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Symmetry Properties for First Integrals

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Abstract

This is the study of Lie algebraic properties of first integrals of scalar second-, third- and higher-order ordinary differential equations (ODEs). The Lie algebraic classification of such differential equations is now well-known from the works of Lie [10] as well as recently Mahomed and Leach [19]. However, the algebraic properties of first integrals are not known except in the maximal cases for the basic first integrals and some of their quotients. Here our intention is to investigate the complete problem for scalar second-order and maximal symmetry classes of higher-order ODEs using Lie algebras and Lie symmetry methods. We invoke the realizations of low-dimensional Lie algebras.

Symmetries of the fundamental first integrals for scalar second-order ODEs which are linear or linearizable by point transformations have already been obtained. Firstly we show how one can determine the relationship between the point symmetries and the first integrals of linear or linearizable scalar ODEs of order two. Secondly, a complete classification of point symmetries of first integrals of such linear ODEs is studied. As a consequence, we provide a counting theorem for the point symmetries of first integrals of scalar linearizable second-order ODEs. We show that there exists the 0, 1, 2 or 3

point symmetry cases. It is proved that the maximal algebra case is unique.

By use of Lie symmetry group methods we further analyze the relationship between the first integrals of the simplest linear third-order ODEs and their point symmetries. It is well-known that there are three classes of linear third-order ODEs for maximal and submaximal cases of point symmetries which are 4, 5 and 7. The simplest scalar linear third-order equation has seven point symmetries. We obtain the classifying relation between the symmetry and the first integral for the simplest equation. It is shown that the maximal Lie algebra of a first integral for the simplest equation $y''' = 0$ is unique and four-dimensional. Moreover, we show that the Lie algebra of the simplest linear third-order equation is generated by the symmetries of the two basic integrals. We also obtain counting theorems of the symmetry properties of the first integrals for such linear third-order ODEs of maximal type. Furthermore, we provide insights into the manner in which one can generate the full Lie algebra of higher-order ODEs of maximal symmetry from two of their basic integrals.

The relationship between first integrals of sub-maximal linearizable third-order ODEs and their symmetries are investigated as well. All scalar linearizable third-order equations can be reduced to three classes by point transformations. We obtain the classifying relations between the symmetries and the first integral for sub-maximal cases of linear third-order ODEs. It is known, from the above, that the maximum Lie algebra of the first integral is achieved for the simplest equation. We show that for the other two classes they are not unique. We also obtain counting theorems of the symmetry properties of the first integrals for these classes of linear third-order ODEs. For the 5 symmetry class of linear third-order ODEs, the first integrals can have 0, 1, 2 and 3 symmetries and for the 4 symmetry class of linear third-order ODEs they

are 0, 1 and 2 symmetries respectively. In the case of sub-maximal linear higher-order ODEs, we show that their full Lie algebras can be generated by the subalgebras of certain basic integrals. For the $n+2$ symmetry class, the symmetries of the first integral I_2 and a two-dimensional subalgebra of I_1 generate the symmetry algebra and for the $n+1$ symmetry class, the full algebra is generated by the symmetries of I_1 and a two-dimensional subalgebra of the quotient I_3/I_2.

Finally, we completely classify the first integrals of scalar nonlinear second-order ODEs in terms of their Lie point symmetries. This is performed by first obtaining the classifying relations between point symmetries and first integrals of scalar nonlinear second-order equations which admit 1, 2 and 3 point symmetries. We show that the maximum number of symmetries admitted by any first integral of a scalar second-order nonlinear (which is not linearizable by point transformation) ODE is one which in turn provides reduction to quadratures of the underlying dynamical equation. We provide physical examples of the generalized Emden-Fowler, Lane-Emden and modified Emden equations.

Declaration

I declare that this is my own work except where due references have been made to the literature. It is being submitted for the degree of Doctor of Philosophy by thesis to the University of the Witwatersrand. It has not been submitted before for any degree to any other university.

 (29 Sept 2014)

Komal S Mahomed

Dedication

To my wonderful parents and loving family

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Chapter 1

Introduction

First integrals of ordinary differential equations (ODEs) are quite an active and interesting area of research at the present time. Whenever one is dealing with differential equations and especially with their solutions, one has to invariably deal with first integrals. In fact, they are in general the first primary steps towards finding the reduction and solutions of ordinary differential equations. First integrals have great importance in classical mechanics as it deals with second-order systems of equations and conserved quantities of the motion such as energy and momentum. There have been several contributions in the search for first integrals of many equations from applications such as the Emden-Fowler equation and its various generalizations, the Kepler problem and the Ermakov systems [1, 2, 3, 4]. In the context of applications such as in mechanics, first integrals are often referred to as constants of the motion. Sometimes they are even called exact invariants. Moreover, essential studies in the theoretical development of first integrals have also been made over the years. These can be accessed in several

textbooks and papers (see e.g., [5, 6, 7, 8, 9]).

Lie theory was initiated by the great Norwegian mathematician Marius Sophus Lie (see, e.g., [10, 11, 12]). Since the publications of his landmark works on the theory and applications of continuous groups to differential equations, several important areas in the vast field of differential equations have opened. Many of these are becoming well-known with continually increasing appeal. Some classical works on the subject have been contributed in monographs and books by Ovsiannikov [13], Olver [5], Stephani [14], Ibragimov [15] and Bluman and Anco [16]. An essential advantage of Lie theory is that it applies to both linear and non-linear problems. It provides a unification of the many ad hoc methods that exist for the reduction and solution of differential equations. This is the prime reason for its attraction and attention in recent decades. However, there are some differential equations that do not admit Lie symmetries and in those cases a generalization of Lie's theory have also proved successful (see e.g. [17]).

Another important aspect in the area of Lie theory and its applications is first integrals. The most basic approach in the calculation of first integrals is the direct method. This has been extensively used. There are other common approaches too, viz. the characteristic or multiplier method, the variational derivative approach, the celebrated Noether theorem, the partial Noether theorem and the method of adding a symmetry condition with the direct method [6, 7, 8, 9]. In the last three approaches symmetry and operators become of great utility. We utilize symmetries in our study of first integrals.

The existence for the maximum number of symmetries for scalar nth-order ODEs

were investigated by Lie [10] (see, e.g. Mahomed [18]). Lie showed that scalar first-order ODEs have infinite number of point symmetries. In the case of scalar second-order ODEs, Lie proved that the maximum is eight and this is achieved by the free particle and indeed linearizable by point transformation equations. In a recent work by Mahomed and Leach [19], they discovered the symmetries of the maximal cases of scalar linear nth-order ODEs, $n \geq 3$. These cases are $n+1, n+2$ and $n+4$. Thus for scalar linear third-order equations these corresponds to 4, 5 and 7 symmetries. There is yet another contribution by Leach and Mahomed [20], in which they have found that the Lie algebra of the fundamental first integrals and their quotient of scalar linear second-order ODEs are three-dimensional and have very interesting properties. This also applies to linearizable by invertible transformations second-order ODEs which are given as examples in their paper. So far, none of these authors consider the classification of the symmetries of first integrals of scalar linear nth-order ODEs, $n \geq 1$, nor even investigate what could be the maximal numbers of symmetries for the first integrals of these linear or linearizable equations. They do however give insights into the algebraic structure of the fundamental first integrals and in some cases their quotients.

Govinder and Leach [21] provided the algebraic structure of the basic first integrals for scalar third-order linear ODEs. They showed that the three equivalence classes each has certain first integrals with a specific number of point symmetries. They followed on the initial investigation of Leach and Mahomed [20] who considered the point symmetries of the basic first integrals of linear second-order ODEs. Then in the work [22] Flessas et al. attempted the symmetry structure for the first integrals of higher-order equations of maximal symmetry. However, this is incomplete. We extend this study and provide a complete analysis on the Lie point symmetries and

first integrals for the simplest third-order ODE including the maximal algebra case.

Algebraic properties of first integrals have been pursued as mentioned in recent works [23, 21, 22, 20]. The first integrals of the free particle equation have remarkable Lie algebraic properties [20]. Since this initial work, other authors [23, 21, 22] have studied the symmetry properties of first integrals of second- and higher-order ODEs which possess maximal symmetry. A few nonlinear equations were considered too. In this thesis we wish to systematically analyze the Lie algebraic properties of first integrals of second-order linear and nonlinear ODEs as well as third-order and higher-order ODEs of maximal and submaximal symmetry. The Lie algebraic classification of such equations is well-known [18, 24, 25, 19].

In chapter 2 we briefly present the notation as well as an overview of the common approaches utilized for the construction of first integrals of scalar ODEs.

In chapter 3 we give the complete classification of point symmetries for the first integrals of scalar linear second-order ODEs and the relationship between the symmetries and first integrals. For this purpose we use the projective transformations to find the different cases of symmetries for the first integrals of scalar second-order ODEs which are linear or linearizable by point transformations. Since all scalar second-order ODEs which are linear or linearizable by point transformations are transformable to the free particle equation [10], we utilize this as our base ODE. We find that there are: the 0 symmetry, 1 symmetry, 2 symmetry and unique 3 symmetry cases.

Then in chapter 4 we investigate the Lie algebraic properties of first integrals of scalar linear third-order ODEs of the maximal class which is represented by $y''' = 0$. We remind the reader that for the simplest class there has been some analysis made

in Flessas et al. [23]. This is in regards to the maximal algebra possessed by an integral of $y''' = 0$ which is listed in Table 4.1, chapter 4 of this thesis. However, this is incomplete. We extend this study and provide a complete analysis of the Lie point symmetries and first integrals for the simplest third-order ODE including the maximal algebra case. We firstly deduce the classifying relation between the point symmetries and first integrals for this simple class. Then we use this to study the point symmetry properties of the first integrals of $y''' = 0$ which also represents all linearizable by point transformations third-order ODEs that reduce to this class. We begin by noting the condition for symmetries of the first integrals of scalar linear ODEs of order one. Then for completeness we review briefly the results of chapter 3 which discusses the relationship between the point symmetries of the first integrals of scalar linear second-order ODEs. These two cases are shown to be distinct in terms of their algebraic properties of their integrals when compared to higher-order ODEs of maximal symmetry.

Chapter 5 commences with the classifying relation between the point symmetries and first integrals for the submaximal classes of scalar linear third-order equations. Then by using this we find the point symmetry properties of the first integrals of the submaximal classes of third-order equations $y''' - y' = 0$ and $y''' + f(x)y'' - y' - f(x)y = 0$ which also represent all linearizable by point transformations third-order ODEs that reduce to these classes. We obtain counting theorems for the number of point symmetries possessed by an integral of such equations. Noteworthy is that the maximal algebra is not unique. In the next section of this chapter, we study the point symmetry properties of the integrals of the 4 symmetry class represented by $y''' - y' = 0$. We remind the reader under what conditions point symmetries of first integrals of scalar linear third-order ODEs exist (see chapter 4). Then in Section 5.3 we analyze the

class $y''' + f(x)y'' - y' - f(x)y = 0$ which has four point symmetries for the symmetry structure of its first integrals. In Section 5.4 we focus on the generation of the full algebra by subalgebras of certain basic integrals.

In chapter 6 we obtain the complete classification of the first integrals of scalar nonlinear second-order ODEs in terms of their symmetry algebras. It is shown in chapter 3 that the maximum symmetry algebra admitted by a first integral of linearizable second-order ODEs is three. We also obtain a counting theorem which gives the interesting result that a first integral of a scalar second-order linearizable ODE can have 0, 1, 2 or 3 point symmetries. In this chapter we provide an extension of these and focus our attention on scalar nonlinear second-order ODEs which admit 1, 2 or 3 symmetries. We use the result [10] of Lie who classified all scalar second-order ODEs in terms of their point symmetries.

Finally, in chapter 7 we present our conclusion and mention some open problems too.

Chapter 2

Mathematical Preliminaries: First Integrals

This chapter presents the notation and results that are used for the rest of the thesis. An overview of different methods for the construction of first integrals of ODEs is given. The most basic approach in the calculation of first integrals is the direct method which is presented in the first part of this chapter. A brief review of other common approaches are discussed in the remaining parts. These approaches are: the characteristic or multiplier method, the variational derivative approach, the celebrated Noether theorem, the partial Noether theorem and the method of adding a symmetry condition with the direct method [6, 7, 8, 9].

2.1 Algebraic Properties of First Integrals

First integrals are important in the reduction and solution of ODEs. There have been several contributions in the search for first integrals of many equations from applications such as the Emden-Fowler equation, the Kepler problem, and the Ermakov systems [1, 2, 3, 4]. In the context of applications such as in classical mechanics, first integrals are often referred to as constants of the motion. Sometimes they are even called exact invariants. Moreover, essential studies in the theoretical development of algebraic properties of first integrals have also been made over the years. These can be accessed in textbooks and papers (see, e.g. [5, 6, 7, 8, 9]).

Algebraic properties of first integrals of paradigm ODEs have been pursued in recent works [23, 21, 22, 20]. The first integrals of the free particle equation have remarkable Lie algebraic properties [20]. Since this initial work, other authors [23, 21, 22] have studied the symmetry properties of the fundamental first integrals of higher-order ODEs which possess maximal symmetry. A few nonlinear equations were considered too.

In this thesis we wish to systematically analyze the Lie algebraic properties of first integrals of second- and higher-order ODEs of maximal symmetry. The Lie algebraic classification of such equations are now well-known [24, 25, 19, 18]. However, the algebraic properties of first integrals are not known except in the maximal cases and those of the fundamental first integrals as pointed out earlier. Here our intention is to investigate the complete problem for second- and higher-order ODEs with maximal symmetry using Lie algebras and Lie symmetry techniques. We also invoke the realizations of low-dimensional Lie algebras when needed in chapters 3 and 4.

2.2 Main Problems

We investigate the Lie algebraic properties of first integrals of scalar second-order and linearizable third- and higher-order ODEs. The Lie algorithm is used to calculate the symmetries of first integrals (see [6] and below for an example). We then obtain a classification of the symmetries of first integrals of such equations. The symmetries form a Lie algebra (which is a subalgebra of the equation [7]). The classification of low-dimensional Lie algebras into different types is well-known (see e.g. [25, 18]). We use this to achieve our goal for second-order ODEs and the maximal symmetry case of the third-order ODE. In the cases of submaxiaml symmetry classes of second- and higher-order ODEs, we invoke optimal systems of Lie algebras.

As an example, for the first integral $I = y'$ of the free particle equation $y'' = 0$, the point symmetries of I (see below) span the three-dimensional Lie algebra $A_{3,3}$ (see [20]). We use another notation for the algebra that will be encountered in the next chapter.

Now we mention the common approaches to construct first integrals of scalar ODEs.

2.3 Direct Method

Consider the scalar nth-order ($n \geq 2$) ODE

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}), \tag{2.1}$$

where $y^{(i)} = d^i y/dx^i$. A first integral of equation (2.1) is a differential function $I \in \mathcal{A}$ which satisfies the equation

$$\frac{dI}{dx} = 0, \quad \frac{d}{dx} = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + \cdots + y^{(n)} \frac{\partial}{\partial y^{(n-1)}}, \tag{2.2}$$

for all solutions of the equation (2.1). Equation (2.2) is referred to as the conservation law of the ODE (2.1). If one uses the form (2.2) to construct the first integrals I, then this is called the direct method. The solution using this approach is quite complicated for many equations, especially nonlinear.

2.4 Characteristic or Multiplier Method

One can write the conservation law in the form

$$\frac{dI}{dx} = Q(y^{(n)} - F(x, y, y', \ldots, y^{(n-1)})), \tag{2.3}$$

in which $Q \in \mathcal{A}$ is the characteristic or multiplier. Equation (2.3) is referred to as the characteristic form of the conservation law (2.2). In this manner one can obtain the first integrals I by first computing the multipliers Q. This gives another approach for the calculation of first integrals.

2.5 Variational Derivative Method

If we take the variational derivative $\delta/\delta y$ of (2.3), then we have

$$\frac{\delta}{\delta y} Q \left(y^{(n)} - F(x, y, y', \ldots, y^{(n-1)}) \right) = 0, \tag{2.4}$$

where the variational derivative is defined as

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} + \sum_{s\geq 1}(-\frac{d}{dx})^s \frac{\partial}{\partial y^{(s)}}. \tag{2.5}$$

In this form (2.4), one calculates the characteristics Q first and then tries to deduce the first integrals.

2.6 The Noether Theorem

Yet another approach is the classical Noether theorem which applies for even order ODEs (2.1) when they have Lagrangian formulations. If equation (2.1), of order $n = 2k$, admits a Lagrangian $L \in \mathcal{A}$, then it can be written as the Euler-Lagrange ODE

$$\frac{\delta L}{\delta y} = 0, \tag{2.6}$$

where $\delta/\delta y$ is the variational derivative as defined in (2.5).

Suppose that

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{2.7}$$

is a generator of Noether point symmetry of the Lagrangian $L(x, y, \ldots, y^{(k)})$, i.e.

$$X^{[k]}L + D_x(\xi)L = D_x B, \tag{2.8}$$

is satisfied, where $D_x = d/dx$, $B \in \mathcal{A}$ is the gauge term and $X^{[k]}$ is the kth prolongation of X given by

$$X^{[k]} = X + \sum_{i=1}^{k} \zeta_i \frac{\partial}{\partial y^{(i)}}, \tag{2.9}$$

in which

$$\begin{aligned} \zeta_1 &= D_x(\eta) - y' D_x(\xi), \\ \zeta_i &= D_x(\zeta_{i-1}) - y^{(i)} D_x(\xi), \quad i = 2, \ldots, k. \end{aligned} \tag{2.10}$$

Then the Noether first integral is given by (Noether's theorem)

$$I = B - N(L), \tag{2.11}$$

where N is the Noether operator

$$N = \xi + W \frac{\delta}{\delta y'} + \sum_{s=1}^{k-1} D_x^s(W) \frac{\delta}{\delta y^{(s+1)}}. \tag{2.12}$$

Here $W = \eta - \xi y'$ is the Lie characteristic function and the variational derivatives are obtained from (2.5) by replacing y by the required derivatives.

2.7 The Partial Noether Theorem

The partial Noether theorem also has first integral as in (2.11). However, the determining equations (2.8) has the extra term $W \frac{\delta L}{\delta y}$ in it as $\frac{\delta L}{\delta y} \neq 0$ (see [8]). That is we have

$$X^{[k]} L + D_x(\xi) L = W \frac{\delta L}{\delta y} + D_x B, \tag{2.13}$$

2.8 Adding a Symmetry Condition to the Direct Method

Another approach is that of adding a symmetry condition to the direct method. So one has in addition

$$X^{[p]}(I) = 0, \tag{2.14}$$

in which p is the order of the integral.

We will use this condition (2.14) to classify first integrals according to their symmetry properties in this thesis. As an example, consider the first integral $I = y'$ of the free particle equation $y'' = 0$. Upon using (2.14) we arrive at

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y},\ X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{2.15}$$

The algebraic properties of such first integrals were investigated in [20]. This was for equations having maximal symmetry. The question arises as to what happens for submaximal cases?

We now commence our study on the symmetry classification of the first integrals of second- and maximal symmetry classes of higher-order ODEs in terms of the algebras they admit. In chapter 3 we begin with the symmetries of first integrals for scalar second-order ODEs which are linearizable by point transformations. Then in chapters 4 and 5 we investigate the symmetry structures of first integrals of third-order ODEs for the maximal and submaximal symmetry cases. In chapter 6 our focus turns to scalar nonlinear second-order ODEs and the symmetry classification of first integrals that they possess. We also comment on the symmetries of first integrals for higher-order linear ODEs. Chapter 7 deals with concluding remarks and some open questions.

