 6^{11}\right)$, peel off 7 copies and so we have the decomposition $\left(\left(13^8, 10^6, 9^4, 6^4, 3^6\right), \left(10^7, 7^7, 6^7\right)\right)$.

We begin by transforming the sub-partition with even multiplicities $\left(13^8, 10^6, 9^4, 6^4, 3^6\right)$. Firstly, we need to determine the order of the parts.

$\sigma_1 = \mathrm{ord}_5(13) = 0, \quad \sigma_2 = \mathrm{ord}_5(10) = 1, \quad \sigma_3 = \mathrm{ord}_5(9) = 0,$
$\sigma_4 = \mathrm{ord}_5(6) = 0, \quad \sigma_5 = \mathrm{ord}_5(3) = 0.$

Thus

$13^8 \mapsto 26^4, \quad 10^6 \mapsto 4^{15}, \quad 9^4 \mapsto 18^2, \quad 6^4 \mapsto 12^2, \quad 3^6 \mapsto 6^3.$

Now applying the Glaisher map to the distinct sub-partition $(10, 7, 6)$, we get

$$10 = 2^1 \times 5, \qquad 7 = 2^0 \times 7, \qquad 6 = 2^1 \times 3\,.$$

Thus

$$10 \mapsto 5^2, \qquad 7 \mapsto 7, \qquad 6 \mapsto 3^2.$$

Multiplying each part by 7 gives $(49, 35^2, 21^2)$. Now taking the union of all image parts, we obtain $\big(49, 35^2, 26^4, 21^2, 18^2, 12^2, 6^3, 4^{15}\big)$ which a partition enumerated by $D_{5,3}(403)$.

In the following chapter, we present the results of our study.

Chapter 4

Contributions

In this chapter, we present the results of our study. In Section 4.1, we give a new bijection of Theorem 1.1.3. In Section 4.2, we provide a generalisation of Theorem 1.1.4 and finally in Section 4.3, we deduce parity formulas for partition functions of Subbarao type. All these sections have been written up as a research paper.

4.1 A New Bijection for Theorem 1.1.3

In this section, we give a new bijection of Theorem 1.1.3 (see MacMahon [13]). George Andrews gave a bijective mapping for the theorem (see [6]); however, the one we give in this section is different. Our approach is to prove a series of partition identities bijectively, and as a consequence, the theorem is proved. We declare the following notation:

$A(n)$: the set of partitions of n into parts appearing at least twice.

$B(n)$: the set of partitions of n in which parts are not congruent to $\pm1, \pm5 \pmod{12}$.
$A_1(n)$: the set of partitions of n into odd parts appearing at least twice.
$A_2(n)$: the set of partitions of n where parts are congruent to $\pm2, \pm3 \pmod{12}$.
$A_3(n)$: the set of partitions of n where even parts are congruent to $2 \pmod 4$ and odd parts appear exactly 3 times.
$E_1(n)$: the set of partitions of n into even parts appearing at least twice.
$E_2(n)$: the set of partitions of n where parts are congruent to $0, \pm4, 6 \pmod{12}$.

Lemma 4.1.1. *For* $n \geq 0$

$$|A_2(n)| = |A_3(n)|.$$

Proof. Define a map $\theta_{23} : A_2(n) \longrightarrow A_3(n)$ as follows. Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \lambda_3^{m_3}, \ldots) \in A_2(n)$.

If $\lambda_i \equiv \pm2 \pmod{12}$, then

$$\lambda_i^{m_i} \mapsto \lambda_i^{m_i}.$$

If $\lambda_i \equiv \pm3 \pmod{12}$, then

$$\lambda_i^{m_i} \mapsto \begin{cases} \left(\frac{\lambda_i}{3}\right)^3, & m_i = 1; \\ (2\lambda_i)^{\frac{1}{6}\left[3m_i - \frac{3}{2}(1-(-1)^{m_i})\right]}, \left(\frac{\lambda_i}{3}\right)^{\frac{3}{2}(1-(-1)^{m_i})}, & m_i > 1. \end{cases}$$

We show that the image of $\lambda \in A_2(n)$ is indeed in $A_3(n)$. Parts congruent to $\pm 2 \pmod{12}$ are fixed under the mapping, and this is clear. If $\lambda_i \equiv \pm 3 \pmod{12}$, then $\lambda_i = 3 + 12t$ or $9 + 12t$ for some $t \in \mathbb{Z}_{\geq 0}$. In the image $(m_i = 1)$, we have $\frac{\lambda_i}{3} = 1 + 4t$ or $3 + 4t$ so that the image part is odd and occurs three times. If $m_i > 1$, the image part $2\lambda_i$ is such that $2\lambda_i = 6 + 24t$ or $18 + 12t$, which is even and congruent to $2 \pmod 4$.

The map θ_{23} is a bijection whose inverse $\theta_{23}^{-1} : A_3(n) \longrightarrow A_2(n)$ is given as follows. Let $\mu = (\mu_1^{m_1}, \mu_2^{m_2}, \ldots) \in A_3(n)$. If $\mu_i \equiv 2 \pmod 4$, then

$$\mu_i^{m_i} \mapsto \begin{cases} \mu_i^{m_i}, & \mu_i \equiv 2, 10 \pmod{12}; \\ \left(\frac{\mu_i}{2}\right)^{2m_i}, & \mu_i \equiv 6 \pmod{12}. \end{cases}$$

If $\mu_i \equiv 1 \pmod 2$, then

$$\mu_i^{m_i} \mapsto (3\mu_i)^{\frac{m_i}{3}}.$$

We show that the image of $\mu \in A_3(n)$ under θ_{23}^{-1} is indeed in $A_2(n)$. Parts congruent to $2 \pmod 4$ are also congruent to $\pm 2, 6 \pmod{12}$. Thus partitions with parts congruent to $\pm 2 \pmod{12}$ are fixed. If $\mu_i \equiv 6 \pmod{12}$, then $\mu_i = 6 + 12t$ for some $t \in \mathbb{Z}_{\geq 0}$. In the image, we have $\frac{\mu_i}{2} = 3 + 6t$, which is odd and congruent to $\pm 3 \pmod{12}$. If $\mu_i \equiv 1 \pmod 2$, then $\mu_i = 1 + 2t$ for some $t \in \mathbb{Z}_{\geq 0}$. In the image, we have $3\mu_i = 3 + 6t$, which is odd and congruent to $\pm 3 \pmod{12}$.

$\square$

Lemma 4.1.2. *For $n \geq 0$,*

$$|A_1(n)| = |A_3(n)|.$$

Proof. Define a map $\theta_{13} : A_1(n) \longrightarrow A_3(n)$ as follows. Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, , \ldots) \in A_1(n)$. Then

$$\lambda_i^{m_i} \mapsto \begin{cases} \lambda_i^{m_i}, & m_i = 3; \\ (\lambda_i)^3, (2\lambda_i)^{\frac{m_i-3}{2}}, & m_i > 3 \text{ and } m_i \equiv 1 \pmod 2; \\ (2\lambda_i)^{\frac{m_i}{2}}, & m_i \equiv 0 \pmod 2. \end{cases}$$

We show that the image of $\lambda \in A_1(n)$ is indeed in $A_3(n)$. If $m_i = 3$, then the image part is odd and has multiplicity 3. So this is clear. If $m_i > 3$ and $m_i \equiv 1 \pmod 2$, the image parts are λ_i and $2\lambda_i$. Since λ_i has multiplicity 3, this is clear. Now $2\lambda_i = 2 + 4t$ for some $t \in \mathbb{Z}_{\geq 0}$, which is even and congruent to $2 \pmod 4$ and has multiplicity $\frac{m_i-3}{2} \geq 1$. If $m_i \equiv 0 \pmod 2$, the image part is $2\lambda_i$ which is congruent to $2 \pmod 4$ and has multiplicity $\frac{m_i}{2} \geq 1$. Indeed the image parts satisfy the description of parts of partitions in $A_3(n)$.

