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Solving multiplication and division word problems – strategies used by Grade 6 learners

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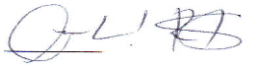
A research report submitted to the Wits School of Education, Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements of the degree of Master of Science (Science Education).

Johannesburg, 2019

Declaration

I, Tumelo Ramaoka, hereby declare that this research is my own work. It is being submitted for the Degree of Master of Science (Science Education) at the University of the Witwatersrand, Johannesburg. This report has not been submitted for any other degree or examination at any other university.



24 July 2019

Tumelo Ramaoka Date

Abstract

This study was aimed at exploring the models and strategies a class of Grade 6 learners used to solve multiplicative word problems prior to, and following, a series of four one-hour intervention lessons based on the literature on multiplicative reasoning using a Realistic Mathematics Education (RME) – based theoretical – framework. The sample of the study were a group of 36 learners from a fee-paying government school in Johannesburg East district that has been classified as a low performing school based on the Annual National Assessment (ANA) for Mathematics. Outcomes based on pre- and post-tests indicated substantial gains related to the content of the intervention lessons as learners were able to make better sense of multiplicative situations and use more efficient strategies to solve multiplicative situations. Pre-test data pointed out that learners struggled to make sense of multiplicative situations especially division items and used a limited range of models and strategies. Post-test data indicated that there were substantial shifts to using a broader range of multiplicative models and strategies including the emergent use of ratio models. This use of a broader range of models and strategies increased success in learners' multiplicative problem-solving performance.

Dedication

To my parents, siblings and friends who have been supportive throughout the M.Sc journey, this one is for you. When nothing made sense and I felt this is not for me, you continuously saw the best in me and encouraged me to keep pushing. Through your unwavering love and support I would have not got this far.

This journey had me at the edge of everything, this is also dedicated to students who did not get the opportunity to complete their studies due to mental illness or finding the course too challenging to complete. May you gather strength to go back and complete what you started. Take it a step at a time, eventually you will get there.

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First, I would like to give my acknowledgement to God, for giving me strength and guidance throughout this journey. For teaching me to trust the process and His timing is forever perfect. Through His grace I was able to complete this study. This is all because of His favour, O! mohau wa Modimo.

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Chapter One: Introduction

1.1. Background of the study

International and local studies on South African learners' mathematical knowledge and proficiency constantly reveal that South African learners perform poorly on international and local assessments (DBE, 2013a; 2015; Reddy *et al*, 2016; Spaull, 2013). In the international TIMSS[1] 2015 assessment, South Africa ranked 47th out of 48 countries, with an overall average performance below the minimum level of mathematical competence of Grade 5 learners internationally (Reddy *et al*, 2016). The TIMSS 2015 report identified three factors within the poor performance that was seen: (1) weak language proficiency, (2) foundational skills lacking, and (3) poor higher order problem solving skills and abstract thought (Jaun *et al*, 2017).

The SACMEQ[2] III 2007, which is a regional assessment for 14 African countries, assessed Grade 6 learners' mathematical competence. South Africa was ranked eighth in Mathematics, with poorer countries such as Kenya, Swaziland and Tanzania performing better (Spaull, 2013). The SACMEQ III 2007 study also revealed that many South African Grade 6 Mathematics educators' mathematical competence was below basic levels of content knowledge, with high proportions of educators being unable to answer questions aimed at their learners (Spaull, 2013).

The Annual National Assessment (ANA), a local assessment aimed at assessing learners' grade-related competency in Mathematics and English, also revealed that learners performed poorly nationally. For the Intermediate Phase (Grade 4-6) in 2014, the average performance for Mathematics was 39 % (DBE, 2015). Grade 6 had the highest average performance at 43 %, whereas Grade 5 had an overall performance of 38 % and Grade 4 had 36 % (DBE, 2015). Poor performance in international and national assessments suggests that there is a need for intervention in South African Mathematics education in the middle grades to encourage effective learning and teaching of mathematics.

Beneath the surface of overall performance, both the TIMSS 2015 and ANA 2012 – 2014 assessments results revealed that South African learners experienced difficulties with basic

[1] TIMSS - Trends in International Mathematics and Science Study
[2] SACMEQ - Southern and Eastern African Consortium for Monitoring Educational Quality (Grade 6 numeracy and literacy)

operations especially multiplication and division, and problem solving (DBE, 2013a; 2015; Juan *et al*, 2017). In the TIMSS 2015 assessment, learners had two multiplicative problem situations to solve. For both multiplicative problem situations, 35 % of the responses were correct, 64 % of the responses were incorrect solutions and 1 % of the responses omitted the items (Juan *et al*, 2017). Incorrect responses suggested that learners might have not read the questions correctly, interpreted the multiplicative problem situation as additive problem situation and/or directly took numbers from the word problem text as it appears (Juan *et al*, 2017). ANA diagnostic reports show further that learners experienced difficulties with executing division and multiplication algorithms, and with solving problems involving grouping and sharing versions of division. Learners use inefficient unit counting, repeated addition and repeated subtraction for multiplication and division even when working with large numbers (DBE, 2013b; 2015). The use of highly inefficient strategies for multiplication and division by South African learners has also been observed internationally among lower attaining learners (Anghileri, 2006; Juan *et al*, 2017; Nunes & Bryant, 2009).

This evidence reflects problems I have observed as a Grade 6 Mathematics educator over three years in urban South African classrooms. For multiplicative word problems, most learners use unit counting methods, or repeated addition and repeated subtraction even when working with large numbers, and many are unable to use symbolic methods appropriately for multiplication and division calculations. Difficulties with sense making of multiplicative word problems are also prevalent. Taken together, the broader evidence coupled with my own experiences suggests a need for intervention in learners' multiplicative reasoning.

In this study I explore the models and strategies Grade 6 learners use to solve multiplication and division word problems prior to, and following, a sequence of intervention lessons based on the literature on multiplicative reasoning using Realistic Mathematics Education (RME) theory. This study is part of the Multiplicative Reasoning in the Intermediate Phase (M-RIP) research program under the umbrella of Wits Maths Connect-Primary (WMC-P) project. The WMC-P project is aimed at researching and developing initiatives to address issues resulting in poor performance in Mathematics. As a part-time Masters student, I was also a teacher in one of the project's partner schools running the M-RIP intervention trial with my Grade 6 class. As the teacher-researcher, I analysed the pre- and post-test data from a test designed in the broader project, and taught the intervention lessons, with support from a project researcher. In this study my attention is on identifying and interpreting differences in learners' approaches to multiplicative reasoning problems across the pre- and post-test, in

order to understand any contributions that the intervention might have played in supporting learners' work on multiplicative reasoning problems.

1.2. Literature based rationale

Given that this study is focused on multiplicative reasoning, I briefly introduce the concept of multiplicative reasoning, and aspects that have been described as important for supporting learning of these ideas: problem-solving based approaches and key representations.

1.2.1. Multiplicative reasoning

Multiplicative reasoning is described as the capacity to work flexibly with the concepts, strategies and representations of multiplication and division as they occur in different contexts (Siemon, Breed & Virgona, 2006; Siemon *et al*, 2013). Multiplicative thinking is an important kind of thinking as it focuses primarily on problem structure and how learners make sense of multiplicative relations between quantities rather than simply just performing arithmetic operations (Nunes & Bryant, 2009). Multiplicative thinking is a prerequisite for understanding of rational numbers and proportional reasoning (Askew, 2018; Nunes & Bryant, 2009). Nunes & Bryant (2009, p.8) make three assumptions in relation to multiplicative thinking:

- To understand multiplication and division in depth, learners must also understand the inverse relation between multiplication and division.
- Links between multiplication and division are conceptual and relate to connections within the domains of reasoning.
- Despite procedural links between addition and multiplication these forms of reasoning are distinct conceptual domains.

These three assumptions emphasise that multiplication and division should not be taught as separate operations. Learners need to be given a chance to see that multiplication and division are interrelated and calculations that emerge are related (Askew, 2018; Nunes & Bryant, 2009). Overviewing an extensive literature base, Askew (2018) notes that use of ratio models and arrays can assist with creating a link between multiplication and division, and these models were therefore strongly in focus in the intervention lessons, as described in Venkat & Mathews' (2018) paper looking at the Grade 7 learning outcomes of the M-RIP intervention. Section 2.7 provides an in-depth description of ratio models and arrays as integrated multiplicative models.

1.2.2. Problem solving

One of the ways of developing multiplicative thinking skills is through word problems. Word problems in mathematics are textual descriptions of situations within which mathematical questions can be contextualised (Sepeng & Webb, 2012). Studies show that many learners dislike and fear word problems (Murray, 2012), which I had also observed from learners I taught. The South African Curriculum and Assessment Policy Statement (CAPS) curriculum emphasises that problem solving should be included in mathematics lessons (DBE, 2011). However, learners' performance in word problems suggests that many learners do not have necessary skills for problem solving as learners are unable to interpret word problems, and use the given information appropriately, translate mathematical language and write correct mathematical sentences (DBE, 2013b; Murray, 2012).

Problem solving is one of the central aspects of mathematics, as it enables learners to make sense of how school mathematics can be used in their everyday lives (Sepeng & Sigola, 2013). Therefore, problem solving should be integrated into mathematics teaching to support learning mathematics with understanding. Problem solving enables learners to use their informal knowledge to make sense of school mathematics and gain experience with a variety of mathematical structures and procedures that can be developed and refined (Murray, 2012). Solving mathematical word problems is a process that requires multistep skills and strategies, which can be developed through guidance of the educator (Sepeng & Sigola, 2013). Effective problem solving thus can equip learners with necessary skills to deal with real-life problems they are confronted with in everyday life experiences.

1.2.3. The role of representations and models in mathematics

Representations are required to access mathematical ideas in a meaningful way and are used to explicitly communicate mathematical relations and meanings (Siemon *et al*, 2013). Barmby, Harries, Higgins & Suggate (2009) categorise representations as either internal or external. Internal representations are mental images or models. External representations are physical, embodied and observable images or models that represent an object or concept.

Mathematical understanding has been described as observable when learners use mental representations as part of a network of representations (Barmby *et al*, 2009; Siemon *et al*, 2013). For learners to thoroughly understand mathematical ideas, procedures or facts, representations are linked to existing networks with more extensive or more numerous connections (Barmby *et al*, 2009). Mathematical modelling involves using and applying

mathematics in the real, or realistic, world. Representations that can be used to find solutions of word problems are part of mathematical modelling (Haylock, 2010).

Models are used to make formal abstract mathematics accessible to learners (Askew, 2012). Models play a significant role in bridging the gap between the informal mathematics and more formal mathematics. To fulfil the bridging function, models must shift from a model of a situation to a model that can be used for classes of situations (Askew, 2012; Van den Heuvel-Panhuizen & Drijvers, 2014). As noted already, arrays and ratio tables are described as particularly useful models for multiplicative situations.

1.2.4. Multiplicative reasoning intervention

The focus of my study is on learning of multiplication and division in the context of an intervention focused on developing learners' ability to model multiplicative situations. Studies show that a large number of South African learners experience difficulties with modelling multiplicative situations and using inefficient strategies to work out multiplication and division calculations (Askew, 2018; DBE, 2013b; Venkat & Mathews, 2018). The literature introduced above notes that models play an important role in learning mathematics in a meaningful manner as they serve as the bridging gap between informal mathematics and more formal mathematics.

Learning mathematics is often associated with the process of learning ready-made algorithms, mathematical procedures and rules, where learners are not given the opportunity to use their own informal tools to construct their own models (Fauzan, 2002). This results in learners frequently not making sense of the mathematical procedures and algorithm they are taught in class, and therefore becoming unable to reconstruct them flexibly when needed. It was to deal with this issue that Realistic Mathematics Education researchers argued that learners should be given the opportunity to develop models of problem situations and strategies for solving problems, with the guidance of the educator (Van den Heuvel-Panhuizen, 2003).

I followed this approach in my intervention lessons, offering learners opportunities to share and discuss models and strategies rather than using the traditional method of memorising mathematical algorithms. Learners were given the opportunity to develop and explore various multiplicative strategies. Therefore, I found RME a suitable theoretical framework for my study. RME emphasises the importance of mathematising subject-matter from problem situations that learners can imagine and that learners should be given the opportunity to re-invent mathematics under the expert guidance of the educator (Fauzan, 2002; Van den

Heuvel-Panhuizen & Drijvers, 2014). The word problems used in the study were problem situations that learners could relate to or easily imagine. To develop multiplicative thinking skills, learners were also given the opportunity to produce their own models when solving word problems. In Chapter 3 (page 34) I discuss the RME theoretical framework in-depth.

1.3. Research questions

In order to understand the kinds of models and strategies used by Grade 6 learners across a range of multiplicative problem situations, I focused on the following aspects:

1. What changes, if any, are seen in learners' overall performance in the post-test?
2. What models and strategies for solving multiplicative problems do Grade 6 learners use prior to the multiplicative reasoning intervention?
3. What models and strategies for solving multiplicative problems do Grade 6 learners use post the multiplicative reasoning intervention?

1.4. Methodology

To address my research questions, I worked with 36 learners in Grade 6 from a fee-paying government school located in the Johannesburg East district. The school is classified as a low performing school based on the ANA Mathematics performance and is part of the broader WMC-P project. The case site was selected based on convenience because it is easily accessible to me as the researcher, as I was a Grade 6 Mathematics educator in the school at the time of data collection. My specific focus on Grade 6 was also motivated by learners I had taught in past years experiencing difficulties with making sense of multiplicative problem situations and using inefficient strategies as the number range increases, such as unit counting. As noted already, similar findings were also reported in ANA 2013 and 2014 diagnostic reports: most Grade 6 learners were unable to execute formal multiplication and division procedures, unable to interpret word problems and/or use the given information to formulate the correct solution (DBE, 2013b; 2015).

The intervention lessons of my study were structured around the multiplicative reasoning literature and the RME theoretical framework. RME advocates that learning mathematics should not involve ready-made procedures, but rather that learners construct knowledge under the expert guidance of the educator (Fauzan, 2002; Van den Heuvel-Panhuizen & Drijvers, 2014). The intervention lessons materials and tests were designed by WMC-P team as part of their larger project on multiplicative reasoning and I describe these instruments later in Chapter 4. The intervention consisted of four multiplicative reasoning lessons over a

period of four weeks. Lessons were designed to focus on different multiplicative situations, which are described in the literature review in Chapter 2. One hour lessons were conducted weekly. Prior and post the four intervention lessons, learners wrote pre- and post-tests, respectively. The first and second intervention lessons focused on multiplicative problem situations that involved equal groups of objects. The third lesson was on equal measures problem situations and the fourth lesson focused on array problem situations. Variety on the focus of the lessons was implemented to expose learners to different kinds of multiplicative situations as various studies note that learners must be exposed to different multiplicative situations (Askew, 2018; DBE, 2013b; Nunes & Bryant, 2009, Venkat & Mathews, 2018).

Each lesson was built around three parts, which were: (1) practicing rapid recall of multiplication and division facts, (2) solving a linked set of three problems and (3) identifying the links between the problems. Three types of representations were worked on during intervention lessons: double number lines, ratio tables and arrays. This was to help learners use more efficient representations and make sense of multiplicative situations and multiplicative relationships.

A pre- and repeated post-test was the key research instrument used to measure the effect of intervention lessons. My focus was on understanding performance differences across these two tests and, subsequently, to analyse the models and strategies presented in learners' mathematisation attempts across their written responses from the pre- and post-tests. The pre- and post-tests were the same, made up of fifteen multiplicative word problems which focused on equivalent groups, arrays and multiplicative comparison. Of the fifteen multiplicative word problems, 10 were division related word problems and 5 were multiplication problems. Greater focus on division in this study was deliberate since prior studies conducted by Dlamini (2014) and Hansa (2017) had largely focused on multiplication, and ANA 2014 diagnostic report mentioned that many learners performed poorly when solving problems involving the relational division skills of grouping and sharing (DBE, 2015). The pre-test was administered before the intervention lessons, to identify the models and strategies learners used prior to intervention. The post-test was conducted five weeks after the pre-test to help identify shifts, if any, in learners' approaches post intervention. The contribution of the intervention lessons was understood by comparing the pre- and post-test performance.

For data analysis and interpretation, test responses were marked and coded by me using and adapting some of the approaches developed by the WMC-P team for the broader M-RIP

study (see Venkat & Mathews, 2018), as well as some of the approaches used in prior project studies (Dlamini, 2014; Hansa, 2017). Scores were used to examine overall performance and coding was used to categorise the models and strategies learners used to respond to word problems. A code book was developed based on the multiplicative reasoning literature reviewed in Chapter 2 combined with the RME theoretical framework. The coding of responses helped me to answer the research questions. A detailed description of the methodology of my study can be found in Chapter 4.

1.5. Structure of the research report

This research report consists of six chapters.

This introductory chapter of the research report introduces and provides a summary of the study and the research problem. It contains the background, literature rationale, guiding research questions and brief description of the methodology that has been used in this study.

The second chapter presents a range of literature that I reviewed prior to conducting the study. The literature focuses on multiplicative reasoning, problem solving, the role of representations and models, and models and strategies for multiplication and division.

The RME theoretical framework is discussed in the third chapter as the guiding theoretical framework of the study.

Chapter 4 outlines the methodology used in my research. Included in this chapter, is the research design, sampling procedure, research instrument, procedure of data collection and data analysis.

In Chapter 5 I present and discuss findings of the study in relation to the research questions and literature reviewed.

The conclusions and recommendations of this study are presented in Chapter 6. This chapter includes concluding remarks, limitations of my study and recommendations for future research.

Chapter 2: Literature Review

2.1 Introduction

Various international and national studies suggest that learners experience difficulties with performing multiplication and division algorithms and making sense of multiplicative word problems (Anghileri, 2006; DBE, 2013a; 2013b; Dlamini, 2014; Hansa, 2017). In South Africa, the 2013 ANA Grade 6 performance results (DBE, 2013a) indicated a particularly low overall mean performance of 19 % for multiplicative items (both 'bald' questions[3] and word problems). For bald multiplication question the average performance was 18 % with responses showing that learners had partial understanding of formal multiplication procedures – especially the vertical column algorithm. Further, learners with responses showing evidence of understanding the multiplication algorithm were unable to execute vertical column method multiplication, and frequently, no reasoning or working out was shown to justify their answers. Learners' responses also showed that learners had poor understanding of place value and struggled with the basic multiplication facts (DBE, 2013a). For the division algorithm, an average of 26 % of responses were correct. From the diagnostic report, it can be seen many learners had limited knowledge of long division processes. For multiplicative word problems the average performance was lower at 13 %, indicating difficulties with understanding the word problem seen sometimes in lack of appropriate models of problem situation and in further difficulties with executing the calculation procedures (DBE, 2013a). As noted already, the issues highlighted above reflected problems areas I experienced and dealt with as a Grade 6 educator.

In this chapter, I provide a literature review on multiplicative thinking divided into four parts. First, I discuss the nature of multiplication and division, then continue to multiplicative reasoning. This is followed by an overview of various multiplicative problem situations and problem solving. I conclude by considering key multiplicative models.

2.2 Multiplication and division

Multiplication and division are two of the four basic operations of mathematics. Both arithmetic operations are often taught separately after the learning of addition and subtraction, with multiplication commonly represented as repeated addition and division

[3] Bald questions – refers to questions that are not presented as word problems, they are presented in symbolic form.

represented as repeated subtraction. Barmby *et al* (2009) argue that this view of multiplication and division as an extension of addition and subtraction is limited. Instead, they argue that multiplication and division require a different kind of arithmetic thinking.

Multiplication refers to situations involving the iteration of equal groups of objects or equal sized groups. Multiplication can be formally explained as $a \times b$ (read 'a' multiplied by 'b') meaning b groups of a. Given $a \times b = c$ then a is called the multiplicand, b is called the multiplier and c is called the product. Hence, 5×7 is read as '5 multiplied by 7', that is, five taken seven times or $5 + 5 + 5 + 5 + 5 + 5 + 5$. Anghileri (2006) suggests that multiplication is a binary operation with two inputs, the first input being the size of the set (multiplicand) and the second input representing the number for replications (multiplier). Both multiplicand and multiplier therefore represent distinct elements of the multiplication process (Anghileri, 2006; Barmby *et al*, 2009). This marks a key difference with additive situations where two or more addends all represent the same kind of quantity.

Division is the inverse operation of multiplication and involves splitting objects or things into equal parts. Division can be formally explained as $c \div b = a$ where the value of a satisfies the equation $a \times b = c$. The value of c is known as the dividend, b is the divisor and a is the quotient. The value of b must not be zero. Division consists of the number which is to be divided (dividend) and the number by which it is divided (divisor). Division processes then leaves a quotient and a possible remainder, if the latter is not zero.

Both multiplication and division have different properties. Multiplication has three distinct properties: commutative, associative and distributive, whereas division has only one property, that is, the distributive property (Anghileri, 2006; Barmby *et al*, 2009; Haylock, 2010). The commutative law of multiplication states that when multiplying two numbers together the order in which you multiply the two numbers does not make any difference. The product is independent of which of the two numbers is the multiplier and which the multiplicand. The commutative property can be recognised formally by the following generalisation, which is true whatever the numbers a and b are: $a \times b = b \times a$. In teaching, two important aspects of the commutative property of multiplication to bear in mind are that: division does not have this property (for example; $6 \div 2 \neq 2 \div 6$) and that use of the commutative property enables us to simplify some calculations (Anghileri, 2006; Haylock 2010; Wright, Ellemor-Collins & Tabor, 2012). In applying the commutative property, the number of facts that need to be learned in multiplication can be halved (Anghileri, 2006; Haylock, 2010).

