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Lie group analysis and conservation laws for the time-fractional 3D Bateman–Burgers equation

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Abstract
This research delves into analyzing the invariant properties and conservation laws (CLs) governing the (3+1)-dimensional (3D) time-fractional Bateman–Burgers (BB) equation, employing the Riemann Liouville (RL) fractional derivative. The fractional Lie symmetry method is employed to ascertain the symmetry attributes of this equation. Consequently, the equation is transformed into a fractional ordinary differential equation (FODE) using the Erdélyi–Kober (EK) fractional differential operator. Additionally, the paper derives new conserved vectors for the 3D time-fractional BB equation using the formal Lagrangian and a detailed derivation based on the new conservation theorem. The explicit solutions and their stability are also examined and analyzed.

Keywords Lie symmetry method · Time-fractional Bateman–Burgers equation · Riemann Liouville fractional derivative · Power series solutions · Conservation laws · Stability analysis

Mathematics Subject Classification 58J70 · 76M60 · 35A15

1 Introduction

The Lie group method was first introduced by the Norwegian mathematician Sophus Lie [1] in the late nineteenth century. Initially developed to solve ordinary differential equations (ODEs), it was later demonstrated that differential equations exhibit invariance under a Lie group transformation [2, 3]. This method enables the transformation of differential equation systems into other equivalent, reduced systems while preserving the invariance of solution space, even when transforming into a different space using a Lie group of infinitesimal transformations by invariant conditions. Lie symmetry analysis provides a powerful approach for finding invariant solutions to initial value problems, conservation laws, and boundary value problems [2–6]. Although the method has been applied to achieve symmetry reduction

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in integer-order partial differential equations (PDEs), its application to fractional differential equations is relatively novel. As a result, there is a growing interest in researching symmetry analysis for fractional differential equations [7–10].

This investigation analyzes a particular variant of the Burgers equation, first introduced by Bateman [11] and later investigated by Burgers [12]. This equation has a wide range of applications in fields such as fluid dynamics [13–16], nonlinear acoustics [17, 18], ocean engineering [19–23], and medical sciences [24–27]. Here, we consider the 3D time-fractional BB equation as follows

$$D_t^\theta \varphi - \varphi_{xx} - \varphi_{yy} - \varphi_{zz} - \varphi\varphi_x = 0, 0 < \theta \leq 1, \tag{1}$$

where $\varphi = \varphi(x, y, z, t)$. Given the broad range of applications of the BB equations, numerous researchers have shown interest in exploring these equations using various methodologies. When $\theta = 1$, the Eq. (1) simplifies to the 3D BB equation [28]

$$\varphi_t - \varphi_{xx} - \varphi_{yy} - \varphi_{zz} - \varphi\varphi_x = 0. \tag{2}$$

Numerous researchers have directed their attention to the investigation of this equation. Shah et al. [29] derived the analytical solution for the fractional coupled BB equation. The variational iteration method for analyzing BB-type equations was used by the authors Alderremy et al. [30]. Also, Mohanty et al. [31, 32] reported some soliton solutions using the (G'/G)-expansion technique for Schamel Burgers' equation. Additionally, Xia et al. [33] utilized Nucci's reduction technique to explore the Burgers' equations.

The primary objective of this study is to identify similarity solutions and establish novel conservation laws [34–36] for the 3D time-fractional BB equation, employing the fractional version of the Lie symmetry analysis method [7–10]. Additionally, a comprehensive examination of the solution series is conducted. Notably, new conserved vectors for the 3D time-fractional BB equation are formulated using the new conservation law theorem, facilitated by a formal Lagrangian framework.

2 Prelaminaries

Some introductory definitions are given in this section which we will use in this article.

Definition 1 The RL derivative is defined as [37]

$$D^\theta g(t) = \begin{cases} \frac{d^p}{dt^p} I^{p-\theta} g(t), & p-1 < \theta < p, \\ \frac{d^p g}{dt^p}, & \theta = p \in \mathbb{N}, \end{cases} \tag{3}$$

where $I^\theta g(t)$ is integration of order θ of RL given by

$$I^\theta g(t) = \begin{cases} \frac{1}{(\theta-1)!} \int\limits_0^t (t-\varpi)^{\theta-1} g(\varpi) d\varpi, & \theta > 0, \\ g(t), & \theta = 0, \end{cases} \tag{4}$$

we also follow $(\theta - 1)! = \Gamma(\theta)$.

Definition 2 The RL partial derivative [38–40] of the function $\varphi(x,t)$ of order θ is defined as

$$D_t^{\theta} g(t) = \begin{cases} \frac{1}{\Gamma(p-\theta)} \frac{\partial^p}{\partial t^p} \int\limits_0^t (t-\varpi)^{p-\theta-1} g(\varpi) d\varpi, & -1 < \theta < p, \\ \frac{\partial^p \varphi}{\partial t^p}, & \theta = p \in \mathbb{N}. \end{cases} \tag{5}$$

2.1 Layout of the Lie symmetry method

Now, we provide the basic idea of the symmetry method [2, 41]. For this purpose, we consider a 3D FPDE

$$\partial_t^{\theta} \varphi(x,y,z,t) = G(t,x,y,z,\varphi,\varphi_x,\varphi_y,\varphi_z,\varphi_{xx},\ldots), \qquad (0 < \theta \leq 1), \tag{6}$$

and the one parameter (local) Lie group of transformations as follows

$$\begin{aligned} t^{\star} &= t + \varepsilon \psi^1(t,x,y,z,\varphi) + O(\varepsilon^2), \\ x^{\star} &= x + \varepsilon \psi^2(t,x,y,z,\varphi) + O(\varepsilon^2), \\ y^{\star} &= y + \varepsilon \psi^3(t,x,y,z,\varphi) + O(\varepsilon^2), \\ z^{\star} &= y + \varepsilon \psi^4(t,x,y,z,\varphi) + O(\varepsilon^2), \\ \varphi^{\star} &= \varphi + \varepsilon \phi(t,x,y,z,\varphi) + O(\varepsilon^2), \end{aligned} \tag{7}$$

where $\varepsilon << 1$ is its parameter, and associated vector field has the following form

$$\mathcal{X} = \psi^1 \frac{\partial}{\partial t} + \psi^2 \frac{\partial}{\partial x} + \psi^3 \frac{\partial}{\partial y} + \psi^4 \frac{\partial}{\partial z} + \phi \frac{\partial}{\partial \varphi}\cdot \tag{8}$$