The map θ_{13} is bijective and its inverse $\theta_{13}^{-1} : A_3(n) \longrightarrow A_1(n)$ is given as follows. Let $\mu = (\mu_1^{m_1}, \mu_2^{m_2}, \ldots) \in A_3(n)$. Then

$$\mu_i^{m_i} \mapsto \begin{cases} \left(\dfrac{\mu_i}{2}\right)^{2m_i}, & \text{if } \mu_i \equiv 2 \pmod 4; \\ \mu_i^{m_i}, & \text{if } \mu_i \equiv 1 \pmod 2. \end{cases}$$

We show that the image of $\mu \in A_3(n)$ under θ_{13}^{-1} is indeed in $A_1(n)$. Two cases are considered depending on the parity of parts. If μ_i is odd, the image part is μ_i and has multiplicity 3. So this is clear. If $\mu_i \equiv 2 \pmod 4$, the image part $\frac{\mu_i}{2} = 1+2t$ for some $t \in \mathbb{Z}_{\geq 0}$ which is odd and has an even multiplicity. $\square$

Theorem 4.1.1. *For* $n \geq 0$*,*

$$|A_1(n)| = |A_2(n)|.$$

Proof. Via the composition of maps, we have that $\theta_{23}^{-1} \circ \theta_{13} : A_1(n) \longrightarrow A_2(n)$ is a bijection. $\square$

Define a map $\beta_{12} : E_1(n) \longrightarrow E_2(n)$ as follows. Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, , \ldots) \in E_1(n)$. Then
Case 1: $\lambda_i \equiv 0, \pm 4 \pmod{12}$

$$\lambda_i^{m_i} \mapsto \begin{cases} \lambda_i^{m_i}, & m_i \equiv 0 \pmod 3; \\ (\lambda_i)^{m_i-4}, (2\lambda_i)^2, & m_i \equiv 1 \pmod 3; \\ (\lambda_i)^{m_i-2}, (2\lambda_i)^1, & m_i \equiv 2 \pmod 3. \end{cases}$$

Case 2: $\lambda_i \equiv \pm 2, 6 \pmod{12}$

$$\lambda_i^{m_i} \mapsto \begin{cases} (3\lambda_i)^{\frac{m_i}{3}}, & m_i \equiv 0 \pmod 3; \\ (3\lambda_i)^{\frac{m_i-4}{3}}, (2\lambda_i)^2, & m_i \equiv 1 \pmod 3; \\ (3\lambda_i)^{\frac{m_i-2}{3}}, (2\lambda_i)^1, & m_i \equiv 2 \pmod 3. \end{cases}$$

Theorem 4.1.2. *For $n \geq 0$, the map β_{12} is a bijection.*

Proof. It is enough to show that the image of $\lambda \in E_1(n)$ is indeed in $E_2(n)$. Consider cases dependent on the value of the multiplicity m_i modulo 3 and part λ_i modulo 12. We have parts congruent to $0, \pm 2, \pm 4, 6 \pmod{12}$. Parts congruent to $0, \pm 4 \pmod{12}$ are in $E_2(n)$, as such are fixed under the mapping if m_i is congruent to $0 \pmod 3$. For λ_i congruent to $\pm 2, 6 \pmod{12}$, the image part $3\lambda_i = \pm 6 + 36t$ or $18 + 36t$ for some $t \in \mathbb{Z}_{\geq 0}$. These parts are congruent to $6 \pmod{12}$. If m_i is congruent to $1 \pmod 3$, the image parts are λ_i, $3\lambda_i$ and $2\lambda_i$. The part $2\lambda_i = 24t$ or $\pm 8 + 24t$ for some $t \in \mathbb{Z}_{\geq 0}$, which is congruent to $0, \pm 4 \pmod{12}$. If m_i is congruent to $2 \pmod 3$, the image parts are λ_i, $3\lambda_i$ and $2\lambda_i$. So this is clear. $\square$

The inverse map of β_{12}, denoted $\beta_{12}^{-1} : E_2(n) \longrightarrow E_1(n)$ is given as follows. Let $\mu = (\mu_1^{m_1}, \mu_2^{m_2}, \ldots) \in E_2(n)$. Then

$$\mu_i^{m_i} \mapsto \begin{cases} \left(\frac{\mu_i}{3}\right)^{3m_i}, & \mu_i \equiv 6 \pmod{12}; \\ \mu_i^{3\lfloor \frac{m_i}{3} \rfloor}, \ \left(\frac{\mu_i}{2}\right)^{2(m_i - 3\lfloor \frac{m_i}{3} \rfloor)}, & \mu_i \equiv 0, \pm 4 \pmod{12}. \end{cases}$$

We show that the image of $\mu \in E_2(n)$ under β_{12}^{-1} is indeed in $E_1(n)$. If μ_i is congruent to $6 \pmod{12}$, the image part is $\frac{\mu_i}{3} = 2 + 4t$, which is congruent to $\pm 2, 6 \pmod{12}$. If μ_i is congruent to $0, \pm 4 \pmod{12}$, the image parts are μ_i and $\frac{\mu_i}{2}$. Since μ_i is congruent to $0, \pm 4 \pmod{12}$, this is clear. Now $\frac{\mu_i}{2} = 6t$ or $\pm 2 + 6t$ for some $t \in \mathbb{Z}_{\geq 0}$, which is congruent to $0, \pm 2, \pm 4, 6 \pmod{12}$. Multiplicities for the image parts are

$3m_i, 3\lfloor \frac{m_i}{3} \rfloor, 2(m_i - 3\lfloor \frac{m_i}{3} \rfloor)$, which are greater than or equal to 2.

By the description of $A(n)$, it is clear that every $\lambda \in A(n)$ can be decomposed uniquely into (λ_e, λ_o) where λ_e and λ_o are the sub-partitions of λ with even (resp. odd) parts, i.e. $\lambda_e \in E_1(j)$ and $\lambda_o \in A_1(n-j)$ for some $j \geq 0$. Similarly, $\mu \in B(n)$ can be decomposed uniquely into (μ_1, μ_2) where μ_1 is the sub-partition of μ with parts congruent to $0, \pm 4, 6 \pmod{12}$ and μ_2 is the sub-partition with parts congruent to $\pm 2, \pm 3 \pmod{12}$, i.e. $\mu_1 \in E_2(j)$ and $\mu_2 \in A_2(n-j)$ for some $j \geq 0$. Hence, we have the following equality of sets (up to a bijection):

$$A(n) = \bigcup_{j=0}^{n} \left(A_1(j) \times E_1(n-j)\right) \tag{4.1.1}$$

and

$$B(n) = \bigcup_{j=0}^{n} \left(A_2(j) \times E_2(n-j)\right). \tag{4.1.2}$$

Using the decomposition in (4.1.1), we now have a new bijection for Theorem 1.1.3, as stated in the following corollary.

Corollary 4.1.1. *Decompose $A(n)$ and $B(n)$ as in (4.1.1) so that every $\lambda \in A(n)$ is such that $\lambda = (\lambda_o, \lambda_e)$ where $\lambda_o \in A_1(j)$ and $\lambda_e \in E_1(n-j)$ for some $j \geq 0$. The map $\tau : A(n) \longrightarrow B(n)$ given by*

$$\tau \,:\, \lambda \mapsto \left(\theta_{23}^{-1} \circ \theta_{13}\right)(\lambda_o) \cup \beta_{12}(\lambda_e)$$

is a bijection.

Note that τ provides a new bijection for Theorem 1.1.3, which is different from known bijections in the literature (Andrews and Sellers). We give an example for verification .

Example 4.1.1. *For $n = 12$, $\{1^{12}, (2^2, 1^8), (2^3, 1^6), (2^4, 1^4), (2^5, 1^2), 2^6, (3^2, 1^6), (3^2, 2^2, 1^2), (3^2, 2^3), (3^3, 1^3), 3^4, (4^2, 2^2), (4^2, 1^4), 4^3, (5^2, 1^2), 6^2\}$ is the set enumerated by $|A(12)|$.*

Partitions $2^6, 4^3, 6^2$ are in $E_1(12)$ since their parts are even (congruent to $2, 4, 6 \pmod{12}$, respectively), and appear at least twice. As such, we apply the map β_{12}. The multiplicities for 2^6 and 4^3 are congruent to $0 \pmod 3$. Thus

$$2^6 \mapsto 6^2,$$

$$4^3 \mapsto 4^3.$$

The multiplicity for 6^2 is congruent to $2 \pmod 3$. Thus

$$6^2 \mapsto 12.$$

Partitions $12, 6^2, 4^3$ have parts congruent to $0, 6$ and $4 \pmod{12}$, respectively. As such they are partitions in $E_2(12)$.

The partition 3^4 is in $A_1(12)$ since the part is odd and appears at least twice. Applying θ_{13} gives

$$3^4 \mapsto 6^2$$

since $4 \equiv 0 \pmod 2$. The part 6 is both congruent to $2 \pmod 4$ and $6 \pmod{12}$. Thus applying θ_{23}^{-1} gives

$$6^2 \mapsto 3^4,$$

a partition in $A_2(12)$.

For comparison with known maps, see below.

Our bijection: $A(12) \to B(12)$

$$(2^6) \mapsto (6^2),$$
$$(4^3) \mapsto (4^3),$$
$$(6^2) \mapsto (12).$$

Andrews' bijection: $A(12) \to B(12)$

$$(2^6) \mapsto (4^3),$$
$$(4^3) \mapsto (12),$$
$$(6^2) \mapsto (6^2).$$

Fu and Sellers' bijection: $A(12) \to B(12)$

$$(2^6) \mapsto (4^3),$$
$$(4^3) \mapsto (3^4),$$
$$(6^2) \mapsto (12).$$

4.2 A Generalisation of Theorem 1.1.4

Our goal in this section is to give a simple extension of Theorem 1.1.4. We start by formulating the following definition.

