

Chromatic Polynomials and Certain Classes of Graphs

by

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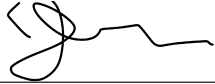
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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree of Doctor of Philosophy in mathematics at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.



____ (Signature of candidate)

01 day of March in 2024

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— Lesiba Joseph Maphakela

Abstract

The chromatic polynomial of a graph has been widely studied in the literature. The focus of this research is on exploring the chromatic polynomial of specific graphs that result from the application of a join operation. The chromatic polynomial of a graph can be expressed in various forms; power form, tree form, factorial form and cycle form. The expressions in various forms, such as power form, tree form, and factorial form, have been subject to comprehensive investigation. However, it should be noted that the cycle form presents relative gaps that necessitate further exploration. This work builds upon the existing literature by engaging in a discussion of the coefficients of the chromatic polynomial of a graph expressed in cycle form. To achieve this objective, we commence by presenting the general formula of the chromatic polynomial in cycle form. Following this, we introduce an algorithm that computes the chromatic polynomial of a graph in cycle form. Additionally, we outline a method for converting the chromatic polynomial of a graph from its tree form into the cycle form. Furthermore, we determine the values of the first and second terms of the chromatic polynomial in its cycle form. This research also complements the well established knowledge of the chromatic polynomial of graphs resulting from the application of a join operation. Of particular interest, we explore the joins of various classes of graphs, including the join of a null graph, N_1 with a graph G , which is known as the vertex join of graph G . Building upon this framework, we extend our analysis to encompass the join of a null graph, N_2 , with graph G . Similarly, we present results pertaining to the join of a complete graph, K_n , with a graph G . Significantly, we conduct a thorough comparative analysis of the chromatic equivalence class among these derived classes of graphs. Lastly, we discuss the chromatic uniqueness of these derived classes of graphs, alongside introducing variations to these derived graphs by deleting their edges and subgraphs.

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Notation

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

$$\mathbb{R} = \text{set of real numbers}$$

$$\mathbb{C} = \text{set of complex numbers}$$

$$|S| = \text{cardinality of a set } S, \text{ the number of elements in finite set, } S$$

$$B \setminus A = B - A = \{x \in B | x \notin A\}$$

$$B \cup A = \text{the union of } A \text{ and } B$$

$$(\lambda)_n = \lambda(\lambda - 1) \cdots (\lambda - n + 1), \text{ the falling factorial}$$

In what follows, G and H are graphs

G : a graph with a vertex set $V = V(G)$ and an edge set $E = E(G)$, or alternatively expressed as $G = (V, E)$

$|V(G)|$: the number of vertices in G or the order of G

$|E(G)|$: the number of edges in G or the size of G

(u, v) : an edge with end vertices, u and v

$d_G(v)$: the degree of v in G where $v \in V(G)$

G' : simplification of G , a graph G having all loops deleted and parallel class replaced with one edge

c = number of connected component on a graph G

$n_G(H)$ = number of subgraphs in G which are isomorphic to H

$t_H(G)$ = number of induced subgraphs in G which are isomorphic to H

$r(G)$ = $|V| - c$, a rank of a graph G .

$cl(X)$ = $X \cup \{e \in E | r(X \cup e) = r(X)\}$, where $G = (V, E)$

$G \cong H$: G is isomorphic to H

K_n : the complete graph of order n

$K_{|V_1|, |V_2|}$: a complete bipartite graph with disjoint vertex sets V_1 and V_2

N_n : the null graph of order n

- T_n : the tree of order n
 P_n : the path of order n
 S_n : the star of order n
 C_n : the cycle of order n
 C_3 : a triangle
 W_n : the wheel of order n
 $W(n, k)$: a minor that remains after deleting all but k consecutive spokes on a wheel, W_n , where $1 \leq k \leq n - 1$ and $n \geq 4$
 $F_{(n, m)}$: the forest with n vertices and m components
 $G \setminus e$: a minor that remains after deleting an edge e from G , where $e \in E(G)$
 $G \setminus v$: a minor that remains after deleting a vertex v with edges incident with v from G , where $v \in V(G)$
 $G + e$: a minor that remains after adding an edge e to G , where $e \notin E(G)$
 $G + k \cdot e$: a graph G with k bridges
 G/e : a minor that remains after contracting an edge e from G , where $e \in E(G)$
 $G \vee H$: the join of G and H , connecting each vertex of G with all vertices of H by a single edge
 $\hat{G}$: the vertex join of G , connecting each vertex of G with an isolated vertex x , by a single edge, where $x \notin V(G)$, alternatively $K_1 \vee G$
 $\hat{\hat{G}}$: the double repeated vertex join of G , connecting each vertex of $\hat{G}$ with y by a single edge, where $y \notin V(\hat{G})$, alternatively $K_2 \vee G$
 $\hat{G}^q$: the q -th repeated vertex join of the graph, adjoining new disjoint vertices $x_1, x_2, \dots, x_q$ to each vertex of G by a single edge and then connecting each pair of vertices $x_1, x_2, \dots, x_q$ by a single edge, alternatively $K_q \vee G$
 $\ddot{G}$: the bivertex join of G , connect two new non-adjacent isolated vertices $\{x_1, x_2\} \notin V(G)$ with each vertex of a graph G by a single edge where $y \notin V(\hat{G})$ i.e. $N_2 \vee G$
 $\ddot{N}_2^{q-1}$: $\underbrace{N_2 \vee N_2 \vee \dots \vee N_2}_{q\text{-times}}$

- $\ddot{G}^q$: the q -th repeated bivertex join of the graph, $G \vee \ddot{N}_2^{q-1}$
- $\ddot{C}(n, k)$: a 2-cone
- $\ddot{C}(n+2, k)$: a 2-cone with $n - k$ missing bispokes
- $\chi(G; \lambda)$: the chromatic polynomial of G in λ
- $\chi(G)$: the chromatic number of G
- $G \stackrel{\chi}{\sim} H$: $\chi(G; \lambda) = \chi(H; \lambda)$, G is chromatically equivalent to H , G is χ -equivalent to H .
- $[G]$ = $\{H | H \sim G\}$
- $G_1 \cup_{K_r} G_2$: the K_r -gluing of G_1 and G_2

Chapter 1

Introduction

In this chapter, we start by giving a brief historical background of the four colour problem. Then, tap into existing literature to motivate the work and lay out the overview of the thesis. Thereafter, we give some basic definitions, classes of graphs and introduce some graph operations. We conclude the chapter with a brief introduction on the graph colouring polynomial known as the chromatic polynomial.

1.1 Historical background

The *Four colour problem* states that any map can be coloured with a minimum of four colours in such a way that any two neighboring regions do not share the same colour. Alternatively, that every loopless planar graph admits a vertex-colouring with at most four different colours. In 1850, Guthrie F. introduced the four colour problem, we refer the reader to [12]. In 1879, Kempe A. published the first proof of the four colour problem, we refer the reader to [34]. In 1890, Heawood P. provided a map that contradicts the methods used by Kempe A. in proving the four colour problem, we refer the reader to [30]. In 1976, Appel K. and Haken W. wrote a series of two papers that amount over 137 pages to prove the four colour problem using computer programming, see [4, 6]. In 1989, the authors revisited the proof for simplification, see [5]. The proof that was provided by Appel K. and Haken W. were not fully accepted because part of the proof uses computer programming and can not be verified by hand and even the part of the proof that is ought to be verified by hand is remarkably complicated and tedious, for these reasons provided, it is alleged that no one has made full check of the proof, see [47]. In 1996, Robertson N., Sanders D. P.,

Seymour R. P. and Thomas R. revisited the proof of the four colour problem and the paper had less than 10 pages to prove the four colour problem also using computer programming, see [47]. However, the aspects of the proof can be verified using a hand and the computation is simplified.

The four colour problem brought to surface the notion of *graph colouring*, we refer the reader to [10]. In graph theory, a *graph colouring* is a process of assigning a colour to each vertex of a graph. Alternatively, a graph colouring is a mapping of vertices of a graph into or onto a particular set of finite colours. A graph colouring is either proper graph colouring or improper graph colouring.

Definition 1.1.1. An *improper colouring* of a graph is the process of assigning a colour to each vertex of a graph such that adjacent vertices may be coloured with the same colour. A *proper graph colouring* is the process of assigning a colour to each vertex of a graph with a restriction that adjacent vertices are not coloured with the same colour.

The notion of proper colouring was explored by Birkoff G. D. in 1912. Birkoff derived a way to determine the number of ways to colour a planar graph properly using a polynomial that is now known as the chromatic polynomial of a graph, we refer the reader to [10]. In 1932, Whitney H. extended the notion of chromatic polynomial from planar graphs to any graph in general, we refer the reader to [51, 52]. The *chromatic polynomial* of a graph determines number of ways to colour a graph properly using a particular finite set of colours. The chromatic polynomial of a graph has been studied more extensively relative to the four colour problem. To study these notions in details we refer the reader to [1, 11, 15, 18, 29, 53].

Chromatic polynomials were given much attention by mathematicians after a paper by Read R. C. in 1968, titled “An introduction to chromatic polynomials”, we refer the reader to [44]. This paper introduces some properties of the chromatic polynomial that give rise to necessary conditions for a polynomial to be the chromatic polynomial of a graph. Some necessary conditions of the chromatic polynomial when it is expressed in power form are; the chromatic polynomial of a graph is monic, the terms in a chromatic polynomial of a graph alternate in sign, the chromatic polynomial has no constant term and the absolute value of the second coefficient is the number of

edges. Since this paper was published, many results on the chromatic polynomials of a graph were discovered due to further investigation on necessary conditions of the chromatic polynomial of a graph, we refer the reader to [21, 31, 44].

In the context of using necessary conditions to derive sufficient conditions, mathematicians have been studying the chromatic polynomials of various families of graphs. Some families of graphs share the same chromatic polynomial and some have formulation to easily compute the chromatic polynomial. Thus, it is easy to conclude on necessary and sufficient conditions of the coefficients of chromatic polynomial for various graphs. Some families of graphs can be derived from graph operations. In this work, we focus on specific families of graphs derived from join graph operations and thereafter determine the properties of their chromatic polynomials.

1.2 Motivation

In this section, we discuss the motivation of this research. Explicit expressions of chromatic polynomials for certain families of graphs has been studied in graph theory literature. Of particular interest are the investigations into the chromatic polynomials of graphs derived from graph operations. Additionally, it was discovered that the chromatic polynomial of the derived graph can be expressed in terms of the chromatic polynomial of the original graphs. One example is the binary graph operation, K_r -gluing of G_1 and G_2 , denoted by $G_1 \cup_{K_r} G_2$. In the study by Read R. C. [44], it established that the chromatic polynomial of the K_r -gluing of two graphs can be expressed in terms of the chromatic polynomial of the two original graphs, $\chi(G_1 \cup_{K_r} G_2) = \frac{\chi(G_1; \lambda)\chi(G_2; \lambda)}{\chi(K_r; \lambda)}$. Another example involves a family of graphs derived from a unary graph operation such as the vertex join of a graph, denoted by $\hat{G}$. According to Tutte's work in [49], the chromatic polynomial of the vertex join of a graph can be determined from the chromatic polynomial of the original graph, $\chi(\hat{G}; \lambda) = \lambda\chi(G; \lambda - 1)$. This raises the question "are there other graph operations that yield a derived graph such that the chromatic polynomial of the derived graph is expressed in terms of the chromatic polynomial of original graphs?"

Furthermore, after determining the chromatic polynomial of the derived graph, we investigate whether the derived graph is chromatically unique or chromatically equiv-

alent to other graphs that are non-isomorphic to it. In this work, we discuss the graph derived from a join of a null graph, N_2 and a graph G . In [17], Chia G. L. posed the question, “For what chromatically unique graphs G and H is $G \vee H$ chromatically unique?” In this work, we identify certain families of derived chromatically equivalent graphs where the derived graph, $G \vee H$, is chromatically unique or it is chromatically equivalent to other graphs that are non-isomorphic to it . In addition, we partially answer the question; “why are certain well known pairs of non-isomorphic graphs chromatically equivalent?” In this cases, we gather and generalize the properties of the chromatic equivalence class. Similarly, we discuss some derived family of graphs that are chromatically unique.

1.3 Thesis overview

In this section, we discuss the outline of the thesis and highlight the main results.

In Chapter 1, Section 1.4, we discuss basic definitions and concepts in graph theory that are relevant to this research. In Section 1.5, we discuss some classes of graphs, namely; complete graph, cycle, tree, null graph and some of their properties. In Section 1.6, we discuss some fundamental graph operations that are relevant to this research. In particular, the vertex join, the join of two graphs and the K_r -gluing of two graphs. In Section 1.7, we discuss the concept of graph polynomials and their roles in graph theory. Thereafter, we discuss various methods of determining the chromatic polynomial of a graph. Additionally, we state the conditions for graphs to be chromatically equivalent.

In Chapter 2, Section 2.2, we discuss the various well known expressions of the chromatic polynomial of a graph in the power form, tree form and factorial form. Thereafter, we state the well known properties of the coefficients of the chromatic polynomial in that given form. In Section 2.3, we study the general formula of the chromatic polynomial of a graph expressed in cycle form. Thereafter, we provide the algorithm that is used to compute the chromatic polynomial of a graph in cycle form. Additionally, we give a method of converting the chromatic polynomial of a graph from tree form into cycle form. Then, we identify the value of the first and the second coefficient of the chromatic polynomial polynomial of a graph in cycle form. Finally, we provide a convenient way to determine the chromatic polynomial of some of the

known special graphs in cycle form.

In Chapter 3, Section 3.2, we start by discussing the well known chromatic polynomial of joins of graphs which uses umbral product operation. We then discuss the chromatic equivalence of some joins of graphs. In Section 3.3, we discuss the well known chromatic polynomial of a special class of join of graphs called the vertex join of a graph which we use to determine the chromatic polynomial of the graph, $K_n \vee G$. Then, we give necessary conditions for a graph to be in the chromatic equivalence class, $[K_n \vee G]$. In Section 3.4, we use the vertex join of a graph, G to construct one of our special join, $G \vee \ddot{N}_2^{q-1}$ where $\ddot{N}_2^{q-1} \cong \underbrace{N_2 \vee N_2 \vee \cdots \vee N_2}_{q\text{-times}}$. We illuminate the structural differences and similarities between $G \vee \ddot{N}_2^{q-1}$ and $\hat{G}^{2q}$. Additionally, we determine the formula for the chromatic polynomial of the graph, $N_2 \vee G$ and we determine necessary conditions for a graph to be in the chromatic equivalence class, $[N_2 \vee G]$. In Section 3.5, we delist some well known pairs of non-isomorphic graphs which are non-trivial chromatic equivalence.

In Chapter 4, we discuss the chromatic equivalence class of the bivertex join of T_n , $\ddot{T}_n$. In Section 4.2, we discuss necessary conditions for a graph to be an element of the chromatic equivalence class, $[\ddot{T}_n]$. In Section 4.3, we discuss the first four coefficients of the chromatic polynomial of a graph, $J \in [\ddot{T}_n]$. In particular, we study the number of subgraphs of $\ddot{T}_n$ which contribute to the first four coefficients of $\chi(\ddot{T}_n; \lambda)$. In Section 4.4, we study in detail the number of subgraphs of $\ddot{T}_n$ which contribute to the fifth coefficient of the chromatic polynomial, $\chi(\ddot{T}_n; \lambda)$. To conclude this chapter, we discuss a well known class of graphs called clique separable in Section 4.5. We use the vertex join and bivertex join operations to construct some classes of clique separable graphs.

In Chapter 5, Section 5.2, we use the vertex join and bivertex join operations to construct the generalized wheel, $\hat{C}_n^t$ and the t -cone, $\ddot{C}_n^t$. Section 5.3, we give the explicit expression of the chromatic polynomial of a 2-cone, $\chi(\ddot{C}_n; \lambda)$. Thereafter, we state the conditions for a wheel to be a chromatically unique graph. Finally, we study conditions under which a 2-cone qualifies as a chromatically unique graph..

In Chapter 6, In Section 6.2, we discuss the wheels with missing spokes. We be-

gin by discussing some of properties of wheels with missing spokes. Thereafter, we discuss the well known results for chromaticity of a wheel with missing spokes. In Section 6.3, we begin by introducing a class of graphs called the 2-cones with missing spokes. Thereafter, we discuss some of the properties of 2-cones with missing spokes. Additionally, we mimic some of the results for the chromaticity of wheels with missing spokes for 2-cones missing spokes. In particular, we determine a sufficient condition for a graph to be in a chromatic equivalence class of a 2-cones with 4 spokes. Finally we determine the cardinality of the chromatic equivalence class, $\check{C}(n+2, 2)$.

1.4 Basic definitions

In this section, we discuss basic definitions and concepts in graph theory that are relevant to this research. For further details, we refer the reader to [7, 44, 49], unless stated otherwise. The results in this chapter are well known and we state them without proofs. A *graph* is an ordered pair $G = (V(G), E(G))$ where a nonempty set $V(G)$ is called the vertex set and $E(G)$ is called the edge set. Each element of $E(G)$ is an ordered pair of elements in $V(G)$, for example an edge $e \in E(G)$ may be expressed as $e = (u, v)$ where $\{u, v\} \subseteq V(G)$. An edge $e = (u, v)$ of a graph G has end vertices u and v that are said to be *adjacent* to each other. Furthermore, the edge e is said to be *incident* with vertices u and v . We refer to the cardinality of the vertex set, $|V(G)|$ as the order of a graph G , and the cardinality of the edge set $|E(G)|$ as the size of a graph G . To simplify notation, we write $V(G)$ as V and $E(G)$ as E . Note that graphs in this work are undirected and finite, that is V and E are always finite.

Two or more edges are said to be *parallel* if they are incident with the same pair of vertices. A set of two or more parallel edges of a graph G is called a parallel class. We say an edge e is a *loop* if it has the same end vertices, that is $e = (u, u)$. The degree of a vertex $v \in V$ is the number of edges incident with v and denoted by $d_G(v)$. That is, the degree of a vertex is equal to the number of edges that are incident to it. The sum of the degrees of all vertices in a finite undirected graph G is equal to twice the number of edges in the graph. This is also referred to as the *degree sum* of G , denoted by $\sum_{v \in V} d_G(v)$. A graph G is a *connected graph* if any pair of vertices in G are connected. A set of vertices V is said to be an *independent set* if none of the vertices in the set are adjacent.

A graph G is *simple* if it has no loops or parallel classes. Furthermore, a graph G is *simplified* if each parallel class is represented by a single edge and its loops are deleted. A graph G' obtained by simplifying the graph G is called the *simplification* of G . A *graph isomorphism* from G to H is a pair $\{\varphi, \theta\}$, where $\varphi : V(G) \rightarrow V(H)$ and $\theta : E(G) \rightarrow E(H)$ are bijections with the property that $e = (u, v) \in E(G)$ if and only if $\theta(e) = (\varphi(u), \varphi(v)) \in E(H)$. A graph H is a *subgraph* of G , $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The number of subgraphs in G which are isomorphic to H is denoted by $n_G(H)$. A subgraph H of G is said to be an *induced subgraph* of G if each edge of G have both its end vertices in $V(H)$ and is also an edge of H . The number of induced subgraphs in G which are isomorphic to H is denoted by $t_H(G)$ or simply t_H . A subgraph H of G is a *spanning subgraph* of G if $V(H) = V(G)$.

A *walk* in a graph G is an alternating sequence, $W = v_0 e_1 v_1 e_2 v_2 \dots e_p v_p$ of vertices and edges such that vertices v_{i-1} and v_i are the end vertices of each edge e_i . The length of a walk is the number of edges in the walk. A *trail* is a walk with distinct edges. A *path* is a walk with distinct vertices and edges. A *closed path* or a *cycle* is a path wherein the first vertex and end vertex are the same, a cycle of order 3 is also called a *triangle*. A *Hamilton cycle* in a graph G , is a walk that forms a cycle and visits each vertex exactly once. Two vertices v and u of a graph G are said to be *connected* if there is a path joining v and u in G . Additionally, if there is no path joining v and u in G , then they are *disconnected*. An edge e of the connected graph G is an *isthmus* or a *bridge* if removing e from G leaves G as a disconnected graph. A vertex v of the connected graph G is a *cut-vertex* if removing v from G leaves G as a disconnected graph. A graph G with $n \geq 3$ is *2-connected* if and only if G contains no cut-vertex. A *block* is a non-trivial connected graph which does not have a cut-vertex. Thus, G is a block if it is 2-connected when $n \geq 3$ and G is isomorphic to K_2 when $n = 2$. A block of a graph G is a subgraph H of G such that H is a maximal 2-connected subgraph.

A subgraph of G is a component of G if it is a maximal connected subgraph of G . We denote the number of components of G by $c(G)$ or simply c . A graph with two or more components is called a disconnected graph. Let $G = (V, E)$ be a graph, the rank of a graph G is given by $r(G) = |V| - c$. The closure of a set $X \subseteq E$ is given by $\text{cl}(X) = X \cup \{x \in E | r(X \cup x) = r(X)\}$. Furthermore, we say X is a closed set if

$$\text{cl}(X) = X.$$

Remark 1.4.1. Let $G = (V, E)$. An edge set $X \subseteq E$ is closed if it is the maximal subset of E that contains X , such that it induces a subgraph of G with a rank value $r(X)$.

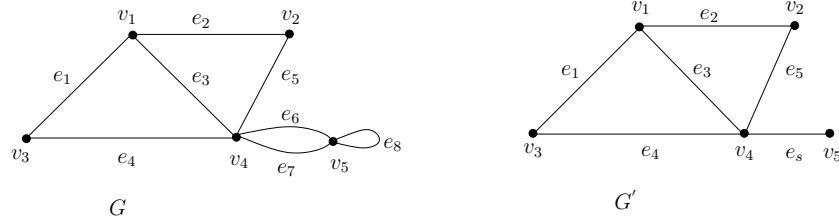


Figure 1.1: A graph G and its simplified graph G'

Example 1.4.2. In this example we use the graph shown on Figure 1.1 to clarify some concepts defined in this section. The graph G' is a simple graph, it has no parallel edges or loops. In fact, G' is the simplification of the connected graph G . In the graph G , a sequence $W = v_1 e_2 v_2 e_5 v_4 e_4 v_3$ is a walk of length 3. In fact, the walk W is a path because it has distinct vertices. In the graph G , a walk, $H = v_1 e_3 v_4 e_4 v_3 e_1 v_1$ is an induced subgraph of G because a set of edges $E(H) = \{e_1, e_3, e_4\}$ in a graph G has all its end vertices in $V(H) = \{v_1, v_3, v_4\}$. In graph G , a walk, $J = v_5 e_6 v_4 e_3 v_1 e_2 v_2 e_5 v_4 e_4 v_3 e_1 v_1$ is a spanning subgraph of G , it is a subgraph of G such that $V(J) = V(G) = \{v_1, v_2, v_3, v_4, v_5\}$. The graphs G and its simplification G' are not isomorphic because $|E(G)| \neq |E(G')|$. The graphs G and G' have a cut vertex given by v_4 and thus G' and G have 2 blocks.

A set $X_1 = \{e_1, e_3, e_4\}$ is closed because $\text{cl}(X_1) = X_1$, that is there exist no $e_i \notin E(G)$ such that $r(X_1 \cup \{e_i\}) = r(X_1)$ for any i . Meanwhile, $X_2 = \{e_1, e_3\}$ is not closed because there exists $e_4 \in E(G)$ such that $r(X_2 \cup \{e_4\}) = r(X_2)$, that is X_2 is not the maximal subset of $E(G)$ that contains X_2 and has rank $r(X_2)$.

1.5 Classes of graphs

In this section, we discuss some classes of graphs. In particular, we discuss complete graph, cycle, tree, null graph and some of their properties. For further details on the concepts discussed in this section, we refer the reader to [7].

A simple graph G is said to be a *complete graph* if every pair of distinct vertices are adjacent in G . A complete graph of order n is denoted by K_n . A graph G is said to be a *null graph* if it has no edges. A null graph of order n is denoted by N_n . A graph G is said to be a *tree* if it is a connected graph with no cycles. A tree of order n is denoted by T_n . A vertex on a tree is said to be a *leaf* or an *external vertex* if it has a degree one. A tree is said to be a *star* if there is only one vertex that is adjacent to all other vertices that are leaves. A star graph of order n is denoted by S_n . We refer to the one vertex on a star that is adjacent to all other leaves, as a root vertex. A tree is said to be a *path* P_n , if it is isomorphic to a walk with distinct vertices and edges. A path of order n is denoted by P_n . A graph G is said to be a *forest* if it is a disjoint union of trees. A forest of order n is denoted by $F_{(n,m)}$ where m is number of components. A graph G is said to be a *cycle* if it is a path wherein the first and end vertices are the same. A cycle of order n is denoted by C_n . A graph G is said to be *unicyclic* if it contains one cycle as a subgraph. A graph G is said to be a *bipartite graph* if it has a set of vertices that can be partitioned into two disjoint and independent sets V_1 and V_2 such that each edge connects a vertex in V_1 with a vertex in V_2 . A graph G is said to be a *complete bipartite graph* if it is a bipartite graph where each vertex in a vertex set V_1 connects to every vertex in vertex set V_2 . A complete bipartite graph is denoted by, $K_{|V_1|,|V_2|}$.