Similarly, the transformations for rest variables becomes

$$\begin{aligned} \frac{\partial^{\theta} \varphi^{\star}}{\partial t^{\theta}} &= \frac{\partial^{\theta} \varphi}{\partial t^{\theta}} + \varepsilon \phi_{\theta}^0(t,x,y,z,\varphi) + O(\varepsilon^2), \\ \frac{\partial \varphi^{\star}}{\partial x} &= \frac{\partial \varphi}{\partial x} + \varepsilon \phi^x(t,x,y,z,\varphi) + O(\varepsilon^2), \\ \frac{\partial^2 \varphi^{\star}}{\partial x^2} &= \frac{\partial^2 \varphi}{\partial x^2} + \varepsilon \phi^{xx}(t,x,y,z,\varphi) + O(\varepsilon^2), \\ \frac{\partial^2 \varphi^{\star}}{\partial y^2} &= \frac{\partial^2 \varphi}{\partial y^2} + \varepsilon \phi^{yy}(t,x,y,z,\varphi) + O(\varepsilon^2), \\ \frac{\partial^2 \varphi^{\star}}{\partial z^2} &= \frac{\partial^2 \varphi}{\partial z^2} + \varepsilon \phi^{zz}(t,x,y,z,\varphi) + O(\varepsilon^2), \\ &\;\,\vdots \end{aligned} \tag{9}$$

The Lie algebra associated to (6) is prolongation of (8) written as

$$\mathcal{X}^{[\theta,n]} = \mathcal{X} + \phi_{\theta}^0 \frac{\partial}{\partial t^{\theta}} + \phi^x \frac{\partial}{\partial \varphi_x} + \phi^{xx} \frac{\partial}{\partial \varphi_{xx}} + \phi^{yy} \frac{\partial}{\partial \varphi_{yy}} + \phi^{zz} \frac{\partial}{\partial \varphi_{zz}} + \cdots, \tag{10}$$

where n is the order of Eq.(6) and

$$\begin{aligned}
\phi^{x} &= D_x(\phi) - \varphi_t D_x(\psi^1) - \varphi_x D_x(\psi^2) - \varphi_y D_x(\psi^3) - \varphi_z D_x(\psi^4),\\
\phi^{xx} &= D_x(\phi^x) - \varphi_{tx} D_x(\psi^1) - \varphi_{xx} D_x(\psi^2) - \varphi_{xy} D_x(\psi^3) - \varphi_{xz} D_x(\psi^4),\\
\phi^{yy} &= D_y(\phi^y) - \varphi_{ty} D_y(\psi^1) - \varphi_{xy} D_y(\psi^2) - \varphi_{yy} D_y(\psi^3) - \varphi_{yz} D_y(\psi^4),\\
\phi^{zz} &= D_z(\phi^z) - \varphi_{tz} D_z(\psi^1) - \varphi_{xz} D_z(\psi^2) - \varphi_{yz} D_z(\psi^3) - \varphi_{zz} D_z(\psi^4),\\
&\;\;.\\
&\;\;.
\end{aligned} \tag{11}$$

From (11), the total differential operator D_i is defined as:

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + \varphi_t \frac{\partial}{\partial \varphi} + \varphi_{tt} \frac{\partial}{\partial \varphi_t} + \varphi_{xt} \frac{\partial}{\partial \varphi_x} \ldots,\\
D_x &= \frac{\partial}{\partial x} + \varphi_x \frac{\partial}{\partial \varphi} + \varphi_{xx} \frac{\partial}{\partial \varphi_x} + \varphi_{xx} \frac{\partial}{\partial \varphi_x} \ldots,
\end{aligned} \tag{12}$$

The operator (8) is known as point symmetry of (6) iff,

$$\mathcal{X}^{[\theta,n]}(\Delta)|_{\Delta=0} = 0, \tag{13}$$

where

$$\Delta := \partial_t^{\theta} \varphi(x, y, z, t) - G(x, y, z, t, \varphi, \varphi_x, \varphi_y, \varphi_{xx}, \ldots).$$

The invariance condition [37] is necessary to the transformation (7), because of the conservative property of above operator

$$\psi^1(x, y, z, t, \varphi)|_{t=0} = 0. \tag{14}$$

Also, the explicit form of θ^{th} extended infinitesimals is given below:

$$\begin{aligned}
\phi_\theta^0 =& \frac{\partial^\theta \phi}{\partial t^\theta} + \left(\phi_\varphi - \theta D_t(\psi^1)\right) \frac{\partial^\theta \varphi}{\partial t^\theta} - \varphi \frac{\partial^\theta \phi_\varphi}{\partial t^\theta}\\
&+ \sum_{p=1}^{\infty} \left[\binom{\theta}{p} \frac{\partial^\theta \phi_\varphi}{\partial t^\theta} - \binom{\theta}{p+1} D_t^{p+1}(\psi^1) \right] D_t^{\theta-p}(\varphi)\\
&- \sum_{p=1}^{\infty} \binom{\theta}{p} D_t^p(\psi^2) D_t^{\theta-p}(\varphi_x) - \sum_{p=1}^{\infty} \binom{\theta}{p} D_t^p(\psi^3) D_t^{\theta-p}(\varphi_y)\\
&- \sum_{p=1}^{\infty} \binom{\theta}{p} D_t^p(\psi^4) D_t^{\theta-p}(\varphi_z) + \lambda,
\end{aligned} \tag{15}$$

where

$$\binom{\theta}{p} = \frac{(-1)^{p-1} \theta \Gamma(p-\theta)}{\Gamma(1-\theta)\Gamma(p+1)},$$

and

$$\lambda = \sum_{p=2}^{\infty} \sum_{l=2}^{p} \sum_{s=2}^{l} \sum_{q=0}^{s-1} \binom{\theta}{p} \binom{p}{l} \binom{s}{q} \frac{1}{s!} \frac{t^{p-\theta}}{\Gamma(p+1-\theta)} (-1)^q \varphi^q \frac{\partial^l}{\partial t^l} [\varphi^{s-q}] \frac{\partial^{p-l+s}}{\partial t^{p-l} \partial \varphi^s} \cdot \tag{16}$$