Definition 4.2.1. *Let $k \geq 2$ and $a_1, a_2, \ldots, a_r \geq 1$ be integers. The tuple $(a_1, a_2, \ldots, a_r)$ is called k-admissible if*

i. $\gcd(a_i, a_j) = 1 \;\; \forall\, i \neq j$*;*

ii. If $\sum_{i=1}^{r} \alpha_i a_i = \sum_{i=1}^{r} \beta_i a_i$ *where* $\alpha_i, \beta_i \in \mathbb{Z}_{\geq 0}$*, then* $\alpha_i = \beta_i \;\; \forall\, i = 1, \ldots, r$*;*

iii. $A_i \cap A_j = \emptyset$ *for* $1 \leq i \neq j \leq r$ *where*

$$A_i = \{x \in \mathbb{Z}_{\geq 1} \; : \; x \equiv -(s+km)a_i \pmod{k \prod_{i=1}^{r} a_i} \; : \; 1 \leq s \leq k-1, \;\; m = 0, 1, \ldots, (\prod_{\substack{j=1 \\ j \neq i}}^{r} a_j) - 1\}.$$

For instance, using the definition, one can easily verify that $(2, 5)$ is 5-admissible.

Let $(a_1, a_2, a_3, \ldots, a_r)$ be k-admissible where $k \geq 2$. Denote by $B(a_1, a_2, \ldots, a_r, k, n)$, the number of partitions of n in which parts occur with multiplicities $\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_r a_r$ where $\alpha_i \in \mathbb{Z}$ and $0 \leq \alpha_i \leq k-1$ for all i. Then we have the following theorem.

Theorem 4.2.1. *Let* $C(a_1, a_2, \ldots, a_r, k, n)$ *be the number of partitions of* n *into parts congruent to* $-(s+mk)a_j \pmod{k \prod_{i=1}^{r} a_i}$ *where* $s = 1, 2, \ldots, k-1,\ \ j = 1, 2, \ldots, r$ *and* $m = 0, 1, 2, \ldots, (\prod_{\substack{i=1\\i\neq j}}^{r} a_i) -\ 1$*. Then*

$$C(a_1, a_2, \ldots, a_r, k, n) = B(a_1, a_2, \ldots, a_r, k, n).$$

Proof. Note that

$$\begin{aligned}
&\sum_{n=0}^{\infty} B(a_1, a_2, \ldots, a_r, k, n)q^n \\
&= \prod_{n=1}^{\infty} \Big(1 + q^{a_1 n} + q^{2a_1 n} + \ldots + q^{(k-1)a_1 n} + q^{a_2 n} + \ldots + q^{(k-1)a_2} \\
&\qquad + q^{(a_1+a_2)n} + q^{(a_1+2a_2)n} + \ldots + q^{((k-1)a_1+(k-1)a_2+\ldots+(k-1)a_r)n}\Big) \\
&= \prod_{n=1}^{\infty} \sum_{\alpha_1=0}^{k-1} \sum_{\alpha_2=0}^{k-1} \sum_{\alpha_3=0}^{k-1} \cdots \sum_{\alpha_r=0}^{k-1} q^{(\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_r a_r)n} \\
&= \prod_{n=1}^{\infty} \left(\sum_{\alpha_1=0}^{k-1} q^{\alpha_1 a_1 n}\right) \left(\sum_{\alpha_2=0}^{k-1} q^{\alpha_2 a_2 n}\right) \cdots \left(\sum_{\alpha_r=0}^{k-1} q^{\alpha_r a_r n}\right) \\
&= \prod_{n=1}^{\infty} \left(\frac{1 - q^{a_1 nk}}{1 - q^{a_1 n}}\right) \left(\frac{1 - q^{a_2 nk}}{1 - q^{a_2 n}}\right) \left(\frac{1 - q^{a_3 nk}}{1 - q^{a_3 n}}\right) \cdots \left(\frac{1 - q^{a_r nk}}{1 - q^{a_r n}}\right) \\
&= \prod_{j=1}^{r} \prod_{n=1}^{\infty} \frac{(1 - q^{a_j nk})}{(1 - q^{a_j n})}
\end{aligned}$$

$$=\prod_{j=1}^{r}\prod_{n=1}^{\infty}\left(\frac{1}{1-q^{ka_jn-a_j}}\right)\left(\frac{1}{1-q^{ka_jn-2a_j}}\right)\left(\frac{1}{1-q^{ka_jn-3a_j}}\right)$$

$$\left(\frac{1}{1-q^{ka_jn-4a_j}}\right)\times\ldots\times\left(\frac{1}{1-q^{ka_jn-(k-1)a_j}}\right)$$

$$=\prod_{j=1}^{r}\prod_{s=1}^{k-1}\prod_{n=1}^{\infty}\left(\frac{1}{1-q^{ka_jn-sa_j}}\right)$$

$$=\prod_{j=1}^{r}\prod_{s=1}^{k-1}\prod_{n=1}^{\infty}\left(\frac{1}{1-q^{k\prod\limits_{i=1}^{r}a_i\,n-sa_j}}\right)\left(\frac{1}{1-q^{k\prod\limits_{i=1}^{r}a_i\,n-sa_j-ka_j}}\right)$$

$$\times\left(\frac{1}{1-q^{k\prod\limits_{i=1}^{r}a_i\,n-sa_j-2ka_j}}\right)\left(\frac{1}{1-q^{k\prod\limits_{i=1}^{k}a_i\,n-sa_j-3ka_j}}\right)$$

$$\times\ldots\times\left(\frac{1}{1-q^{k\prod\limits_{i=1}^{r}a_i\,n-sa_j-((\prod\limits_{\substack{i=1\\i\neq j}}^{k}a_i)-1)ka_j}}\right)$$

$$=\prod_{j=1}^{r}\prod_{s=1}^{k-1}\prod_{m=0}^{(\prod\limits_{\substack{i=1\\i\neq j}}^{k}a_i)-1}\prod_{n=1}^{\infty}\left(\frac{1}{1-q^{k\prod\limits_{i=1}^{r}a_i\,n-sa_j-mka_j}}\right)$$

$$=\sum_{n=0}^{\infty}C(a_1,a_2,\ldots,a_r,k,n)q^n. \qquad \square$$

Lemma 4.2.1. *Let* $k \in \mathbb{Z}_{>1}$. *If* $a_1 \not\equiv a_2 \pmod{k}$, *then* $A_1 \cap A_2 = \emptyset$ *where*

$$A_1 = \{x \in \mathbb{Z}_{\geq 1} : x \equiv -(1+km)a_1 \pmod{ka_1a_2} : 0 \leq m \leq a_2-1\},$$

$$A_2 = \{x \in \mathbb{Z}_{\geq 1} : x \equiv -(1+k\alpha)a_2 \pmod{ka_1a_2} : 0 \leq \alpha \leq a_1-1\}.$$

Proof. Suppose $a_1 \not\equiv a_2 \pmod{k}$. If $A_1 \cap A_2 \neq \emptyset$, then there is $x \in \mathbb{Z}_{\geq 1}$ such that $x \equiv -(1+km)a_1 \pmod{ka_1a_2}$ and $x \equiv -(1+k\alpha)a_2 \pmod{ka_1a_2}$ for some $m \in \{0, 1, \ldots, a_2 - 1\}$ and $\alpha \in \{0, 1, \ldots, a_1 - 1\}$. Thus

$$-(1+km)a_1 + (1+k\alpha)a_2 \equiv 0 \pmod{ka_1a_2}$$

so that

$$(a_2 - a_1) + k(\alpha a_2 - ma_1) \equiv 0 \pmod{ka_1a_2}.$$

From this, it is clear that $a_2 - a_1$ is congruent to $0 \pmod{k}$, which is a contradiction. Thus we must have $A_1 \cap A_2 = \emptyset$. □

Corollary 4.2.1. *If* $a_1 \not\equiv a_2 \pmod{2}$ *and* $\gcd(a_1, a_2) = 1$, *then the number of partitions of* n *in which parts occur with multiplicities* $a_1, a_2, a_1 + a_2$ *is equal to the number of partitions of* n *into parts congruent to* $-(1+2m)a_j \pmod{2a_1a_2}$ *where* $j = 1, 2$ *and* $m = 0, 1, \ldots, (\prod_{\substack{i=1 \\ i \neq j}}^{2} a_i) - 1$.

