C H A P T E R 6

THE BEHAVIOUR OF SPECIAL COMPOUND POISSON DISTRIBUTIONS AS A FUNCTION OF TIME

In this chapter we investigate the Generalized Inverse Gaussian-Poisson family as a function of time. Special attention is given to the relations between the Inverse Gaussian-Poisson and the Negative Binomial model.

The Generalized Inverse Gaussian-Poisson is a new distribution, which does not have yet enough coverage in the statistical literature. It is presented therefore in great details in Section 6.1. The derivation of other distributions from the Generalized Inverse Gaussian-Poisson is presented as well.

6.1 The Generalized Inverse Gaussian-Poisson Family as a Function of Time

The Generalized Inverse Gaussian-Poisson family, called for short the GIGP, was proposed by Sichel (1971,1973 and 1975). This is a compound Poisson family of distributions, where the Generalized Inverse Gaussian was used for mixing. It is an intermediate probability law between Pearson's type III and type V distributions and at the limits it can take on either form. The probability law of the Generalized Inverse Gaussian is

$$\Psi(\lambda) = \frac{(2\sqrt{1-\theta}/\alpha\theta)^{\gamma}}{2K_{\gamma}(\alpha\sqrt{1-\theta})}\,\lambda^{\gamma-1}\,e^{-\left[\frac{1-\theta}{\theta}\right]\lambda - \frac{\alpha^2\theta}{4\lambda}} \qquad (6.1.1)$$

where $\lambda \in (0,\infty)$, $-\infty < \gamma < \infty$, $\alpha \geq 0$, $0 \leq \theta \leq 1$ and

K_{γ} is the modified Bessel function of the second kind of order γ.

Substitution of (6.1.1) into (5.1.1) leads, after solving the convolution integral, to a new family of discrete compound Poisson models:

$$f(x|t) = \frac{(\sqrt{1-\theta_t})^{\gamma}}{K_{\gamma}(\alpha_t\sqrt{1-\theta_t})}\,\frac{(\alpha_t\theta_t/2)^{x}}{x!}\,K_{x+\gamma}(\alpha_t)\;, \text{ where} \qquad (6.1.2)$$

$$\alpha_t = \alpha\sqrt{1+(t-1)\theta}\;,\quad \theta_t = \frac{t\theta}{1+(t-1)\theta} \quad \text{and } x = 0, 1, 2, \ldots$$

and where t stands for the length of the time period which is to be considered.

For $t = 1$ we have

$\alpha_1 = \alpha$, $\theta_1 = \theta$ and $\gamma_1 = \gamma$,

which denote the three parameters for the original period.

The parameter $\gamma_t = \gamma$ is time invariant. The indices

$\alpha_t\sqrt{1-\theta_t} = \alpha\sqrt{1-\theta}$ and

$$\frac{\theta_t}{(1-\theta_t)t} = \frac{\theta}{1-\theta}$$

are also time independent.

The parameters θ and θ_t respectively characterize the length of the tail of the distribution $f(x|t)$. If $\theta \to 1$ the tail is very long, whereas for $\theta \to 0$ it becomes short.

For $t = 1$, and by substituting $\theta = \frac{2\beta}{\alpha}$ into (6.1.2) one obtains $f(x|1)$ in the following form:

$$f(x|1) = \frac{\left[\sqrt{\alpha^2 - 2\alpha\beta}\right]^{\gamma}}{\alpha^{\gamma} K_{\gamma}\left[\sqrt{\alpha^2 - 2\alpha\beta}\right]} \frac{\beta^x}{x!} K_{x+\gamma}(\alpha) \text{ , where} \tag{6.1.3}$$

$\alpha > 0$, $\beta > 0$, $-\infty < \gamma < \infty$ and $x = 0, 1, 2, \ldots$

At times it may be advantageous to express this distribution in terms of parameters a, b and c as:

$$f(x|a,b,c) = \frac{1}{K_a(bc)} \left[\frac{b^2}{b^2+2}\right]^{a/2} \frac{1}{x!} \left[\frac{c}{\sqrt{b^2+2}}\right]^x K_{x+a}\left[c\sqrt{b^2+2}\right]$$

where (6.1.4)

$a = \gamma$, $b = \sqrt{\frac{\alpha}{\beta} - 2}$ and $c = \sqrt{\alpha\beta}$.

From the Generalized Inverse Gaussian-Poisson distribution it is possible to derive some standard distributions, such as the Poisson, Negative Binomial, Logarithmic Series distributions as well as the Inverse Gaussian-Poisson distribution.

In order to develop these distributions from the GIGP by a limiting process it is necessary to define some relations for the modified Bessel function of the second kind.

For large values of z, i.e. $z \gg 1$ and $z \gg \gamma^2$ the modified Bessel functions become:

$$K_{\gamma}(z) \approx \sqrt{\frac{\pi}{2z}}\, e^{-z} \left[1 + \frac{(4\gamma^2-1^2)}{1!\; 8z} + \frac{(4\gamma^2-1^2)(4\gamma^2-3^2)}{2!\; (8z)^2} + \ldots\right]$$

$$\approx \sqrt{\frac{\pi}{2z}}\, e^{-z} \; . \tag{6.1.5}$$

For small values of z, i.e. for $z^2/4 \ll n$, $z > 0$ and a large $n > 0$, they become:

$$K_n(z) \approx \left[\frac{2}{z}\right]^n \frac{\Gamma(n)}{2}\left[1 - \frac{(z/2)^2}{1!(n-1)} + \frac{(z/2)^4}{2!(n-1)(n-2)} + .. + (-1)^{n-1}\frac{(z/2)^{2n-2}}{\left[(n-1)!\right]^2}\right] \tag{6.1.6}$$

For $n \geq 1$ the dominant term is

$$K_n(z) \approx (2/z)^n \frac{\Gamma(n)}{2} . \tag{6.1.7}$$

The modified Bessel function of the second kind is tabulated for orders $n = 0\ (1)\ 20$ and argument $z = 0{,}1\ (0{,}1)\ 20$. For the GIGP distribution one needs the numerical values of Bessel functions of orders much higher than 20 and sometimes of an argument larger than $z = 20$. For numerical calculations of such functions the recurrence formula

$$K_{\gamma+1}(z) = \frac{2\gamma}{z} K_\gamma(z) + K_{\gamma-1}(z) \tag{6.1.8}$$

is particularly useful.

6.1.1 Derivation of the Poisson distribution

If we take in (6.1.3) $\alpha \gg \gamma$ and $\alpha \gg x+\gamma$ and use the approximation from (6.1.5) then we get

$$K_{x+\gamma}(\alpha) \approx \sqrt{\frac{\pi}{2\alpha}}\, e^{-\alpha} \quad \text{and} \tag{6.1.9}$$

$$K_\gamma\left[\sqrt{\alpha^2 - 2\alpha\beta}\right] \approx \sqrt{\frac{\pi}{2\sqrt{\alpha^2-2\alpha\beta}}}\, e^{-\sqrt{\alpha^2-2\alpha\beta}} . \tag{6.1.10}$$

Hence (6.1.3) becomes

$$f(x) \approx \left[1 - \frac{2\beta}{\alpha}\right]^{(\gamma/2+1/4)} e^{-\alpha\left[1-\sqrt{1-\frac{2\beta}{\alpha}}\right]} \frac{\beta^x}{x!} \tag{6.1.11}$$

Now we limit the two expressions of (6.1.11) to infinity:

$$\lim_{\alpha\to\infty} \left[1 - \frac{2\beta}{\alpha}\right]^{(\gamma/2+1/4)} = 1 \quad \text{and} \tag{6.1.12}$$

$$\lim_{\alpha\to\infty} \alpha\left[1 - \sqrt{1-\frac{2\beta}{\alpha}}\right] = \beta \ . \tag{6.1.13}$$

Hence (6.1.11) becomes

$$f(x) \approx e^{-\beta} \frac{\beta^x}{x!} , \tag{6.1.14}$$

which is a Poisson distribution.

The same result can be obtained by letting $\theta_t \to 0$ in (6.1.2).

6.1.2 Derivation of the Negative Binomial distribution

Method 1

The Negative Binomial distribution can be developed from the GIGP for special conditions of parameters γ and α. If in (6.1.3) we take $\gamma > 0$ and $\alpha \ll \gamma$ we might use (6.1.7) and then

$$K_{x+\gamma}(\alpha) \approx \left[\frac{2}{\alpha}\right]^{x+\gamma} \frac{\Gamma(x+\gamma)}{2} \quad \text{and} \tag{6.1.15}$$

$$K_{\alpha}\left[\sqrt{\alpha^2-2\alpha\beta}\right] = \left[\frac{2}{\sqrt{\alpha^2-2\alpha\beta}}\right]^{\gamma} \frac{\Gamma(\gamma)}{2} . \tag{6.1.16}$$

Substitution of (6.1.15) and (6.1.16) into (6.1.3) leads to

$$f(x) = \frac{\left[1 - \frac{2\beta}{\alpha}\right]^{\gamma}}{\Gamma(\gamma)} \frac{\Gamma(x+\gamma)}{\Gamma(x+1)} \left[\frac{2\beta}{\alpha}\right]^{x} . \qquad (6.1.17)$$

Remembering now that $\theta = \frac{2\beta}{\alpha}$, (6.1.17) becomes

$$f(x) \approx \frac{(1-\theta)^{\gamma}}{\Gamma(\gamma)} \frac{\Gamma(x+\gamma)}{\Gamma(x+1)} \theta^{x} , \qquad (6.1.18)$$

which is a standard NBD model.

Method 2

An alternative derivation of the NBD is by mixing the Poisson with the Gamma distribution

$$\varphi(\lambda) = \frac{1}{\Gamma(\gamma)} \left[\frac{1-\theta}{\theta}\right]^{\gamma} \lambda^{\gamma-1} e^{-\left[\frac{1-\theta}{\theta}\right]\lambda} , \quad \text{where} \qquad (6.1.19)$$

$\gamma > 0$ and $0 < \theta < 1$.

Substitution of (6.1.19) into (5.1.1) gives, after solving the convolution integral, the following discrete model:

$$f(x|t) = \frac{1}{\Gamma(\gamma)} \left[1 - \frac{t\theta}{1+(t-1)\theta}\right]^{\gamma} \frac{\Gamma(x+\gamma)}{\Gamma(x+1)} \left[\frac{t\theta}{1+(t-1)\theta}\right]^{x} , \qquad (6.1.20)$$

where $0 \le \theta \le 1$, $t > 0$ and $x = 0, 1, 2, \ldots$,

which is the NBD model.

Finally, by using the reparameterization

$\theta_t = \dfrac{t\theta}{1+(t-1)\theta}$, where $0 < \theta_t < 1$, the function $f(x|t)$ from (6.1.20) obtains the form

$$f(x|t) = \frac{\left[1-\theta_t\right]^{\gamma}}{\Gamma(\gamma)} \frac{\Gamma(x+\gamma)}{\Gamma(x+1)} \theta_t^{\,x} . \qquad (6.1.21)$$

6.1.3 Derivation of the Inverse Gaussian-Poisson distribution

The Inverse Gaussian-Poisson is derived as a special case of the GIGP, where the parameter $\gamma = -1/2$.