3 Application of the method to BB Eq.(1)

Suppose that Eq.(1) remains invariant under the (7) and (9), then we have

$$D_t^\theta \bar{\varrho} - \bar{\varrho}_{xx} - \bar{\varrho}_{yy} - \bar{\varrho}_{zz} - \bar{\varrho}\bar{\varrho}_x = 0, \qquad (0 < \theta \leq 1), \tag{17}$$

such that $\varrho = \varphi(x, y, z, t)$ becomes similar to Eq (1). After that substituting Eq. (7) into Eq. (17), we get the following invariant equation:

$$\phi_\theta^0 - \phi^{xx} - \phi^{zz} - \phi^{zz} - \varphi_x \phi - \varphi\phi^x = 0. \tag{18}$$

By substituting the values of ϕ^x, ϕ^{xx}, ϕ^{yy}, ϕ^{zz} and ϕ_θ^0 into Eq. (18) and after comparing coefficients of all the monomials of derivatives of φ, we obtained the following determining system:

$$\begin{aligned}
&\phi_{\varphi\varphi} = \psi_\varphi^1 = \psi_x^1 = \psi_y^1 = \psi_z^1 = \psi_\varphi^2 = \psi_\varphi^3 = \psi_\varphi^4 = \psi_t^2 = \psi_t^3 = \psi_t^4 = 0,\\
&\psi_{xx}^2 + \psi_{yy}^2 + \psi_{zz}^2 - 2\phi_{x\varphi} + \varphi(\psi_x^2 - \theta\psi_t^1) - \phi = 0,\\
&\psi_{xx}^3 + \psi_{yy}^3 + \psi_{zz}^3 - 2\phi_{y\varphi} + \varphi\psi_x^3 = 0,\\
&\psi_{xx}^4 + \psi_{yy}^4 + \psi_{zz}^4 - 2\phi_{z\varphi} + \varphi\psi_x^4 = 0,\\
&2\psi_x^2 - \theta\psi_t^1 = 0,\\
&2\psi_x^3 - \theta\psi_t^1 = 0,\\
&2\psi_x^4 - \theta\psi_t^1 = 0,\\
&2(\psi_x^3 + \psi_y^2) = 0,\\
&2(\psi_x^4 + \psi_z^2) = 0,\\
&2(\psi_y^4 + \psi_z^3) = 0,\\
&\partial_t^\theta \phi - \phi_{xx} - \phi_{yy} - \phi_{zz} - \varphi\phi_x - \varphi\partial_t^\theta\phi_\varphi = 0,\\
&\binom{\theta}{p}\partial_t^\theta\phi - \binom{\theta}{p+1} D_t^{p+1}(\psi^1) = 0. \quad p = 1, 2, 3, \ldots
\end{aligned} \tag{19}$$

where $(0 < \theta \leq 1)$. Solution of above system yields

$$\psi^1 = \frac{2}{\theta} b_1 t + b_6, \quad \psi^2 = b_1 x + b_2,$$
$$\psi^3 = b_1 y + b_3 z + b_4, \quad \psi^4 = b_1 z - b_3 y + b_5, \quad \phi = -b_1\varphi,$$

where b_1, b_2, b_3, b_4, b_5 and b_6 are constants. Using the time invariance condition (14), we get $b_6 = 0$ and so the infinitesimals are

$$\psi^1 = \frac{2}{\theta} b_1 t, \quad \psi^2 = b_1 x + b_2, \quad \psi^3 = b_1 y + b_3 z + b_4,$$
$$\psi^4 = b_1 z - b_3 y + b_5, \quad \phi = -b_1\varphi.$$

Thus, we have five-dimensional symmetry algebra for the BB Eq. (1) given by

$$\mathcal{X}_1 = \frac{\partial}{\partial x}, \quad \mathcal{X}_2 = \frac{\partial}{\partial y}, \quad \mathcal{X}_3 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad \mathcal{X}_4 = \frac{\partial}{\partial z}, \tag{20}$$

$$\mathcal{X}_5 = \frac{2}{\theta} t\frac{\partial}{\partial t} - \varphi\frac{\partial}{\partial \varphi} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\cdot \tag{21}$$

Reduction by symmetry $\mathcal{X}_1$ For symmetry $\mathcal{X}_1$, one can write

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dy}{0} = \frac{dz}{0} = \frac{d\varphi}{0}.$$

This gives the following similarity transformations

$$\nu = t, \quad \varphi(x, y, z, t) = \mathcal{H}(\nu). \tag{22}$$

Putting these values in Eq. (1), we get

$$D_t^{\theta}\varphi = 0,$$

which gives

$$\varphi(x, y, z, t) = k_1 t^{\theta-1}, \quad k_1 > 0. \tag{23}$$

By reduction of symmetry $\mathcal{X}_2$ and $\mathcal{X}_4$, we get the same results.
Reduction by symmetry $\mathcal{X}_5$ If we consider symmetry $\mathcal{X}_5$, we can write

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{\theta dt}{2t} = \frac{d\varphi}{-\varphi},$$

and this provides below similarity transformations.

$$\nu = (z + y + x)t^{-\frac{\theta}{2}}, \quad \varphi(x, y, z, t) = t^{-\frac{\theta}{2}}\mathcal{H}(\nu), \tag{24}$$

where $\mathcal{H}$ is dependent on variable ν. Substituting Eq. (24) into Eq. (1), we get a nonlinear FODE.