Proof. By Lemma 4.2.1 with $k = 2$, we have $A_1 \cap A_2 = \emptyset$. Since $\gcd(a_1, a_2) = 1$ and that $a_1, a_2, a_1 + a_2$ are all different, by Definition 4.2.1, we conclude that (a_1, a_2) is 2-admissible. Setting $k = r = 2$ in Theorem 4.2.1 yields the result. $\square$

Remark 4.2.1. *Corollary 4.2.1 is a generalisation of Subbarao's partition theorem, Theorem 1.1.4, where $a_1 = 2$ and $a_2 = 3$. Of course Theorem 4.2.1 is a more general extension.*

Bijective proof of Theorem 4.2.1

Let λ be enumerated by $B(a_1, a_2, \ldots, a_r, k, n)$. Write each multiplicity m of a part p as a linear combination of $a_1, a_2, \ldots, a_r$, i.e.

$$m = \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_r a_r.$$

For clarity's sake, we shall call α_i the coefficient of a_i in the multiplicity m of p. Now construct partitions β_j's in this way:

$$\beta_j = ((a_j p_1)^{\alpha_{1j}}, (a_j p_2)^{\alpha_{2j}}, \ldots), \quad 1 \leq j \leq r$$

where $p_1 > p_2 > \ldots$ are the original parts of λ and α_{ij} is the coefficient of a_j in the multiplicity of the part p_i of λ.
Then divide each part in β_j by a_j and then apply the inverse of the Glaisher map ϕ^{-1} (see Theorem 1.1.2). In other words, compute

$$\phi^{-1}(p_1^{\alpha_{1j}}, p_2^{\alpha_{2j}}, \ldots). \tag{4.2.1}$$

Multipy each part in (4.2.1) by a_j to get $a_j \phi^{-1}(p_1^{\alpha_{1j}}, p_2^{\alpha_{2j}}, \ldots)$ and the image of λ is given by taking the union over all j's, i.e.

$$\bigcup_{j=1}^{r} a_j \phi^{-1}(p_1^{\alpha_{1j}}, p_2^{\alpha_{2j}}, \ldots).$$

To invert the process, suppose μ is enumerated by $C(a_1, a_2, \ldots, a_r, k, n)$. Decompose μ into $(\mu_1, \mu_2, \ldots, \mu_r)$ where u_j is the sub-partition consisting of parts congruent to $-(s+mk)a_j \pmod{k \prod_{i=1}^{r} a_i}$. For each μ_j, perform the following steps:

(a) Divide each part by a_j.

(b) Apply ϕ to the result in (a).

(c) Repeat each part in (b) a_j times, and call the resulting partition $\bar{\mu}_j$.

Then the image of μ is given by

$$\bigcup_{j=1}^{r} \bar{\mu}_j.$$

We include the following example to illustrate our bijection.

Example 4.2.1. *Let* $\lambda = (27^3, 22^6, 19^3, 15^5, 12^6, 10^8, 7^{11}, 5^{10}, 4^{13}, 2^{16})$, $k = 3$, $a_1 = 3$ *and* $a_2 = 5$.

Clearly, λ is enumerated by $B(3, 5, 3, 708)$. Executing the first step of the bijection, we have

$$\begin{array}{llll}
27^3 = 27^{1.3+0.5}, & p_1 = 27, & \alpha_1 = 1, & \alpha_2 = 0; \\
22^6 = 22^{2.3+0.5}, & p_2 = 22, & \alpha_1 = 2, & \alpha_2 = 0; \\
19^3 = 22^{1.3+0.5}, & p_3 = 19, & \alpha_1 = 1, & \alpha_2 = 0; \\
15^5 = 15^{0.3+1.5}, & p_4 = 15, & \alpha_1 = 0, & \alpha_2 = 1;
\end{array}$$

$$
\begin{aligned}
12^6 &= 12^{2.3+0.5}, &\quad p_5 &= 12, &\quad \alpha_1 &= 2, &\quad \alpha_2 &= 0;\\
10^8 &= 10^{1.3+1.5}, & p_6 &= 10, & \alpha_1 &= 1, & \alpha_2 &= 1;\\
7^{11} &= 7^{2.3+1.5}, & p_7 &= 7, & \alpha_1 &= 2, & \alpha_2 &= 1;\\
5^{10} &= 5^{0.3+2.5}, & p_8 &= 5, & \alpha_1 &= 0, & \alpha_2 &= 2;\\
4^{13} &= 4^{1.3+2.5}, & p_9 &= 4, & \alpha_1 &= 1, & \alpha_2 &= 2;\\
2^{16} &= 22^{2.3+2.5}, & p_{10} &= 2, & \alpha_1 &= 2, & \alpha_2 &= 2.
\end{aligned}
$$

Thus

$$\beta_1 = (81, 66^2, 57, 36^2, 30, 21^2, 12, 6^2),$$
$$\beta_2 = (75, 50, 35, 25^2, 20^2, 10^2).$$

so that

$$\frac{\beta_1}{a_1} = (27, 22^2, 19, 12^2, 10, 7^2, 4, 2^2) \text{ and } \frac{\beta_2}{a_2} = (15, 10, 7, 5^2, 4^2, 2^2).$$

We now apply the Glaisher map ϕ^{-1} to $\frac{\beta_1}{a_1}$ and $\frac{\beta_2}{a_2}$ and obtain

$$\phi^{-1}\left(\frac{\beta_1}{a_1}\right) = (22^2, 19, 10, 7^2, 4^7, 2^2, 1^{27}),$$
$$\phi^{-1}\left(\frac{\beta_2}{a_2}\right) = (10, 7, 5^5, 4^2, 2^2).$$

and

$$a_1\phi^{-1}\left(\frac{\beta_1}{a_1}\right) = (66^2, 57, 30, 21^2, 12^7, 6^2, 3^{27}),$$
$$a_2\phi^{-1}\left(\frac{\beta_2}{a_2}\right) = (50, 35, 25^5, 20^2, 10^2).$$

Therefore, the image of λ is

$$(66^2, 57, 50, 35, 30, 25^5, 21^2, 20^2, 12^7, 10^2, 6^2, 3^{27}),$$

a partition enumerated by $C(3, 5, 3, 708)$.

We now invert the process. The residue set is

$$\{3, 5, 6, 10, 12, 15, 20, 21, 24, 25, 30, 33, 35, 39, 40, 42\}.$$

Thus

$$\mu_1 = (66^2, 57, 30, 21^2, 12^7, 6^2, 3^{27}),$$
$$\mu_2 = (50, 35, 25^5, 20^2, 10^2)$$

so that

$$\frac{\mu_1}{a_1} = (22^2, 19, 10, 7^2, 4^7, 2^2, 1^{27}) \text{ and } \frac{\mu_2}{a_2} = (10, 7, 5^5, 4^2, 2^2).$$

Applying the Glaisher map ϕ to $\frac{\mu_1}{a_1}$ and $\frac{\mu_2}{a_2}$ with

$1 = 1.3^0,\ 2 = 2.3^0,\ 5 = 2.3^0{+}1.3^1,\ 7 = 1.3^0{+}2.3^1$ and $27 = 1.3^3$

we obtain:

$$\phi\left(\frac{\mu_1}{a_1}\right) = (27, 22^2, 19, 10, 7^2, 4, 2^2),$$
$$\phi\left(\frac{\mu_2}{a_2}\right) = (15, 10, 7, 5^2, 4^2, 2^2).$$

Repeating each part a_j times and taking union of the resulting partitions results in

$$\lambda = (27^3, 22^6, 19^3, 15^5, 12^6, 10^8, 7^{11}, 5^{10}, 4^{13}, 2^{16}).$$

Suppose we relax some of the conditions in k-admissibility definition, what would happen? For instance, if the third condition (iii) is relaxed in Definition 4.2.1, then Theorem 4.2.1 does not hold. However one always has a partition identity, in terms of color partitions. To demonstrate this, we provide an example (see Theorem 4.2.2).