By substitution of $\gamma = -1/2$ into the equation (6.1.2) and use of the properties of the modified Bessel function of the second kind

$$K_{1/2}(\alpha_t) = \sqrt{\frac{\pi}{2\alpha_t}}\, e^{-\alpha_t} \quad \text{and} \tag{6.1.22}$$

$$K_{-\gamma}(z) = K_{\gamma}(z)\ , \tag{6.1.23}$$

one obtains $f(x|t)$ in the form

$$f(x|t) = \sqrt{\frac{2\alpha_t}{\pi}}\, e^{\left[\alpha_t(1-\alpha_t)^{1/2}\right]}\, \frac{\left[\alpha_t\theta_t/2\right]^x}{x!}\, K_{x-1/2}(\alpha_t)\ , \tag{6.1.24}$$

where $x = 0,1,2,\dots$.

This is the Inverse Gaussian-Poisson (IGP).

The IGP can also be obtained directly from the convolution process in the compound Poisson, where the Inverse Gaussian is used as the mixing distribution. This was introduced by Tweedie (1957) and a general review was presented by Folks and Chikara (1978).

The following characteristics of the compound Poisson distributions are discussed below: shape, mean, variance, coefficient of variation denoted CV, coefficient of dispersion denoted COD, coefficient of skewness denoted $\sqrt{\beta_1}$, coefficient of kurtosis denoted β_2 and Yule characteristic.

6.2 A Discriminating Ratio Function

In order to distinguish between different types of discrete distributions $f(x)$, Katz (1965) introduced a useful ratio function

$$R(x) = \frac{f(x+1)}{f(x)}\,, \quad \text{where} \tag{6.2.1}$$

$x = 0,1,2\ldots$.

The three above mentioned discrete distributions are examined with the help of (6.2.1). At $x = 0$ the ratio function is used as an indicator for the shape of non-multimodal discrete distributions.

At $x = 0$ the ratio function is

$$R(0) = \frac{f(1)}{f(0)}\,. \tag{6.2.2}$$

If $R(0) < 1$ then the distribution is of a reverse J-shape type;

if $R(0) = 1$ then the distribution has equiprobable modes; and

if $R(0) > 1$ then the distribution is unimodal. (6.2.3)

6.2.1 GIGP

The ratio function of the GIGP distribution at $x = 0$ for any period of length t is given by

$$R_{GIGP}(0|t) = \frac{\alpha_t \theta_t K_{\gamma+1}(\alpha_t)}{2K_\gamma(\alpha_t)}\,. \tag{6.2.4}$$

If $\dfrac{\alpha_t \theta_t K_{\gamma+1}(\alpha_t)}{2K_\gamma(\alpha_t)} < 1$ then the distribution is of a reverse

J-shape;

if $\dfrac{\alpha_t\theta_t K_{\gamma+1}(\alpha_t)}{2K_\gamma(\alpha_t)} = 1$ then the distribution has equiprobable modes; and

if $\dfrac{\alpha_t\theta_t K_{\gamma+1}(\alpha_t)}{2K_\gamma(\alpha_t)} > 1$ then the distribution is unimodal.

If $t \to \infty$ the GIGP distribution is unimodal for any γ.

6.2.2 IGP

The ratio function of the IGP distribution at x = 0 for any period of length t is given by

$$R_{IGP}(0|t) = \frac{\alpha_t\theta_t}{2} . \qquad (6.2.5)$$

If $\dfrac{\alpha_t\theta_t}{2} < 1$ the distribution is of a reverse J-shape;

if $\dfrac{\alpha_t\theta_t}{2} = 1$ the distribution has equiprobable modes; and

if $\dfrac{\alpha_t\theta_t}{2} > 1$ the distribution is unimodal.

If $t \to \infty$ the distribution is always unimodal.

6.2.3 NBD

(a) For the observational period $t = 1$ one finds

$$\theta_1 = \theta \text{ and } R_{NBD}(0|1) = \gamma\theta . \qquad (6.2.6)$$

Using the conditions of (6.2.3) one obtains that if $0 < \theta < 1$, then $0 < \gamma \le 1$ and the NBD will be reverse J-shaped. If $\gamma > 1/\theta$ we obtain that the NBD i unimodal.

(b) For a period of any length t, the ratio function for NBD is

$$R_{NBD}(0|t) = \gamma\theta_t = \frac{t\gamma\theta}{1+(t-1)\theta} \quad \text{and} \qquad (6.2.7)$$

$$\lim_{t\to\infty} R_{NBD}(0|t) = \gamma \ .$$

Using the conditions of (6.2.3) one obtains that for a large t, the distribution is reverse J-shaped if $\gamma < 1$. It has equiprobable modes if $\gamma = 1$ and is unimodal if $\gamma > 1$.

6.3 Statistical Indices

6.3.1 Mean

(a) GIGP

At t one has:

$$\mu_1'(x|t) = \sum_{x=0}^{\infty} x\ f(x|t) = \frac{(1-\theta_t)^{\gamma}(\alpha_t\theta_t/2)}{K_\gamma(\alpha_t\sqrt{1-\theta_t})} \sum_{j=0}^{\infty} \frac{(\alpha_t\theta_t/2)^j}{j!} K_{j+1+\gamma}(\alpha_t)$$

$$= \frac{\alpha_t\theta_t}{2\sqrt{1-\theta_t}} \frac{K_{\gamma+1}(\alpha_t\sqrt{1-\theta_t})}{K_\gamma(\alpha_t\sqrt{1-\theta_t})} \ . \qquad (6.3.1)$$

By applying the definitions of α_t and θ_t given in (6.1.2) one obtains the following relation between the mean at any period of length t and the mean at the observational period $t = 1$:

$$\mu_1'(x|t) = \frac{t\alpha\theta}{2\sqrt{1-\theta}} \frac{K_{\gamma+1}(\alpha\sqrt{1-\theta})}{K_\gamma(\alpha\sqrt{1-\theta})} \tag{6.3.2}$$

If we denote

$$\mu_1'(x|1) = \frac{\alpha\theta}{2\sqrt{1-\theta}} \frac{K_{\gamma+1}(\alpha\sqrt{1-\theta})}{K_\gamma(\alpha\sqrt{1-\theta})} \tag{6.3.3}$$

then

$$\mu_1'(x|t) = t\,\mu_1'(x|1) .$$

(b) IGP

Substitution of $\gamma = -1/2$ into (6.3.2) and (6.3.3) leads to

$$\mu_1'(x|t) = \frac{t\alpha\theta}{2\sqrt{1-\theta}} \quad \text{and} \tag{6.3.4}$$

$$\mu_1'(x|1) = \frac{\alpha\theta}{2\sqrt{1-\theta}} . \tag{6.3.5}$$

Therefore $\mu_1'(x|t) = t\,\mu_1'(x|1)$.

(c) NBD

Using the moment generating function of the NBD

$$M(y) = \left[1 + \frac{t\theta}{1-\theta}(1 - e^y)\right]^{-\gamma} \tag{6.3.6}$$

one obtains the mean at t:

$$\mu_1'(x|t) = \frac{t\gamma\theta}{1-\theta} \tag{6.3.7}$$

and for the observational period $t = 1$

$$\mu_1'(x|1) = \frac{\gamma\theta}{1-\theta} . \tag{6.3.8}$$

Hence

$\mu_1'(x|t) = t\ \mu_1'(x|1)$.

6.3.2 Variance

(a) GIGP

At t the second moment around the origin is:

$$\mu_2'(x|t) = \sum_{x=0}^{\infty} x^2\ f(x|t) = \tag{6.3.9}$$

$$= \frac{\alpha_t^2 \theta_t^2}{4(1-\theta_t)}\ \frac{K_{\gamma+2}(\alpha_t\sqrt{1-\theta_t})}{K_\gamma(\alpha_t\sqrt{1-\theta_t})} + \frac{\alpha_t \theta_t}{2\sqrt{1-\theta_t}}\ \frac{K_{\gamma+1}(\alpha_t\sqrt{1-\theta_t})}{K_\gamma(\alpha_t\sqrt{1-\theta_t})} .$$

Considering that

$$\mu_2(x|t) = \mu_2'(x|t) - [\mu_1'(x|t)]^2 , \tag{6.3.10}$$

one obtains:

$$\mu_2(x|t) = \frac{\alpha_t^2 \theta_t^2}{4(1-\theta_t)} \left[\frac{K_{\gamma+2}(\alpha_t\sqrt{1-\theta_t})}{K_\gamma(\alpha_t\sqrt{1-\theta_t})} - \frac{K_{\gamma+1}^2(\alpha_t\sqrt{1-\theta_t})}{K_\gamma^2(\alpha_t\sqrt{1-\theta_t})}\right]$$

$$+ \frac{\alpha_t \theta_t}{2\sqrt{1-\theta_t}}\ \frac{K_{\gamma+1}(\alpha_t\sqrt{1-\theta_t})}{K_\gamma(\alpha_t\sqrt{1-\theta_t})} . \tag{6.3.11}$$

By using the definitions of α_t and θ_t given in (6.1.2) one obtains the following relation for the variance as a function of the parameters of the observational period α and θ:

$$\mu_2(x|t) = \left[\frac{\alpha t \theta}{2\sqrt{1-\theta}}\right]^2 \left[\frac{K_{\gamma+2}(\alpha\sqrt{1-\theta})}{K_\gamma(\alpha\sqrt{1-\theta})} - \frac{K_{\gamma+1}^2(\alpha\sqrt{1-\theta})}{K_\gamma^2(\alpha\sqrt{1-\theta})}\right]$$

$$+ \frac{\alpha t \theta}{2\sqrt{1-\theta}}\ \frac{K_{\gamma+1}(\alpha\sqrt{1-\theta})}{K_\gamma(\alpha\sqrt{1-\theta})} . \tag{6.3.12}$$

At t = 1 the variance is

$$\mu_2(x|1) = \frac{\alpha^2\theta^2}{4(1-\theta)}\left[\frac{K_{\gamma+2}(\alpha\sqrt{1-\theta})}{K_\gamma(\alpha\sqrt{1-\theta})} - \frac{K^2_{\gamma+1}(\alpha\sqrt{1-\theta})}{K^2_\gamma(\alpha\sqrt{1-\theta})}\right]$$

$$+ \frac{\alpha\theta}{2\sqrt{1-\theta}}\,\frac{K_{\gamma+1}(\alpha\sqrt{1-\theta})}{K_\gamma(\alpha\sqrt{1-\theta})}\,. \tag{6.3.13}$$

(b) IGP

Substitution of $\gamma = -1/2$ into (6.3.12) and (6.3.13) and use of the recurrence formula (6.1.8) leads to

$$\mu_2(x|t) = \frac{t\alpha\theta(2+t\theta-2\theta)}{4(1-\theta)^{3/2}} \quad \text{and} \tag{6.3.14}$$

$$\mu_2(x|1) = \frac{\alpha\theta(2-\theta)}{4(1-\theta)^{3/2}}\,. \tag{6.3.15}$$