The associative property of multiplication states that when multiplying three (or more) numbers, it does not matter whether you start by multiplying the first and second numbers, or by multiplying the second and third numbers. The product is independent of the order in which we perform the multiplication. For all real numbers *a*, *b* and *c* the associative law states $(a \times b) \times c = a \times (b \times c)$. As before, the associative property does not hold for division as $(a \div b) \div c \neq a \div (b \div c)$ (Anghileri, 2006; Haylock, 2010; Wright *et al*, 2012).

Both division and multiplication have a distributive property. The distributive property of multiplication states that for all numbers *a*, *b* and *c* in a system of whole numbers, then $(a + b) \times c = (a \times c) + (b \times c)$ and $(a - b) \times c = (a \times c) – (b \times c)$. Therefore, *a* can be distributed across both *b* and *c*. The distributive property of division states that *c* can be distributed across both *a* and *b*, if *c* does not equal to 0 (zero). That is, $(a + b) \div c = (a \div c) + (b \div c)$ and $(a - b) \div c = (a \div c) - (b \div c)$ (Haylock, 2010; Wright *et al*, 2012). It must be noted that multiplication is both left and right distributive over addition and subtraction, whereas division is only right distributive over addition and subtraction. This means that, 14×5 can be calculated as $(14 \times 2) + (14 \times 3)$ or $(10 \times 5) + (4 \times 5)$, however $56 \div 8$ is equivalent to $(40 \div 8) + (16 \div 8)$ but is not equivalent to $(56 \div 6) + (56 \div 2)$. For multiplication both factors (multiplier and multiplicand) can be partitioned, whereas for division only the dividend can be partitioned, not the divisor (Thompson, 2008).

2.3 Multiplicative reasoning

Studies show that many learners have limited access to a broader range of relational ideas for multiplication and division (Nunes & Bryant, 2009; Siemon *et al*, 2013). Difficulties with developing efficient mental strategies results in relying on memorised procedures for multiplying and dividing larger whole numbers and decimals. Difficulties with developing efficient, and therefore flexible, mental strategies can be attributed to the following (Siemon *et al*, 2013, p.189):

- an inability to trust the count and see numbers as countable units in their own right,
- poorly developed or non-existent addition and subtraction mental strategies,
- limited or no exposure to different models of multiplication and division, and
- over-reliance on physical models to solve simple multiplicative problems.

Multiplicative reasoning is the capacity to work flexibly with the concepts, strategies and representations of multiplication and division as they occur in different contexts (Siemon *et al*, 2006; 2013). Multiplicative reasoning is characterised by the capacity to work flexibly and

efficiently with an extended range of numbers, the ability to recognise and solve a range of multiplication and division problems and to effectively communicate multiplication and division in a variety of ways (Siemon *et al*, 2006). Multiplicative reasoning is an important kind of thinking as it focuses primarily on relational problem structures. The main goal is to understand how learners make sense of multiplicative relations between quantities rather than merely duplicating arithmetic operations.

Multiplicative reasoning is a pre-requisite for understanding of rational numbers and proportional reasoning topics: fractions, percentage, decimals, ratio and proportion (Askew, 2018; Nunes & Bryant, 2009). Variation in acquisition of multiplicative reasoning skills is described as one of the most important reasons why there is a range of achievement in mathematics in Grade 5 - 9 (Siemon *et al*, 2006; 2013).

Researchers suggest that multiplicative reasoning does not develop naturally, but requires deliberate and focused instruction from the educator (Siemon *et al*, 2006; 2013). Unlike additive thinking which can be developed over a short time, multiplicative thinking may take many years to develop as it is more complex than additive thinking (Nunes & Bryant, 2009; Siemon *et al*, 2013). Additive thinking involves actions of joining, separating, and placing sets in one-to-one correspondence, whereas multiplicative thinking involves actions of putting two (or more) variables in one-to-many correspondence, an action that keeps the ratio between the variables constant (Nunes & Bryant, 2009). Multiplicative thinking also includes situations which involve sharing, division, and splitting (Nunes & Bryant, 2009). In additive situations invariant difference between two values is considered, thus addition and subtraction can be used to find an unknown member of another pair (Van Dooren, De Bock & Verschaffel, 2010). For example: *Mpho has 17 marbles and Karabo has 25 marbles. How many more counters does Karabo have than Mpho?* The given numbers in this situation are linked by addition and subtraction. There are two variables in this situation (Mpho's marbles and Karabo's marbles) and the difference between these variables is invariant: subtracting Karabo's marbles by Mpho's marbles will provide the difference of the number of marbles. For multiplicative situations, it is an invariant ratio between values that is considered. Thus, an unknown number in another related pair can be found using multiplication and division (Van Dooren *et al*, 2010). For example: *One mango cost R6. Tumi buys 8 mangoes. How much does Tumi spend?* The two variables in the problem situation (mango and cost) are linked by multiplication and division with an invariant: knowing that each mango costs R6,

multiplying the number of mangoes by R6 gives the price, and dividing the price of mangoes by R6 gives the number of mangoes bought (Van Dooren *et al*, 2010).

Nunes & Bryant's (2009, p.8) assumptions in relation to multiplicative thinking, restated below, emphasise that multiplication and division should not be taught as separate operations. Learners should be given a chance to see that the two concepts are interrelated and calculations which emerge are related:

- To develop an in-depth understanding of multiplication and division, learners must also understand the inverse relation between the two.
- Links between multiplication and division are conceptual and relate to connections within the domains of reasoning.
- Despite procedural links between addition and multiplication, these forms of reasoning are distinct conceptual domains.

Various situations give rise to learners' understanding relational connections of multiplicative thinking. Greer (1992, p.276) classifies these multiplicative situations into four main categories:

- equivalent groups (e.g. 3 bags of cupcakes with 4 cupcakes in each bag),
- rectangular arrays (e.g. 3 rows of cupcakes with 4 cupcakes in each row),
- multiplicative comparison (e.g. chocolate cupcakes are 3 times as many as vanilla cupcakes), and
- cartesian product (e.g. the number of different possibilities for chocolate-vanilla cupcakes pairs from 3 chocolate cupcakes and 4 vanilla cupcakes).

Figure 2.1 illustrates visual representations of the four main categories of multiplicative situations.

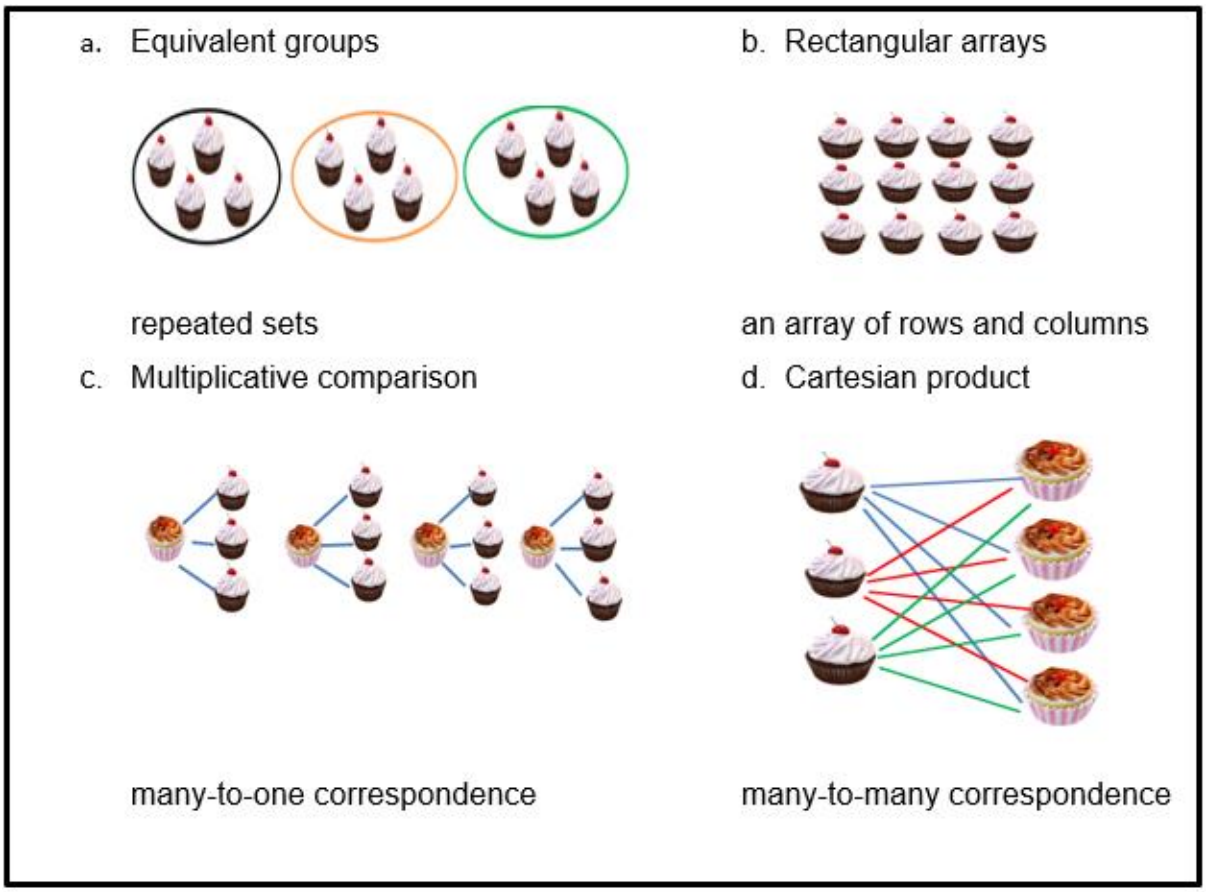


Figure 2.1: Four main categories of multiplicative situations

Anghileri (2006) states that each practical situation can be represented in different ways that show repeated sets (equivalent groups), many-to-one correspondence (multiplicative comparison), an array of rows and columns (rectangular arrays), and many-to-many correspondence (Cartesian product), as is illustrated in Figure 2.1. Multiplicative situations always have two (or more) variables with a fixed ratio between them (Nunes & Bryant, 2009). In Section 2.5 I look in-depth into multiplicative situations, drawing from Haylock's (2010) range of multiplicative situation types.

Multiplicative situations involve four numbers: they are quaternary in nature. The four numbers are: the number of objects in each set, the number of sets, the total number and the hidden number 1, which is implied using words like 'each' or 'per' (Askew, 2018; Greer, 1992). Multiplication requires that we find the total number (the product) while division requires that we find either the number of groups or the number in each group or size of a group (Anghileri, 2006; Carpenter *et al*, 1999). In division situations, there are two types of division depending on the missing number: measurement/grouping (quotitive) and sharing (partitive). In quotitive problem situations the total number and the number of objects in each set are known, so that the problem requires finding the number of groups. In partitive

problem situations the total number and the number of groups are known, the problem requires finding the number of objects in each set. The quaternary nature of multiplicative situations is more explicit when set up using ratio models (double number line and ratio table) rather than using the direct model (symbolic methods) of multiplicative situations (Askew, 2018). I further discuss the ratio models in Section 2.7.2.

The ideas discussed above indicate that developing multiplicative thinking is not an easy process. It is important for educators to develop the concept of multiplicative reasoning in depth, rather than placing the responsibility on educators in the next grade or assuming previous grade educators have already developed the concept (Siemon *et al*, 2006). In this study, I investigate learners' multiplicative reasoning when dealing with multiplicative word problems.

2.4 Problem Solving

Problem solving of word problems is one of the approaches that can be used to develop relational multiplicative reasoning skills. Word problems in mathematics are textual descriptions of situations within which mathematical questions can be contextualised (Sepeng & Webb, 2012). Sepeng & Sigola (2013) define mathematical word problems as mathematical tasks that are 'dressed up' in real-world situations which require learners to 'undress' the tasks and solve them. Undressing the tasks refers to learners modelling the problem situation.

Problem solving is considered one of the main objectives of teaching school mathematics. The South African curriculum (CAPS, 2010) emphasises that learners should 'acquire and apply knowledge and skills in ways that are meaningful to their own lives' and with the mathematics curriculum specifically aiming at 'acquisition of specific knowledge and skills for application of mathematics to physical, social and mathematical problems' (DBE, 2011, pp.3,8). Educators are also encouraged to set assessments (exams and tests) which have 10% weighting for problem solving (DBE, 2011, p.296). Problem solving is a central aspects of mathematics as it enables learners to make sense of how school mathematics can be used in their daily lives. Through problem solving tasks learners are encouraged to make use of their informal knowledge, relate to and make sense of school mathematics, and gain experience with a variety of mathematical structures and procedures that can be developed and refined (Murray, 2012).

Word problems are often associated with poor performance, with many writers arguing that the most common difficulty learners experience is reading problem situations with understanding in order to decide what is required of them (Daroczy, Wolska, Meurers & Nuerk, 2015; Murray, 2012; Sepeng & Webb, 2012). Various authors and studies highlight that little time is spent on teaching and learning problem solving in mathematics lessons, resulting in learners having some anxiety when dealing with word problems (Murray, 2012; Sepeng & Sigola, 2013). It is proposed that more time needs to be spent on word problems to help develop effective strategies learners can use to solve word problems (DBE, 2013b; 2015; Juan *et al*, 2017).

For a learner to be an effective problem solver, the learner is expected to (1) understand the language and information given in the word problem, (2) translate the problem into mathematical equation and (3) devise a plan that can be used to solve the word problem (Sepeng & Sigola, 2013). Haylock (2010) discusses the effective problem-solving skills using the process of mathematical modelling, which consists of four steps. I discuss the process of mathematical modelling in Section 2.6 (page 22). Effective problem solving equip learners with necessary skills to deal with real-life problems they are confronted with everyday life experiences.

In the next section, I discuss the different multiplicative situations that can be used to develop a deeper understanding of multiplicative thinking.

2.5 Multiplicative problem situations

In this section, I elaborate on Greer's (1992) four main categories of multiplicative situations discussed in Section 2.3 (page 13). I draw from Haylock's (2010) discussions of a range of multiplicative situation types and discuss literature on different kinds of multiplicative problem situations that I worked with. These multiplicative problem situations were:

- Multiplication as repeated addition
- Multiplication as rate
- Multiplication as scale
- Multiplication as array
- Division as sharing
- Division as grouping

2.5.1 Multiplication as repeated addition

Multiplication as repeated addition refers to situations where several groups of the same value or size are to be added together. The repeated addition structure is an extension of the addition structure (Fischbein, Deri, Nello & Marino, 1985; Haylock, 2010). For example: *Matome collects cans for a school recycling project. He collected 6 cans on Monday, 6 cans on Tuesday, 6 cans on Wednesday and 6 cans of Thursday. How many cans will Matome have collected?* (Hansa, 2017).

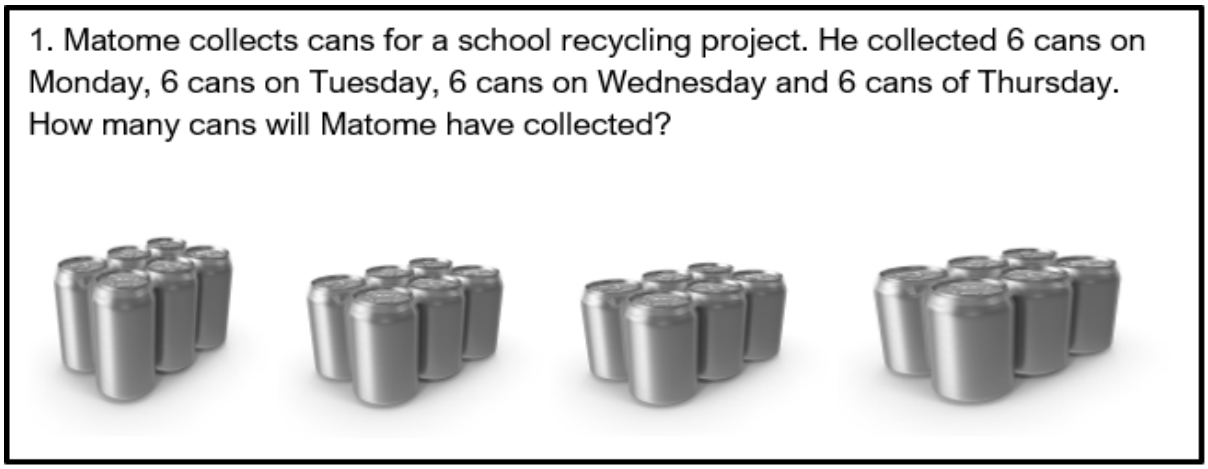


Figure 2.2: Multiplication as repeated as repeated addition (6 + 6 + 6 +6 or 6 × 4 = 24)

Multiplication as repeated addition is the elementary idea that multiplication means 'so many sets of' or 'so many lots of', which is an extension of the aggregation structure of addition (Haylock, 2010). In relation to the above problem (Figure 2.2), Matome collected the same number of cans for four days. The number of collected cans can be recorded mathematically as 6 + 6 + 6 + 6 = 24 or 6 × 4 = 24. The salient feature in this word problem is that there are four sets of six. Multiplication as repeated addition problem situations involves one factor that describes the number of items in each group and the other describing the number of groups with the product being the number of items in total.

2.5.2 Multiplication as rate

Multiplication as rate problem situation involves conditions where there is an implicit ratio. Keywords such as 'each', 'per' and 'every' are used to refer to multiplication as rate. These keywords can be presented in the context of cost per unit measurement. Rate problem situations may implicitly or explicitly state that there is a 'per' in the context. Rate problem situations involve comparing two quantities or a constant relation of one-to-many correspondence between two sets (Greer, 1992; Haylock, 2010; Nunes & Bryant, 2009). For

example: *Nthabeleng is putting out plates on tables. She puts 8 plates on each table. How many plates will she need for 7 tables?* (Hansa, 2017).

Multiplication as rate problem situations express a pair of numbers that remain invariant even if the set varies. In relation to the example, there are eight plates per table, and each time we add an extra table we must add eight plates to the set of plates. There is one-to-many correspondence in rate problem situations, as can be seen with the example above. In rate problem situations, the constant one-to-many correspondence is the invariant which is not present in additive thinking (Nunes & Bryant, 2009).

Multiplication as rate problem situations occur when we deal with particular numbers of some given quantity. This can be usually found in problem situations involving mass, liquid volume, length, time and shopping (Haylock, 2010; Van Dooren *et al*, 2010).

2.5.3 Multiplication as scale

Multiplication as scale refers to problem situations where there is an increase in size of quantity by a scaling factor (Haylock, 2010; Haylock & Cockburn, 2013). Scale situations involve a number of replications relating two set sizes of the same type. A key phrase usually used for multiplication as scale situations is '*n* times as many as', which expresses the two sets involved in the problem situation (Greer, 1992). For example: *Maki and Lebo are collecting bottles for recycling. Maki collected 9 bottles. Lebo collected 3 times as many bottles as Maki. How many bottles did Lebo collect?* (Hansa, 2017).

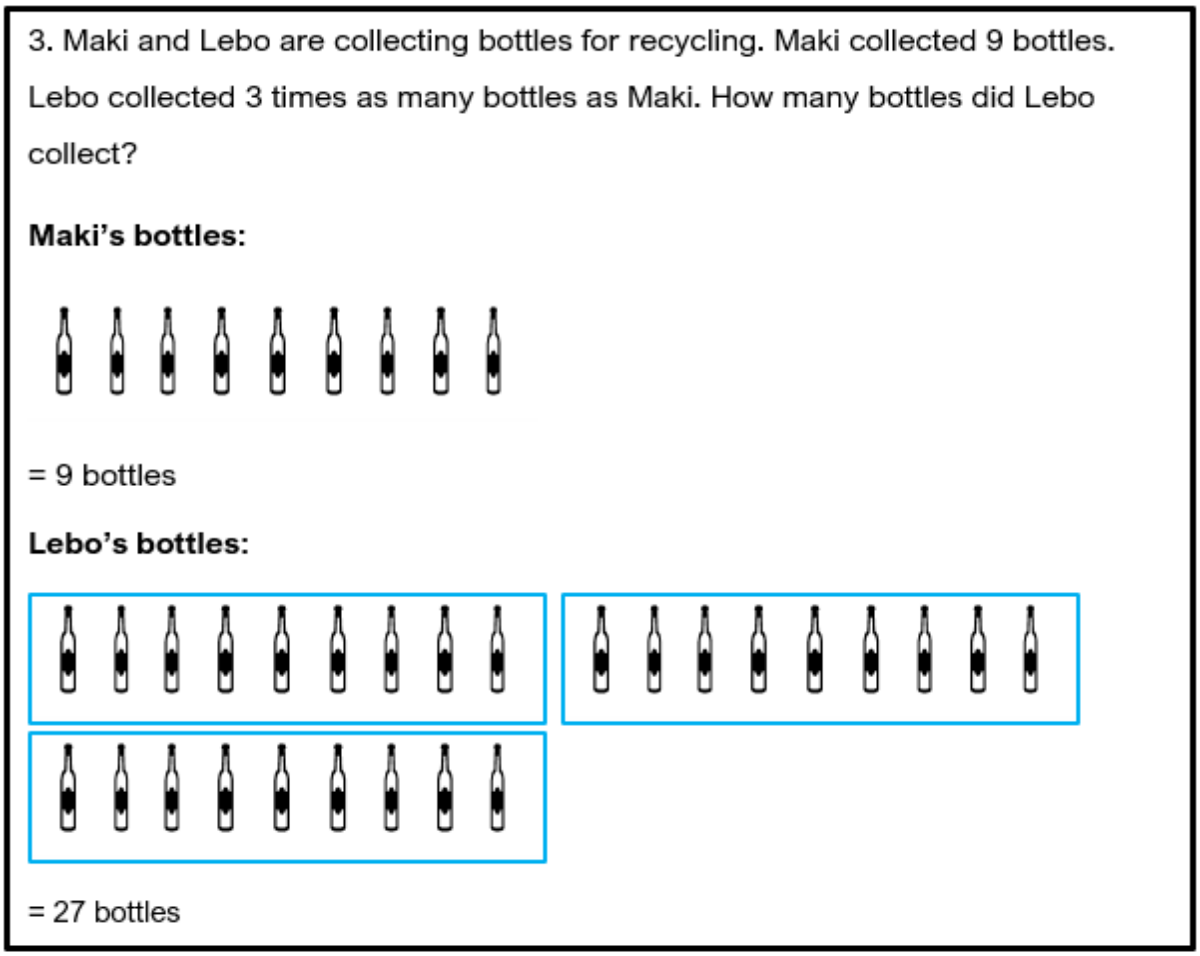


Figure 2.3: Multiplication as scale (9 × 3 = 27)

In the problem situation in Figure 2.3, there are two sets of bottles – Maki's bottles and Lebo's bottles. One set of bottles has nine bottles (Maki's bottles) and the second set of bottles is three times as many (Lebo's bottles). Learners often misunderstand the 'times as many' idea because they fail to understand the relationship between the two sets of numbers in the problem situation. Learners tend to default to addition when they misunderstand the '*n* times as many as' idea (Hart, 1981). It is important for learners to be aware of the scaling factor as an enlargement by a factor of *n* and not how much more, or bigger, one set of objects is than the other. Multiplication as scale is a process of enlargement by a scaling factor, therefore a process of making a quantity 'so many times bigger' (Carpenter *et al*, 1999; Haylock & Cockburn, 2013).