Theorem 4.2.2. *The number of partitions of* n *in which parts occur with multiplicities* $2\alpha_1 + 7\alpha_2 + 15\alpha_3$ *where* $0 \leq \alpha_i \leq 2$ *is equal to the number of partitions of* n *into parts congruent to* $-(s+3m)a_j \pmod{630}$ *where* $s = 1, 2,\ \ j = 1, 2, 3$ *and* $m = 0, 1, \ldots, (\prod_{\substack{i=1 \\ i \neq j}}^{3} a_i) - \ 1$ *and those parts congruent to* ± 14, ± 28, ± 56, ± 70, ± 98, ± 112, ± 140, ± 154, ± 182, ± 196, ± 224, ± 266, $\pm 308 \pmod{630}$ *appearing in* 2*-colors.*

Proof.

$$\prod_{n=1}^{\infty} \left(1 + q^{2n} + q^{4n} + q^{7n} + q^{14n} + q^{15n} + \ldots + q^{(4+14+30)n}\right)$$

$$= \prod_{n=1}^{\infty} \left(1 + q^{2n} + q^{4n}\right)\left(1 + q^{7n} + q^{14n}\right)\left(1 + q^{15n} + q^{30n}\right)$$

$$= \prod_{n=1}^{\infty} \left(\frac{1 - q^{6n}}{1 - q^{2n}}\right)\left(\frac{1 - q^{21n}}{1 - q^{7n}}\right)\left(\frac{1 - q^{45n}}{1 - q^{15n}}\right)$$

$$= \prod_{n=1}^{\infty} \frac{1}{\left(1-q^{630n-2}\right)\left(1-q^{630n-4}\right)\left(1-q^{630n-7}\right)\left(1-q^{630n-8}\right)}$$

$$\times \frac{1}{\left(1-q^{630n-10}\right)\left(1-q^{630n-14}\right)^2\left(1-q^{630n-15}\right)\left(1-q^{630-16}\right)}$$

$$\times \frac{1}{\left(1-q^{630-20}\right)\left(1-q^{630n-22}\right)\left(1-q^{630n-26}\right)\left(1-q^{630n-28}\right)^2}$$

$$\times \frac{1}{\left(1-q^{630n-30}\right)\left(1-q^{630n-32}\right)\left(1-q^{630n-34}\right)\left(1-q^{630n-35}\right)}$$

$$\times \frac{1}{\left(1-q^{630n-38}\right)\left(1-q^{630n-40}\right)\left(1-q^{630n-44}\right)\left(1-q^{630n-46}\right)}$$

$$\times \frac{1}{\left(1-q^{630n-49}\right)\left(1-q^{630n-50}\right)\left(1-q^{630n-52}\right)\left(1-q^{630n-56}\right)^2}$$

$$\times \frac{1}{\left(1-q^{630n-58}\right)\left(1-q^{630n-50}\right)\left(1-q^{630n-52}\right)\left(1-q^{630n-56}\right)^2}$$

$$\times \frac{1}{\left(1-q^{630n-58}\right)\left(1-q^{630n-60}\right)\left(1-q^{630n-62}\right)\left(1-q^{630n-64}\right)}$$

$$\times \frac{1}{\left(1-q^{630n-68}\right)\left(1-q^{630n-70}\right)^2\left(1-q^{630n-74}\right)\left(1-q^{630n-75}\right)}$$

$$\times \frac{1}{\left(1-q^{630n-76}\right)\left(1-q^{630n-77}\right)\left(1-q^{630n-80}\right)\left(1-q^{630n-82}\right)}$$

$$\times \frac{1}{\left(1-q^{630n-86}\right)\left(1-q^{630n-88}\right)\left(1-q^{630n-91}\right)\left(1-q^{630n-92}\right)}$$

$$\times \frac{1}{\left(1-q^{630n-94}\right)\left(1-q^{630n-98}\right)^2\left(1-q^{630n-100}\right)\left(1-q^{630n-104}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-105}\right)\left(1-q^{630n-106}\right)\left(1-q^{630n-110}\right)\left(1-q^{630n-112}\right)^2}$$
$$\times \frac{1}{\left(1-q^{630n-116}\right)\left(1-q^{630n-118}\right)\left(1-q^{630n-119}\right)\left(1-q^{630n-120}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-122}\right)\left(1-q^{630n-124}\right)\left(1-q^{630n-128}\right)\left(1-q^{630n-130}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-133}\right)\left(1-q^{630n-134}\right)\left(1-q^{630n-136}\right)\left(1-q^{630n-140}\right)^2}$$
$$\times \frac{1}{\left(1-q^{630n-142}\right)\left(1-q^{630n-146}\right)\left(1-q^{630n-148}\right)\left(1-q^{630n-152}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-154}\right)^2\left(1-q^{630n-158}\right)\left(1-q^{630n-160}\right)\left(1-q^{630n-161}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-164}\right)\left(1-q^{630n-165}\right)\left(1-q^{630n-166}\right)\left(1-q^{630n-170}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-1172}\right)\left(1-q^{630n-175}\right)\left(1-q^{630n-176}\right)\left(1-q^{630n-178}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-182}\right)^2\left(1-q^{630n-184}\right)\left(1-q^{630n-188}\right)\left(1-q^{630n-190}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-196}\right)^2\left(1-q^{630n-200}\right)\left(1-q^{630n-203}\right)\left(1-q^{630n-206}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-208}\right)\left(1-q^{630n-210}\right)\left(1-q^{630n-212}\right)\left(1-q^{630n-214}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-217}\right)\left(1-q^{630n-218}\right)\left(1-q^{630n-220}\right)\left(1-q^{630n-224}\right)^2}$$

$$\times \frac{1}{\left(1-q^{630n-226}\right)\left(1-q^{630n-230}\right)\left(1-q^{630n-232}\right)\left(1-q^{630n-236}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-238}\right)\left(1-q^{630n-240}\right)\left(1-q^{630n-242}\right)\left(1-q^{630n-244}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-245}\right)\left(1-q^{630n-248}\right)\left(1-q^{630n-250}\right)\left(1-q^{630n-254}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-255}\right)\left(1-q^{630n-256}\right)\left(1-q^{630n-259}\right)\left(1-q^{630n-260}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-262}\right)\left(1-q^{630n-266}\right)^2\left(1-q^{630n-268}\right)\left(1-q^{630n-272}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-274}\right)\left(1-q^{630n-278}\right)\left(1-q^{630n-280}\right)\left(1-q^{630n-284}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-285}\right)\left(1-q^{630n-286}\right)\left(1-q^{630n-287}\right)\left(1-q^{630n-290}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-292}\right)\left(1-q^{630n-296}\right)\left(1-q^{630n-298}\right)\left(1-q^{630n-300}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-301}\right)\left(1-q^{630n-302}\right)\left(1-q^{630n-304}\right)\left(1-q^{630n-308}\right)^2}$$
$$\times \frac{1}{\left(1-q^{630n-310}\right)\left(1-q^{630n-314}\right)\left(1-q^{630n-150}\right)\left(1-q^{630n-194}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-195}\right)\left(1-q^{630n-320}\right)\left(1-q^{630n-316}\right)\left(1-q^{630n-322}\right)^2}$$
$$\times \frac{1}{\left(1-q^{630n-326}\right)\left(1-q^{630n-328}\right)\left(1-q^{630n-329}\right)\left(1-q^{630n-330}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-332}\right)\left(1-q^{630n-334}\right)\left(1-q^{630-338}\right)\left(1-q^{630-340}\right)}$$

$$\times \frac{1}{\left(1-q^{630n-343}\right)\left(1-q^{630n-344}\right)\left(1-q^{630n-345}\right)\left(1-q^{630n-346}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-350}\right)\left(1-q^{630n-352}\right)\left(1-q^{630n-356}\right)\left(1-q^{630n-358}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-362}\right)\left(1-q^{630n-364}\right)^2\left(1-q^{630n-368}\right)\left(1-q^{630n-370}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-371}\right)\left(1-q^{630n-374}\right)\left(1-q^{630n-375}\right)\left(1-q^{630n-376}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-380}\right)\left(1-q^{630n-382}\right)\left(1-q^{630n-385}\right)\left(1-q^{630n-386}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-388}\right)\left(1-q^{630n-390}\right)\left(1-q^{630n-392}\right)\left(1-q^{630n-394}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-398}\right)\left(1-q^{630n-400}\right)\left(1-q^{630n-404}\right)\left(1-q^{630n-406}\right)^2}$$
$$\times \frac{1}{\left(1-q^{630n-410}\right)\left(1-q^{630n-412}\right)\left(1-q^{630n-413}\right)\left(1-q^{630n-416}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-418}\right)\left(1-q^{630n-420}\right)\left(1-q^{630n-422}\right)\left(1-q^{630n-424}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-427}\right)\left(1-q^{630n-430}\right)\left(1-q^{630n-434}\right)^2\left(1-q^{630n-435}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-436}\right)\left(1-q^{630n-440}\right)\left(1-q^{630n-442}\right)\left(1-q^{630n-446}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-448}\right)^2\left(1-q^{630n-452}\right)\left(1-q^{630n-454}\right)\left(1-q^{630n-455}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-458}\right)\left(1-q^{630n-460}\right)\left(1-q^{630n-464}\right)^2\left(1-q^{630n-465}\right)}$$