(c) NBD

Using the moment generating function (6.3.6) and the definition of the variance (6.3.10) one obtains

$$\mu_2(x|t) = \frac{t\gamma\theta}{1-\theta}\left[1 + \frac{t\theta}{1-\theta}\right] \quad \text{and} \tag{6.3.16}$$

$$\mu_2(x|1) = \frac{\gamma\theta}{(1-\theta)^2}\,. \tag{6.3.17}$$

6.3.3 Coefficients of dispersion and variation

(a) GIGP

Substitution of (6.3.2) and (6.3.12) into (5.1.8) leads to

$$\mathrm{COD}(x|t) = \frac{t\alpha\theta}{2\sqrt{1-\theta}}\left[\frac{K_{\gamma+2}(\alpha\sqrt{1-\theta})}{K_{\gamma+1}(\alpha\sqrt{1-\theta})} - \frac{K_{\gamma+1}(\alpha\sqrt{1-\theta})}{K_{\gamma}(\alpha\sqrt{1-\theta})}\right] + 1 \; . \qquad (6.3.18)$$

Hence

$$\lim_{t\to\infty} \mathrm{COD}(x|t) = \infty \quad \text{and}$$

$$\lim_{t\to 0} \mathrm{COD}(x|t) = 1 \; .$$

Therefore for $t > 0$ one has $1 < \mathrm{COD}(x|t) < \infty$. (6.3.19)

Substitution of (6.2.2) and (6.2.12) into (5.1.11) leads to

$$\mathrm{CV}(x|t) = \left[\left[\frac{t\alpha\theta}{2\sqrt{1-\theta}}\right]^{-1} \frac{K_{\gamma}(\alpha\sqrt{1-\theta})}{K_{\gamma+1}(\alpha\sqrt{1-\theta})} + \frac{K_{\gamma+2}(\alpha\sqrt{1-\theta})K_{\gamma}(\alpha\sqrt{1-\theta})}{K^2_{\gamma+1}(\alpha\sqrt{1-\theta})} - 1\right]^{1/2} .$$

(6.3.20)

Hence

$$\lim_{t\to\infty} \mathrm{CV}(x|t) = \left[\frac{K_{\gamma+2}(\alpha\sqrt{1-\theta})\, K_{\gamma}(\alpha\sqrt{1-\theta})}{K^2_{\gamma+1}(\alpha\sqrt{1-\theta})} - 1\right]^{1/2} \quad \text{and} \qquad (6.3.21)$$

$$\lim_{t\to 0} \mathrm{CV}(x|t) = \infty \; . \qquad (6.3.22)$$

For $\gamma+i \gg \alpha\sqrt{1-\theta}$ and for substitution of (6.1.7) into (6.3.21) one obtains the following result:

$$\lim_{t\to\infty} \mathrm{CV}(x|t) = \gamma^{-1/2} \; . \qquad (6.3.23)$$

(b) IGP

Substitution of $\gamma = -1/2$ into (6.2.18) and use of the two properties of the modified Bessel function of the second kind given by (6.1.8) and (6.1.23) leads to

$$COD(x|t) = 1 + \frac{t\theta}{2(1-\theta)} , \tag{6.3.24}$$

$\lim_{t\to\infty} COD(x|t) = \infty$ and

$\lim_{t\to 0} COD(x|t) = 1$.

The coefficient of variation is

$$CV(x|t) = \left[\frac{2(1-\theta) + t\theta}{t\alpha\theta}\right]^{1/2} (1-\theta)^{-1/4} , \tag{6.3.25}$$

$\lim_{t\to\infty} CV(x|t) = (1-\theta)^{-1/4} \alpha^{-1/2}$ and

$\lim_{t\to 0} CV(x|t) = \infty$.

Hence for $t > 0$

$1 < COD(x|t) < \infty$ and

$$(1-\theta)^{-1/4} \alpha^{-1/2} < CV(x|t) < \infty . \tag{6.3.26}$$

(c) NBD

Substitution of (6.3.7) and (6.3.16) into (5.1.8) leads to

$$COD(x|t) = 1 + \frac{t\theta}{1-\theta} , \tag{6.3.27}$$

$\lim_{t\to\infty} COD(x|t) = \infty$ and

$\lim_{t\to 0} COD(x|t) = 1$.

Hence for $t > 0$ one has $1 < COD(x|t) < \infty$. (6.3.28)

Substitution of (6.3.7) and (6.3.16) into (5.1.11) leads to

$$CV(x|t) = \left[\frac{1}{t}\left[\frac{1-\theta}{\gamma\theta}\right] + \frac{1}{\gamma}\right]^{1/2}, \tag{6.3.29}$$

$\lim_{t\to\infty} CV(x|t) = \gamma^{-1/2}$ and

$\lim_{t\to 0} CV(x|t) = \infty$.

Hence for $t > 0$ one has

$$\gamma^{-1/2} < CV(x|t) < \infty \ . \tag{6.3.30}$$

6.3.4 Coefficients of skewness and kurtosis

The calculation of the coefficient of skewness and kurtosis requires the calculation of the second, the third and the fourth central moments.

(a) GIGP

The third and the second central moments are

$$\mu_3(x|t) = (mt)^3\left[\frac{K_{\gamma+3}(u)}{K_\gamma(u)} - \frac{3K_{\gamma+2}(u)\ K_{\gamma+1}(u)}{K_\gamma^2(u)} + \frac{2K_{\gamma+1}^3}{K_\gamma^3}\right]$$

$$+ (mt)^2\left[\frac{3K_{\gamma+2}(u)}{K_\gamma(u)} - \frac{K_{\gamma+1}^2(u)}{K_\gamma^2(u)}\right] - mt\ \frac{K_{\gamma+1}(u)}{K_\gamma(u)} \quad \text{and} \tag{6.3.31}$$

$$\mu_2(x|t) = (mt)^2\left[\frac{K_{\gamma+2}(u)}{K_\gamma(u)} - \frac{K_{\gamma+1}^2(u)}{K_\gamma^2(u)}\right] + mt\ \frac{K_{\gamma+1}(u)}{K_\gamma(u)}, \quad \text{where} \tag{6.3.32}$$

$u = \alpha\sqrt{1-\theta}$ and

$$m = \frac{\alpha\theta}{2\sqrt{1-\theta}} \quad . \tag{6.3.33}$$

In the limiting process the coefficient of skewness becomes

$$\lim_{t\to\infty} \sqrt{\beta_1(x|t)} = \frac{K_\gamma^2(u)K_{\gamma+3}(u) - 3K_\gamma(u)K_{\gamma+1}(u)K_{\gamma+2}(u) - 2K_{\gamma+1}^3(u)}{\left[K_{\gamma+2}(u)K_\gamma(u) - K_{\gamma+1}^2(u)\right]^{3/2}} \,. \tag{6.3.34}$$

The fourth central moment $\mu_4(x|t)$ can be expressed only in a very cumbersome form as a function of the modified Bessel function of the second kind.

In the limiting process the coefficient of kurtosis $\beta_2(x|t)$ becomes

$$\lim_{t\to\infty} \beta_2(x|t) = \tag{6.3.35}$$

$$= \frac{K_\gamma^3(u)K_{\gamma+4}(u) - 4K_\gamma^2(u)K_{\gamma+1}(u)K_{\gamma+3}(u) + 6K_\gamma(u)K_{\gamma+1}^2(u)K_{\gamma+2}(u) - 3K_{\gamma+1}^4(u)}{[K_{\gamma+2}(u)K_\gamma(u) - K_{\gamma+1}^2(u)]^2} \,.$$

What we must notice is that in the limiting process when $t \to \infty$ the coefficients of skewness and kurtosis of the GIGP are the same as of the mixing distribution GIG .

(b) IGP

The formulae for the coefficients of skewness and kurtosis are obtained by substitution of $\gamma = -1/2$ and the use of the recurrence formula (6.1.8).

Hence

$$\lim_{t\to\infty} \sqrt{\beta_1(x|t)} = 3(\alpha\sqrt{1-\theta})^{-1/2} \quad \text{and} \tag{6.3.36}$$

$$\lim_{t\to\infty} \beta_2(x|t) = 3\left[\frac{5}{\alpha\sqrt{1-\theta}} + 1\right]. \tag{6.3.37}$$

(c) NBD

The coefficient of skewness for the NBD is given as a function of t by

$$\sqrt{\beta_1(x|t)} = \frac{2t^2\theta^2 + 3t\theta(1-\theta) + (1-\theta)^2}{(t\gamma\theta)^{1/2}(1-\theta+t\theta)^{3/2}}. \tag{6.3.38}$$

In the limiting process (6.3.38) becomes

$$\lim_{t\to\infty} \sqrt{\beta_1(x|t)} = 2\gamma^{-1/2}. \tag{6.3.39}$$

This is the coefficient of skewness of the Gamma distribution as a function of the parameter γ. This result was expected. As shown in Theorem 5.1, the coefficient of skewness of the compound Poisson approaches the coefficient of skewness of the mixing distribution. As proved previously, the NBD is a compound Poisson with the mixing taken as a Gamma distribution.

Another interesting result is obtained from the limiting process of the coefficient of kurtosis $\beta_2(x|t)$

$$\lim_{t\to\infty} \beta_2(x|t) = 3 + 6/\gamma, \tag{6.3.40}$$

which is the coefficient of kurtosis of the Gamma distribution.

6.4 The Ratio of the Coefficients of Skewness as a Function of t

The general formulae for the moments around the origin for the GIG in terms of the parameters of (6.1.4) are

$$\mu_i'(\lambda) = \frac{K_{\gamma+i}(b)}{K_\gamma(b)}(bc/2)^i . \tag{6.4.1}$$

Equation (5.1.15) expresses the coefficient of skewness of a compound Poisson in terms of the central moments of the mixing distribution. The coefficient of skewness of the compound Poisson can be written as

$$g(t) = \sqrt{\beta_1(x|t)} = \frac{At^3 + 3Bt^2 + Ct}{(Bt^2 + Ct)^{3/2}} , \quad \text{where} \tag{6.4.2}$$

$A = \mu_3(\lambda)$, $B = \mu_2(\lambda)$ and $C = \mu_1'(\lambda)$.

The function $g(t)$ reaches the minimum at the same point as the function $R_{Sk}(t)$ given in (5.1.16) because the coefficient of skewness of the mixing distribution is constant with respect to t. The minimum is reached at the point t, given by the general solution

$$t_{1,2} = \frac{BC \pm C\sqrt{12AC - 23B^2}}{6(AC - 2B^2)} . \tag{6.4.3}$$

For the IGP the minimum is achieved at

$$t = \left[\frac{1 + \sqrt{13}}{3}\right]c^{-1} . \tag{6.4.4}$$

The ratio of coefficients of skewness between the IGP and the Inverse Gaussian distributions is

$$R_{Sk}(t) = \left[\frac{\beta_1(x|t)}{\beta_1(\lambda)}\right]^{1/2} = \frac{3(c/2)^3t^3 + 3(c/2)^2t^2 + (c/2)t}{3[(c/2)^2t^2 + (c/2)t]^{3/2}} . \tag{6.4.5}$$

This ratio does not depend on the parameter b. The graphs of Figures 6.1, 6.2 and 6.3 show the behaviour of ratio (6.4.5). From the graphs and from the theory developed in Theorem 5.1 one finds that the ratio (6.4.5) approaches unity if $t \to \infty$.