2.5.4 Multiplication as array

An array is a set of objects laid out in a rectangle compromising of rows and columns, which provides visual representation of grouped count situations (Barmby *et al*, 2009; Dlamini *et al*, 2015; Stott, 2016). Arrays are useful tools for developing learners' conceptual understanding

of multiplication because they can be used to deepen learners' relational understanding of distributive and commutative properties (Barmby *et al*, 2009; Stott, 2016). Thus, the array is considered as a key to represent multiplication. For example: *A Grade 6 class is planting rows of mango trees. There are 8 rows. They plant 6 trees in each row. How many mango trees will they plant?*

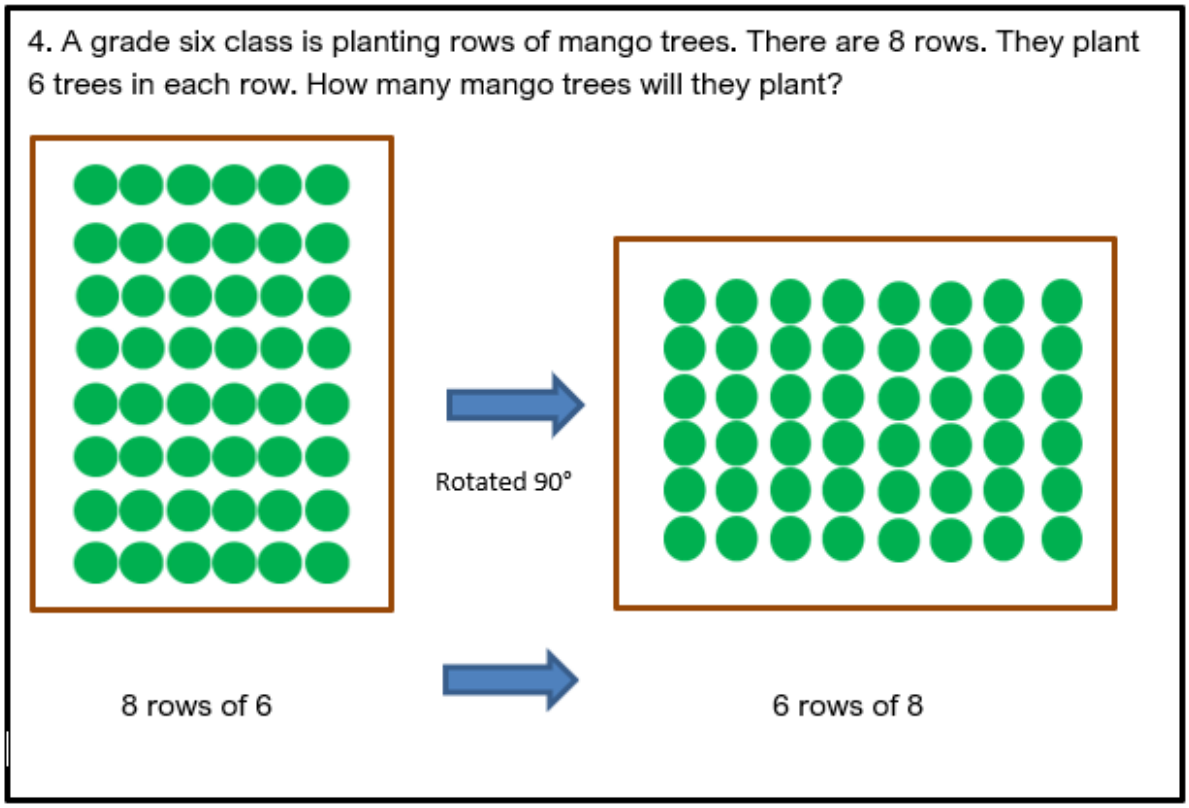


Figure 2.4: Multiplication as array (8 × 6 or 6 × 8 = 48)

Barmby *et al* (2009) argue that arrays encourage learners to develop their thinking about multiplication as a binary operation with rows and columns representing the two distinctive inputs – the size of each group and the scaling ratio. Unlike other multiplicative situations, the two inputs (factors) in an array situation do not have distinctly different roles. The multiplier and the multiplicand are exchangeable; they essentially fulfil the same function in calculating total of an array. In relation to Figure 2.4, the commutative property can be seen as either, six rows of eight mango trees (6 × 8) or 8 columns of six mango trees (8 × 6). The multiplier and multiplicand are interchangeable which demonstrates the commutative property for multiplication.

2.5.5 Division as sharing

Division as sharing refers to problem situations where the number in each share is not known. This problem situation fundamentally requires we allocate one object at a time into each of the groups (Barmby *et al*, 2009; Haylock, 2010). Division as sharing is also known as partitive division. In partitive division, the total objects and number of groups/shares are known, and the question requires that we determine how many or how much there is in each group or share. Thus, the solution requires we find the number of objects in the group, rather than the number of groups (Barmby *et al*, 2009; Carpenter *et al*,1999; Fischbein *et al*, 1985; Haylock, 2010).

For example: *A class of 42 learners goes on a field trip. They get into 6 teams. How many members are in each team?* In this problem situation, 42 learners (dividend) need to be shared equally into six teams (divisor). Sharing problems can be modelled by constructing groups of objects. The sharing division can be related to splitting as you literally split the set into a given number (Haylock, 2010). The idea of division as sharing is not new to learners as they are likely to work with this idea of sharing in their everyday context. For instance, sharing sweets amongst a group of friends, children are likely to split the sweets into correct number of friends one at a time until there are no sweets left over (Nunes & Bryant, 2009). Anghileri (2006) goes on to say research studies shows that children as young as four have the capacity to perform division as sharing using concrete objects. However, Carpenter *et al* (1999) note that young learners sometimes struggle with distinguishing the difference between the object they might use to represent the group and the objects in the group.

Division as sharing is the structure teachers most connect with division as it is strongly associated with the language of *sharing* and *how many (or how much) each?* (Haylock, 2010). However, it is important to note the difference between the action for sharing and the operation of division. Understanding division goes beyond actions of sharing, as division can involve a diversity of situations (Anghileri, 2006).

Counting strategies for sharing division often progress slowly beyond unit distribution. As the number of share/objects in each group is the unknown, learners may use trial-and-error to figure out which number to skip-count by or to add or subtract (Carpenter *et al*, 1999; Nunes & Bryant, 2009). The divisor tells us how many numbers there should be in the counting sequence, and the dividend tells us where the counting sequence should stop. The challenge is deciding what to count by (the number in each group) (Carpenter *et al*, 1999).

2.5.6 Division as grouping

Division as grouping situation uses a different interpretation of division. This situation involves finding how many (number of) equal groups of a particular size are in a given amount. In division as grouping, the total objects and the size of each group are known, the question requires finding the number of groups (how many groups there is). The solution requires learners to construct a given number of sets, each containing the specified number of objects (Barmby *et al*, 2009; Fischbein *et al*, 1985; Haylock, 2010). Division as grouping is also known as repeated subtraction or quotitive division.

For example: *Loniah bakes cupcakes for a school cake and candy sale. She bakes 54 cupcakes and packs 6 cakes in each pack. How many packs of cupcakes will Loniah have?* In this problem situation, 54 cupcakes (dividend) will be packed into groups of six cupcakes (divisor). Grouping problems can be modelled by allocating groups or shares at a time. The grouping division can be related to repeated subtraction as you repeatedly subtract each group/share (set collection) until there is nothing left over (Barmby *et al*, 2009; Carpenter *et al*,1999; Haylock, 2010). In relation to the example, interpreting this problem situation might be thought as of how many times can I take six away from 54 until there is nothing left over? The skip-count strategy can be used to add or subtract grouped units to solve grouping problems. In the cupcakes situation, learners can skip count in sixes {6, 12, 18, 24, …, 54} until they reach the given number or repeatedly subtract sixes from 54 until there is nothing left to subtract from {54, 48, 42, 36, …, 0}. The solution is the number of times learners have counted, added or subtracted the sixes.

The distinction between division as sharing and division as grouping situations are important because learners solve the two problems in different ways, reflecting the different information given in the problem (Carpenter *et al*, 1999).

2.6 The role of representations and models in mathematics

Mathematical ideas are abstract, so representations are required to access mathematical ideas in a meaningful way. Representations enable us to communicate mathematical relations and meanings, and they support our understanding of all mathematical tasks (Siemon *et al*, 2013). Representations play a significant role in mathematical competence, and can be classified as either internal or external (Barmby *et al*, 2009; Siemon *et al*, 2013).

Internal representations refer to mental models or mental images. These include verbal/syntactic, symbolic, imagistic, visual images and affective representations (Barmby *et al*, 2009; Siemon *et al*, 2013). External representations are physical, embodied and observable configurations that represent an object or concept. External representations include traditional representations such as number lines, equations, graphs, grids and tables of values, and other concrete materials used in the teaching and learning of mathematics (Siemon *et al*, 2013).

Learners' understanding of mathematical problems can be affected by two types of representation, which are instructional representations and cognitive representations, constructed by learners as they attempt to find a solution to the problem situation or make sense of the mathematical concept. Instructional representations include definitions, examples and models that support the instruction of mathematics. Learners create mental images of mathematical relationships described to them by educators, and use representations to communicate their mathematical ideas (Siemon *et al*, 2013)

Siemon *et al* (2013) argues that foundations of mathematics lie in the core representation that we have internalised. Linking representations enables us to see complex mathematical ideas in a new way and apply these complex ideas more effectively. Mathematical understanding refers to when the mental representations is part of a network of representations (Barmby *et al*, 2009; Siemon *et al*, 2013). The degree of mathematical understanding is determined by the number and strength of its connections (Barmby *et al*, 2009). Thus, for learners to thoroughly understand mathematical ideas, procedures or facts it must be linked to existing networks with stronger or more numerous connections (Barmby *et al*, 2009).

Mathematical modelling involves using and applying mathematics to make sense of the real world. Representations that can be used to find solutions of word problems are part of mathematical modelling (Haylock, 2010). A *model* can be defined as a representation used to represent a problem situation whereas a *strategy* is the process undertaken to get a solution (which can be either correct or incorrect) (Van den Heuvel-Panhuizen & Drijvers, 2014). Models are often used to make formal abstract mathematics accessible to learners (Askew, 2012). Haylock (2010) illustrates the process through which mathematical modelling can occur using four steps. The four steps of mathematical modelling are; (1) set up the mathematical model, (2) obtain the mathematical solutions, (3) interpret the mathematical solution in the real world and (4) compare the solution with the reality of the original

situation (Haylock, 2010). Figure 2.5 below illustrates the four steps of mathematical modelling.

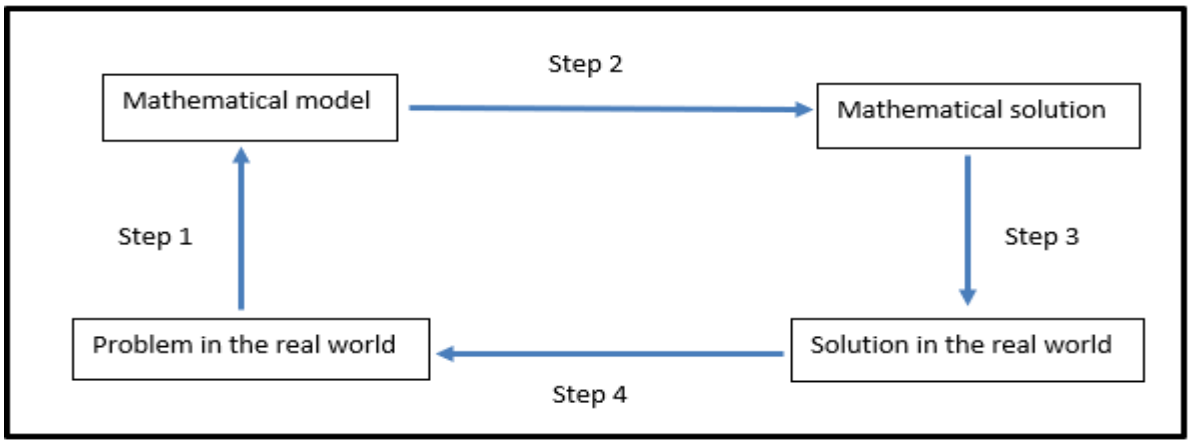


Figure 2.5: The process of mathematical modelling (Haylock, 2010)

Step One of the mathematical modelling process refers to when a word problem is translated into mathematical symbols and relations. For example, for: *52 people are going on a car trip. Each car can carry 8 people. How many cars will the people need*? (Hansa, 2017) one of the ways of translating the word problem into a mathematical model would be by the mathematical expression 52 ÷ 8 or a drawing which groups the 52 people into groups of eight. This step involves a shift from the real world to the mathematical representation of the real world (Fauzan, 2002; Haylock, 2010; Van den Heuvel-Panhuizen & Drijvers, 2014). Step Two involves manipulating mathematical relationships to obtain mathematical solution. This could be by means of a mental or written calculation (Haylock, 2010). This step concerns moving within the abstract world of symbols (Fauzan, 2002; Van den Heuvel-Panhuizen & Drijvers, 2014). In relation to the example, this could be doing the calculation of 52 ÷ 8 resulting in a solution 6,5 or by counting the groups in the diagram. The third step is then interpretation of the mathematical solution back into the real world (Haylock, 2010). Based on the example 7 cars would be needed to transport all the people. The final step compares the result with the reality of the original situation (Haylock, 2010). In this case, while the mathematical solution is 6,5 it would be illogical to say that 6,5 cars are needed to transport 52 people. An extra car would be needed to take the remaining 4 people; therefore, seven cars are needed to transport 52 people. For my study, the nature of the problem situations focused on the first two steps of the modelling process.

Models play a significant role in bridging the gap between the informal, context-related mathematics and the more formal mathematics. Van den Heuvel-Panhuizen & Drijvers (2014) argues that to fulfil the bridging function, models must shift from a *model of* a particular situation to a *model for* all kinds of other situations. As learners work more frequently with these models for, they will eventually become mathematical tools which they can use for thinking with (Askew, 2012). However, it is important to note that the shift from a *model of* to a *model for* to *tool for thinking* takes time and learners will not all experience the shift through these stages at the same pace (Askew, 2012).

2.7 Key multiplicative models

In this section I will look at key models used to represent multiplicative problem situations and strategies emanating from these models. Middleton & Van den Heuvel-Panhuizen (1995) argues that middle school educators should use a variety of tools in instruction to allow learners different relational access routes to the mathematics. These key multiplicative models treat multiplication and division as linked operations. Askew (2018) refers to the key multiplicative models as mediating models, he argues that these models provide a stronger ground for later work in ratio and proportion. The multiplicative models I focus on for my study are arrays and ratio models – double number line and ratio table.

Carpenter *et al* (1999) notes that children begin to solve multiplication and division problems by directly modelling the actions and relationships described in the problems. For example, direct models of equivalent groups situations can involve a diagram of 3 bags containing 4 cookies each. Array situations can be represented in a systematic way by constructing a given number of rows of counters with the same number of counters in each row (Carpenter *et al*, 1991). An increase in number ranges results in direct modelling being time consuming and inefficient. With an increase in number ranges there is a shift from direct modelling strategies to more compressed symbolic number relation-based models (Carpenter *et al*, 1999; Venkat & Mathews, 2018).

2.7.1 Arrays

Arrays are situations where the number in each group and number of equal groups are aligned as discrete elements in rows and columns. Array provide visual representations of grouped count situations. Several researchers argued that array images help learners to develop multiplicative thinking and to distinguish multiplicative thinking from additive thinking (Barmby *et al*, 2009; Stott, 2016, Venkat & Mathews, 2018). Barmby *et al* (2009) mention

that the two-dimensional nature (rectangular) of an array image reflects the two-variable structure of multiplicative situations, the size of groups and the number of groups of this size, contrasting the one-dimensional nature of number line that reflects the one variable structure of additive situations. Arrays can be interpreted as ‘times as many’ or ‘for each’ (Siemon *et al*, 2013).

Stott (2016) highlights that the array serves as a bridging tool between modelling multiplication with physical objects and using more algorithmic models. One of the advantages of the array idea is that all four aspects of the multiplicative situation; the number of groups (rows), equal size of each group (columns), the hidden 1 and the total, can be observed at once. Thus, arrays provide a way to make close connections between multiplication and division (Siemon *et al*, 2013; Stott, 2016) as it supports both structures of multiplication and division. For example, Figure 2.6 (a) is an array image of $8 \times 7 = 7 \times 8 = 56$. The array images show that the product of eight and seven is 56. For division, the discussion around the number of groups and number in each group would emerge drawing on grouping and equal sharing (Stott, 2016). In relation to example $8 \times 7 = 56$, for division the total would be given (56) with either the number of groups ($56 \div 8$) or number in each group ($56 \div 7$). Figure 2.6 (b) show arrays set ups of division with the value of one of the sides missing. To find the value of the missing side multiplication facts will be used.

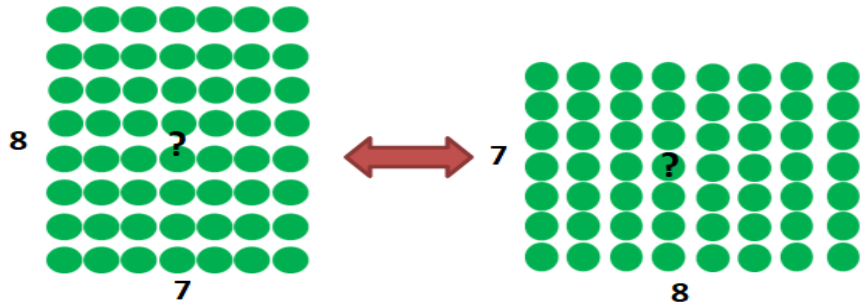


Figure 2.6 (a): An array representation of (8 × 7) and (7 × 8)

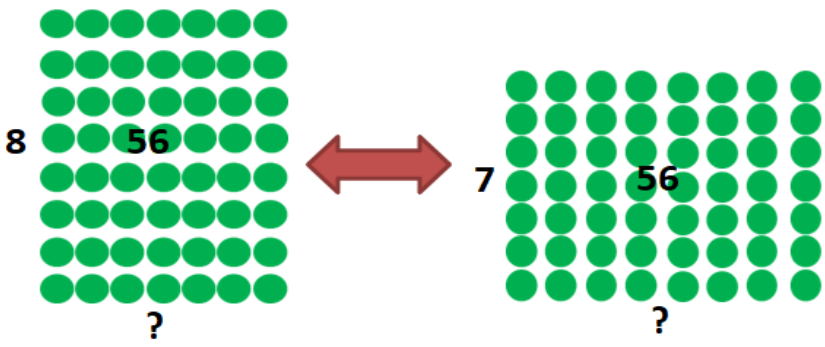


Figure 2.6 (b): Array representation of (56 ÷ 7) and (56 ÷ 8)

The Intermediate Phase curriculum (CAPS, 2010) for Mathematics specifies that learners should be able to recognise and use the distributive and commutative properties of multiplication (DBE, 2011). I noted earlier that the other advantage of using arrays is that both the commutative and distributive properties of multiplication can be demonstrated. Intermediate Phase CAPS for Grade 4 encourages educators to use arrays for teaching learners both the distributive and commutative properties of multiplication. The commutative property of multiplication can be demonstrated by rotating the array region to show that the number in *a* rows of *b* is equivalent to the number in *b* rows of *a* (Siemon *et al*, 2013). Figure 2.6 (a) demonstrates the commutative property of multiplication $8 \times 7 = 7 \times 8 = 56$ by re-orientating the array. The distributive property of multiplication can be demonstrated using spacing between the rows and columns of the array. For instance, using the five spacing vertically and horizontally, 8×7 represented as $(5 + 3) \times (5 + 2)$. Therefore, $8 \times 7 = (5 + 3) \times (5 + 2) = ((5 \times 5) + (5 \times 2)) + ((3 \times 5) + (3 \times 2)) = 25 + 10 + 15 + 6 = 56$ (Barmby *et al*, 2009).