$$\times \frac{1}{\left(1-q^{630n-466}\right)\left(1-q^{630n-469}\right)\left(1-q^{630n-470}\right)\left(1-q^{630n-472}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-476}\right)^{2}\left(1-q^{630n-478}\right)\left(1-q^{630n-480}\right)\left(1-q^{630n-482}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-484}\right)\left(1-q^{630n-488}\right)\left(1-q^{630n-490}\right)^{2}\left(1-q^{630n-494}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-496}\right)\left(1-q^{630n-497}\right)\left(1-q^{630n-500}\right)\left(1-q^{630n-502}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-506}\right)\left(1-q^{630n-508}\right)\left(1-q^{630n-510}\right)\left(1-q^{630n-511}\right)^{2}}$$
$$\times \frac{1}{\left(1-q^{630n-512}\right)\left(1-q^{630n-514}\right)\left(1-q^{630n-518}\right)^{2}\left(1-q^{630n-520}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-524}\right)\left(1-q^{630n-525}\right)\left(1-q^{630n-526}\right)\left(1-q^{630n-530}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-532}\right)^{2}\left(1-q^{630n-536}\right)\left(1-q^{630n-538}\right)\left(1-q^{630n-539}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-542}\right)\left(1-q^{630n-544}\right)\left(1-q^{630n-548}\right)\left(1-q^{630n-550}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-553}\right)\left(1-q^{630n-554}\right)\left(1-q^{630n-555}\right)\left(1-q^{630n-556}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-560}\right)^{2}\left(1-q^{630n-562}\right)\left(1-q^{630n-566}\right)\left(1-q^{630n-568}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-570}\right)\left(1-q^{630n-572}\right)\left(1-q^{630n-574}\right)^{2}\left(1-q^{630n-578}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-580}\right)\left(1-q^{630n-581}\right)\left(1-q^{630n-584}\right)\left(1-q^{630n-586}\right)}$$

$$\times \frac{1}{\left(1-q^{630n-590}\right)\left(1-q^{630n-592}\right)\left(1-q^{630n-595}\right)\left(1-q^{630n-596}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-598}\right)\left(1-q^{630n-600}\right)\left(1-q^{630n-602}\right)^2\left(1-q^{630n-604}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-608}\right)\left(1-q^{630n-610}\right)\left(1-q^{630n-614}\right)\left(1-q^{630n-615}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-616}\right)^2\left(1-q^{630n-620}\right)\left(1-q^{630n-622}\right)\left(1-q^{630n-623}\right)}$$
$$\times \frac{1}{\left(1-q^{630n-626}\right)\left(1-q^{630n-628}\right)}.$$

□

Example 4.2.2. *Consider* $n = 14$.

Partitions of 14 with the specified multiplicities are:

$$(1^{14}), (2^7), (3^2, 2^2, 1^4), (3^2, 2^4), (3^4, 1^2), (4^2, 2^2, 1^2), (2^2, 3^2),$$
$$(5^2, 1^4), (6^2, 1^2), (7^2).$$

The other partitions are:

$$(2^7), (4, 2^5), (4^2, 2^3), (4^3, 2), (7^2), (8, 2^3), (8, 4, 2), (10, 2^2),$$
$$(10, 4), (14_1), (14_2).$$

Note that 14 appears in two colors which are represented by the subscripts 1 and 2.

4.3 Parity Formulas

In this section, we give parity formulas for certain partition functions related to Subbarao's partition theorem.

In Theorem 1.1.4, observe that the parts occur with multiplicities $a_1, a_2, a_1 + a_2$ where $a_1 = 2$ and $a_2 = 3$. In a related way, we want to apply the condition separately on even and odd parts.

We start with the partition function $e(n)$, which denotes the number of partitions of n in which odd parts occur with multiplicities 2, 3 and 5, and even parts occur with multiplicities 1,3 and 4. Then we obtain a parity formula for $e(n)$ as follows.

Theorem 4.3.1. *For all $n \geq 1$,*

$$e(n) \equiv \begin{cases} 1 \pmod 2, & n = \frac{3j(3j\pm1)}{2}, j \geq 0; \\ 0 \pmod 2, & \textit{otherwise}. \end{cases}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} e(n)q^n &= \prod_{n=1}^{\infty}(1 + q^{2n} + q^{3(2n)} + q^{4(2n)}) \\ &\qquad \times (1 + q^{2(2n-1)} + q^{3(2n-1)} + q^{5(2n-1)}) \\ &= \prod_{n=1}^{\infty} \left(1 + q^{2n}\right)\left(1 + q^{3(2n)}\right)\left(1 + q^{2(2n-1)}\right)\end{aligned}$$

$$
\begin{aligned}
&\quad\quad \times \left(1+q^{3(2n-1)}\right) \\
&\equiv \prod_{n=1}^{\infty} \left(1+q^{2n}\right)\left(1+q^{3(2n)}\right)\left(1-q^{2(2n-1)}\right) \\
&\quad\quad \times \left(1-q^{3(2n-1)}\right) \pmod 2 \\
&\equiv \prod_{n=1}^{\infty} \frac{\left(1+q^{2n}\right)\left(1+q^{6n}\right)}{\left(1+q^{2n}\right)\left(1+q^{3n}\right)} \pmod 2 \\
&\equiv \prod_{n=1}^{\infty} (1+q^{3n}) \pmod 2 \\
&\equiv 1+\sum_{n=1}^{\infty} q^{\frac{3n}{2}(3n\pm 1)} \text{ (by (1.2.3))}.
\end{aligned}
$$

□

Example 4.3.1. *Consider* $n \in \{1,2,3,\ldots,8\}$.

n	$e(n)$	
1	0	-
2	2	$2, 1^2$
3	1	1^3
4	2	$4, (2, 1^2)$
5	2	$(2, 1^3), 1^5$
6	5	$6, (4, 2), (4, 1^2), 3^2, 2^3$
7	2	$(4, 1^3), (2, 1^5)$
8	8	$8, (6, 2), (6, 1^2), (4, 2, 1^2), (3^2, 2), 2^4, (2^3, 1^2)$

Table 4.1: Partitions enumerated by $e(n)$

Theorem 4.3.1 suggests that we could look at a slightly more general case. Let $g(m_1, m_2, a_1, a_2, n)$ denote the number of partitions of n in which even parts occur with multiplicities $m_1, m_2, m_1 + m_2$ and odd parts occur with multiplicities a_1, a_2, a_1+a_2. The generating function for $g(m_1, m_2, a_1, a_2, n)$ is:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} g(m_1, m_2, a_1, a_2, n) q^n \\
&= \prod_{n=1}^{\infty} \left(1 + q^{m_1(2n)} + q^{m_2(2n)} + q^{(m_1+m_2)(2n)}\right) \\
&\qquad \times \prod_{n=1}^{\infty} \left(1 + q^{a_1(2n-1)} + q^{a_2(2n-1)} + q^{(a_1+a_2)(2n-1)}\right) \\
&= \prod_{n=1}^{\infty} \left(1 + q^{m_1(2n)}\right)\left(1 + q^{m_2(2n)}\right) \\
&\qquad \times \prod_{n=1}^{\infty} \left(1 + q^{a_1(2n-1)}\right)\left(1 + q^{a_2(2n-1)}\right) \\
&\equiv \prod_{n=1}^{\infty} \left(1 + q^{m_1(2n)}\right)\left(1 + q^{m_2(2n)}\right) \\
&\qquad \times \prod_{n=1}^{\infty} \left(1 - q^{a_1(2n-1)}\right)\left(1 - q^{a_2(2n-1)}\right) \pmod{2} \\
&\equiv \prod_{n=1}^{\infty} \frac{\left(1 + q^{2m_1 n}\right)\left(1 + q^{2m_2 n}\right)}{\left(1 + q^{a_1 n}\right)\left(1 + q^{a_2 n}\right)} \pmod{2}.
\end{aligned}
$$

We now look at some special cases.

$$\begin{aligned}\sum_{n=0}^{\infty} g(m_1, m_2, m_1, 2m_2, n)q^n \equiv & \prod_{n=1}^{\infty} \frac{\left(1+q^{2m_1 n}\right)\left(1+q^{2m_2 n}\right)}{\left(1+q^{m_1 n}\right)\left(1+q^{2m_2 n}\right)} \\ \equiv & \prod_{n=1}^{\infty} \left(1-q^{m_1 n}\right) \pmod 2 \\ = & 1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{m_1 n}{2}(3n\pm 1)} \quad \text{(by (1.2.3))}\end{aligned}$$

and so

$$g(m_1, m_2, m_1, 2m_2, n) \equiv \begin{cases} 1 \pmod 2, & if\, n = \frac{m_1 j(3j\pm 1)}{2}; \\ 0 \pmod 2, & otherwise. \end{cases} \tag{4.3.1}$$

Similarly,

$$\begin{aligned}& \sum_{n=0}^{\infty} g(m_1, m_2, 2m_1, 2m_2, n)q^n \\ \equiv & \prod_{n=1}^{\infty} \frac{\left(1+q^{2m_1 n}\right)\left(1+q^{2m_2 n}\right)}{\left(1+q^{2m_1 n}\right)\left(1+q^{2m_2 n}\right)} \pmod 2 \\ \equiv & 1 \pmod 2.\end{aligned}$$

This yields

$$g(m_1, m_2, 2m_1, 2m_2, n) \equiv 0 \pmod 2 \,\forall\, n \geq 1. \tag{4.3.2}$$

It is easy to see that

$$g(m_1, m_2, 2m_1, 2m_2, 2n-1) = 0 \,\forall\, n \geq 1.$$

Denote by $G(m_1, m_2, 2m_1, 2m_2, n)$ the set of partitions enumerated by $g(m_1, m_2, 2m_1, 2m_2, n)$. Let $h(m_1, m_2, 2m_1, 2m_2, n)$ be the number of partitions in $G(m_1, m_2, 2m_1, 2m_2, n)$ satisfying the following property:
parts are congruent to $0 \pmod 4$ *or if for every odd part occurring with multiplicity* $2j$*, there is an even part that is twice the odd part and has multiplicity* j*, and for every even part congruent to* $2 \pmod 4$ *occurring with multiplicity* l*, there is an odd part which is half the even part and has multiplicity* $2l$.