After this procedure, we have the following special result:

Theorem 1 *The transformation (24) converts the BB Eq. (1) to following nonlinear FODE*

$$\left(\mathcal{P}_{\frac{2}{\theta}}^{1-\frac{3\theta}{2},\theta}\mathcal{H}\right)(\nu) - \mathcal{H}\mathcal{H}_{\nu} - 3\mathcal{H}_{\nu\nu} = 0, \tag{25}$$

where the generalized form of EK fractional differential operator [41, 42] is defined by

$$\left(\mathcal{P}_{\beta}^{\tau,\theta}\mathcal{H}\right)(\nu) = \prod_{j=0}^{p-1}\left(\tau + j - \frac{1}{\beta}\nu\frac{d}{d\nu}\right)\left(\mathcal{K}_{\beta}^{\tau+\theta,p-\theta}\mathcal{H}\right)(\nu), \tag{26}$$

$$p = \begin{cases} [\theta] + 1, & \theta \notin \mathbb{N}, \\ \theta, & \theta \in \mathbb{N}, \end{cases} \tag{27}$$

and

$$\left(\mathcal{K}_{\beta}^{\tau,\theta}\mathcal{H}\right)(\nu) = \begin{cases} \frac{1}{\Gamma(\theta)}\int\limits_1^{\infty}(\varphi - 1)^{\theta-1}\varphi^{-(\tau+\theta)}\mathcal{H}(\nu\varphi^{\frac{1}{\beta}})d\varphi, & \theta > 0 \\ \mathcal{H}(\nu), & \theta = 0, \end{cases} \tag{28}$$

where $\mathcal{K}_{\beta}^{\tau,\theta}$ is the EK fractional integral operator.

Proof Suppose that $p - 1 < \theta < p$, $p = 1, 2, 3, \ldots$ and order(p)=order(r), here $r \in \mathbb{R}$. After using RL definition for Eq. (24), we get

$$\frac{\partial^{\theta}\varphi}{\partial t^{\theta}} = \frac{\partial^r}{\partial t^r}\left[\frac{1}{\Gamma(r-\theta)}\int\limits_0^t (t-\alpha)^{r-\theta-1}\alpha^{\frac{\theta}{2}}\mathcal{H}\left((x+y+z)\alpha^{\frac{-\theta}{3}}\right)ds\right]. \tag{29}$$

Substitute $\gamma = t/\alpha \Rightarrow d\alpha = -(t/\gamma^2)d\gamma$, then Eq. (29) can be written as:

$$\frac{\partial^\theta \varphi}{\partial t^\theta} = \frac{\partial^r}{\partial t^r}\left[t^{r-\frac{3\theta}{2}} \frac{1}{\Gamma(r-\theta)} \int_1^\infty (\gamma-1)^{r-\theta-1}\gamma^{-(r-\theta+1-\frac{\theta}{2})}\mathcal{H}\left(\nu\gamma^{\frac{\theta}{2}}\right)d\gamma\right], \tag{30}$$

By following (28), above equation becomes:

$$\frac{\partial^\theta \varphi}{\partial t^\theta} = \frac{\partial^r}{\partial t^r}\left[t^{r-\frac{3\theta}{2}}\left(\mathcal{K}_{\frac{2}{\theta}}^{1-\frac{\theta}{2},r-\theta}\mathcal{H}\right)(\nu)\right]. \tag{31}$$

After simplification the right hand side of Eq. (31) changes as:

$$\begin{aligned}
\frac{\partial^\theta \varphi}{\partial t^\theta} &= \frac{\partial^{r-1}}{\partial t^{r-1}}\left[\frac{\partial}{\partial t}\left(t^{r-\frac{3\theta}{2}}\left(\mathcal{K}_{\frac{2}{\theta}}^{1-\frac{\theta}{2},r-\theta}\mathcal{H}\right)(\nu)\right)\right],\\
&= \frac{\partial^{r-1}}{\partial t^{r-1}}\left[t^{r-1-\frac{3\theta}{2}}\left(r-\frac{3\theta}{2}-\frac{\theta}{2}\nu\frac{d}{d\nu}\right)\left(\mathcal{K}_{\frac{2}{\theta}}^{1-\frac{\theta}{2},r-\theta}\mathcal{H}\right)(\nu)\right],\\
&\;\cdot\\
&\;\cdot\\
&\;\cdot\\
&= t^{-\frac{3\theta}{2}}\prod_{j=0}^{p-1}\left[\left(1-\frac{3\theta}{2}+j-\frac{\theta}{2}\nu\frac{d}{d\nu}\right)\left(\mathcal{K}_{\frac{2}{\theta}}^{1-\frac{3\theta}{2},r-\theta}\mathcal{H}\right)(\nu)\right].
\end{aligned} \tag{32}$$

By following (26), we reach at the main result, i.e.

$$\frac{\partial^r}{\partial t^r}\left[t^{r-\frac{3\theta}{2}}\left(\mathcal{K}_{\frac{2}{\theta}}^{1-\frac{3\theta}{2},r-\theta}\mathcal{H}\right)(\nu)\right] = t^{-\frac{3\theta}{2}}\left(\mathcal{P}_{\frac{2}{\theta}}^{1-\frac{3\theta}{2},\theta}\mathcal{H}\right)(\nu). \tag{33}$$

Therefore Eq. (31) becomes

$$\frac{\partial^\theta \varphi}{\partial t^\theta} = t^{-\frac{3\theta}{2}}\left(\mathcal{P}_{\frac{2}{\theta}}^{1-\frac{3\theta}{2},\theta}\mathcal{H}\right)(\nu). \tag{34}$$

Thus, we get this final result.

$$\left(\mathcal{P}_{\frac{2}{\theta}}^{1-\frac{3\theta}{2},\theta}\mathcal{H}\right)(\nu) - \mathcal{H}\mathcal{H}_\nu - 3\mathcal{H}_{\nu\nu} = 0. \tag{35}$$

So, the proof is completed.