Carpenter *et al* (1999) note that learners only use arrays when the problem requires them to construct arrays, although arrays provide natural representations for many multiplicative situations.

2.7.2 Ratio models

Ratio models assist learners to develop mental strategies for solving word problems that require sound proportional thinking. Ratio models include the double number line and the ratio table. The advantage of using ratio models is that the two quantities involved are represented at the same time (Askew, 2018; Venkat & Mathews, 2018), this helps to illustrate the scalar relationship within quantities and functional relationship between quantities (Askew, 2018).

2.7.5.1 Double number line

Double number lines provide a visual way of coordinating the two values in each ratio. Küchemann, Hodgen & Brown (2011) argue that the double number line provides a neat way of representing multiplicative relations. The model is often presented horizontally, and all possible ratios are set out (Küchemann *et al*, 2011; Orrill & Brown, 2012).

The double number line is one important model for multiplication as it offers a gradual shift from additive approach to multiplication. Repeated addition modelled by skips along the number lines provides a reliable model for the multiplication of whole numbers. The double number line also serves as a model of a variety of problem situations (Küchemann *et al*, 2011).

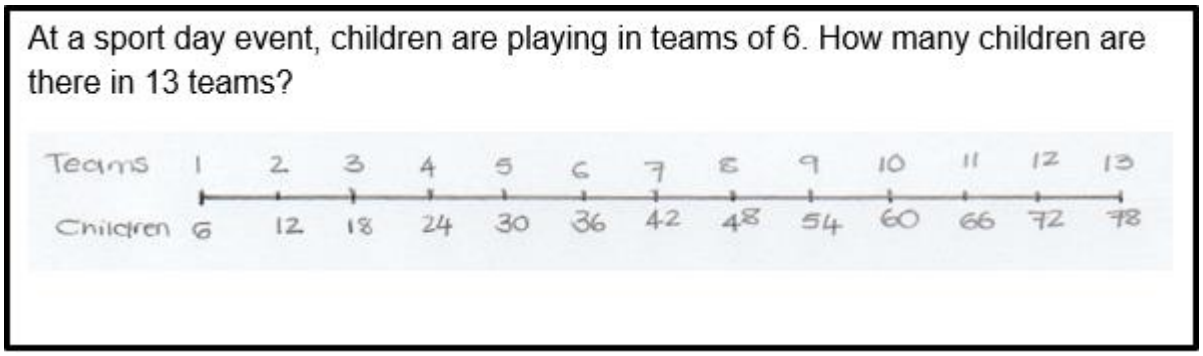


Figure 2.7: The double number line

The double number line coordinates values by setting up two scales, as it can be seen on Figure 2.7. The top and the bottom lines are not arranged on the same scale, but they coordinate the relationship between the values (Orrill & Brown, 2012). For instance, the top number line in Figure 2.7 is arranged on the scale of one (team) and the bottom number line is arranged on the scale of six (number of children). Orrill & Brown (2012) mention that the coordination of the two values forms a visual relationship of the composed unit as the quantities align and are connected.

The double number line acts as a precursor to the ratio table (Askew, 2018; Orrill & Brown, 2012, Venkat & Mathews, 2018). It can support learners who have basic knowledge of proportions to reason about the covariation of the quantities as they move toward understanding the resulting rate. Therefore, double number line has potential to be used by educators as a tool to scaffold learners in learning to reason about composed quantities (Orrill & Brown, 2012) embedded in the notion of co-variance.

2.7.5.2 Ratio table

The ratio table is one way of assisting learners to develop mental strategies for solving rate and proportion problems (Dole, 2008; Ercole, Frantz & Ashline, 2011; Lamon, 2012; Nasution, 2014). The ratio table is a vertical or horizontal table constructed to show the relationship between two quantities and their values. The basic idea behind the ratio table is that one can generate equivalent ratios by successive manipulation of numbers to reach the ratio that best fits the problem situations. Multiplicative strategies can be used to work out the equivalent ratios (Middleton & Van den Heuvel-Panhuizen, 1995). Unlike the double number line, the ratio table does not set out all possible ratios. It encourages learners to look for relationships that avoid filling in all the possible steps, e.g. halving, doubling, multiplying or dividing by 5, 10, 100 and so on (Askew, 2018; Ercole *et al*, 2011; Lamon, 2012; Middleton & Van den Heuvel-Panhuizen, 1995). The ratio table is also referred to as T-table or proportion table (Lamon, 2012; Venkat & Mathews, 2018).

The ratio table is one of the tools available for Intermediate Phase educators to use for instruction. For example: *Mandla is baking scones. He puts 7 scones into each bag. How many scones will Mandla use to fill 12 bags*? Step 1: construct the ratio table and display all the given information from the word problem (Dole, 2008).

Bags	Scones
1	7
12	?

Figure 2.8: Ratio table Step 1

Step 2: select successive operations to determine the solution. For instance; multiplying or dividing by 5, 10, halving or doubling.

Bags	Scones
1	7
10	70
2	14
12	? (84)

Given values in the problem (1 bag = 7 scones)

Breaking up the number into 10 and 2

Figure 2.9 (a): Ratio table Step 2

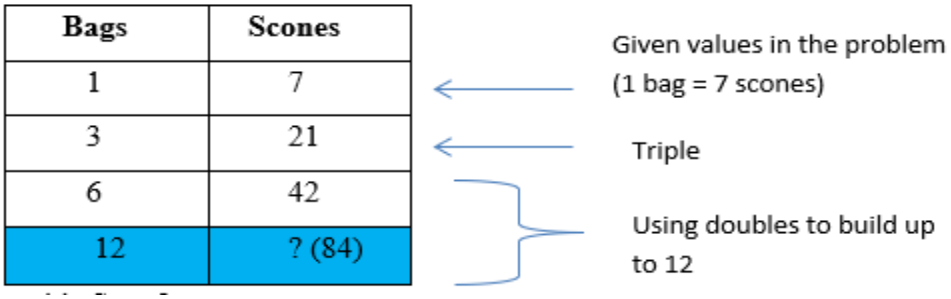


Figure 2.9 (b): Ratio table Step 2

In the example above, the pathways to the solution differed. The first ratio table (Figure 2.9 (a)) displays that the number 12 has been broken up into 10 and 2, as the two numbers are easy to multiply. The second ratio table (Figure 2.9 (b)) displays the use of tripling and then doubling to build up to finding how many scones will be needed for 12 bags.

Ratio tables cannot be taught directly as they are based on individual constructions of personal thought processes. As seen from the example different pathways can be taken to determine the solution of the problem situation (Dole, 2008; Lamon, 2012; Middleton & Van den Heuvel-Panhuizen, 1995). Various authors emphasise that educators should encourage discussion of the solution strategies to assist learners in seeing that ratio tables are flexible tools as different pathways can be followed (Dole, 2008; Lamon, 2012). The advantage of using ratio tables is they are tools which help to organise numbers and keep track of operations and their results (Dole, 2008; Middleton & Van den Heuvel-Panhuizen, 1995).

'Building up' and 'breaking down' strategy can be used to solve rate and proportion problem situations. Lamon (2012) mentions that building up and breaking down strategy is one strategy that learners use spontaneously and works in many situations as learners develop their reasoning processes. Building up strategy involves reasoning up from some fraction to unit; these are multiplication problem situations (Ercole *et al*, 2011; Lamon, 2012). With the building up approach the size of the set and number of sets are given, the building up approach is used to work out the total.

The building up approach requires learners establish a ratio and then extend it to another ratio using addition (Lamon, 2012). Adding, doubling and multiplying are often used for building up strategy as it requires determining a larger unit measurement (Ercole *et al*, 2011; Lamon, 2012; Nasution, 2014). In relation to the example above, the building up approach was used to find the solution to the word problem by working with the notion of doubling and

multiplying by 10 for the first solution (Figure 2.9 (a)), and tripling and doubling for the second solution (Figure 2.9 (b)) as explained in detail above.

Breaking down approach involves reasoning back down from the unit to another fraction (Lamon, 2012). For problem situations where the total and the number of sets are given, the breaking down approach could be used to find the size of each set. These are the division problem situations. Halving, dividing and subtracting are used when working with breaking down approach as it involves determining smaller unit measurement (Nasution, 2014). For example: *Zweli is putting out chairs in rows. He puts out 8 equal rows of chairs. He puts out 96 chairs altogether. How many chairs does Zweli put in each row*?

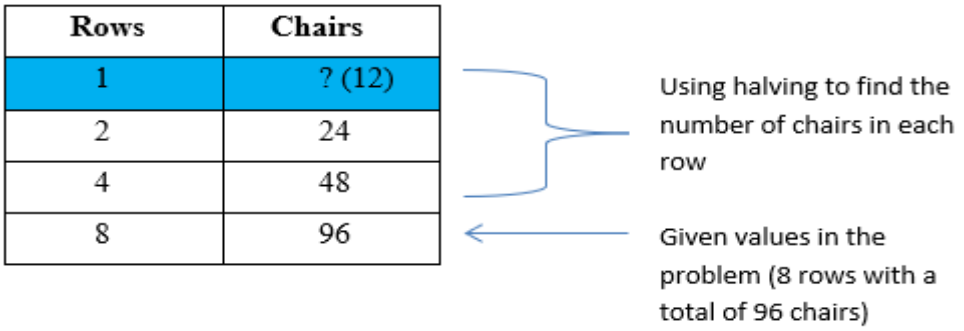


Figure 2.10: Example of breaking down strategy

In the example above, the problem situation deals with division, therefore, to get the number of chairs in each row halving is used to break down until we get to one row (chairs per row).

Multiplication builds up numbers in stages, division can break down numbers in equivalent stages. Thus, division is reinforced as the inverse operation to multiplication (Anghileri, 2006). Lamon (2012) argues that encouraging learners to use reasoning up and down process results in powerful and highly desirable ways of thinking. Furthermore, learners develop strategic planning, number sense and confidence in problem solving when using reasoning up and down process. At the beginning of the process, learners rely on doubling and halving, and use many steps to get to the answer. As learners make progress, they plan (in advance) how they will use the givens to get to the target quantity, and simultaneously use fewer steps (Lamon, 2012).

Dole (2008) identifies four issues with using ratio tables. First, ratio tables can be time-consuming. Neatness on constructing regular columns may have to be compromised; focus should be directed to number patterns and exploring relationships in the situation. Second,

ratio tables can be extended infinitely until the solution is reached. Educators should remind learners that to arrive at a solution, learners are in control of constructing and using the table. Thirdly, calculations do not have to be sequenced from smallest to largest. The ratio tables are meant to show the sequence of calculations. Learners must be reassured constantly that their record of calculations does not need to be ordered from smallest to largest. Lastly, learners need to be reminded that each progressive calculation corresponds with the given ratio. This reinforces the multiplicative relationship within the situation and promotes proportional reasoning (Dole, 2008).

For ratio table, the setting up of the problem situation is the model and how the learner works out the solution is the strategy. The latter will depend on the procedure the learner uses, and evidence for it may not be visible in the written response. Given the possibility for flexible working, responses in my study which used ratio tables were classified into two categories. In a case where a ratio table has all possible values, the response was classified as skip counting as the learner was skip counting using a specific factor. However, when a learner used relationships such as doubling, halving, multiplying by 5, 10, 100 and so on the responses were classified as symbolic methods, as here more compressed number relations were used to work towards the solution.

2.8 Concluding remarks

This literature review highlighted that learners experience difficulties with making sense of multiplicative word problems and performing multiplication and division algorithms. For learners to have access to a broader range of multiplication and division ideas, teaching and learning multiplication and division should be interrelated by educators. Multiplicative thinking is not about performing arithmetic operations; it is about learners making sense of multiplicative relations. Therefore, it is important for educators to fully develop the concept of multiplicative thinking taking into consideration the theoretical aspects discussed in this review.

Literature also reveals that problem solving is one of the ways that can be used to develop multiplicative thinking. Solving word problems requires multistep and strategies, which must be developed under the expert guidance of the educator. Models serve as the bridging gap between the informal and more formal mathematics. Therefore, models should shift from a model of a particular situation to a model for all kinds of other situations. The intervention lessons were intended at shifting learners from informal representations to more formal

representations of different multiplicative situations using mathematical symbols, algorithms and ratio modelling, through the guidance of the educator. However, it must be noted that some learners would continue to use the more traditional operational algorithms that they have seen previously (Nunes & Bryant, 2009).

In the next chapter I discuss the RME theory as the theoretical framework of my study.

Chapter 3: Theoretical Framework

3.1. Introduction

The focus of my study is on models and strategies Grade 6 learners use to solve multiplicative word problems. In this study I broadly focus on how learners can model multiplicative word problems using a wide range of mathematical representations and tools. The theoretical framework I chose to use for my study was Realistic Mathematics Education (RME), which focuses on how learners can be assisted by the educator to develop and explore models to solve mathematical word problems (Van den Heuvel-Panhuizen & Drijvers, 2014). As noted in the previous chapter, models play an important role in learning mathematics, therefore I found the RME framework relevant to my study.

In this chapter I outline my reasons for selecting the RME framework as the theoretical framework in my study. The RME framework emphasises that formal mathematical knowledge can be developed from learners' informal knowledge. Learners should be provided with the opportunity to reinvent mathematics under the expert guidance of an educator (Gravemeijer & Doorman, 1999; Van den Heuvel-Panhuizen, 2003). Word problems used in my study were based on problem situations that are experientially real (or relevant) to learners. They were given the opportunity to produce their own models to solve the word problems during the intervention lessons. First, I introduce RME theory and then I look at the key principles of RME for instructional design. Last, I discuss the six guiding principles of RME that educators need to be aware of.

3.2. The theory of Realistic Mathematics Education (RME)

RME roots lie in Hans Freudenthal's interpretation of mathematics as a human activity. RME is a domain-specific instruction theory for mathematics education (Van den Heuvel-Panhuizen, 2003) which purports that mathematics should be learnt by mathematising subject matter from problem situations that are experientially real, rather than the traditional view of mathematics as a ready-made system. In RME realistic situations are given a prominent position in the learning process. The realistic situations serve as starting point for developing mathematical concepts, tools, and procedures. The process is completed by reflecting the solution back to the real world (Fauzan, 2002; Van den Heuvel–Panhuizen & Drijvers, 2014). Realistic situations in RME are commonly associated with real-world problem situations which are relevant and real to learners. Therefore, problems presented to learners can come

from real world, fantasy world of fairy tales, or the formal world of mathematics. 'Realistic' does not mean that mathematics should be real in every aspect, the problem situations should be imaginable to learners (Van den Heuvel-Panhuizen, 2003; Van den Heuvel–Panhuizen & Drijvers, 2014).

RME mainly focuses on mathematising or organising subject matter from realistic contexts through a process of modelling the relations between quantities of the situation to produce a model with which to solve the problem (Fauzan, 2002; Van den Heuvel–Panhuizen & Drijvers, 2014). Through the process of progressive mathematisation, learners are expected to learn through reinventing mathematical insights, relationships and procedures. Learners should be active participants in learning mathematics rather than being receivers of ready-made mathematics knowledge (Gravemeijer & Doorman, 1999; Van den Heuvel–Panhuizen & Drijvers, 2014). This allows learners to regard the knowledge they acquire as their own private knowledge.

RME distinguishes between two processes of mathematisation, namely, horizontal and vertical mathematisation. Horizontal mathematisation refers to processes in which learners describe or represent given problem situations in mathematical terms. Learners use mathematical tools to organise and solve problems in realistic situations. This is the movement from the world of life into the world of symbols (Gravemeijer & Doorman, 1999; Van den Heuvel–Panhuizen, 2003). This takes place when learners describe contextual problems using their informal strategies to solve contextual problems (Fauzan, 2002).

Vertical mathematisation refers to 'all kinds of re-organisations and operations done by learners within the mathematical system itself' (Van den Heuvel-Panhuizen, 2003, p.12) resulting in shortcuts by using connections between concepts and strategies. If the informal strategies lead learners to solve the problems using mathematical language or apply an appropriate algorithm, then the process of movement shows a vertical mathematisation. Vertical mathematisation is concerned with the shift within the abstract world of symbols (Van den Heuvel-Panhuizen, 2003; Van den Heuvel-Panhuizen & Drijvers, 2014). Through the process of vertical mathematisation the learner reaches a higher level of mathematics (Gravemeijer & Doorman, 1999).

For this study, the way the learner represented the word problem was regarded as horizontal mathematisation and the process the learner used to workout the solution, using their selected strategy, this was regarded as vertical mathematisation. In relation to the previous chapter, I

use the terminology of the *models* learners presented as horizontal mathematisation and the *strategies* learners used to solve the word problems as vertical mathematisation. Figure 3.1 below illustrates the process of mathematisation, dotted lines indicate horizontal mathematisation and arrows indicate vertical mathematisation.

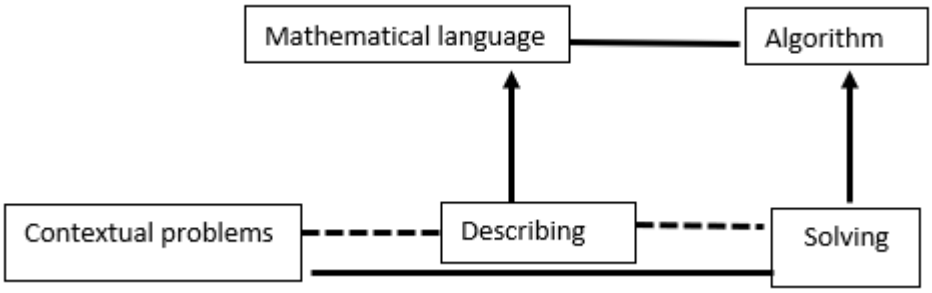


Figure 3.1: Representation of the process of mathematisation (Gravemeijer in Barnes, 2005, p.40)

RME advocates both processes of mathematisation. The RME approach is characterised by realistic problem situations, where learners are guided by the educator to develop their own informal mathematical models and strategies into more formal mathematical models over time (Fauzan, 2002; Van den Heuvel-Panhuizen & Drijvers, 2014). It is in the process of mathematisation that learners construct mathematics (Gravemeijer &Doorman, 1999; Van den Heuvel-Panhuizen, 2003).

Following this approach, word problems were used as the basis for mathematisation in the intervention lessons. During intervention lessons learners were encouraged to develop their own models, which were used to build towards the concept of ratio tables. It was expected that different models and strategies would emerge as learners engage with contextual problems. While engaging with the problems in teaching, I introduced the relational multiplicative reasoning models described in Chapter 2 (arrays, double number lines and ratio tables) in order to focus on multiplicative structure embedded in the problems.

3.3. RME key principles of instructional design

Three key principles of RME in terms of instructional design are: guided re-invention, didactical phenomenology and self-developed or emergent models.

3.3.1 Guided re-invention through progressive mathematisation

Guided reinvention principle requires that learners be presented with well-chosen contextual problems which present learners with the opportunities to develop informal solution strategies. These informal solution strategies function as the inventions for formalisation and

(Barnes, 2005). This process is referred as progressive mathematising (Barnes, 2005; Gravemeijer & Doorman, 1999).

When learning mathematics, the RME position is that learners should be given the opportunity to experience mathematics process in a similar practice by which mathematics was invented by mathematicians (Gravemeijer & Doorman, 1999). Guided re-invention principle refers to the process whereby the instructional designer (educator) maps out the learning route to allow learners to discover the intended mathematics by themselves under the guidance of the educator. In the process of designing the learning route, the instructional designer starts with imagining a route by which he or she could have arrived at a solution independently (Fauzan, 2002). This tenet of RME affords learners the opportunity to develop and gain knowledge that is perceived by them to be the product of their personal enterprise which, resultantly, becomes their own private knowledge (Barnes, 2005; Gravemeijer & Doorman, 1999).

For my study, at the beginning of the lesson learners were given the opportunity to solve different multiplicative word problems using their own personalised models. Once learners had worked out the problem, three learners were selected to present their models to the whole class. These models were then refined through classroom discussion with the aim of helping learners to make sense of the multiplicative structure of the word problems situations (horizontal mathematisation) and use efficient strategies to solve the word problems (vertical mathematisation).

3.3.2 Didactical phenomenology

In the didactical phenomenology principle problem contexts are used to lead learners to discover mathematical structures and concepts (Fauzan, 2002). This implies that in the process of learning mathematics, learning should start from phenomena (contexts) that are relevant and meaningful to learners, which are then used to organise and stimulate the learning processes. An implication of this principle is that the instructional developer is to provide learners with contextual problems taken from phenomena that are real and accessible to learners (Fauzan, 2002).

For this study, problem situations used in the intervention lessons and tests were situations that were experientially real to the learners. The problem situations were designed by the WMC-P team as part of their bigger study on multiplicative reasoning.