We now prove a corollary to (4.3.2), as stated below.

Corollary 4.3.1.

$$h(m_1, m_2, 2m_1, 2m_2, n) \equiv 0 \pmod 2 \,\forall\, n \geq 1. \tag{4.3.3}$$

Proof. Define a map
$\psi : G(m_1, m_2, 2m_1, 2m_2, n) \longrightarrow G(m_1, m_2, 2m_1, 2m_2, n)$ as follows:
For $\lambda = (\lambda_1^{f_1}, \lambda_2^{f_2}, \ldots, \lambda_l^{f_l}) \in G(m_1, m_2, 2m_1, 2m_2, n)$,

$$\lambda_i^{f_i} \mapsto \begin{cases} (2\lambda_i)^{\frac{f_i}{2}}, & \text{if } \lambda_i \equiv 1 \pmod 2; \\ \left(\frac{\lambda_i}{2}\right)^{2f_i}, & \text{if } \lambda_i \equiv 2 \pmod 4; \\ \lambda_i^{f_i}, & \text{otherwise.} \end{cases}$$

Then $\psi(\lambda) = \cup_{i=1}^{l} \psi(\lambda_i^{f_i})$. We claim that ψ is an involution.

To see this, proceed as follows.
If $\lambda_i \equiv 0 \pmod 4$, then $\psi(\psi(\lambda_i^{f_i})) = \psi(\lambda_i^{f_i}) = \lambda_i^{f_i}$.
If $\lambda_i \equiv 1 \pmod 2$, then $\psi(\psi(\lambda_i^{f_i})) = \psi((2\lambda_i)^{\frac{f_i}{2}}) = \left(\frac{2\lambda_i}{2}\right)^{2(\frac{f_i}{2})}$ (since $2\lambda_i \equiv 2 \pmod 4$). We have that $\psi(\psi(\lambda_i^{f_i})) = \lambda_i^{f_i}$.
If $\lambda_i \equiv 2 \pmod 4$, then $\psi(\psi(\lambda_i^{f_i})) = \psi\left(\left(\frac{\lambda_i}{2}\right)^{2f_i}\right) = \left(2\left(\frac{\lambda_i}{2}\right)\right)^{\frac{2f_i}{2}}$ (since $\frac{\lambda_i}{2}$ is odd). Thus $\psi(\psi(\lambda_i^{f_i})) = \lambda_i^{f_i}$.
Therefore, $\psi(\psi(\lambda)) = \lambda$. Indeed ψ is an involution on $G(m_1, m_2, 2m_1, 2m_2, n)$. A careful look into the set reveals that the $h(m_1, m_2, 2m_1, 2m_2, n)$-partitions are precisely those partitions fixed under the involution. For this reason, their number must have the same parity as the number of all partitions in $G(m_1, m_2, 2m_1, 2m_2, n)$. $\square$

We conclude this section by looking at cases where subbarao's condition is applied to even parts only or odd parts only, and some restrictions on the other parts.

Theorem 4.3.2. *Let $b(n)$ denote the number of partitions of n where odd parts appear with multiplicities $1, 4, 5$ and even parts appear exactly 4 times. Then*

$$b(n) \equiv \begin{cases} 1 \pmod 2, & n = \frac{j(j+1)}{2}; \\ 0 \pmod 2, & otherwise. \end{cases}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} b(n)q^n &= \prod_{n=1}^{\infty}\left(1+q^{4(2n)}\right)\left(1+q^{2n-1}+q^{4(2n-1)}+q^{5(2n-1)}\right)\\
&= \prod_{n=1}^{\infty}\left(1+q^{4(2n)}\right)\left(1+q^{(2n-1)}\right)\left(1+q^{4(2n-1)}\right)\\
&\equiv \prod_{n=1}^{\infty}\left(1-q^{8n}\right)\left(1-q^{(2n-1)}\right)\left(1-q^{4(2n-1)}\right) \pmod{2}\\
&\equiv \prod_{n=1}^{\infty}\frac{\left(1-q^{8n}\right)}{\left(1-q^{n}\right)\left(1-q^{4n}\right)} \pmod{2}\\
&\equiv \prod_{n=1}^{\infty}\frac{\left(1-q^{4n}\right)}{\left(1-q^{n}\right)} \pmod{2}\\
&\equiv \prod_{n=1}^{\infty}\frac{\left(1-q^{n}\right)^4}{\left(1-q^{n}\right)} \pmod{2}\\
&\equiv \prod_{n=1}^{\infty}\left(1-q^{n}\right)^3 \pmod{2}\\
&= \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}} \quad (\text{by } (1.2.2))\\
&\equiv \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}.
\end{aligned}$$

□

Example 4.3.2. *Consider* $n \in \{1, 2, 3, \ldots, 8\}$.

n	$b(n)$	
1	1	1
2	0	-
3	1	3
4	2	$(3,1), 1^4$
5	2	$5, 1^5$
6	1	$(5,1)$
7	2	$7, (3, 1^4)$
8	4	$(7,1), (5,3), (3,1^5), 2^4$

Table 4.2: Partitions enumerated by $b(n)$

In Theorem 4.3.2, if we interchange the multiplicity roles of even and odd parts, we obtain the following theorem. In the theorem, $\overline{b}(n)$ is the number of partitions of n in which even parts occur 1, 4 and 5 times, and odd parts occur exactly 4 times.

Theorem 4.3.3. *For all* $n \geq 1$,

$$\overline{b}(n) \equiv \begin{cases} 1 \pmod 2, & n = j(j+1); \\ 0 \pmod 2, & \textit{otherwise}. \end{cases}$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{b}(n) q^n &= \prod_{n=1}^{\infty} \left(1 + q^{4(2n-1)}\right) \left(1 + q^{(2n)} + q^{4(2n)} + q^{5(2n)}\right) \\ &= \prod_{n=1}^{\infty} \left(1 + q^{4(2n-1)}\right) \left(1 + q^{(2n)}\right) \left(1 + q^{4(2n)}\right) \end{aligned}$$

$$\equiv \prod_{n=1}^{\infty} \left(1 - q^{4(2n-1)}\right) \left(1 - q^{2n}\right) \left(1 - q^{8n}\right) \pmod{2}$$

$$\equiv \prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \left(1 - q^{4n}\right) \pmod{2}$$

$$\equiv \prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \left(1 - q^{2n}\right)^2 \pmod{2}$$

$$\equiv \prod_{n=1}^{\infty} \left(1 - q^{2n}\right)^3 \pmod{2}$$

$$\equiv \sum_{n=0}^{\infty} (-1)^n (2n+1) \left(q^2\right)^{\frac{n(n+1)}{2}} \quad \text{(by (1.2.2))}$$

$$\equiv \sum_{n=0}^{\infty} q^{n(n+1)}.$$

$\square$

Example 4.3.3. *Consider* $n \in \{1, 2, 3, \ldots, 8\}$.

n	$\overline{b}(n)$	
1	0	-
2	1	2
3	0	-
4	2	$4, 1^4$
5	0	-
6	3	$6, (4, 2), (2, 1^4)$
7	0	-
8	4	$8, (6, 2), (4, 1^4), 2^4$

Table 4.3: Partitions enumerated by $\overline{b}(n)$

Chapter 5

Conclusion

In this dissertation, we reviewed relevant theorems in the literature. Theorems 1.1.2, 1.1.5 and 1.2.2 are some of the important ones. In Chapter 2, we studied bijective proofs for MacMahon's theorem, Theorem 1.1.3 (see [13]), due to G. Andrews, S. Fu and J. Sellers. In Chapter 3, we studied bijective proofs for generalisations of Theorem 1.1.3 (see [2, 16]). Theorem 3.1.1 places a lower bound on the odd multiplicities and Theorem 3.2.1 places an upper bound on the even multiplicities, and lower and upper bounds on the odd multiplicities.