4 Series solutions for the BB Eq. (1)

In this current section, we aim to find series solutions for the BB Eq. (1). To achieve this goal, let us consider the following series solutions of nonlinear FODE (25).

$$\mathcal{H}(\nu) = \sum_{u=0}^{\infty} h_u \nu^u, \tag{36}$$

where h_u are coefficients which we will evaluate later. Now, by using Eq. (36) in Eq. (25), one reaches at the following relation

$$\sum_{u=0}^{\infty}\frac{\Gamma(\frac{4-3\theta}{2}-\frac{n\theta}{2})}{\Gamma(\frac{4-\theta}{2}-\frac{n\theta}{2})}h_u\nu^u-3\sum_{u=0}^{\infty}(u+1)(u+2)h_{u+2}\nu^u \\ -\sum_{u=0}^{\infty}\sum_{w=0}^{u}(u+1-w)h_w h_{u+1-w}\nu^u=0. \tag{37}$$

So, by comparing coefficients for $u=0$, we get

$$h_2=\frac{1}{6}\left[\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}h_0-h_0h_1\right], \tag{38}$$

where h_0 and h_1 are an arbitrary constants. Generally, for $u\geq 1$, we can get the following recursive relation

$$h_{u+2}=\frac{1}{3(u+1)(u+2)}\left[\frac{\Gamma(\frac{4-3\theta}{2}-\frac{n\theta}{2})}{\Gamma(\frac{4-\theta}{2}-\frac{n\theta}{2})}h_u-\sum_{w=0}^{u}(u+1-w)h_w h_{u+1-w}\right]. \tag{39}$$

It follows from Eqs. (38) and (39) that all coefficients h_u, for $u\geq 3$ can be expressed in terms of both h_0 and h_1. Now, to simplify our case we can first choose $h_0\neq 0, h_1=0$, this gives coefficients for one solution expressed entirely in terms of h_0. Next, if we chose $h_1\neq 0, h_0=0$, then coefficients for other solutions can expressed in terms of h_1.
Using Eq. (38) in both cases.

Case: 1 When $h_0\neq 0, h_1=0$ then (38) becomes

$$h_2=\frac{1}{6}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}h_0\right). \tag{40}$$

Now, the recurrence relation (39) for $u=1,2,3,\ldots$ gives.

$$\begin{aligned} h_3&=\frac{-1}{54}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}h_0^2\right),\\ h_4&=\frac{1}{216}\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\left(\frac{\Gamma(\frac{4-5\theta}{2})}{\Gamma(\frac{4-3\theta}{2})}h_0+\frac{1}{18}h_0^3\right),\\ h_5&=\left(\frac{-1}{3240}\frac{\Gamma(2)}{\Gamma(2-2\theta)}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\right)-\frac{1}{3240}\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\left(\frac{\Gamma(\frac{4-5\theta}{2})}{\Gamma(\frac{4-3\theta}{2})}\right)\right.\\ &\quad\left.-\frac{1}{1080}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\right)\right)h_0^2-\frac{1}{58320}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\right)h_0^4. \end{aligned} \tag{41}$$

We get the following solution in this case

$$
\begin{aligned}
\mathcal{H}_0(\nu) &= h_0 + \frac{1}{6}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}h_0\right)\nu^2 - \frac{1}{54}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}h_0^2\right)\nu^3 + \frac{1}{216}\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\left(\frac{\Gamma(\frac{4-5\theta}{2})}{\Gamma(\frac{4-3\theta}{2})}h_0\right.\\
&\quad \left. + \frac{1}{18}h_0^3\right)\nu^4\\
&\quad - \left[\left(\frac{1}{3240}\frac{\Gamma(2)}{\Gamma(2-2\theta)}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\right) + \frac{1}{3240}\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\left(\frac{\Gamma(\frac{4-5\theta}{2})}{\Gamma(\frac{4-3\theta}{2})}\right)\right.\right.\\
&\quad \left. + \frac{1}{1080}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\right)\right)h_0^2\\
&\quad \left. + \frac{1}{58320}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\right)h_0^4\right]\nu^5 + \cdots,
\end{aligned}
\tag{42}
$$

where h_0 is any arbitrary non-zero constant.
Case: 2 When $h_1 \neq 0$, $h_0 = 0$ then (38) becomes

$$
h_2 = 0. \tag{43}
$$

Now, the recurrent relation (39) for $u = 1, 2, 3, \ldots$ gives

$$
\begin{aligned}
h_3 &= \frac{1}{18}\left(\frac{\Gamma(2-2\theta)}{\Gamma(2-\theta)}h_1 - h_1^2\right),\\
h_4 &= 0,\\
h_5 &= \frac{1}{1080}\left(\frac{\Gamma(2)}{\Gamma(2-2\theta)}\left(\frac{\Gamma(2-2\theta)}{\Gamma(2-\theta)}\right)\right)h_1 - \frac{1}{60}\left(\frac{\Gamma(2)}{\Gamma(2-2\theta)}\right)h_1^2.
\end{aligned}
\tag{44}
$$

We get the following solution in this case

$$
\begin{aligned}
\mathcal{H}_1(\nu) = h_1\nu + \tfrac{1}{18}\left(\tfrac{\Gamma(2-2\theta)}{\Gamma(2-\theta)}h_1 - h_1^2\right)\nu^3 + \left[\tfrac{1}{1080}\left(\tfrac{\Gamma(2)}{\Gamma(2-2\theta)}\left(\tfrac{\Gamma(2-2\theta)}{\Gamma(2-\theta)}\right)\right)h_1\right.\\
\left. - \tfrac{1}{60}\left(\tfrac{\Gamma(2)}{\Gamma(2-2\theta)}\right)h_1^2\right]\nu^5 + \cdots,
\end{aligned}
\tag{45}
$$

where h_1 is any arbitrary non-zero constant. Hence, by using (42) and (45) the final form of Eq. (36) is written as:

$$
\mathcal{H}(\nu) = \mathcal{H}_0(\nu) + \mathcal{H}_1(\nu) \tag{46}
$$

$$
\begin{aligned}
\mathcal{H}(\nu) &= h_0 + h_1\nu + \frac{1}{6}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}h_0\right)\nu^2 + \left(\frac{-1}{54}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}h_0^2\right) + \frac{1}{18}\left(\frac{\Gamma(2-2\theta)}{\Gamma(2-\theta)}h_1 - h_1^2\right)\right)\nu^3\\
&\quad + \frac{1}{216}\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\left(\frac{\Gamma(\frac{4-5\theta}{2})}{\Gamma(\frac{4-3\theta}{2})}h_0 + \frac{1}{18}h_0^3\right)\nu^4
\end{aligned}
$$

$$+\Bigg[\Bigg(\frac{-1}{3240}\frac{\Gamma(2)}{\Gamma(2-2\theta)}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\right)-\frac{1}{3240}\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\left(\frac{\Gamma(\frac{4-5\theta}{2})}{\Gamma(\frac{4-3\theta}{2})}\right)$$
$$-\frac{1}{1080}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\right)\Bigg)h_0^2-\frac{1}{58320}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\right)h_0^4+\frac{1}{1080}\left(\frac{\Gamma(2)}{\Gamma(2-2\theta)}\left(\frac{\Gamma(2-2\theta)}{\Gamma(2-\theta)}\right)\right)h_1$$
$$-\frac{1}{60}\left(\frac{\Gamma(2)}{\Gamma(2-2\theta)}\right)h_1^2\Bigg]\nu^5+\cdots. \tag{47}$$

Then, the explicit solution of (25) can be expressed in the form of:

$$\varphi(x,y,z,t)=h_0+h_1(x+y+z)t^{-\theta}+\frac{1}{6}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}h_0\right)(x+y+z)^2t^{-\frac{3\theta}{2}}$$
$$+\left(\frac{-1}{54}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}h_0^2\right)+\frac{1}{18}\left(\frac{\Gamma(2-2\theta)}{\Gamma(2-\theta)}h_1-h_1^2\right)\right)(x+y+z)^3t^{-2\theta}$$
$$+\frac{1}{216}\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\left(\frac{\Gamma(\frac{4-5\theta}{2})}{\Gamma(\frac{4-3\theta}{2})}h_0+\frac{1}{18}h_0^3\right)(x+y+z)^4t^{-\frac{5\theta}{2}}$$
$$+\Bigg[\Bigg(\frac{-1}{3240}\frac{\Gamma(2)}{\Gamma(2-2\theta)}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\right)$$
$$-\frac{1}{3240}\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\left(\frac{\Gamma(\frac{4-5\theta}{2})}{\Gamma(\frac{4-3\theta}{2})}\right)-\frac{1}{1080}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\right)\Bigg)h_0^2-\frac{1}{58320}\left(\frac{\Gamma(\frac{4-3\theta}{2})}{\Gamma(\frac{4-\theta}{2})}\right)h_0^4$$
$$+\frac{1}{1080}\left(\frac{\Gamma(2)}{\Gamma(2-2\theta)}\left(\frac{\Gamma(2-2\theta)}{\Gamma(2-\theta)}\right)\right)h_1$$
$$-\frac{1}{60}\left(\frac{\Gamma(2)}{\Gamma(2-2\theta)}\right)h_1^2\Bigg](x+y+z)^5t^{-3\theta}+\ldots \tag{48}$$

4.1 Convergence analysis of the explicit structures

This analysis makes us able to examine how the explicit solution (36), with coefficients provided by Eqs. (38) and (39), converges. Eq. (39) can be written as:

$$|h_{u+2}|\le\frac{|\Gamma(\frac{4-3\theta}{2})|}{|\Gamma(\frac{4-\theta}{2})|}|h_u|+\sum_{w=0}^{u}|h_w||h_{u+1-w}|. \tag{49}$$

But $\frac{\Gamma(n)}{\Gamma(m)}<1$ for any $n(m)$ numbers, therefore (49) becomes

$$|h_{u+2}|\le|h_u|+\sum_{w=0}^{u}|h_w||h_{u+1-w}|. \tag{50}$$

Now, consider another power series suggestion as

$$\mathcal{C}(\nu)=\sum_{u=0}^{\infty}c_s\nu^u, \tag{51}$$

with $c_i=|h_i|, i=0,1$ so, one can write

$$c_{u+2}=c_u+\sum_{w=0}^{u}c_wc_{u+1-w}, \tag{52}$$

where $u = 0, 1, 2, \cdots$ and it is clear that

$$|h_{u+2}| \leq c_{u+2} \quad \Rightarrow \quad |h_u| \leq c_u.$$

By following above relations, we reach a result that Eq. (51) is a majorant series of Eq. (36). Now, we need to prove that series (51) has a positive radius of convergence. So, Eq. (51) takes the following form

$$\begin{aligned} \mathcal{C}(\nu) &= c_0 + c_1\nu + \sum_{u=2}^{\infty} c_u \nu^s, \\ &= c_0 + c_1\nu + \sum_{u=0}^{\infty}\left[c_u + \sum_{w=0}^{u} c_w c_{u+1-w}\right]\nu^{u+2}, \\ &= c_0 + c_1\nu + \sum_{u=0}^{\infty} c_u\nu^{u+2} + \sum_{u=0}^{\infty}\sum_{w=0}^{u} c_w c_{u+1-w}\nu^{u+2}. \end{aligned} \tag{53}$$

Now, considering an implicit functional theorem concerning ν, we get

$$\delta(\nu, \mathcal{C}) = \mathcal{C} - c_0 - c_1\nu - [\mathcal{C} - c_0]\nu^2 - [\mathcal{C}^2 - c_0^2]\nu^2, \tag{54}$$

- since δ is an analytic function in the neighborhood of $(0, c_0)$,
- $\delta(0, c_0) = 0$ and
- $\frac{\partial\delta}{\partial\mathcal{C}}(0, c_0) \neq 0$.