1.6 Graph operations

In this section, we discuss some fundamental graph operations that are relevant to this research. In particular, the join of two graphs and the K_r -gluing of two graphs, we refer the reader to [7, 44, 49] for further details.

A *graph operation* is a process of deriving a new graph from original graph or graphs. *Unary graph operations* derive a new graph from the original graph. *Binary graph operations* derive a new graph from two original graphs.

Let $G = (V(G), E(G))$ be a graph with a vertex set $V(G) = \{v_1, v_2, v_3 \dots v_n\}$ and an edge set $E(G) = \{e_1, e_2, e_3 \dots e_k\}$. Let $e_2 = (v_1, v_2)$, when we remove an edge e_2 from a graph G we obtain a graph $G \setminus e_2$ and we say that e_2 has been deleted from a graph G . Further, $V(G \setminus e_2) = V(G)$ and $E(G \setminus e_2) = E(G) - \{e_2\}$. Let $X \subseteq E$ be a set of edges, when we delete X from a graph G we obtain a graph $G \setminus X$. Further, $V(G \setminus X) = V(G)$ and $E(G \setminus X) = E(G) - X$. When we connect the two vertices in G with a new edge e_s , we obtain a graph $G + e_s$ and we say that e_s has been added to a graph G . Further, $V(G + e_s) = V(G)$ and $E(G + e_s) = E(G) + \{e_s\}$. When we remove a vertex $v_2 \in V(G)$ with all the edges incident with v_2 from a graph G , we obtain a graph $G \setminus v_2$, and we say that vertex v_2 has been deleted from a graph G . Further, $V(G \setminus v_2) = V(G) - \{v_2\}$ and $E(G \setminus v_2) = E(G) - Y$ where Y is a set of edges incident with v_2 on a graph G . When we delete $e_2 = (v_1, v_2)$ and merge together the end vertices of e_2 in G , we obtain G/e_2 , and we say that e_2 has been contracted from the graph G . Furthermore, $E(G/e_2) = E(G) - \{e_2\}$ and $V(G/e_2) = V(G) - \{v_1\}$ or $V(G/e_2) = V(G) - \{v_2\}$. When we contract $X \subseteq E$ from a graph G we obtain a graph G/X . Further, $V(G/X) = V(G) - Z$ where Z is a set of vertices that were deleted from G during the contraction process and $E(G/X) = E(G) - X$.

Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be disjoint graphs. When we connect each vertex of a graph G_1 with all vertices of a graph G_2 by a single edge, we obtain a *join of two graphs* G_1 and G_2 , denoted by $G_1 \vee G_2$. The graph $G_1 \vee G_2$ will have an edge set, $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{(u, v) | u \in V(G_1) \text{ and } v \in V(G_2)\}$ and vertex set, $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$.

Proposition 1.6.1. *Let $G = G_1 \vee G_2$ and the order of G_1 is n and the order of G_2 is m . Then, the order of $G_1 \vee G_2$ is $n + m$. Furthermore, the size of $G_1 \vee G_2$ is $|E(G_1)| + |E(G_2)| + mn$.*

Definition 1.6.2. Let $G = (V(G), E(G))$ with a vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, e_3 \dots e_k\}$. When we connect an isolated vertex $\{x\} \notin V(G)$ with each vertex of a graph G by a single edge we obtain a *vertex join* of a graph G , denoted by $\hat{G} = (V(\hat{G}), E(\hat{G}))$. That is, $\hat{G} \cong G \vee K_1$. Further, $V(\hat{G}) = V(G) \cup \{x\}$ and $E(\hat{G}) = E(G) \cup \{(x, v_1), (x, v_2), \dots, (x, v_n)\}$ where $x \notin V(G)$. A graph $\hat{G}$ has order $n + 1$ and has size $|E(G)| + n$. Furthermore, a vertex join of a path, $\hat{P}_n$ is called a *fan*. A vertex join of a cycle, $\hat{C}_n = W_{n+1}$ is called a *wheel*. A vertex join of a complete graph K_n is a complete graph K_{n+1} .

Let G be a graph with subgraphs G_1 and G_2 such that G_1 and G_2 intersect on a complete graph K_r , then G is said to be a K_r -gluing of G_1 and G_2 . We denote K_r -gluing of G_1 and G_2 by $G_1 \cup_{K_r} G_2$. Further, $V(G_1 \cup_{K_r} G_2) = V(G_1) \cup V(G_2) - V(K_r)$ and $E(G_1 \cup_{K_r} G_2) = E(G_1) \cup E(G_2) - E(K_r)$. Thus, if the order of G_1 is n and the order of G_2 is m then the order of $G_1 \cup_{K_r} G_2$ is $n + m - r$. Furthermore, the size of $G_1 \cup_{K_r} G_2$ is $|E(G_1)| + |E(G_2)| - \binom{r}{2}$, where $E(K_r) = \binom{r}{2}$.

1.7 The chromatic polynomial

In this section, we first discuss the concept of graph polynomials and their roles in graph theory. Thereafter, we discuss various methods of determining the chromatic polynomial of a graph. Additionally, we state the properties of graphs that are chromatically equivalent.

A *graph invariant* is a property that describes abstract structure and not graphical representations, such as specific labeling or drawings of the graph. A *graph polynomial* is a graph invariant whose values are polynomials. Some of the roles of graph polynomials are; to determine the chromatic number, number of bad coloured edges, the order of a graph, the size of a graph, number of connected components and number of spanning trees on a graph. Some of the graph polynomials that have been studied are characteristic polynomial, chromatic polynomial, Tutte polynomial, bad colouring polynomial, k -defect polynomial and rank generating polynomial. In this work we discuss the chromatic polynomial. For further details on graph polynomials we refer the reader to [40, 41, 44].

Recall Definition 1.1.1, a proper graph colouring is the process of assigning a colour to each vertex of a graph with a restriction that adjacent vertices are not coloured with the same colour. The chromatic polynomial of a graph G is a polynomial in λ that counts the number of ways to colour a graph G properly. The formula for the *chromatic polynomial* of a graph G is given by,

$$\chi(G; \lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{r(E) - r(X) + c}$$

where $r(X)$ is the rank of X and c is the number of components of G , we refer the reader to [50]. The chromatic number of a graph G determines minimum number of

colours that will colour vertices of a graph G properly using λ colours. The chromatic number of a graph is denoted by $\chi(G)$.

Proposition 1.7.1. [51] *Let G be a graph, then, the chromatic polynomial, $\chi(G; \lambda) = \chi(G + e; \lambda) + \chi(G/e; \lambda)$, where $G + e$ a minor that remains after adding an edge e to G , and G/e is the contraction of e in G .*

Alternatively, the chromatic polynomial of a graph G can be computed by the recursive method where we delete and contract edges of a graph G .

Proposition 1.7.2. [21] *Let G be a graph with the chromatic polynomial $\chi(G; \lambda)$, then*

- (i) *if e is a bridge of G , then $\chi(G; \lambda) = (\lambda - 1)\chi(G/e; \lambda)$;*
- (ii) *if G has a loop, then $\chi(G; \lambda) = 0$;*
- (iii) *if e is neither a loop nor a bridge in G , then $\chi(G; \lambda) = \chi(G \setminus e; \lambda) - \chi(G/e; \lambda)$.*
- (iv) *if the graph G is a disjoint union of connected graphs $G_1, G_2, \dots, G_k$,*

$$\chi(G; \lambda) = \chi(G_1; \lambda)\chi(G_2; \lambda) \cdots \chi(G_k; \lambda).$$

The recursive method shown in Proposition 1.7.2 could be tedious in computing the chromatic polynomial of some graphs that have large sizes and orders. In graph theory literature there has been explicit expressions that we may use to ease computations for some graphs. In following proposition we list some of the explicit expressions of the chromatic polynomial of some classes of graphs. In particular, the chromatic polynomial of a null graph, tree, cycle and complete graph.

Proposition 1.7.3. *Let N_n be a null graph, T_n be a tree, C_n be a cycle and K_n be a complete graph. Then,*

- (i) $\chi(N_n; \lambda) = \lambda^n$.
- (ii) $\chi(T_n; \lambda) = \lambda(\lambda - 1)^{n-1}$.
- (iii) $\chi(C_n; \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$.
- (iv) $\chi(K_n; \lambda) = \lambda(\lambda - 1)(\lambda - 1) \cdots (\lambda - n + 1)$.

Corollary 1.7.4. *Let $F_{n,m}$ be a forest on order n and size m . Then,*
 $\chi(F_{n,m}; \lambda) = \lambda^m(\lambda - 1)^{n-m}$.

The explicit expressions of the chromatic polynomial of some non-isomorphic graphs are equal. Two graphs, G and H are said to be *chromatically equivalent* or χ -*equivalent* if and only if $\chi(G; \lambda) = \chi(H; \lambda)$. The chromatic equivalence of G and H is denoted by $G \stackrel{\chi}{\sim} H$. The relation “ $\sim$ ” of being χ -equivalent is an equivalence relation on the family $\mathcal{G}$ of graphs, thus $\mathcal{G}$ is partitioned into *chromatic equivalence classes*. The chromatic equivalence class determined by a graph G will be given by $[G] = \{H \in \mathcal{G} | H \stackrel{\chi}{\sim} G\}$. A graph G is said to be χ -*unique* if $[G] = \{G\}$. That is, G is χ -unique if it is distinctively identified by its chromatic polynomial. In graph theory, the *chromaticity* of a graph G refers to the study of G being chromatically unique or not chromatically unique, and what other characteristics can be studied regarding $[G]$. Various classes of chromatically equivalent graphs have been studied and most of these classes of graphs were constructed using certain graph operations, see [8, 13, 21, 39].

Proposition 1.7.5. *Let G and H be graphs with chromatic polynomials $\chi(G; \lambda)$ and $\chi(H; \lambda)$ respectively. Then,*

- (a) *if graph H and graph G are isomorphic, $\chi(G; \lambda) = \chi(H; \lambda)$.*
- (b) *if G' is the simplification of a graph G , $\chi(G; \lambda) = \chi(G'; \lambda)$.*

The following proposition provide one of the convenient ways of computing the chromatic polynomial of a graph J which is isomorphic to a K_r -gluing of two graphs.

Proposition 1.7.6. *Let $J \cong G \cup_{K_r} H$. Then, $\chi(J) = \frac{\chi(G; \lambda)\chi(H; \lambda)}{\chi(K_r; \lambda)}$.*

In [21], Dong F. M., Koh K. M. and Teo K. L studied conditions for two graphs to be chromatically equivalent stated in the following.

Proposition 1.7.7. *Let graphs G and H be two chromatically equivalent graphs. Then,*

- (a) $|V(G)| = |V(H)|$
- (b) $|E(G)| = |E(H)|$
- (c) $\chi(G) = \chi(H)$

- (d) $n_G(C_3) = n_H(C_3)$
- (e) $i_G(C_4) - 2n_G(K_4) = i_H(C_4) - 2n_H(K_4)$
- (f) $c(G) = c(H)$
- (g) $b(G) = b(H)$
- (h) G is connected if and only if H is connected
- (i) G is 2-connected if and only if H is 2-connected

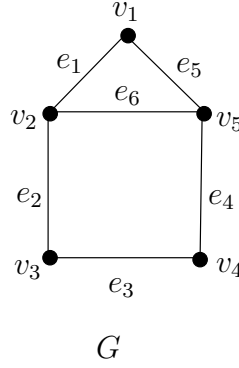


Figure 1.2: A graph, G

Example 1.7.8. In this example we calculate the chromatic polynomial of the graph G shown in Figure 1.2 using the deletion and contraction method given in Proposition 1.7.2 and then verify Proposition 1.7.3.

$$\begin{aligned}
\chi(G; \lambda) &= \chi(G \setminus e_1; \lambda) - \chi(G/e_1; \lambda) \\
&= (\lambda - 1)\chi(C_4; \lambda) - \chi(C_4; \lambda) \\
&= (\lambda - 2)\chi(C_4; \lambda) \\
&= (\lambda - 2)((\lambda - 1)^4 + (\lambda - 1)), \text{ by Proposition 1.7.3} \\
&= \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda.
\end{aligned}$$

Note that the graph $G \cong C_3 \cup_{K_2} C_4$. Thus,

$$\begin{aligned}
\chi(G; \lambda) &= \frac{\chi(C_3; \lambda)\chi(C_4; \lambda)}{\chi(K_2; \lambda)}, \text{ by Proposition 1.7.6} \\
&= \frac{((\lambda - 1)^3 + (-1)^3(\lambda - 1))((\lambda - 1)^4 + (-1)^4(\lambda - 1))}{\lambda(\lambda - 1)} \\
&= \frac{(\lambda^3 - 3\lambda^2 + 2\lambda)(\lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda)}{(\lambda^2 - \lambda)} \\
&= \lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 3\lambda + 3) \\
&= \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda.
\end{aligned}$$

The two methods of computations are now verified.

Chapter 2

Coefficients of the chromatic polynomial in various forms

2.1 Introduction

In this chapter, we discuss the coefficients of the chromatic polynomial of a graph expressed in well-known forms: the power form, tree form, and factorial form. For each of these forms we first state the chromatic polynomial expression and then the well known properties of the coefficients of the chromatic polynomial in that given form. Thereafter, we give the expression of the chromatic polynomial of a graph in cycle form and mimic the results of other three forms for cycle form. Finally, we give the explicit expression of special classes of graph in cycle form.

2.2 Coefficients of the chromatic polynomial

In this section, we discuss some coefficients of the chromatic polynomial of a graph that is expressed in each of these forms. For further details on the properties of the coefficients of the chromatic polynomial expressed in power form, tree form and factorial form, we refer the reader to [21, 31, 44, 50].

2.2.1 Power form

The chromatic polynomial of a graph expressed in a power form is given by,

$$\chi(G; \lambda) = \sum_{i=1}^n (-1)^{n-i} a_i \lambda^i \quad (2.1)$$

where a_i 's are natural numbers. We state without proof some of the properties of the coefficients of a chromatic polynomial of a graph G when it is expressed in power form. The following results are discussed in detail by Farrell E. J. in [23].

Proposition 2.2.1. *Let G be a graph of order n , size m and c number of components.*

If the graph G has chromatic polynomial, $\chi(G; \lambda) = \sum_{i=1}^n (-1)^{n-i} a_i \lambda^i$. Then,

- (i) *the coefficient of λ^n is 1.*
- (ii) *the coefficient of λ^{n-1} is $a_{n-1} = -m$.*
- (iii) *the coefficient of λ^{n-2} is $\binom{m}{2} - n_G(C_3)$.*
- (iv) *the coefficient of λ^{n-3} is $-\binom{m}{3} + (m-2)n_G(C_3) + n_G(C_4) - 2n_G(K_4)$.*
- (v) *the coefficient of λ^{n-4} is $\binom{m}{4} - \binom{m-2}{2}t_{C_3}(G) + \binom{t_{C_3}(G)}{2} - (m-3)t_{C_4}(G) + (2m-9)t_{K_4}(G) - t_{C_5}(G) - 6t_{K_5}(G) + t_{K_{2,3}}(G) + 2t_{W(5,4)}(G) + 3t_{W_5}(G)$ where $t_{C_3}(G)$ is the number of triangles C_3 , $t_{C_4}(G)$ is the number of induced cycles C_4 , $t_{K_4}(G)$ is the number of complete graphs K_4 , $t_{C_5}(G)$ is the number of pentagons C_5 , $t_{K_5}(G)$ is the number of K_5 subgraph, $t_{K_{2,3}}(G)$ is the number of $K_{2,3}$ subgraphs, $t_7(W(5,4))$ is the number of 5-vertex wheels with one spoke missing and $t_8(W_5)$ is the number of wheel W_5 .*
- (vi) *$\chi(G; \lambda)$ has no constant term.*
- (vii) *the terms in $\chi(G; \lambda)$ alternate in sign.*
- (viii) $\sum_{i=1}^n (-1)^{n-i} a_i = 0$.
- (ix) *the lowest subscript i such that $a_i > 0$ is given by $i = c$.*

In 1968, Read [44] gave a conjecture on absolute values of the coefficients of the chromatic polynomial of a graph in power form. In 1974, Hoggar S. G. [32] strengthened the unimodal conjecture to the log concave conjecture. In 2012, June H. [33] proved the unimodal and log concave conjecture. These results are stated below without proof.

Proposition 2.2.2. *Let G be a connected graph of order n with a chromatic polynomial, $\chi(G; \lambda) = \sum_{i=1}^n (-1)^{n-i} a_i \lambda^i$. Then,*

(i) *the sequence $a_1, \dots, a_n$ is unimodal; that is, for some $2 \leq j \leq n-1$,*

$$a_1 \leq a_2 \leq \dots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \dots \geq a_n.$$

(ii) *the sequence $a_1, \dots, a_n$ is log concave; that is, for $1 \leq i \leq n-1$,*

$$a_i^2 \geq a_{i-1} a_{i+1}.$$

Recall, the chromatic polynomial of a graph G in Figure 1.2, $\chi(G; \lambda) = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda$. The coefficients of this chromatic polynomial satisfies conditions in Proposition 2.2.2 because the coefficients are unimodal since $a_1 = 1 < a_2 = 6 < a_3 = 14 > a_4 = 15 > a_5 = 6$. The coefficients are also log concave because $a_4^2 \geq a_3 a_5 \Rightarrow 255 \geq 84$, $a_3^2 \geq a_2 a_4 \Rightarrow 196 \geq 90$ and $a_2^2 \geq a_1 a_3 \Rightarrow 36 \geq 14$.

2.2.2 Tree form

The chromatic polynomial of a graph expressed in a tree form is given by,

$$\chi(G; \lambda) = \sum_{i=1}^{n-1} t_i \lambda (\lambda - 1)^i \quad (2.2)$$

where t_i 's are integers. This method was introduced by Nijenhuis A. and Wilf H.S. in 1978, we refer the reader to [43]. We state without proof some of the properties of the coefficients of a chromatic polynomial of a graph G when it is expressed in tree form. The results are discussed in detail by Frank, S. and Shier, D, [24]. Recall, a block of a graph is a non-trivial connected graph or subgraph which does not have a cut-vertex.

Proposition 2.2.3. *Let G be a graph of order n and size m with b number of blocks.*

If a graph G has the chromatic polynomial, $\chi(G; \lambda) = \sum_{i=1}^{n-1} t_i \lambda (\lambda - 1)^i$. Then,

(i) *the coefficient of t_{n-1} is 1.*

- (ii) the coefficient of $|t_{n-2}|$ is $m - n + 1$.
- (iii) $\chi(G; \lambda)$ has no constant term.
- (iv) the terms in $\chi(G; \lambda)$ alternate in sign.
- (v) the lowest subscript i such that $t_i = 0$ is given by $i < b$.
- (vi) $|t_i| > 0$ for any $i = b, b + 1, \dots, n - 1$.

In 1976, Leorinc, B. and Whitehead Jr., E.G. [2] gave a conjecture on absolute values of the coefficients of the chromatic polynomial of a graph in tree form. In 1988, Read, R.C. and Tutte, W.T. [46] strengthened the unimodal conjecture to the log concave conjecture. In 2012, June H. [33] proved the unimodal and log concave conjecture. These results are stated below without proof.

Proposition 2.2.4. *Let G be a connected graph of order n with a chromatic polynomial, $\chi(G; \lambda) = \sum_{i=1}^{n-1} t_i \lambda(\lambda - 1)^i$. Then,*

- (i) the sequence $|t_1|, |t_2|, \dots, |t_{n-1}|$ is unimodal; that is, for some $2 \leq j \leq n - 2$,

$$|t_1| \leq |t_2| \leq \dots \leq |t_{j-1}| \leq |t_j| \geq |t_{j+1}| \geq \dots \geq |t_{n-1}|.$$

- (ii) the sequence $|t_1|, |t_2|, \dots, |t_{n-1}|$ is log concave; that is, for $1 \leq i \leq n - 1$,

$$t_i^2 \geq |t_{i-1}| |t_{i+1}|.$$

Recall, the chromatic polynomial of a graph G in Figure 1.2,

$$\chi(G; \lambda) = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda$$

This can be expressed in tree form as,

$$\chi(G; \lambda) = \lambda(\lambda - 1)^4 - 2\lambda(\lambda - 1)^3 + 2\lambda(\lambda - 1)^2 - \lambda(\lambda - 1).$$

The coefficients of this chromatic polynomial in tree form satisfies conditions in Proposition 2.2.4. That is, the coefficients are unimodal since $|t_4| = 1 < |t_3| = 2 = |t_2| > |t_1| = 1$. The coefficients are also log concave because $t_3^2 \geq |t_2| |t_4| \Rightarrow 4 > 2$ and $t_2^2 \geq |t_1| |t_3| \Rightarrow 4 > 2$.

2.2.3 Factorial form

The chromatic polynomial of a graph expressed in a factorial form is given by,

$$\chi(G; \lambda) = \sum_{i=\chi(G)}^n b_i \chi(K_i, \lambda) = \sum_{i=\chi(G)}^n b_i (\lambda)_i \quad (2.3)$$

where b_i 's are natural numbers, $(\lambda)_i = \chi(K_i, \lambda)$ and $\chi(G)$ is the chromatic number. This method was introduced by Whitney [51] in 1932. We state without proof some of the properties of the coefficients of a chromatic polynomial of a graph G when it is expressed in factorial form.

Proposition 2.2.5. *Let G be a graph of order n , size m and with the chromatic polynomial, $\chi(G; \lambda) = \sum_{i=\chi(G)}^n b_i (\lambda)_i$. Then,*

(i) *the coefficient of b_n is 1.*

(ii) *the coefficient of b_{n-1} is $\binom{n}{2} - |m|$.*

(iii) *$\chi(G; \lambda)$ has no constant term.*

(iv) *the lowest subscript i such that $b_i > 0$ is given by $i = \chi(G)$.*

(v) *$b_i > 0$ for all $i = \chi(G), \dots, n$.*

In 1981, Leorinc, B. and Whitehead Jr., E.G. [2] gave a conjecture on absolute values of the coefficients of the chromatic polynomial of a graph in factorial form. In 1988, Read, R.C. and Tutte, W.T. [46] strengthened the unimodal conjecture to the log concave conjecture. In 2012, June H. [33] proved the unimodal and log concave conjecture. These results are stated below without proof.

Proposition 2.2.6. *Let G be a connected graph of order n with a chromatic polynomial, $\chi(G; \lambda) = \sum_{i=\chi(G)}^n b_i (\lambda)_i$. Then,*

(i) *the sequence $b_{\chi(G)}, \dots, b_{n-1}$ is unimodal; that is, for some $\chi(G) + 1 \leq j \leq n - 1$,*

$$b_{\chi(G)} \leq \dots \leq b_j \leq b_{j+1} \geq \dots \geq b_{n-1}.$$

(ii) the sequence $b_{\chi(G)}, \dots, b_{n-1}$ is log concave; that is, for $\chi(G) \leq i \leq n-1$,

$$b_i^2 \geq b_{i-1}b_{i+1}.$$

The chromatic polynomial in Example 1.7.8 given by,

$$\chi(G; \lambda) = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda$$

This can be expressed in factorial form as,

$$\chi(G; \lambda) = (\lambda)_5 + 4(\lambda)_4 + 3(\lambda)_3.$$

Note that this form of the chromatic polynomial of G satisfies conditions in Proposition 2.2.6. That is, the coefficients are unimodal since $b_5 = 1 < b_4 = 4 > b_3 = 3$ and thus the coefficients are also log concave. The coefficients are also log concave because $b_4^2 > b_3b_5 \Rightarrow 16 > 3$.

2.3 Cycle form

In this section, we extend on the literature of forms of chromatic polynomial expression by discussing the chromatic polynomial of a graph in cycle form. Firstly, we provide an algorithm that is used to determine the chromatic polynomial of a graph in cycle form. Then, we give the general formula of a chromatic polynomial of a graph in cycle form. Thereafter, we determine the relationship between the tree form and the cycle form. Additionally, we determine some of the values of the coefficients of the chromatic polynomial of a graph in cycle form. Finally, we determine the chromatic polynomial of some of the special graphs in cycle form.