In Figure 6.1 the parameters of the IGP are $b = 0{,}1$ and $c = 0{,}4$. The minimum is reached at $t = 3$.

In Figure 6.2 the parameters of the IGP are $b = 0{,}1$ and $c = 0{,}04$. The minimum is reached at $t = 38$.

In Figure 6.3 the parameters of the IGP are $b = 0{,}1$ and $c = 0{,}004$. The minimum is reached at $t = 383$.

For the GIGP distribution the expressions of the moments $\mu_1'(\lambda)$, $\mu_2(\lambda)$ and $\mu_3(\lambda)$ can be written as

$$\mu_1'(\lambda) = (bc/2)\,\frac{K_{\gamma+1}(b)}{K_\gamma(b)} = (bc/2)\; f_1(\gamma,b)\ , \tag{6.4.6}$$

$$\mu_2(\lambda) = (bc/2)^2\left[\frac{K_{\gamma+2}(b)}{K_\gamma(b)} - \frac{K^2_{\gamma+1}(b)}{K^2_\gamma(b)}\right] = (bc/2)^2\; f_2(\gamma,b) \tag{6.4.7}$$

and

$$\mu_3(\lambda) = (bc/2)^3\left[\frac{K_{\gamma+3}(b)}{K_\gamma(b)} - \frac{3K_{\gamma+2}(b)}{K_\gamma(b)}\,\frac{K_{\gamma+1}(b)}{K_\gamma(b)} + \frac{2K^3_{\gamma+1}(b)}{K^3_\gamma(b)}\right]$$
$$= (bc/2)^3\; f_3(\gamma,b). \tag{6.4.8}$$

Therefore the minimum point for $g(t)$ is obtained at

$$t_{1,2} = \frac{f_1(\gamma,b)\left[f_2(\gamma,b) \pm \sqrt{12f_1(\gamma,b)f_3(\gamma,b) - 23f_2^2(\gamma,b)}\right]}{3bc\left[f_1(\gamma,b)f_3(\gamma,b) - 2f_2^2(\gamma,b)\right]}\ . \tag{6.4.9}$$

It is difficult to develop a simpler formula for the minimum point because of the dependency of $g(t)$ on b, c and γ, through the modified Bessel funct.on of the second kind of order γ .

In Figure 6.4 the coefficient of skewness ratio is presented for the GIGP with the parameters $\gamma = 0$, $b = 0{,}1$ and $c = 0{,}04$. From the graphical display we conclude that the minimum point is reached at $t = 57$.

The same solution is obtained from equation (6.4.9). The second solution represents a point of inflexion which is also presented in the graph of Figure 6.4.

<u>Conclusions</u>

1. The function $R_{Sk}(t)$ has always a point of intersection with $y = 1$.
2. The function has a minimum point.
3. The function tends assymptotically to 1.

modified Bessel function of the second kind of order γ .

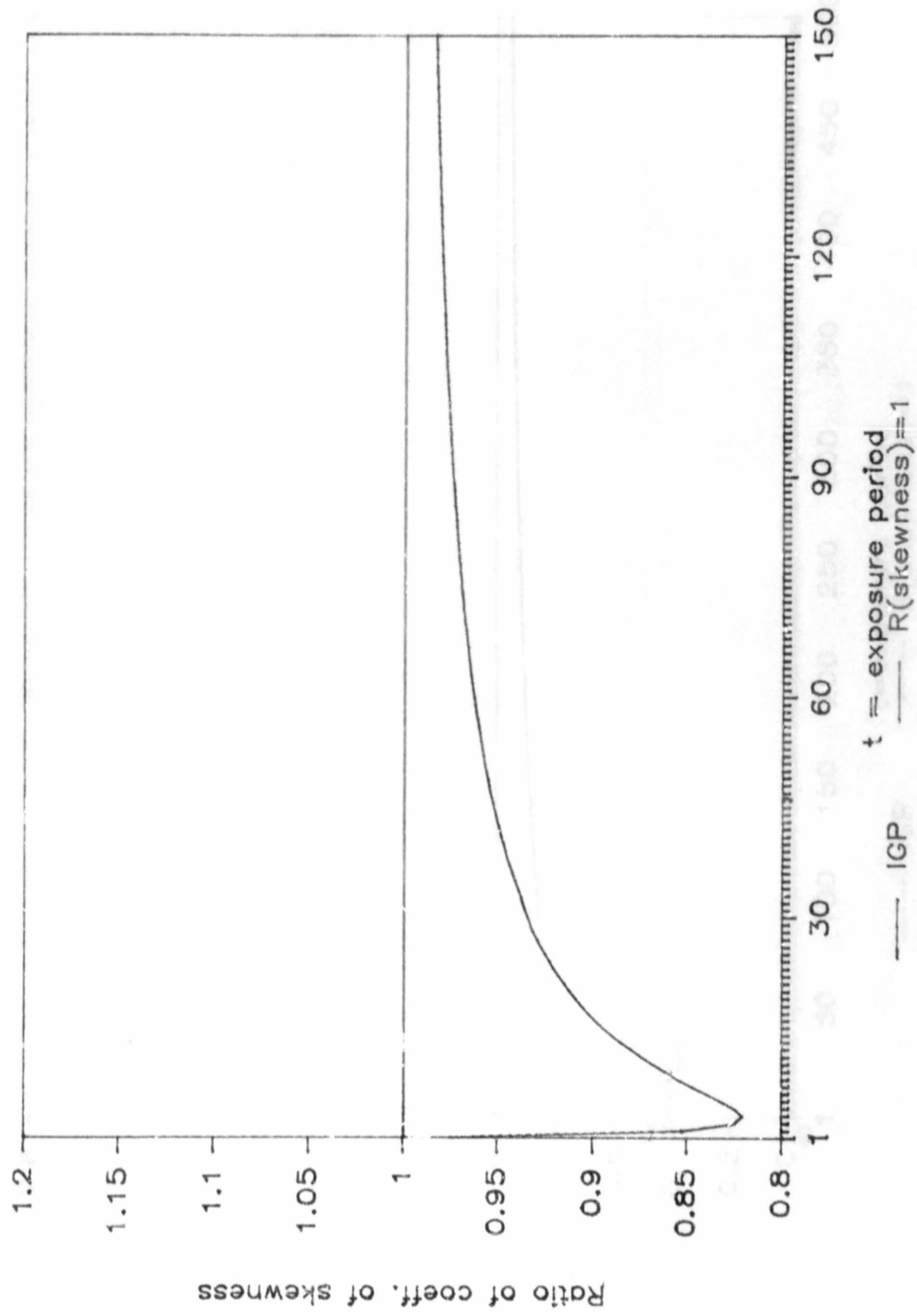
In Figure 6.4 the coefficient of skewness ratio is presented for the GIGP with the parameters $\gamma = 0$, $b = 0,1$ and $c = 0,04$. From the graphical display we conclude that the minimum point is reached at $t = 57$.

The same solution is obtained from equation (6.4.9). The second solution represents a point of inflexion which is also presented in the graph of Figure 6.4.

Conclusions

1. The function $R_{Sk}(t)$ has always a point of intersection with $y = 1$.
2. The function has a minimum point.
3. The function tends assymptotically to 1.

<u>Figure 6.1</u> The coefficients of skewness ratio for the IGP with b = 0,1 and c = 0,4



<u>Figure 6.2</u> The coefficients of skewness ratio for the IGP with b = 0,1 and c = 0,04

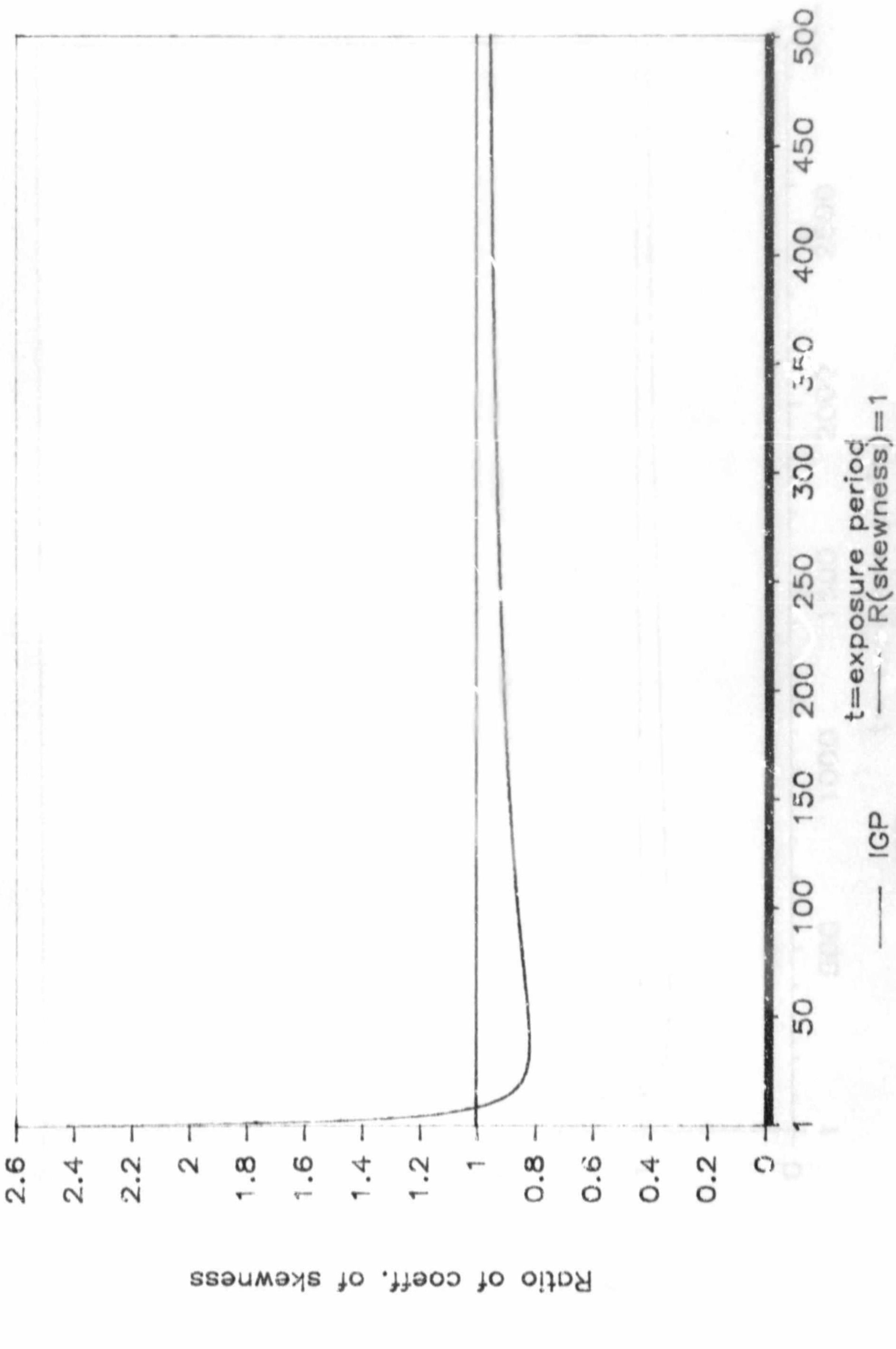


Figure 6.3 The coefficients of skewness ratio for the IGP with b = 0,1 and c = 0,004