Chapter 3

Symmetry Classification of First Integrals for Scalar Linearizable Second-Order ODEs

3.1 Introduction

This chapter gives the complete classification of point symmetries for the first integrals of scalar linear second-order ODEs and the relationship between the symmetries and first integrals. For this purpose we use the projective transformations to find the different cases of symmetries for the first integrals of scalar second-order ODEs which are linear or linearizable by point transformations. Since all scalar second-order ODEs which are linear or linearizable by point transformations are transformable to the

free particle equation, we utilize this as our base ODE. We find that there are: the no symmetry, one symmetry, two symmetry and unique three symmetry cases. This chapter constitutes new work which we have published in [26].

It is well-known that the second-order ODE (see e.g. [14])

$$E(x, y, y', y'') = 0 \tag{3.1}$$

is invariant under the infinitesimal generator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{3.2}$$

if and only if

$$X^{[2]}E|_{E=0} = 0, \tag{3.3}$$

where

$$X^{[2]} = X + \zeta_1\frac{\partial}{\partial y'} + \zeta_2\frac{\partial}{\partial y''}, \tag{3.4}$$

with (see prolongation formula in section 2.6)

$$\begin{aligned} \zeta_1 &= D_x(\eta) - y' D_x(\xi), \\ \zeta_2 &= D_x(\zeta_1) - y'' D_x(\xi) \end{aligned} \tag{3.5}$$

in which D_x is the total differentiation operator, is called the second prolongation of the generator X.

Now we can say that (3.2) is the point symmetry of (3.1), whereas, in the case of first integrals, the first integral

$$I = f(x, y, y'), \tag{3.6}$$

of the ODE (3.1), is annihilated by X, i.e. (3.2) is the symmetry generator of (3.6) if and only if (from section 2.8)

$$X^{[1]}I = 0. \tag{3.7}$$

Here X annihilates I and does not leave it invariant as in the case of symmetries of equations. Note that the procedure for finding symmetries of ODEs is different to that of finding symmetries of first integrals. In fact the symmetries of the first integrals is a subalgebra of the symmetries of the equation itself (see Kara and Mahomed [6]).

It is essential to point out that equation (3.1) is linearizable by point transformation to the free particle equation if and only if it is cubic in the first derivatives as

$$y'' = A(x,y)y'^3 + B(x,y)y'^2 + C(x,y)y' + D(x,y), \tag{3.8}$$

where the functions A to D satisfy the invariant conditions (see Tressé [27] and also [28])

$$\begin{aligned} &3A_{xx} + 3A_xC - 3A_yD + 3AC_x + C_{yy} - 6AD_y + BC_y - 2BB_x - 2B_{xy} = 0,\\ &6A_xD - 3B_yD + 3AD_x + B_{xx} - 2C_{xy} - 3BD_y + 3D_{yy} + 2CC_y - CB_x = 0. \end{aligned} \tag{3.9}$$

As an example we revisit the well-known modified Emden equation which has eight point symmetries [29]

$$y'' + 3yy' + y^3 = 0. \tag{3.10}$$

This ODE satisfies (3.8) and (3.9) and is reducible to the free particle equation $\bar{y}'' = 0$ via the map (see [29])

$$\bar{x} = x - \frac{1}{y}, \quad \bar{y} = \frac{1}{2}x^2 - \frac{x}{y}.$$

Therefore in the sequel we consider the free particle equation as representative of all linearizable by point transformations scalar second-order ODEs.

In the next section we give the classifying relation for the symmetries of the first integrals of the free particle equation.

3.2 Symmetries of the Fundamental First Integrals

We consider the free particle equation

$$y'' = 0 \tag{3.11}$$

which has the maximum number of symmetries, viz. eight given by (we list them here as we use these in what follows)

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} \\ X_2 &= \frac{\partial}{\partial y} \\ X_3 &= x\frac{\partial}{\partial x} \\ X_4 &= y\frac{\partial}{\partial y} \\ X_5 &= x\frac{\partial}{\partial y} \\ X_6 &= y\frac{\partial}{\partial x} \\ X_7 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \\ X_8 &= xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \end{aligned} \tag{3.12}$$

It is clear that the free particle equation (3.11) has two functionally independent first integrals

$$\begin{aligned} I_1 &= y' \\ I_2 &= xy' - y. \end{aligned} \tag{3.13}$$

The first integral (3.13a) has three symmetries [20]

$$X_1 = \frac{\partial}{\partial x}$$

$$X_2 = \frac{\partial}{\partial y}$$
$$X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{3.14}$$

and (3.13b) also has three symmetries [20]

$$G_1 = x\frac{\partial}{\partial x}$$
$$G_2 = x\frac{\partial}{\partial y}$$
$$G_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{3.15}$$

We observe that the symmetries of the first integral of (3.13a) are the same as that of (3.13b) if we multiply the symmetries of (3.13a) by x which is the multiplier or characteristic of the free particle equation that results in the integral I_2.

Let us see what happens if we find the symmetries of the quotient of the first integrals (3.13), viz.

$$\frac{I_2}{I_1} = x - \frac{y}{y'}. \tag{3.16}$$

As shown in [20], (3.16) possesses three symmetries as well. These are

$$Y_1 = y\frac{\partial}{\partial x}$$
$$Y_2 = y\frac{\partial}{\partial y}$$
$$Y_3 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} \tag{3.17}$$

which are the same as the symmetries (3.14) if we multiply the symmetries of (3.13a) by y. However, this is not a multiplier of our equation.

It was demonstrated in the seminal paper [20], that the Lie algebras of the symmetries of the first integrals I_1, I_2 and their quotient I_2/I_1 are isomorphic. Also each triplet

(3.14), (3.15) and (3.17) can be mapped into the other by a projective transformation. Furthermore, it was noted in [20] that the three triplets together generate the Lie algebra $sl(3, R)$ of the free particle equation.

3.3 Classifying Relation for the Symmetries

We know (see [20]) the symmetries of the functionally independent first integrals I_1 and I_2 or their quotient of the free particle equations. These we briefly reviewed and commented on in the previous section. Now the question arises if we want to know the symmetry properties of say the product $I_1 I_2$. We then need to compute them from first principles by using the symmetry condition. Instead of doing this each time from the beginning principles, can one obtain the relationship between the symmetries and first integrals? This is what we do here. The benefit of having such a relation enables us to also classify the first integrals of the free particle equation in terms of their point symmetries.

Let then F be an arbitrary function of I_1 and I_2, viz. $F = F(I_1, I_2)$. The symmetry of this general function of the first integrals is

$$X^{[1]}F = X^{[1]}I_1 \frac{\partial F}{\partial I_1} + X^{[1]}I_2 \frac{\partial F}{\partial I_2} = 0, \tag{3.18}$$

where

$$\begin{aligned} X^{[1]}I_1 &= [\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta_x \frac{\partial}{\partial y'}]y' = \zeta_x \\ X^{[1]}I_2 &= [\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta_x \frac{\partial}{\partial y'}](xy' - y) = \xi y' + x\zeta_x - \eta. \end{aligned} \tag{3.19}$$

Now ξ, η and ζ_x are

$$\begin{aligned}\xi &= a_1 + xa_3 + ya_6 + x^2a_7 + xya_8\\ \eta &= a_2 + ya_4 + xa_5 + xya_7 + y^2a_8\\ \zeta_x &= -y'a_3 + y'a_4 + a_5 - y'^2a_6 + (y - xy')a_7 + (yy' - xy'^2)a_8. \end{aligned} \tag{3.20}$$

These are the coefficients of $X^{[1]}$ which are obtained by setting

$$X^{[1]} = \sum_{i=1}^{8} a_i X_i^{[1]} \tag{3.21}$$

where X_is are the free particle symmetries as given in (3.12) and the a_is are constants. The reason for this is that the symmetries of the first integrals are always the symmetries of the free particle equation (see [6, 7]).

After substituting the values of $X^{[1]}I_1$, $X^{[1]}I_2$ as in (3.19), with ξ, η and ζ_x as in (3.20), in equation (3.18), we get after some calculations

$$\begin{aligned}&[-y'a_3 + y'a_4 + a_5 - y'^2a_6 + (y - xy')a_7 + (yy' - xy'^2)a_8]\frac{\partial F}{\partial I_1}\\ &\qquad +[(a_1 + xa_3 + ya_6 + x^2a_7 + xya_8)y'\\ &+(-y'a_3 + y'a_4 + a_5 - y'^2a_6 + (y - xy')a_7 + (yy' - xy'^2)a_8)x\\ &\qquad -(a_2 + ya_4 + xa_5 + xya_7 + y^2a_8)]\frac{\partial F}{\partial I_2} = 0, \end{aligned} \tag{3.22}$$

Then by using the relations $I_1 = y'$ and $I_2 = xy' - y$ from (3.13), we finally arrive at the classifying relation

$$\begin{aligned}&(-I_1a_3 + I_1a_4 + a_5 - I_1^2a_6 - I_2a_7 - I_1I_2a_8)\frac{\partial F}{\partial I_1}\\ &\quad +(I_1a_1 - a_2 + I_2a_4 - I_1I_2a_6 - I_2^2a_8)\frac{\partial F}{\partial I_2} = 0. \end{aligned} \tag{3.23}$$

The relation (3.23) provides the relationship between the symmetries and first integrals of the free particle equation. We remind the reader that any symmetry of a first integral

of the free particle equation is contained in the condition (3.23). We use this to classify the first integrals according to their symmetries.

3.4 Symmetry Structure of First Integrals

We invoke the classifying relation (3.23) to establish the number of symmetries possessed by the first integrals of the free particle equation.

There arise four cases.

Case 1. **No symmetry**.

If F is any arbitrary function of I_1 and I_2, then F_{I_1} and F_{I_2} are not related to each other. In this case we have from (3.23) that

$$-I_1a_3 + I_1a_4 + a_5 - I_1^2a_6 - I_2a_7 - I_1I_2a_8 = 0, \tag{3.24}$$

and

$$I_1a_1 - a_2 + I_2a_4 - I_1I_2a_6 - I_2^2a_8 = 0. \tag{3.25}$$

It is easy to see from (3.24) and (3.25) that all the a's are zero. Therefore there exists no symmetry for this case.

As an illustrative example, if we take $F = I_1 \ln I_2$, then equation (3.23) yields

$$\begin{aligned}(-I_1a_3 + I_1a_4 + a_5 - I_1^2a_6 - I_2a_7 - I_1I_2a_8)I_2 \ln I_2 \\ +(I_1a_1 - a_2 + I_2a_4 - I_1I_2a_6 - I_2^2a_8)I_1 = 0.\end{aligned} \tag{3.26}$$

This straightforwardly results in all the a's being zero.

The results here are quite unexpected and surprising. One will not have imagined a zero symmetry case for a first integral of the simplest equation! The consequence of this is as follows.

If we set the first integral to be a constant as in the example, we have

$$y' \ln(xy' - y) = C. \tag{3.27}$$

To integrate this kind of messy integral (3.27) and find the solution of the free particle equation from it is not easy. But this difficulty is avoidable. One does not usually obtain complicated first integral such as (3.27) in ones computation in the first place by using the approaches such as the direct method, Noether's theorem, multiplier approach etc. (see chapter 2 for these methods).

Case 2. One Symmetry.

Firstly we notice that if F satisfies the classifying relation (3.23), then X which is a linear combination of the free particle generators, is a symmetry of this classifying relation. We also observe from (3.23) that if one has any of the free symmetry generators X_i as a symmetry of a first integral of the equation, then one ends up with three symmetries! That is one can have more than one symmetry.

Say if we take a_2 arbitrary, i.e. $X = \partial/\partial y$, then (3.23) yields (since $\partial F/\partial I_2 = 0$ and $\partial F/\partial I_1 \neq 0$)

$$-I_1 a_3 + I_1 a_4 + a_5 - I_1^2 a_6 - I_2 a_7 - I_1 I_2 a_8 = 0,$$

which in turn implies that a_1 is arbitrary and $a_3 = a_4$ as well. Thus we get more than one symmetry.

We in fact arrive at the three symmetries given in (3.14). The same applies for the other symmetries taken one at a time.

However, we do have several cases when exactly one symmetry occurs.

If we take $F = I_1 I_2$ or any function of the product, then the relation (3.23) gives rise to exactly the symmetry

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{3.28}$$

If we let $F = \exp(I_1^2 I_2)$, then (3.23) results in only

$$X = 3x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}. \tag{3.29}$$

As another simple example, if we set $F = I_1 \exp(-I_2)$, then (3.23) implies the one symmetry

$$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}. \tag{3.30}$$

As a matter of interest there are infinitely many one symmetry cases.

To see this we consider the first integral

$$F = \frac{1}{2}I_1^2 - aI_2, \quad a \neq 0. \tag{3.31}$$

The relation (3.23) then yields

$$\begin{aligned}(-I_1 a_3 + I_1 a_4 + a_5 - I_1^2 a_6 - I_2 a_7 - I_1 I_2 a_8) I_1 \\ +(I_1 a_1 - a_2 + I_2 a_4 - I_1 I_2 a_6 - I_2^2 a_8)(-a) = 0.\end{aligned} \tag{3.32}$$

Separation with respect to powers of I_1 and I_2 gives rise to $aa_1 = a_5$. Therefore we have the parameter dependent symmetry

$$X = X_1 + aX_5. \tag{3.33}$$

Yet a more complicated one is

$$X = X_1 + aX_5 + aX_6, \quad a \neq 0. \tag{3.34}$$

This symmetry is associated with the first integral

$$F = \frac{(I_2 a - 1)^2}{I_1^2 a - a} \tag{3.35}$$

which can be constructed just as before. Similarly, there are many possibilities for one symmetry.

Therefore the one symmetry case is not unique. Next we discuss the two symmetry case.

Case 3. Two Symmetries.

We have already seen from Case 2 that the translations in x and y symmetries further imply the uniform scaling symmetry. Thus one cannot have two symmetries of translations alone that are associated with a first integral of the free particle equation. Likewise the same applies for the translations in y and the uniform scaling symmetries.

Further if we have the symmetries

$$X = \frac{\partial}{\partial y} \text{ and } Y = x\frac{\partial}{\partial y}$$

which forms the two-dimensional Abelian algebra, then a_2 and a_5 are arbitrary in (3.23). This directly gives

$$\frac{\partial F}{\partial I_1} = \frac{\partial F}{\partial I_2} = 0,$$

and hence no integral. This means that one does not have these type of symmetries admitted by any first integral of the free particle equation.

The same argument applies if we consider

$$X = \frac{\partial}{\partial y} \text{ and } Y = y\frac{\partial}{\partial y}$$

which forms a two-dimensional non-Abelian algebra. Here again this two-dimensional algebra is not admitted by any integral of the equation.

So when do two symmetries occur for a first integral of the free particle equation? From the above it is clear that the simple type of symmetry combinations do not form two symmetries of an integral. Thus there have to be combinations of the symmetries. One such combination is

$$\begin{aligned} X &= \frac{\partial}{\partial x} - x\frac{\partial}{\partial x}, \\ Y &= \frac{\partial}{\partial y} - x\frac{\partial}{\partial y}. \end{aligned} \tag{3.36}$$

The Lie algebra formed by (3.36) is two-dimensional with commutator $[X, Y] = -Y$. Here the combination of symmetries means that $a_3 = -a_1$ and $a_5 = -a_2$. The use of these in the relation (3.23) forces F to satisfy the one condition

$$\frac{\partial F}{\partial I_1} + \frac{\partial F}{\partial I_2} = 0, \tag{3.37}$$

which gives the independent integral

$$F = I_2 - I_1. \tag{3.38}$$

Hence this F admits two symmetries.

We now look at a case in which at least one of the symmetries has a parameter in it. This is provided by the operators

$$X = X_1 + aX_6, \quad a \neq 0$$
$$Y = X_2 + aX_4. \tag{3.39}$$

The symmetries (3.39) span a two-dimensional algebra with

$$[X, Y] = -aX. \tag{3.40}$$

Here F is given by

$$F = a\frac{I_2}{I_1} - \frac{1}{I_1}. \tag{3.41}$$

We conclude by saying that the two symmetry case is not unique.

Case 4. Three Symmetry.

We now present a detailed study of possible three-dimensional algebra of symmetries admitted by first integrals of the free particle equation. Two essential deductions come out of our analysis. Firstly we show that the three-dimensional algebra admitted by a first integral is unique. Secondly we prove that three is the maximal dimension admitted by any integral.

We utilize the realizations of three-dimensional Lie algebras in the real plane given by Mahomed and Leach [24]. However, we use the notation given in Ibragimov and Mahomed [25] (see also Mahomed [18]), i.e. $L^{\alpha}_{3;i}$, where 3 refers to the dimension of the algebra, i to the number of the algebra in some given ordering and α is the realization as an algebra may have more than one realization. For example, $L^{II}_{3;4}$ denotes the second realization of the fourth Lie algebra of dimension 3.

All canonical forms of three-dimensional real Lie algebras in the plane is given in Table 3.1. This is taken from [18].

Table 3.1

Realizations of three-dimensional algebras in the real plane

$p = \partial/\partial x$ and $q = \partial/\partial y$	
Algebra	Realizations in (x, y) plane
$L_{3;1}$	$X_1 = q, X_2 = xq, X_3 = h(x)q$
$L_{3;2}$	$X_1 = q, X_2 = p, X_3 = xq$
$L^I_{3;3}$	$X_1 = q, X_2 = p, X_3 = xp + (x + y)q$
$L^{II}_{3;3}$	$X_1 = q, X_2 = xq, X_3 = p + yq$
$L^I_{3;4}$	$X_1 = p, X_2 = q, X_3 = xp$
$L^{II}_{3;4}$	$X_1 = q, X_2 = xq, X_3 = xp + yq$
$L^I_{3;5}$	$X_1 = p, X_2 = q, X_3 = xp + yq$
$L^{II}_{3;5}$	$X_1 = q, X_2 = xq, X_3 = yq$
$L^I_{3;6}$	$X_1 = p, X_2 = q, X_3 = xp + ayq,\ a \neq 0, 1$
$L^{II}_{3;6}$	$X_1 = q, X_2 = xq, X_3 = (1 - a)xp + yq,\ a \neq 0, 1$
$L^I_{3;7}$	$X_1 = p, X_2 = q, X_3 = (bx + y)p + (by - x)q$
$L^{II}_{3;7}$	$X_1 = xq, X_2 = q, X_3 = (1 + x^2)p + (x + b)yq$
$L^I_{3;8}$	$X_1 = q, X_2 = xp + yq, X_3 = 2xyp + y^2q$
$L^{II}_{3;8}$	$X_1 = q, X_2 = xp + yq, X_3 = 2xyp + (y^2 - x^2)q$
$L^{III}_{3;8}$	$X_1 = q, X_2 = xp + yq, X_3 = 2xyp + (y^2 + x^2)q$
$L^{IV}_{3;8}$	$X_1 = q, X_2 = yq, X_3 = y^2q$
$L_{3;9}$	$X_1 = (1 + x^2)p + xyq, X_2 = xyp + (1 + y^2)q,$
	$X_3 = yp - xq$

Remark 3.1. We point out that the Lie algebras $L_{3;1}$ and $L_{3;8}^{IV}$ are not admitted by any scalar second-order ODE. Hence we do not consider these hereafter (see [24]).