3.3.3 Self-developed or emergent models

The emergent models tenet plays a significant role in bridging the gap between the informal and the formal mathematical knowledge by shifting from a *model of* to a *model for* (Van den Heuvel-Panhuizen, 2003) a particular situation. In RME approaches, learners are given the opportunity to develop and use their own models when they are solving problems. At the beginning of the process, it is expected of learners to develop a model which they are familiar with, which is known as *model of* a particular situation (Fauzan, 2002; Gravemeijer & Doorman, 1999). Over time, the models are refined into more formal mathematical reasoning tools, which become *models for* more formal mathematical reasoning (Barnes, 2005; Gravemeijer & Doorman, 1999) that can be applied flexibly across similar and different contexts. The shift in the purpose of models is referred to as the process of a transition from model-of to model-for. Post transition the model may be used as a model for mathematical reasoning in general (Fauzan, 2002; Gravemeijer & Doorman, 1999).

3.4. Core teaching and learning principles of RME

Once curriculum material has been developed, it is important to consider how teaching and learning should take place to align with curriculum specifications, how educators should present the curriculum material that is developed, and how learners can learn from the developed curriculum material. Van den Heuvel-Panhuizen & Drijvers (2014) proposed six core principles for teaching mathematics. Six guiding principles of RME that educators need to be aware of are; (1) activity principle, (2) reality principle, (3) level principle, (4) intertwinement principle, (5) interactivity principle and (6) guidance principle. I discuss these shortly in the paragraphs that follow.

3.4.1 Activity principle

In RME the activity principle suggests that learners should be treated as active participants in the learning process (Fauzan, 2002; Van den Heuvel- Panhuizen & Drijvers, 2014). Learning mathematics is a constructive activity, it is not just absorption of knowledge presented or transmitted (Fauzan, 2002). Learning mathematics is best achieved by doing mathematics. Therefore, learners need to be encouraged to develop their own models and participate actively in the process of constructing knowledge (Van den Heuvel- Panhuizen & Drijvers, 2014).

3.4.2 Reality principle

Reality principle can be recognised in RME in two ways. Firstly, it expresses the importance of including learners' ability to apply mathematics in solving 'real-life' problems. Secondly, it means that mathematics education should start from problem situations that are meaningful to learners, which affords learners the opportunity to attach meaning to the mathematical constructs they develop while solving problems (Van den Heuvel- Panhuizen & Drijvers, 2014).

In RME, teaching starts with concrete ideas or problems in rich contexts. Teaching should start with posing relevant problems in rich contexts that require mathematical organisation. These problems should then go through the process mathematisation in order to place learners on the track of informal context-related solution strategies as a first step in the learning process (Fauzan, 2002; Van den Heuvel-Panhuizen & Drijvers, 2014).

3.4.3 Level principle

Fauzan (2002, p.44) states that in the level principle, 'the learning of a mathematical concept or skill is viewed as a process which is often stretched out over the long term and which moves through various levels of abstractions'. Thus, learning mathematics should allow learners to pass through various levels of understanding. The levels of understanding through which mathematising takes place, range from informal to formal and from the intuitive level to the level of subject-matter systematics (Fauzan, 2002; Van den Heuvel-Panhuizen & Drijvers, 2014). Models play a significant role for bridging the gap between the informal, context related and the more formal mathematics. To fulfil the bridging function, models learners' constructs should shift from a model of a particular situation to model for related situations (Van den Heuvel-Panhuizen & Drijvers, 2014).

In the relatively short timeframe of my study, many learners shifted from concrete models (diagrammatic representations - unit count), and structured repeated group models and arrays towards more formal models such as ratio tables.

3.4.4 Intertwinement principle

The intertwinement principle emphasises that content domains cannot be dealt with as separate entities; instead, an intertwining of content domains is exploited in problem solving. Learners should be offered problems which allow them to use various mathematical tools and knowledge when solving such problems. The educator's role is to ensure that the content

domains in mathematics relate to each other (Fauzan, 2002; Van den Heuvel-Panhuizen & Drijvers, 2014).

As pointed out in Chapter 2 (page 9), multiplicative reasoning is a prerequisite for understanding of rational numbers and proportional reasoning. The skill of multiplicative reasoning has a close connection with number sense, algorithmic thinking, mental arithmetic and estimation skills (Askew, 2018; Nunes & Bryant, 2009).

3.4.5 Interactivity principle

Interactivity principle emphasises the importance of social contexts. Learning mathematics is not an individual activity but it occurs in a society and is directed and stimulated by the socio-cultural context (Fauzan, 2002; Van den Heuvel-Panhuizen & Drijvers, 2014). Mathematics education should by nature be explicit negotiations, discussion, intervention, cooperation and evaluation (Fauzan, 2002). RME favours whole-class discussions and collaborative work which offers learners opportunities to share their approaches with others (Fauzan, 2002; Van den Heuvel-Panhuizen & Drijvers, 2014). Interaction with others' ideas and contributions encourages learners to reflect on the learning process because they share and interrogate ideas and strategies. This can lead learners to higher level of understanding (Van den Heuvel-Panhuizen & Drijvers, 2014).

For my study, learners had three problem situations to solve individually. For each problem situation, two or three learners were carefully selected to share their models and strategies on the board with the rest of the class. Before learners presented their models and strategies, the problem situation was analysed through whole class discussion to help learners make better sense of the problem situation. For each model presented on the board there would be a discussion around the model, and learners were asked whether the model was appropriate or inappropriate, and to explain why. This was aimed at encouraging whole class participation. Towards the end of each lesson learners were also given the opportunity to discuss what they had observed/learnt from the focal problem situations they solved, that is what is the same and different about the problems. Class discussion took place in order to broaden reflection on key aspects of multiplicative situations and models.

3.4.6 Guidance principle

The guidance principle emphasises that educators should facilitate the learning process by guiding learners. This implies that the educators should be proactive in the learning process and that the educational programs should contain contexts which are accessible to learners'

understanding (Van den Heuvel-Panhuizen & Drijvers, 2014). Van den Heuvel-Panhuizen & Drijvers (2014, p.532) state that for this to happen 'the teaching and the programs should be based on coherent long-term teaching-learning trajectories'.

3.5. Concluding remarks

The RME theoretical framework opposes the traditional view of teaching and learning mathematics as ready-made systems with general applicability (Van den Heuvel-Panhuizen, 2003). The framework emphasises that learners should learn mathematics by mathematising subject matter from real situations and re-invent the mathematics in the same way it was discovered by mathematicians (Barnes, 2005; Van den Heuvel-Panhuizen, 2003). Through a process of progressive mathematisation, learners are given the opportunity to re-invent mathematical insight, procedures and relationships. Learners go through stages of horizontal mathematisation and vertical mathematisation in the progressive mathematisation (Gravemeijer & Doorman, 1999; Van den Heuvel-Panhuizen, 2003). The RME approach affords learners with many opportunities to make use of horizontal mathematisation, whereas the traditional teaching and learning mathematics starts from vertical mathematisation (Barnes, 2005).

The RME framework was proposed as a possible theory to drive the design and implementation of this intervention on multiplicative reasoning. Firstly, RME draws on everyday contexts. Learning should start from contexts that are relevant and meaningful for learners to develop their formal mathematical knowledge (Fauzan, 2002; Van den Heuvel-Panhuizen & Drijvers, 2014). Secondly, the theory highlights the importance of developing and exploring models in learning mathematics. Models should shift from *model of* a particular situation to *model for* general applicability, in the process of learning mathematics (Van den Heuvel-Panhuizen, 2003). Thirdly, the theory emphasises that to develop relational understanding, learners should be offered opportunities to re-invent mathematics with the expert guidance of the educator (Gravemeijer& Doorman, 1999). Lastly, the theory encourages social interaction in mathematics classroom. The theory favours interactivity as learners can work collaboratively with others to share ideas and reflect on the learning process (Van den Heuvel-Panhuizen & Drijvers, 2014).

Chapter Four: Research Methodology

4.1. Introduction

This study aimed to determine the effect of a series of four intervention lessons on multiplicative reasoning in a Grade 6 classroom. The study analyses learners' models and strategies when dealing with multiplicative problem situations prior and post intervention lessons. This section outlines the research methodologies which were used to collect and analyse the data to achieve the aims of the study.

4.2. Research design

The research design describes the ways and means for carrying out research. The purpose of research design is to specify a plan of generating evidence that will be used to answer the research questions. It indicates how the research is set up, what happens to research subjects and what methods of data collection are used for the study (McMillan & Schumacher, 2010).

My study is concerned with determining the effects of multiplicative reasoning intervention lessons in a Grade 6 classroom. This study takes the form of a mixed methods study. A mixed methods study is a procedure for collecting, analysing, and incorporating both quantitative and qualitative methods in a single study to understand a research problem (Creswell, 2012). Quantitative methods emphasise objectivity in measuring and describing phenomena whereas qualitative methods emphasise gathering data in an inductive way and establishing patterns or themes of the phenomena occurring (McMillan & Schumacher, 2010).

Explanatory sequential design was the type of mixed methods approach I used in my study. In explanatory sequential design, quantitative data is collected and analysed then followed by qualitative collection and analysis. Qualitative data is used to explain or elaborate on the quantitative findings (Creswell, 2012; McMillan & Schumacher, 2010). In relation to my study, qualitative data (document analysis) were used to elucidate the quantitative findings.

My research design used an experimental research design, which is one of the quantitative research designs. In experimental research designs, the researcher manipulates what the subjects will experience by systematically imposing or withholding specified interventions (McMillan & Schumacher, 2010). The main purpose of experimental design is to investigate the cause-and-effect relationships between interventions and measured outcomes (Basit, 2010; Gall et al, 2007; McMillan & Schumacher, 2010). My study investigates the effect of

multiplicative reasoning intervention, which uses a different approach from the traditional anti-didactical inversion of teaching mathematics.

My research study involved testing (pre- and post-tests) with a group of Grade 6 learners who were participants in an intervention study aimed at understanding multiplicative reasoning. This approach draws on a single group pretest - posttest experimental design as a single group of subjects were given a pre-test, then the treatment (intervention), and then the post-test (Cohen, Manion & Morrison, 2011; Gall *et al*, 2007). Both the pre- and post-tests were identical. The effects of the intervention lessons were explored by comparing the pre- and post-tests performance and learners' responses to word problems.

The intervention lessons were taught by the researcher under the guidance of the WMC-P team members. My role was that of researcher and an educator (described in detail in Appendix A). Throughout the study, my role as a researcher was to carefully analyse whether learners had made any gains that could be associated with intervention lessons; whereas as an educator my role was to guide learners in the process of re-invention of mathematics and assist learners to gain an improved understanding of different modelling approaches to multiplicative problem situations.

4.3 Research participants

4.3.1 The sample

The sample of the study were a class of Grade 6 learners at a government fee-paying school located in Johannesburg East district. The case site was in a suburban area, with the majority of learners from the neighbouring township. The case site was classified by the district as a low performing school, based on 2013 ANA performance results. A total of 36 learners who were present for the pre- and post-test were selected as participants in the study, with 19 females and 17 males in the sample. Participants were aged between 11 and 13 years at the time of data collection. The data collection took place from May to mid-June 2017, with one 1-hour lesson conducted per week (including pre- and post-tests), at the same time of the day. The study took place during the normal Mathematics lesson where content of the study was aligned to the CAPS requirements.

The 36 participants were well distributed on their Mathematics performance based on their Term 1 Mathematics results. Of the 36 participants; 12 were high performers with Term 1

results from 65 % and above, 13 were average performers with results ranging from 45 % to 64 %, and the remaining 11 participants obtained results below 45 %.

4.3.2 Sampling method

A convenience sampling method was used to select research participants. This method of sampling is a common type in educational research as it may not be feasible to select participants from larger group (McMillan & Schumacher, 2010). The research sample was selected as they were easily accessible to the researcher, due to the learners being located in the school where I work.

The case site has three Grade 6 classes (all of which I teach), but only one class was used for the study. My selection of the class was also based on convenience sampling method as this class had a double period of Mathematics on Tuesdays between 09:00 and 10:00, making this longer lesson particularly useful for intervention lessons.

4.4. Research instrument

Paper-and-pencil tests were used to identify participants' multiplicative reasoning skills prior and post intervention lessons. The test was designed by WMC-P team for multiplicative reasoning project in Intermediate Phase. The test content was focused on basic proportion problems across equivalent groups, arrays and multiplicative comparison categories.

The test consisted of 22 tasks, with 18 word problems and 4 bald calculations across all four basic operations. For this study, my focus was on the multiplicative word problems. Of the 18 word problems, 15 were multiplicative problem situations and 3 were additive problem situations. Additive problem situations were used to explore participants' awareness of distinctions between additive problem situations and multiplicative problem situations.

The test content comprised of three multiplication types and two division types of word problems. The types of multiplicative word problems were:

i. Multiplication as rate
ii. Multiplication as scale
iii. Multiplication as array
iv. Division as sharing
v. Division as grouping

All the word problems used in the tests were drafted using the RME approach with contexts of the word problems imaginable to South African Grade 6 learners. Table 4.1 (below) provides the summary of the word problems used in the pre- and post-tests classified according to multiplicative problem types.

Classification	Question number	Question
Multiplication as rate	1	Learners sell bags of mangoes during market day. They put 8 mangoes in each bag. How many mangoes will they need to fill 19 bags?
	14	Nomonde makes bead bracelets. Each bracelet contains 12 beads. She makes 19 bracelets. How many beads did Nomonde use?
	9	One bar of chocolate costs R 12. How much will 18 bars of this chocolate cost?
Multiplication as scale	6	Zwi collected 8 bottles for recycling. Her friend collected 6 times as many bottles as Zwi. How many bottles did the friend collect?
Multiplication as array	5	Hamsa is counting how many tiles cover the bathroom floor. She counts 12 rows of tiles. There are 11 tiles in each row. How many tiles cover the floor?
Division as grouping	2	A class of 56 children go out to play games. They get into teams of 8. How many teams do they make?
	8	96 people are going on a car trip. Each car can carry 6 people. How many cars will the people need?
	13	A company packs pencil into boxes. Each box contains 15 pencils. How many boxes are needed to pack 195 pencils?
	16	Denozo cycles at 14 km per hour. How many hours will it take for Denozo to travel 84 km?
	17	A farmer is planting rows of apple trees. He has 252 apple trees. He plants 12 trees in each row. How many rows will there be?
	18	A red hat costs R 15 and a blue hat costs R 90. How many times as much does the red hat cost as the blue hat?
Division as sharing	3	The total mass of 7 same-sized bags of sugar is 63 kg. What is the mass of one bag of sugar?
	7	Mother puts out 180 muffins equally on 12 plates. How many muffins does she put on each plate?
	10	A baker put pies in rows on a tray. He puts the same number of pies in each of 8 rows. The baker has 128 pies altogether. How many pies must he put in each row?
	11	Joey bought 108 metres of wire to put up a fence. This is 6 times more wire than Peter bought. How much wire did Peter buy?

Table 4.1: Summary of pre- and post-tests multiplicative word problems

As mentioned earlier, the pre- and post-tests were kept the same to determine the effects of the intervention lessons by comparing the pre-test and post-test performance. The pre-test

was used to identify models and strategies that Grade 6 learners used prior to intervention lessons. The post-test was used to identify models and strategies Grade 6 learners used, to ascertain if there were any differences post intervention lessons.

4.5. Process of data collection

Data collection for this study took place over a period of six weeks, with 2 test periods (pre- and post-tests) and 4 intervention lessons. The same time of the day was used, Tuesday 09:00 to 10:00. Figure 4.1 (above) illustrates the focus of each lesson.

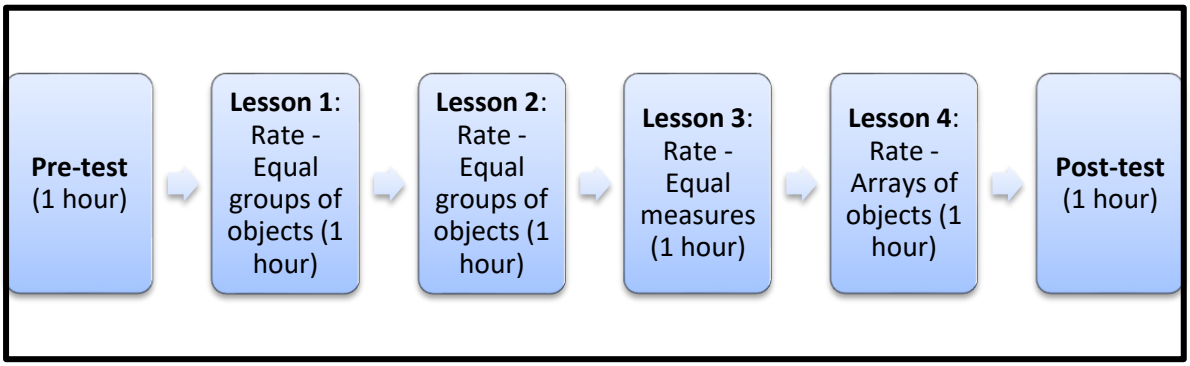
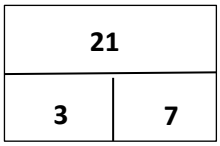


Figure 4.1: Process of data collection

The pre- and repeated post-test were used to collect data. Both tests were administered by WMC-P team members during contact time. To address poor literacy issues, each test question was read out twice to the class. Participants had 2 minutes to $2\frac{1}{2}$ minutes to answer questions in the space provided, using whatever strategy they preferred, but using the most efficient strategies they could, and showing working out where they felt this was needed.

Three principles underpinned the design of each lesson. First, there was attention on building fluency in rapid recall of multiplication and division facts. Triple cards were used to build fluency in multiplication and division facts (see Table 4.2 for image of triple card). Triple cards are made up of three related numbers e.g. 7, 3 and 21. Learners are required to make multiplicative number sentences using the three numbers. Number sentences using the three numbers, would be $3 \times 7 = 21$, $7 \times 3 = 21$, $21 \div 3 = 7$ and $21 \div 7 = 3$. In this intervention, the educator would show all three numbers on the card. Then cover up the product (21), ask learners to say the number sentence linking the two numbers they can see and to provide the answer (seven multiplied by three is 21). After multiplication facts, the shift would move to division. Covering of factor, ask learners to say the number sentence linking the two numbers.

The second lesson principle was that participants needed to be able to identify different multiplicative problem types and have the language to describe these problem types. This was aimed at helping participants to see multiplication and division problems as linked. The last lesson principle emphasised that learners needed to work with key representations to help them be more efficient problem solvers. Arrays, double number lines and ratio tables were introduced during intervention lessons. Table 4.2 shows the structure of each intervention lesson into four sections.

Section	Description
1. Revision of multiplication and division facts.	Involve working with 'Triple cards' to form word and number sentences. Example:  Number bonds: 3×7 = 21, 7×3 = 21, 21÷3 = 7 and 21÷7 = 3
2. Solving a linked set of three problems in pairs.	Learners worked in pairs to solve three word problems. This was followed by teacher-led whole class discussion based on the three word problems. Emphasis was made on eliciting, presentation and discussion of structural commonalities underlying multiplicative structure.
3. Individually solving six word problems.	Learners work individually to solve six word problems. Five of the word problems were multiplicative situations and one was additive situation. Learners worked independently to set up models of the word problem situations and using models to work towards an answer. Solutions to the six word problems were discussed during the lesson. Different versions of ratio tables were selected to be presented on the board by learners. Discussions on the different problem situations were aimed at helping learners distinguish between additive and multiplicative situations.

4. Bald multiplication and division number sentences for independent practice.	Once learners completed the individual task (section 3), they worked independently on a set of bald number multiplication and division problems. This was aimed at building fluency with mental and written calculations.

Table 4.2: Structure of each intervention lesson

See Appendix A for detailed lesson plan for each lesson. This includes the word problems used in the lesson, structure of the lesson, educator's role and the practice or homework tasks.

4.6. Data analysis and interpretation

Scores were assigned to each response: correct responses were recorded as 1 and incorrect responses were recorded as 0. Blank responses were also recorded as 0. Scores were aimed at identifying correct and incorrect responses, and the extent of these. To identify specific models and strategies learners that used to respond to word problems, learners' responses were coded. The scores and codes for responses were then recorded on Microsoft Excel worksheet to organise and simplify the data. This helped me to obtain useable and useful information.

Single-item scores were used to provide a detailed analysis for each word problem. Summed scores (for multiplicative items) focused on the total scores of all fifteen multiplicative word problems. Unlike single-item scores, summed scores across the focal items capture a participant's perspective (Creswell, 2012). Summed scores were used to develop an understanding of participants' performances from pre- and post-test. To quantify any shifts in participants' performances for both tests, differences in scores were measured. My focus was on understanding patterns of difference across the class, using the literature base and theory to make claims about whether these differences could be interpreted as development in multiplicative reasoning thinking. With the results showing development in these terms, I then went on to analyse the models and strategies (following the RME approach) that underpinned these shifts.