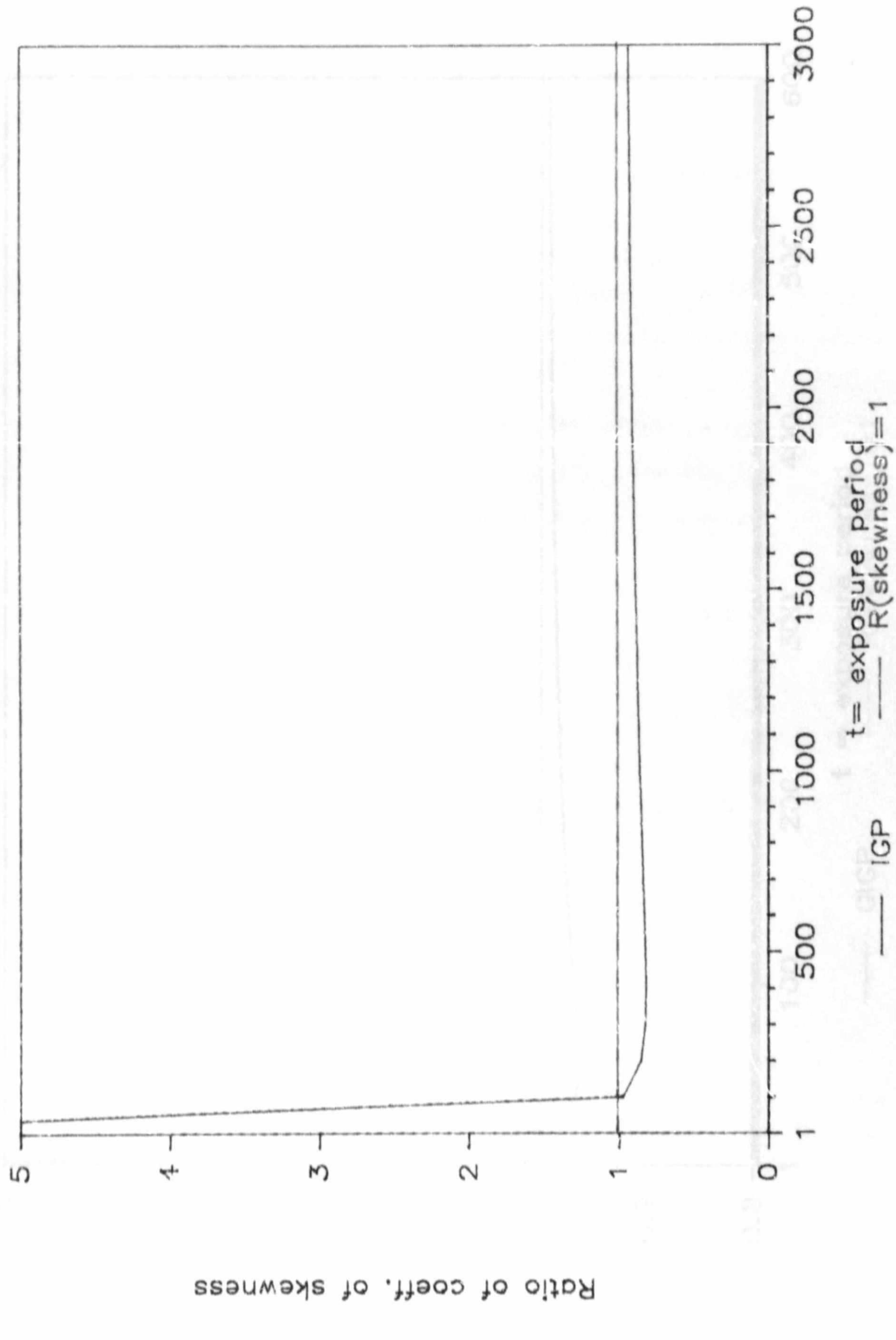
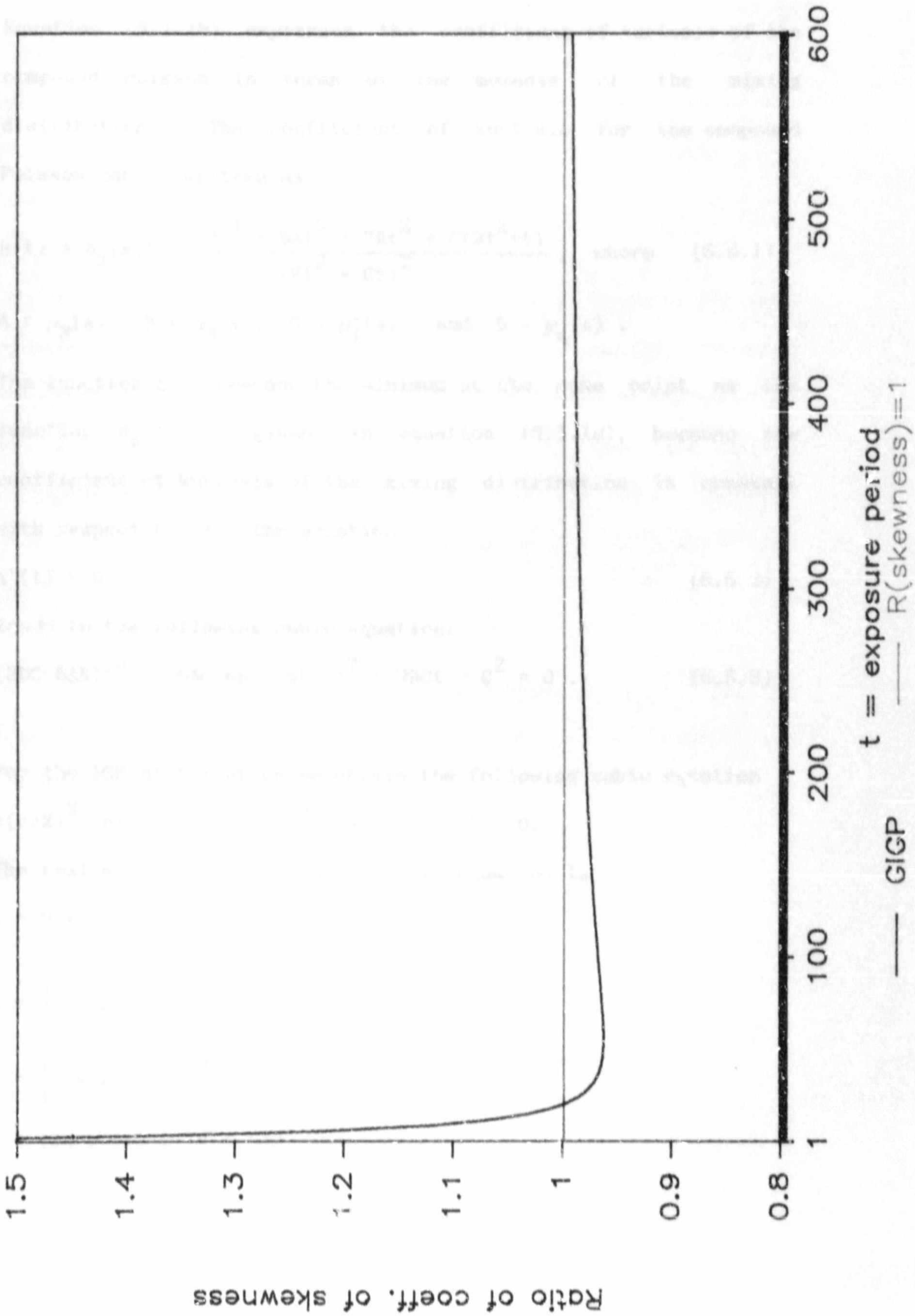


Figure 6.4 The coefficients of skewness ratio for the GIGP with $\gamma = 0$, $b = 0,1$ and $c = 0,04$



6.5 The Ratio of the Coefficients of Kurtosis as a Function of t

Equation (5.1.18) expresses the coefficient of kurtosis of the compound Poisson in terms of the moments of the mixing distribution. The coefficient of kurtosis for the compound Poisson can be written as

$$h(t) = \beta_2(x|t) = \frac{Dt^4 + 6At^3 + 7Bt^2 + C(3t^2+t)}{(Bt^2 + Ct)^2}, \quad \text{where} \qquad (6.5.1)$$

$A = \mu_3(\lambda)$, $B = \mu_2(\lambda)$, $C = \mu_1'(\lambda)$ and $D = \mu_4(\lambda)$.

The function h(t) reaches the minimum at the same point as the function $R_K(t)$ given in equation (5.1.18), because the coefficient of kurtosis of the mixing distribution is constant with respect to t. The equation

$$h'(t) = 0 \qquad (6.5.2)$$

leads to the following cubic equation:

$$(2DC-6AB)t^3 + (6AC-6BC-14B^2)t^2 - 3BCt - C^2 = 0 . \qquad (6.5.3)$$

For the IGP distribution we obtain the following cubic equation

$$6(c/2)^3 (b+2)t^3 + c(c-3)t^2 - 3(c/2)t - 1 = 0.$$

The real solution for the above cubic equation is

$$t = U + V - \frac{4(c-3)}{9c^2(b+2)}, \quad \text{where}$$

$$U = \left[-(v/2) + \sqrt{(v^2/4)+(u^3/27)}\right]^{1/3},$$

$$V = \left[-(v/2) - \sqrt{(v^2/4)+(u^3/27)}\right]^{1/3},$$

$$u = (3q-p^2)/3 , \quad v = (2p^3-9pq+27r)/27 ,$$

$$p = \frac{4(c-3)}{3c^2(b+2)}, \quad q = \frac{-2}{c^2(b+2)} \quad \text{and} \quad r = \frac{4}{3c^3(b+2)}.$$

The coefficient of kurtosis ratio $R_K(t)$ for the IGP distribution depends on both parameters b and c. In Figures 6.5, 6.6 and 6.7 we obtain different shapes of $R_K(t)$ for various values of the parameters b and c. For $b = 0,1$ and $c = 0,04$ the minimum point is achieved at $t = 1175$.