2.3.1 The chromatic polynomial of a graph in cycle form

The notion of expressing the chromatic polynomial of a graph in cycle form is not new. In 1978, Whitehead Jr. and Chao C.Y introduced the expression of chromatic polynomial of a graph using cycle form, refer to [13]. In the same text, they also discussed the relationship between power form, tree form, cycle form and factorial form. That is, converting $(\lambda)_n$ into power form, λ^n into factorial form, $\chi(C_n, \lambda)$ into power form, power form into cycle form, $\chi(T_n, \lambda)$ into power form and $\chi(T_n, \lambda)$ into

factorial form. We note that the authors were interested in the conversion methods and did not provide the explicit expression of the chromatic polynomial in cycle form neither did they discuss the coefficients of a chromatic polynomial in cycle form. The authors also did not relate the tree form to the cycle form.

In 1990, Adam A. A. [3], in his PhD thesis' appendix made relative comparison of the explicit expression of the chromatic polynomial of some classes of graphs that are expressed in power form, tree form, cycle form and factorial form. It is evident that Adam used the recursive method of deleting edges and adding edges as shown in Proposition 1.7.1 and Proposition 1.7.2, to manually compute explicit expression the chromatic polynomial of a graph in cycle form. The author has not discussed the general expression of a chromatic polynomial of a graph in cycle form neither has he discussed the coefficients of the chromatic polynomial of a graph in cycle form. In the following discussions we aim at filling these gaps.

Algorithm 2.3.1. In the following we discuss an iterative algorithm for computing the chromatic polynomial of a graph in cycle form by applying both methods of deleting and adding of edges. Let G be a graph with set of edges E and set of vertices V .

- (i) If a graph G has a Hamilton cycle of order $|V|$, then apply deletion and contraction method on edges that are not part of the Hamilton cycle of order $|V|$. Repeat the deletion and contraction method until we have a cycle of order $|V|$.
- (ii) If a graph G does not have a Hamilton cycle of order $|V|$, then repeat the addition and contraction method on G until we have a Hamilton cycle of order $|V|$. Then, proceed by applying (i).
- (iii) Apply methods (i) and (ii) to the other minors obtained, until all minors become cycles.

Remark 2.3.2. Let G be a connected graph of order $n \geq 2$. The computation of a chromatic polynomial of a graph in a power form, tree form and factorial form have one property in common. That is, only using Proposition 1.7.1 or Proposition 1.7.2, we either only add edges or delete edges to derive the respective form. In the literature, the method of adding and deleting edges are not combined, [38, 44]. Nonetheless,

with the computation of the chromatic polynomial of a graph in a cycle form both methods of adding and deleting edges may be combined depending on the existence of the Hamilton cycle. When we determine the chromatic polynomial of a graph in cycle form and in tree form, the edges are deleted in such a way that the graph does not get disconnected. Frank S. and Shier D., [24] refer to this edge selection process as the connectivity preserving selection strategy (CPSS).

We are now in a position to state some of the results of this chapter without proof. The proof of the results follows by applying Algorithm 2.3.1 on a graph, G .

Theorem 2.3.3. *Let G be a connected graph with order $n \geq 2$. The chromatic polynomial of a graph G in cycle form is given by,*

$$\chi(G; \lambda) = \sum_{i=1}^{n-1} d_i \chi(C_{i+1}, \lambda)$$

where the d_i 's are integers.

Proposition 2.3.4. *The chromatic polynomial of T_n ,*

$$\chi(T_n; \lambda) = \chi(C_n; \lambda) + \chi(C_{n-1}; \lambda).$$

Proof. Let $\chi(C_n; \lambda)$ be the chromatic polynomial of a cycle of order n with an edge e . We express the chromatic polynomial of C_n by using the deletion and contraction method or recursive method given in Proposition 1.7.2 as follows,

$$\begin{aligned} \chi(C_n; \lambda) &= \chi(C_n \setminus e; \lambda) - \chi(C_n / e; \lambda) \\ &= \chi(T_n; \lambda) - \chi(C_{n-1}; \lambda) \end{aligned}$$

Therefore, when we rearrange the terms and expressing the results with respect to $\chi(T_n; \lambda)$, we have:

$$\chi(T_n; \lambda) = \chi(C_n; \lambda) + \chi(C_{n-1}; \lambda).$$

□

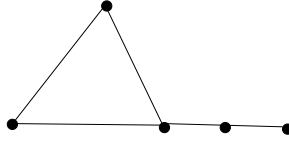


Figure 2.1: The graph G .

Example 2.3.5. In this example, we verify the results of Algorithm 2.3.1 using well known results from Proposition 2.3.4. Let G be a graph in Figure 2.1. The chromatic polynomial of a graph G in a cycle form using the recursive method in Algorithm 2.3.1 is given in Figure 2.2 where each diagram represents the chromatic polynomial of that graph. Similarly, the chromatic polynomial of a graph G in a tree form is given in Figure 2.3.

$$\begin{aligned}
& \text{Graph } G = \text{Triangle} + \text{Path of 3 edges} \\
&= \text{Graph 1} + \text{Graph 2} \\
&= (\text{Graph 3} - \text{Graph 4}) + \text{Graph 5} \\
&= \text{Graph 6} - (\text{Graph 7} + \text{Graph 8}) + \text{Graph 9} \\
&= \text{Graph 10} - \text{Graph 11} \\
&= \chi(C_5; \lambda) - \chi(C_3; \lambda)
\end{aligned}$$

Figure 2.2: The chromatic polynomial of a graph G in cycle form.

The chromatic polynomial of a graph G in cycle form is $\chi(G; \lambda) = \chi(C_5; \lambda) - \chi(C_3; \lambda)$ and in tree form is $\chi(G; \lambda) = \chi(T_5; \lambda) - \chi(T_4; \lambda)$, as shown in Figure 2.2 and Figure 2.3, respectively. Applying Proposition 2.3.4 on the tree form,

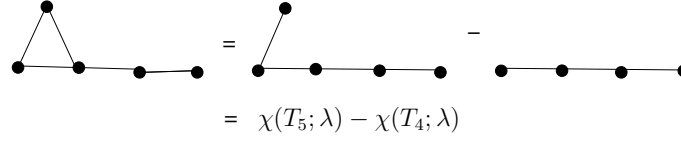


Figure 2.3: The chromatic polynomial of a graph G in tree form

$$\begin{aligned}
 \chi(G; \lambda) &= \chi(T_5; \lambda) - \chi(T_4; \lambda) \\
 &= (\chi(C_5; \lambda) + \chi(C_4; \lambda)) - (\chi(C_4; \lambda) + \chi(C_3; \lambda)) \\
 &= \chi(C_5; \lambda) - \chi(C_3; \lambda).
 \end{aligned}$$

Corollary 2.3.6. *Let G be a connected graph with order $n \geq 2$ and its chromatic polynomial expressed in cycle form as,*

$$\chi(G; \lambda) = \sum_{i=1}^{n-1} d_i \chi(C_{i+1}; \lambda)$$

and its chromatic polynomial is expressed in tree form as,

$$\chi(G; \lambda) = \sum_{i=1}^{n-1} t_i \chi(T_{i+1}; \lambda).$$

Then, $d_i = t_i + t_{i+1}$ for $1 \leq i \leq n-2$ and $t_{n-1} = d_{n-1}$.

Proof. Let G be a connected graph of order $n \geq 2$. We note that, $\chi(T_2; \lambda) = \chi(C_2; \lambda)$. By Proposition 2.3.4, the chromatic polynomial of G in tree form,

$$\chi(G; \lambda) = \sum_{i=1}^{n-1} t_i \chi(T_{i+1}, \lambda) \tag{2.4}$$

where the t_i 's are integers. Substituting results from Proposition 2.3.4,

$\chi(T_n; \lambda) = \chi(C_n; \lambda) + \chi(C_{n-1}; \lambda)$ into Equation (2.4),

$$\begin{aligned}
\chi(G; \lambda) &= \sum_{i=1}^{n-1} t_i \chi(T_{i+1}; \lambda) \\
&= t_1 \chi(T_2; \lambda) + \sum_{i=2}^{n-1} t_i \chi(T_{i+1}; \lambda) \\
&= t_1 \chi(C_2; \lambda) + \sum_{i=2}^{n-1} t_i (\chi(C_{i+1}; \lambda) + \chi(C_i; \lambda)) \\
&= t_1 \chi(C_2; \lambda) + t_2 (\chi(C_3; \lambda) + \chi(C_2; \lambda)) + t_3 (\chi(C_4; \lambda) + \chi(C_3; \lambda)) \\
&\quad + t_4 (\chi(C_5; \lambda) + \chi(C_4; \lambda)) + t_5 (\chi(C_6; \lambda) + \chi(C_5; \lambda)) + \cdots \\
&\quad + t_{n-2} (\chi(C_{n-1}; \lambda) + \chi(C_{n-2}; \lambda)) + t_{n-1} (\chi(C_n; \lambda) + \chi(C_{n-1}; \lambda)) \\
&= \chi(C_2; \lambda)(t_1 + t_2) + \chi(C_3; \lambda)(t_2 + t_3) + \chi(C_4; \lambda)(t_3 + t_4) \\
&\quad + \chi(C_5; \lambda)(t_4 + t_5) + \cdots + \chi(C_{n-1}; \lambda)(t_{n-2} + t_{n-1}) + \chi(C_n; \lambda)t_{n-1} \\
&= \chi(C_n; \lambda)t_{n-1} + \chi(C_{n-1}; \lambda)(t_{n-2} + t_{n-1}) + \cdots + \chi(C_5; \lambda)(t_4 + t_5) \\
&\quad + \chi(C_4; \lambda)(t_3 + t_4) + \chi(C_3; \lambda)(t_2 + t_3) + \chi(C_2; \lambda)(t_1 + t_2)
\end{aligned}$$

Therefore, when replacing $d_i = t_i + t_{i+1}$ for $1 \leq i \leq n-2$ and $t_{n-1} = d_{n-1}$.

$$\begin{aligned}
\chi(G; \lambda) &= \chi(C_n; \lambda)d_{n-1} + \chi(C_{n-1}; \lambda)d_{n-2} + \cdots + \chi(C_5; \lambda)d_4 + \chi(C_4; \lambda)d_3 \\
&\quad + \chi(C_3; \lambda)d_2 + \chi(C_2; \lambda)d_1, \text{ as required.}
\end{aligned}$$

□

Example 2.3.7. In this example, we verify the results of Corollary 2.3.6 by using a graph G in Figure 2.1. Recall Example 2.3.5, chromatic polynomial of a graph G in cycle form is $\chi(G; \lambda) = \chi(C_5; \lambda) - \chi(C_3; \lambda)$ and in tree form is $\chi(G; \lambda) = \chi(T_5; \lambda) - \chi(T_4; \lambda)$. Note, $t_4 = 1, t_3 = -1$ and $t_1, t_2 = 0$ then $d_4 = t_4 = 1, d_3 = t_4 + t_3 = 0$ and $d_2 = t_1 + t_2 = -1$, as required.

We have now verified the relationship between the chromatic polynomial of a graph in cycle form and in tree form. Furthermore, we verify Corollary 2.3.6, since $t_4 = 1, t_3 = -1$ and $t_1, t_2 = 0$ then $d_4 = t_4 = 1, d_3 = t_4 + t_3 = 0$ and $d_2 = t_1 + t_2 = -1$.

2.3.2 Coefficients of the chromatic polynomial of a graph in cycle form

In this section, we mimic the results of the coefficients of a chromatic polynomial of a graph in power form, tree form and factorial form for a cycle form.

Proposition 2.3.8. *Let G be a graph of order n and size $m > n$. If $\chi(G; \lambda)$ is a chromatic polynomial of a graph G that is expressed in cycle form then, the sign of the second coefficient is negative.*

Proof. Let G be a graph of order n and size $m > n$. We prove the proposition by using a two way mathematical induction on n and m . It is verified by computation in Figure 2.4 and Figure 2.5, respectively, that the proposition is true for all graphs of order $n = 4$ and $m = 5, 6$.

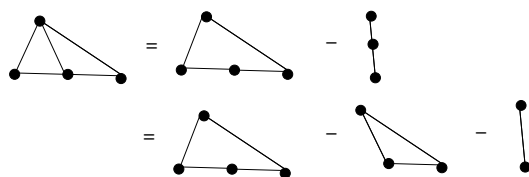


Figure 2.4: The chromatic polynomial of a graph of order $n = 4$ and size $m = 5$.

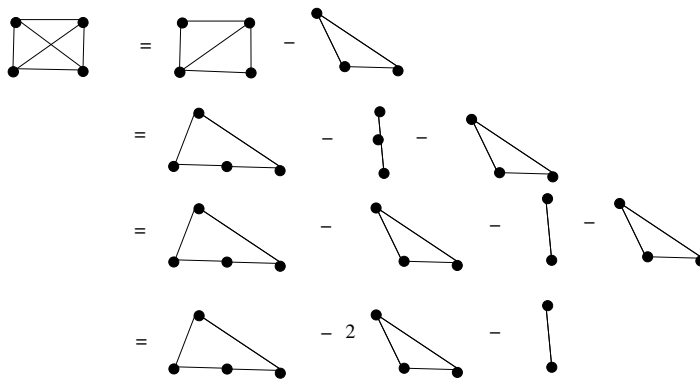


Figure 2.5: The chromatic polynomial of a graph of size $n = 4$ and size $m = 6$.

Assume that the proposition is true for a graph of order n or less. We consider a graph of order $n + 1$. We now assume that the proposition is true for a graph of order

$n + 1$ with k or less edges.

Now, consider G^* of order $n + 1$ and $k + 1$ edges. We apply algorithm 2.3.1 on G^* to get the chromatic polynomial,

$$\chi(G^*; \lambda) = \chi(G^* \setminus e; \lambda) - \chi(G^* / e; \lambda)$$

where $G^* \setminus e$ has $n + 1$ vertices and k edges and G^* / e has n vertices and k edges. By induction assumption

$$\chi(G^* \setminus e; \lambda) = \chi(C_{n+1}; \lambda) - \chi(C_n; \lambda)p_n + \cdots + \chi(C_3; \lambda)p_3 + \chi(C_2; \lambda)p_2$$

and

$$\chi(G^* / e; \lambda) = \chi(C_n; \lambda) - \chi(C_{n-1}; \lambda)p_{n-1} + \cdots + \chi(C_3; \lambda)p_3 + \chi(C_2; \lambda)p_2.$$

Therefore the sign of the second coefficient of $\chi(G^*; \lambda)$ is negative and the proposition is true for all graphs of any order n . $\square$

Note 2.3.9. By proof of Proposition 2.3.8, every deletion of an edge added 1 to d_{n-2} . Similarly, every addition of an edge subtracted 1 to d_{n-2} .

Proposition 2.3.10. *Let G be a graph of order n and size m . If a graph G has the chromatic polynomial, $\chi(G; \lambda) = \sum_{i=1}^{n-1} d_i \chi(C_{i+1}, \lambda)$. Then, the second term of $\chi(G; \lambda)$,*

$$d_{n-2} = n - m.$$

Proof. Let $\chi(G; \lambda)$ be a chromatic polynomial of a graph G of order n and size m . To compute the second coefficient of the chromatic polynomial of a graph G in cycle form, we consider two cases; when G contains a Hamilton cycle of order n and when G does not contain a Hamilton cycle of order n .

Case 1. G contains a Hamilton cycle of order n . By applying Algorithm 2.3.1, we delete all edges except the n edges on a Hamilton cycle of G . Therefore, there are $m - n$ deleted edges. By applying Note 2.3.9, $d_{n-2} = -(m - n) = n - m$.

Case 2. G does not contain a Hamilton cycle of order n . By applying Method 2.3.1, add k edges on a graph G such that it result to the desired graph with Hamilton cycle

of order n . This results to a minor G^* with $(m + k)$ edges. Moreover, by applying Note 2.3.9, we added k to the second coefficient of $\chi(G, \lambda)$. Since G^* has a Hamilton cycle with n edges, then by Case 1, the second coefficient of G^* , $d_{n-2}^* = -((m+k)-n)$. Therefore, the second coefficient of $\chi(G, \lambda)$, $d_{n-2} = d_{n-2}^* + k = -(m+k-n) + k = -(m-n) = n-m$.

□

Proposition 2.3.11. *Let G be a connected graph of order n and size m . If $\chi(G; \lambda)$ is a chromatic polynomial of a graph G that is expressed in cycle form then, $d_{n-2} \leq 1$.*

Proof. Let G be a connected graph of order n and size m . We consider 3 cases; $m > n$, and $m = n$. We note from Proposition 2.3.8 that when $m > n$, we have that $d_{n-2} < 0$. Now, consider a case where $m = n$, thus $d_{n-2} = n - m = 0$. In conclusion, for any connected graph G with order n and size m , $d_{n-2} \leq 1$.

□

2.3.3 Cycle form for special graphs

In this section, we discuss the chromatic polynomial of some special classes of graphs expressed in cycle form. We aim at generalising some coefficients of the chromatic polynomial of these classes graphs in cycle form.

Proposition 2.3.12. *Let G be a connected graph with $n \geq 3$ and the chromatic polynomial, $\chi(G; \lambda) = \sum_{i=1}^{n-1} d_i \chi(C_{i+1}, \lambda)$. Then, $G \cong T_n$ if and only if $d_{n-1} = 1$, $d_{n-2} = 1$ and $d_i = 0$ for $1 \leq i \leq n-3$.*

Proof. Let G be a connected graph with $n \geq 3$. If $G = T_n$ then $\chi(T_n; \lambda) = \chi(C_n; \lambda) + \chi(C_{n-1}; \lambda)$ and therefore $d_{n-2} = 1$, $d_{n-1} = 1$ and $d_i = 0$ for $1 \leq i \leq n-3$. Now consider the chromatic polynomial of a connected graph G that is expressed in cycle form such that $d_{n-2} = 1$, $d_{n-1} = 1$ and $d_i = 0$ for $1 \leq i \leq n-3$. That is, $\chi(G; \lambda) = \chi(C_n; \lambda) + \chi(C_{n-1}; \lambda)$. By Theorem 2.3.3 the chromatic polynomial of a tree of order n is given by $\chi(T_n; \lambda) = \chi(C_n; \lambda) + \chi(C_{n-1}; \lambda)$. Therefore, by Proposition 2.3.4 a graph G and T_n have the same chromatic polynomial and thus $G \cong T_n$ since the chromatic polynomial of T_n is unique to its family of graphs, [21].

□

Proposition 2.3.13. [24] *A graph G is a bipartite graph if and only if $\chi(G; 2) \neq 0$.*

Proposition 2.3.14. *Let G be a graph with a chromatic polynomial $\chi(G; \lambda) = \sum_{i=1}^{n-1} d_i \chi(C_{i+1}; \lambda)$. Then, a graph G is a bipartite graph if and only if $\sum_{i=1}^{n-1} d_i (1 + (-1)^{i+1}) \neq 0$.*

Proof. Let G be a graph with a chromatic polynomial,

$$\chi(G; \lambda) = \sum_{i=1}^{n-1} d_i \chi(C_{i+1}; \lambda)$$

By Proposition 2.3.13, a graph G is a bipartite graph if and only if $\chi(G; 2) \neq 0$. Therefore,

$$\chi(G; 2) \neq 0 \Leftrightarrow \sum_{i=1}^{n-1} d_i (1 + (-1)^{i+1}) \neq 0.$$

□

Example 2.3.15. In this example we verify the results of Proposition 2.3.14. In Figure 2.6, each graph represent the chromatic polynomial of that graph. Thus, $\chi(K_4; \lambda) = \chi(C_4; \lambda) - 2\chi(C_3; \lambda) - \chi(C_2; \lambda)$. When we apply Proposition 2.3.14 on $\chi(K_4; \lambda)$, we note that $2d_1 + 2d_3 = 2(-1) + 2(1) = 0$. Thus, K_4 is not a bipartite graph. Similarly, by Proposition 2.3.14, $\chi(T_n; \lambda) = \chi(C_n; \lambda) + \chi(C_{n-1}; \lambda)$, we note that $d_{n-1} = 1$ and $d_{n-2} = 1$. Thus, $d_{n-1}(1 + (-1)^n) + d_{n-2}(1 + (-1)^{n-1}) = 2 + (-1)^n + (-1)^{n-1} = 2 \neq 0$. Therefore, T_n is a bipartite graph when n is even or odd.

Definition 2.3.16. Let G be a graph with n vertices and m edges. The graph $G + k$ is obtained by adjoining k vertices and then connecting them to a vertex or vertices of a graph G in no order. We refer to $G + k$ as a *graph G with k bridges*. The number of edges of $G + k$ is given by $|E(G + k)| = m + k$ and the number of vertices are given by $|V(G + k)| = n + k$.

Example 2.3.17.

Recall Proposition 1.7.2, the chromatic polynomial of a graph G with k bridges in general is given by $\chi(G + k, \lambda) = (\lambda - 1)^k \chi(G, \lambda)$.

Proposition 2.3.18. *Let $C_n + k$ be a cycle C_n with k bridges. The chromatic polynomial of $C_n + k$ in cycle form is given by,*

$$\chi(C_n + k; \lambda) = \chi(C_{n+k}; \lambda) + (-1)^n \chi(C_{k+1}; \lambda).$$

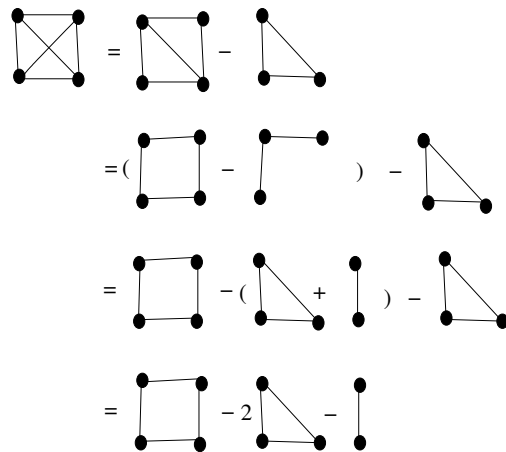


Figure 2.6: K_4 in cycle form.

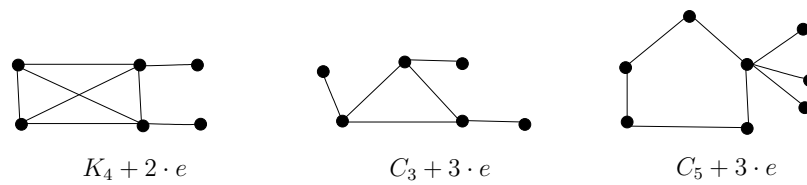


Figure 2.7: Graphs with bridges.

Proof. Let $C_n + k$ be a cycle C_n with k bridges. We recall from Proposition 1.7.2 that the chromatic polynomial of a graph $C_n + k$ with k bridges can be expressed as:

$$\begin{aligned}
\chi(C_n + k; \lambda) &= (\lambda - 1)^k \chi(C_n; \lambda) \\
&= (\lambda - 1)^k ((\lambda - 1)^n + (-1)^n (\lambda - 1)) \\
&= (\lambda - 1)^{n+k} + (-1)^n (\lambda - 1)^{k+1} \\
&= (\lambda - 1)^{n+k} + (-1)^n (\lambda - 1)^{k+1} + (-1)^{n+k} (\lambda - 1) + (-1)(-1)^{n+k} (\lambda - 1) \\
&= (\lambda - 1)^{n+k} + (-1)^{n+k} (\lambda - 1) + (-1)^n (\lambda - 1)^{k+1} + (-1)(-1)^{n+k} (\lambda - 1) \\
&= (\lambda - 1)^{n+k} + (-1)^{n+k} (\lambda - 1) + (-1)^n (\lambda - 1)^{k+1} + (-1)^n (-1)^{k+1} (\lambda - 1) \\
&= ((\lambda - 1)^{n+k} + (-1)^{n+k} (\lambda - 1)) + (-1)^n ((\lambda - 1)^{k+1} + (-1)^{k+1} (\lambda - 1)) \\
&= \chi(C_{n+k}; \lambda) + (-1)^n \chi(C_{k+1}; \lambda)
\end{aligned}$$

□

Example 2.3.19. In this example we verify the results of Proposition 2.3.18 by determining the chromatic polynomial of $C_4 + 4$. Recall Proposition 1.7.2, the chromatic polynomial of $C_4 + 4$ can be expressed as,

$$\begin{aligned}
\chi(C_4 + 4; \lambda) &= (\lambda - 1)^4 \chi(C_4; \lambda) \\
&= (\lambda - 1)^4 ((\lambda - 1)^4 + (-1)^4 (\lambda - 1)) \\
&= (\lambda - 1)^{4+4} + (\lambda - 1)^{4+1} \\
&= ((\lambda - 1)^8 + (\lambda - 1)) + ((\lambda - 1)^5 - (\lambda - 1)) \\
&= \chi(C_8; \lambda) + \chi(C_5; \lambda).
\end{aligned}$$

Now, we determine the chromatic polynomial of G by applying Proposition 2.3.18 where $n, k = 4$.

$$\begin{aligned}
\chi(C_4 + 4; \lambda) &= \chi(C_{4+4}; \lambda) + \chi(C_{4+1}; \lambda) \\
&= \chi(C_8; \lambda) + \chi(C_5; \lambda),
\end{aligned}$$

verifying Proposition 2.3.18.