Instead of using the realizations $L_{3;8}^{I}$, $L_{3;8}^{II}$ and $L_{3;8}^{III}$ given in Table 3.1, we use the free particle generators (see [30])

$$X_1 = p,\ X_2 = xp + \frac{1}{2}yq,\ X_3 = x^2p + xyq, \tag{3.42}$$

$$X_1 = p + xq,\ X_2 = xp + 2yq,\ X_3 = 2(x^2 - y)p + 2xyq, \tag{3.43}$$

$$X_1 = p - xq,\ X_2 = -xp + 2yq,\ X_3 = 2(x^2 + y)p + 2xyq. \tag{3.44}$$

Therefore the realizations of three-dimensional algebras given in Table 3.1 by replacement of $L_{3;8}^{I}$, $L_{3;8}^{II}$ and $L_{3;8}^{III}$ by their free particle operators (3.42), (3.43) and (3.44) above, except $L_{3;1}$ and $L_{3;8}^{IV}$, are free particle symmetry generators. We utilize these in our analysis below.

As $L_{3;1}$ is not admitted by the free particle equation, we begin with $L_{3;2}$. We want this algebra to be admitted by a first integral of the free particle equation. We utilize the classifying relation (3.23). Therefore a_1, a_2 and a_5 are arbitrary which imply that F is constant. Hence this algebra is not admitted by any first integral.

The same applies to the algebras $L_{3;3}^{I}$, $L_{3;3}^{II}$, $L_{3;4}^{I}$, $L_{3;4}^{II}$, $L_{3;5}^{II}$, $L_{3;6}^{I}$, $L_{3;6}^{II}$ $L_{3;7}^{I}$, $L_{3;7}^{II}$ and $L_{3;8}^{I}$.

We separately consider the algebra $L_{3;8}^{II}$. We show that this algebra is not admitted by an integral as well. For what follows we utilize the free particle representation (3.43). We find that these operators correspond to $a_1 = a_5$, $2a_3 = a_4$ and $a_6 = -a_7$. The

substitution of the latter in the relation (3.23) result in the three conditions on F, viz.

$$\begin{aligned} I_1\frac{\partial F}{\partial I_1}+2I_2\frac{\partial F}{\partial I_2}&=0,\\ \frac{\partial F}{\partial I_1}+I_1\frac{\partial F}{\partial I_2}&=0,\\ (I_1^2-I_2)\frac{\partial F}{\partial I_1}+I_1I_2\frac{\partial F}{\partial I_2}&=0. \end{aligned} \tag{3.45}$$

The first two imply that F is constant which satisfy the third. Thus there is no algebra of this type admitted by a first integral of the free particle equation. The analysis for $L_{3;8}^{III}$ is similar and this algebra too is not admitted.

For $L_{3;9}$, the operators imply that $a_6=-a_5$, $a_7=a_1$ and $a_8=a_2$, the insertion of which into the relation (3.23) gives a condition on F with a_1, a_2 and a_5 arbitrary. Then the result that F must be constant arises. Thus this algebra is not admitted as well.

In the case of $L_{3;5}^{I}$ we have that I_1 has this algebra. This is precisely the algebra of the symmetries given in (3.14). We mention that the symmetries given in (3.15) and (3.17) also form the algebra $L_{3;5}^{I}$ as a projective transformation (see [20]) maps each of the representations to the one given in (3.14).

In conclusion, we have that the only three-dimensional algebra admitted by a first integral of the free particle equation is $L_{3;5}^{I}$.

We can state the following theorems.

Theorem 3.1. A first integral of the free particle or any scalar linearizable, by point transformation, second-order ODE admits a three-dimensional algebra if and only if the algebra is $L_{3;5}^{I}$.

The proof follows from the preceding discussion. Also we note that this algebra $L^I_{3;5}$ is admitted by the integrals I_1, I_2 or I_2/I_1.

Theorem 3.2. The maximum dimension of the algebra admitted by any first integral of the free particle or any scalar linearizable, by point transformation, second-order ODE is three and the algebra is $L^I_{3;5}$.

Proof. A first integral of the free particle or scalar second-order ODE which is linearizable by point transformation, cannot admit a maximal algebra of dimension more than three since the functionally independent integrals or their quotient has the unique three-dimensional algebra which corresponds to $L^I_{3;5}$. The other integrals possess lower dimensional algebras.

Chapter 4

Algebraic Properties of First Integrals for Scalar Linear Third-Order ODEs of Maximal Symmetry

4.1 Introduction

The subject of the present chapter is the investigation of the Lie algebraic properties of first integrals of scalar linear third-order ODEs of the maximal class which is represented by $y''' = 0$. We remind the reader that for the simplest class there has been some analysis made in Flessas et al. [22]. This is in regards to the maximal algebra

possessed by an integral of $y''' = 0$ which is listed in Table II of the paper cited. However, this is incomplete. We extend this study and provide a complete analysis on the Lie point symmetries and first integrals for the simplest third-order ODE including the maximal algebra case. We firstly deduce the classifying relation between the point symmetries and first integrals for this simple class. Then we use this to study the point symmetry properties of the first integrals of $y''' = 0$ which also represents all linearizable by point transformations third-order ODEs that reduce to this class. The contents of this chapter are original and we have published these in the paper [31].

We begin by noting the condition for symmetries of the first integrals of scalar linear ODEs of order one. Then for completeness we review briefly the results of the previous chapter (see also Mahomed and Momoniat [26]) which discusses the relationship between the point symmetries of the first integrals of scalar linear second-order ODEs. These two cases are shown to be distinct in terms of their algebraic properties of their integrals when compared to higher-order ODEs of maximal symmetry.

Linear First-Order Equations

Consider the simplest first-order ODE

$$y' = 0. \tag{4.1}$$

It is easy to see that

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{4.2}$$

is a point symmetry generator of (4.1) if

$$X^{[1]}y'|_{y'=0} = 0, \tag{4.3}$$

where as before

$$X^{[1]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'} \tag{4.4}$$

with

$$\zeta_x = D_x(\eta) - y'D_x(\xi), \tag{4.5}$$

in which D_x is the total differentiation operator and $X^{[1]}$ is the first prolongation of the generator X. We quickly see that

$$\eta_x = 0, \eta = \eta(y), \tag{4.6}$$

where η is an arbitrary function of y. Therefore,

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(y)\frac{\partial}{\partial y}. \tag{4.7}$$

Thus there is an infinite number of point symmetries. We now show that only $X = \xi(x, y)\partial/\partial x$ are symmetries of the first integral.

This forms an infinite-dimensional subalgebra of the Lie algebra of the equation (4.1).

Now $I = y$ is a first integral of (4.1). It has point symmetry X as in (4.2) if

$$XI = 0. \tag{4.8}$$

This implies thus $\eta = 0$ which immediately results in

$$X = \xi(x, y)\frac{\partial}{\partial x}. \tag{4.9}$$

There is an infinite number of symmetries of the first integral $I = y$ of (4.1).

Let F be an arbitrary function of I, viz. $F = F(I)$. The symmetry of this general function of the first integral is

$$XF = XI\frac{\partial F}{\partial I} = 0. \tag{4.10}$$

Therefore X as in (4.9) is a symmetry of $I = y$ and also any function of $F(y)$. Since any scalar first-order ODE is equivalent to the simplest one (4.1), this means that a first integral of a nonlinear first-order ODE has infinitely many symmetries too.

As an example we consider the nonlinear first-order Riccati equation

$$y' + y^2 = 0 \tag{4.11}$$

the first integral of which is

$$I = \frac{1}{y} - x. \tag{4.12}$$

A symmetry of the first integral (4.12) is

$$X = \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y}. \tag{4.13}$$

In fact, we have an infinite number of symmetries given by

$$X = \xi(x, y)(\frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y}), \tag{4.14}$$

where ξ is an arbitrary function in its arguments.

Therefore we note here that the symmetries of the first integrals of a first-order equation forms a proper subalgebra of the Lie algebra of the equation itself. We cannot generate the full algebra as is the case for linear scalar second-order ODEs [20] by use of the algebra of the basic integral alone.

The symmetries of the first integrals of scalar linear second-order ODEs have interesting properties as we have seen in Chapter 3 (see [20, 26]). The first integrals of such linear equations can have 0, 1, 2 and the maximum 3 symmetries. The Lie algebra of the maximum case is unique. Peculiar to such equations is the other remarkable

property that their Lie algebra is generated by the symmetry properties of the basic integrals and their quotient [20].

Below we study the symmetry properties of first integral for the simplest scalar linear third-order ODEs of maximal point symmetry. In the case of the basic first integrals, the algebraic properties are known from the work [21]. Here we pursue the relationship between symmetries and first integrals of scalar linear third-order ODEs for the simplest and maximal class. We obtain the classifying relation for this class and invoke this to arrive at counting theorems and the result on the maximal case of symmetries of the first integrals.

In the following we look at algebraic properties of first integrals for the seven point symmetry case by deriving the classifying relation between the symmetries and their first integrals. We use this relation to arrive at interesting properties which appear for the first time in the literature.

4.2 Algebraic Properties of the First Integrals of $y''' = 0$

We consider the simplest third-order ODE

$$y''' = 0 \tag{4.15}$$

which as is well-known has the seven symmetries (Lie [10] and e.g. [18])

$$X_1 = \frac{\partial}{\partial y}$$

$$\begin{aligned} X_2 &= x\frac{\partial}{\partial y} \\ X_3 &= x^2\frac{\partial}{\partial y} \\ X_4 &= y\frac{\partial}{\partial y} \\ X_5 &= \frac{\partial}{\partial x} \\ X_6 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ X_7 &= x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}. \end{aligned} \tag{4.16}$$

We have listed the symmetries in the order of the solution symmetries being first, then the homogeneity symmetry and the remaining three which form the algebra $sl(2, R)$. This ODE (4.15) also represents all linearizable third-order ODEs reducible to it via point transformation. The order in which the symmetries appear in (4.16) is used in what follows. It is obvious that (4.15) has three functionally independent first integrals

$$\begin{aligned} I_1 &= y'' \\ I_2 &= xy'' - y' \\ I_3 &= \frac{1}{2}x^2y'' - xy' + y. \end{aligned} \tag{4.17}$$

We use the ordering of the integrals as given in [21]. The first integral (4.17a) has four symmetries [21]

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} \\ X_2 &= \frac{\partial}{\partial y} \\ X_3 &= x\frac{\partial}{\partial y} \\ X_4 &= x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}, \end{aligned} \tag{4.18}$$

from which we observe that there are two solution symmetries, one translation in x symmetry and a scaling symmetry. The translation in x symmetry is a subset of the $sl(2,R)$ symmetries with X_4 being a combination of the uniform scaling symmetry in both variables contained in the $sl(2,R)$ symmetries together with the homogeneity symmetry. Part of this fact was also noted in [22]. The second first integral (4.17b) has three symmetries [21]

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial y} \\ Y_2 &= x^2\frac{\partial}{\partial y} \\ Y_3 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \end{aligned} \tag{4.19}$$

with two solution symmetries and Y_3 being part of the $sl(2,R)$ symmetries. The third first integral (4.17c) also has four symmetries [21]

$$\begin{aligned} G_1 &= x\frac{\partial}{\partial x} \\ G_2 &= x\frac{\partial}{\partial y} \\ G_3 &= x^2\frac{\partial}{\partial y} \\ G_4 &= x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}. \end{aligned} \tag{4.20}$$

Again one can see the solution symmetries, scaling and the symmetry G_4 which is contained in the $sl(2,R)$ symmetries. Note further that the symmetries in (4.20) are found by multiplying those of (4.18) by the factor x. In fact these two sets are equivalent via a point transformation [21]. The other important properties are discussed in the next section.

Below we obtain the classifying relation.

Classifying relation for the symmetries and integrals

Now let F be an arbitrary function of the integrals (4.17), I_1, I_2 and I_3, viz.

$$F = F(I_1, I_2, I_3).$$

The symmetry of this general function of the first integrals is

$$X^{[2]}F = X^{[2]}I_1\frac{\partial F}{\partial I_1} + X^{[2]}I_2\frac{\partial F}{\partial I_2} + X^{[2]}I_3\frac{\partial F}{\partial I_3} = 0, \tag{4.21}$$

where

$$\begin{aligned}
X^{[2]}I_1 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'} + \zeta_{xx}\frac{\partial}{\partial y''}]y'' \\
&= \zeta_{xx} \\
X^{[2]}I_2 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'} + \zeta_{xx}\frac{\partial}{\partial y''}](xy'' - y') \\
&= \xi y'' - \zeta_x + x\zeta_{xx} \\
X^{[2]}I_3 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'} + \zeta_{xx}\frac{\partial}{\partial y''}](\frac{1}{2}x^2y'' - xy' + y) \\
&= \xi(xy'' - y') + \eta - x\zeta_x + \frac{1}{2}x^2\zeta_{xx}.
\end{aligned} \tag{4.22}$$

The coefficient functions ξ, η, ζ_x and ζ_{xx} are

$$\begin{aligned}
\xi &= a_5 + xa_6 + x^2a_7 \\
\eta &= a_1 + xa_2 + x^2a_3 + ya_4 + ya_6 + 2xya_7 \\
\zeta_x &= a_2 + 2xa_3 + y'a_4 + 2ya_7 \\
\zeta_{xx} &= 2a_3 + y''a_4 - y''a_6 + (2y' - 2xy'')a_7.
\end{aligned} \tag{4.23}$$

These are obtained by setting

$$X^{[2]} = \sum_{i=1}^{7} a_iX_i^{[2]}, \tag{4.24}$$

where the X_i are the symmetry generators as given in (4.16) and the a_i are constants. The reason being that the symmetries of the first integrals are always the symmetries of the equation (see [6]).

After substitution of the values of $X^{[2]}I_1$, $X^{[2]}I_2$, $X^{[2]}I_3$ as in (4.22), with ξ, η, ζ_x, ζ_{xx} given in (4.23), as well as by use of the first integrals $I_1 = y''$, $I_2 = xy'' - y'$, $I_3 = \frac{1}{2}x^2y'' - xy' + y$ in equation (4.21), we arrive at the classifying relation

$$\begin{aligned}[2a_3 + (a_4 - a_6)I_1 - 2a_7I_2]\frac{\partial F}{\partial I_1} \\ +(-a_2 + a_4I_2 + a_5I_1 - 2a_7I_3)\frac{\partial F}{\partial I_2} \\ +[a_1 + (a_4 + a_6)I_3 + a_5I_2]\frac{\partial F}{\partial I_3} = 0.\end{aligned} \tag{4.25}$$

The relation (4.25) explicitly provides the relationship between the symmetries and the first integrals of the simple third-order equation (4.15). We invoke this to classify the first integrals according to their symmetries in what follows.

We use the classifying relation (4.25) to establish the number and property of symmetries possessed by the first integrals of the simplest third-order equation (4.15).

There arise five cases. We deal with each below.

Case 1: No symmetry

If F is any arbitrary function of I_1, I_2 and I_3 then F_{I_1}, F_{I_2} and F_{I_3} are not related to each other. In this case we have from (4.25) that

$$2a_3 + (a_4 - a_6)I_1 - 2a_7I_2 = 0, \tag{4.26}$$

$$-a_2 + a_4I_2 + a_5I_1 - 2a_7I_3 = 0, \tag{4.27}$$

$$a_1 + (a_4 + a_6)I_3 + a_5 I_2 \;\; = \;\; 0. \tag{4.28}$$

It is easy to see from (4.26), (4.27) and (4.28) that all the a's are zero. Therefore there exists no symmetry for this case.

As an illustrative example, if we take $F = I_1 I_2 \ln I_3$, then equation (4.25) straightforwardly yields

$$\begin{aligned} &[2a_3 + (a_4 - a_6)I_1 - 2a_7 I_2]I_2 I_3 \ln I_3 \\ +&(-a_2 + a_4 I_2 + a_5 I_1 - 2a_7 I_3)I_1 I_3 \ln I_3 \\ +&[a_1 + (a_4 + a_6)I_3 + a_5 I_2]I_1 I_2 = 0. \end{aligned} \tag{4.29}$$

This easily results in all the a's being zero.

Case 2: One Symmetry

If F satisfies the relation (4.25), then there exists one symmetry. For the simple symmetries of (4.15) one obtains further symmetries except for X_6 which we consider below.

If we take $F = I_1 I_2 I_3$ or any function of this product, then the relation (4.25) becomes

$$\begin{aligned} &[2a_3 + (a_4 - a_6)I_1 - 2a_7 I_2]I_2 I_3 \\ +&(-a_2 + a_4 I_2 + a_5 I_1 - 2a_7 I_3)I_1 I_3 \\ +&[a_1 + (a_4 + a_6)I_3 + a_5 I_2]I_1 I_2 = 0. \end{aligned} \tag{4.30}$$

In (4.30), a_1 to a_7 are zeros except a_6 which gives the one symmetry

$$X_6 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{4.31}$$

Consider $J = I_1 I_3 - \frac{1}{2} I_2^2$ and $I_2 = xy'' - y'$. Now let $F = J/I_2$ (cf. [22]). Then relation (4.25) becomes

$$\begin{aligned}
&[2a_3 + (a_4 - a_6)I_1 - 2a_7 I_2]2I_2 I_3 \\
&+(-a_2 + a_4 I_2 + a_5 I_1 - 2a_7 I_3)(-2I_1 I_3 - I_2^2) \\
&+[a_1 + (a_4 + a_6)I_3 + a_5 I_2]2I_1 I_2 = 0.
\end{aligned} \tag{4.32}$$

We see here that all the a's are zero except a_6. Therefore there exists one symmetry which again is (4.31).

In fact similar to the free particle equation as in chapter 3 (see also [26]), there are many one symmetry cases.

Case 3: Two Symmetries

Here there are many cases as well. We begin by utilizing the Lie table for the classification of the two-dimensional algebras in the plane which are given e.g. in [25]. They are

$$\begin{aligned}
Y_1 = \frac{\partial}{\partial y}, \quad Y_2 = \frac{\partial}{\partial x}, \\
Y_1 = \frac{\partial}{\partial y}, \quad Y_2 = x\frac{\partial}{\partial y}, \\
Y_1 = \frac{\partial}{\partial y}, \quad Y_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \\
Y_1 = \frac{\partial}{\partial y}, \quad Y_2 = y\frac{\partial}{\partial y}.
\end{aligned} \tag{4.33}$$

These form subalgebras of the Lie algebra of symmetries of (4.15) as can clearly be observed.

We take the first realization listed above. If a_1 is arbitrary in (4.25), then $X_1 = \partial/\partial y$ implies that F is independent of I_3. Further $X_5 = \partial/\partial x$ yields that F does not depend on I_2 as well. Since we require that $\partial F/\partial I_1 \neq 0$, we have

$$2a_3 + I_1(a_4 - a_6) - 2a_7 I_2 = 0$$

from which it follows that $a_3 = a_7 = 0$ and $a_4 = a_6$. Thus we end up with two more symmetries X_2 and $X_4 + X_6$. These turn out to be the four symmetries of the integral I_1 given in (4.18).

Likewise for the second realization we obtain (4.18) again. For the third realization listed above we find the symmetries of I_2 as in (4.19).

Hence the first three realizations listed do not provide maximal two symmetries for the first integrals of (4.15).

In fact the fourth realization results in a two symmetry case as a_1 and a_4 arbitrary give rise to

$$\frac{\partial F}{\partial I_3} = 0, \quad I_1 \frac{\partial F}{\partial I_1} + I_2 \frac{\partial F}{\partial I_2} = 0$$

which has solution

$$F = H(I_2/I_1).$$

The further substitution of this form into the relation (4.25) constrains all the a's to be zero except for a_1 and a_4. This result prompts the following simple products and quotients that do give two symmetries.