The implicit function theorem states that if $\delta(\mathcal{C}, c)$ is continuously differentiable and $\frac{\partial\delta}{\partial\mathcal{C}}(0, c_0)$ is non-zero at $(0, c)$, then there exists a function $\mathcal{C} = \mathcal{C}(\nu)$ such that $\delta(\mathcal{C}(c), c) = 0$, for c near c_0. Thus, we reach at convergence.

5 Conservation laws

Now, we calculate CLs of Eq. (1) which satisfy the below form:

$$\left[D_t(\hat{S}^t) + D_x(\hat{S}^x) + D_y(\hat{S}^y) + D_z(\hat{S}^z)\right]_{Eq(1)} = 0, \tag{55}$$

where $\hat{S}^t$, $\hat{S}^x$, $\hat{S}^y$ and $\hat{S}^z$ are conserved quantities. By using the results given in [10, 35], formal Lagrangian of Eq. (1) has this form:

$$\mathcal{L} = \rho(x, y, z, t)[D_t^\theta\varphi - \varphi_{xx} - \varphi_{yy} - \varphi_{zz} - \varphi\varphi_x], \qquad (0 < \theta \leq 1), \tag{56}$$

where $\rho(x, y, z, t)$ is a new dependent variable. By using Eq. (56), we follow the action integral as

$$\int_0^t \int_{\Omega_1} \int_{\Omega_2} \int_{\Omega_3} \mathcal{L}(x, y, z, t, D_t^\theta\varphi, \varphi_x, \varphi_{xx}, \varphi_{yy}, \varphi_{zz}) dxdydzdt. \tag{57}$$

And, the Euler-Lagrange operator is defined as

$$\frac{\delta}{\delta\varphi} = \frac{\partial}{\partial\varphi} - (D_t^\theta)^* \frac{\partial}{\partial D_t^\theta\varphi} - D_x\frac{\partial}{\partial\varphi_x} + D_x^2\frac{\partial}{\partial\varphi_{xx}} + D_y^2\frac{\partial}{\partial\varphi_{yy}} + D_z^2\frac{\partial}{\partial\varphi_{zz}}, \tag{58}$$

where $(D_t^\theta)^*$ denotes adjoint operator in the Fre´chet sense of (D_t^θ) and the adjoint equation for (1) is given as

$$\frac{\delta \mathcal{L}}{\delta \varphi} = 0. \tag{59}$$

The operator (adjoint) $(D_t^\theta)^*$ from (58) in the sense of RL fractional operator is given by

$$(D_t^\theta)^* = (-1)^p I_T^{p-\theta}(D_t^p) = {}_t^d D_T^\theta, \tag{60}$$

where $I_T^{p-\theta}$ is written as:

$$I_T^{p-\theta} g(x,t) = \frac{1}{\Gamma(p-\theta)} \int_t^\kappa \frac{g(x,\kappa)}{(\kappa-t)^{\theta-p+1}} d\kappa. \tag{61}$$

Considering one dependent variable φ and four independent variables $t,\ x,\ y$ and z we have this form:

$$\begin{aligned}&\hat{\mathcal{X}} + D_t(\psi^1)\hat{\mathcal{I}} + D_x(\psi^2)\hat{\mathcal{I}} + D_y(\psi^3)\hat{\mathcal{I}} + D_z(\psi^4)\hat{\mathcal{I}} \\ &\quad = W\frac{\delta}{\delta\varphi} + D_t(\hat{S}^t) + D_x(\hat{S}^x) + D_y(\hat{S}^y) + D_z(\hat{S}^z)\end{aligned} \tag{62}$$

where the operator $\hat{\mathcal{I}}$ is known as identity operator and $\frac{\delta}{\delta\varphi}$ is Euler-Lagrange operator. So, $\hat{\mathcal{X}}$ has below form

$$\begin{aligned}\hat{\mathcal{X}} = \psi^1\frac{\partial}{\partial t} + \psi^2\frac{\partial}{\partial x} + \psi^3\frac{\partial}{\partial y} + \psi^4\frac{\partial}{\partial z} + \phi_\theta^0\frac{\partial}{\partial D_t^\theta\varphi} + \phi^x\frac{\partial}{\partial\varphi_x} \\ + \phi^{xx}\frac{\partial}{\partial\varphi_{xx}} + \phi^{yy}\frac{\partial}{\partial\varphi_{yy}} + \phi^{zz}\frac{\partial}{\partial\varphi_{zz}},\end{aligned} \tag{63}$$

and the Lie characteristic function (W) given by
$W = \phi - \psi^1\varphi_t - \psi^2\varphi_x - \psi^3\varphi_y - \psi^4\varphi_z$.
By considering RL-fractional derivative, the $\hat{S}^t$ component of conserved quantities is given by:

$$\hat{S}^t = \psi^1\mathcal{L} + \sum_{v=0}^{p-1}(-1)^v_0 D_t^{\theta-1-v}(W_m)D_t^v\frac{\partial\mathcal{L}}{\partial_0 D_t^\theta\varphi} - (-1)^p J\left(W_m, D_t^p\frac{\partial\mathcal{L}}{\partial_0 D_t^\theta\varphi}\right), \tag{64}$$

where $J(.)$ is defined as:

$$J(a,b) = \frac{1}{\Gamma(p-\theta)}\int_0^t\int_t^T \frac{a(T,x,y,z)b(\kappa,x,y,z)}{(\kappa-T)^{\theta-p+1}} d\kappa dT, \tag{65}$$

and $\hat{S}^i$ for independent variables $x,\ y$ and z yields:

$$\begin{aligned}\hat{S}^i =& \psi^i\mathcal{L} + W_m\left[\frac{\partial\mathcal{L}}{\partial\varphi_i^m} - D_j(\frac{\partial\mathcal{L}}{\partial\varphi_{ij}^m})\right. \\ &\left. + D_jD_k(\frac{\partial\mathcal{L}}{\partial\varphi_{ijk}^m}) - \dots\right] + D_j(W_m)\left[\frac{\partial\mathcal{L}}{\partial\varphi_{ij}^m} - D_k(\frac{\partial\mathcal{L}}{\partial\varphi_{ijk}^m}) + \dots\right] \\ & + D_jD_k(W_m)\left[\frac{\partial\mathcal{L}}{\partial\varphi_{ijk}^m} - \dots\right] + \dots,\end{aligned} \tag{66}$$

where $i,\ j, k = 1, 2, 3$ and $m = 1, 2, 3$.