Proposition 2.3.20. *Let G be a connected graph of order n . Then, if $\chi(G; \lambda) = \sum_{i=1}^{n-1} d_i \chi(C_{i+1}; \lambda)$ and $G + k$ is a graph G with k bridges then*

$$\chi(G + k; \lambda) = \sum_{i=1}^{n-1} d_i \chi(C_{i+k+1}; \lambda) + \chi(C_{k+1}; \lambda) \sum_{i=1}^{n-1} d_i (-1)^{i+1}$$

Proof. Let $G+k$ be a graph with k bridges and $\chi(G; \lambda) = \chi(C_n; \lambda) + \chi(C_{n-1}; \lambda)d_{n-2} + \cdots + \chi(C_3; \lambda)d_2 + \chi(C_2; \lambda)d_1$. Recall Proposition 2.3.18, $(\lambda-1)^k \chi(C_n; \lambda) = \chi(C_{n+k}; \lambda) + (-1)^n \chi(C_{k+1}; \lambda)$. We know that a graph G with k bridges has a chromatic polynomial,

$$\begin{aligned}
\chi(G+k; \lambda) &= (\lambda-1)^k \chi(G; \lambda) \\
&= (\lambda-1)^k (\chi(C_n; \lambda) + \chi(C_{n-1}; \lambda)d_{n-2} + \cdots + \chi(C_3; \lambda)d_2 + \chi(C_2; \lambda)d_1) \\
&= (\lambda-1)^k \chi(C_n; \lambda) + (\lambda-1)^k \chi(C_{n-1}; \lambda)d_{n-2} + \cdots + (\lambda-1)^k \chi(C_3; \lambda)d_2 \\
&\quad + (\lambda-1)^k \chi(C_2; \lambda)d_1 \\
&= \chi(C_{n+k}; \lambda) + (-1)^n \chi(C_{k+1}; \lambda) + \chi(C_{n+k-1}; \lambda) + (-1)^{n-1} \chi(C_{k+1}; \lambda)d_{n-2} \\
&\quad + \cdots + (\chi(C_{3+k}; \lambda) + (-1)^3 \chi(C_{k+1}; \lambda))d_2 + \chi(C_{2+k}; \lambda) + (-1)^2 \chi(C_{k+1}; \lambda)d_1 \\
&= \chi(C_{n+k}; \lambda) + \chi(C_{n+k-1}; \lambda)d_{n-2} + \cdots + \chi(C_{3+k}; \lambda)d_2 + \chi(C_{2+k}; \lambda)d_1 \\
&\quad + \chi(C_{k+1}; \lambda)((-1)^n + (-1)^{n-1}d_{n-2} + \cdots + (-1)^3d_2 + (-1)^2d_1) \\
&= \sum_{i=1}^{n-1} d_i \chi(C_{i+k+1}; \lambda) + \chi(C_{k+1}; \lambda) \sum_{i=1}^{n-1} d_i (-1)^{i+1}
\end{aligned}$$

□

2.4 Conclusion

In this chapter, we gave an algorithm for computing the general expression of the chromatic polynomial in cycle form and stated the relationship between cycle form and tree form. Furthermore, we calculated the values of the first and second coefficients of the chromatic polynomial in cycle form. Finally, we gave the explicit expression of the chromatic polynomial of some special graphs in cycle form.

There are other questions that arise from these discussion: What are values for other coefficients besides the first and second coefficients? What is the sum of the coefficients and does it have a special meaning? Could other coefficients be zero besides the second coefficient? We note that, $\chi(K_3 \cup K_1, \lambda) = \chi(C_4, \lambda) + \chi(C_3, \lambda) - \chi(C_2, \lambda)$, then what is the general expression of the chromatic polynomial of disconnected graphs in cycle form, since in power form we multiply the chromatic polynomials of the components if the graph is disconnected?

Chapter 3

On the Join of Graphs and Chromatic Equivalence

3.1 Introduction

In this chapter, we discuss a well-known formula for the chromatic polynomial of the join of graphs. In particular, we focus on the application of the vertex join operation to derive other families of joins of graphs, namely bivertex join of a graph. Thereafter, we study the chromatic equivalence on repeated vertex join and bivertex join of a graph. In addition, we give certain conditions for a graph to be in the chromatic equivalence class of these families. Finally, we study a list of well-known pairs of non-trivial chromatic equivalent graphs and remove some pairs from the list.

3.2 Chromatic polynomial of join of graphs

In this section, we start by discussing the well known chromatic polynomial of joins of graphs which uses umbral product operation. We then discuss the chromatic equivalence of some joins of graphs.

One natural question that arises in the study of chromatic polynomials is whether, a graph constructed using a graph operation, can have the chromatic polynomial expressed in terms of the chromatic polynomials of the original graphs. This question was partially addressed by Read [44], for one operation, join of two graphs. We now state this result without proof.

Proposition 3.2.1. [44] *Let G_1 and G_2 be any two graphs where $\chi(G_1; \lambda)$ and*

$\chi(G_2; \lambda)$ are expressed in factorial form. Then,

$$\chi(G_1 \vee G_2; \lambda) = \chi(G_1; \lambda) \otimes \chi(G_2; \lambda)$$

where $\otimes$ is the umbral product and $(\lambda)_i \otimes (\lambda)_j = (\lambda)_{i+j} = \chi(K_{i+j}, \lambda)$.

Example 3.2.2. In this example, we apply Proposition 3.2.1 to compute the chromatic polynomial of the join of $T_2 \vee T_2$.

$$\begin{aligned} \chi(T_2 \vee T_2; \lambda) &= \chi(T_2; \lambda) \otimes \chi(T_2; \lambda) \\ &= (\lambda)_2 \otimes (\lambda)_2 \\ &= (\lambda)_4 \\ &= \chi(K_4; \lambda). \end{aligned}$$

This result can also be visualised by the diagrams in Figure 3.1.

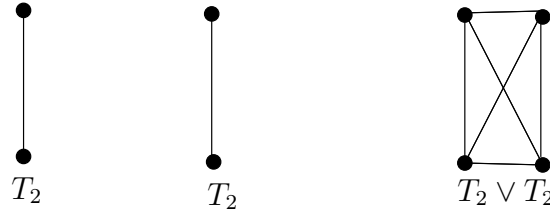


Figure 3.1: The join, $T_2 \vee T_2$.

In 2015, Boon Leong Ng and Fengming Dong, further explored chromatic polynomials of joins of graphs, see [42]. The authors applied Proposition 3.2.1 to study chromatic equivalence of join of graphs, which we now state without proof.

Proposition 3.2.3. *Let G , H_1 and H_2 be graphs. Then the graphs $G \vee H_1$ and $G \vee H_2$ are chromatically equivalent if and only if H_1 and H_2 are chromatically equivalent.*

3.3 Chromatic equivalence class of the repeated vertex join of a graph

In this section, we recall from literature, the chromatic polynomial of a special class of join of graphs called the vertex join of a graph. Thereafter, we use the chromatic

polynomial of the vertex join of the graph to determine the chromatic polynomial of the graph, $K_n \vee G$. Finally, we give necessary conditions for a graph to be in the chromatic equivalence class, $[K_n \vee G]$.

Recall Definition 1.6.1, a vertex join of a graph G is a join of K_1 and G . That is, a vertex join of a graph is given by $\hat{G} \cong K_1 \vee G$. It is known that the chromatic polynomial of $\hat{G}$ is expressed in terms of $\chi(G; \lambda)$ and $\chi(K_1; \lambda)$. We now state this result in the following proposition without proof.

Proposition 3.3.1. [49] *Let $\hat{G}$ be the vertex join of a graph G . Then the chromatic polynomial of $\hat{G}$,*

$$\chi(\hat{G}; \lambda) = \lambda \chi(G; \lambda - 1).$$

Corollary 3.3.2. *Let $[G]$ be a chromatic equivalence class of a graph G and let $H \in [G]$. Then, $\hat{H} \in [\hat{G}]$.*

Definition 3.3.3. The *repeated vertex join* of a graph G is obtained by adjoining a new isolated vertex x_2 and then connecting it to each vertex of $\hat{G}$ by a single edge. We denote the repeated vertex join of a graph G by $\hat{\hat{G}}$ or simply $\hat{G}^2$. Thus, $\hat{\hat{G}} \cong K_2 \vee \hat{G}$.

Definition 3.3.4. The *q -th repeated vertex join* of the graph G , is the graph obtained by adjoining new disjoint vertices $x_1, x_2, \dots, x_q$ to each vertex of G by a single edge and then connecting each pair of vertices $x_1, x_2, \dots, x_q$ by a single edge. We denote the q -th repeated vertex join of a graph G by $\hat{G}^q$.

Proposition 3.3.5. *Let G be a graph and K_q be a complete graph of order q . Then, $\hat{G}^q \cong K_q \vee G$.*

Proof. Let G be a graph and K_q be a complete graph of order q . In construction of $\hat{G}^q$, we adjoin new disjoint vertices $x_1, x_2, \dots, x_q$ to each vertex of G by a single edge and then connecting each pair of vertices $x_1, x_2, \dots, x_q$ by a single edge. In a graph $\hat{G}^q$, the vertices $x_1, x_2, \dots, x_q$ induce a complete subgraph graph K_q . Thus, in a graph $\hat{G}^q$, each vertex of the induced subgraph K_q is joined to each vertex of the graph G by an edge. Therefore, $\hat{G}^q \cong K_q \vee G$. □

Proposition 3.3.6. *Let G be a graph with n vertices and m edges, $\hat{G}^q$ be a q -th repeated vertex join of G and $\hat{\hat{G}}^q$ be the q -th repeated bivertex join of G . Then,*

$$(i) \quad |V(\hat{G}^q)| = n + q.$$

$$(ii) \quad |E(\hat{G}^q)| = \frac{q(q-1)}{2} + m + nq.$$

Proof. Let G be a graph with n vertices and m edges and $\hat{G}^q$ be the q -th repeated bivertex join of G .

$$(i) \quad |V(\hat{G}^q)| = |V(G)| + |V(K_q)| = n + q.$$

(ii) In determining the number of edges of $\hat{G}^q$, $E(\hat{G}^q) = E(K_q \vee G)$, we consider the number of edges of G , $|E(G)| = m$, the number of edges of $|E(K_q)| = \frac{q(q-1)}{2}$ and the number of edges that results from the join operation by Proposition 1.6.1. Since the join operation join every vertex of G with every vertex of K_q by an edge and we are given n vertices from G and q vertices from K_q , we have nq edges. Thus, $|E(\hat{G}^q)| = \frac{q(q-1)}{2} + m + nq$.

□

We are now in a position to discuss some results of this chapter; the explicit expression of chromatic polynomial of $\hat{G}^q$, and necessary conditions for a graph to be in the chromatic equivalence class $[\hat{G}^q]$.

Theorem 3.3.7. *Let $\hat{G}^q$ be a q -th repeated vertex join of the graph G . Then, the chromatic polynomial, $\chi(\hat{G}^q; \lambda) = \chi(K_q; \lambda)\chi(G; \lambda - q)$.*

Proof. Let $\hat{G}^q$ be a q -th repeated vertex join of the graph G . By Proposition 3.3.1, $\chi(\hat{G}; \lambda) = \lambda\chi(G; \lambda - 1)$. Therefore, the vertex join of $\hat{G}$ is given by $\chi(\hat{G}^2; \lambda) = \lambda(\lambda - 1)\chi(G; \lambda - 2)$. Repeating this q times we get,

$$\begin{aligned} \chi(\hat{G}^q; \lambda) &= \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - q)\chi(G; \lambda - q) \\ &= \chi(K_q; \lambda)\chi(G; \lambda - q). \end{aligned}$$

□

Corollary 3.3.8. *Let G and H be graphs. Then, $H \in [G]$ implies that $\hat{H}^q \in [\hat{G}^q]$.*

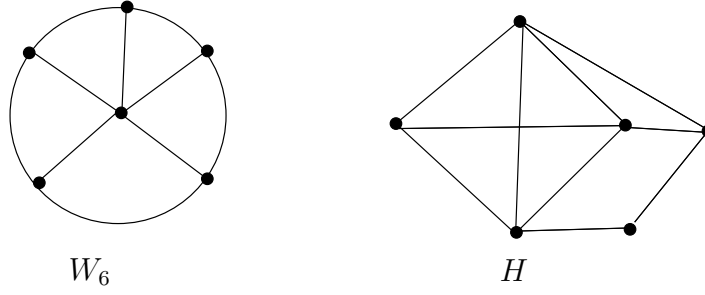


Figure 3.2: Chromatic equivalent pair.

We note from Corollary 3.3.8 that some of the elements in $[\hat{G}^q]$ are derived by applying q -th vertex join operation on elements in the chromatic equivalence class, $[G]$. However, a question arises; could we have elements in $[\hat{G}^q]$ that are not derived from q -th vertex join operation? When we observe the graphs, W_6 and H in Figure 3.2, we note that $W_6 \cong \hat{C}_5$, is derived from a vertex join operation. However, $H \not\cong \hat{J}$ for any J , but $H \in [\hat{C}_5]$. Therefore, there may exist some graphs in $[\hat{G}^q]$ that are not derived from q -th vertex operation.

Proposition 3.3.9. *Let $\hat{G}^q$ be a q -th repeated vertex join of the graph G . Then, the lower bound value for the number of elements in $[\hat{G}^q]$ is given by $|[G]|$. That is, $|[G]| \leq |[\hat{G}^q]|$.*

Proof. Let $\hat{G}^q$ be a q -th repeated vertex join of the graph G . We note that when we apply the q -th repeated vertex join operation on elements in the chromatic equivalence class, $[G]$ we get elements of the chromatic equivalence class $[\hat{G}^q]$. That is, the minimum number of elements in $[\hat{G}^q]$ is $|[G]|$. Furthermore, we also note that in some chromatic equivalence classes, $[\hat{G}^q]$, there are elements that are not derived from the q -th repeated vertex join operation. Therefore, in this case, $|[G]| < |[\hat{G}^q]|$. Similarly, if all elements in $[\hat{G}^q]$ are derived from the q -th repeated vertex join operation then, $|[G]| = |[\hat{G}^q]|$. $\square$

We discuss necessary conditions for a graph to be in the chromatic equivalence class, $[\hat{G}^q]$. To prove these conditions, we state the following lemma without proof.

Lemma 3.3.10. *[9] Let G be a graph. If there exists a bridge on a graph G . Then $\chi(G; \lambda)$ is divisible by $(\lambda - 1)^2$.*

Proposition 3.3.11. *Let $J \in [\hat{G}^q]$ where G is a graph with $n \geq 2$ vertices, m edges and a chromatic number, $\chi(G) = p$. Then,*

$$(i) \quad |V(J)| = n + q.$$

$$(ii) \quad |E(J)| = \frac{q(q-1)}{2} + m + nq.$$

$$(iii) \quad \chi(J) = p + q.$$

(iv) J has no bridge.

(v) J is 2-connected.

$$(vi) \quad \text{The number of triangles in } J \text{ is given by, } n_J(C_3) = n_G(C_3) + n_{K_q}(C_3) + mq + \frac{nq(q-1)}{2}.$$

Proof. Let $J \in [\hat{G}^q]$ where G is a graph with n vertices, m edges and a chromatic number, $\chi(G) = p$. By Proposition 3.3.5, $\hat{G}^q \cong K_q \vee G$.

(i) $|V(\hat{G}^q)| = n + q$, by Proposition 3.3.6. Therefore, we deduce that $|V(J)| = n + q$.

(ii) $|E(J)| = \frac{q(q-1)}{2} + m + nq$, by Proposition 3.3.6. We deduce that $|E(J)| = \frac{q(q-1)}{2} + m + nq$.

(iii) Every vertex of K_q connects to every vertex of G . That is, the minimum number of colours needed to colour the graph $\hat{G}^q$ properly is the sum of $\chi(G) = p$ and $\chi(K_q) = q$. Thus, $\chi(\hat{G}^q) = p + q$. We deduce that $\chi(J) = p + q$.

(iv) $\hat{G}^q$ has no bridge because each of its vertices has a degree greater or equal to $q + 1$, $q \geq 1$, since $\hat{G}^q \cong K_q \vee G$. That is, by Lemma 3.3.10, the chromatic polynomial, $\chi(\hat{G}^q, \lambda)$ is not divisible by $(\lambda - 1)^2$. Thus, the chromatic polynomial, $\chi(J, \lambda)$ is not divisible by $(\lambda - 1)^2$. We deduce that J does not have a bridge.

(v) $\hat{G}^q$ is 2-connected because each vertex of G is connected to all vertices of K_q , since $\hat{G}^q \cong K_q \vee G$, and thus, removing or deleting any vertex of $\hat{G}^q$ will result to a minor that is still connected. Therefore, by applying Proposition 1.7.7, J is also 2-connected.

(vi) In determining the number of triangles of $\hat{G}^q$, $n_{\hat{G}^q}(C_3)$, we consider the number of triangles of G , $n_G(C_3)$, the number of triangles of K_q , $n_{K_q}(C_3)$ and the number of triangles that results from the join operation. The join operation join every

vertex of G with every pair of vertices that belong to an edge of K_q . This process results to $n|E(K_q)| = \frac{nq(q-1)}{2}$ number of triangles. Similarly, the join operation join every vertex of $\hat{K}_q$ with every pair of vertices that belong to an edge of G . This process results to mq number of triangles. Adding up all triangles, we get, $n_{\hat{G}^q}(C_3) = n_G(C_3) + n_{K_q}(C_3) + mq + \frac{nq(q-1)}{2}$. We deduce that $n_J(C_3) = n_G(C_3) + n_{K_q}(C_3) + mq + \frac{nq(q-1)}{2}$.

□

Recall Proposition 3.2.3, the chromatic equivalence of G_1 and G_2 implies chromatic equivalence of $K_n \vee G_1$ and $K_n \vee G_2$. By Proposition 1.7.3, all non-isomorphic trees with the same order are chromatically equivalent. Therefore, all the graphs, $K_q \vee T_n$, for any tree T_n , are chromatically equivalent. In the following example, we use $[P_6]$ to construct some of the elements in $[\hat{P}_6^2]$.

Example 3.3.12. The set of non-isomorphic trees of order 6 is given in Figure 3.3.

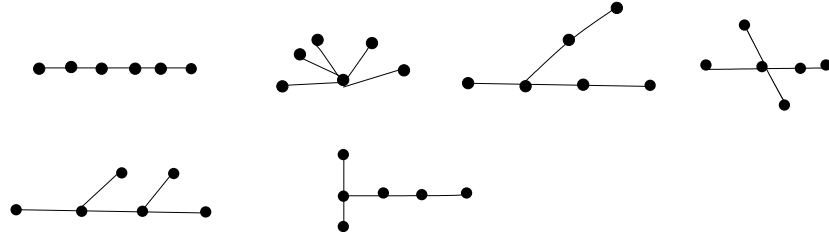


Figure 3.3: Elements of the chromatic equivalent class, $[P_6]$

Applying Corollary 3.3.8 to the non-isomorphic trees of order 6, presented by graphs in Figure 3.3, we get the derived the graphs shown in Figure 3.4.

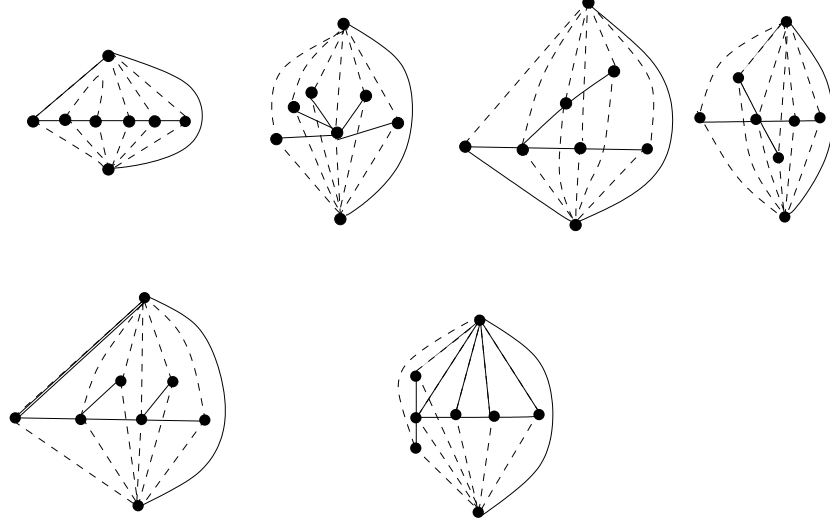


Figure 3.4: Elements of $[\hat{P}_6^2]$.

3.4 Chromatic equivalence class of the bivertex join of a graph

In this section, we use the vertex join of a graph, G to construct one of our special join, $G \vee \ddot{N}_2^{q-1}$ where $\ddot{N}_2^{q-1} \cong \underbrace{N_2 \vee N_2 \vee \cdots \vee N_2}_{q\text{-times}}$. We illuminate the structural differences and similarities between $G \vee \ddot{N}_2^{q-1}$ and $K_{2q} \vee G$. Additionally, we determine the chromatic polynomial of the graph, $N_2 \vee G$. Moreover, we determine necessary conditions for a graph to be in the chromatic equivalence class, $[N_2 \vee G]$.

Recall Definition 1.6.1, a vertex join of a graph G is given by $\hat{G} \cong K_1 \vee G$.

Definition 3.4.1. The *bivertex join* of a graph G , is obtained by adjoining new isolated pair of vertices $\{x_1, x_2\} \notin V(G)$ with each vertex of a graph G by a single edge. Thus, $\ddot{G} \cong N_2 \vee G$.

Definition 3.4.2. Let G be a graph and $\{x_1, x_2\}, \{x_3, x_4\}, \dots, \{x_{2q-1}, x_{2q}\}$ be q number of pairs of vertices that are disjoint and disconnected, such that each pair of vertices induces a null graph, N_2 . Begin by connecting each vertex in $\{x_1, x_2\}$ to every vertex in G by an edge, resulting in a new graph, $\ddot{G}$. Next, proceed by connecting each vertex within $\{x_3, x_4\}$ to all vertices in $\ddot{G}$ with unique edges, forming $\ddot{G}^2$. Continuing in this approach of adjoining pairs of vertices, repeating for q iterations. This process results in the q -th repeated bivertex join of the initial graph G , denoted as $\ddot{G}^q$.

Proposition 3.4.3. Let G be a graph and N_2 be a null graph of order 2. Then, $\ddot{G}^q \cong G \vee \ddot{N}_2^{q-1}$.

Proof. Let G be a graph and N_2 be a null graph of order 2. In construction of $\ddot{G}^q$, we sequentially adjoin, through iterations, one of the q number of pairs of vertices $\{x_1, x_2\}, \{x_3, x_4\}, \dots, \{x_{2q-1}, x_{2q}\}$ to a derived graph, such that the i -th pair of vertices adjoin to every vertex in the $(i-1)$ -th derived graph where $i \in \{1, 2, \dots, q\}$. In a graph $\ddot{G}^q$, the q number of pairs of vertices $\{x_1, x_2\}, \{x_3, x_4\}, \dots, \{x_{2q-1}, x_{2q}\}$, induce a graph $\ddot{N}_2^{q-1}$ which joins to a graph, G . Therefore, $\ddot{G}^q \cong G \vee \ddot{N}_2^{q-1}$. $\square$

Lemma 3.4.4. Let N_2 be a null graph of order 2. Then,

$$(i) \quad |V(\ddot{N}_2^{q-1})| = 2q.$$

$$(ii) \quad |E(\ddot{N}_2^{q-1})| = 2q(q-1).$$

Proof. Let N_2 be a null graph of order 2.