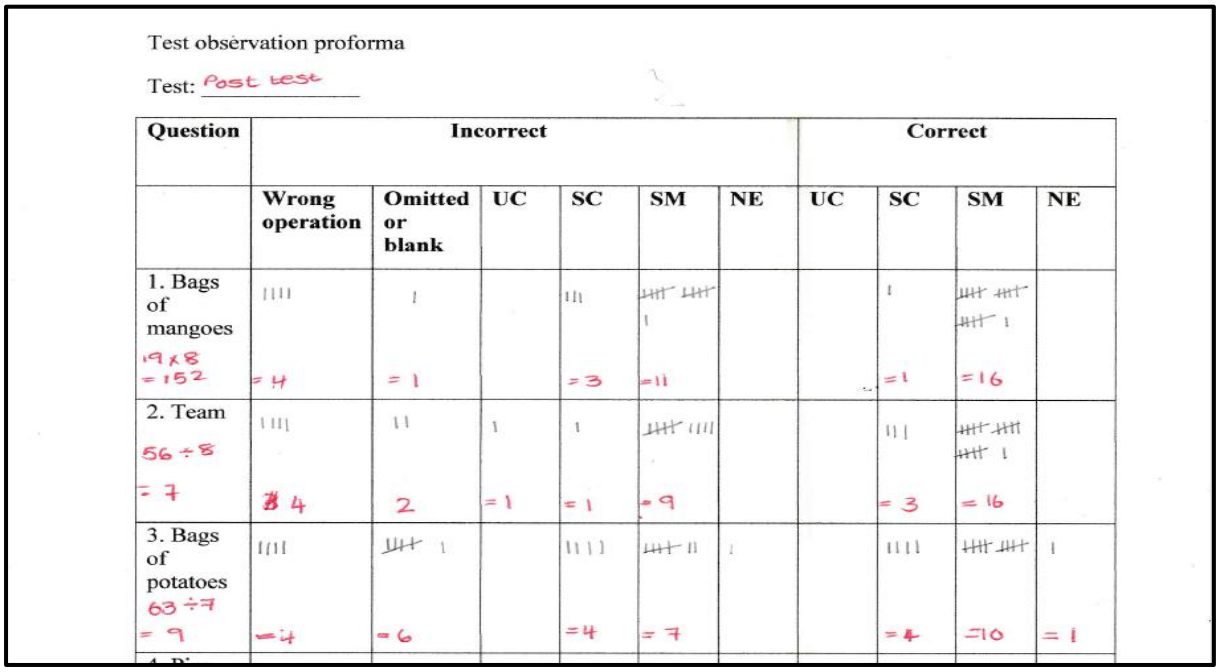
A codebook was developed to record models and strategies learners used to respond to word problems to help answer the research questions. Table 4.3 summarises how learners' responses were coded. Coding started with identifying the model used and then coding the strategy used to solve the problem. Creswell (2012) refers to this process of coding as of narrowing data into few themes. The codes were developed according to literature based on multiplicative reasoning and the RME theoretical framework.

Model		Strategy	
Key	**Description**	**Key**	**Description**
1	Inappropriate model e.g. adding two numbers in the question, rather than multiplying them.	1	*Unit counting* – diagrammatic representation of multiplicative problems using tallies, sticks, dots, circles or other shapes to depict everyday object.
2	Appropriate model with incorrect answer e.g. multiplying 6 × 4 but getting 26 as an answer instead of 24.	2	*Skip counting* – counting in multiples e.g. 2s, 3s, 4s, 5s …
3	Appropriate model with correct answer	3	*Symbolic strategy* – e.g. compressed ratio strategy, long division, column multiplication, lattice method etc. In this strategy the learner may uses an algorithm to work out the solution. This strategy involves using compressed number relationships to calculate the unknown.
0	Blank or omitted response	4	No visible working.

Table 4.3: Codes for models and strategies

In a case of a correct answer, the coding started with 3 for the model since the model is appropriate with correct answer and then followed by the strategy used to work out the problem. For example, 3/3 meant that the answer is correct, and the learner used symbolic strategy for calculations. For appropriate model with incorrect answer, coding started with 2, followed by the correct strategy used and then the incorrect answer. For instance, 2/1/8 for 56 ÷ 8 word problem means that the answer is incorrect with appropriate model. The learner used unit counting strategy and got an answer of 8 for 56 ÷ 8, which is incorrect as 56 ÷ 8 = 7. Incorrect answers were included in coding to help identify emerging trends in learners' incorrect responses and see errors in learners' responses. Inappropriate models started with 1 and followed by the incorrect answer the learner obtained. For example, 1/64 means the learner had an inappropriate model of 56 ÷ 8 as 56 + 8, resulting in 64. In this case the learner added the two numbers.

Test response proforma was designed to provide a picture of how learners were performing as a class. The test response proforma captured a sense of what most learners were doing. Figure 4.2 is a snapshot of the post-test response proforma. Responses were coded using Table 4.3, abbreviations used on the test response proforma were UC – unit counting, SC – skip counting, SM – symbolic strategy and NE – no evidence of working out shown.



Test observation proforma

Test: Post test

Question	Incorrect						Correct			
	Wrong operation	Omitted or blank	UC	SC	SM	NE	UC	SC	SM	NE
1. Bags of mangoes 19 x 8 = 152	4	1		3	11			1	16	
2. Team 56 ÷ 8 = 7	4	2	1	1	9			3	16	
3. Bags of potatoes 63 ÷ 7 = 9	4	6		4	7	1		4	10	1

Figure 4.2: Snapshot of the post-test response proforma

Double bar graphs were used to display performance for all multiplicative word problems across the pre- and post-tests. The use of double bar graphs assisted with comparing patterns of shifts in overall performance across all the word problems. Microsoft Excel and Fathom were used to obtain the descriptive statistical analysis for overall performance of pre- and post-tests.

To analyse the data on the test proforma I also used Venkat & Mathews' (2018) item level approach, which they used for their study on multiplicative reasoning. The item level approach uses facility bands by classifying items by number of correct responses of the class population. Item facility bands classifies items in the pre- and post-tests into three categories, which are:

- low facility items (0 – 25 % of the class population getting the item correct),
- moderate facility items (26 – 49 % of the class population getting the item correct), and
- high facility items (50 – 100 % of the class population getting the item correct).

While these facility bands are not equal sized bands and the definition of high facility band is generous, these bands have been chosen because they allow us to analyse change on the

ground in the context of low performance (Venkat & Mathews, 2018). Data was then analysed based on facility items to see emerging patterns within facility bands.

In a case where examples of learners' responses are used in Chapter 5, these were selected based on most clearly illustrated common phenomena.

4.7. Limitations of the study

This study relied on the pre- and post-tests to explore intervention effects with the focal class. The lack of inclusion of a comparison group means I do not have access to data on what 'normal' teaching might have achieved in the same time frame, and this is a limitation. Secondly, due to time constraints, a delayed post-test could not be administered, so there is no rigorous research access to longer-term retention gains, although, as the class teacher, I did see learners use ratio tables models in both formal and informal assessments during the remaining period of the year. Lastly, my data collection was limited to test responses. Since the broader study did not include other research instruments such as learner interviews, I did not seek ethical permission for learner interviews, which in retrospect, would have been useful to probe further some learners' responses to get a better understanding of approaches used and in cases of no evidence of working out to get how the problem was worked out. However, the test responses provided a rich dataset to work with to gain an insight into learners' models and strategies for multiplicative problem situations and the effect of multiplicative intervention lessons.

4.8. Validity and reliability

Validity refers to the extent which the scientific explanations of the phenomena match reality. This is the truthfulness of findings and conclusions of the study on models and strategies Grade 6 learners use to solve multiplicative word problems (Creswell, 2012; McMillan & Schumacher, 2010). To ensure that the issue of validity is dealt with, careful attention was considered when sampling, development of research instrument and statistical analysis of data. Reliability is the measure of consistency (Basit, 2010; Creswell, 2012; McMillan & Schumacher, 2010). Reliability of this study was achieved through standardisation of the test and crosschecking the data. In this section, I discuss how the study is valid and reliable.

Careful consideration was taken in the process of designing this study to reduce sources of error and to provide results that is credible.

4.8.1. Internal validity

Internal validity is defined as the extent which the researcher controls extraneous variables so that the observed effect of the research is attributed to the treatment variable (Gall *et al*, 2007). Therefore, it focuses on the viability of causal links between dependent and independent variables (McMillan & Schumacher, 2010). Data gathered does not ensure internal validity, however the internal validity relies on claims and process of gathering data which measures what is intended to measure (Gall *et al*, 2007).

As this study adopted the experimental approach, careful consideration was made to reduce certain risk factors. Potential threats of internal validity of single group pretest – posttest design are history, statistical regression, instrumentation, attrition, testing and maturation (Gall *et al*, 2007; McMillan & Schumacher, 2010). To ensure internal validity in the data collection process, the test and intervention worksheet were non-biased, non-discriminatory, level appropriate for Grade 6 and used relevant contexts to South African learners. The focus of the study was on models and strategies Grade 6 learners use for multiplicative word problems. The test content had different types of multiplicative word problems based on classes of multiplicative situations discussed earlier in literature and varied in number ranges.

Often South African classrooms use a traditional and authoritarian approach to teaching mathematics (Barnes, 2005). Multiplication and division are traditionally taught as separate topics in South African classroom. In this study an RME approach was used, whereby relevant and real contexts and learners were given the opportunity to reinvent the mathematics with the guidance of the researcher as an educator. Multiplication and division were taught as integrated concepts with the use of arrays, double number lines and ratio tables models. Prior to intervention, learners had experience of using expanded column multiplication and division, as well as column multiplication and long division algorithms. The use of double number lines and ratio tables post-intervention lessons would likely be related to take up from the four intervention lessons.

The use of the same test for the pre- and post-tests posed a threat to internal validity of claims made. Pre- and post-tests were identical to avoid the difference in nature of instrument possibly impacting on performance and directly measure the effectiveness of intervention lessons by comparing the learners' pre- and post-tests which were administered five weeks apart. Learners were not aware that the tests content was kept the same. The test administrator for both pre- and post-tests was the same WMC-P researcher. This was to

eliminate any possible bias in the administration of the tests. McMillan & Schumacher (2010) mention that change in test administrator might affect performance, therefore keeping the administrator the same was to avoid compromising internal validity.

Withdrawal of participants in a study can pose as a threat of internal validity. Only learners who completed both the pre- and post-tests were included for statistical analysis (Gall et al, 2007; McMillan & Schumacher, 2010). A total of 36 learners wrote both tests and these learners were present for all four intervention lessons.

4.8.2. External validity

External validity refers to the generalisability of the results and conclusions to specified, large population samples (Basit, 2010; Gall *et al*, 2007; McMillan & Schumacher, 2010). The results in this study are not generalisable to a large population as the sample of the study was selected based on convenience sampling and the small sample size of 36 learners. McMillan & Schumacher (2010) argue that generalisability of the findings of convenient sample are limited to the characteristics of the subjects. The aim of my study was not to make generalisations, but was intended at gaining insight into the multiplicative reasoning of Grade 6 learners. Thus, findings can be used to inform the extent to which Grade 6 educators in similar contexts can use the outcomes of this study for teaching multiplication and division.

4.8.3. Reliability

According to Basit (2010) consistency and replicability over time, instruments and research participants serve the purposes of reliability in quantitative approach. Reliability in quantitative research can be achieved through standardisation of research instruments, by crosschecking the data and by using different instruments to examine the same thing (Basit, 2010).

The pre- and repeated post-test were used to collect the research data from a Grade 6 class. Using the same test content for both tests required that there was enough time between the tests so that learners do not memorise the test questions (Gall *et al*, 2007). As mentioned earlier in the chapter, the tests were administered five weeks apart. During the process of writing tests, learners were provided with clear instructions for writing the test. To reduce the test anxiety, learners were assured that tests would not count for their Mathematics formal assessments. This was also highlighted in the parents' and learners' consent forms.

The data (learners' written responses) were coded and scored by me as a researcher and WMC-P team to ensure that the codes and scores are reliable.

In conclusion, the study used a convenient small-scale sample, therefore claims made in the study are limited. However, the test instrument used was valid and reliable, which makes the claims in the study valid and reliable. Therefore, this study is appropriate to tell a story on models and strategies Grade 6 learners use to solve multiplication and division word problems in the South African context.

4.9. Ethical considerations

Participants of the study were learners within an age range of 11 to 13 years, and the case site was a learning institution. To ensure safety of participants, ethics clearance from the Wits School of Education Human Research Ethical Committee was granted in early 2017. The protocol number 2017ECE033M was provided, the ethics clearance letter is provided in Appendix B.

Since the case site was a government public school, an application was made with the Gauteng Department of Education. Permission was granted to conduct the study in the leaning institution. Information letters were used to inform participants about the study and ask them to participate in the study. Information letters were given to participants, parents (as participants were under 18 years) and the principal of the learning institution. In the information letters it was highlighted that participation was voluntary and that anyone could withdraw from participating in the study at any time, without being disadvantaged in any way. Since the study was conducted during school hours, participants and relevant parties were informed that intervention lessons and tests aligned with the CAPS specifications to ensure that curriculum coverage time was not lost. Copies of information letters and consent forms is included in Appendices C and D.

As mentioned in ethics application, it is important to maintain confidentiality and anonymity of the participants. Confidentiality was achieved by using pseudonyms in recording data and research report. This enabled me to protect the participants from being known publicly. Anonymity could be assured between the researcher and participants. The research method required face to face interaction with participants. All participants agreed to participate in the study and none of the participants decided to withdraw from participating in the study during the data collection process.

Finally, the data was stored in a safe and secure environment with only the researcher and the WMC-P research team having access to the data. This data will furthermore be destroyed after a period of 5 years have lapsed, as per university regulation.

Chapter Five: Findings and Analysis

5.1 Introduction

The purpose of this study was to support learners with making sense of multiplicative situations (multiplication and division) and to broaden their models and strategies for working with multiplicative situations through a small-scale intervention focused on a lesson sequence based on problem solving.

In this chapter, I present and discuss the findings of my study in relation to the models and strategies Grade 6 learners used across the pre- and post-intervention tests. I begin by providing an overview of changes in learners' overall performance from both tests. Then I look at the mathematisation approaches underlying these changes in performance.

Data was gathered from 36 Grade 6 learners' written responses from the pre- and post-intervention tests. Both tests were intended to establish the kinds of models and strategies learners used for multiplicative word problems prior to, and following, the intervention, and to investigate whether these shifts were associated with any shifts in overall performance. During the intervention lessons learners explored arrays and ratio models (double number lines and then ratio tables), both of which are described in the literature as useful for emphasising the multiplicative reasoning (MR) correspondence relationship that underlies multiplicative situations, with the aim to develop learners' multiplicative reasoning. The post-test was used to identify if there were any shifts in models and strategies, and gains in overall performance following the series of four one-hour multiplicative reasoning intervention lessons.

To analyse the results from pre- and post-tests, statistical graphs in the form of bar graphs, box plots and tables are used. Both models and strategies used by learners were coded and organised in the categories described in the literature and summarised in the methodology chapter. The focus was to examine the patterns and shifts regarding models and strategies emerging between the pre- and post-tests.

As noted earlier, in a case where illustrative examples of learners' work were used, the learner's work was selected based on a response that clearly illustrated common phenomena.

5.2 Overview of learners' performance for both pre- and post- tests

Data from the pre- and post-tests suggested that there were positive gains in learner performance from the intervention lessons. The class average for the pre-test was 24,07 % (3,61 out of 15 multiplicative items). In the post-test results, there was a substantial improvement in learners' performance. At the overview level, the class average for the post-test was 41,30 %, (6,19 out of 15 multiplicative word problems dealt with correctly). Underlying this gain, and detailed later in this chapter, learners were more commonly able to use more formal mathematical models and strategies to solve the word problems and interpret the word problems.

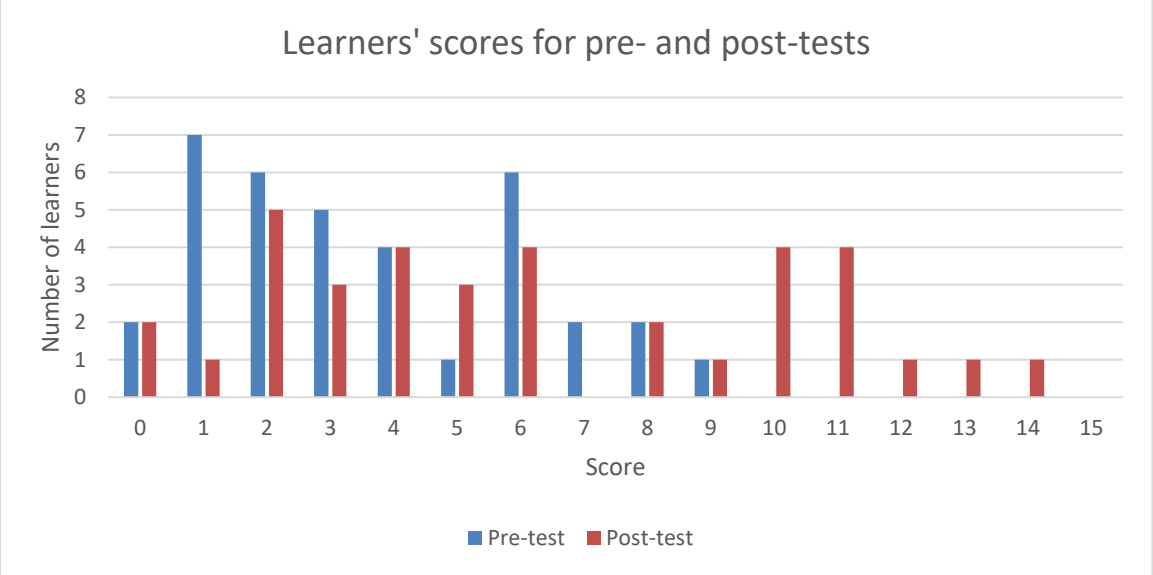


Figure 5.1: Learners' scores per item for the pre- and post- tests

Figure 5.1 depicts the learners' scores with regards to the number of correct responses to the 15 multiplicative word problems for both tests. The pre-test data is skewed to the right, which illustrates that most of the learners scored lower marks, in the pre-test than in the post-test. Of the 36 learners in the sample, 22 scored between 1/15 and 4/15 for multiplicative word problems. The highest score for the pre-test was 9/15 correct responses, with the lowest score being 0/15 correct responses. Only one learner was able to score 9/15 correct responses for the pre-test. The pre-test range was 9 (min0; max9), and the median score was 3/15 correct responses. There was substantial improvement in the profile of performance, as illustrated in the Figure 5.2 illustrating the box plot of the pre- and post-tests performance.

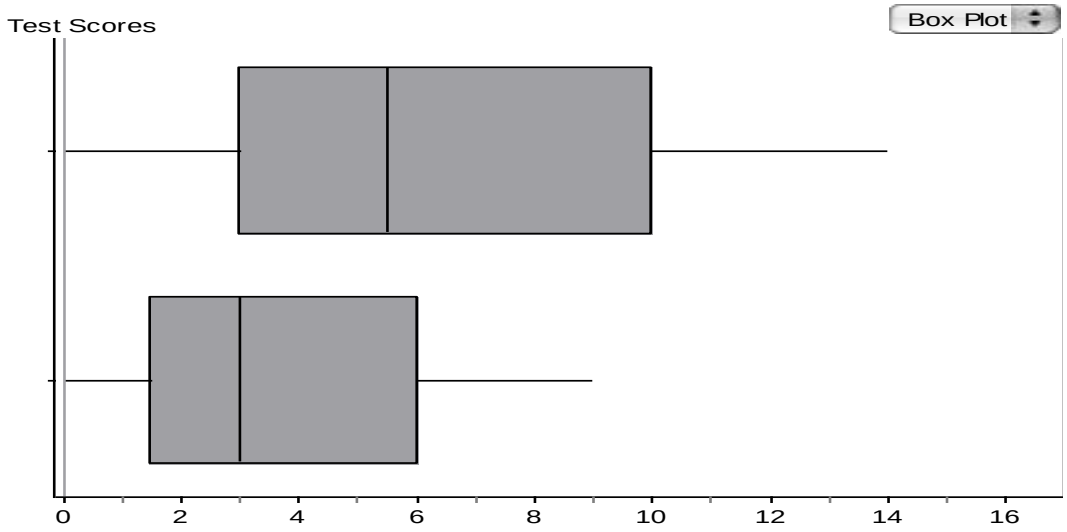


Figure 5.2: Parallel box plots of the pre- and post- intervention tests scores

In the post-test there were substantial improvements in whole class performance, the post-test data is skewed to the left. As shown on Figure 5.1 and 5.2, the highest score of correct responses for post-test was 14/15, an improvement by 5 marks in the highest scores obtained on the pre-test scores. The lowest score remained as 0/15 word problems, with two learners obtaining this score for the post-test. Figure 5.1 shows that the distribution of test scores improved with 13 learners scoring above 9/15 word problems for the post-test, which was the highest score prior to the intervention. The median score increased to 5/15 correct responses, with a range of 14 (min0; max14) for the whole class performance. These post-test scores reflect improvements in learners' performance.

In drilling down to the item level, I followed the approach taken in Venkat & Mathews' (2018) analysis to look at a more in-depth analysis based on item facility by classifying the items in the pre- and post-tests into facility bands that reflected performance on the ground:

- low facility items (0 – 25 % of the class population getting the item correct),
- moderate facility items (26 – 49 % of the class population getting the item correct), and
- high facility items (50 – 100 % of the class population getting the item correct).

The analysis produced the results presented in Figure 5.3. Several features of these overview patterns of change are of interest.