If $F = I_1 I_2$, then relation (4.25) yields

$$[2a_3 + (a_4 - a_6)I_1 - 2a_7 I_2]I_2$$

$$+(-a_2 + a_4 I_2 + a_5 I_1 - 2a_7 I_3) I_1 = 0 \tag{4.34}$$

Here we observe that a_2, a_3, a_5 and a_7 are zeros whereas a_1 is arbitrary and $a_6 = 2a_4$ and therefore we obtain the two symmetries

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y} \\ Y &= 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} \end{aligned} \tag{4.35}$$

which form a two-dimensional algebra with

$$[X_1, Y] = 3X_1.$$

If we set $F = I_1 I_3$, then we end up getting a_1, a_3, a_4, a_5, a_7 equal to zero. Since a_2 and a_6 are arbitrary so they result in two symmetries

$$\begin{aligned} X_2 &= x\frac{\partial}{\partial y} \\ X_6 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \end{aligned} \tag{4.36}$$

with Lie bracket

$$[X_2, X_6] = 0.$$

If we take $F = I_2 I_3$, then we see that a_3 is arbitrary and $a_6 = -2a_4$ which then give rise to the two symmetries

$$\begin{aligned} X_3 &= x^2\frac{\partial}{\partial y} \\ Y &= 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \end{aligned} \tag{4.37}$$

with

$$[X_3, Y] = -3X_3.$$

Consider now $F = I_3/I_1$. This shows that a_2 and a_4 are arbitrary and the resulting two symmetries are

$$\begin{aligned} X_2 &= x\frac{\partial}{\partial y} \\ X_4 &= y\frac{\partial}{\partial y} \end{aligned} \tag{4.38}$$

with commutation relation

$$[X_2, X_4] = X_2.$$

If we let $F = I_3/I_2$, then here a_3 and a_4 are arbitrary and therefore the two symmetries are

$$\begin{aligned} X_3 &= x^2\frac{\partial}{\partial y} \\ X_4 &= y\frac{\partial}{\partial y} \end{aligned} \tag{4.39}$$

with

$$[X_3, X_4] = X_3.$$

If $J = I_1 I_3 - \frac{1}{2}I_2^2$, $I_1 = y''$ and $F = J/I_1$ (cf [22]), then we have from the relation (4.25),

$$\begin{aligned} &[2a_3 + (a_4 - a_6)I_1 - 2a_7 I_2]\frac{1}{2}I_2^2 \\ +&[-a_2 + a_4 I_2 + a_5 I_1 - 2a_7 I_3](-I_1 I_2) \\ +&[a_1 + (a_4 + a_6)I_3 + a_5 I_2]I_1^2 = 0. \end{aligned} \tag{4.40}$$

This results in a_1, a_2, a_3 and a_7 being zero whereas a_5 arbitrary and $a_4 = -a_6$, which give rise to the two symmetries

$$X_5 = \frac{\partial}{\partial x}$$

$$Y = x\frac{\partial}{\partial x}, \tag{4.41}$$

with

$$[X_5, Y] = X_5.$$

If $J = I_1 I_3 - \frac{1}{2} I_2^2$, $I_3 = \frac{1}{2} x^2 y'' - xy' + y$ and $F = J/I_3$ (cf [22]), then relation (4.25) yields

$$\begin{aligned} &[2a_3 + (a_4 - a_6)I_1 - 2a_7 I_2]2I_3^2 \\ &+(-a_2 + a_4 I_2 + a_5 I_1 - 2a_7 I_3)(-2I_3 I_2) \\ &+[a_1 + (a_4 + a_6)I_3 + a_5 I_2]I_2^2 = 0. \end{aligned} \tag{4.42}$$

The above relation shows that a_1, a_2, a_3, a_5 are zeros, a_7 is arbitrary and $a_4 = a_6$. This imply two symmetries

$$\begin{aligned} X_7 &= x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} \\ Y &= x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} \end{aligned} \tag{4.43}$$

together with

$$[X_7, Y] = -X_7.$$

There are thus many two symmetry cases. One could obtain more. Also they could arise as different combinations of the seven symmetries (4.16). So one concludes that the two-dimensional algebra cases are not unique. We have seen the occurrence of both Abelian and non-Abelian Lie algebras.

Case 4: Three Symmetries

Here we use the three-dimensional real realizations of Mahomed and Leach [24]. The notation used is that given in [18]. Since we adapt these realizations as symmetries of third-order equations, the entries for the first entry $L_{3;1}$ and the non-solvable algebra $L^{I}_{3;8}$ are those which are symmetries of such equations. The table below is similar to Table 3.1 except for these entries. To avoid confusion we present the table of realizations here suited to our needs.

Note that for $L^{II}_{3;8}$, $L^{III}_{3;8}$ and $L^{IV}_{3;8}$ one can use the realizations as given in Table 4.1 or the ones obtained by interchanging x and y in the realizations given. The reason for this is that one still obtains third-order representative equations for the latter realizations.

Table 4.1

Realizations of three-dimensional algebras in the real plane

$p = \partial/\partial x$ and $q = \partial/\partial y$	
Algebra	Realizations in (x, y) plane
$L_{3;1}$	$X_1 = q, X_2 = xq, X_3 = x^2 q$
$L_{3;2}$	$X_1 = q, X_2 = p, X_3 = xq$
$L^I_{3;3}$	$X_1 = q, X_2 = p, X_3 = xp + (x + y)q$
$L^{II}_{3;3}$	$X_1 = q, X_2 = xq, X_3 = p + yq$
$L^I_{3;4}$	$X_1 = p, X_2 = q, X_3 = xp$
$L^{II}_{3;4}$	$X_1 = q, X_2 = xq, X_3 = xp + yq$
$L^I_{3;5}$	$X_1 = p, X_2 = q, X_3 = xp + yq$
$L^{II}_{3;5}$	$X_1 = q, X_2 = xq, X_3 = yq$
$L^I_{3;6}$	$X_1 = p, X_2 = q, X_3 = xp + ayq,\ a \neq 0, 1$
$L^{II}_{3;6}$	$X_1 = q, X_2 = xq, X_3 = (1 - a)xp + yq,\ a \neq 0, 1$
$L^I_{3;7}$	$X_1 = p, X_2 = q, X_3 = (bx + y)p + (by - x)q$
$L^{II}_{3;7}$	$X_1 = xq, X_2 = q, X_3 = (1 + x^2)p + (x + b)yq$
$L^I_{3;8}$	$X_1 = p, X_2 = xp + yq, X_3 = x^2 p + 2xyq$
$L^{II}_{3;8}$	$X_1 = q, X_2 = xp + yq, X_3 = 2xyp + (y^2 - x^2)q$
$L^{III}_{3;8}$	$X_1 = q, X_2 = xp + yq, X_3 = 2xyp + (y^2 + x^2)q$
$L^{IV}_{3;8}$	$X_1 = q, X_2 = yq, X_3 = y^2 q$
$L_{3;9}$	$X_1 = (1 + x^2)p + xyq, X_2 = xyp + (1 + y^2)q,$
	$X_3 = yp - xq$

As $L_{3;1}$ is the first three-dimensional algebra in Table 4.1, we start with that. We want this Abelian algebra to be admitted by a first integral of the equation (4.15). We utilize the classifying relation (4.25). Therefore a_1, a_2 and a_3 are arbitrary which imply that F is constant. Hence this algebra is not admitted by any first integral of (4.15) although, if it is admitted by a nonlinear third-order ODE, it implies linearization (see Mahomed and Leach [19]).

It is not difficult to deduce that the same applies to the algebras $L_{3;2}$, $L^{I}_{3;3}$, $L^{II}_{3;3}$, $L^{I}_{3;4}$, $L^{II}_{3;4}$, $L^{I}_{3;5}$, $L^{II}_{3;5}$, $L^{I}_{3;6}$ when $a \neq 2$ and $L^{II}_{3;6}$ when $a \neq 1/2, -1$. If $a = 2$ for $L^{I}_{3;6}$, then one ends up with four symmetries of $F = F(I_1)$ which are (4.18). The same applies to $L^{II}_{3;6}$, $a = 1/2$. For both these cases, the algebras $L^{I}_{3;6}$, $a = 2$ and $L^{II}_{3;6}$, $a = 1/2$ are admitted by a first integral $F = F(I_1)$ but these are not maximal and contained in a four-dimensional algebra, spanned by operators (4.18).

Now we focus on the three symmetry case which is admitted by $F = F(I_2)$. These three symmetries are given in (4.19). The Lie algebra of the generators (4.19) has nonzero commutators

$$[X_1, X_3] = X_1, [X_2, X_3] = -X_2 \tag{4.44}$$

with the elements thus forming the Lie algebra $L^{II}_{3;6}$, $a = -1$. In fact the transformation

$$X = x^2, \quad Y = x + y$$

maps it to the canonical form of Table 4.1, viz.

$$\bar{X}_1 = \frac{\partial}{\partial Y}, \bar{X}_2 = X\frac{\partial}{\partial Y}, \bar{X}_3 = 2X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y}.$$

Therefore the symmetries of $F = F(I_2)$ has the Lie algebra $L^{II}_{3;6}$, $a = -1$.

There is yet another three-dimensional algebra which is admitted by a first integral of (4.15). This occurs for $L^{I}_{3;8}$. We use the classifying relation (4.25). Here a_5, a_6 and a_7 are arbitrary. Making these constants one at a time unity and the rest zero yield

$$\begin{aligned} I_1\frac{\partial F}{\partial I_2}+I_2\frac{\partial F}{\partial I_3} &= 0,\\ -I_1\frac{\partial F}{\partial I_1}+I_3\frac{\partial F}{\partial I_3} &= 0,\\ I_2\frac{\partial F}{\partial I_1}+I_3\frac{\partial F}{\partial I_2} &= 0. \end{aligned} \tag{4.45}$$

The solution to this system (4.45) yields

$$F = F(I_1I_3-\frac{1}{2}I_2^2). \tag{4.46}$$

Thus the basic first integral $J = I_1I_3 - \frac{1}{2}I_2^2$ has the algebra $L^{I}_{3;8}$.

The Lie algebras $L^{II}_{3;8}$, $L^{III}_{3;8}$ and $L^{IV}_{3;8}$ are not admitted by any first integral of (4.15) as these are subalgebras of the maximal six-dimensional algebras (see Ibragimov and Mahomed [25]) which are admitted by nonlinear third-order ODEs not reducible to the simplest equation (4.15).

In the case of the algebra $L_{3;9}$ one has the situation that this algebra is not a subalgebra of the seven-dimensional algebra of equation (4.15) (Wafo and Mahomed [32]).

In conclusion of this discussion, we have two three-dimensional algebras admitted by a first integral of equation (4.15) which are $L^{I}_{3;8}$ and $L^{II}_{3;6}$, $a = -1$.

We state the following theorem.

Theorem 4.1. If a first integral of the simplest third-order ODE, $y''' = 0$, admits a three-dimensional algebra, then it is one of the two three-dimensional algebras $L^{II}_{3;6}$,

$a = -1$, or $L^I_{3;8}$.

The proof follows from the previous discussions. Note here that the three-dimensional algebra admitted is not unique.

Case 5: Four Symmetries

In the four symmetry case we have that only I_1 and I_3 possess four symmetries. They are given by (4.18) and (4.20). Both are similar by a point transformation [21]. The Lie algebra of (4.18) is $L^I_{4;9}$ (see [22]). This can be seen by interchanging X_1 and X_2 in (4.18). Thus the nonzero commutation relations are

$$\begin{aligned} [X_2, X_3] = X_1, [X_1, X_4] = 2X_1 \\ [X_2, X_4] = X_2, [X_3, X_4] = X_3 \end{aligned} \tag{4.47}$$

We have the following theorem.

Theorem 4.2. The maximum dimension of the Lie algebra admitted by any first integral of the simplest third-order ODE, $y''' = 0$ or any third-order ODE linearizable by point transformation to the simplest ODE, is $L^I_{4;9}$.

Proof. Any first integral of $y''' = 0$ or third-order ODE reducible to the simplest ODE by point transformation cannot admit a maximal algebra of dimension greater then four since the basic integrals I_1 and I_3 has the unique four dimension algebra $L^I_{4;9}$. The other integrals have lower dimensional Lie algebras in the classification obtained above.

Finally, we have the following counting theorem.

Theorem 4.3. A first integral of the simplest third-order ODE, $y''' = 0$ or any linearizable third-order ODE by point transformation to the simplest ODE, can have 0, 1, 2, 3 or the maximum 4 symmetries. The four symmetry case is unique.

4.3 Symmetry Properties of First Integrals of Higher-Order ODEs: Some Remarks

In the case of symmetries of the simplest first-order ODE $y' = 0$ we have seen that the algebra of any first integral constitute a proper subalgebra of the equation itself. One cannot generate the full algebra of $y' = 0$ via the algebras of any integral. This result also applies to any scalar first-order ODE due to equivalence of this with the simplest equation.

What occurs for scalar linear second-order ODEs is very different to the first-order ODE case. Here as has been shown in [20], the Lie algebra of $y'' = 0$ which represents any linear or linearizable second-order ODE can be generated by the three-dimensional algebras of the triplets of the basic integrals and their quotient which are isomorphic to each. Thus in this case one requires not only the basic integrals, say I_1 and I_2, but a functionally dependent quotient integral $J = I_2/I_1$.

Another important point to make is that the full Lie algebra of the simplest third-order equation (4.15) is generated by the four symmetries (4.18) as well as the three symmetries G_2, G_3 and G_4 of (4.20). Hence one requires only the symmetries of the basic integrals I_1 and I_3 to generate the full algebra of our equation (4.15). One should

contrast this with what happens for $y' = 0$ and $y'' = 0$ discussed above. So one has the seven symmetries of our equation (4.15) being generated by four symmetries of I_1 together with three symmetries of I_3. The natural question then is: what occurs for higher-order ODEs of maximal symmetry? Patterns emerge, some of which are discussed in three propositions in the paper of Flessas et al. [22]. We discuss another important property, viz. that of generation of the full algebra via integrals now.

Consider the nth-order ODE of maximal symmetry

$$y^{(n)} = 0, \quad n \geq 3. \tag{4.48}$$

This ODE (4.48) has $n+4$ symmetries as is well-known. The n first integrals of (4.48) are easily constructible and we focus on the first and last which are

$$I_1 = y^{(n-1)} \tag{4.49}$$

and

$$I_n = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{(n-i)!} x^{n-i} y^{(n-i)}. \tag{4.50}$$

The first integral (4.49) has $n+1$ symmetries which are

$$\begin{aligned} X_i &= x^{i-1}\frac{\partial}{\partial y}, \quad i = 1, \ldots, n-1 \\ X_n &= \frac{\partial}{\partial x}, \\ X_{n+1} &= x\frac{\partial}{\partial x} + (n-1)y\frac{\partial}{\partial y}. \end{aligned} \tag{4.51}$$

This forms an $n+1$-dimensional subalgebra of the equation (4.48). Now the first integral (4.50) has symmetries

$$Y_i = x^i \frac{\partial}{\partial y}, \quad i = 1, \ldots, n-1$$

$$\begin{aligned} Y_n &= x\frac{\partial}{\partial x}, \\ Y_{n+1} &= x^2\frac{\partial}{\partial x} + (n-1)xy\frac{\partial}{\partial y} \end{aligned} \tag{4.52}$$

which one can see comes from multiplying the symmetries of (4.51) by x. We can observe from these two sets (4.51) and (4.52) that the full Lie algebra of our equation (4.48) is generated from the $n+1$ symmetries of (4.51) and 3 symmetries of (4.52), viz. Y_{n-1}, Y_n and Y_{n+1}. Further the two sets (4.51) and (4.52) are equivalent to each other by means of the point transformation

$$\bar{x} = \frac{1}{x}, \quad \bar{y} = \frac{y}{x^{n-1}}. \tag{4.53}$$

This is an extension of the transformation given in [21] and is for higher-order ODEs. In [21] it was given for third-order ODEs.

We have the following theorem the proof of which is evident from the above.

Theorem 4.4. The full Lie algebra of the nth-order ODE $y^{(n)} = 0, \quad n \geq 3$, is generated by two subalgebras, viz. the $n+1$-dimensional algebra $< X_j : j = 1, \ldots, n+1 >$ of I_1 and the three-dimensional subalgebra $< Y_{n-1}, Y_n, Y_{n+1} >$ of I_n.

Hence the picture is quite distinct for the manner in which the full Lie algebra is generated for the ODEs $y' = 0$, $y'' = 0$ and $y^{(n)} = 0,\ n \geq 3$. This is also consistent with the properties of their symmetry algebra which are different (see, e.g. [18]).

Chapter 5

Characterization of Symmetry Properties of First Integrals for Submaximal Linearizable Third-Order ODEs

5.1 Introduction

The discussion of this chapter is about the Lie algebraic properties of first integrals of scalar linearizable third-order ODEs of the submaximal classes which are represented by $y''' - y' = 0$ and $y''' + f(x)y'' - y' - f(x)y = 0$, where $f(x)$ is an arbitrary function of x. The former has four point symmetries and the latter five. As we mentioned earlier

that there was some work [22] commenced by Flessas et al. for the simplest class and which we extended in the previous chapter (see also Mahomed and Momoniat in [31]) to provide a complete analysis on the symmetries and first integrals for this simplest class of ODEs which included the maximal algebra case being generated by algebras of two basic integrals of the equation. In the present chapter we deduce the classifying relation between the point symmetries and first integrals for the sub-maximal classes of scalar linear third-order equations. Then by using this we find the point symmetry properties of the first integrals of the sub-maximal classes of third-order equations $y''' - y' = 0$ and $y''' + f(x)y'' - y' - f(x)y = 0$ which also represent all linearizable by point transformations third-order ODEs that reduce to these classes. We obtain counting theorems for the number of point symmetries possessed by an integral of such equations. Noteworthy is that the maximal algebra is not unique. This chapter appears in our published work [33].

In the next section we study the point symmetry properties of the integrals of the 4 symmetry class represented by $y''' - y' = 0$. This section is to remind the reader under what conditions point symmetries of first integrals of scalar linear third-order ODEs exist (see chapter 4 and [31]). Then in Section 5.3 we analyze the class $y''' + f(x)y'' - y' - f(x)y = 0$ which has four point symmetries for the symmetry structure of its first integrals. In Section 5.4 we focus on the generation of the full algebra by subalgebras of certain basic integrals.

5.2 Lie Algebraic Properties of the Integrals of $y''' - y' = 0$

We consider the representative third-order ODE

$$y''' - y' = 0 \tag{5.1}$$

which has five point symmetries

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} \\ X_2 &= \frac{\partial}{\partial y} \\ X_3 &= e^x \frac{\partial}{\partial y} \\ X_4 &= e^{-x} \frac{\partial}{\partial y} \\ X_5 &= y \frac{\partial}{\partial y}. \end{aligned} \tag{5.2}$$

The ordering of these is the translation in x followed by the three solution symmetries and then the homogeneity symmetry. It is easy to see here that (5.1) has three functionally independent first integrals

$$\begin{aligned} I_1 &= y'' - y \\ I_2 &= e^x y'' - e^x y' \\ I_3 &= e^{-x} y'' + e^{-x} y'. \end{aligned} \tag{5.3}$$

The order of the integrals are dictated by their algebraic properties which come at the end of this section.