- (i) When constructing $\ddot{N}_2^{q-1}$, on each of the q iterations, we adjoin a pair of vertices. Therefore, $|V(\ddot{N}_2^{q-1})| = 2q$.
- (ii) When constructing $\ddot{N}_2^{q-1}$, all vertices are adjacent except for the q number of pairs of vertices $\{x_1, x_2\}, \{x_3, x_4\}, \dots, \{x_{2q-1}, x_{2q}\}$. Therefore,

$$\begin{aligned} |E(\ddot{N}_2^{q-1})| &= \binom{2q}{2} - q \\ &= \frac{2q(2q-1)}{2} - q = 2(q^2 - q) = 2q(q-1). \end{aligned}$$

$\square$

Proposition 3.4.5. *Let $\ddot{G}^q$ be the q -th repeated bivertex join of G where G has n vertices and m edges. Then,*

$$(i) \quad |V(\ddot{G}^q)| = n + 2q.$$

$$(ii) \quad |E(\ddot{G}^q)| = m + 2q(q - 1) + 2nq.$$

Proof. Let $\ddot{G}^q$ be the q -th repeated bivertex join of G where G is a graph with n vertices and m edges.

$$(i) \quad \text{When applying Proposition 1.6.1, } |V(\ddot{G}^q)| = |V(G \vee \ddot{N}_2^{q-1})| = |V(G) + V(\ddot{N}_2^{q-1})| = n + 2q.$$

(ii)

$$\begin{aligned} |E(\ddot{G}^q)| &= |E(G \vee \ddot{N}_2^{q-1})| \\ &= |E(G) + E(\ddot{N}_2^{q-1})| + |V(G)||V(\ddot{N}_2^{q-1})|, \text{ by Proposition 1.6.1} \\ &= m + 2q(q - 1) + n(2q) \\ &= m + 2q(q - 1) + 2nq. \end{aligned}$$

□

Before stating the next result of this section, we illustrate the difference and similarities in the derived graphs resulting from q -th repeated vertex join, $\hat{G}^q$ and q -th repeated bivertex join of a graph G .

In Figure 3.5 we illustrate the q -th repeated vertex join of a graph G , $\hat{G}^q$ for $q = 1, 2$.

In Figure 3.6 we illustrate the q -th repeated vertex join of a graph G , $\hat{G}^q$ for $q = 2, 4$.

In Figure 3.7 we illustrate the q -th repeated bivertex join of a graph G , $\ddot{G}^q$ for $q = 3, 4$.

Proposition 3.4.6. *Let G be a graph with n vertices and m edges, $\hat{G}^q$ be a q -th repeated vertex join of G and $\ddot{G}^q$ be the q -th repeated bivertex join of G . Then,*

$$(i) \quad |V(\ddot{G}^q)| = |V(\hat{G}^{2q})|.$$

$$(ii) \quad |E(\ddot{G}^q)| = |E(\hat{G}^{2q})| + q.$$

Proof. Let G be a graph with n vertices and m edges, $\hat{G}^q$ be a q -th repeated vertex join of G and $\ddot{G}^q$ be the q -th repeated bivertex join of G .

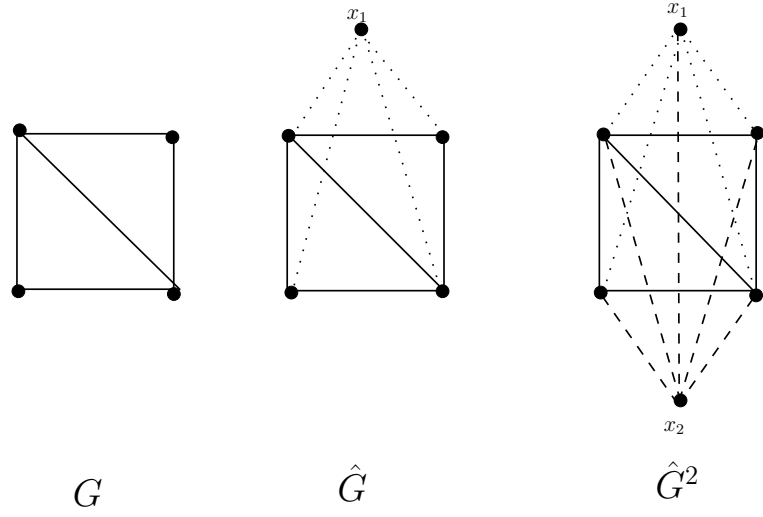


Figure 3.5: A graph G with its q -th repeated vertex join graphs; $\hat{G}$, $\hat{G}^2$.

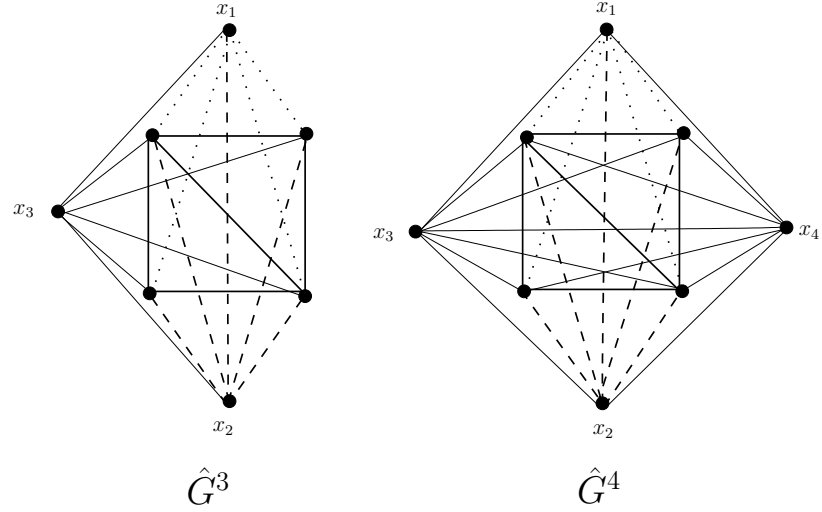


Figure 3.6: A graph G with its q -th repeated vertex join graphs; $\hat{G}^3$, $\hat{G}^4$.

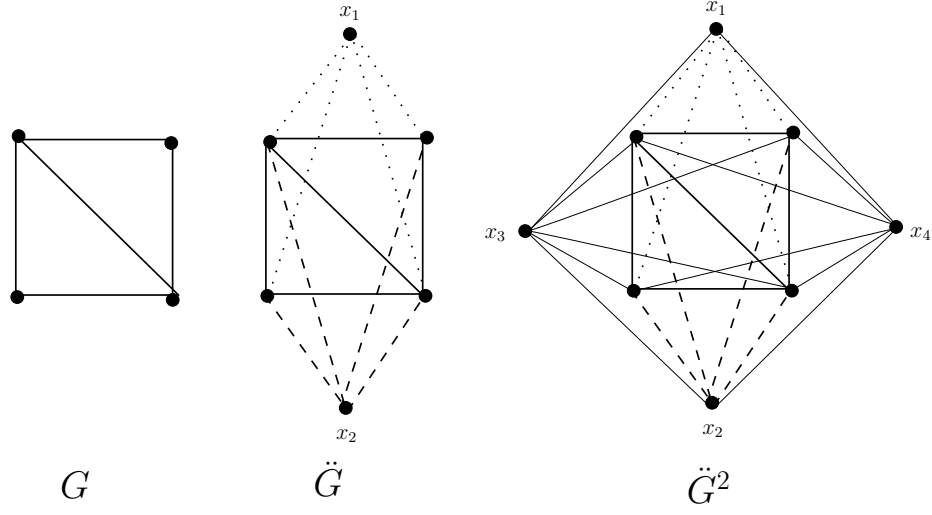


Figure 3.7: A graph G with its q -th repeated bivertex join graphs; $\tilde{G}$ and $\tilde{G}^2$.

(i) By Proposition 3.3.6, $|V(\hat{G}^q)| = n + q$. That is, $|V(\hat{G}^{2q})| = n + 2q$. By Proposition 3.4.5 $|V(\ddot{G}^q)| = n + 2q$. Therefore, $|V(\ddot{G}^q)| = |V(\hat{G}^{2q})|$.

(ii) By Proposition 3.3.6, $|E(\hat{G}^q)| = \frac{q(q-1)}{2} + m + nq$. That is,

$$\begin{aligned}
 |E(\hat{G}^{2q})| &= \frac{2q(2q-1)}{2} + m + 2nq \\
 &= m + q(2q-1) + 2nq \\
 &= m + 2q^2 - q + 2nq.
 \end{aligned}$$

By Proposition 3.4.5

$$\begin{aligned}
 |E(\ddot{G}^q)| &= m + 2q(q-1) + 2nq \\
 &= m + 2q^2 - 2q + 2nq \\
 &= (m + 2q^2 - q + 2nq) + q \\
 &= |E(\hat{G}^{2q})| + q.
 \end{aligned}$$

□

We now state and prove one of the results in this chapter, the chromatic polynomial of a bivertex join of a graph G .

Theorem 3.4.7. *Let G be a graph with the chromatic polynomial $\chi(G; \lambda)$. Then,*

$$\chi(\ddot{G}; \lambda) = \lambda(\lambda - 1)\chi(G; \lambda - 2) + \lambda\chi(G; \lambda - 1).$$

Proof. Let $\hat{G}$ be a vertex join and $\hat{\hat{G}}$ be a repeated vertex join of a graph G . Let x_1 be the new vertex in the construction of $\hat{G}$ and x_2 be the new vertex in the construction of $\hat{\hat{G}}$. Let $e = (x_1, x_2) \in E(\hat{\hat{G}})$, then $\hat{\hat{G}} \setminus e$ is isomorphic to $\ddot{G}$ and $\hat{\hat{G}}/e$ is isomorphic to $\hat{G}$ up to simplification. Hence by Proposition 1.7.2 and Proposition 1.7.5,

$$\begin{aligned} \chi(\hat{\hat{G}}; \lambda) &= \chi(\hat{\hat{G}} \setminus e; \lambda) - \chi(\hat{\hat{G}}/e; \lambda) \\ &= \chi(\ddot{G}; \lambda) - \chi(\hat{G}; \lambda). \end{aligned} \tag{3.1}$$

Rearranging Equation 3.1, we get

$$\chi(\ddot{G}; \lambda) = \chi(\hat{\hat{G}}; \lambda) + \chi(\hat{G}; \lambda).$$

Applying Corollary 3.3.7 and Proposition 3.3.1, we get

$$\chi(\ddot{G}; \lambda) = \lambda(\lambda - 1)\chi(G; \lambda - 2) + \lambda\chi(G; \lambda - 1).$$

□

Corollary 3.4.8. *Let $[G]$ be the chromatic equivalence class of graph G and let graph $H \in [G]$. Then $\ddot{H}^q \in [\ddot{G}^q]$.*

We now discuss one of our results of this chapter; necessary conditions for a graph to be in the chromatic equivalence class, $[\ddot{G}^q]$.

Proposition 3.4.9. *Let $J \in [\ddot{G}^q]$ where G is a graph with n vertices, m edges and a chromatic number, $\chi(G) = p$. Then,*

$$(i) \quad |V(J)| = n + 2q$$

$$(ii) \quad |E(J)| = \frac{q(q-1)}{2} + m + 2nq - q$$

$$(iii) \quad \chi(J) = p + q$$

$$(iv) \quad J \text{ has no bridge}$$

$$(v) \quad J \text{ is 2-connected}$$

Proof. Let $J \in [\ddot{G}^q]$ where G is a graph with n vertices, m edges and a chromatic number, $\chi(G) = p$.

- (i) By Proposition 3.4.5, $|V(\ddot{G}^q)| = n + 2q$. Therefore, $|V(J)| = n + 2q$.
- (ii) By Proposition 3.4.5, $|E(\ddot{G}^q)| = m + 2q(q - 1) + 2nq$. Therefore, $|E(J)| = m + 2q(q - 1) + 2nq$.
- (iii) By Proposition 3.4.3, $\ddot{G}^q \cong G \vee \ddot{N}_2^{q-1}$, therefore, every vertex of $\ddot{N}_2^{q-1}$ connects to every vertex of G . That is, the minimum number of colours needed to colour the graph $\hat{G}^q$ is the sum of $\chi(G) = p$ and $\chi(\ddot{N}_2^{q-1}) = q$. Thus, $\chi(\hat{G}^q) = p + q$. We then deduce that $\chi(J) = p + q$.
- (iv) $\ddot{G}^q$ has no bridge because each of its vertices has a degree greater or equal to $2q, q \geq 1$, because by Proposition 3.4.3, $\ddot{G}^q \cong G \vee \ddot{N}_2^{q-1}$. That is, by Lemma 3.3.10, the chromatic polynomial, $\chi(\ddot{G}^q, \lambda)$ is not divisible by $(\lambda - 1)^2$, since $\ddot{G}^q$ has no bridge. Thus, the chromatic polynomial, $\chi(J, \lambda)$ is not divisible by $(\lambda - 1)^2$. We deduce that J does not have a bridge.
- (v) $\ddot{G}^q$ is 2-connected because by Proposition 3.4.3, $\ddot{G}^q \cong G \vee \ddot{N}_2^{q-1}$, therefore, each of vertex of G is connected to all vertices of $\ddot{N}_2^{q-1}$, and thus, removing or deleting any vertex of $\ddot{G}^q$ will result to a minor that is still connected. Then by applying Proposition 1.7.7, J is also 2-connected.

□

3.5 Application of a bivertex join of a graph to known pairs

In this section, we delist two pairs of well known non-isomorphic graphs from the list of non-trivial chromatic equivalent graphs found in the literature. We then give an example of a class of chromatic equivalent graphs. We end by posing a few questions on chromatic equivalence on this class.

There are known graph operations in the literature which one can use to construct chromatic equivalent graphs. Hence in the absence of a known graph operation in constructing chromatic equivalent, non-isomorphic graphs, the graphs are said to be non-trivial chromatically equivalent. A list of such graphs can be found in [21].

Note 3.5.1. Let $G = P_2 \cup P_2$ and $H = P_3 \cup P_1$. These forests have the same order and the same number of components and hence by Proposition 1.7.4, they are chromatically equivalent. Figure 3.8 present the construction of the bivertex joins, $\ddot{G}$ and $\ddot{H}$. The graphs, $\ddot{G}$ and $\ddot{H}$ are chromatically equivalent, by Corollary 3.4.8. We note that this pair of derived graphs is the pair discussed, in 1968 by Read [44]. Moreover, it is stated that this pair is not trivially chromatic equivalent. Since this pair can be constructed using the bivertex join operation, we have now removed it from a list of non-trivial chromatically equivalent graphs that is provided in [21].

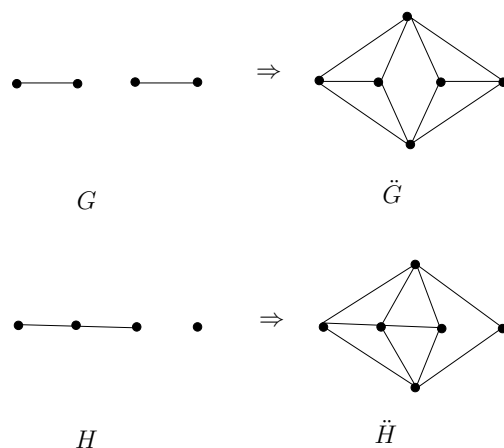


Figure 3.8: Bivertex join of forests.

Note 3.5.2. Similarly, let P_4 and S_4 be a path and star of order 4, respectively. These trees have the same order and hence by Proposition 1.7.3, they are chromatically equivalent. Figure 3.9 present the construction of the bivertex joins, $\ddot{P}_4$ and $\ddot{S}_4$. The graphs, $\ddot{P}_4$ and $\ddot{S}_4$ are chromatically equivalent, by Corollary 3.4.8. We note that this pair of derived graphs is the pair discussed, in 2011, Morgan [39] as not trivially chromatic equivalent.

Since this pair can be constructed using the bivertex join operation, we have now removed it from a list of non-trivial chromatically equivalent graphs that is provided in the literature.

Example 3.5.3. In this example, we construct a chromatic equivalence class of graphs using the bivertex join operation on trees of order 6, $[\ddot{T}_6]$, see Figure 3.10.

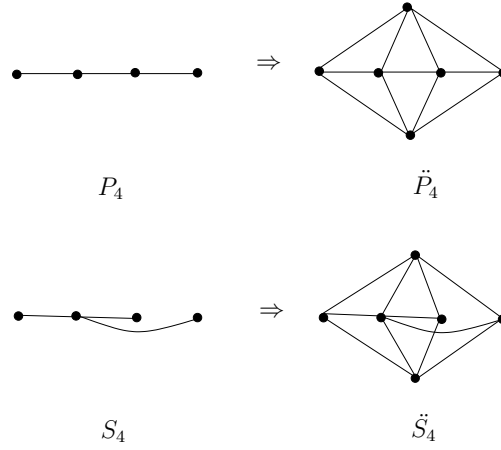


Figure 3.9: Bivertex join of trees.

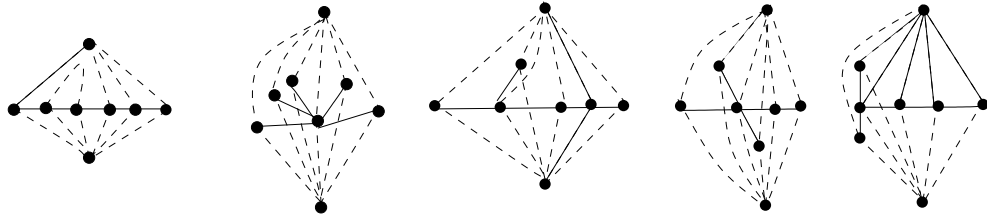


Figure 3.10: Some graphs in $[T_6]$.

Now, we can ask further questions for investigation. Is the number of non-isomorphic trees of order n , $|[T_n]|$ equal to the cardinality of the equivalence class $[\ddot{P}_n]$? Alternatively, can we find a graph which is chromatically equivalent to $\ddot{P}_n$ but it is not a bivertex join of a certain tree T_n . The following result, answers this question partially.

Theorem 3.5.4. *Let $G \cong N_2 \vee J$ such that $G \in [N_2 \vee T_n]$ then $J \cong T_n$.*

Proof. Let $G \cong N_2 \vee J$ such that $G \in [N_2 \vee T_n]$. By Proposition 3.2.3, $L \vee H_1$ is chromatically equivalent to $L \vee H_2$ if and only if H_1 is chromatically equivalent to H_2 . Then we have that J is chromatically equivalent to T_n . Since, only trees of the same order are chromatically equivalent to each other, therefore, $J \cong T_n$. □

3.6 Conclusion

In this chapter, we listed the necessary conditions for a graph to be in the chromatic equivalence classes, $\hat{G}^q$ and $\ddot{G}^q$. Thereafter, we showed that $\hat{G}^{2q}$ and $\ddot{G}^q$ have the same order and the difference in their sizes is q . Finally, we applied the knowledge of the bivertex join of graphs to delist two pairs of well known non-isomorphic graphs from the list of non-trivial chromatic equivalent graphs found in the literature.

We raise the following questions: What is the cardinality of the chromatic equivalence classes, $[[K_q \vee G]]$ and $[[N_q \vee G]]$? In chromatic equivalence classes, $[K_q \vee G]$ and $[N_q \vee G]$, is there a relationship among the elements that are not derived from a join operation? What are sufficient conditions for a graph to belong to chromatic equivalence classes, $[K_q \vee G]$ and $[N_q \vee G]$?

Chapter 4

Chromatic equivalence class of the bivertex join of T_n

4.1 Introduction

In this chapter, we begin by listing the necessary conditions for a graph to be an element of the chromatic equivalence class, $[\ddot{T}_n]$. Subsequently, we calculate the first four coefficients of the chromatic polynomial of a graph, J in the chromatic equivalence class, $[\ddot{T}_n]$. Thereafter, we examine the number of subgraphs of $\ddot{T}_n$ that contribute to the first four coefficients of the chromatic polynomial, $\chi(\ddot{T}_n; \lambda)$. We then extend the discussion to determine the number of subgraphs of $\ddot{T}_n$ that contribute to the fifth coefficient of the chromatic polynomial, $\chi(\ddot{T}_n; \lambda)$, thereby determining its value. Lastly, we use the vertex join and bivertex join operations to introduce some classes of clique-separable graphs.

4.2 Conditions for chromatic equivalence class, $[\ddot{T}_n]$

In this section, we discuss necessary conditions for a graph to be an element of the chromatic equivalence class, $[\ddot{T}_n]$. We follow the discussion on necessary condition for a graph to be an element of $[\ddot{G}^q]$, given by Proposition 3.4.9.

Corollary 4.2.1. *Let a graph, $\ddot{T}_n$ be a bivertex join of a tree of order n . Then,*

$$(i) \quad |V(\ddot{T}_n)| = n + 2$$

$$(ii) \quad |E(\ddot{T}_n)| = 3n - 1$$

- (iii) The chromatic number, $\chi(\ddot{T}_n) = 3$
- (iv) $\ddot{T}_n$ has no bridge
- (v) $\ddot{T}_n$ is 2-connected
- (vi) The number of triangles in $\ddot{T}_n$ is given by, $t_{C_3} = 2(n-1)$

4.3 First four coefficients of $\chi(T_n; \lambda)$

In this section, we discuss the first four coefficients of the chromatic polynomial of a graph, $J \in [\ddot{T}_n]$. In particular, we study the number of subgraphs of $\ddot{T}_n$ which contribute to the first four coefficients of $\chi(T_n; \lambda)$.

We recall and restate part of Proposition 2.2.1 as a remark.

Remark 4.3.1. Let G be a graph of order n and size m , with c number of components and the chromatic polynomial, $\chi(G; \lambda) = \sum_{i=1}^n a_i \lambda^i$ where a_i is the $(n-i+1)$ -th coefficient. Then,

- (i) the first coefficient, $a_n = 1$.
- (ii) the second coefficient, $a_{n-1} = -m$.
- (iii) the third coefficient, $a_{n-2} = \binom{m}{2} - t_{C_3}$.
- (iv) the forth coefficient, $a_{n-3} = -\binom{m}{3} + (m-2)t_{C_3} + t_{C_4} - 2t_{K_4}$.

where t_{C_3} : the number of triangles C_3 , t_{C_4} : the number of induced cycles C_4 , t_{K_4} : the number of complete subgraphs K_4 in G .

Note that in Corollary 4.2.1, we already determined $|E(\ddot{T}_n)|$ is number of edges and t_{C_3} is the number of induced triangles. Thus, we can compute a_n , a_{n-1} and a_{n-2} , the first three coefficients of the chromatic polynomial, $\chi(T_n; \lambda)$.

Proposition 4.3.2. Let $\ddot{T}_n$ be a bivertex join of the graph, T_n . Then,

- (i) the number of complete subgraphs K_4 , $t_{K_4} = 0$.

(ii) the number of induced cycles C_4 , $t_{C_4} = \binom{n-1}{2}$.

Proof. Let $\ddot{T}_n$ be a bivertex join of the graph T_n . In this proof we use bivertex join of a star graph given in Figure 4.1. Trees share the same chromatic polynomial and hence the result applies to all trees.

(i) Corollary 4.2.1 states that $\chi(\ddot{T}_n) = 3$ and the chromatic number of the complete graph, $\chi(K_4) = 4$. Since the chromatic number of a graph can not be less than the chromatic number of its subgraph then, the number of subgraphs K_4 of $\ddot{T}_n$ is given by $t_{K_4} = 0$.

(ii) We consider the subgraph H of G that is induced by $\{v_1, v_2, \dots, v_n\}$ which is isomorphic to S_n . The subgraph induced by $\{v_i, v_j, u, w\}$ is isomorphic to C_4 where v_i and v_j are not adjacent and $i \neq j$. Hence, the number of induced C_4 , $t_{C_4} = \binom{n-1}{2}$.

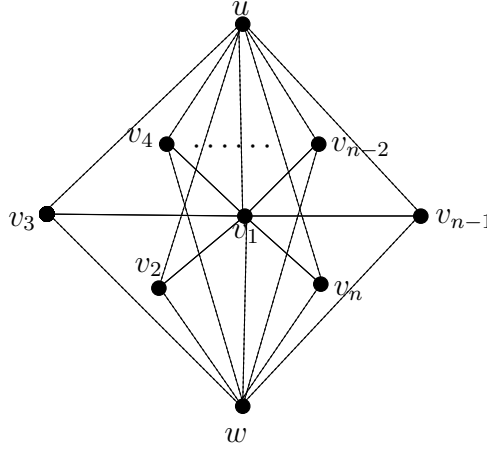


Figure 4.1: The graph, $\ddot{S}_n$

□

We are now in a position to state and prove one of the results of this chapter.

Theorem 4.3.3. Let $\chi(\ddot{T}_n, \lambda) = \sum_{i=1}^k a_i \lambda^i$ be a chromatic polynomial of $\ddot{T}_n$ where $|V(\ddot{T}_n)| = k$ and $k = n + 2$ and the a_i is the $(k - i + 1)$ -th coefficient. Then,

(i) the first coefficient, $a_k = 1$.

(ii) the second coefficient, $a_{k-1} = -3n + 1$.

(iii) the third coefficient, $a_{k-2} = \frac{9n^2 - 13n + 6}{2}$.

(iv) the forth coefficient, $a_{k-3} = \frac{(n-1)(9n^2 - 22n + 16)}{2}$.