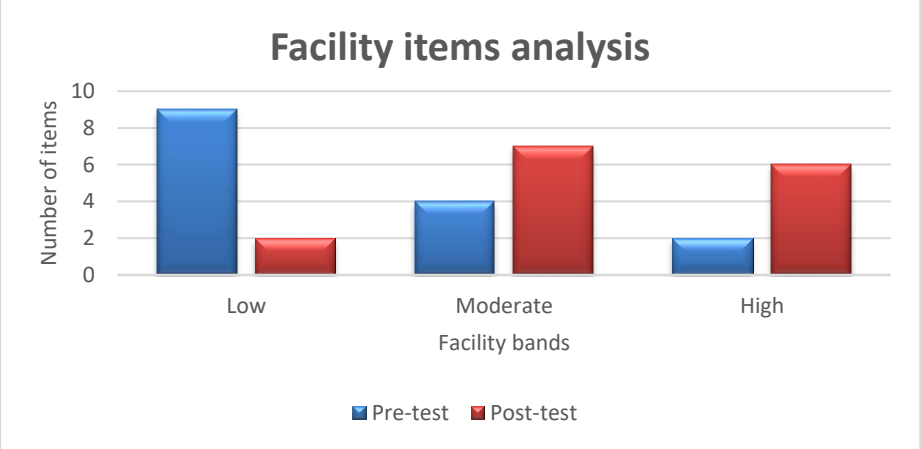


Figure 5.3: Number of items in each facility band across the pre- and post-tests

Low facility items were prevalent for the pre-test, with 9/15 items falling in this category with only 2/15 items showing 50 % and above correct responses. A further, 4/15 items had moderate facility responses, which suggests that pre-test, the overall profile of test performance was skewed towards low facility. In the post-test there was a positive shift in performance, with 6/15 items being high facility items. Low facility items dropped by seven items to 2/15 items, indicating that learners gained better understanding of multiplicative situations. Moderate response increased by three items to 7/15 items. This marked shift in performance in pre- and post-tests created an interest in looking in-depth at performance in particular item clusters and changes in learners' approaches to working with these item clusters.

With marked shifts in facility items for the post-test, I also looked at items which had high improvements when comparing the pre- and post-tests performances. High improvements items showed progress by ten or more correct learner responses, or 27,78 percentage points. I began by looking at the pre-test high, moderate and low facility item categories. In Section 5.2.4 (page 64) I provide detailed analysis of high improvement items, to highlight shifts in learners' performance. I also look at items that remained in the low facility category post intervention lessons in Section 5.2.5 (page 66), to gain an insight on why performance remained low for these test items.

5.2.1 Pre-test high facility items

Two items were in the high facility item cluster for the pre-test, these problem situations are (in paraphrased form):

- Q2: 56 children getting into teams of 8 children each. (66,67 % correct)

- Q8: 96 people traveling in cars that carry 6 people each. (50% correct)

Both problem situations involved two digits by one digit division, which suggest that the number range may possibly be one of the factors that influenced performance. However, there was a two digits by one digit division problem situation in the low facility item category in the pre-test (Q3: 7 same sized bags of sugar, weighing 63 kg in total). The literature suggests two possible causes of poor performance for this test item (Q3). The first possible cause is that this problem is a division as sharing situation whereas the other two problems were division as grouping problem situations. Studies show that learners experience difficulties in solving division as sharing problem situations because they require learners to find the number of objects in each group rather than the number of groups (Carpenter *et al*, 1999). Division as sharing becomes difficult when using unit counting strategies because they are cumbersome and error-prone. Skip counting can also be cumbersome because learners try to figure out the number of objects in each group using trial and error (Carpenter *et al*, 1999). The second possible cause is that the order in which the numbers appear in the test item are not consistent with the number order in the calculation in Q3, whereas, in Q2 and Q8, the numbers appear in the same order as they would in the symbolic version (dividend appears first, then followed by the divisor). In Q3 the numbers are in inverted order (divisor appears before the dividend).

I began, noting, in the subsequent analyses, the problem situation class and whether the order of presentation of numbers in the word problem mirrored the order of the numbers in the parallel symbolic version, to see if these factors also contributed to the facility levels seen.

	Number of correct responses		Appropriate models [4]of word problem situation										Inappropriate models		Omitted (#)	
	Pre	Post	Pre-test (#)					Post-test (#)					Pre	Post	Pre	Post
			UC	SC	SM	NE	Incorrect answers	UC	SC	SM	NE	Incorrect answers				
Q2	24/36	19/36	18	5	8	2	9	1	4	25	0	11	3/36	4/36	0	2
Q8	18/36	21/36	12	11	8	2	15	0	6	26	1	12	3/36	3/36	0	0

Table 5.1: High facility item analysis (pre-test)

[4] Appropriate model – Refers to when the learner interprets the word problem correctly. For example; One avocado costs R6. Thabang buys 7 avocados. How much does Thabang spend? When the learner interprets the word problem as 6 × 7, this learner uses an appropriate model of the problem (whether their working out is correct or incorrect).

* Keys: UC – unit counting, SC – skip counting, SM – symbolic strategies and NE – no evidence of working out shown.

Table 5.1 shows that unit counting-oriented strategies were the most commonly used strategy by learners in the pre-test for high facility items, with 30/72 responses using this approach. Of the 30 unit counting responses, 17 responses were correct and the remaining 13 responses had incorrect answers. Both skip counting and symbolic strategies were the second most popular strategy, with item having 16/72 responses using these strategies.

In the post-test there was a greater shift to using symbolic strategies, with 51/72 responses using symbolic strategies and 10/72 response using skip counting strategies. There was a greater shift towards using more efficient strategies as only 1 of the 72 responses using a unit counting strategy to respond to the word problem. This signals that the focus of the intervention lessons was successfully attained. However, it is also worth noting that, in this move towards using more efficient strategies, the overall facility on Q2 dropped, from 24/36 correct responses to 19/36 correct responses in the post-test. While the vast majority of responses showed appropriate models, this result pointed to fragility with calculating answers once an appropriate model was set up. This further pointed to calculating errors in the context of more efficient strategies.

5.2.2 Pre-test moderate facility items

Four items were moderate facility items, the word problems:

- Q1: 8 mangoes in each bag, 19 bags need to be filled. (27,78 % correct)
- Q6: Zwi collects 8 bottles, her friend collects 6 times as many. (44,44 % correct)
- Q10: 8 rows of pies, 128 pies baked. (33,33 % correct)
- Q13: 15 pencil per box, 195 pencils altogether. (36,11 % correct)

In the pre-test, 4/15 items were moderate facility items, with 51/144 correct responses (35,44 %). Two of the moderate facility items were multiplication word items and the other two were division items. The division items both had numbers in inverted order. For the pre-test, the majority of responses had appropriate models of the problem situations but most of these responses had incorrect working out. Responses with the appropriate models of the problem situations tallied as 115/144 responses (79,86 %), with 64/115 of the appropriate models (55,65 %) having incorrect answers. Unit counting was the most popular strategy used by learners, with 45/144 (31,25 %) responded using this strategy. The symbolic strategy was the

second most popular strategy with 34/144 responses (23,61 %), followed by skip counting at 32/144 responses (22,22 %). Responses with no evidence of working out (correct or incorrect) were at 4/144 responses (2,78 %). Inappropriate model responses were 27/144 responses (18,75 %), which shows that in the majority of cases, learners were able to appropriately model the four moderate facility items.

	Number of correct responses		Appropriate models of word problem situation										Inappropriate models		Omitted (#)	
	Pre	Post	Pre-test (#)					Post-test (#)					Pre	Post	Pre	Post
			UC	SC	SM	NE	Incorrect answers	UC	SC	SM	NE	Incorrect answers				
Q1	10/36	17/36	23	6	3	1	23	0	4	27	0	14	2/36	4/36	1	1
Q6	16/36	18/36	5	11	14	1	15	0	12	17	1	12	4/36	1/36	1	5
Q10	12/36	18/36	10	6	10	1	15	0	1	24	2	9	9/36	8/36	0	1
Q13	13/36	11/36	7	9	7	1	11	0	8	17	0	14	12/36	8/36	0	3

Table 5.2: Pre-test moderate facility item analysis

Post intervention lessons there were shifts in performance, two of the pre-test moderate facility items improved to high facility items (Q6 and Q10), whereas Q1 and Q13 remained in moderate facility items band. However, it must be noted that Q1 experienced a positive shift in performance. Prior to intervention 10/36 responses (27,78 %) were correct, which improved by 7 correct responses to 17/36 responses (47,22 %) post intervention. Q13 showed a small drop in correct responses from 13/36 to 11/36 responses (30,56 %) being correct in the post-test. The responses to Q13 show improved sense making of the problem situation, but learners experienced problems with performing calculations as 14/25 appropriate model responses (56 %) had incorrect answers. Table 5.2 shows that there were greater shifts towards using symbolic strategies (17/36 responses) post-test for Q13. It appears that the drop in correct responses was caused by learners experiencing difficulties with executing the calculations using symbolic strategies. Of 17 symbolic strategies, 9 had incorrect working out (52,94 %) and 8 had correct solution (47,06 %).

Overall the pre-test moderate facility items showed positive shifts in performance for the post-test. Correct responses were 64/144 (44,44 %), an increase by 13 responses (25,47 % increase). Responses with inappropriate models of four problem situations decreased to 21/144 responses (14,58 %). There was a greater shift towards using more efficient strategies, 85/144 responses (59,03 %) used symbolic strategies which was more than double the pre-test responses. Skip counting strategies were the second most popular strategy with 35/144

responses (24,31 %) using this strategy. Responses with appropriate models of problem situations were 113/144 responses (78,47 %), a slight decrease by 2 responses in comparison to pre-test. For post-test 49/113 responses (43,36 %) were incorrect solutions, which was a drop by 23,42 % in comparison to the pre-test responses.

5.2.3 Pre-test Low facility items

Low facility items incorporated 9/15 items with 37/324 correct responses (11,42 %). The nine low facility items were:

- Q3: 7 same-sized bags of sugar is 63 kg. (22,22 % correct)
- Q5: 12 rows of tiles with 11 tiles in each row. (19,44 % correct)
- Q7: 180 muffins equally on 12 plates. (22,22 % correct)
- Q9: 1 bar of chocolate costs R12, price of 18 chocolate. (11,11 % correct)
- Q11: Joey bought 108 m wire, which is 6 times more than Peter's wire. (0 % correct)
- Q14: Nomonde makes 19 bracelets, each bracelet contains 12 beads. (11,11 % correct)
- Q16: Denozo cycles at 14 km per hour, how long to cycle 84 km. (2,78 % correct)
- Q17: 252 apple trees, 12 trees in each row. (5,56 % correct)
- Q18: Red hat R15, blue hat R90. (8,33 % correct)

Poor performance for these problem situations was largely due to learners inappropriately modelling multiplicative situations. The responses with inappropriate models were 46,6 % (151/324) and responses with incorrect answers were 40,12 % (130/324). Inappropriate models of multiplication word problems mostly interpreted the multiplication problems as involving addition, whereas with division the problems were often interpreted as involving subtraction and/or multiplication. Of the low facility items, three of the test items were multiplication word problems, which all required two digits by two digits multiplication. Skip counting and symbolic strategies (column multiplication) were most favoured strategies for multiplication word problems, however, most answers were incorrect as learners experienced difficulties with executing column multiplication and/or had incorrect multiples. As identified earlier in literature, learners experience difficulties with executing column multiplication as they do not understand the procedure (Anghileri, 2006; Ma, 2010; Haylock, 2006). Omitted or blank responses featured in 6/324 responses (1,85 %).

	Number of correct responses	Appropriate models of word problem situation (#)					Inappropriate models	Omitted (#)
		UC	SC	SM	NE	Incorrect answers		
Pre-test	37/324	40	63	51	13	130	151/324	6/324

Table 5.3: Pre-test low facility items analysis

The remaining two thirds (6/9) low facility items were division problem situations, with five of the six division low facility items involving division by a two-digit divisor. Half of the division low facility items (3/6) were number order consistent, with the other half as number order inconsistent. Unit counting was the most favoured strategy with 31/87 of responses (35,63 %) with appropriate models of division using this strategy. Of the unit counting responses, 24/31 (77,42 %) of the responses had incorrect answers. Incorrect unit counting responses commonly had incomplete representations and/or incorrect set up of problem situation. Symbolic strategies (long division) were the second most favoured strategy with 21/87 of the responses (29,89 %) to low facility division items using this strategy. Like unit counting responses, majority of symbolic strategies responses (84,62 %) had incorrect answers. Common errors in symbolic strategies responses showed evidence of a lack in understanding the long division procedure. Here incorrect multiples were common errors identified in incorrect solutions. With the division of low facility items there was also an emergent use of responses with no evidence of working out, 9/87 of the responses (10,24 %) showed no evidence of working out.

For the pre-test low facility items, post-test information was not included as items moved to moderate and high performance categories which I deal with in the next section as high improvements items.

5.2.4 High improvement items

Six items were high improvement items in the terms outlined earlier. High improvement facility items were items that improved with 10 or more correct responses, which is 27,78 percentage points. These word problems were:

- Q.5: 12 rows of tiles with 11 tiles in each row. (33,33 percentage points)[5]

[5] Improvement of responses in percentage points

- Q.7: 180 muffins equally shared on 12 plates. (33,34 percentage points)
- Q.9: 1 bar of chocolate costs R12, price of 18 chocolates. (30,56 percentage points)
- Q.14: Nomonde makes 19 bracelets, each bracelet contains 12 beads. (27,78 percentage points)
- Q.16: Denozo cycles at 14 km per hour, how long to cycle 84km. (36,11 percentage points)
- Q.17: 252 apples with 12 trees in each row. (36,11 percentage points)

All six high improvement items were low facility items for pre-test. These items were either multidigit multiplication or division by multidigit divisor. Three of the high improvement items were two-digit by two-digit multiplication items and the other three word problems required division by a two digit divisor. Prior to intervention lessons, learners experienced difficulties with multidigit calculations for both multiplication and division word problems. In the pre-test skip counting and symbolic strategies were popular approaches used for these problem situations, while unit counting was rarely used, except for Q7. However, in the context of the most commonly used strategies for pre-test, most of the responses had incorrect answers. Learners had trouble with executing multiplication and division multidigit procedures across the multiple steps, with responses including incorrect multiplication and/or division facts. Table 5.4 provides a detailed summary of high improvement word problems and shifts in models and strategies.

	Number of correct responses		Appropriate models of word problem situation										Inappropriate models		Omitted (#)	
	Pre	Post	Pre-test (#)					Post-test (#)					Pre	Post	Pre	Post
			UC	SC	SM	NE	Incorrect answers	UC	SC	SM	NE	Incorrect answers				
Q5	7/36	19/36	8	9	8	2	20	0	10	23	0	14	9/36	1/36	0	2
Q7	8/36	20/36	14	5	7	2	20	0	3	25	0	8	7/36	8/36	1	0
Q9	4/36	15/36	1	20	7	0	24	0	6	28	0	19	8/36	2/36	0	0
Q14	4/36	14/36	0	13	10	2	21	0	0	26	1	13	11/36	8/36	0	1
Q16	1/36	14/36	3	3	3	0	8	0	11	9	0	6	26/36	12/36	1	4
Q17	2/36	15/36	2	5	7	1	13	0	3	18	0	6	20/36	14/36	1	1

Table 5.4: High improvement item analysis

As can be seen on Table 5.4 in the post-test there was improved sense making of multiplicative problem situations as there was a great decline in the use of inappropriate models of problem situations. For instance, Q16 and Q17, which involved division as grouping, were associated with inappropriate models in the pre-test for most learners. Post intervention lessons the majority of the learners were able to model the two word problems appropriately. While it is impossible to make causal claims, the improved setting up of appropriate models involving division as grouping reflected the focus of the intervention lessons on sense making of different multiplicative problem situations. There was also a greater shift to using abstract strategies post intervention lessons and learners showed improved fluency with basic multiplication facts. None of the responses to high improvement items used unit counting strategies. However, it appears that many learners were still experiencing difficulties with multiplication of two digits numbers by two digits numbers. The number of incorrect answers of two digits by two digits multiplication items was higher when compared to lower multiplication number ranges. In Section 5.3.2 (page 70) I provide in depth analysis of some difficulties learners experienced with two digits by two digits multiplication for both tests.

5.2.5 Post-test low facility items

Two items were commonly in the low facility items for both tests, detailed here in paraphrased form:

- Q.11: Joey bought 108 metres of wire, the wire is 6 times more wire than Peter bought. (8,33 % correct)
- Q.18: Red hat costs R15 and a blue hat costs R90. How many times more is the blue hat. (11,11 % correct)

Table 5.5 provides a detailed summary of performance and shifts in models in the pre- and post-tests low facility test items.

	Number of correct responses		Appropriate models of word problem situation										Inappropriate models		Omitted (#)	
	Pre	Post	Pre-test (#)					Post-test (#)					Pre	Post	Pre	Post
			UC	SC	SM	NE	Incorrect answers	UC	SC	SM	NE	Incorrect answers				
Q11	0/36	3/36	1	1	3	0	5	0	2	4	0	3	30/36	27/36	1	3
Q18	3/36	4/36	0	3	4	1	5	0	2	2	0	0	28/36	32/36	0	0

Table 5.5: Post-test low performance items

Both problems are division problem situations, and most of the responses had inappropriate models of word problems for both tests. As can be seen in Table 5.5, for the pre-test 58/72 responses (80,56 %) to division items had inappropriate models. The division items were commonly interpreted as multiplication problem situations. The literature signals that this misinterpretation stems from language inconsistency. As can be seen in the paraphrased form of Q11 and Q18 above, both division word problems use the keyword ‘times’, which is often associated with multiplication. For both problem situations, the inappropriate models were mostly presented as multiplication word problems (108 × 6) and (90 × 15). Bestgen (2009) refers to the use of ‘times’ in division problem situations as an instance of language inconsistency. Language inconsistent problems are word problems where the relational terms prime (or cue) the use of an inappropriate arithmetical operation (Bestgen, 2009). For instance, in relation to Q11 and Q18, the relational term ‘times’ primes the use of multiplication when division is required to solve the problems. The use of inappropriate arithmetical operation, multiplication in this case, is called reversal error. Reversal errors are identified as more likely to occur with inconsistent problems in various studies conducted in elementary schools and even undergraduate university programs (Bestgen, 2009). This suggests that Q11 and Q18 performances were affected by the wording of both pre- and post-tests resulting in low performance for both tests.

In the post-test the majority of learners were still interpreting the two division items as multiplication word problems. Correct answers slightly increased from 3/72 (4,17 %) to 7/72 (9,72 %) in the post-test.

Q6 word problem (Zwi collects 8 bottles, her friend collects 6 times as many) used the keyword 'times', however this word problem required multiplication. For the pre-test, 31/36 (86,11 %) of responses to Q6 were able to interpret the word problem involving multiplication and for the post-test 30/36 responses (83,33 %) were able to model the word problem appropriately. With this mentioned, this shows that the use of 'times' in Q11 and Q18 might have triggered learners to use the inappropriate arithmetical operation for both test items.

In relation to number consistency, the two division items had different number orders. Q11 had a consistent number order since the order of numbers in the word problem was the same as the symbolic version of the question, 108 ÷ 6. The use of multiplication for Q11 therefore seems to be mainly influenced by language inconsistency in the wording. For Q18, the word problem had both language and number order inconsistencies, and predictably perhaps, saw most learners interpreting the word problem as two digits by two digits multiplication. The performance in these two division items suggests that intervention lessons should also focus more on language and number order inconsistency problems types.

Findings and analysis based on facility items reveals that the number range for multiplication word problem, type of division word problems, number order consistency of division word problems, language inconsistency and approaches learners used for calculations may possibly be factors influencing learners' performance. Therefore, in the next sections, I will provide detailed findings and analysis of data based on number ranges of multiplication items (Section 5.3.2), division problem types (Section 5.4.1), number order consistency of division items (Section 5.4.2) and shifts in strategies learners use (Section 5.5).

Given that there were differences in overall performance levels between the multiplication and division items, and given too, the evidence of particular problems with carrying out multiplication calculations, I now focus on the multiplication and division problems separately, dividing each of these item groups into a range of sub-categories that the analysis above has indicated as of potential interest.

5.3 Performance and shifts on multiplication items

In this section I provide findings and analysis on multiplication word problems. Facility band items findings and analysis suggest that learners' horizontal modelling skills are stronger than vertical modelling skills for multiplication items. Learners struggled more with executing

calculations involving multiplication by two digits multiplier, therefore the number range of the multiplier might have influenced learners' performance.

First, I provide an overview of learners' performance on multiplication items and then I look at performance based on the number ranges of multiplication word problems.

5.3.1 Overview of learners' performance for multiplication items

The five multiplication items, in paraphrased form, were:

- Q1: 8 mangoes in each bag, 19 bags need to be filled.
- Q5: 12 rows of tiles with 11 tiles in each row.
- Q6: Zwi collects 8 bottles, her friend collects 6 times as many bottles than Zwi.
- Q9: 1 bar of chocolate costs R12, price of 18 chocolates.
- Q14: Nomonde makes 19 bracelets, each bracelet contains 12 beads.

Two of the multiplication items were multiplication by one digit multiplier (Q1 and Q6) and three of the multiplication items were multiplication by two digits multiplier (Q5, Q9 and Q14). Table 5.6 provides a summary of overall performance for all five multiplication items (total of 180 responses) based on the learners' models.

Test	**Appropriate models of multiplication items**		**Inappropriate models**	**Blank responses**
	Correct answers	**Incorrect answers**		
Pre	41/180	103/180	34/180	2/180
Post	83/180	72/180	16/180	9/180

Table 5.6: Overall performance for pre- and post-tests across multiplication items

As shown in Table 5.6, for the pre-test 41/180 responses (22,78 %) were correct solutions to five multiplication items. The majority of responses (103/108) had the appropriate model representations of word problems with incorrect solutions. These incorrect solutions responses show that learners experienced difficulties with carrying out multiplication procedures and/or had incorrect multiples. Similar findings were identified in the ANA and TIMSS diagnostic reports, as highlighted earlier in this report (DBE, 2013b; 2015; Juan *et al*, 2017; Murray, 2012). Responses with inappropriate models of multiplication problem situations were 34/180 responses (18,89 %), which supports the earlier findings that learners made better sense of multiplication problem situations in comparison to division problem situations. Only 2/180 responses (1,11 %) were either omitted or left blank.