Classifying relation for the symmetries of $y''' - y' = 0$

Let F be an arbitrary function of I_1, I_2 and I_3, viz. $F = F(I_1, I_2, I_3)$. The symmetry of this general function of the first integrals is

$$X^{[2]}F = X^{[2]}I_1\frac{\partial F}{\partial I_1} + X^{[2]}I_2\frac{\partial F}{\partial I_2} + X^{[2]}I_3\frac{\partial F}{\partial I_3} = 0, \tag{5.4}$$

where

$$\begin{aligned}
X^{[2]}I_1 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'} + \zeta_{xx}\frac{\partial}{\partial y''}](y'' - y) \\
&= -\eta + \zeta_{xx} \\
X^{[2]}I_2 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'} + \zeta_{xx}\frac{\partial}{\partial y''}](e^x y'' - e^x y') \\
&= (e^x y'' - e^x y')\xi - e^x\zeta_x + e^x\zeta_{xx} \\
X^{[2]}I_3 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'} + \zeta_{xx}\frac{\partial}{\partial y''}](e^{-x} y'' + e^{-x} y') \\
&= (-e^{-x} y'' - e^{-x} y')\xi + e^{-x}\zeta_x + e^{-x}\zeta_{xx}.
\end{aligned} \tag{5.5}$$

Now ξ, η, ζ_x and ζ_{xx} are

$$\begin{aligned}
\xi &= a_1 \\
\eta &= a_2 + e^x a_3 + e^{-x} a_4 + y a_5 \\
\zeta_x &= e^x a_3 - e^{-x} a_4 + y' a_5 \\
\zeta_{xx} &= e^x a_3 + e^{-x} a_4 + y'' a_5.
\end{aligned} \tag{5.6}$$

These are the coefficients of $X^{[2]}$ which are obtained by

$$X^{[2]} = \sum_{i=1}^{5} a_i X_i^{[2]}, \tag{5.7}$$

where X_is are the symmetry generators as given in (5.2) and the a_is are constants. The reason for taking a linear combination is that the symmetries of the first integrals are always the symmetries of the equation (see [34] for a general result on this).

After substitution of the values of $X^{[2]}I_1$, $X^{[2]}I_2$, $X^{[2]}I_3$ given in (5.5), with ξ, η, ζ_x, ζ_{xx} as in (5.6) and together using the first integrals $I_1 = y'' - y$, $I_2 = e^x y'' - e^x y'$, $I_3 = e^{-x}y'' + e^{-x}y'$ in equation (5.4), we finally arrive at the classifying relation

$$\begin{aligned}(-a_2 + I_1 a_5)\frac{\partial F}{\partial I_1} + [(a_1 + a_5)I_2 + 2a_4]\frac{\partial F}{\partial I_2} \\ +[(a_5 - a_1)I_3 + 2a_3]\frac{\partial F}{\partial I_3} = 0.\end{aligned} \tag{5.8}$$

The relation (5.8) provides the relationship between the symmetries and first integrals of the third-order equation (5.1). We use this relation in order to classify the first integrals according to their symmetries.

Symmetry structure of the first integrals of $y''' - y' = 0$

We utilize the classifying relation (5.8) to investigate the number and properties of the symmetries of the first integrals of the ODE (5.1).

In the first instance we see that if F is arbitrary, then by use of (5.8) we immediately see that

$$\begin{aligned}-a_2 + I_1 a_5 &= 0, \\ (a_1 + a_5)I_2 + 2a_4 &= 0, \\ (a_5 - a_1)I_3 + 2a_3 &= 0.\end{aligned} \tag{5.9}$$

The relations (5.9) imply that all the a's are zero. Hence there is no symmetry for this case, i.e, for F an arbitrary function.

In order to effectively and systematically study the one and higher symmetry cases of first integrals we obtain optimal systems of one-dimensional subalgebra spanned

by (5.2). Then we invoke the classifying relation (5.8). So the strategy followed here is different to that employed for the simplest third-order ODE, $y''' = 0$. The reason being that we do not have in a simple manner subalgebra structure of the symmetries of (5.1), as we had for $y''' = 0$.

The Lie algebra of the operators (5.2) is five-dimensional and has commutator relations given in the table below.

Table 5.1: The commutation relations for the symmetries of equation (5.1)

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	X_3	$-X_4$	0
X_2	0	0	0	0	X_2
X_3	$-X_3$	0	0	0	X_3
X_4	X_4	0	0	0	X_4
X_5	0	$-X_2$	$-X_3$	0	0

In order to calculate the adjoint representation, we utilize the Lie series (see Olver [5]).

$$\mathrm{Ad}(\exp(\epsilon X))Y = Y - \epsilon[X,Y] + \frac{1}{2!}\epsilon^2[X,[X,Y]] - \frac{1}{3!}\epsilon^3[X,[X,[X,Y]]] + ... \qquad (5.10)$$

together with the commutator table, viz. Table 5.1. As an example,

$$\begin{aligned} \mathrm{Ad}(\exp(\epsilon X_1))X_3 &= X_3 - \epsilon[X_1, X_3] + \frac{1}{2!}\epsilon^2[X_1,[X_1,X_3]] - ... \\ &= X_3 - \epsilon X_3 + \frac{1}{2!}\epsilon^2 X_3 - \frac{1}{3!}\epsilon^3 X_3 + ... \\ &= e^{-\epsilon} X_3. \end{aligned} \qquad (5.11)$$

In like manner, we obtain the other entries of the adjoint table and we have the adjoint representation given by the table below.

Table 5.2: The adjoint table for the symmetries (5.2)

Ad	X_1	X_2	X_3	X_4	X_5
X_1	X_1	X_2	$e^{-\epsilon}X_3$	$e^{\epsilon}X_4$	X_5
X_2	X_1	X_2	X_3	X_4	$X_5 - \epsilon X_2$
X_3	$X_1 + \epsilon X_3$	X_2	X_3	X_4	$X_5 - \epsilon X_3$
X_4	$X_1 - \epsilon X_4$	X_2	X_3	X_4	$X_5 - \epsilon X_4$
X_5	X_1	$e^{\epsilon}X_2$	$e^{\epsilon}X_3$	$e^{\epsilon}X_4$	X_5

Here the (i, j) entry represents $\mathrm{Ad}(\exp(\epsilon X_i))X_j$. For a nonzero vector

$$X = a_1X_1 + a_2X_2 + ... + a_5X_5, \tag{5.12}$$

we need to simplify the coefficients a_i as far as possible through adjoint maps to X. The computations are straightforward and we find an optimal system of one-dimensional subalgebras spanned by

$$\begin{aligned} X_1, \\ X_2, \\ X_1 \pm X_2, \\ X_2 \pm X_4, \\ aX_1 + X_5, \\ X_1 + X_5 \pm X_3, \\ -X_1 + X_5 \pm X_4. \end{aligned} \tag{5.13}$$

The discrete symmetry transformation $y \mapsto -y$ will map $X_1 - X_2$ to $X_1 + X_2$ and that of $x \mapsto -x$ will transform the last entry in (5.13) to $X_1 + X_5 \pm X_3$. Also $X_1 + X_5 - X_3$ will go to $X_1 + X_5 + X_3$ under $y \mapsto -y$. Therefore the above list (5.13) is reduced by four.

We now invoke each of the operators of (5.13) in the classifying relation (5.8) to systematically work out the symmetry structure of the first integrals of (5.1).

Firstly we consider X_1. Since a_1 is arbitrary, we have

$$I_2 \frac{\partial F}{\partial I_2} - I_3 \frac{\partial F}{\partial I_3} = 0 \tag{5.14}$$

and hence

$$F = F(I_1, I_2 I_3) \tag{5.15}$$

which possesses X_1 as symmetry. After the substitution of (5.14) into (5.8) and taking into account (5.15) we arrive at

$$(-a_2 + a_5 I_1)\frac{\partial F}{\partial I_1} + 2(a_5 \alpha + a_3 I_2 + a_4 I_3)\frac{\partial F}{\partial \alpha} = 0, \tag{5.16}$$

where $\alpha = I_2 I_3$. This at once gives $a_3 = a_4 = 0$.

Note that for a_3, a_4, nonzero we have $\partial F/\partial \alpha = 0$ in which case we further have that $a_2 = a_5 = 0$. This results in $F = F(I_1)$ which has symmetry generators X_1, X_3, X_4 which is the maximal case.

We systematically consider the cases when (5.16) imply two generators. These arise as follows.

(i) Suppose that a_1, a_2 are arbitrary. Then (5.16) gives $\partial F/\partial I_1 = 0$ and

$$2\alpha a_5 \frac{\partial F}{\partial \alpha} = 0.$$

For F not a constant we must have that $a_5 = 0$ and we get

$$F = F(I_2 I_3) \tag{5.17}$$

which has X_1 and X_2 as symmetries.

(ii) Suppose that a_1, a_5 are arbitrary. Then (5.16) implies that

$$I_1 \frac{\partial F}{\partial I_1} + 2\alpha \frac{\partial F}{\partial I_1} = 0 \tag{5.18}$$

from which we arrive at

$$F = F(I_1 (I_2 I_3)^{-1/2}). \tag{5.19}$$

This integral (5.19) has X_1 and X_5 as symmetry generators.

We do not obtain any further three symmetry cases from (5.16) apart from the earlier for I_1 as it gives F a constant and hence no integral.

Next we focus on X_2. The use of the classifying relation (5.8) gives rise to

$$[(a_1 + a_5)I_2 + 2a_4]\frac{\partial F}{\partial I_2} + [(a_5 - a_1)I_3 + 2a_3]\frac{\partial F}{\partial I_3} = 0, \tag{5.20}$$

and therefore

$$F = F(I_2, I_3) \tag{5.21}$$

admits X_2. In a similar manner as for X_1 we have the following cases.

(i) If a_1, a_2 are arbitrary, then we obtain F as in (5.15).

(ii) If a_2, a_3 are arbitrary, then we have X_2, X_3, $X_1 - X_5$ and $F = F(I_2)$.

(iii) If a_2, a_4 are arbitrary, then we have X_2, X_4, $X_1 + X_5$ and $F = F(I_3)$.

(iv) If a_2, a_5 are arbitrary, then X_2, X_5 result in $F = F(I_3/I_2)$.

We do not get any three symmetry case here.

The pattern is now clear. Instead of going through each of the remaining cases which are quite tedious albeit straightforward, we present our findings in a table. For completeness this table also includes the cases X_1 and X_2 together with the corresponding first integrals.

Table 5.3: One symmetry cases and the integrals of (5.1)

One symmetry	*First integral*
X_1	$F = F(I_1, I_2 I_3)$
X_2	$F = F(I_2, I_3)$
$X_1 + X_2$	$F = F(I_2 I_3, I_2 \exp(I_1))$
$X_2 \pm X_4$	$F = F(I_2 \pm I_1, I_3)$
X_5	$F = F(I_3/(I_2 \pm I_1))$
$X_5 + aX_1, a \neq 0$	$F = F(I_2 I_1^{-1-a}, I_3 I_1^{a-1})$
$X_1 + X_5 + X_3$	$F = F(I_2 I_1^{-2}, I_3 - \ln I_2)$

Table 5.4: Two symmetry cases and the integrals of (5.1)

Two symmetries	*First integral*
X_1, X_2	$F = F(I_2 I_3)$
X_1, X_5	$F = F(I_1 (I_2 I_3)^{-1/2})$
X_2, X_5	$F = F(I_3/I_2)$
$X_2 \pm X_4, X_5$	$F = F(I_3/(I_2 \pm I_1))$
X_3, X_5	$F = F(I_2/I_1)$
$X_5 + aX_1, X_2, a \neq 0$	$F = F(I_3 I_2^{a-1/a+1})$
$X_5 + aX_1, X_3$	$F = F(I_2 I_1^{-1-a})$
$X_2, X_1 + X_5 + X_3$	$F = F(I_3 - \ln I_2)$
$X_4, X_1 + X_5 + X_3$	$F = F(I_3 - \ln I_1^2)$

Finally we look at the three symmetry cases.

For I_1 there are three symmetries

$$\begin{aligned} X_1 &= \exp x \frac{\partial}{\partial y}, \\ X_2 &= \exp(-x) \frac{\partial}{\partial y}, \\ X_3 &= \frac{\partial}{\partial x} \end{aligned} \tag{5.22}$$

which has nonzero commutation relations

$$[X_1, X_3] = -X_1, \quad [X_2, X_3] = X_2. \tag{5.23}$$

The Lie algebra is $L_{3;4}$. In the case of the first integral I_2, the symmetries are

$$X_1 = \exp x \frac{\partial}{\partial y},$$

$$\begin{aligned} X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= \frac{1}{2}y\frac{\partial}{\partial y} - \frac{1}{2}\frac{\partial}{\partial x} \end{aligned} \tag{5.24}$$

which have nonzero Lie brackets

$$[X_1, X_3] = X_1, \quad [X_2, X_3] = \frac{1}{2}X_2 \tag{5.25}$$

and constitute the Lie algebra $L_{3;5}$, $a = 1/2$. The Lie algebra of the symmetries of I_3 is isomorphic to that of I_2 by means of the discrete transformation $\bar{x} = -x$.

Thus there are two Lie algebras of dimension three, viz. $L_{3;4}$ and $L_{3;5}$, $a = 1/2$. There are no four symmetry cases. Therefore we have the following result.

Theorem 5.1. The maximal dimension of the Lie algebra admitted by a first integral of $y''' - y' = 0$ or a third-order ODE linearizable by point transformation to this linear ODE is three. The maximal Lie algebras are $L_{3;4}$ and $L_{3;5}$, $a = 1/2$.

The proof follows easily from the preceding discussion.

We also have the following counting theorem.

Theorem 5.2. The Lie algebra admitted by a first integral of $y''' - y' = 0$ or a third-order ODE linearizable by point transformation to this linear ODE is 0, 1, 2 or 3.

The proof follows from equation (5.9), the above Tables 5.3, 5.4 and Theorem 5.1.

5.3 Algebraic Properties of the Integrals of $y''' + fy'' - y' - fy = 0$

We consider the representative third-order ODE

$$y''' + f(x)y'' - y' - f(x)y = 0, \tag{5.26}$$

where f is an arbitrary function of x. This equation possesses four symmetries

$$\begin{aligned} X_1 &= e^x \frac{\partial}{\partial y} \\ X_2 &= e^{-x} \frac{\partial}{\partial y} \\ X_3 &= \alpha(x) \frac{\partial}{\partial y} \\ X_4 &= y \frac{\partial}{\partial y}, \end{aligned} \tag{5.27}$$

where again we commenced with the three solution symmetries and then the homogeneity symmetry. Here $\alpha = \frac{1}{2} e^x \int e^{(-x - \int f(x)dx)} dx - \frac{1}{2} e^{-x} \int e^{(x - \int f(x)dx)} dx$ is a solution of the equation (5.26). The third-order equation (5.26) has the three functionally independent first integrals

$$\begin{aligned} I_1 &= (y'' - y) e^{\int f(x)dx} \\ I_2 &= y e^{-x} + y' e^{-x} - [\int e^{(-x - \int f(x)dx)} dx](y'' - y) e^{\int f(x)dx} \\ I_3 &= y e^{x} - y' e^{x} + [\int e^{(x - \int f(x)dx)} dx](y'' - y) e^{\int f(x)dx}. \end{aligned} \tag{5.28}$$

The first in this list is the simplest followed by the other two for which the order does not matter.

Classifying relation for the symmetries of $y''' + fy'' - y' - fy = 0$.

Let F be an arbitrary function of I_1, I_2 and I_3, viz. $F = F(I_1, I_2, I_3)$. The symmetry of this general function of the first integrals is

$$X^{[2]}F = X^{[2]}I_1\frac{\partial F}{\partial I_1} + X^{[2]}I_2\frac{\partial F}{\partial I_2} + X^{[2]}I_3\frac{\partial F}{\partial I_3} = 0, \tag{5.29}$$

where

$$\begin{aligned}
X^{[2]}I_1 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'} + \zeta_{xx}\frac{\partial}{\partial y''}](y'' - y)e^{\int f(x)dx} \\
&= \xi[(y'' - y)e^{\int f(x)dx}f(x)] - \eta e^{\int f(x)dx} + \zeta_{xx}e^{\int f(x)dx} \\
X^{[2]}I_2 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'} + \zeta_{xx}\frac{\partial}{\partial y''}] \\
&ye^{-x} + y'e^{-x} - [\int e^{(-x-\int f(x)dx)}dx](y'' - y)e^{\int f(x)dx} \\
&= [-(y + y')e^{-x} - (\int e^{(-x-\int f(x)dx)}dx)(y'' - y)e^{\int f(x)dx}f(x) \\
&-(y'' - y)e^{\int f(x)dx}e^{(-x-\int f(x)dx)}]\xi + [e^{-x} + (e^{\int f(x)dx})(\int e^{(-x-\int f(x)dx)}dx)]\eta + e^{-x}\zeta_x \\
&-[(\int e^{(-x-\int f(x)dx)}dx)e^{\int f(x)dx}]\zeta_{xx} \\
X^{[2]}I_3 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'} + \zeta_{xx}\frac{\partial}{\partial y''}] \\
&ye^{x} - y'e^{x} + [\int e^{(x-\int f(x)dx)}dx](y'' - y)e^{\int f(x)dx} \\
&= [(y - y')e^{x} + (\int e^{(x-\int f(x)dx)}dx)(y'' - y)e^{\int f(x)dx}f(x) \\
&+(y'' - y)e^{\int f(x)dx}e^{(x-\int f(x)dx)}]\xi + [e^{x} - (e^{\int f(x)dx})(\int e^{(x-\int f(x)dx)}dx)]\eta - e^{x}\zeta_x \\
&+[(\int e^{(x-\int f(x)dx)}dx)e^{\int f(x)dx}]\zeta_{xx}
\end{aligned} \tag{5.30}$$

Now ξ, η, ζ_x and ζ_{xx} are

$$\begin{aligned}
\xi &= 0 \\
\eta &= e^{x}a_1 + e^{-x}a_2 + \alpha(x)a_3 + ya_4 \\
\zeta_x &= e^{x}a_1 - e^{-x}a_2 + \alpha'(x)a_3 + y'a_4 \\
\zeta_{xx} &= e^{x}a_1 + e^{-x}a_2 + \alpha''(x)a_3 + y''a_4.
\end{aligned} \tag{5.31}$$

These are the coefficients functions of $X^{[2]}$ are obtained by setting

$$X^{[2]} = \sum_{i=1}^{4} a_i X_i^{[2]} \tag{5.32}$$

where X_is are the symmetry generators as given in (5.27) and the a_is are constants. The reason for taking a linear combination mentioned earlier is that the symmetries of the first integrals are always the symmetries of the equation (see [34] for a general result).

After insertion of the values of $X^{[2]}I_1$, $X^{[2]}I_2$, $X^{[2]}I_3$ as in (5.30), with ξ, η, ζ_x, ζ_{xx} as in (5.31), first integrals $I_1 = (y''-y)e^{\int f(x)dx}$, $I_2 = ye^{-x} + y'e^{-x} - [\int e^{(-x-\int f(x)dx)}dx](y''-y)e^{\int f(x)dx}$, $I_3 = ye^{x} - y'e^{x} + [\int e^{(x-\int f(x)dx)}dx](y''-y)e^{\int f(x)dx}$ as well as use of

$$\begin{aligned}
\alpha'' - \alpha &= e^{-\int f(x)dx} \\
\alpha' + \alpha &= e^{x}\int e^{(-x-\int f(x)dx)}dx \\
\alpha - \alpha' &= -e^{-x}\int e^{(x-\int f(x)dx)}dx \\
y'' - y &= I_1 e^{-\int f(x)dx} \\
(y'+y)e^{-x} &= I_2 + I_1\int e^{(-x-\int f(x)dx)}dx \\
(y-y')e^{x} &= I_3 - I_1\int e^{(x-\int f(x)dx)}dx
\end{aligned} \tag{5.33}$$

in equation (5.29), we eventually find the classifying relation

$$\begin{aligned}
(a_3 + I_1 a_4)\frac{\partial F}{\partial I_1} + (2a_1 + I_2 a_4)\frac{\partial F}{\partial I_2} \\
+(2a_2 + I_3 a_4)\frac{\partial F}{\partial I_3} = 0.
\end{aligned} \tag{5.34}$$

The relation (5.34) provides the relationship between the symmetries and first integrals of the third-order equation (5.26). We utilize this to classify the first integrals in terms of their symmetries.