Case: 1 For symmetry $\mathcal{X}_1$, we get $W_1 = -\varphi_x$. Putting these values in above equations, we find

$$\begin{aligned}
\hat{S}^t &= \rho D_t^{\theta-1}(-\varphi_x) + J(-\varphi_x, \rho_t),\\
\hat{S}^x &= \rho[D_t^{\theta}\varphi - \varphi_{yy} - \varphi_{zz}] - \rho_x\varphi_x,\\
\hat{S}^y &= -\varphi_x\rho_y + \rho\varphi_{xy},\\
\hat{S}^z &= -\varphi_x\rho_z + \rho\varphi_{xz}.
\end{aligned}$$

Case: 2 For $\mathcal{X}_2$, we get $W_2 = -\varphi_y$. Putting these values in Eqs. (64–66), we have

$$\begin{aligned}
\hat{S}^t &= \rho D_t^{\theta-1}(-\varphi_y) + J(-\varphi_y, \rho_t),\\
\hat{S}^x &= \varphi_y(\varphi\rho - \rho_x) + \rho\varphi_{xy},\\
\hat{S}^y &= \rho[D_t^{\theta}\varphi - \varphi_{xx} - \varphi_{zz} - \varphi\varphi_x] - \rho_y\varphi_y,\\
\hat{S}^z &= -\varphi_y\rho_z + \rho\varphi_{yz}.
\end{aligned}$$

Case: 3 For $\mathcal{X}_3$, $W_3 = -z\varphi_y + y\varphi_z$. Putting these values in Eqs. (64–66), we get

$$\begin{aligned}
\hat{S}^t &= \rho D_t^{\theta-1}(-z\varphi_y + y\varphi_z) + J(-z\varphi_y + y\varphi_z, \rho_t),\\
\hat{S}^x &= (-z\varphi_y + y\varphi_z)(\varphi + \rho_x) - \rho(-z\varphi_{xy} + y\varphi_{xz}),\\
\hat{S}^y &= z\mathcal{L} + \rho_y(-z\varphi_y + y\varphi_z) - \rho(-z\varphi_{yy} + \varphi_z + y\varphi_{yz}),\\
\hat{S}^z &= -y\mathcal{L} + \rho_z(-z\varphi_y + y\varphi_z) - \rho(-z\varphi_{zy} - \varphi_y + y\varphi_{zz}).
\end{aligned}$$

Case: 4 For symmetry $\mathcal{X}_4$, we get $W_4 = -\varphi_z$. By following cases 1 and 2, we have

$$\begin{aligned}
\hat{S}^t &= \rho D_t^{\theta-1}(-\varphi_z) + J(-\varphi_z, \rho_t),\\
\hat{S}^x &= \varphi_z(\varphi\rho - \rho_x) + \rho\varphi_{xz},\\
\hat{S}^y &= -\varphi_z\rho_y + \rho\varphi_{yz},\\
\hat{S}^z &= \rho[D_t^{\theta}\varphi - \varphi_{xx} - \varphi_{yy} - \varphi\varphi_x] - \rho_z\varphi_z.
\end{aligned}$$

Case: 5 For $\mathcal{X}_5$, we get $W_5 = -\varphi - \frac{2}{\theta}t\varphi_t - x\varphi_x - y\varphi_y - z\varphi_z$. By following the same pattern as above, we have

$$\begin{aligned}
\hat{S}^t &= \frac{2}{\theta}t\mathcal{L} + \rho D_t^{\theta-1}(-\varphi - \frac{2}{\theta}t\varphi_t - x\varphi_x\\
&\quad -y\varphi_y - z\varphi_z) + J((-\varphi - \frac{2}{\theta}t\varphi_t - x\varphi_x - y\varphi_y - z\varphi_z), \rho_t),\\
\hat{S}^x &= x\mathcal{L} + (-\varphi - \frac{2}{\theta}t\varphi_t - x\varphi_x - y\varphi_y - z\varphi_z)(-\varphi + \rho_x)\\
&\quad -\rho(-\varphi_x - \frac{2}{\theta}t\varphi_{xt} - \varphi_x - x\varphi_{xx}\\
&\quad -y\varphi_{xy} - z\varphi_{xz}),\\
\hat{S}^y &= y\mathcal{L} + \rho_y(-\varphi - \frac{2}{\theta}t\varphi_t - x\varphi_x - y\varphi_y - z\varphi_z)\\
&\quad -\rho(-\varphi_y - \frac{2}{\theta}t\varphi_{yt} - x\varphi_{xy} - \varphi_y - y\varphi_{yy} - z\varphi_{yz}),\\
\hat{S}^z &= z\mathcal{L} + \rho_z(-\varphi - \frac{2}{\theta}t\varphi_t - x\varphi_x - y\varphi_y - z\varphi_z)
\end{aligned}$$

$$-\rho(-\varphi_z - \frac{2}{\theta}t\varphi_{zt} - x\varphi_{xz} - y\varphi_{yz} - \varphi_z - z\varphi_{zz}).$$

6 Conclusions

In this paper, the Lie point symmetry properties of the 3D time-fractional BB equation have been studied using the RL fractional derivative. These symmetries have been employed to derive CLs for the 3D time-fractional BB equation. The study utilizes a new conservation law theorem in conjunction with a formal Lagrangian approach to obtain new components of conserved vectors, enabling the construction of CLs for the 3D time-fractional BB equation. The proposed analysis is an effective and reliable approach for developing CLs and obtaining similarity solutions for fractional differential equations.

Data availability For the current study, no data sharing is applicable because no data were investigated or developed for this article.

Declarations

Conflict of interest The authors professed no conflict of interest.

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