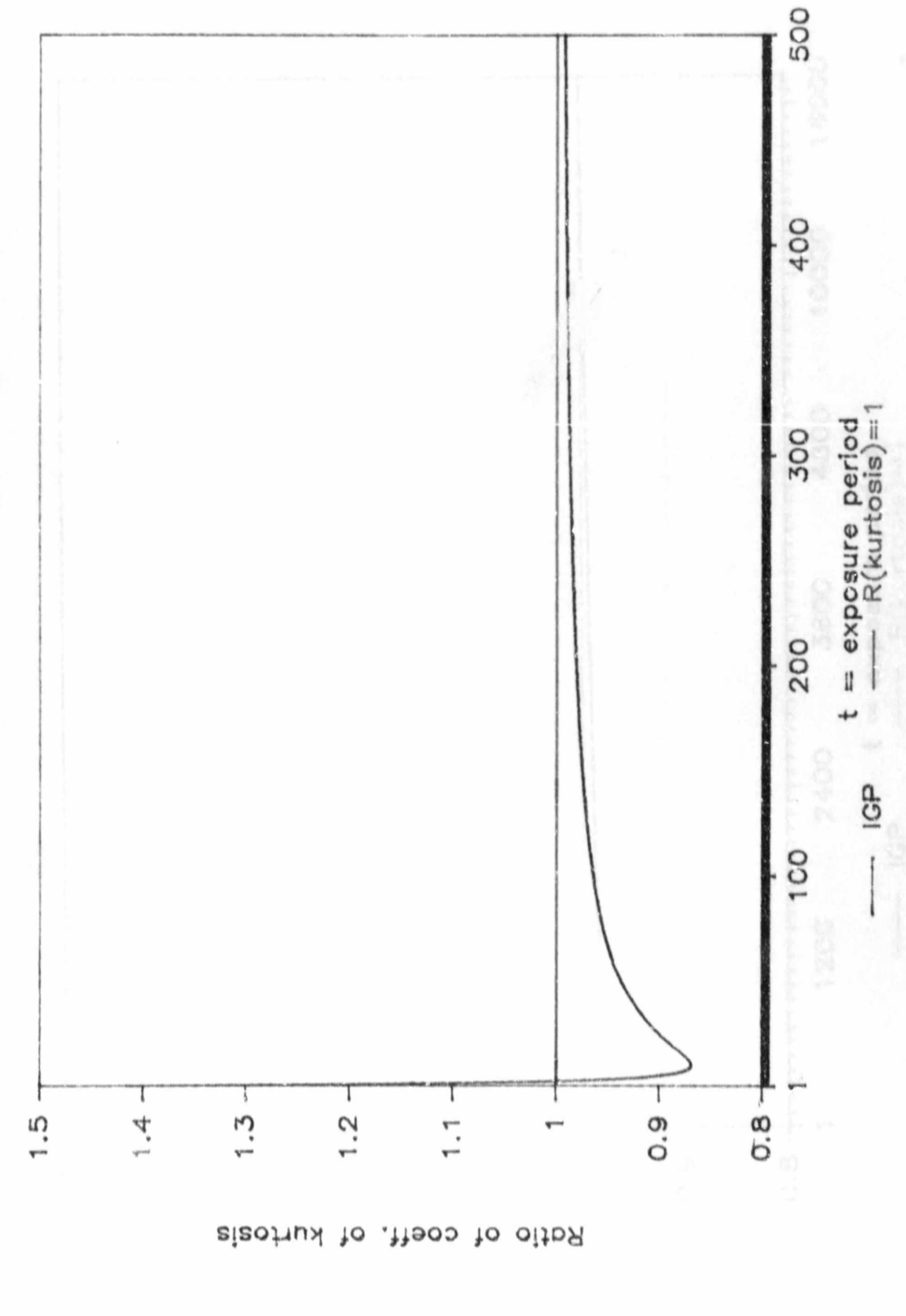
For the GIGP distribution it is necessary to solve the [illegible] equation (6.5.3). The coefficients of t will depend, however, not only on the parameters b, c and γ, but also on the modified Bessel function of the second kind of order γ. This fact implies a complex expression for the real solution of the cubic equation (6.5.3). For specific values of the parameters b, c and γ one can find a real solution to (6.5.3).

In Figure 6.8 one can see the graphical display of $R_K(t)$ for the GIGP distribution with the parameters $b = 0,1$, $c = 0,04$ and $\gamma = 0$.

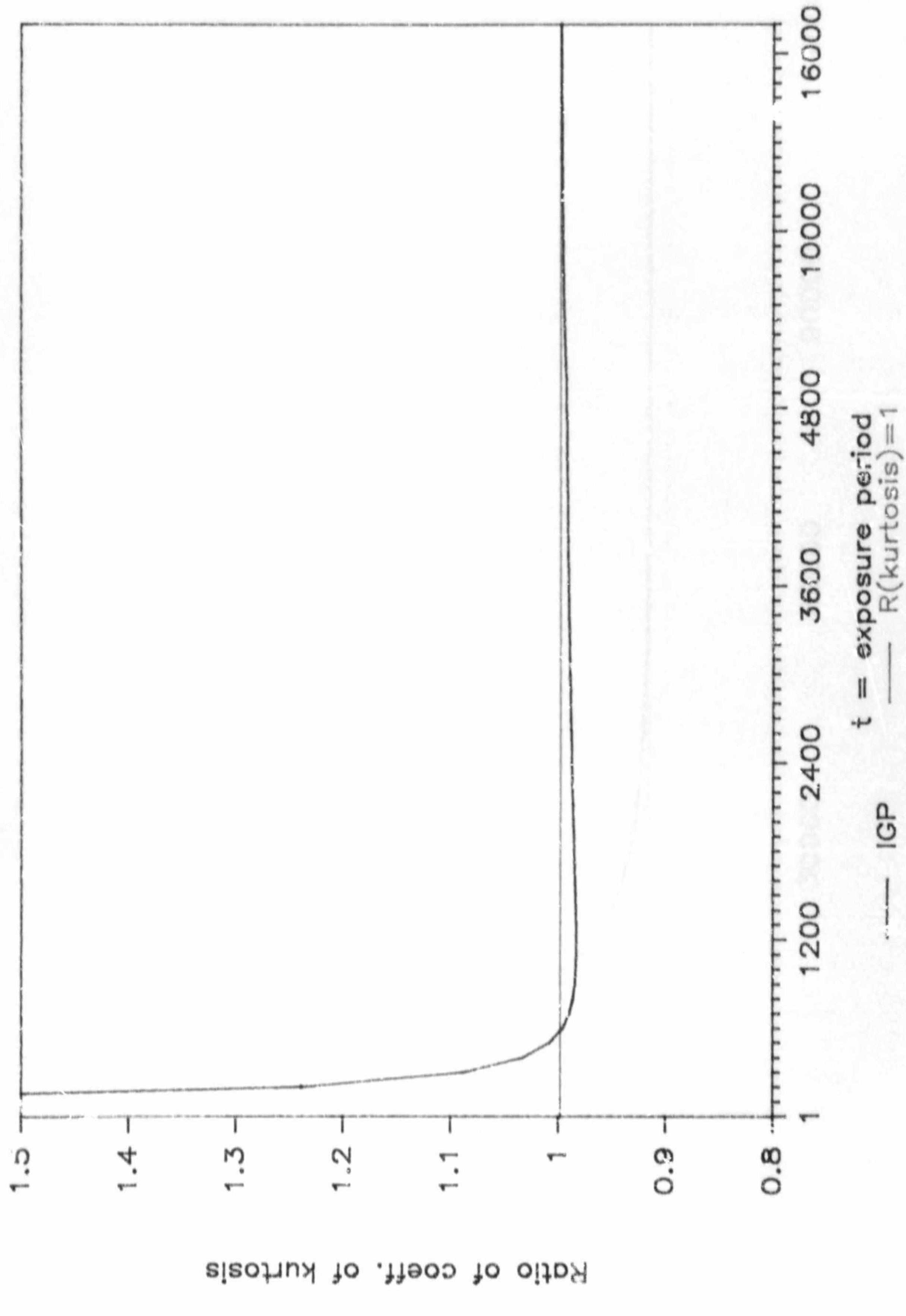
<u>Conclusions</u>

1. The function $R_K(t)$ has always a point of intersection with $y = 1$.
2. The function has a minimum point.
3. The function tends assymptotically to 1.

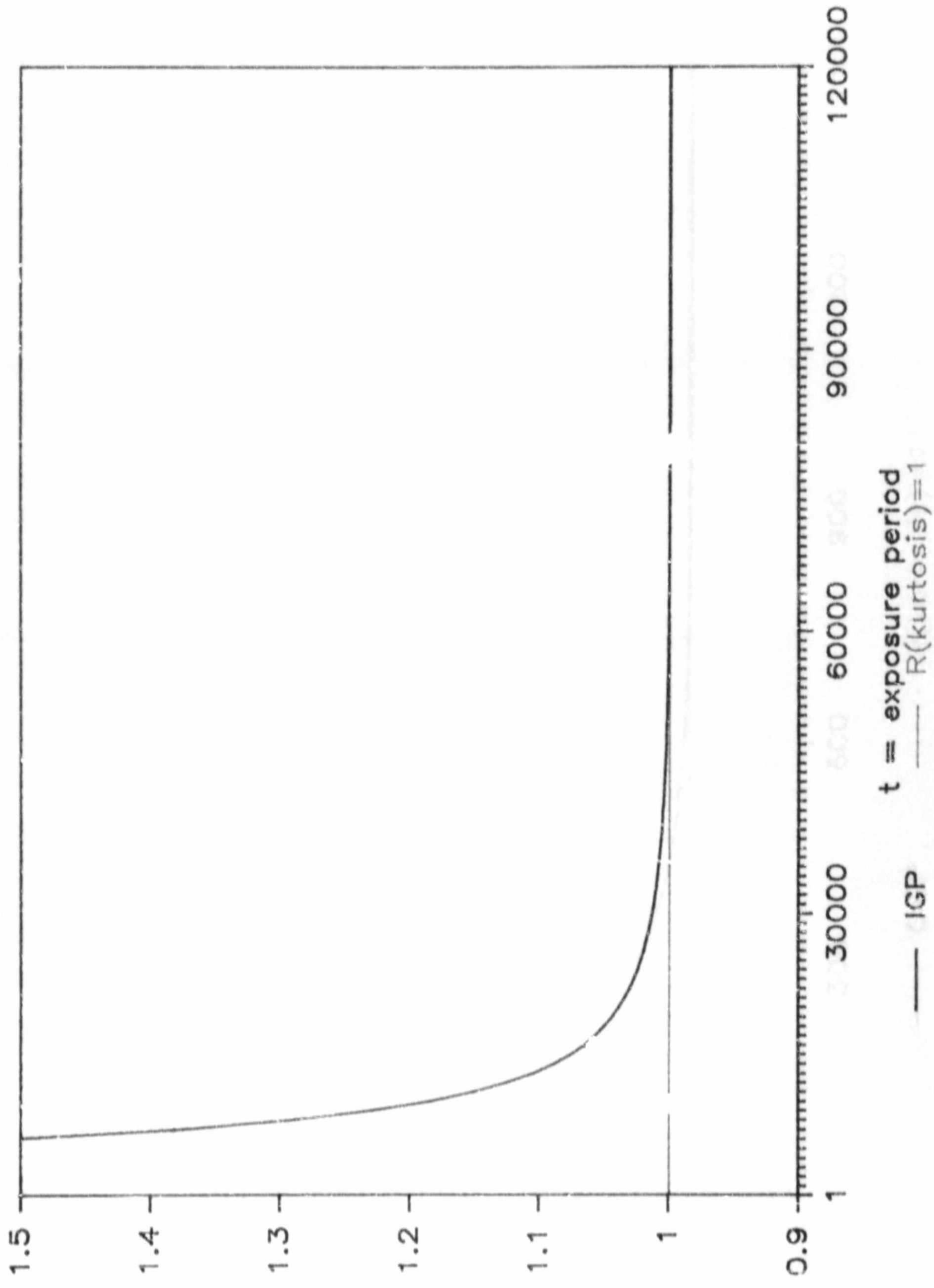
<u>Figure 6.5</u> The coefficient of kurtosis ratio for the IGP with b = 0,1 and c = 0,4