Proof. Let $\chi(\ddot{T}_n, \lambda) = \sum_{i=1}^k a_i \lambda^i$ be a chromatic polynomial of $\ddot{T}_n$ where $|V(\ddot{T}_n)| = k$ and $k = n + 2$ and the a_i is the $(k - i + 1)$ -th coefficient.

(i) The coefficient of λ^k , a_k is 1, by Corollary 4.2.1.

(ii) By Corollary 4.2.1 and Remark 4.3.1, $a_{k-1} = -|E(\ddot{T}_n)| = -3n + 1$.

(iii) By Proposition 2.2.1, the coefficient of λ^{k-2} is given by,

$$\begin{aligned}
a_{k-2} &= \binom{|E(\ddot{T}_n)|}{2} - t_{C_3} \\
&= \binom{3n-1}{2} - 2(n-1), \text{ by Corollary 4.2.1} \\
&= \frac{(3n-1)!}{2!(3n-1-2)!} - 2(n-1) \\
&= \frac{(3n-1)!}{2!(3n-3)!} - 2(n-1) \\
&= \frac{(3n-1)(3n-2)(3n-3)!}{2!(3n-3)!} - 2(n-1) \\
&= \frac{(3n-1)(3n-2)}{2} - 2(n-1) \\
&= \frac{9n^2 - 9n + 2}{2} - 2(n-1) \\
&= \frac{9n^2 - 9n + 2}{2} - \frac{4(n-1)}{2} \\
&= \frac{9n^2 - 13n + 6}{2}.
\end{aligned}$$

(iv) By Proposition 2.2.1, the coefficient of λ^{k-3} is given by,

$$\begin{aligned}
a_{k-3} &= -\binom{|E(\ddot{T}_n)|}{3} + (|E(\ddot{T}_n)| - 2)t_{C_3} + t_{C_4} - 2t_{K_4} \\
&= -\binom{3n-1}{3} + (3n-1-2)(2(n-1)) + \binom{n-1}{2} \\
&= -\binom{3n-1}{3} + (3n-3)(2n-2) + \binom{n-1}{2}
\end{aligned}$$

Simplifying terms separately,

$$\begin{aligned}
-\binom{3n-1}{3} &= -\frac{(3n-1)(3n-2)(n-1)}{2}, \\
(3n-3)(2n-2) &= 6n^2 - 12n + 6, \\
\binom{n-1}{2} &= \frac{(n-1)(n-2)}{2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_{k-3} &= -\frac{(3n-1)(3n-2)(n-1)}{2} + 6n^2 - 12n + 6 + \frac{(n-1)(n-2)}{2} \\
&= -\frac{9n^3 - 18n^2 + 11n - 2}{2} + \frac{12n^2 - 24n + 12}{2} + \frac{n^2 - 3n + 2}{2} \\
&= \frac{-9n^3 + 31n^2 - 38n + 16}{2} \\
&= \frac{(n-1)(9n^2 - 22n + 16)}{2}
\end{aligned}$$

□

4.4 Fifth coefficient of $\chi(\ddot{T}_n, \lambda)$

In this section, we study in detail the number of subgraphs of $\ddot{T}_n$ which contribute to the fifth coefficient of the chromatic polynomial, $\chi(\ddot{T}_n, \lambda)$. We use a bivertex of star graph, $\ddot{T}_n = \ddot{S}_n$ to study the coefficients of $\chi(\ddot{T}_n, \lambda)$. Trees share the same chromatic polynomial and hence the result applies to all trees.

Definition 4.4.1. Let a graph W_n be a wheel with n vertices. A *wheel with missing spoke*, $W(n, k)$ is given by deleting all but k consecutive spokes, where $1 \leq k \leq n-1$ and $n \geq 4$. The graph $W(n, n-1)$ is a wheel.

We now restate part of Proposition 2.2.1 as a remark.

Remark 4.4.2. Let G be a graph of order n and size m , with c number of components and the chromatic polynomial, $\chi(G; \lambda) = \sum_{i=1}^n a_i \lambda^i$ where the a_i is the $(n - i + 1)$ -th coefficient. Then, the fifth coefficient is given by,

$$a_{n-4} = \binom{m}{4} - \binom{m-2}{2} t_{C_3} + \binom{t_{C_3}}{2} - (m-3)t_{C_4} + (2m-9)t_{K_4} - t_{C_5} - 6t_{K_5} \\ + t_{K_{2,3}} + 2t_{W(5,3)} + 3t_{W_5},$$

where t_{C_3} is the number of triangles C_3 , t_{C_4} is the number of induced cycles C_4 , t_{K_4} is the number of complete graphs K_4 , t_{C_5} is the number of induced cycles C_5 , t_{K_5} the number of K_5 subgraphs, $t_{K_{2,3}}$ is the number of induced complete bipartite graphs $K_{2,3}$, $t_{W(5,3)}$ is the number of induced 5-vertex wheels with one spoke deleted and t_{W_5} is the number of induced wheels W_5 .

Proposition 4.4.3. Let $\ddot{T}_n$ be a bivertex join of the graph, T_n . Then;

- (i) the number of induced C_5 , $t_{C_5} = 0$
- (ii) the number of induced K_5 subgraph, $t_{K_5} = 0$
- (iii) the number of induced $K_{2,3}$ subgraphs, $t_{K_{2,3}} = \binom{n-1}{3}$
- (iv) the number of induced 5-vertex wheels with one spoke deleted, $t_{W(5,3)} = 0$
- (v) the number of induced wheels, W_5 , $t_{W_5} = \binom{n-1}{2}$

Proof. Let $\ddot{S}_n$ be a bivertex join of a star as shown in Figure 4.1. Consider a subgraph H of $\ddot{S}_n$ that is induced by $\{v_1, v_2, \dots, v_n\}$, which is isomorphic to S_n .

- (i) Note, all subgraphs, C_5 in $\ddot{T}_n$ contains a pair of vertices that are adjacent, but not along the cycle. Therefore, there are no induced C_5 . That is, $t_{C_5} = 0$.
- (ii) The chromatic number, $\chi(\ddot{T}_n) = 3$ and the chromatic number of the complete graph, $\chi(K_5) = 5$. Since the chromatic number of a graph can not be less than the chromatic number of its subgraph then, the number of subgraphs K_5 of $\ddot{T}_n$ is given by $t_{K_5} = 0$.

- (iii) The subgraph induced by $\{v_i, v_j, v_k, u, w\}$ is isomorphic to $K_{2,3}$ where $v_i, v_j, v_k \neq v_1$ and i, j, k are not equal to each other. Thus, the total number of induced $K_{2,3}$ of $\ddot{T}_n$ is given by $t_{K_{2,3}} = \binom{n-1}{3}$, because from all the $n-1$ leaves of H , we choose only 3 leaves, combined with vertices $\{u, w\}$ to induce a $K_{2,3}$ subgraph.
- (iv) The subgraph induced by $\{v_i, v_j, v_k, u, w\}$ is isomorphic to $W(5, 3)$ if v_i, v_j, v_k induce a subgraph $P_2 \cup P_1$ where i, j, k are not equal to each other. Any pair of vertices in H connect to v_1 and thus inducing a subgraph P_3 . Therefore, it is impossible to have a subgraph $P_2 \cup P_1$ induced by v_i, v_j, v_k . Therefore, $t_{W(5,3)} = 0$.
- (v) The subgraph induced by $\{v_i, v_j, v_1, u, w\}$ is isomorphic to W_5 where v_i and v_j are not adjacent and $i \neq j$. Thus, the total number of induced W_5 of $\ddot{T}_n$ is given by $t_{W_5} = \binom{n-1}{2}$.

□

We now state and prove one of the results of this chapter.

Theorem 4.4.4. *Let $\chi(\ddot{T}_n, \lambda) = \sum_{i=1}^k a_i \lambda^i$ be the chromatic polynomial of $\ddot{T}_n$ where $|V(\ddot{T}_n)| = k$ and $k = n + 2$. Then, the coefficient of λ^{k-4} is given by*

$$a_{k-4} = \frac{(n-1)(n-2)(81n^2 - 275n + 264)}{24}.$$

Proof. Let $\chi(\ddot{T}_n, \lambda) = \sum_{i=1}^k a_i \lambda^i$ be a chromatic polynomial of $\ddot{T}_n$ where $|V(\ddot{T}_n)| = k$ and $k = n + 2$. Then, from Remark 4.4.2, the coefficient of λ^{k-4} is given by,

$$\begin{aligned} a_{k-4} = & \binom{m}{4} - \binom{m-2}{2} t_{C_3} + \binom{t_{C_3}}{2} - (m-3) t_{C_4} - (2m-9) t_{K_4} - t_{C_5} \\ & - 6t_{K_5} + t_{K_{2,3}} + 2t_{W(5,3)} + 3t_{W_5} \end{aligned}$$

By Corollary 4.2.1, Theorem 4.3.3 and Proposition 4.4.3,

$$\begin{aligned} |E(\ddot{T}_n)| = 3n-1, t_{C_3} = 2(n-1), t_{C_4} = \binom{n-1}{2}, t_{K_4} = 0, t_{C_5} = 0, t_{K_5} = 0, t_{W(5,3)} = \\ 0, t_{K_{2,3}} = \binom{n-1}{3} \text{ and } t_{W_5} = \binom{n-1}{2}. \end{aligned}$$

$$\begin{aligned}
a_{k-4} &= \binom{|E(\ddot{T}_n)|}{4} - \binom{|E(\ddot{T}_n)|-2}{2} t_{C_3} + \binom{t_{C_3}}{2} - (|E(\ddot{T}_n)|-3) t_{C_4} \\
&\quad - (2|E(\ddot{T}_n)|-9) t_{K_4} - t_{C_5} - 6t_{K_5} + K_{2,3} + 2t_{W(5,3)} + 3t_{W_5} \\
&= \binom{3n-1}{4} - \binom{3n-1-2}{2} (2(n-1)) + \binom{2(n-1)}{2} \\
&\quad - (3n-1-3) \binom{n-1}{2} + \binom{n-1}{3} + 3 \binom{n-1}{2} \\
&= \binom{3n-1}{4} - \binom{3n-3}{2} (2(n-1)) + \binom{2(n-1)}{2} - (3n-4) \binom{n-1}{2} \\
&\quad + \binom{n-1}{3} + 3 \binom{n-1}{2} \\
&= \frac{(3n-4)(3n-3)(3n-2)(3n-1)}{24} - (n-1)(3n-4)(3n-3) \tag{4.1}
\end{aligned}$$

$$+ (2(n-1)-1)(n-1) - \frac{(n-1)(n-2)(3n-4)}{2} \tag{4.2}$$

$$+ \frac{(n-1)(n-2)(n-3)}{6} + \frac{3(n-2)(n-1)}{2} \tag{4.3}$$

When we simplify parts labeled (4.1), (4.2) and (4.3) separately,

$$\begin{aligned}
(4.1) &= \frac{(3n-4)(3n-3)(3n-2)(3n-1)}{24} - (n-1)(3n-4)(3n-3) \\
&= \frac{(3n-4)(n-1)(9n^2-33n+26)}{8},
\end{aligned}$$

$$\begin{aligned}
(4.2) &= (2(n-1)-1)(n-1) - \frac{(n-1)(n-2)(3n-4)}{2} \\
&= -\frac{(n-1)(3n^2-14n+14)}{2},
\end{aligned}$$

$$\begin{aligned}
(4.3) &= \frac{(n-1)(n-2)(n-3)}{6} + \frac{3(n-2)(n-1)}{2} \\
&= \frac{(n-2)(n-1)(n+6)}{6}.
\end{aligned}$$

Therefore, by substitution

$$\begin{aligned}
a_{k-4} &= \frac{(3n-4)(n-1)(9n^2-33n+26)}{8} - \frac{(n-1)(3n^2-14n+14)}{2} \\
&\quad + \frac{(n-2)(n-1)(n+6)}{6} \\
&= \frac{(n-1)(n-2)(81n^2-275n+264)}{24}.
\end{aligned}$$

□

The question arises, do all the number of subgraphs that contribute to the coefficients, the same for all $J \in [\ddot{T}_n]$? We discuss partial solution in the following proposition.

Proposition 4.4.5. *Let $J \in [\ddot{T}_n]$ where J is derived from a bivertex join of a tree.*

$$\text{Then, } t_{K_{2,3}}(J) + 2t_{W(5,3)}(J) + 3t_{W_5}(J) = \frac{(n-1)(n-2)(n+6)}{6}.$$

Proof. Let $J \in [\ddot{T}_n]$ where J is derived from a bivertex join of a tree. Assume that

$$\chi(J, \lambda) = \sum_{i=1}^k b_i \lambda^i \text{ and } \chi(\ddot{S}_n, \lambda) = \sum_{i=1}^k a_i \lambda^i \text{ where } b_i \text{'s and } a_i \text{'s are coefficients of } \chi(J, \lambda)$$

and $\chi(\ddot{S}_n, \lambda)$ respectively. We note from discussing the first four coefficients that the following conditions hold for $\ddot{S}_n$ and J ; $|E(\ddot{T}_n)| = 3n-1$, $t_{C_3} = 2(n-1)$, $t_{C_4} = \binom{n-1}{2}$, $t_{K_4} = 0$, by Remark 4.3.1 and Proposition 4.4.3. Furthermore, simplifying $b_{n-4} = a_{n-4}$ implies,

$$-t_{C_5}(J) + t_{K_{2,3}}(J) + 2t_{W(5,3)}(J) + 3t_{W_5}(J) = -t_{C_5}(\ddot{S}_n) + t_{K_{2,3}}(\ddot{S}_n) + 2t_{W(5,3)}(\ddot{S}_n) + 3t_{W_5}(\ddot{S}_n).$$

Thus, $b_{n-4} = a_{n-4}$ implies,

$$t_{K_{2,3}}(J) + 2t_{W(5,3)}(J) + 3t_{W_5}(J) = \binom{n-1}{3} + 3\binom{n-1}{2}.$$

Since by Proposition 4.4.3, $t_{C_5}(\ddot{S}_n) = 0$, $t_{K_5}(\ddot{S}_n) = 0$, $t_{W(5,3)}(\ddot{S}_n) = 0$, $t_{K_{2,3}}(\ddot{S}_n) = \binom{n-1}{3}$ and $t_{W_5}(\ddot{S}_n) = \binom{n-1}{2}$. Therefore, $t_{K_{2,3}}(J) + 2t_{W(5,3)}(J) + 3t_{W_5}(J) = \frac{(n-1)(n-2)(n+6)}{6}$. □

Example 4.4.6. In this example, we compute the number of subgraphs, $t_{K_{2,3}}$, $t_{W(5,3)}$ and t_{W_5} and thereafter, verify Proposition 4.4.5. Let $\ddot{P}_6$ be bivertex join of a path, P_6 as displayed by the graph in Figure 4.2. We note that $K_{2,3}$ is induced by independent set of P_6 of order 3 and a vertex set $\{u, w\}$. That is, the following sets induce

$K_{2,3}$; $\{v_1, v_3, v_5, u, w\}$, $\{v_1, v_3, v_6, u, w\}$, $\{v_1, v_4, v_6, u, w\}$, $\{v_2, v_4, v_6, u, w\}$. Therefore, $t_{W_5} = 4$. The subgraph, $W(5, 3)$ is induced by three vertices of P_6 where two of them are adjacent and the other is not, and vertex set $\{u, w\}$. That is, the following sets induce $W(5, 3)$; $\{v_1, v_2, v_4, u, w\}$, $\{v_1, v_2, v_5, u, w\}$, $\{v_1, v_2, v_6, u, w\}$, $\{v_2, v_3, v_5, u, w\}$, $\{v_2, v_3, v_6, u, w\}$, $\{v_1, v_3, v_4, u, w\}$, $\{v_3, v_4, v_6, u, w\}$, $\{v_1, v_4, v_5, u, w\}$, $\{v_2, v_4, v_5, u, w\}$, $\{v_1, v_5, v_6, u, w\}$, $\{v_2, v_5, v_6, u, w\}$, $\{v_3, v_5, v_6, u, w\}$. Therefore, $t_{W(5,3)} = 12$. The subgraph, W_5 is induced by three vertices of P_6 that induce a path P_3 , and vertex set $\{u, w\}$. That is, the following sets induce W_5 ; $\{v_1, v_2, v_3, u, w\}$, $\{v_2, v_3, v_4, u, w\}$, $\{v_3, v_4, v_5, u, w\}$, $\{v_4, v_5, v_6, u, w\}$. Therefore, $t_{W_5} = 4$. Thus, $t_{K_{2,3}} + 2t_{W(5,3)} + 3t_{W_5} = 4 + 2(12) + 3(4) = 40$. We also note that, $\frac{(n-1)(n-2)(n+6)}{6} = \frac{(5)(4)(12)}{6} = 40$, verifying Proposition 4.4.5.

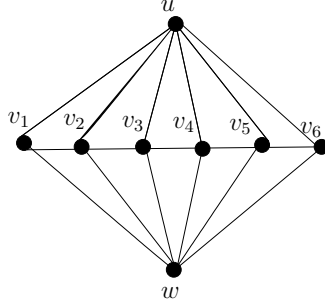


Figure 4.2: The graph $\ddot{P}_6$.

4.5 Clique Separable

In this section, we use the vertex join and bivertex join operations to construct some classes of clique separable graphs.

Gavril. F, [26] introduced and studied the clique separable graphs, in 1977.

Definition 4.5.1. Let G be a graph given by $G = (E, V)$. A graph G is a *clique separable* graph if it has either one of the following:

Type 1 If $V(G) = V_1 \cup V_2$ where V_1 and V_2 are disjoint subsets such that $|V_1| \geq 3$. A subgraph induced by a vertex set V_1 is a connected bipartite graph, a subgraph

induced by a vertex set V_2 is a complete graph and every vertex of V_1 is adjacent to every vertex of V_2 .

Type 2 If $V(G) = V_1 \cup V_2 \cup V_3 \cup \dots \cup V_k$ where each vertex set is an independent set and each vertex of V_i is adjacent to every vertex of V_j where $i \neq j$. That is, when G is a complete k -partite graph of some k .

Type 3 If G is derived from a K_r -gluing operation, $G_1 \cup_{K_r} G_2$ such that G_1 and G_2 are clique separable graphs.

In Definition 4.5.1, Type 3 is inductively defined by Type 1 and Type 2 where G_1 and G_2 are clique separable. A natural question which arises is “if a graph G is clique separable, are the q -th vertex join and bivertex join of a graph G also clique separable?” Our primary focus is to answer this question, so we determine the conditions under which the q -th vertex join and bivertex join of a graph are clique-separable.

We now state and prove the results of this section by considering the 3 types of Definition 4.5.1.

Theorem 4.5.2. *Let $\hat{G}^q$ be a q -th vertex join of G where $q \geq 1$. Then, $\hat{G}^q$ is a clique separable graph if G satisfies either one of the conditions below:*

- (i) *If G is a bipartite graph and $|V(G)| \geq 3$*
- (ii) *If G is a k -partite graph.*

Proof. Let $\hat{G}^q$ be a q -th vertex join of G where $q \geq 1$.

- (i) Let G be a bipartite graph with $|V(G)| \geq 3$. Consider a graph $\hat{G}^q \cong K_q \vee G$ where $V(G)$ induce a bipartite subgraph and $V(K_q)$ induce a complete subgraph. Therefore, $V(G)$ and $V(K_q)$ are disjoint set of vertices and every vertex of $V(G)$ is adjacent to every vertex of $V(K_q)$. Thus, $\hat{G}^q$ is a clique separable graph by Type 1.
- (ii) Let G be a k -partite graph where $V(G) = V_1 \cup V_2 \cup V_3 \cup \dots \cup V_k$. When we construct $\hat{G}^1$ from G , we adjoin an independent vertex, v_s and connect it to all vertices of G . Thus, the vertex set of $\hat{G}^1$ can be partitioned into $k + 1$ union of independent vertex sets that are disjoint where v_s is the only element in its vertex set. That is, $V(\hat{G}^1) = V_1 \cup V_2 \cup V_3 \cup \dots \cup V_k \cup V_{k+1}$ where $V_{k+1} = \{v_s\}$.

It is trivial that $\hat{G}^1$ is a clique separable graph. When we repeat this q times, we note that $V(\hat{G}^q) = V_1 \cup V_2 \cup V_3 \cup \dots \cup V_k \cup V_{k+1} \cup \dots \cup V_{k+q}$ such that each $|V_{k+i}| = 1$ for $1 \leq i \leq q$. The results follows that $\hat{G}^q$ is a clique separable graph by Type 2 since $\hat{G}^q$ is a $k + q$ -partite graph. That is, $\hat{G}^q$ is a clique separable graph by Type 2.

□

Theorem 4.5.3. *Let $\ddot{G}$ be a bivertex join of G where $q \geq 1$. Then, $\ddot{G}$ is a clique separable graph if G satisfies either one of the conditions below:*

- (i) *If G is a bipartite graph and $|V(G)| \geq 3$.*
- (ii) *If G is a k -partite graph.*

Proof. Let $\ddot{G}$ be a bivertex join of G where $q \geq 1$ and G is a k -partite graph for some k .

- (i) Let G be a bipartite graph with $|V(G)| \geq 3$, then $V(G) = V_1 \cup V_2$ where V_1 and V_2 are independent sets. Consider a graph $\ddot{G} \cong N_2 \vee G$ where $V(N_2) = V_3$ is a independent sets. Then, $V(\ddot{G}) = V_1 \cup V_2 \cup V_3$ and the vertex sets V_1, V_2 and V_3 are independent sets. Each vertex of V_i is adjacent to every vertex of V_j where $i \neq j$. Thus, $\ddot{G}$ is a 3-partite graph and hence it satisfy Type 2. Therefore, $\ddot{G}$ is a clique separable graph.
- (ii) Let G be a k -partite graph where $V(G) = V_1 \cup V_2 \cup V_3 \cup \dots \cup V_k$. When we construct $\ddot{G}$ from G , we adjoin a pair of independent vertices, v_{s1} and v_{s2} then connect them to all vertices of G . Thus, the vertex set of $\ddot{G}^1$ can be partition into $k + 1$ union of independent vertex sets that are disjoint where v_{s1} and v_{s2} are in their own vertex set. That is, $V(\ddot{G}^1) = V_1 \cup V_2 \cup V_3 \cup \dots \cup V_k \cup V_{k+1}$ where $V_{k+1} = \{v_{s1}, v_{s2}\}$. It is trivial that $\ddot{G}^1$ is a clique separable graph by Type 2. When we repeat this for q times we note that $V(\ddot{G}^q) = V_1 \cup V_2 \cup V_3 \cup \dots \cup V_k \cup V_{k+1} \cup \dots \cup V_{k+q}$ such that each $|V_{k+i}| = 2$ for $1 \leq i \leq q$. The results follows that $\ddot{G}^q$ is a clique separable graph by Type 2.

□

Example 4.5.4. In this example we construct a repeated bivertex join of the graph C_4 as shown by the diagram a graph in Figure 4.3. Clearly, $C_4 \subset \ddot{C}_4^2$, $V(C_4) =$

$V_1 \cup V_2$ where $V_1 = \{z_1, z_4\}$ and $V_2 = \{z_3, z_2\}$ are disjoint. The vertex sets V_1 and V_2 are independent sets with V_1 joining with every vertex of V_2 . Hence, C_4 is a clique separable graph. In addition, $V(\ddot{C}_4^2) = V_1 \cup V_2 \cup V_3 \cup V_4$ where $V_1 = \{z_1, z_4\}$, $V_2 = \{z_2, z_3\}$, $V_3 = \{y_1, y_2\}$ and $V_4 = \{x_1, x_2\}$ are disjoint and independent sets with each vertex of V_i is adjacent to every vertex of V_j where $i \neq j$. Hence by Type 2, $\ddot{C}_4^2$ is a clique separable graph.

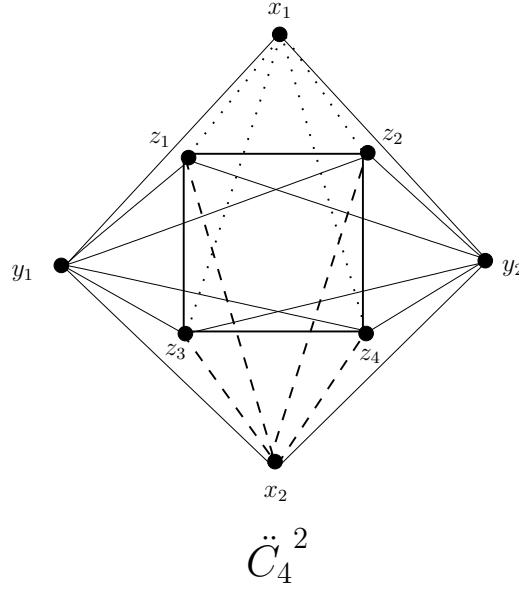


Figure 4.3: Repeated bivertex join of $C_4, \ddot{C}_4^2$.