Symmetry structure of the first integrals of $y''' + fy'' - y' - fy = 0$

We use the relation (5.34) to systematically study the relationship between the symmetries and first integrals of (5.26).

We quickly note that if F is arbitrary, then (5.34) implies

$$\begin{aligned} a_3 + I_1 a_4 &= 0, \\ 2a_1 + I_2 a_4 &= 0, \\ 2a_4 + I_3 a_4 &= 0, \end{aligned} \tag{5.35}$$

which in turn give that the a's are zero. Thus there results no symmetry for this case.

As in the previous section on the constant coefficient ODE, we obtain the optimal system of one-dimensional subalgebras of the four-dimensional algebra symmetry algebra of our ODE spanned by (5.27).

The Lie algebra of the symmetries (5.27) is represented by the following table.

Table 5.5: The commutation relations for the symmetries of equation (5.26)

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	X_1
X_2	0	0	0	X_2
X_3	0	0	0	X_3
X_4	$-X_1$	$-X_2$	$-X_3$	0

By use of this table we can construct the adjoint representation which we present in

the following table.

Table 5.6: The adjoint table for the symmetries (5.27)

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	$X_4 - \epsilon X_1$
X_2	X_1	X_2	X_3	$X_4 - \epsilon X_2$
X_3	X_1	X_2	X_3	$X_4 - \epsilon X_3$
X_4	$e^{\epsilon} X_1$	$e^{\epsilon} X_2$	$e^{\epsilon} X_3$	X_4

We then obtain an optimal system of one-dimensional subalgebras spanned by

$$\begin{aligned} X_3, \\ X_4, \\ X_2 + aX_3, \\ X_1 + aX_2 + bX_3. \end{aligned} \tag{5.36}$$

For each of these operators, we are systematically able to compute the corresponding first integrals by using the classifying relation (5.34).

Below we tabulate the symmetries and the corresponding first integrals.

Table 5.7: One symmetry cases and the integrals of (5.26)

One symmetry	*First integral*
X_3	$F = F(I_2, I_3)$
X_4	$F = F(I_2/I_1, I_3/I_2)$
$X_2 + aX_3$	$F = F(I_2 I_3, I_2 \exp(I_1))$
$X_2 \pm X_4$	$F = F(I_1 - \frac{1}{2}aI_3, I_2)$
$X_1 + aX_2$	$F = F(I_3 - aI_2, I_1)$
$X_1 + aX_2$	$F = F(I_1 - \frac{1}{2}aI_2, I_3)$
$X_1 + aX_2 + bX_3, a, b \neq 0$	$F = F(bI_3 - 2aI_1, I_2)$

Table 5.8: Two symmetry cases and the integrals of (5.26)

Two symmetries	*First integral*
X_2, X_3	$F = F(I_2)$
X_3, X_4	$F = F(I_3/I_2)$
$X_1, X_2 + aX_3$	$F = F(I_1 - \frac{1}{2}aI_3)$
$X_2 + aX_3, X_4$	$F = F(I_2/(I_1 - \frac{1}{2}aI_3))$
$X_1 + aX_2, X_3$	$F = F(I_3 - aI_2)$
$X_1 + aX_2, X_4$	$F = F(I_3 - aI_2/I_2)$
$X_1 + aX_3, X_2, a \neq 0$	$F = F(I_1 - \frac{1}{2}aI_2)$
$X_1 + aX_3, X_4, a \neq 0$	$F = F(I_1 - \frac{1}{2}aI_2/I_3)$
$X_2 + aX_3, X_1, a \neq 0$	$F = F(I_1 - \frac{1}{2}aI_3)$
$X_1 + aX_2 + bX_3, X_4, a, b \neq 0$	$F = F(bI_3 - 2aI_1/I_2)$

It follows that there are no three symmetry cases. Moreover, we note that the maximal case of symmetries of the first integrals for (5.26) is two and these are listed in Table 5.8.

We therefore have the following result.

Theorem 5.3. The Lie algebra admitted by a first integral of $y'''+f(x)y''-y'-f(x)y = 0$ or a third-order ODE linearizable by point transformation to this linear ODE is 0, 1 or 2.

The proof follows from equation (5.35) and the above Tables 5.7 and 5.8.

5.4 Further Considerations: Symmetries of First Integrals of Submaximal Higher-Order ODEs

We know that one cannot generate the full Lie algebra of any scalar first-order ODE via the algebras of any of its integrals (see previous chapter 4 and [31]). Also for scalar linear second-order ODEs it has been shown in [20] that the full Lie algebra of $y'' = 0$ which represents any linear or linearizable second-order ODE can be generated by three isomorphic triplets of three-dimensional algebras of the basic integrals and one of their quotient which have the interesting property that the algebras are isomorphic to each other. In chapter 4 and indeed our recent work [31] we have pointed out that the full Lie algebra of the simplest third-order equation $y''' = 0$ is generated by the point symmetries of only two of the basic integrals I_1 and I_3 from the three

$$\begin{aligned} I_1 &= y'' \\ I_2 &= xy'' - y' \\ I_3 &= \frac{1}{2}x^2y'' - xy' + y. \end{aligned} \tag{5.37}$$

This is indeed very different to what happens for the classes $y' = 0$ and $y'' = 0$. One has that the seven symmetries of our the simplest third-order ODE is generated by four symmetries of I_1 together with three symmetries of I_3. In the case of higher-order ODEs of maximal symmetry it was shown in chapter 4 (see also [31]) that similar properties persist. That is the full Lie algebra of $y^{(n)} = 0, \quad n \geq 3$ is generated by two subalgebras, viz. the $n+1$-dimensional algebra of the integral $I_1 = y^{(n-1)}$ and the three-dimensional subalgebra of the integral $I_n = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{(n-i)!} x^{n-i} y^{(n-i)}$.

What occurs for higher-order ODEs of submaximal symmetry? We discuss this below.

Consider the nth-order ODE of sub-maximal symmetry

$$y^{(n)} - y^{(n-2)} = 0, \quad n \geq 3. \tag{5.38}$$

This ODE (5.38) can be taken as a representative of higher-order ODEs which has $n+2$ point symmetries. We have chosen this in a way that reduces to the third-order case focused on earlier. The n first integrals of (5.38) have the same pattern as for the third-order case and are thus easily constructible and we focus on the first and second which are

$$I_1 = y^{(n-1)} - y^{(n-3)} \tag{5.39}$$

and

$$I_2 = e^x (y^{(n-1)} - y^{(n-2)}). \tag{5.40}$$

The first integral (5.39) has n point symmetries

$$\begin{aligned} X_1 &= e^x \frac{\partial}{\partial x}, \; X_2 = e^{-x} \frac{\partial}{\partial y}, \; X_3 = \frac{\partial}{\partial x}, \\ X_i &= x^{i-4} \frac{\partial}{\partial y}, \quad i = 4, \ldots, n. \end{aligned} \tag{5.41}$$

This forms an n-dimensional subalgebra of the symmetry algebra of equation (5.38). The nonzero commutation relations are

$$[X_1, X_3] = -X_1,\ [X_2, X_3] = X_2,\ [X_3, X_i] = (i-4)X_{i-1},\ i = 4, \ldots, n. \tag{5.42}$$

The first integral (5.40) has point symmetries

$$\begin{aligned} Y_1 &= e^x \frac{\partial}{\partial y}, \\ Y_i &= x^{i-2} \frac{\partial}{\partial y}, \quad i = 2, \ldots, n-1, \\ Y_n &= y\frac{\partial}{\partial y} - \frac{\partial}{\partial x}. \end{aligned} \tag{5.43}$$

These generators have nonzero commutation relations

$$[Y_1, Y_n] = 2Y_1,\ [Y_i, Y_n] = Y_i + (i-2)Y_{i-1},\ i = 2, \ldots, n-1. \tag{5.44}$$

We see that these two sets of symmetries (5.41) and (5.43) are easy to deduce as it is clear that (5.41) form symmetries of (5.39) since they are translation in x and solution symmetries with maximum degree power $x^{(n-4)}$. Also for $n = 3$ they reduce to the third-order case of the previous section. The full Lie algebra of equation (5.38) is generated from the n symmetries of (5.41) and two symmetries of (5.43), viz. Y_{n-1} and Y_n of (5.43). However, the latter does not close due to the commutation relations (5.44). However, if we exclude Y_1, then $< Y_2, \ldots, Y_n >$ does span an $(n-1)$-dimensional algebra. Alternatively, a simpler way to generate the full algebra of equation (5.38) is to utilize the symmetries (5.43) together with the two symmetries X_2 and X_3 of (5.41).

We therefore have the theorem the proof of which follows from the above discussion.

Theorem 5.4. The full Lie algebra of the linear nth-order ODE $y^{(n)} - y^{(n-1)} = 0, \quad n \geq 3$ which is $n+2$ dimensional, is generated by two subalgebras, viz. the

n-dimensional algebra $< Y_j : j = 1, \ldots, n >$ of $I_2 = e^x(y^{(n-1)} - y^{(n-2)})$ and the two-dimensional subalgebra $< X_2, X_3 >$ of $I_1 = y^{(n-1)} - y^{(n-3)}$.

We now study the generation of the full algebra of a representative nth-order, $n \geq 3$ of sub-maximal symmetries $n+1$. A natural extension of the third-order ODE (5.26) is

$$y^{(n)} - y^{(n-2)} + f(x)(y^{(n-1)} - y^{(n-3)}) = 0, \ n \geq 3, \tag{5.45}$$

where $f(x)$ is an arbitrary function of x. Following the pattern of the integrals in (5.28), we can write the corresponding three out of n immediately. They are

$$\begin{aligned} I_1 &= (y^{(n-1)} - y^{(n-3)})e^{\int f(x)dx}, \\ I_2 &= e^{-x}(y^{(n-2)} + y^{(n-3)}) - [\int e^{(-x-\int f(x)dx)}dx](y^{(n-1)} - y^{(n-3)})e^{\int f(x)dx}, \\ I_3 &= e^{x}(y^{(n-3)} - y^{(n-2)}) + [\int e^{(x-\int f(x)dx)}dx](y^{(n-1)} - y^{(n-3)})e^{\int f(x)dx}. \end{aligned} \tag{5.46}$$

We show that the symmetries of these integrals are sufficient to generate the full algebra. From the previous section Table 5.8 we notice that X_1 and X_2 of (5.27) are symmetries of the integral I_1 in (5.28). Further that X_3 and X_4 of (5.27) are symmetries of the quotient integral I_3/I_2. In a similar fashion we have these algebraic properties persisting for the linear higher-order equation (5.45). The equation (5.45) has the $n+1$ point symmetries

$$\begin{aligned} X_1 &= e^x\frac{\partial}{\partial y}, \ X_2 = e^{-x}\frac{\partial}{\partial y}, \\ X_i &= x^{i-3}\frac{\partial}{\partial y}, \ i = 3, \ldots, n-1, \\ X_n &= \alpha(x)\frac{\partial}{\partial y}, \\ X_{n+1} &= y\frac{\partial}{\partial y}, \end{aligned} \tag{5.47}$$

where α is a solution to (5.45) and satisfies similar properties to that of the corresponding linear third-order equation, viz.

$$\begin{aligned}
\alpha^{(n-1)} - \alpha^{(n-3)} &= e^{-\int f(x)dx}, \\
\alpha^{(n-1)} - \alpha^{(n-2)} &= -e^{x} \int e^{x-\int f(x)dx} dx + e^{-\int f(x)dx}, \\
\alpha^{(n-2)} + \alpha^{(n-3)} &= e^{x} \int e^{-x-\int f(x)dx} dx,
\end{aligned} \tag{5.48}$$

It is evident that the first n are solution symmetries and the $(n+1)$th is the homogeneity symmetry which are straightforward to observe. The first integral I_1 in (5.46) has the $n-1$ symmetries $X_1, \ldots, X_{n-1}$ which is clear. The algebra constituted is Abelian. This fact can also be seen for I_1 of (5.28). Now we analyze what occurs for the quotient integral I_3/I_2 of (5.46). It is noticed that the homogeneity symmetry X_{n+1} is a symmetry of I_3/I_2 as if we replace y by γy in the quotient, it is left invariant. Moreover, for X_n we have the invariance condition

$$\begin{aligned}
& X_n^{(n-1)}(\frac{I_3}{I_2}) \\
= \; & \frac{1}{I_2}\{[e^{x}(\alpha^{(n-3)} - \alpha^{(n-2)}) + \int e^{x-\int f(x)dx} e^{\int f(x)dx} dx(\alpha^{(n-1)} - \alpha^{(n-3)})] \\
& -\frac{I_3}{I_2}[e^{-x}(\alpha^{(n-2)} + \alpha^{(n-3)}) - \int e^{-x-\int f(x)dx} e^{\int f(x)dx} dx(\alpha^{(n-1)} - \alpha^{(n-3)})]\} \\
= \; & 0.
\end{aligned} \tag{5.49}$$

The terms in the square brackets vanish due to the relations (5.48). Thus X_n is a symmetry of this quotient integral. In view of the above, we have the following theorem.

Theorem 5.5. The full Lie algebra of the linear nth-order ODE $y^{(n)} - y^{(n-2)} + f(x)(y^{(n-1)} - y^{(n-3)}) = 0, \quad n \geq 3$ which is $n+1$ dimensional, is generated by two

subalgebras, viz. the $(n-1)$-dimensional algebra $< X_j : j = 1, \ldots, n-1 >$ of I_1 as given in (5.46) and the two-dimensional subalgebra $< X_n, X_{n+1} >$ of I_3/I_2 as in (5.46).

Hence the manner in which the full Lie algebra is generated for the ODEs $y'' = 0$ [20], $y^{(n)} = 0,\ n \geq 3$ (chapter 4 and [31]) and the two submaximal linear cases investigated in the foregoing are quite interesting. This also conforms with the properties of their symmetry algebra which are different (see, e.g. [25, 18]).

Chapter 6

Symmetry Classification of First Integrals for Scalar Dynamical Equations

6.1 Introduction

The aim of this chapter is to provide an extension of the classification of the first integrals for scalar second-order ODEs linearizable by point transformations (see chapter 3 and [26]) and focus our attention on scalar nonlinear second-order ODEs which admit 1, 2 or 3 symmetries. We completely classify first integrals of scalar nonlinear second-order which have submaximal Lie algebras of dimensions 1, 2 and 3. This constitutes completely new work as symmetries of first integrals of scalar nonlinear second-order

ODEs which admit 1, 2 or 3 symmetries have not been studied before. The essence of this chapter appears in our published work [35].

Recall from chapter 2, that the first integral

$$I = I(x, y, y'), \tag{6.1}$$

of the scalar second-order ODE $E(x, y, y', y'') = 0$ is said to be invariant under the infinitesimal generator $X = \xi(x, y)\partial/px + \eta(x, y)\partial/\partial y$ if and only if

$$X^{[1]}I = 0, \tag{6.2}$$

where X is the first prolonged generator

$$X^{[1]} = X + \zeta_1 \frac{\partial}{\partial y'}, \tag{6.3}$$

with the usual first prolongation formula

$$\zeta_1 = D_x(\eta) - y' D_x(\xi), \tag{6.4}$$

in which D_x is the total differentiation operator. Now the reduced first-order ODE $I = C$ also has the symmetry X as is known (see [6]).

This chapter is organized as follows. In section 6.2, we determine the classifying relations between point symmetries and first integrals of scalar nonlinear second-order equations which admit one and two point symmetries. Then we provide the symmetry structure of the first integrals of such nonlinear equations. In section 6.3, we obtain the classifying relations between symmetries and first integrals of scalar nonlinear second-order equations which have three symmetries. There are four equations in Lie's classification that have three symmetries. We investigate each in turn for the symmetry properties of their first integrals. Applications to generalized Emden-Fowler, Lane-Emden and modified Emden equations are given.

6.2 Nonlinear Equations with One and Two Symmetries

In Lie's classification of scalar second-order ODEs in the complex domain [10] there exists one general class of equations with a single point symmetry.

Consider the scalar nonlinear second-order ODE in Lie's classification

$$y'' = F(x, y'), \tag{6.5}$$

where F is an arbitrary function. It is indeed easy to see that equation (6.5) has in general one point symmetry

$$X = \frac{\partial}{\partial y} \tag{6.6}$$

and the corresponding first integral for (6.5) is

$$I = I(x, y'). \tag{6.7}$$

We observe that the first integral (6.7) has the same symmetry as given in (6.6). Hence the symmetry of the first integral is the same as that of the equation itself.

We next investigate the symmetries of the first integrals of scalar second-order ODEs in Lie's classification which possess two point symmetries. There are two equations, Type I and Type II. We study each in turn for the symmetry structure of their first integrals.

The scalar nonlinear equation of Type I

$$y'' = g(y'), \tag{6.8}$$

where g is an arbitrary function of its argument, has two point symmetries

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y} \tag{6.9}$$

and the two functionally independent first integrals

$$I_1 = K(y') - x,\ I_2 = y - y'K(y') + \int K(y')dy', \tag{6.10}$$

where

$$K(y') = \int \frac{1}{g(y')}dy'.$$

Classifying relation for the symmetries of the first integrals of $y'' = g(y')$

We let F be an arbitrary function of I_1 and I_2, viz. $F = F(I_1, I_2)$. Then the symmetry of this general function of the first integrals is

$$X^{[1]}F = X^{[1]}I_1\frac{\partial F}{\partial I_1} + X^{[1]}I_2\frac{\partial F}{\partial I_2} = 0, \tag{6.11}$$

where

$$\begin{aligned} X^{[1]}I_1 = [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'}][K(y') - x] &= -\xi \\ X^{[1]}I_2 = [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'}][y - y'K + \int Kdy'] &= \eta. \end{aligned} \tag{6.12}$$

Here ξ, η and ζ_x are given by

$$\xi = a_1,\ \eta = a_2,\ \zeta_x = 0. \tag{6.13}$$

These are the coefficients of $X^{[1]}$ which are arrived at by setting

$$X^{[1]} = \sum_{i=1}^{2} a_i X_i^{[1]}, \tag{6.14}$$

where the X_is are the symmetries as given in (6.9) and the a_is are constants. The symmetries of a first integral are always the symmetries of the equation itself. This is a general result proved in [6] (see also [7]). Therefore we have taken here and in what follows $X^{[1]}$ to be a linear combination of the symmetries of the equation under consideration.

After substitution of the values of $X^{[1]}I_1$, $X^{[1]}I_2$ as in (6.12), with ξ, η, ζ_x as in (6.13) and together by using the first integrals $I_1 = K(y') - x$, $I_2 = y - y'K + \int K dy'$ in equation (6.11), we deduce after some calculations

$$-a_1 \frac{\partial F}{\partial I_1} + a_2 \frac{\partial F}{\partial I_2} = 0, \tag{6.15}$$

The relation (6.15) provides the relationship between the symmetries and the first integrals of the nonlinear equation (6.8). We use this to classify the first integrals of (6.8) according to their symmetries.

Symmetry structure of the first integrals of $y'' = g(y')$

We utilize the classifying relation (6.15) to investigate the number and properties of the symmetries of the first integrals of the ODE (6.8).

For a_1 and a_2 arbitrary, it is seen that the relation (6.15) implies that F be a constant. This clearly means that there is no first integral of (6.8) which has two symmetries (6.9). If one of a_1 or a_2 is arbitrary in (6.15), we end up with either the first integral I_2 or I_1 or an arbitrary function of either. In general we have that $F = F(a_2 I_1 + a_1 I_2)$ has the symmetry $X = a_1 X_1 + a_2 X_2$.