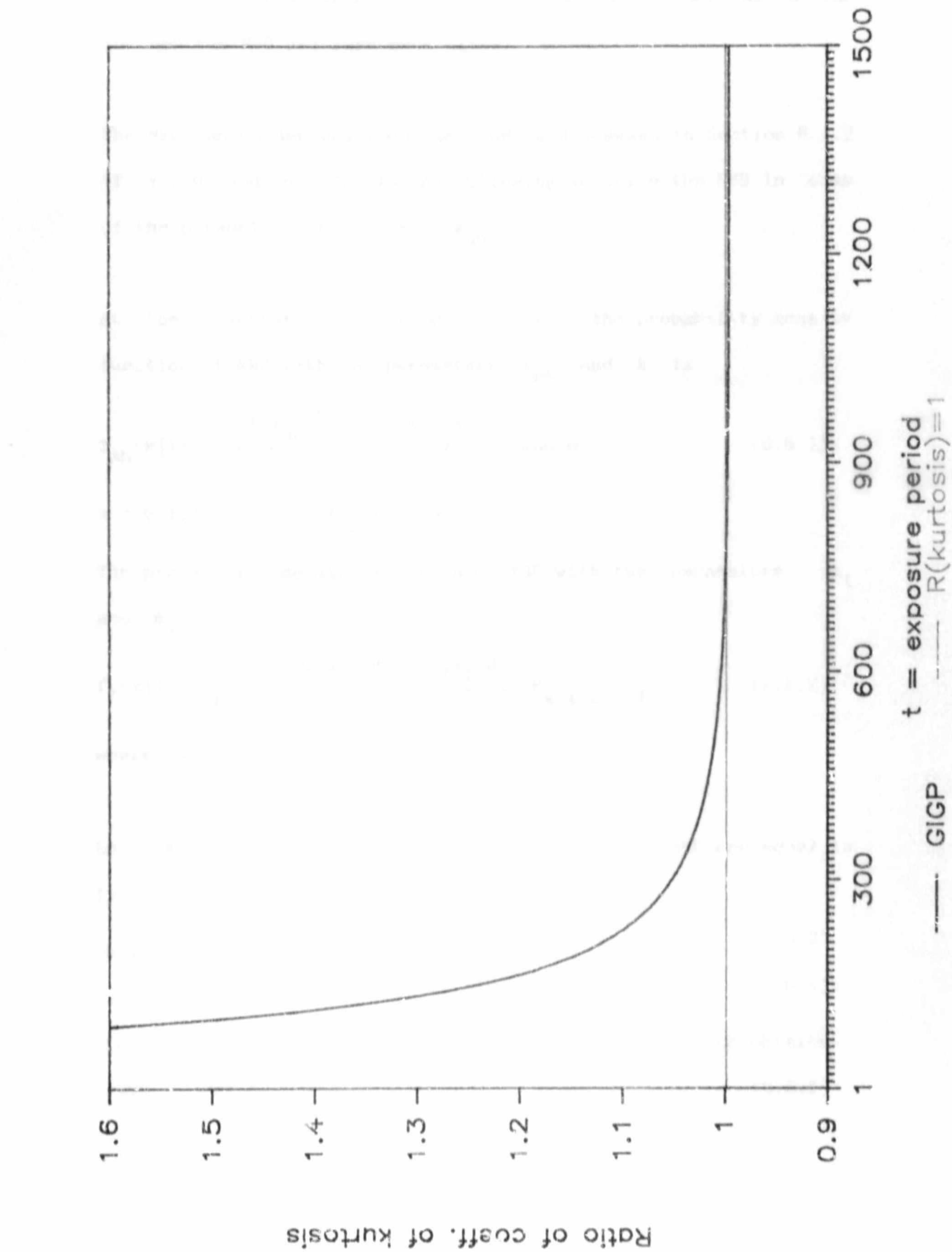
<u>Figure 6.6</u> The coefficient of kurtosis ratio for the IGP with b = 0,1 and c = 0,94



<u>Figure 6.7</u> The coefficient of kurtosis ratio for the IGP with $b = 0,1$ and $c = 0,004$



<u>Figure 6.8</u> The coefficient of kurtosis ratio for the GIGP with $\gamma = 0$, $b = 0{,}1$ and $c = 0{,}04$



6.6 Relations between the IGP Model and the NBD Model

In this section we investigate the circumstances under which the IGP and the NBD are very much alike.

The NBD can be derived from the GIGP as discussed in Section 6.1.2 if $\gamma > 0$ and $\alpha = 0$. In the following we write the NBD in terms of the parameters $k = \gamma$ and θ_{nb}.

At the observational period $t = 1$ the probability density function of NBD with the parameters θ_{nb} and k is

$$f_{nb}(x|1) = \frac{(1-\theta_{nb})^k}{\Gamma(k)} \frac{\Gamma(x+k)}{\Gamma(x+1)} \theta_{nb}^x, \text{ where} \tag{6.6.1}$$

$x = 0,1,2,\ldots,$ $0 \le \theta_{nb} \le 1$ and $k > 0$.

The probability density function of IGP with the parameters α_I and θ_I is

$$f_I(x|1) = \sqrt{\frac{2\alpha_I}{\pi}}\, e^{\alpha_I\sqrt{1-\theta_I}} \frac{(\alpha_I\theta_I/2)^x}{x!} K_{x-1/2}(\alpha_I), \tag{6.6.2}$$

where $x = 0,1,2,\ldots,$ $0 \le \theta_I \le 1$ and $\alpha_I > 0$.

Let us assume that the first two moments of the NBD are equal to the respective ones of the IGP. In this case

$${}_{nb}\mu_1' = {}_I\mu_1' \quad \text{and} \tag{6.6.3}$$

$${}_{nb}\mu_2 = {}_I\mu_2 . \tag{6.6.4}$$

From the definition of the coefficient of dispersion one obtains

$${}_{nb}COD = {}_I COD . \tag{6.6.5}$$

Hence one gets

$$\frac{1}{1-\theta_{nb}} = \frac{2-\theta_I}{2(1-\theta_I)} . \qquad (6.6.6)$$

This results in the following relation between θ_{nb} and θ_I:

$$\theta_{nb} = \frac{\theta_I}{2-\theta_I} , \quad \text{or} \quad \theta_I = \frac{2\theta_{nb}}{\theta_{nb}+1} . \qquad (6.6.7)$$

Because θ_I and θ_{nb} must satisfy the constraints

$$0 \leq \theta_I \leq 1 \quad \text{and} \quad 0 \leq \theta_{nb} \leq 1 , \qquad (6.6.8)$$

it follows that

$$\theta_I \geq \theta_{nb} . \qquad (6.6.9)$$

From the assumption (6.6.3) and from equation (6.6.7) one obtains

$$\alpha_I = k \left[\frac{1+\theta_{nb}}{1-\theta_{nb}} \right]^{1/2} . \qquad (6.6.10)$$

Substitution of the constraint $0 \leq \theta_{nb} \leq 1$ into (6.6.10) leads to the following inequality:

$$\alpha_I \geq k . \qquad (6.6.11)$$

The conclusion is that under the above constraints, for $\theta_I \geq \theta_{nb}$ and $\alpha_I \geq k$, the first two moments of the NBD and the IGP are equal.

In what follows we show that the NBD and the IGP have different shapes, in spite of having the same means and variances.

For $k = 0,9$ and $\theta_{nb} = 0,985$ the NBD is reverse J-shaped. The respective values of the parameters for the IGP distribution, when subjected to the constraints (6.6.7) and (6.6.10) are: $\alpha_I = 0,9 \sqrt{397/3} \approx 10,35$ and $\theta_I = 394/397 \approx 0,992$. For these parameters the IGP is unimodal.

The means and the variances of the NBD and the IGP are equal:

$_{nb}\mu'_1 = {_I}\mu'_1 = 59,1$ and $_{nb}\mu_2 = {_I}\mu_2 = 3940$.

The coefficients of variation and dispersion are consequently equal:

$_{nb}CV = {_I}CV = 1,0621$ and $_{nb}COD = {_I}COD = 66,67$.

The coefficients of skewness of the two distributions are however different:

$\sqrt{\beta_1} = 2,10825$ for the NBD and

$\sqrt{\beta_1} = 3,13871$ for the IGP.

The coefficients of kurtosis are also different:

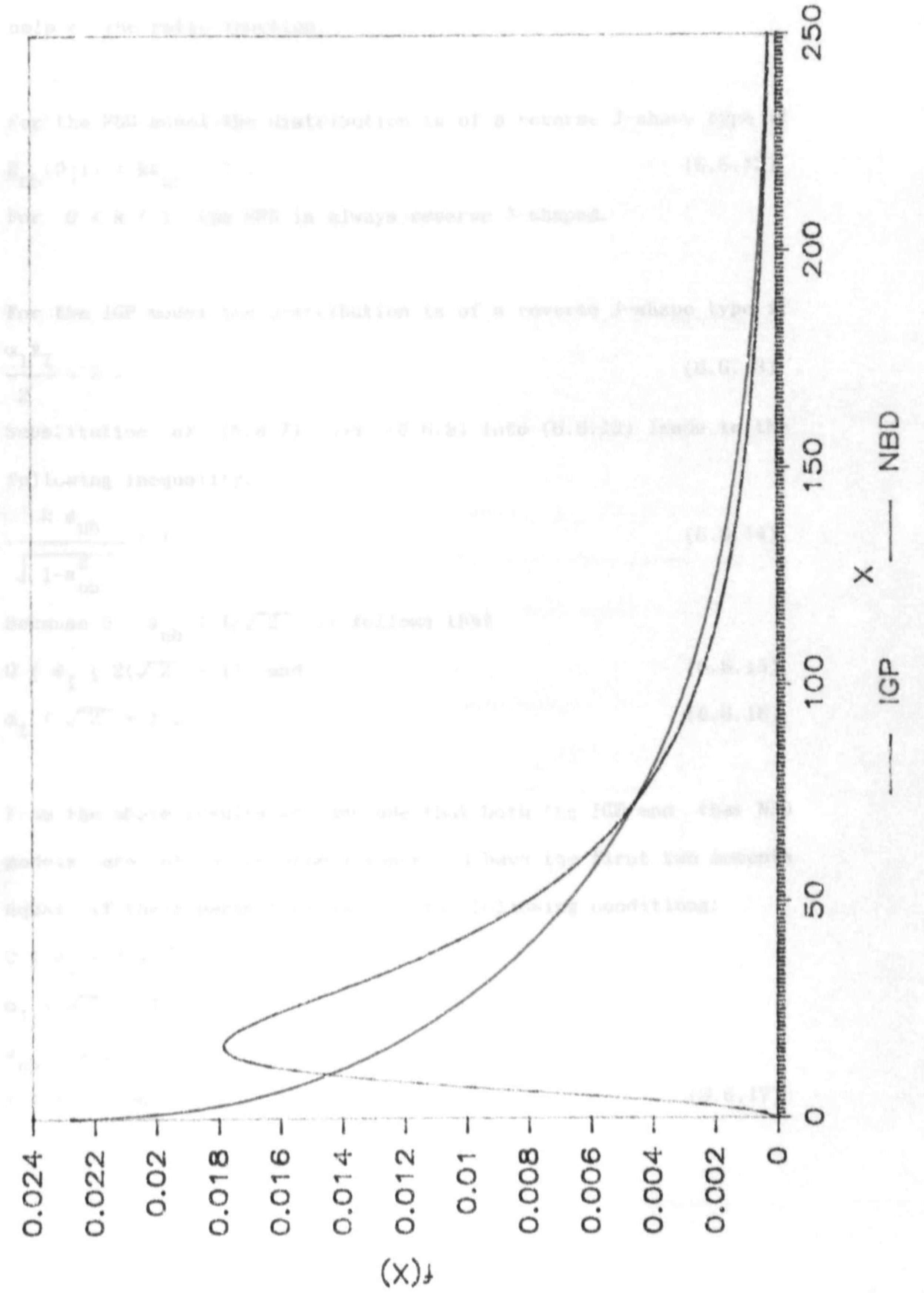
$\beta_2 = 9,6669$ for the NBD and

$\beta_2 = 19,4677$ for the IGP.

The NBD and the IGP with the above parameters are presented in Figure 6.9.

From the graphical display one could expect that the coefficient of skewness of the NBD will be higher than the coefficient of skewness of the IGP. The above calculations show however that the coefficient of skewness of the IGP is higher. This is explained by the fact that in the far tail the probabilities of the IGP are higher than the probabilities of the NBD.

<u>Figure 6.9</u> **The NBD and the IGP with equal means and variances for $k = 0,9$, $\theta_{nb} = 0,985$, $\alpha_I = 10,35$ and $\theta_I = 0,992$**



The next factor under investigation is the shape of the distributions. As mentioned in Section 6.2 this is done with the help of the ratio function.

For the NBD model the distribution is of a reverse J-shape type if

$$R_{nb}(0|1) = k\theta_{nb} < 1 . \tag{6.6.12}$$

For $0 < k < 1$ the NBD is always reverse J-shaped.

For the IGP model the distribution is of a reverse J-shape type if

$$\frac{\alpha_I \theta_I}{2} < 1 . \tag{6.6.13}$$

Substitution of (6.6.7) and (6.6.9) into (6.6.13) leads to the following inequality:

$$\frac{k\,\theta_{nb}}{\sqrt{1-\theta_{nb}^2}} < 1 . \tag{6.6.14}$$

Because $0 < \theta_{nb} < 1/\sqrt{2}$ it follows that

$$0 \le \theta_I \le 2(\sqrt{2} - 1) \quad \text{and} \tag{6.6.15}$$

$$\alpha_I < \sqrt{2} + 1 . \tag{6.6.16}$$

From the above results we conclude that both the IGP and the NBD models are of a reverse J-shape and have the first two moments equal, if their parameters satisfy the following conditions:

$0 \le \theta_I \le 2(\sqrt{2} - 1)$,

$\alpha_I < \sqrt{2} + 1$,

$\theta_{nb} < 1/\sqrt{2}$,

$k < 1$ and (6.6.17)

$$\theta_I = \frac{2\theta_{nb}}{\theta_{nb} + 1}, \quad \alpha_I = k \left[\frac{1 + \theta_{nb}}{1 - \theta_{nb}}\right]^{1/2} \tag{6.6.18}$$

If we take for example $k = 1/2$ and $\theta_{nb} = 1/2$ for the NBD model and apply (6.6.18) we obtain the following parameters for the IGP model:

$\gamma_I = -1/2$, $\theta_I = 2/3$ and $\alpha_I = \sqrt{3}/2$.

In Table 6.1 the NBD and IGP models are presented as a function of the above parameters at $t = 1$.

Table 6.1 The NBD model with parameters $k(1) = 1/2$ and $\theta_{nb}(1) = 1/2$ and the IGP model with parameters $\gamma_I(1) = -1/2$, $\theta_I(1) = 2/3$ and $\alpha_I(1) = \sqrt{3}/2$ at $t = 1$.

x	$f_{nb}(x\|1)$	$f_I(x\|1)$
0	0,707	0,693
1	0,177	0,200
2	0,066	0,060
3	0,028	0,024
4	0,012	0,010

If t increases then the constraints on the parameters in

(6.6.17) are not satisfied anymore and the differences between the two probability laws increases as tabulated in Table 6.2.

Table 6.2 The NBD model with parameters $k(2) = 1/2$ and $\theta_{nb}(2) = 2/3$ and the IGP model with parameters $\gamma_I(2) = -1/2$, $\theta_I(2) = 4/5$ and $\alpha_I(2) = \sqrt{5}/2$ at $t = 2$.

x	$f_{nb}(x\|2)$	$f_I(x\|2)$
0	0,577	0,539
1	0,192	0,241
2	0,096	0,102
3	0,053	0,049
4	0,010	0,026

As shown in Table 6.3, there are big differences between the two models when t increases. One could expect these differences because the parameters $\theta_{nb}(3)$, $\theta_I(3)$ and $\alpha_I(3)$ do not satisfy the constraints (6.6.17).

Table 6.3 The NBD model with parameters $k(3) = 1/2$ and $\theta_{nb}(3) = 3/4$ and the IGP model with parameters $\gamma_I(3) = -1/2$, $\vartheta_I(3) = 6/7$ and $\alpha_I(3) = \sqrt{7}/2$ at $t = 3$.

x	$f_{nb}(x\|3)$	$f_I(x\|3)$
0	0,500	0,439
1	0,188	0,249
2	0,105	0,124
3	0,066	0,066
4	0,043	0,039

Lemma 6.1

For very small values of θ the NBD and the IGP models are very much alike.

Proof

In order to ensure that the distributions are very close to each other one must look at the equality of

(a) their probabilities at $x = 0$,

(b) their ratio functions at $x = 0$ and

(c) their first two moments.

The following conditions must be therefore satisfied

$$f_{nb}(0|1) = f_I(0|1), \tag{6.6.19}$$

$$R_{nb}(0|1) = R_I(0|1), \tag{6.6.20}$$

$$_{nb}\mu_1' = {}_I\mu_1' \quad \text{and} \tag{6.6.21}$$

$$_{nb}\mu_2 = {}_I\mu_2 \; . \tag{6.6.22}$$

From the equality of the NBD and the IGP probabilities at $x = 0$ one obtains:

$$(1-\theta_{nb})^k = \exp\left[-\alpha_I\left(1-\sqrt{1-\vartheta_I}\right)\right] . \tag{6.6.23}$$

In (6.6.23) we substitute the expressions for α_I and ϑ_I from (6.6.18), which were obtained under the constraint of equality of the means and the variances of the IGP and the NBD. This leads to:

$$(1-\theta_{nb})^k = \exp\left[-k\left[\sqrt{\frac{1+\theta_{nb}}{1-\theta_{nb}}} - 1\right]\right] . \tag{6.6.24}$$

Using McLaurin series $(1-\theta_{nb})^k$ is expanded to:

$$(1-\theta_{nb})^k = 1 - k\theta_{nb} + \frac{k(k-1)}{2!}\theta_{nb}^2 - \frac{k(k-1)(k-2)}{3!}\theta_{nb}^3 + \dots \tag{6.6.25}$$

and $\exp\left[-k\left[\sqrt{\frac{1+\theta_{nb}}{1-\theta_{nb}}} - 1\right]\right]$ is expanded to:

$$\exp\left[-k\left[\sqrt{\frac{1+\theta_{nb}}{1-\theta_{nb}}} - 1\right]\right] = 1 - k\theta_{nb} + \frac{k(k-2)}{2!}\theta_{nb}^2 - \dots \tag{6.6.26}$$

The condition (6.6.19) is satisfie mall values of θ_{nb}.

Also the condition (6.6.21) is sat

$$k\theta_{nb} \approx \frac{k\theta_{nb}}{\sqrt{1-\theta_{nb}^2}} . \tag{6.6.27}$$

Hence $1 \approx \dfrac{1}{\sqrt{1-\theta_{nb}^{2}}}$ exists only if θ_{nb} is very small.

This result was expected, because when θ is very small we deal with distributions which have short tails. In Section 6.1.1 it was proved that when $\theta \to 0$ the GIGP tends to Poisson distribution. The Poisson distribution is known for its short tail. The NBD and the IGP are both special cases of the GIGP and when θ becomes very small the distributions have very short tails. Most of the weight is therefore concentrated in the first cells, causing them to be very much alike.

<u>Lemma 6.2</u>

If ${}_{nb}\mu_1'(x|1) = {}_{I}\mu_1'(x|1)$ and (6.6.28)

${}_{nb}\mu_2(x|1) = {}_{I}\mu_2(x|1)$, for (6.6.29)

$$\alpha_I = k\sqrt{\frac{1+\theta_{nb}}{1-\theta_{nb}}} \quad \text{and} \quad \theta_I = \frac{2\theta_{nb}}{\theta_{nb}+1}\,, \tag{6.6.30}$$

then

${}_{nb}\mu_1'(x|t) = {}_{I}\mu_1'(x|t)$ and (6.6.31)

${}_{nb}\mu_2(x|t) = {}_{nb}\mu_2(x|t)$.

Proof

From *Lemma* 5.2 we obtain that

${}_{nb}\mu_1'(x|t) = t\;{}_{nb}\mu_1'(x|1)$ and (6.6.32)

$${}_{I}\mu_1'(x|t) = t \; {}_{I}\mu_1'(x|1)$$

Because (6.6.28) is satisfied under the conditions (6.6.30) it follows that

$${}_{nb}\mu_1'(x|t) = {}_{I}\mu_1'(x|t) \; .$$

From Lemma 5.2 we obtain also that

$${}_{nb}\mu_2(x|t) = t^2 \; {}_{nb}\mu_2(x|1) - t(t-1) \; {}_{nb}\mu_1'(x|1) \quad \text{and}$$

$${}_{I}\mu_2(x|t) = t^2 \; {}_{I}\mu_2(x|1) - t(t-1) \; {}_{I}\mu_1'(x|1) \; .$$

Because (6.6.28) and (6.6.29) are satisfied under the conditions (6.6.30) it follows that

$${}_{nb}\mu_2(x|t) = {}_{I}\mu_2(x|t) \; .$$