The following proposition states some conditions where $\hat{G}^q$ and $\ddot{G}^q$ are not clique separable.

Proposition 4.5.5. *Let $G = G_1 \cup_{K_r} G_2$, a K_r -gluing of a clique separable graphs G_1 and G_2 . Then, $\ddot{G}$ and $\hat{G}$ are not clique separable graph.*

Proof. Let G be derived from a K_r -gluing operation, $G_1 \cup_{K_r} G_2$ such that G_1 and G_2 are clique separable graphs. The join $\ddot{G} = N_2 \vee G$ results in a graph that is no longer a K_r -gluing. Thus, $\ddot{G}$ does not satisfy Type 3. Similarly, $K_1 \vee G$ is not a K_r -gluing of any graph. Thus, this does not satisfy Type 3. Furthermore, $V(G_1 \cup_{K_r} G_2)$ does not have disjoint and independent subsets that completely connect with each other.

Thus, $\hat{G}$ and $\ddot{G}$ does not satisfy Type 1 and Type 2. Therefore, $\hat{G}$ and $\ddot{G}$ are not clique separable graph. $\square$

4.6 Conclusion

In this chapter, we gave the necessary conditions for a graph to be in the chromatic equivalence class, $[\ddot{T}_n]$. Thereafter, we determined the values of the first 5 coefficients of the chromatic polynomial, $\chi(\ddot{T}_n, \lambda)$. Additionally, we showed that the q -th repeated vertex join of G , $\hat{G}^q$ and the q -th repeated bivertex join of G , $\ddot{G}^q$ are clique separable graph if G is a bipartite or k -partite graph. However, $\hat{G}^q$ and $\ddot{G}^q$ are not clique separable if G is derived from a K_r -gluing operation.

We raise the following questions: Are all elements in $\ddot{T}_n$ derived from only bivertex join of elements in $[T_n]$? Thus, is the cardinality, $|\ddot{T}_n|$ equal to the cardinality, $|[T_n]|$? What are the sufficient conditions for graphs to be in the chromatic equivalence class, $[\ddot{T}_n]$?

Chapter 5

The chromaticity of bivertex of C_n

5.1 Introduction

In this chapter, we discuss the well-known results on chromaticity of wheels and generalised wheels, alternatively 1-cones and repeated vertex join of cycles, respectively. Then, we identify the bivertex join of a graph which is isomorphic to a 2-cone. Thereafter, we state the conditions for a wheel to be a chromatically unique graph. Finally, we study conditions under which a 2-cone qualifies as a chromatically unique graph.

5.2 t -cone graphs

In this section, we discuss the vertex join and bivertex join of a cycle.

Definition 5.2.1. A *generalised wheel* is q -th repeated vertex join of a cycle given by $W_n^t = C_n \vee K_t = \hat{C}_n^t$ where $n \geq 3, t \geq 1$ and K_t is a complete graph with t -vertices. On a generalised wheel, W_n^t , an edge that is connecting a vertex of K_t and a vertex of C_n is called a *spoke* of a generalised wheel. The sub-graph of W_n^t induced by $V(C_n)$ is called the rim of a generalised wheel.

We now state some of the trivial results of this chapter without proof. The results follow from Proposition 3.3.11.

Proposition 5.2.2. Let W_n^t be a generalised wheel. Then,

$$(i) \quad V(W_n^t) = n + t$$

$$(ii) \quad E(W_n^t) = \frac{2n(t+1) + t(t-1)}{2}$$

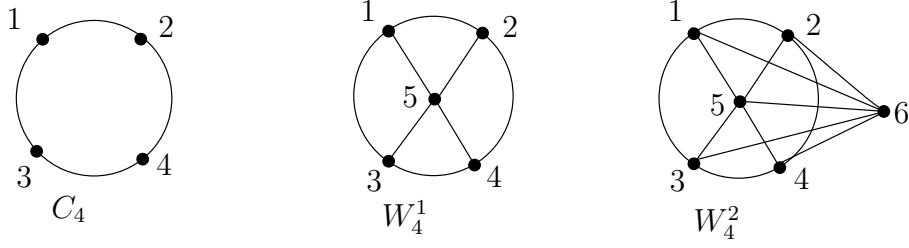


Figure 5.1: Generalised wheels.

(iii) W_n^t has a set of spokes, $E(W_n^t) - E(C_n) - E(K_t)$.

(iv) The number of triangles in W_n^t ,

$$t_{C_3}(W_n^t) = \begin{cases} 9 & \text{if } n = 3 \\ \frac{t(3nt + 3n + t^2 - 3t + 2)}{6} & \text{if } n \geq 4 \text{ is odd} \end{cases}$$

(v) W_n^1 is a wheel.

Definition 5.2.3. A t -cone or a t -point suspension is given by $C_n \vee N_t$, where $n \geq 3, t \geq 1$ and N_t is a null graph with t vertices. On a t -cone, $C_n \vee N_t$, an edge that is connecting a vertex of N_t and a vertex of C_n is called a spoke of a t -cone. A 2-cone or the double cone of C_n is given by $C_n \vee N_2$, which is a bivertex of C_n . The sub-graph of $C_n \vee N_t$ induced by $V(C_n)$ is called the rim of a t -cone.

The word t -cone was coined by Gallian in [25].

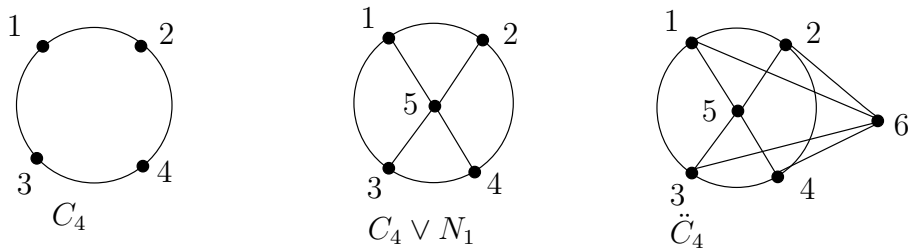


Figure 5.2: t -cones.

We now state some of the trivial results of this chapter without proof. The results follow from Proposition 3.4.9.

Proposition 5.2.4. *Let $C_n \vee N_t$ be a t -cone. Then,*

$$(i) \ V(C_n \vee N_t) = n + t.$$

$$(ii) \ E(C_n \vee N_t) = n(t + 1).$$

$$(iii) \ C_n \vee N_t \text{ has a set of spokes, } E(C_n \vee N_t) - E(C_n).$$

$$(iv) \ \text{The number of triangles in } C_n \vee N_t,$$

$$t_{C_3}(C_n \vee N_t) = \begin{cases} 9 & \text{if } n = 3 \\ nt & \text{if } n \geq 4 \text{ is odd} \end{cases}$$

$$(v) \ C_n \vee N_1 \text{ is a wheel.}$$

Example 5.2.5. In this example, we use the graphs in Figure 5.1 and Figure 5.2 to clarify the properties of a generalised wheel and a 2-cone. The graph labeled W_4^2 in Figure 5.1 is a generalised wheel of order 6, $C_4 \vee K_2$ with K_2 as the subgraph induced by vertex set $\{5,6\}$ and C_4 as a subgraph induced by vertex set $\{1,2,3,4\}$, a set of spokes on W_4^2 is given by $\{(1,6), (2,6), (3,6), (4,6), (1,5), (2,5), (3,5), (4,5)\}$. An edge, $(5,6)$ is not a spoke of W_4^2 . The graph labeled $\ddot{C}_4$ in Figure 5.2 is a 2-cone, with N_2 as the disconnected subgraph induced by vertex set $\{5,6\}$ and C_4 as a subgraph induced by vertex set $\{1,2,3,4\}$, a set of spokes on $\ddot{C}_4$ is given by $\{(1,6), (2,6), (3,6), (4,6), (1,5), (2,5), (3,5), (4,5)\}$.

5.3 The chromaticity of a 2-cone

In this section, we give an explicit expression of the chromatic polynomial of a 2-cone, $\chi(\ddot{C}_n; \lambda)$. Thereafter, we state some known conditions for a wheel to be a chromatically unique graph. These conditions for chromatic uniqueness of a wheel influence this work. We study conditions under which a 2-cone, $\ddot{C}_n$ qualifies as a chromatically unique graph.

The chromaticity of a graph G relates to the study of chromatic uniqueness of G , and if G is not chromatically unique, subsequently what other aspects can be studied regarding $[G]$. Mathematicians have investigated the concept of chromaticity of graphs in their own right and also the chromaticity of join of some graphs. The discussion on the chromaticity of a wheel which is a vertex join of a cycle, was initiated by Chao

C. Y. and Whitehead E. G. in [14]. Other discussions on the chromaticity of join of some graphs are found in [21, 36, 45, 48, 54]. In [35], Koh K.M and Teo K.L asked a question; “For what chromatically unique graphs G and H is $G \vee H$ chromatically unique?”.

Proposition 5.3.1. *[21] The empty graph N_t , complete graph K_r and cycle C_n where $r, t \geq 1$ and $n \geq 3$, are chromatically unique.*

Given the results expressed in Proposition 5.3.1 and Chapter 3, it is worth exploring the chromatic uniqueness and chromatic equivalence of joins, $\hat{C}_n^t$ and $N_t \vee C_n$. That is, we intend to discuss the chromatic uniqueness and chromatic equivalence of wheels, generalised wheels and 2-cones.

The following proposition is compiled from [14, 36, 45, 48, 54]

Proposition 5.3.2. *Let W_n be wheel. Then,*

- (i) W_n is chromatically unique if n is odd and W_8 is not chromatically unique.
- (ii) W_6 are not chromatically unique.
- (iii) W_{10} is chromatically unique.
- (iv) W_{12} is chromatically unique.

The following proposition is compiled from [19, 27].

Proposition 5.3.3. *Let W_n^t be generalised wheel. Then*

- (i) W_n^t is chromatically unique if $t \geq 1$ and $n \geq 4$ is even.
- (ii) W_5^t is not chromatically unique.

From Proposition 1.7.3, Theorem 3.4.7 and Corollary 3.3.7, we recall

$$\begin{aligned}\chi(C_n; \lambda) &= \lambda((\lambda - 1)^n + (-1)^n(\lambda - 1)), \\ \chi(G \vee N_2; \lambda) &= \lambda(\lambda - 1)\chi(G; \lambda - 2) + \lambda\chi(G; \lambda - 1) \text{ and} \\ \chi(G \vee K_t; \lambda) &= \chi(K_t; \lambda)\chi(G; \lambda - t).\end{aligned}$$

Proposition 5.3.4. *The chromatic polynomial of $\ddot{C}_n$,*

$$\chi(\ddot{C}_n; \lambda) = \lambda(\lambda - 1)(\lambda - 3)((\lambda - 3)^{n-1} + (-1)^{n+1}) + \lambda(\lambda - 2)((\lambda - 2)^{n-1} + (-1)^{n+1}).$$

Proof. By Definition 3.4.1, $\ddot{C}_n \cong C_n + N_2$, thus,

$$\begin{aligned} \chi(\ddot{C}_n; \lambda) &= \lambda(\lambda - 1)\chi(C_n; \lambda - 2) + \lambda\chi(C_n; \lambda - 1) \\ &= \lambda(\lambda - 1)((\lambda - 3)^n - (-1)^n(\lambda - 3)) + \lambda((\lambda - 2)^n - (-1)^n(\lambda - 2)) \\ &= \lambda(\lambda - 1)(\lambda - 3)((\lambda - 3)^{n-1} + (-1)^{n+1}) + \lambda(\lambda - 2)((\lambda - 2)^{n-1} + (-1)^{n+1}) \end{aligned}$$

□

Corollary 5.3.5. *The chromatic number of a 2-cone, $\ddot{C}_n$,*

$$\chi(\ddot{C}_n) = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd.} \end{cases}$$

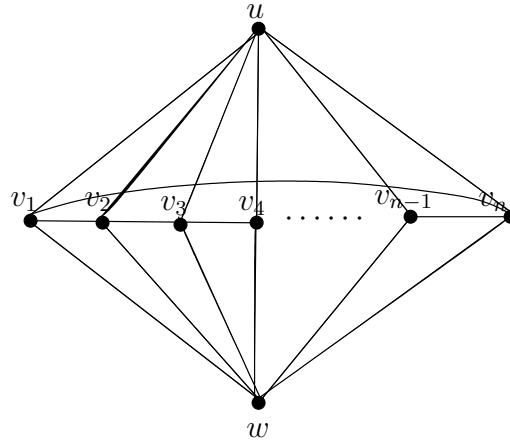


Figure 5.3: A graph, $\ddot{C}_n$.

Lemma 5.3.6. *Let $\ddot{C}_n$ be a 2-cone of C_n of order $n + 2$ and t_G be the number of induced subgraphs, G in $\ddot{C}_n$. Then,*

(i)

$$t_{C_3} = \begin{cases} 7 & \text{if } n = 3 \\ 2n & \text{if } n \geq 4 \end{cases}$$

(ii)

$$t_{C_4} = \begin{cases} 3 & \text{if } n = 4 \\ \frac{n(n-3)}{2} & \text{if } n \geq 5 \end{cases}$$

(iii)

$$t_{K_4} = \begin{cases} 2 & \text{if } n = 3 \\ 0 & \text{if } n \geq 4. \end{cases}$$

Proof. Without loss of generosity, consider the graph $\ddot{C}_n$ given in Figure 5.3. Let H be a subgraph of $\ddot{C}_n$ induced by $\{v_1, v_2, \dots, v_n\}$. Then, $H \cong C_n$.

(i) If $n = 3$, then, H induces a triangle, therefore, $t_{C_3} = 2(3) + 1 = 7$. If $n \geq 3$, the subgraph induced by $\{v_i, v_j, u\}$ or $\{v_i, v_j, w\}$ is isomorphic to C_3 where v_i and v_j are adjacent and $i \neq j$. Hence, the number of induced C_3 , $t_{C_3} = 2n$ if $n \geq 4$.

(ii) If $n = 4$ then, H induces a graph isomorphic to C_4 , therefore, $t_{C_4} = \frac{4(1)}{2} + 1 = 3$. If $n \geq 3$, the subgraph induced by $\{v_i, v_j, u, w\}$ is isomorphic to C_4 where v_i and v_j are not adjacent and $i \neq j$. Therefore, for $n \geq 5$ and excluding double counting, the number of induced C_4 associated with the vertices $v_1, v_2, v_3, \dots, v_{n-1}, v_n$ are $n-3, n-3, n-4, n-5, \dots, 1$ respectively. Summing up these numbers,

$$\begin{aligned} t_{C_4} &= \sum_{i=3}^{n-1} (n-i) + (n-3) \\ &= \frac{(n-3)(n-2)}{2} + (n-3) \\ &= \frac{n(n-3)}{2}. \end{aligned}$$

(iii) Assume $n = 3$, then H is induced by $\{v_1, v_2, v_3\}$ which is isomorphic to a triangle. The subgraphs induced by $\{v_1, v_2, v_3, u\}$ or $\{v_1, v_2, v_3, w\}$ are isomorphic to K_4 . Hence, for $n = 3$, the number of K_4 , $t_{K_4} = 2$. When $n \geq 4$, the subgraph H is not a complete graph and does not have a complete subgraph, K_4 . Therefore, joining u or w to H will also not result to a complete subgraph. Hence, for $n \geq 4$, $t_{K_4} = 0$.

□

Lemma 5.3.7. [54] *Let G be a graph containing subgraph which is a triangle, C_3 . If there exist a vertex of degree 2 on C_3 . Then $\chi(G; \lambda)$ is divisible by $(\lambda - 2)^2$.*

Proposition 5.3.8. *The 2-cone, $\check{C}_3$ is chromatically unique.*

Proof. To show that 2-cone, $\check{C}_3$ is chromatically unique, we assume that Y is a graph that is chromatically equivalent to $\check{C}_3$. Then,

$$\begin{aligned}
\chi(Y; \lambda) &= \chi(\check{C}_3; \lambda) \\
&= \lambda(\lambda - 1)\chi(C_3; \lambda - 2) + \lambda\chi(C_3; \lambda - 1) \\
&= \lambda(\lambda - 1)((\lambda - 3)^3 - (-1)^3(\lambda - 3)) + \lambda((\lambda - 2)^3 - (-1)^3(\lambda - 2)) \\
&= \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^2 \\
&= \lambda^5 - 9\lambda^4 + 29\lambda^3 - 39\lambda^2 + 18\lambda.
\end{aligned}$$

By Proposition 1.7.7 and Lemma 5.3.7, the chromatic equivalent graphs share certain properties. We deduce the following as properties of Y ; Y is connected, Y has 5 vertices, 9 edges, 7 triangles, $2|E| = 18$ as a degree sum. Since Y is connected then it has no degree that is equal to zero. Y has no bridge or a vertex with a degree that is equal to 1, since $\chi(Y; \lambda)$ is not divisible by $(\lambda - 1)^2$, by Lemma 3.3.10. We note that Y has no vertex of degree 2 on subgraph that is a triangle, since $\chi(Y; \lambda)$ is not divisible by $(\lambda - 2)^2$, by Lemma 5.3.7. That is, vertices of Y that belong to subgraphs that are triangles, have degrees ≥ 3 . Assume there are k such vertices that are covered by triangles. We recall that Y has 5 vertices and thus, $k \leq 5$. Since, Y has 7 triangles and we know that K_4 has 4 triangles which is less than triangles in Y , then $k = 5$. That is, each vertex of Y belong to one of the 7 triangles. Assume Y is a graph such that all vertices have degree 3, then the degree sum of such a graph is $3(5) = 15$. Which is a contradiction because the degree sum is an even number. Now, assume that Y has 5 vertices and its degree sum is 16(8 edges). Construct the cases where a graph has 5 vertices and 8 edges and then later adjoin the 9-th edge. The graphs in Figure 5.4 represent two cases; G_1 and G_2 where a graph has 5 vertices and 8 edges. We notice that in Figure 5.4, G_1 contradicts that vertices that belong to a triangle have degree ≥ 3 . When we connect any two vertices of G_2 with degree 3 we result to a graph that is isomorphic to G_3 . It can also be easily proven by construction that any graph of order 5 and size 9 is isomorphic to G_3 . Therefore, Y is represented by a graph G_3 which is isomorphic to 2-cone, $\check{C}_3$. Thus, 2-cone, $\check{C}_3$ is chromatically unique.

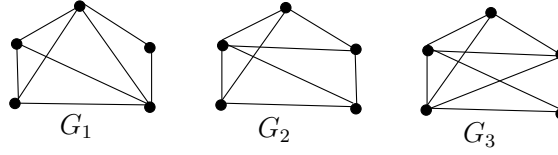


Figure 5.4:

□

Proposition 5.3.9. *The 2-cone, $\ddot{C}_4$ is chromatically unique.*

Proof. To show that 2-cone, $\ddot{C}_4$ is chromatically unique, we assume that Y is a graph that is chromatically equivalent to $\ddot{C}_4$. Thus,

$$\begin{aligned}
 \chi(Y; \lambda) &= \chi(\ddot{C}_4; \lambda) \\
 &= \lambda(\lambda - 1)\chi(C_4; \lambda - 2) + \lambda\chi(C_4; \lambda - 1) \\
 &= \lambda(\lambda - 1)((\lambda - 3)^4 - (-1)^4(\lambda - 3)) + \lambda((\lambda - 2)^4 - (-1)^4(\lambda - 2)) \\
 &= \lambda(\lambda - 1)(\lambda - 2)(\lambda^3 - 9\lambda^2 + 29\lambda - 32) \\
 &= \lambda^6 - 12\lambda^5 + 58\lambda^4 - 137\lambda^3 + 154\lambda^2 - 64\lambda.
 \end{aligned}$$

We recall from Proposition 1.7.7 and Lemma 5.3.7, the chromatic equivalent graphs share certain properties. Thus, Y is connected, Y has 6 vertices, 12 edges, 8 triangles, $2|E| = 24$ as a degree sum. Since Y is connected then it has no degree that is equal to zero. Y has no bridge or a vertex with a degree that is equal to 1, since $\chi(Y; \lambda)$ is not divisible by $(\lambda - 1)^2$, see Lemma 3.3.10. We note that Y has no vertex of degree 2 on subgraph which is a triangle, since $\chi(Y; \lambda)$ is not divisible by $(\lambda - 2)^2$, see Lemma 5.3.7. That is, vertices of Y that belong to subgraphs that are triangles, have degrees ≥ 3 . Assume there are k such vertices that are covered by triangles. We recall that Y has 6 vertices and thus, $k \leq 6$. By Lemma 5.3.6, Y does not have K_4 as a subgraph since $\chi(K_4) = 4$ and $\chi(Y) = 3$ since n is even. If $k = 6$ and all vertices of Y belong to a triangle and have a degree ≥ 3 . In construction of graphs with 6 vertices, 12 edges and 24 as degree sum, we result to the following cases:

- (i) A graph with 6 vertices of degree 4. This is represented by a graph G_1 in Figure 5.5.
- (ii) A graph with 2 vertices of degree 3, 2 vertices of degree 4 and 2 vertices of degree 5. This is represented by a graph G_2 in Figure 5.5.

- (iii) A graph with 1 vertices of degree 3, 4 vertices of degree 4 and 1 vertices of degree 5. This is represented by a graph G_3 in Figure 5.5.
- (iv) A graph with 3 vertices of degree 3 and 3 vertices of degree 5. This is represented by a graph G_4 in Figure 5.5.

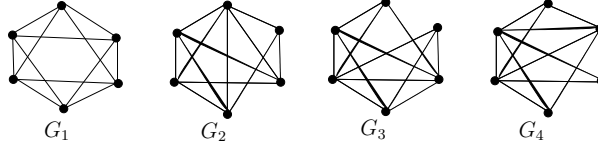


Figure 5.5:

Graphs, G_2, G_3 and G_4 have K_4 as a subgraph and therefore, these graphs are not chromatically equivalent to Y . The graph, G_1 , is isomorphic to $\ddot{C}_4$. Thus, 2-cone, $\ddot{C}_4$ is chromatically unique.

□

Proposition 5.3.10. *The 2-cone, $\ddot{C}_5$ is chromatically unique.*

Proof. To show that 2-cone, $\ddot{C}_5$ is chromatically unique, we assume that Y is a graph that is chromatically equivalent to $\ddot{C}_5$. Thus,

$$\begin{aligned}
\chi(Y; \lambda) &= \chi(\ddot{C}_5; \lambda) \\
&= \lambda(\lambda - 1)\chi(C_5; \lambda - 2) + \lambda\chi(C_5; \lambda - 1) \\
&= \lambda(\lambda - 1)((\lambda - 3)^5 - (-1)^5(\lambda - 3)) + \lambda((\lambda - 2)^5 - (-1)^5(\lambda - 2)) \\
&= \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda^3 - 9\lambda^2 + 30\lambda - 35) \\
&= \lambda^7 - 15\lambda^5 + 95\lambda^5 - 320\lambda^4 + 594\lambda^3 - 565\lambda^2 + 210\lambda.
\end{aligned}$$

By Proposition 1.7.7 and Lemma 5.3.7, the chromatic equivalent graphs share certain properties. Thus, Y has 7 vertices, 15 edges, 10 triangles and $2|E| = 30$ as a degree sum. As Y is connected then it has no degree that is equal to zero. The graph Y has no bridge or a vertex with a degree that is equal to 1, since $\chi(Y; \lambda)$ is not divisible by $(\lambda - 1)^2$, by Lemma 3.3.10. We note that Y has no vertex of degree 2 on subgraph which is a triangle, since $\chi(Y; \lambda)$ is not divisible by $(\lambda - 2)^2$, see Lemma 5.3.7. That is, vertices of Y that belong to subgraphs that are triangles,

have degrees ≥ 3 . By Lemma 5.3.7, $t_{C_4} = \frac{5(2)}{2} = 5$ and $t_{K_4} = 0$. The graphs in Figure 5.6 are all the graphs with the same number of vertices, edges and triangles as Y . Applying Proposition 1.7.7, graphs that are chromatically equivalent to Y have the property $t_{C_4}(Y) - 2t_{K_4}(Y) = 5$. We also note that,

$$\begin{aligned} t_{C_4}(G_1) - 2t_{K_4}(G_1) &= 3 - 2(0) = 3 \\ t_{C_4}(G_2) - 2t_{K_4}(G_2) &= 6 - 2(2) = 2 \\ t_{C_4}(G_3) - 2t_{K_4}(G_3) &= 6 - 2(3) = 0 \\ t_{C_4}(G_4) - 2t_{K_4}(G_4) &= 5 - 2(1) = 3 \\ t_{C_4}(G_5) - 2t_{K_4}(G_5) &= 6 - 2(2) = 2 \\ t_{C_4}(G_6) - 2t_{K_4}(G_6) &= 5 - 2(0) = 5. \end{aligned}$$

Therefore, Y is chromatically equivalent to G_6 and G_6 is isomorphic to $\ddot{C}_5$. Therefore, Y is chromatically unique.