Now we focus our attention on the nonlinear equation of Type II

$$xy'' = h(y'). \tag{6.16}$$

This is the second ODE in Lie's classification with two symmetries. We find that (6.16) has two Lie symmetries

$$X_1 = \frac{\partial}{\partial y},\ X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{6.17}$$

The two first integrals of (6.16) are

$$I_1 = x^{-1}K(y'),\ I_2 = \int K(y')dy' + (y - xy')x^{-1}K(y'), \tag{6.18}$$

where

$$K(y') = \exp(\int \frac{dy'}{h}). \tag{6.19}$$

Classifying relation for the symmetries of the first integrals of $xy'' = h(y')$

We again let F to be an arbitrary function of I_1 and I_2, viz. $F = F(I_1, I_2)$. The symmetry of this function of the first integrals is (6.11), where

$$\begin{aligned} X^{[1]}I_1 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'}][x^{-1}K(y')] \\ &= [\frac{-1}{x^2}K(y')]\xi + [\frac{1}{x}K'(y')]\zeta_x \\ X^{[1]}I_2 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'}][\int K dy' + (y - xy')x^{-1}K(y')] \\ &= [\frac{-y}{x}K(y')]\xi + [\frac{1}{x}K(y')]\eta \\ &+ [\frac{\partial}{\partial y'}(\int K dy') + \frac{y}{x}K'(y') - y'K'(y') - K(y')]\zeta_x. \end{aligned} \tag{6.20}$$

Now ξ, η and ζ_x are

$$\xi = xa_2,\ \eta = a_1 + ya_2,\ \zeta_x = 0. \tag{6.21}$$

These are the coefficients of $X^{[1]}$ which are obtained by (6.14), where X_i's are the symmetries as given in (6.17) and the a_i's are constants.

After insertion of the values of $X^{[1]}I_1$, $X^{[1]}I_2$ as in (6.20), with ξ, η, ζ_x as in (6.21) and by invoking the first integrals $I_1 = x^{-1}K(y')$, $I_2 = \int K dy' + (y - xy')x^{-1}K(y')$ in equation (6.11), we have

$$-a_2\frac{\partial F}{\partial I_1} + a_1\frac{\partial F}{\partial I_2} = 0. \tag{6.22}$$

The relation (6.22) provides the relationship between the symmetries and first integrals of the nonlinear equation (6.16). We use this to classify the first integrals of (6.16) in terms of their symmetries.

Symmetry structure of the first integrals of $xy'' = h(y')$

We utilize the classifying relation (6.22) to investigate the number and properties of the symmetries of the first integrals of the ODE (6.16). The reasoning is the same as for the Type I Lie equation. It is evident that for arbitrary a_1 and a_2, (6.22) has no nontrivial integral which has symmetry. For one of X_1 or X_2 we have that $F(I_2)$ or $F(I_1)$ are the corresponding first integrals. Of course the linear combination $a_1X_1 + a_2X_2$ is possessed by the integral $F(a_1I_1 + a_2I_2)$.

In conclusion, we have proved the following theorem.

Theorem 6.1. The full Lie algebra of a scalar nonlinear second-order ODE which is one- or two-dimensional is generated by the one-dimensional subalgebras of the independent first integrals of the equation.

6.3 Nonlinear Equations with Three Symmetries

In Lie's classification, there are four scalar second-order ODEs which admit three point symmetries. In this section we study the first integrals of the four Lie types of nonlinear second-order equations which admit three symmetries.

Consider the first equation in Lie's classification, viz.

$$y'' = Ae^{-y'}, A \neq 0, \tag{6.23}$$

which possesses the three symmetries

$$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial y}, X_3 = x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}, \tag{6.24}$$

and has the two first integrals ($A \neq 0$)

$$I_1 = x - A^{-1}e^{y'}, I_2 = A^{-1}y'e^{y'} - A^{-1}e^{y'} - y. \tag{6.25}$$

Classifying relation for the symmetries of the first integrals of $y'' = Ae^{-y'}$

We let $F = F(I_1, I_2)$. Then the symmetry of this general function is given by (6.11), where

$$\begin{aligned} X^{[1]}I_1 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'}][x - A^{-1}e^{y'}] \\ &= \xi - [A^{-1}e^{y'}]\zeta_x \\ X^{[1]}I_2 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'}][A^{-1}y'e^{y'} - A^{-1}e^{y'} - y] \\ &= -\eta + [A^{-1}y'e^{y'}]\zeta_x. \end{aligned} \tag{6.26}$$

The coefficient functions ξ, η and ζ_x are

$$\xi = a_1 + xa_3,$$

$$\begin{aligned} \eta &= a_2 + (x+y)a_3, \\ \zeta_x &= a_3. \end{aligned} \tag{6.27}$$

These are the coefficients of $X^{[1]}$ which are found from

$$X^{[1]} = \sum_{i=1}^{3} a_i X_i^{[1]}, \tag{6.28}$$

where X_is are the symmetries as given in (6.24) and the a_is are constants. Again the subalgebra property of the first integrals allows us the use of the relation (6.28).

After substitution of the values of $X^{[1]}I_1$, $X^{[1]}I_2$ in (6.26), with ξ, η, ζ_x of (6.27) and together use of the first integrals $I_1 = x - A^{-1}e^{y'}$, $I_2 = A^{-1}y'e^{y'} - A^{-1}e^{y'} - y$ in equation (6.11), we get after some calculations

$$[a_1 + I_1 a_3]\frac{\partial F}{\partial I_1} + [-a_2 + (I_2 - I_1)a_3]\frac{\partial F}{\partial I_2} = 0. \tag{6.29}$$

The relation (6.29) provides the relationship between the symmetries and first integrals of the nonlinear equation (6.23). We use this to classify the first integrals according to their symmetries.

Symmetry structure of the first integrals of $y'' = Ae^{-y'}$

In order to effectively study the one and higher symmetry cases possessed by the integrals (6.25) we obtain optimal systems of one-dimensional subalgebras spanned by (6.24) and then utilize these in the relation (6.29).

The Lie algebra of the operators (6.24) is three-dimensional and has commutator relations given in Table 6.1 below.

Table 6.1: The commutation relations for the symmetries (6.24) of equation (6.23)

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$X_1 + X_2$
X_2	0	0	X_2
X_3	$-(X_1 + X_2)$	$-X_2$	0

In order to compute the adjoint representation, we again utilize the Lie series

$$\mathrm{Ad}(\exp(\epsilon X))Y = Y - \epsilon[X, Y] + \frac{1}{2!}\epsilon^2[X, [X, Y]] - \frac{1}{3!}\epsilon^3[X, [X, [X, Y]]] + ... \quad (6.30)$$

together with the commutator table, viz. Table 6.1. As a simple example, it is straightforward to determine

$$\begin{aligned} \mathrm{Ad}(\exp(\epsilon X_1))X_3 &= X_3 - \epsilon[X_1, X_3] + \frac{1}{2!}\epsilon^2[X_1, [X_1, X_3]] - ... \\ &= X_3 - \epsilon(X_1 + X_2) + \frac{1}{2!}\epsilon^2[X_1, (X_1 + X_2)] - ... \\ &= X_3 - \epsilon(X_1 + X_2). \end{aligned} \quad (6.31)$$

In similar fashion, we find the other entries of the adjoint table and have the adjoint representation given by the Table 6.2 below.

Table 6.2: The adjoint table for the symmetries (6.24)

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \epsilon(X_1 + X_2)$
X_2	X_1	X_2	$X_3 - \epsilon X_2$
X_3	$X_1 e^{\epsilon} + \epsilon e^{\epsilon} X_2$	$X_2 e^{\epsilon}$	X_3

Here the (i, j) entry represents $\mathrm{Ad}(\exp(\epsilon X_i))X_j$. For a nonzero vector

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3, \quad (6.32)$$

we are required to simplify the coefficients a_i as far as possible through adjoint maps to X. The calculations are not difficult and we arrive at an optimal system of one-dimensional subalgebras spanned by

$$\{X_1,\, X_2,\, X_3\} \tag{6.33}$$

We invoke each of the operators in (6.33) together with the classifying relation (6.29) to construct the Table 6.3.

Table 6.3: One symmetry cases for the integrals of (6.23)

One symmetry	*First integral*
X_1	$F = F(I_2)$
X_2	$F = F(I_1)$
X_3	$F = F(I_2/I_1 + \ln I_1)$

From Table 6.3, we observe that the first integrals of (6.23) admit maximum one symmetry. We do not find cases for higher symmetry. It can happen that a first integral has no symmetry – e.g., $F = I_1 I_2$. This clearly follows by substitution of this F into (6.29) and working out the values of a_i which in this case turns out to be zero.

We turn to the second equation in Lie's classification given by

$$xy'' = Ay'^3 - \frac{1}{2}y',\ A \neq 0, \tag{6.34}$$

which has the three symmetries

$$X_1 = \frac{\partial}{\partial y},\ X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\ X_3 = 2xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} \tag{6.35}$$

and equation (6.34) has the two functionally independent first integrals

$$\begin{aligned} I_1 &= \frac{A}{x} - \frac{1}{2xy'^2}, \\ I_2 &= \frac{1}{y'} + \frac{Ay}{x} - \frac{y}{2xy'^2}. \end{aligned} \tag{6.36}$$

Classifying relation for the symmetries of the first integrals of $xy'' = Ay'^3 - \frac{1}{2}y'$

We set F be $F = F(I_1, I_2)$. The invariance of this general function of the first integrals is (6.11) where

$$\begin{aligned} X^{[1]}I_1 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'}][\frac{A}{x} - \frac{1}{2xy'^2}] \\ &= [-\frac{A}{x} + \frac{1}{2x^2y'^2}]\xi + [\frac{1}{xy'^3}]\zeta_x \\ X^{[1]}I_2 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'}][\frac{1}{y'} + \frac{Ay}{x} - \frac{y}{2xy'^2}] \\ &= [\frac{-Ay}{x^2} + \frac{y}{2x^2y'^2}]\xi + [\frac{A}{x} - \frac{1}{2xy'^2}]\eta + [\frac{-1}{y'^2} + \frac{y}{xy'^3}]\zeta_x. \end{aligned} \tag{6.37}$$

Now ξ, η and ζ_x are given by

$$\begin{aligned} \xi &= xa_2 + 2xya_3, \\ \eta &= a_1 + ya_2 + y^2a_3, \\ \zeta_x &= -2xy'^2a_3. \end{aligned} \tag{6.38}$$

These are the coefficients of $X^{[1]}$ which are obtained by (6.28), where X_is are the symmetries as given in (6.35) and the a_i's are constants.

After substitution of the values of $X^{[1]}I_1$, $X^{[1]}I_2$ in (6.37) with ξ, η, ζ_x as in (6.38) and together by use of the first integrals $I_1 = \frac{A}{x} - \frac{1}{2xy'^2}$, $I_2 = \frac{1}{y'} + \frac{Ay}{x} - \frac{y}{2xy'^2}$ in equation

(6.11), we arrive at the classifying relation

$$[-I_1^2 a_2 - 2I_1 I_2 a_3]\frac{\partial F}{\partial I_1} + [I_1^2 a_1 + (2A - I_2^2)a_3]\frac{\partial F}{\partial I_2} = 0. \tag{6.39}$$

The relation (6.39) relates the symmetries and first integrals of the nonlinear equation (6.34). We use this to classify the first integrals according to their symmetries.

Symmetry structure of the first integrals of $xy'' = Ay'^3 - \frac{1}{2}y'$

To systematically pursue the one and higher symmetry cases we construct optimal systems of one-dimensional subalgebras spanned by the symmetries (6.35).

The Lie algebra of the generators (6.35) is three-dimensional and has commutator relations given in the table below. This constitutes the familiar algebra $sl(2, R)$.

Table 6.4: The commutation relations for the symmetries (6.35) of equation (6.34)

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	X_1	$2X_2$
X_2	$-X_1$	0	X_3
X_3	$-2X_2$	$-X_3$	0

As before, we utilize the Lie series (6.30). For example, here

$$\begin{aligned} \mathrm{Ad}(\exp(\epsilon X_2))X_3 &= X_3 - \epsilon[X_2, X_3] + \frac{1}{2!}\epsilon^2[X_2, [X_2, X_3]] - ... \\ &= X_3 - \epsilon X_3 + \frac{1}{2!}\epsilon^2 X_3 - ... \\ &= X_3 e^{-\epsilon}. \end{aligned} \tag{6.40}$$

In similar manner, we obtain the other entries of the adjoint table and we have the adjoint representation given by the Table 6.5 below.

Table 6.5: The adjoint table for the symmetries (6.35)

Ad	X_1	X_2	X_3
X_1	X_1	$X_2 - \epsilon X_1$	$X_3 - 2\epsilon X_2 + \epsilon^2 X_1$
X_2	$e^{\epsilon} X_1$	X_2	$e^{-\epsilon} X_3$
X_3	$X_1 + 2\epsilon X_2 + \epsilon^2 X_3$	$X_2 + \epsilon X_3$	X_3

This has been calculated before in [36] where it was also found the optimal system of one-dimensional subalgebras. This necessitated the construction of an invariant of the adjoint group which is $J = a_2^2 - 4a_1a_3$ [36]. An optimal system of one-dimensional subalgebras is spanned by [36]

$$\{X_1,\, X_2,\, X_1 + X_3\}. \tag{6.41}$$

For each entry of (6.41), we determine its first integral via the classifying relation (6.39). We tabulate our results in Table 6.6.

Table 6.6: One symmetry cases and the integrals of (6.34)

One symmetry	*First integral*
X_1	$F = F(I_1)$
X_2	$F = F(I_2)$
$X_1 + X_3$	$F = F\left((I_1^2 + I_2^2 - 2A)/I_1\right)$

We observe from this table that each of the operators in the optimal system has in general one integral. So the maximum number of symmetries admitted by any integral in this case is one.

Consider now the third equation in Lie's classification

$$xy'' = -y'^3 + y' + A(1 - y'^2)^{3/2},\ A \neq 0 \tag{6.42}$$

which possesses the three point symmetries

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y},\ X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \\ X_3 &= 2xy\frac{\partial}{\partial x} + (y^2 - x^2)\frac{\partial}{\partial y}, \end{aligned} \tag{6.43}$$

and the two first integrals

$$\begin{aligned} I_1 &= Ax^{-1} + x^{-1}y'(1+y'^2)^{-1/2}, \\ I_2 &= (1 + x^{-1}yy')(1+y'^2)^{-1/2} + Ax^{-1}y. \end{aligned} \tag{6.44}$$

Classifying relation for the symmetries of $xy'' = -y'^3 + y' + A(1-y'^2)^{3/2}$

Let F be an arbitrary function of I_1 and I_2, viz. $F = F(I_1, I_2)$. Then the symmetry of this function is (6.11), where

$$\begin{aligned} X^{[1]}I_1 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'}][Ax^{-1} + x^{-1}y'(1+y'^2)^{-1/2}] \\ &= [-\frac{A}{x^2} - \frac{y'(1+y'^2)^{-1/2}}{x^2}]\xi + [\frac{-y'^2(1+y'^2)^{-3/2}}{x} + \frac{(1+y'^2)^{-1/2}}{x}]\zeta_x \\ X^{[1]}I_2 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'}][(1 + x^{-1}yy')(1+y'^2)^{-1/2} + Ax^{-1}y] \\ &= [-\frac{Ay}{x^2} - \frac{yy'(1+y'^2)^{-1/2}}{x^2}]\xi + [\frac{y'(1+y'^2)^{-1/2}}{x} + \frac{A}{x}]\eta \\ &\quad + [\frac{y(1+y'^2)^{-1/2}}{x} - y'(1+y'^2)^{-3/2}(1 + \frac{yy'}{x})]\zeta_x. \end{aligned} \tag{6.45}$$

Now ξ, η and ζ_x are

$$\begin{aligned} \xi &= xa_2 + 2xya_3, \\ \eta &= a_1 + ya_2 + (y^2 - x^2)a_3, \\ \zeta_x &= -2x(1+y'^2)a_3. \end{aligned} \tag{6.46}$$

These are the coefficients of $X^{[1]}$ which are obtained from (6.28), where X_is are the symmetries as given in (6.43) and the a_i's are constants.

After inserting the values of $X^{[1]}I_1$, $X^{[1]}I_2$ in (6.45), with ξ, η, ζ_x as in (6.46) as well as using the first integrals $I_1 = Ax^{-1} + x^{-1}y'(1+y'^2)^{-1/2}$, $I_2 = (1+x^{-1}yy')(1+y'^2)^{-1/2} + Ax^{-1}y$ in equation (6.11), we get after some calculations

$$[-I_1a_2 - 2I_2a_3]\frac{\partial F}{\partial I_1} + [I_1a_1 + (\frac{1 - A^2 - I_2^2}{I_1})a_3]\frac{\partial F}{\partial I_2} = 0. \tag{6.47}$$

The relation (6.47) gives the relationship between the symmetries and first integrals of the nonlinear equation (6.42). We utilize this to classify the first integrals according to their symmetries.

Symmetry structure of the first integrals of $xy'' = -y'^3 + y' + A(1-y'^2)^{3/2}$

In order to effectively study the one and higher symmetry cases we obtain optimal systems of one-dimensional subalgebra spanned by (6.43).

The Lie algebra of the operators (6.43) is three-dimensional and has commutator relations given in the Table 6.4. It forms the well-known algebra $sl(2, R)$.

An optimal system has already been obtained for this algebra and is given by (6.41).

For each operator entry of (6.41) we work out the associated first integral. These are given in Table 6.7 below.

Table 6.7: One symmetry cases and the integrals of (6.42)

One symmetry	*First integral*
X_1	$F = F(I_1)$
X_2	$F = F(I_2)$
$X_1 + X_3$	$F = F\left((I_1^2 + I_2^2 + A^2 - 1)/I_1\right)$

We remark that the maximum number of symmetries admitted by a first integral in this case is again one.

We lastly focus on the fourth second-order ODE in Lie's classification. This is

$$y'' = Ay'^{\frac{(a-2)}{(a-1)}},\ A \neq 0,\ a \neq 0, 1, 1/2, 2 \tag{6.48}$$

and possesses the three symmetries

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y}, \\ X_3 &= x\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y} \end{aligned} \tag{6.49}$$

and the two first integrals

$$\begin{aligned} I_1 &= (\frac{a-1}{a})y'^{\frac{a}{a-1}} - Ay, \\ I_2 &= Ax + (1-a)y'^{\frac{1}{a-1}}. \end{aligned} \tag{6.50}$$

Classifying relation for the symmetries of $y'' = Ay'^{\frac{(a-2)}{(a-1)}}$, $a \neq 0$, 1, 1/2, 2

If F is an arbitrary function of I_1 and I_2, $F = F(I_1, I_2)$, then the symmetry of this function of the integrals is (6.11), where

$$X^{[1]}I_1 = [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'}][(\frac{a-1}{a})y'^{\frac{a}{a-1}} - Ay]$$

$$\begin{aligned} &= -A\eta + [y'^{\frac{1}{a-1}}]\zeta_x \\ X^{[1]}I_2 &= [\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta_x\frac{\partial}{\partial y'}][Ax + (1-a)y'^{\frac{1}{a-1}}] \\ &= A\xi - [y'^{\frac{(2-a)}{(a-1)}}]\zeta_x. \end{aligned} \tag{6.51}$$

Now ξ, η and ζ_x are

$$\begin{aligned} \xi &= a_1 + xa_3, \\ \eta &= a_2 + yaa_3, \\ \zeta_x &= y'(a-1)a_3. \end{aligned} \tag{6.52}$$

These are the coefficients of $X^{[1]}$ which are obtained from (6.28), where X_is are the symmetries as given in (6.49) and the a_i's are constants.