In Chapter 4, we gave a new bijection for MacMahon's partition theorem (Theorem 1.1.3), and generalized Subbarao's partition theorem (Theorem 1.1.4). Not only did we generalize Subbarao's theorem, but also gave a Glaisher based bijective proof of the generalisation. Furthermore, we proved parity formulas for some partition functions of Subbarao type.

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Appendix

In the following SAGE codes, we use the following notation:

$A1(n)$: the set of partitions of n into odd parts appearing at least twice.
$A2(n)$: the set of partitions of n where parts are congruent to $\pm 2, \pm 3 \pmod{12}$.
$A3(n)$: the set of partitions of n where even parts are congruent to $2 \pmod 4$ and odd parts appear exactly 3 times.
$E1(n)$: the set of partitions of n into even parts appearing at least twice.
$E2(n)$: the set of partitions of n where parts are congruent to $0, \pm 4, 6 \pmod{12}$.
$B3(n, a1, a2)$: the set of partitions of n in which parts occur with multiplicities $\alpha_1 a_1 + \alpha_2 a_2$ where $0 \leq \alpha_i \leq 2$.
$B2(n, a1, a2)$: the set of partitions of n in which parts occur with multiplicities $\alpha_1 a_1 + \alpha_2 a_2$ where $0 \leq \alpha_i \leq 1$.
$C3(n, a1, a2)$: the set of partitions of n into parts congruent to $-(s+3m)a_j \pmod{3a_1a_2}$ where $s = 1, 2,\ \ j = 1, 2$ and $m = 0, 1, 2, \ldots, (\prod_{\substack{i=1 \\ i \neq j}}^{2} a_i) -\ 1.$
$C2(n, a1, a2)$: the set of partitions of n into parts congruent to $-(1+2m)a_j \pmod{2a_1a_2}$ where $j = 1, 2$ and $m = 0, 1, 2, \ldots, (\prod_{\substack{i=1 \\ i \neq j}}^{2} a_i) -\ 1.$

$e(n)$: the number of partitions of n in which odd parts occur with multiplicities $2, 3$ and 5, and even parts occur with multiplicities $1, 3$ and 4.
$b(n)$: the number of partitions of n where odd parts appear with multiplicities $1, 4$ and 5 and even parts appear exactly 4 times.
$bbar(n)$: the number of partitions of n in which even parts occur $1, 4$ and 5 times, and odd parts occur exactly 4 times.
$G(n, m1, m2, a1, a2)$: the set of partitions of n in which even parts occur with multiplicities m_1, m_2, m_1+m_2 and odd parts occur with multiplicities $a_1, a_2, a_1 + a_2$.

```
def E1(n):
    L=[]
    L2=[]
    for j in Partitions(n).list():
        E=[p for p in j if p%2==1] #odd parts of j
        EO=[p for p in j if p%2==0] #even parts of j
        for k in EO:
            if list(j).count(k)>1:
                L2.append(k)
        if len(L2)==len(j):
            L.append(j)
    return L


def E2(n):
    L=[]
    L1=[]
    for j in Partitions(n).list():
        E=[p for p in j if p%2==1] #odd parts of j
        EO=[p for p in j if p%2==0] #even parts of j
```

```
        for k in EO:
            if k%12 in [0,4,6,8]:
                L1.append(k)
        if len(L1)==len(j):
            L.append(j)
    return L


def A1(n):
    L=[]
    L1=[]
    for j in Partitions(n).list():
        E=[p for p in j if p%2==1] #odd parts of j
        EO=[p for p in j if p%2==0] #even parts of j
        for k in EO:
            if list(j).count(k) > 1:
                L1.append(k)
        if len(L1) == len(j):
            L.append(j)
    return L


def A2(n):
    L=[]
    for j in Partitions(n).list():
        L1=[]
        for i in range(len(j)):
            c=j[i]%12
            if c in [2,3,9,10]:
                L1.append(c)
        if len(L1)==len(j):
            L.append(j)
    return L
```

```
def A3(n):
    L=[]
    L2=[]
    L1=[]
    for j in Partitions(n).list():
        E=[p for p in j if p%2==1] #odd parts of j
        EO=[p for p in j if p%2==0] #even parts of j
        for k in E:
            if k%4 == 2:
                L1.append(k)
        for k in EO:
            if list(j).count(k) == 3:
                L2.append(k)
        if len(L1) + len(L2) == len(j):
            L.append(j)
    return L


def B2(n, a1, a2):
    L = []
    RES = [((-(1 + 2*m))*a1)%(2*a1*a2)
          for m in [0..(a2-1)]] +
          [((-(1 + 2*m))*a2)%(2*a1*a2)
          for m in [0..(a1-1)]] ;

    for j in Partitions(n).list():
        g = [i for i in j if i%(2*a1*a2) in RES]
        if len(j) == len(g):
            L.append(j)
    return L
```

```
def C2(n, a1, a2):
    L = []
    L1= []
    for a in [0,1]:
        for b in [0,1]:
            L1.append(a*a1 + b*a2) ;
    #checks linear combinations
    for j in Partitions(n).list():
        h = [p for p in j if list(j).count(p) in L1]
        if len(j) == len(h):
            L.append(j)
    return L


def B3(n, a1, a2):
    L = []
    RES1 = [((-(1 + 3*m))*a1)%(3*a1*a2)
           for m in [0..(a2-1)] +
           [((-(1 + 3*m))*a2)%(3*a1*a2)
           for m in [0..(a1-1)]] ;
    RES2 = [((-(2 + 3*m))*a1)%(3*a1*a2)
           for m in [0..(a2-1)]] +
           [((-(2 + 3*m))*a2)%(3*a1*a2)
           for m in [0..(a1-1)]] ;
    RES = RES1 + RES2

    for j in Partitions(n).list():
        g = [i for i in j if i%(3*a1*a2) in RES]
        if len(j) == len(g):
            L.append(j)
    return L
```

```
def C3(n, a1, a2):
    L = []
    L1= []
    for a in [0,1,2]:
        for b in [0,1,2]:
            L1.append(a*a1 + b*a2) ;
    #check linear combinations
    for j in Partitions(n).list():
        h = [p for p in j if list(j).count(p) in L1]
        if len(j) == len(h):
            L.append(j)
    return L



def e(n):
    L=[]
    L2=[]
    L1=[]
    for j in Partitions(n).list():
        E=[p for p in j if p%2==1] #odd parts of j
        EO=[p for p in j if p%2==0] #even parts of j
        for k in EO:
            if list(j).count(k) in [1,3,4]:
                L1.append(k)
        for h in E:
            if list(j).count(h) in [2,3,5]:
                L2.append(h)
        if len(L1) + len(L2)==len(j):
            L.append(j)
    return len(L)
```

```
def b(n):
    L=[]
    L2=[]
    L1=[]
    for j in Partitions(n).list():
        E=[p for p in j if p%2==1] #odd parts of j
        EO=[p for p in j if p%2==0] #even parts of j
        for k in EO:
            if list(j).count(k) == 4:
                L1.append(k)
        for h in E:
            if list(j).count(h) in [1,4,5]:
                L2.append(h)
        if len(L1) + len(L2)==len(j):
            L.append(j)
    return  len(L)

def bbar(n):
    L=[]
    L2=[]
    L1=[]
    for j in Partitions(n).list():
        E=[p for p in j if p%2==1] #odd parts of j
        EO=[p for p in j if p%2==0] #even parts of j
        for k in EO:
            if list(j).count(k) in [1,4,5]:
                L1.append(k)
        for h in E:
            if list(j).count(h) == 4:
                L2.append(h)
        if len(L1) + len(L2)==len(j):
            L.append(j)
    return  len(L)
```

```
def G(n, m1, m2, a1, a2):
    L = []
    L1 = []
    L2 = []
    L3 = []
    for j in Partitions(n).list():
        O =[p for p in j if p%2==1] #odd parts of j
        E =[p for p in j if p%2==0] #even parts of j
        E1 = [p for p in E if p%4 == 0] #E%0 mod 4
        E2 = [p for p in E if p%4 == 2] #E%2 mod 4
        for k in E1:
            if list(j).count(k) in [m1, m2, m1 + m2]:
                L1.append(k)
        for k in E2:
            if list(j).count(k) in [m1, m2, m1 + m2]:
                L2.append(k)
        for k in O:
            if list(j).count(k) in [a1, a2, a1 + a2]:
                L3.append(k)
        if len(L1) + len(L2) + len(L3) == len(j):
            L.append(j)
    return L
```