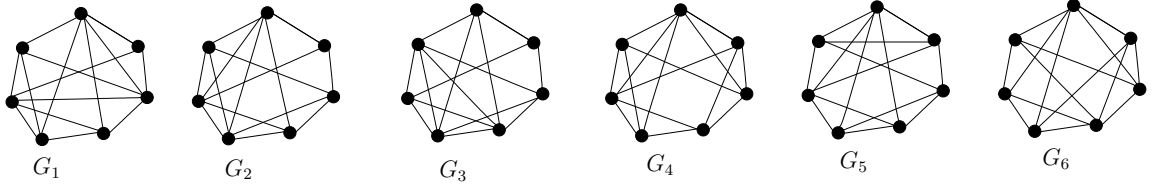


Figure 5.6: Graphs with the same t_{C_3} .

□

Proposition 5.3.11. *The 2-cone, $\ddot{C}_6$ is not chromatically unique.*

Proof. The chromatic polynomial of $\ddot{C}_6$ is given by,

$$\begin{aligned} \chi(\ddot{C}_6; \lambda) &= \lambda(\lambda - 1)\chi(C_6; \lambda - 2) + \lambda\chi(C_6; \lambda - 1) \\ &= \lambda(\lambda - 1)((\lambda - 3)^6 - (-1)^6(\lambda - 3)) + \lambda((\lambda - 2)^6 - (-1)^6(\lambda - 2)) \\ &= \lambda(\lambda - 1)(\lambda - 2)(\lambda^5 - 15\lambda^4 + 94\lambda^3 - 303\lambda^2 + 498\lambda - 332) \\ &= \lambda^8 - 18\lambda^7 + 141\lambda^6 - 615\lambda^5 + 1595\lambda^4 - 2432\lambda^3 + 1992\lambda^2 - 664\lambda. \end{aligned}$$

We show that 2-cone, $\ddot{C}_6$ is not chromatically unique by providing a list of graphs that are chromatically equivalent to it. The list of the graphs in the chromatic equivalence

class, $[\ddot{C}_6]$ is provided in Figure 5.7. This list was compiled using a programming tool called SageMath. We notice that G_1 is isomorphic to $\ddot{C}_6$.

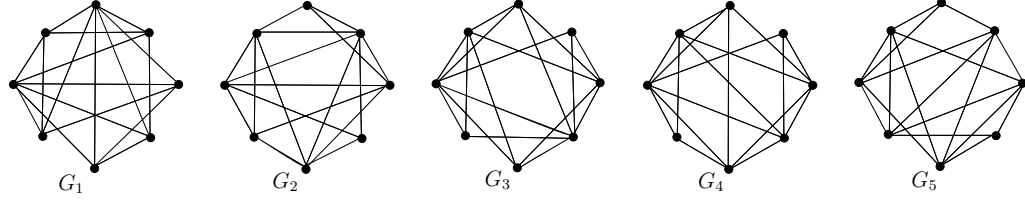


Figure 5.7: $[\ddot{C}_6]$

□

Conjecture 5.3.12. *The 2-cone, $\ddot{C}_n$ is chromatically unique if n is odd.*

5.4 Conclusion

In this chapter, we gave an explicit expression of the chromatic polynomial of a 2-cone on n -vertices. Then, we showed that the 2-cone is chromatically unique for some values of n .

There are other questions that arise from these discussion: In general, which values of n are the 2-cones chromatically unique. How do we identify the 2-cones that are not chromatically unique?

Chapter 6

The chromaticity of 2-cone with missing spokes

6.1 Introduction

In this chapter, we begin by discussing the wheels with missing spokes, basically, 1-cone with missing spokes. Thereafter, we state the well known results for chromaticity of wheels with missing spokes. In addition, we extend on the literature of 1-cone with missing spoke to 2-cones with missing spokes. Thus, we mimic some of the results for the chromaticity of 1-cone with missing spokes for 2-cones missing spokes. In particular, we determine a sufficient conditions for a graph to be in a chromatic equivalence class of a 2-cone of order $n + 2$ with $2(n - 2)$ missing spokes.

6.2 Wheels with deleted spokes

In this section, we begin by discussing some of properties of wheels with missing spokes. Thereafter, we discuss the well known results for chromaticity of a wheel with missing spokes.

The concepts of missing spokes was introduced by Whitehead Jr. and Chao C. in 1979. The authors discussed the chromaticity of a wheel with missing spokes, refer to [14]. A sequel to this literature was authored by Whitehead Jr. and Li N. in 1992 where they proved that a wheel of order n with 6 spokes is chromatically unique, refer to [37]. Thereafter, several other authors contributed to the literature of the

chromaticity of wheels with missing spokes, refer to [16, 20, 22, 28].

We recall and restate Definition 4.4.1 as a remark.

Remark 6.2.1. Let a graph W_n be a wheel given by a vertex set $\{v_1, v_2, \dots, v_n\} \cup \{u\}$ and edge set $\{(v_1, v_2), (v_2, v_3), (v_3, v_4), \dots, (v_{n-1}, v_n), (v_n, v_1)\} \cup \{(v_1, u), (v_2, u), (v_3, u), \dots, (v_{n-1}, u), (v_n, u)\}$ where u is a hub. A *wheel with missing spoke*, $W(n, k)$ is given by deleting all but k consecutive spokes, where $1 \leq k \leq n - 1$ and $n \geq 4$. Thus, $W(n, k)$ has $n - k - 1$ missing spokes and $W(n, n - 1)$ is a wheel.

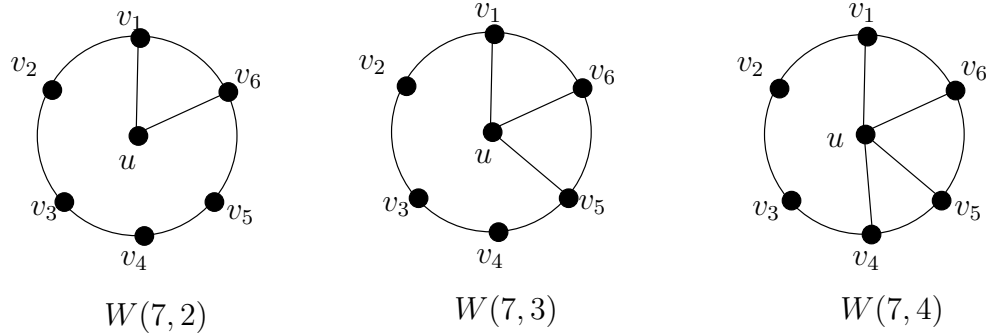


Figure 6.1: Wheels with missing spokes.

We now state some trivial properties of wheels with missing spokes without proof.

Proposition 6.2.2. *Let a graph $W(n, k)$ be a wheel with n vertices and $2 \leq k \leq n - 2$ spokes. Then,*

- (i) $|V(W(n, k))| = n$.
- (ii) $|E(W(n, k))| = n + k$.
- (iii) $\chi(W(n, k)) = 3$.
- (iv) $W(n, k)$ has no bridge.
- (v) $W(n, k)$ is 2-connected.
- (vi) The number of triangles in $W(n, k)$ is given by, $t_{C_3} = k - 1$.

We state some well known results for wheels with missing spokes without proof. These results are discussed in details in [14, 16, 20, 37].

Proposition 6.2.3. [14] *Let $W(n, k)$ be a wheel of order n with k spokes and $n - k - 1$ missing spokes. Then,*

- (i) $W(n, 3)$ is chromatically unique for $n \geq 5$.
- (ii) $W(n, 4)$ is chromatically unique for $n \geq 6$.
- (iii) $W(7, 5)$ is not chromatically unique.
- (iv) $W(n, 6)$ is chromatically unique for $n \geq 8$.
- (v) $W(n, n - 3)$ is chromatically unique for $n \geq 6$.
- (vi) $W(n, n - 2)$ is chromatically unique when $n \geq 9$ is odd.

6.3 The 2-cones with missing spokes

In this section, we discuss some of the properties of 2-cones with missing spokes. Additionally, we mimic some of the results for the chromaticity of wheels with deleted spokes for 2-cones. In particular, we determine a sufficient condition for a graph to be in a chromatic equivalence class of a 2-cone of order $n + 2$ with $2(n - 2)$ missing spokes.

Definition 6.3.1. Let a graph $\ddot{C}_n$ be a wheel given by a vertex set $\{v_1, v_2, \dots, v_n\} \cup \{u, w\}$ and edge set $\{(v_1, v_2), (v_2, v_3), (v_3, v_4), \dots, (v_{n-1}, v_n), (v_n, v_1)\} \cup \{(v_1, u), (v_2, u), (v_3, u), \dots, (v_{n-1}, u), (v_n, u)\} \cup \{(v_1, w), (v_2, w), (v_3, w), \dots, (v_{n-1}, w), (v_n, w)\}$. A pair of edges $\{(v_i, u), (v_i, w)\}$ where $i \in \{1, 2, 3, \dots, n\}$ is called a *bispoke*, of $\ddot{C}_n$. A 2-cone with missing bispokes, $\ddot{C}(n + 2, k)$ is given by deleting all, but k consecutive bispokes, where $1 \leq k \leq n - 1$ and $n \geq 4$. Thus, $\ddot{C}(n + 2, k)$ has $n - k$ missing bispokes and $\ddot{C}(n + 2, n)$ is a 2-cone, $\ddot{C}_n$.

Example 6.3.2. Let $\ddot{C}(8, 2)$, $\ddot{C}(8, 3)$ and $\ddot{C}(8, 4)$ be 2-cones with missing bispokes as represented by graphs in Figure 6.2. The graph, $\ddot{C}(8, 2)$ has $k = 2$ bispokes given by $\{(v_1, u), (v_1, w)\}$ and $\{(v_6, u), (v_6, w)\}$. Moreover, $\ddot{C}(8, 2)$ has $n - k - 1 = 8 - 2 - 1 = 4$ missing bispokes.

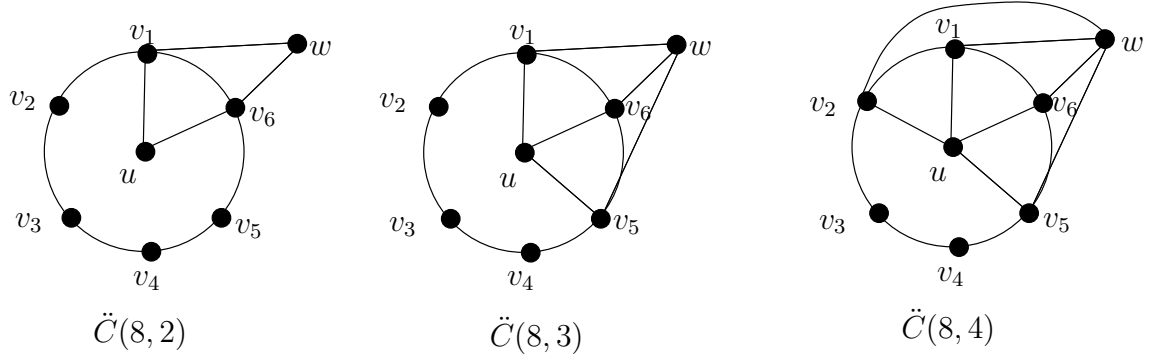


Figure 6.2: 2-cones with missing spokes.

We state some trivial results and some known results without proof in the following two propositions.

Proposition 6.3.3. *Let a graph $\check{C}(n+2, k)$ be a 2-cone with k bispokes and $n-k-1$ missing bispokes where $k \leq n-2$. Then,*

- (i) $|V(\check{C}(n+2, k))| = n+2$.
- (ii) $|E(\check{C}(n+2, k))| = n+2k$.
- (iii) $\check{C}(n+2, k)$ has no bridge.
- (iv) $\check{C}(n+2, k)$ is 2-connected.
- (v) The number of triangles in $\check{C}(n+2, k)$ is given by, $t_{C_3} = 2(k-1)$.

Proposition 6.3.4. [21] *Let G be a K_2 -gluing of a cycle C_n and a complete graph K_q , $C_n \cup_{K_2} K_q$. Then, G is chromatically unique.*

Lemma 6.3.5. *Let G be a K_2 -gluing of a cycle C_n and a complete graph K_3 , $G = C_n \cup_{K_2} K_3$. Then, G is chromatically unique and*

$$\chi(G, \lambda) = (\lambda - 2)\chi(C_{n+1}; \lambda).$$

Proof. Let $G = K_3, C_{n+1} \cup_{K_2} K_3$. By Proposition 6.3.4, G is chromatically unique and the chromatic polynomial of G ,

$$\begin{aligned}
\chi(G; \lambda) &= \chi(C_{n+1} \cup_{K_2} K_3, \lambda) \\
&= \frac{\chi(C_{n+1}; \lambda) \chi(K_3; \lambda)}{\chi(K_2; \lambda)} \\
&= \frac{\chi(C_{n+1}; \lambda) \chi(K_3; \lambda)}{\lambda(\lambda - 1)} \\
&= \frac{\chi(C_{n+1}; \lambda) \lambda(\lambda - 1)(\lambda - 2)}{\lambda(\lambda - 1)} \\
&= \chi(C_{n+1}; \lambda)(\lambda - 2)
\end{aligned}$$

□

Corollary 6.3.6. *Let $\ddot{C}(n+2, 2)$ be 2-cone with $n-2$ missing bispokes. Then,*

$$\chi(\ddot{C}(n+2, 2), \lambda) = (\lambda - 2)^2 \chi(C_{n+1}; \lambda).$$

Theorem 6.3.7. *Let $\ddot{C}(n+2, 2)$ be 2-cone with $n-2$ missing bispokes. Then, elements of the chromatic equivalence class $[\ddot{C}(n+2, 2)]$ are only derived from a K_2 -gluing operation, $K_3 \cup_{K_2} C_n \cup_{K_2} K_3$.*

Proof. We assume that a graph G is chromatically equivalent to $\ddot{C}(n+2, 2)$ and then show that G can only be derived from a K_2 -gluing, $K_3 \cup_{K_2} C_n \cup_{K_2} K_3$. Since G is chromatically equivalent to $\ddot{C}(n+2, 2)$ then $\chi(G, \lambda) = (\lambda - 2)^2 \chi(C_{n+1}; \lambda)$, by Corollary 6.3.6. Recall Proposition 1.7.7, graphs that are chromatically equivalent also share certain properties. Then G shares the following properties with $\ddot{C}(n+2, 2)$;

1. G has $n+2$ vertices
2. G has $n+4$ edges, and the degree sum $2|E| = 2(n-2) = 2n-4$
3. G has 2 triangles
4. G has 1 component
5. $\chi(G) = 3$
6. G has no cut vertex and is 2-connected
7. G has no vertex of degree 1 or no bridge since $(\lambda - 1)^2$ does not divide $\chi(G, \lambda)$, refer to Lemma 3.3.10

8. G has vertex of degree 2 that belongs to a triangle since $(\lambda - 2)^2$ divide $\chi(G, \lambda)$, refer to Lemma 5.3.7
9. Given that G has vertex of degree 2 that belongs to a triangle then we express the chromatic polynomial of $\chi(G, \lambda)$ in a way that we apply delete and contraction method on the aforesaid triangle. That is, $\chi(G, \lambda) = (\lambda - 2)\chi(H; \lambda)$, for some minor H that results from G when we contract one of the edges of G that connects a vertex v of degree 2 and also belongs to one of the two triangles. We have expressed the chromatic polynomial of G in two ways; $\chi(G; \lambda) = (\lambda - 2)((\lambda - 2)\chi(C_{n+1}; \lambda))$ and $\chi(G; \lambda) = (\lambda - 2)(\chi(H; \lambda))$. This implies that $\chi(H; \lambda) = (\lambda - 2)\chi(C_{n+1}; \lambda)$. We recall from Lemma 6.3.5 that, $\chi(C_n \cup_{K_2} K_3, \lambda) = (\lambda - 2)\chi(C_{n+1}; \lambda)$ is chromatically unique. That is, $H = C_n \cup_{K_2} K_3$. We also note that, $\chi(G; \lambda) = (\lambda - 2)(\chi(C_n \cup_{K_2} K_3; \lambda))$. Thus, G has a subgraph $H = C_n \cup_{K_2} K_3$ which contains one triangle, and the second triangle has a vertex of order 2 and hence $\chi(G; \lambda)$ has a factor $(\lambda - 2)$ on $\chi(G; \lambda) = (\lambda - 2)(\chi(C_n \cup_{K_2} K_3; \lambda))$. Then, G has 2 triangles such that each triangle has a vertex of order 2 and also G has a subgraph, G . Therefore, $G = K_3 \cup_{K_2} C_n \cup_{K_2} K_3$.

□

Example 6.3.8. The graphs in Figure 6.3 present elements of the chromatic equivalence class $[\ddot{C}(7, 2)]$.

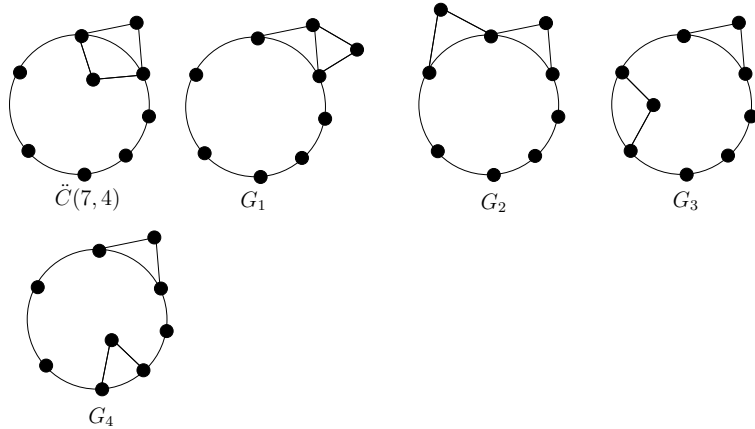


Figure 6.3: Elements in the chromatic equivalent class, $[\ddot{C}(7, 2)]$.

Corollary 6.3.9. *Let $\ddot{C}(n+2, 2)$ be 2-cone with $n-2$ missing bispokes. Then, the cardinality of the chromatic equivalence class,*

$$|[\ddot{C}(n+2, 2)]| = \begin{cases} \frac{n+4}{2}, & \text{when } n \text{ is even} \\ \frac{n+3}{2}, & \text{when } n \text{ is odd} \end{cases}$$

Proof. By Theorem 6.3.7, elements of $[\ddot{C}(n+2, 4)]$ are derived from a K_2 -gluing operation. We initiate construction process with a graph that we discussed in Lemma 6.3.5, $C_n \cup_{K_2} K_3$. We aim to find number of ways to add an additional triangle with a vertex of order 2 on $C_n \cup_{K_2} K_3$. The additional triangle can be adjoined to any of the edges of $C_n \cup_{K_2} K_3$. We note that $C_n \cup_{K_2} K_3$ has $n+2$ edges. Now, consider constructing a graph $K_3 \cup_{K_2} C_n \cup_{K_2} K_3$ from a graph $C_n \cup_{K_2} K_3$. Since $C_n \cup_{K_2} K_3$ has $n+2$ edges then there are $n+2$ ways to construct $K_3 \cup_{K_2} C_n \cup_{K_2} K_3$ through K_2 -gluing operation. We note that from $n+2$ graphs that are constructed, there are isomorphic graphs and since elements of a chromatic equivalence class are non-isomorphic, we only consider the non-isomorphic graphs. The number of non-isomorphic graphs depends on whether n is odd or even. We note that, there are $\frac{n}{2}$ isomorphic graphs when n is even and there are $\frac{n+1}{2}$ isomorphic graphs when n is odd. Thus, when counting the non-isomorphic graphs we note that there are $n+2 - \frac{n}{2} = \frac{n+4}{2}$ non-isomorphic graphs when n is even and there are $n+2 - \frac{n+1}{2} = \frac{n+3}{2}$ non-isomorphic graphs when n is odd. $\square$

Note that the results of Corollary 6.3.9 are verified by graphs in Figure 6.3, since the cardinality of $[\ddot{C}(7, 2)]$ is $\frac{n+3}{2} = \frac{7+3}{2} = 5$.

6.4 Conclusion

In this chapter, we have extended the notion of missing spokes of 1-cone to 2-cones. Then shown that all the elements of one specific 2-cone of order $n+2$ with $2(n-2)$ missing spokes can be constructed from the K_r -gluing operation and hence not chromatically unique. In addition, for this class we have also given an explicit number for

the cardinality of its equivalence class.

We raise the following questions: Is there a relationship in the chromaticity of $[\ddot{C}(n, k)]$ for various values of k ? What is the chromaticity of $[\ddot{C}(n, 2)]$ if the missing bispokes are not consecutive. What values of k is $\ddot{C}(n, k)$ chromatically unique?

Chapter 7

Conclusion

In this thesis, we introduced an algorithm for computing the general expression of the chromatic polynomial in cycle form and stated the relationship between cycle form and tree form. We calculated the values of the first and second coefficients of the chromatic polynomial in cycle form and we gave an explicit expression of the chromatic polynomial of some special graphs in cycle form.

We then proceeded by listing the necessary conditions for a graph to be in the chromatic equivalence classes, $\hat{G}^q$ and $\ddot{G}^q$. We showed that $\hat{G}^{2q}$ and $\ddot{G}^q$ have the same order and the difference in their sizes is q . We applied the knowledge of the bivertex join of graphs to delist two pairs of well known non-isomorphic graphs from the list of non-trivial chromatic equivalent graphs found in the literature.

We added the necessary conditions for a graph to be in the chromatic equivalence class, $[\ddot{T}_n]$. We determined the values of the first 5 coefficients of the chromatic polynomial, $\chi(\ddot{T}_n, \lambda)$. We showed that the q -th repeated vertex join of G , $\hat{G}^q$ and the q -th repeated bivertex join of G , $\ddot{G}^q$ are clique separable graph if G is a bipartite or k -partite graph. However, $\hat{G}^q$ and $\ddot{G}^q$ are not clique separable if G is derived from a K_r -gluing operation.

We then continued by giving an explicit expression of the chromatic polynomial of a 2-cone on n -vertices. We showed that the 2-cone is chromatically unique for some values of n .

In conclusion, we extended the notion of missing spokes of 1-cone to 2-cones. We

showed that all the elements of 2-cone of order $n + 2$ with $2(n - 2)$ missing spokes can be constructed from the K_r -gluing operation and hence not chromatically unique. Finally, we gave an explicit number for the cardinality of its equivalence class.

In this thesis we raised the following questions:

In chapter 2 we asked: What are values for other coefficients besides the first and second coefficients? What is the sum of the coefficients and does it have a special meaning? Could other coefficients be zero besides the second coefficient? We note that, $\chi(K_3 \cup K_1, \lambda) = \chi(C_4, \lambda) + \chi(C_3, \lambda) - \chi(C_2, \lambda)$, then what is the general expression of the chromatic polynomial of disconnected graphs in cycle form, since in power form we multiply the chromatic polynomials of the components if the graph is disconnected?

In chapter 3 we asked: What is the cardinality of the chromatic equivalence classes, $[[K_q \vee G]]$ and $[[N_q \vee G]]$? In chromatic equivalence classes, $[K_q \vee G]$ and $[N_q \vee G]$, is there a relationship among the elements that are not derived from a join operation? What are sufficient conditions for a graph to belong to chromatic equivalence classes, $[K_q \vee G]$ and $[N_q \vee G]$?

In chapter 3 we asked: What is the cardinality of the chromatic equivalence classes, $[[K_q \vee G]]$ and $[[N_q \vee G]]$? In chromatic equivalence classes, $[K_q \vee G]$ and $[N_q \vee G]$, is there a relationship among the elements that are not derived from a join operation? What are sufficient conditions for a graph to belong to chromatic equivalence classes, $[K_q \vee G]$ and $[N_q \vee G]$?

In chapter 4 we asked: Are all elements in $\ddot{T}_n$ derived from only bivertex join of elements in $[T_n]$? Thus, is the cardinality, $|\ddot{T}_n|$ equal to the cardinality, $|[T_n]|$? What are the sufficient conditions for graphs to be in the chromatic equivalence class, $[\ddot{T}_n]$?

In chapter 5 we asked: In general, which values of n are the 2-cones chromatically unique. How do we identify the 2-cones that are not chromatically unique?

In chapter 6 we asked: Is there a relationship in the chromaticity of $[\ddot{C}(n, k)]$ for various values of k ? What is the chromaticity of $[\ddot{C}(n, 2)]$ if the missing bispokes are

not consecutive. What values of k is $\ddot{C}(n, k)$ chromatically unique?

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