After substituting the values of $X^{[1]}I_1$, $X^{[1]}I_2$ as in (6.51), with ξ, η, ζ_x as in (6.52) and together using the first integrals $I_1 = (\frac{a-1}{a})y'^{\frac{a}{a-1}} - Ay$, $I_2 = Ax + (1-a)y'^{\frac{1}{a-1}}$ in equation (6.11), we obtain

$$[-Aa_2 + I_1aa_3]\frac{\partial F}{\partial I_1} + [Aa_1 + I_2a_3]\frac{\partial F}{\partial I_2} = 0. \tag{6.53}$$

The relation (6.53) is the relationship between the symmetries and the first integrals of the equation (6.48). We use this to classify the first integrals in terms of their symmetries.

Symmetry structure of the first integrals of $y'' = Ay'^{\frac{(a-2)}{(a-1)}}$, $a \neq 0$, 1, 1/2, 2

To study the one and higher symmetry cases we again obtain optimal systems of one-dimensional subalgebras spanned by (6.49).

The Lie algebra of the operators (6.49) is three-dimensional and has commutator relations given in the Table 6.8 below.

Table 6.8: The commutation relations for the symmetries of equation (6.48)

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	X_1
X_2	0	0	aX_2
X_3	$-X_1$	$-aX_2$	0

The calculation of the adjoint representation is as before. We utilize the Lie series. As an example,

$$\begin{aligned} \mathrm{Ad}(\exp(\epsilon X_3))X_2 &= X_2 - \epsilon[X_3, X_2] + \frac{1}{2!}\epsilon^2[X_3,[X_3,X_2]] - ... \\ &= X_2 + \epsilon a X_2 + \frac{1}{2!}\epsilon^2 a^2 X_2 + ... \\ &= e^{\epsilon a} X_2. \end{aligned} \tag{6.54}$$

We obtain the other entries of the adjoint table easily and we have the adjoint representation given by the Table 6.9 below.

Table 6.9: The adjoint table for the symmetries (6.49)

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \epsilon X_1$
X_2	X_1	X_2	$X_3 - \epsilon a X_2$
X_3	$e^{\epsilon} X_1$	$e^{a\epsilon} X_2$	X_3

We utilize the above adjoint table to compute an optimal system. The computations again are straightforward and we find an optimal system of one-dimensional subalgebras spanned by

$$\{X_3,\ X_1 + bX_2\}. \tag{6.55}$$

We use the optimal system (6.55) to deduce the relation given in the following table.

Table 6.10: One symmetry cases and the integrals of (6.48)

$One \quad symmetry$	$First \quad integral$
X_3	$F = F(I_1^{-1} I_2^a)$
X_1	$F = F(I_1)$
$X_1 + bX_2, b \neq 0$	$F = F(I_1 + bI_2)$

As a consequence of the preceding discussions, we have the following result.

Theorem 6.2. The full Lie algebra of a scalar nonlinear second-order ODE which is three-dimensional is generated by the one-dimensional subalgebras of the first integrals of the equation as given in Tables 6.3, 6.6, 6.7 and 6.10.

Moreover, as a conclusion of the the analysis of sections 6.2 and 6.3, we have the following theorem.

Theorem 6.3. The maximum dimension of the Lie algebra admitted by a first integral of a scalar nonlinear second-order ODE which has symmetry is one.

We now illustrate our main focus.

6.4 Physical Applications

We consider three physical examples in order to illustrate our results.

1. The Lane-Emden equation

$$y'' + \frac{y'}{x} + e^y = 0, \quad x \neq 0 \tag{6.56}$$

has been studied on many occasions (see, e.g. [37, 38, 39, 40]). Equation (6.56) has two point symmetries

$$\begin{aligned} X_1 &= \frac{1}{2}x\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \\ X_2 &= (x\ln x - x)\frac{\partial}{\partial x} - 2\ln x\frac{\partial}{\partial y} \end{aligned} \tag{6.57}$$

which constitute a two-dimensional Lie algebra with $[X_1, X_2] = X_1$. One can easily effect the reduction of (6.57) to the Lie canonical form

$$\bar{X}_1 = \frac{\partial}{\partial \bar{y}}, \quad \bar{X}_2 = \bar{x}\frac{\partial}{\partial \bar{x}} + \bar{y}\frac{\partial}{\partial \bar{y}} \tag{6.58}$$

by means of the transformation

$$\bar{x} = x^{-1}e^{-y/2}, \quad \bar{y} = 2\ln x - 2 \tag{6.59}$$

which transforms (6.56) to the type II Lie equation

$$\bar{x}\bar{y}'' = -4\bar{y}' - \frac{1}{8}\bar{y}'^3. \tag{6.60}$$

Now equation (6.60) has the two functionally independent first integrals (see (6.18) subject to (6.19))

$$I_1 = \bar{x}^{-1}K(\bar{y}'), \quad I_2 = \int K(\bar{y}')d\bar{y}' + (\bar{y} - \bar{x}\bar{y}')\bar{x}^{-1}K(\bar{y}'), \tag{6.61}$$

where

$$K(\bar{y}') = \exp\int \frac{d\bar{y}'}{-4\bar{y}' - \bar{y}'^3/8}. \tag{6.62}$$

The relationship between the symmetries and first integrals of (6.60) is (6.22) with F now a function of I_1 and I_2 in (6.61). Thus we obtain one symmetry of a first integral of (6.60) if F is a function of I_1, I_2 or $a_1I_1 + a_2I_2$. However, if F is an arbitrary function of the integrals, then there exists no symmetry.

As an example, we have in original coordinates that

$$I_1 = 2xy' + \frac{1}{2}x^2y'^2 + x^2e^y \tag{6.63}$$

is the first integral of (6.56) and (6.63) admits X_1 as given in (6.57). This single symmetry can be used to effect a quadrature of the Lane-Emden equation (6.56) (for a general result on reductions of equations, the reader is referred to the work [6]). One uses the invariant $u = x^2e^y$ corresponding to X_1. Then the reduced ODE is first-order $I_1 = C_1$, viz.

$$\frac{1}{2}x^2\frac{u'^2}{u^2} - 2 + u = C_1, \quad C_1 = const, \tag{6.64}$$

which is variables separable.

2. The generalized Emden-Fowler equation

$$y'' = f(x)y^n \tag{6.65}$$

occur in several applications (see [41, 42, 43, 44, 45]). These works utilized the Lie and Noether symmetry approaches in analyzing (6.65). The solutions of (6.65) for various forms of $f(x)$ and constant n are known in the papers cited. Here we take the form

$$y'' + x^{-20/7}y^2 = 0 \tag{6.66}$$

for which the solutions are known (see [42]). A first integral of (6.66) is

$$I_1 \;=\; \frac{1}{2}\left(x^{4/7}y' - \frac{4}{7}x^{-3/7}y - \frac{12}{343}x^{3/7}\right)^2$$

$$+ \quad \frac{1}{3}\left(x^{-4/7}y - \frac{6}{49}x^{2/7}\right)^3. \tag{6.67}$$

The integral (6.67) has symmetry generator

$$X = 343x^{8/7}\frac{\partial}{\partial x} + (196x^{1/7}y + 12x)\frac{\partial}{\partial y} \tag{6.68}$$

which is also the symmetry of equation (6.66) as given in [42]. An invariant of (6.68) is [42]

$$u = x^{-4/7}y - \frac{6}{49}x^{2/7}. \tag{6.69}$$

We write $I_1 = C_1$ which in terms of u and du/dx is

$$\frac{1}{2}x^{16/7}u'^2 + \frac{1}{3}u^3 = C_1. \tag{6.70}$$

Equation (6.70) is variables separable and results in the same solution as given in [42].

3. The modified Emden equation of index 3, viz.

$$y'' + \frac{3x}{k+x^2}y' + y^3 = 0, \tag{6.71}$$

where k is a constant, was investigated in [46] in which the first integral

$$I = (k+x^2)^2y'^2 + 2x(k+x^2)yy' + \frac{1}{2}(k+x^2)^2y^4 - ky^2 \tag{6.72}$$

was found. The equation (6.71) is subsumed under a general class by Berkovich in [47] who studied the Lie group properties and exact solutions of the generalized Emden-Fowler type equation

$$y'' + a_1(x)y' + a_0(x)y + f(x)y^n = 0, n \neq 0, n \neq 1. \tag{6.73}$$

He showed that equation (6.73) is reducible to the autonomous canonical form

$$\ddot{z} + b_1\dot{z} + b_0z + cz^n = 0, \tag{6.74}$$

for constants b_0, b_1 and c, by the Kummer-Liouville (KL) transformation $y = v(x)z, dt = u(x)dx$ for specific forms of the function $f(x)$. In the paper cited, viz. [47], he gives all the cases of $f(x)$ for which (6.73) admits one, two or three Lie point symmetries. The particular exact solutions of the form

$$y = \rho v(x), b_0\rho + c\rho^n = 0, \tag{6.75}$$

are also presented once $v(x)$ is found (see Theorem 3.1 in [47]). Furthermore in Theorem 4.2 (see [47]) which applies to subclasses of equations of (6.73), viz. those for which $a_0 = 0$ and $f(x) = p$, p a constant, the necessary and sufficient condition that such ODEs admit the symmetry generator

$$X = \frac{1}{u}\frac{\partial}{\partial x} + y\frac{v'}{uv}\frac{\partial}{\partial y} \tag{6.76}$$

is that a_1 satisfies the nonlinear second-order ODE

$$a_1'' + \frac{4n}{n+3}a_1a_1' + \frac{2(n^2-1)}{(n+3)^2}a_1^3 = 0. \tag{6.77}$$

This condition (6.77) on the coefficient a_1 was previously obtained in Leach [46] as a condition for the existence of a Noether symmetry. The complete integrability of (6.71) was not provided in [46] (see also references cited in this paper). Now by using the approach of Berkovich [47], one can find the KL transformation $y = (k+x^2)^{-1/2}z$, $dt = (k + x^2)^{-1/2}dx$ that reduces our modified Emden ODE (6.71) to the canonical form (6.74) with $b_0 = -1$, $b_1 = 0$, $c = 1$ and $n = 3$. Then nontrivial particular exact solutions of the form (6.75) are determined as $y = \pm(k + x^2)^{-1/2}$ which are easily verified by substitution into (6.71).

We now focus on the general solution here by imposing a symmetry on the integral (6.72). By use of the symmetry condition (6.2), X is a symmetry generator of the

integral (6.72) if the determining equations

$$\begin{aligned} \xi_y = 0,\ 2x\xi + (\eta_y - \xi_x)(k + x^2) = 0, \\ y\xi + x\eta + (k + x^2)\eta_x = 0, \\ x(k + x^2)y^4\xi + (k + x^2)^2 y^3\eta - ky\eta + xy(k + x^2)\eta_x = 0 \end{aligned} \tag{6.78}$$

hold. The solution of (6.78) after some calculations result in the generator

$$X = (k + x^2)^{1/2}\frac{\partial}{\partial x} - xy(k + x^2)^{-1/2}\frac{\partial}{\partial y} \tag{6.79}$$

which turns out to be the only Lie point symmetry of the modified Emden equation (6.71). It should be noted that the symmetry operator (6.79) is precisely of the form (6.76) with $u = v = (k + x^2)^{-1/2}$ since it satisfies Theorem 4.2 of [47].

An invariant function of the generator (6.79) is

$$u = y(k + x^2)^{1/2}. \tag{6.80}$$

We utilize u in (6.80) as a new dependent variable in the reduced first-order ODE $I = C$, C a constant, which in turn becomes

$$(k + x^2)u'^2 - u^2 + \frac{1}{4}u^4 = C. \tag{6.81}$$

Equation (6.81) can be re-written as

$$\frac{du}{\pm\sqrt{u^2 - \frac{1}{4}u^4 + C}} = \frac{dx}{(k + x^2)^{1/2}} \tag{6.82}$$

The equation (6.82) easily yields the general solution of our modified Emden equation (6.71) in terms of quadratures.

Chapter 7

Conclusion

In this work we have provided the algebraic structure of first integrals of the free particle or any scalar linearizable, via point transformation, ODE. Firstly, we derived the relationship between the symmetries and the first integrals of the free particle equation. By analyzing this classifying relation, we were able to establish the number of symmetries possessed by any first integral of the free particle equation. We obtained the important result that the symmetries admitted by a first integral can be 0, 1, 2 or 3. It was observed that the zero symmetry case was rather surprising or unexpected as one does not have a route to integration of the equation due to the lack of any symmetry and this too for the simplest equation. The one and two symmetry cases were not unique - there were many first integrals with differing one and two symmetry structures. These were carefully discussed. Finally, we studied completely the situation when a first integral has three symmetries. We used the classification of realizations in the plane adapted as free particle symmetries. We showed that the only

three-dimensional algebra admitted by a first integral of the free particle equation is $L^I_{3;5}$ which is admitted by the functionally independent integrals I_1 and I_2 as well as their quotient I_2/I_1. Although this triplet of symmetries was discovered before in the seminal work of Leach and Mahomed [20], these authors did not prove that it was unique nor the maximum algebra. However, they did emphasize the important result that the algebras of the triplets of symmetries were isomorphic and that the three triplets of symmetries generate the $sl(3, R)$ symmetry of the equation. We showed that the maximum algebra is indeed the three-dimensional algebra $L^I_{3;5}$ by completely analyzing all representations of the three-dimensional algebras.

For the symmetries of the simplest first-order ODE $y' = 0$ we have noted that the algebra of any first integral is a proper subalgebra of the equation itself. Also one cannot generate its full algebra via the algebras of any integral. This result then applies to any scalar first-order ODE. In contrast as has been shown in [20], the Lie algebra of $y'' = 0$ which represents any linear or linearizable second-order ODE can be generated by the three-dimensional algebras of the triplets of the basic integrals and their quotient which turn out isomorphic to each.

We have shown that the full Lie algebra of the simplest third-order equation (4.15) is generated by the four symmetries (4.18) and the three symmetries G_2, G_3 and G_4 of (4.20). Thus here one requires only the symmetries of the basic integrals I_1 and I_3 to generate the full algebra of the equation (4.15). This is indeed different to what happens for $y' = 0$ and $y'' = 0$. For higher-order ODEs of maximal symmetry, certain patterns emerge, some of which are discussed in three propositions in the paper of Flessas et al. [22]. We have seen another important property, viz. that of the generation of the full algebra via integrals. In this case we proved a theorem, viz.

that the full Lie algebra of the nth-order ODE $y^{(n)} = 0, \quad n \geq 3$, is generated by two subalgebras, viz. the $n+1$-dimensional algebra $< X_j : j = 1, \ldots, n+1 >$ of I_1 and the three-dimensional subalgebra $< Y_{n-1}, Y_n, Y_{n+1} >$ of I_n. Therefore, the picture is distinct for the way in which the full Lie algebra is generated for the ODEs $y' = 0$, $y'' = 0$ and $y^{(n)} = 0,\ n \geq 3$. This turns out to be consistent with the properties of their symmetry algebra which are also different (see, e.g. [18]).

The algebraic properties of the first integrals of the 8 symmetry or maximal class were pursued in [20] in which it was shown that the algebra $sl(3, R)$ of the linearizable equations can be generated by three isomorphic triplets of three-dimensional algebras. Then in [21] the authors considered the symmetry properties of the basic first integrals of scalar linear third-order ODEs for which the symmetry structure has been investigated before (see, e.g. the review [25]). In a recent paper we performed a complete study of the symmetry structure of first integrals of the free particle or linearizable second-order ODEs. We showed in chapter 3 (see also [26]) that the first integrals have rich symmetry algebras - we found that they have 0, 1, 2 or 3 dimensional algebras and that the maximal case is unique with algebra $L^I_{3,5}$. Motivated by this and recent works [20, 21, 22], we performed in chapter 4 (see [31] as well) a symmetry classification of the first integrals of the maximal class of linear third-order ODEs represented by $y''' = 0$. Many interesting properties came to light. It was shown that the symmetry structure of the first integrals is also rich and there exits the 0, 1, 2 and 3 symmetry cases. In the case of the maximal algebra of the integrals which is 3, we showed that similar to the free particle case, it is unique. We also proved that the full Lie algebra of the equation for linear third and higher order can be generated by just two basic integrals. This result differs from what happens for the free particle or even first order equations as pointed out chapters 3 and 4 [20, 26, 31].

In this work we also investigated the symmetry properties of the first integrals of scalar linearizable third-order ODEs of submaximal classes, viz. the 4 and 5 symmetry classes (see also [33]). Here we obtained the result that there can be the 0, 1 or 2 symmetry cases for the 4 symmetry class and 0, 1, 2 or 3 symmetry cases for the 5 symmetry class. Also we noted that the maximal cases are not unique as for the free particle or simplest third-order equations. We further studied the generation of the full Lie algebras of the submaximal classes of linear higher-order ODEs and have shown how these are generated by subalgebras of certain basic integrals and a quotient of two integrals.

We have further shown that the full Lie algebra of the linear nth-order ODE $y^{(n)} - y^{(n-1)} = 0, \quad n \geq 3$ which is $n+2$ dimensional, is generated by two subalgebras: the n-dimensional algebra $< Y_j : j = 1, \ldots, n >$ of $I_2 = e^x(y^{(n-1)} - y^{(n-2)})$ and the two-dimensional subalgebra $< X_2, X_3 >$ of $I_1 = y^{(n-1)} - y^{(n-3)}$. In the case of the linear nth-order ODE $y^{(n)} - y^{(n-2)} + f(x)(y^{(n-1)} - y^{(n-3)}) = 0, \quad n \geq 3$ which is $n+1$ dimensional, the full Lie algebra is shown to be generated by two subalgebras: the $(n-1)$-dimensional algebra $< X_j : j = 1, \ldots, n-1 >$ of I_1 as given in (5.46) and the two-dimensional subalgebra $< X_n, X_{n+1} >$ of I_3/I_2, where I_2 and I_3 are as given in (5.46).

Further work could be to study submaximal classes of nonlinear higher-order ODEs for the symmetry properties of their first integrals.

The relationship between symmetries and first integrals are of paramount importance and arises in the famous Noether theorem (see, e.g. [48]). The Noether theorem gives the direct relation between a symmetry and its integral via a formula. Although a

symmetry generator X gives a first integral I by the Noether theorem for a variational problem, the integral may have more than one symmetry generator. Also for equations that are not variational, there exists a relationship between symmetries and first integrals (see [6]).

There have been interesting contributions that provide the link between symmetries and first integrals for linear ODEs. In the case of scalar second- and higher-order linear ODEs there have been important properties (see also [20, 21, 26, 31, 33]) as we have shown in chapters 3 to 5.

Moreover, in this work we have focused on the symmetry properties of first integrals of scalar nonlinear second-order ODEs with sub-maximal symmetry algebras, viz. those having 1, 2 and 3 point symmetries (see as well [35]). We have shown that the maximum number of symmetries possessed by any first integral of a scalar second-order ODE with sub-maximal symmetry is one. It is known that one symmetry is sufficient for complete integrability of the scalar nonlinear second-order ODE. We have provided physical examples of the generalized Emden-Fowler, Lane-Emden and modified Emden equations to illustrate how the relationship between symmetries and first integrals lead to reduction to quadrature of the underlying equation.

It would be of interest to further investigate symmetries of first integrals which are more general as in [49] as well as for linear second-order systems which have been considered for symmetry properties in [50].

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