The post-test results in Table 5.6 show a substantial improvement on learners' performance for multiplication word problems. Correct responses increased by about double the pre-test responses to 83/180 responses (46,11 %). Both responses with appropriate models with incorrect answers and responses with inappropriate models sharply declined in number of responses. Responses with inappropriate models of multiplication items dropped to 16/180 responses (8,89 %) and appropriate models with incorrect solutions were 72/180 responses (40 %). Increase in correct solutions suggests that learners were able to use efficient strategies to perform multiplication calculations and had improved fluency with basic multiplication facts.

5.3.2 Multiplication items and number ranges

Table 5.7 presents pre- and post-tests findings for multiplication items based on number ranges. Data was arranged into two groups, which is, multiplication by single digit multiplier and multiplication by two digits multiplier. The data is also presented in percentage form, to be able to compare the performance of the two groups of number ranges as it was highlighted earlier that the two groups had different performances.

Multiplier type	**Pre-test (%)**				**Post-test (%)**			
	Appropriate models		Inappropriate models	Omitted	Appropriate models		Inappropriate models	Omitted
	Correct	Incorrect			Correct	Incorrect		
Single digit	36,11	52,78	8,33	2,78	48,61	36,11	6,94	8,33
Two digits	13,89	60,19	25,93	0	44,44	42,59	10,19	2,78

Table 5.7: Summary for overall performance in the pre- and post-tests across different number ranges

Pre-test data showed in Table 5.7 reveals that learners were more successful with single digit multiplier calculations than two digits multiplier. Correct responses for multiplication by single digit were 36,11 % of all responses and for multiplication by two digits correct responses were at a low 13,89 %. Appropriate models of multiplication word problems with incorrect solutions were 52,78 % for single digit multiplier and 60,19 % for two digits multiplier. Although the data shows that learners struggled more with two digits multiplier calculations, a high number of incorrect solutions in both number ranges show that learners generally struggled with executing multiplication calculations. With a high number of

incorrect solutions, this suggests that learners needed intervention lessons to help them identify and use more efficient strategies. Single digit multipliers task incorrect solutions indicated incorrect multiplication facts, and learners experienced difficulties with executing column method procedures when multiplying two digits by one digit. For two digits multiplier tasks, incorrect solutions had incorrect multiples when using skip counting, incomplete unit counting strategies and incorrect mathematical procedures when executing the column method. Difficulties with executing column method are both common in national and international studies as the increase in the number range results in learners finding the column method procedure complicated as some steps are often forgotten (DBE, 2013a; Ma, 2010).

Inappropriate models of multiplication items were at a low 8,33 % for single digit multiplier problems and at 23,93 % for two digits multiplier problems. For both number ranges, all inappropriate models represented multiplication items as additive problem situations.

Post-test results showed a significant change in learners' performance, especially for multiplication by two digits multiplier word problems. Correct responses for two digits multiplier increased by a factor of three to 44,44 %, whereas single digit multiplier correct responses were at 48,61 %. Incorrect solutions declined by 16,67 percentage points to 36,11 % for multiplication by single digit multiplier items and declined by 17,6 percentage points to 42,59 % for multiplication by two digits multiplier items. Inappropriate models of multiplication by two digits multiplier problems greatly declined by 15,74 percentage points to 10,19 %, suggesting that learners were making better sense of word problems where this skill was required post intervention lessons. For multiplication by single digit multiplier items, inappropriate models were at a low 6,94 % which shows a decline by 1,39 percentage points. Slight differences between single digit multiplier and two digits multiplier for the post-test suggests that learners had improved sense making and execution of multiplication calculations for both number ranges.

5.4. Performance and shifts on division items

In this section I look at the division items findings and analysis. Division items used for the test were either division as sharing or grouping problem situations and ranged from division of two digits by one digit to division of three digits by two digits items. For findings and analysis, I focus on division problem types and the number order consistency of division

problems, as these were salient factors in facility bands analysis. Table 5.8 provides a summary of performance for division word problems.

Test	Appropriate models of division items		Inappropriate models	Blank responses
	Correct answers	Incorrect answers		
Pre	89/360	115/360	150/360	6/360
Post	140/360	80/360	121/360	19/360

Table 5.8: Summary of the overall performance for pre- and post-tests across division items

The overall pre-test results for division word problems suggest that learners did not perform well on division items. As shown in Table 5.8 the correct responses were 89/360 (24,72 %), which is below a quarter of all responses correct to division items. Appropriate models with incorrect answers were 115/360 responses (31,94 %), with incorrect answers mainly caused by difficulties in executing division procedures. Problems with executing division procedures are also common in international studies (Anghileri, 2006). Responses with inappropriate models of division word problems were 150/360 (41,67 %). These responses showed evidence of learners interpreting division items as additive problem situations or/and multiplication problem situations. The high number of inappropriate models indicates that there was a greater need for intervention on division problem situations. Blank or omitted responses were 6/360 (1,67 %).

Post-test data indicates that there was a good improvement in learners' performance where division procedures were required. The correct answers increased by 51 responses to 140/360 responses (38,89 %), with more learners able to execute division procedures using efficient strategies. Appropriate models of division problem situations with incorrect answers declined by 35 responses to 80/360 responses (22,22 %) in comparison to pre-test scores. These responses mostly had incorrect answers because learners experienced difficulties with using a ratio table to break down to the quotient, as well as incorrect multiplication or division facts. Responses with inappropriate models of division problem situations were 121/360 responses (33,61 %), but there was a decline by 29 responses in comparison to pre-test. The high number of inappropriate model responses suggests that there is a need for further intervention on division problem situations. Post-test results also indicate that blank or omitted responses increased to 19/360 responses (5,28 %).

Overall, shifts in learners' performance suggest that learners benefitted from intervention lessons as there were positive gains evident in the post-test results.

5.4.1 Division word problems findings and analysis based on the type of problem situations

Six of the division word problems were grouping problem situations and four were sharing problem situations. Facility bands findings and analysis revealed that the different types of problem situations might have influenced learners' performance based on the type of problem situation.

Problem type	**Pre-test (%)**				**Post-test (%)**			
	Appropriate models		Inappropriate models	Omitted	Appropriate models		Inappropriate models	Omitted
	Correct	Incorrect			Correct	Incorrect		
Sharing	19,44	37,5	40,28	2,78	38,89	21,53	32,64	6,94
Grouping	28,24	28,24	42,13	0,93	38,89	22,69	33,80	4,62

Table 5.9: Summary of pre- and post-tests based on division problem type

Pre-test data shows that division as grouping items had higher success rate than division as sharing items. For division as grouping items 28,24 % of responses were correct and 19,44 % responses were correct for division as sharing word problems. Studies show that learners struggle mostly with division as sharing problem situations as it requires learners to find how many there is in each group/share, which learners can struggle with when using unit counting or skip counting strategies as they do not know which sequence to use (Carpenter *et al*, 1999). This was also evident with the number of incorrect answers to division as sharing word problems which had 37,5 % of responses with incorrect solutions and division as grouping word problems had 28,24 % of incorrect solutions responses. Inappropriate model responses were 42,13 % for division as grouping word problems and 40,28 % for division as sharing word problems. The high number of inappropriate model responses suggest that learners struggled to make sense of both categories of division word problems.

For the post intervention lessons there were good improvements in both division types of word problems as learners were more confident to solve both problem types. This was reflected by both word problems having 38,89 % of correct responses. Division as sharing

items correct responses had doubled in comparison to pre-test performance and division as grouping items correct responses increased by 10,65 percentage points. Appropriate models with incorrect answers declined for both problem types, division as sharing items declined by 15,97 percentage points to 21,53 % and division as grouping items declined by 5,55 percentage points to 22,69 %. The large increases in correct responses and decline in incorrect responses for division as sharing items suggest that the intervention lessons did equip learners with some of the necessary skills to perform calculations for division as sharing problem situations. Inappropriate model responses for division as grouping items declined by 7,64 percentage points to 33,80 % and division as sharing items declined by 8,33 percentage points to 32,64 %. Small differences between division as sharing and division as grouping items shows that learners have gained a better understanding of both problem types and have gained confidence in dealing with both problems situations.

5.4.2 Findings and analysis based on number order consistency of division items

Facility band items findings and analysis pointed that the order of numbers in division items may be one of the factors that influenced learners' performance. In this section I provide findings and analysis of division items based on order consistency.

Five of the division items had consistent number order and the other five division word problems had inconsistent number order. Consistent order word problems, in paraphrased form, were:

- Q2: 56 children getting into teams of 8 children each.
- Q7: 180 muffins shared equally on 12 plates.
- Q8: 96 people travelling in cars that carry 6 people each.
- Q11: Joey bought 108 m wire, 6 times more than Peter's wire.
- Q17: 252 apple trees, 12 trees in each row.

Inconsistent order word problems were:

- Q3: 7 same sized bags of sugar is 63 kg.
- Q10: 8 rows of pies, 128 pies baked.
- Q13: 15 pencils per box, 195 pencils altogether.
- Q16: Denozo cycles at 14 km per hour, how long to cycle 84 km.
- Q18: Red hat R15, blue hat R90. How many times more is blue hat?

Overall results summarised in Table 5.10 indicate that learners performed better in consistent order division items than inconsistent order division items in both the pre- and post-tests.

Number order	Pre-test (%)				Post-test (%)			
	Appropriate models		Inappropriate models	Omitted	Appropriate models		Inappropriate models	Omitted
	Correct	Incorrect			Correct	Incorrect		
Consistent	28,89	34,44	35,0	1,67	43,33	22,22	31,11	3,33
Inconsistent	20,56	29,44	48,33	1,67	34,44	22,22	35,56	7,78

Table 5.10: Summary of the pre- and post-tests results based on order consistency of numbers of division items

For the pre-test, division items with consistent number order showed 63,33 % of the responses with appropriate model representations and 35,0 % of the responses with inappropriate models. Inconsistent number order items had 50,0 % of the responses with appropriate models and 48,33 % of the responses had inappropriate models. This suggests that learners struggled more with making sense of inconsistent number order word problems as the numbers in the word problems were in inverted order compared to the symbolic version. Thus, this resulted in learners interpreting the division word problem as multiplication and/or additive word problems. Correct responses for consistent items were 28,89 % and for inconsistent items 20,56 % were correct. Daroczy *et al* (2015, p.4) argues that 'the correspondence between the order in which numerical data is presented in the word problem and the order in which it can be used to solve it has been shown to be a major predictive variable'. Therefore, the order in which the numerical data is presented in a word problem is important.

In the post-test there was a positive shift in learners' performance. Learners seemed to be more equipped to deal with inconsistent number order items. However, learners remained stronger in dealing with consistent number order items than inconsistent number order items. Inconsistent number order items showed a 13,88 percentage points increase in correct responses, resulting in 34,44 % of responses to inconsistent items being correct. Appropriate models with incorrect responses of inconsistent order items declined from 29,44 % for pre-test to 22,22 % for the post-test, a decline by 7,22 percentage points. Inappropriate model of inconsistent order items declined by 12,77 percentage points to 35,56 %, suggesting that learners were making better sense of inconsistent number order items. Consistent number order items correct responses increased by 14,44 percentage points to 43,33 % responses post intervention lessons and inappropriate models declined by 3,89 percentage points to 31,11 %

responses. Appropriate models with incorrect responses of consistent order word problems decreased by 12,22 percentage points to 22,22 % responses, which possibly means that learners were using more efficient strategies to perform calculations. These findings on consistency of number orders suggests that the order of numbers in division problem situations has an impact on learners' sense making of word problems. Teaching and learning division word problems should consider exposing learners to both number order problem situations.

5.5. Mathematisation approaches underlying changes in performance

Overall data from previous sections highlighted that there were positive gains and shifts in strategies learners used prior and post intervention lessons. In this section I discuss strategies underlying changes in performance for both tests. Figure 5.4 (below) provides overall strategies for all fifteen multiplicative word problems in pre- and post-tests.

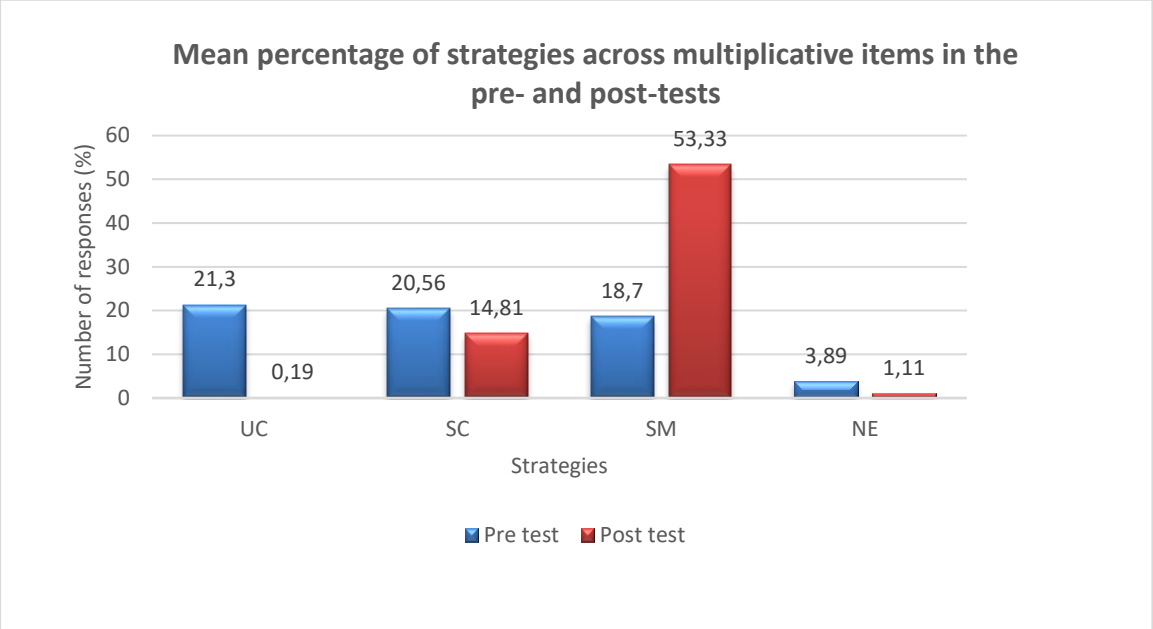


Figure 5.4: Overall strategies in the pre- and post-tests across all fifteen multiplicative items

Figure 5.4 illustrates that there were positive shifts in learners' mathematisation strategies from pre- to post-tests. Responses with appropriate models of multiplicative problem situations (horizontal mathematisation) were 64,44 % for the pre-test, with four strategies used by learners. These strategies were classified as unit counting (UC), skip counting (SC), symbolic strategy (SM) and solutions with no evidence of working out (NE). Note, as mentioned in Chapter 2 (page 9), in a case whereby a learner used ratio table to work out the answer: if the learner listed all multiples in order, I coded the learner's response as skip

counting strategy (SC). However, if a learner used compressed number relationships e.g. using halving, doubling, multiples of 5 or 10 etc, I coded the learner's response as symbolic strategy (SM).

Pre-test results in Figure 5.4 indicate that unit counting and skip counting were the most popular strategies learners used with 21,30 % (115/540) responses and 20,56 % (111/540) responses, respectively. Appropriate model responses which used symbolic strategies were 18,70 % (101/540) and responses with no evidence of working out were 3,89 % (21/540). For this study, responses with no evidence of working out were assumed to have the appropriate models as the solutions were correct or closer to the correct answer. Of the responses with appropriate model representations of multiplicative problem situations, 37,36 % (130/348) of the responses had correct answers and 62,64 % (218/348) of the responses had incorrect answers. A high number of incorrect answers suggested that there was a need for multiplicative reasoning intervention lessons to help learners use efficient strategies to calculate solutions.

The overall post-test results for multiplicative items show that there were substantial shifts towards more efficient strategies. Responses with appropriate models of multiplicative problem situations increased by 5,0 percentage points to 69,44 % (375/540). In the post-test there was a greater shift in using more formal strategies. Symbolic strategies were the most favoured strategy that learners used, which were more than double the pre-test symbolic strategies responses. Symbolic strategies responses were at a high 53,33 % responses (288/540). Use of skip counting strategies declined by 5,75 percentage points to 14,81 % of the responses (80/540). Use of unit counting strategies decreased from 21,30 % (115/540) for pre-test to 0,19 % (1/540) of the responses for post-test. The great decline in use of unit counting and skip counting strategies suggests that learners had shifted to using more efficient strategies, which saw an increase in use of symbolic strategies. The increase by more than double in using correct symbolic strategies responses for post-test may be attributed to intervention lessons as the focus of intervention lessons was on using ratio tables to solve multiplicative problem situations and improving fluency in multiplication and division facts.

There were improvements in both horizontal mathematisation of multiplicative problem situations as well as in vertical mathematisation. Of the 69,44 % responses (375/540) with appropriate models of multiplicative situations, 59,47 % responses (223/375) were able to get correct solutions and 40,53 % of the responses (152/375) were incorrect solutions.

The overall results reveal strong gains for multiplicative items. Strong gains suggest that learners benefitted from intervention lessons as the focus on arrays and ratio models allowed learners to explore and develop different strategies of multiplicative reasoning in the post-test, as can be seen in Figure 5.4. Results also suggest that learners were equipped with skills to identify different problem types as inappropriate models decreased and learners developed fluency in rapid recall of multiplication and division facts.

Upon a closer look at mathematisation data, pre-test results revealed that the use of strategies differed between multiplication and division items. In the pre-test, multiplication items had a high number of skip counting responses, whereas unit counting was the popular strategy learners used for division items. In the next sub-sections I look at strategies for multiplication and division items individually.

5.5.1 Mathematisation approaches for multiplication items

Findings and analysis in previous sections revealed that learners were able to make better sense of multiplication items than division items, and also in performing correct calculations. Figure 5.5 shows that skip counting and symbolic strategies were the most popular strategies for solving multiplication items in both the pre- and post-tests. Figure 5.5 presents the summary of strategies learners used for multiplication items in both tests.

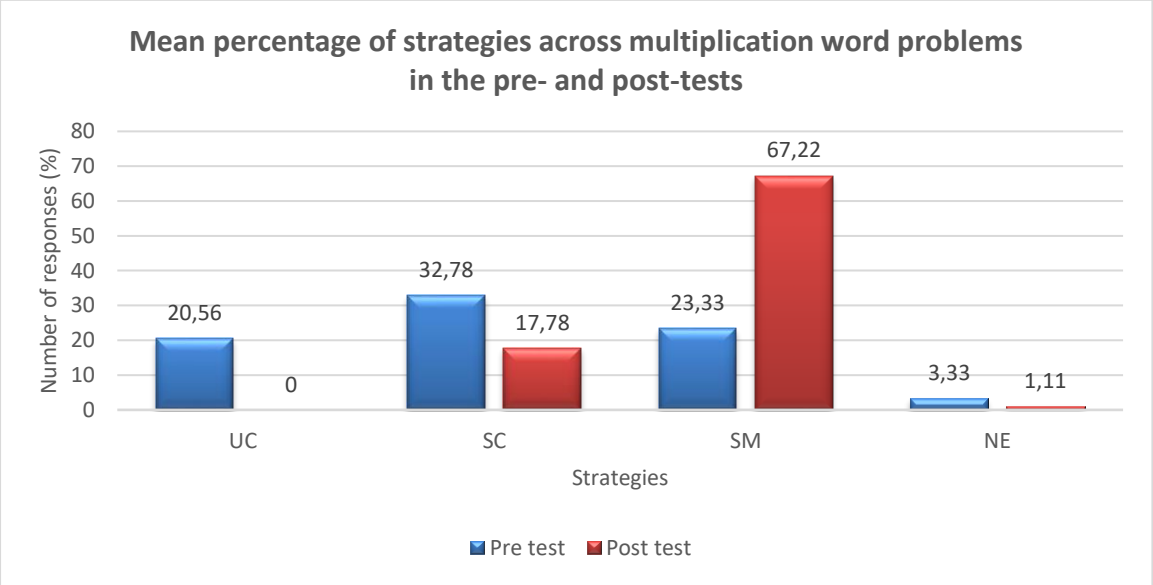


Figure 5.5: Summary of strategies for multiplication items in the pre- and post-tests

In the pre-test skip counting strategies were most popular strategy used, 32,78 % (59/180) of the multiplication items responses used skip counting. Symbolic strategies responses were

23,33 % (42/180) and unit counting responses were 20,56 % (37/180). Responses with no evidence of working out were 3,33 % (6/180). With many responses able to horizontally mathematise multiplication items, responses with correct products were 29,29 % (41/140) and the remaining 70,71 % (99/140) of appropriate model responses were incorrect answers. Table 5.11 provides strategies for each multiplication items in the pre-test.

Question	**Inappropriate model (#)**	**Omitted (#)**	**Appropriate models with correct answers (#)**				**Appropriate models with incorrect answers (#)**			
			UC	**SC**	**SM**	**NE**	**UC**	**SC**	**SM**	**NE**
Q1 (8×19)	2	1	5	3	2	0	18	3	1	1
Q5 (12×11)	9	0	3	2	1	1	5	7	7	1
Q6 (8×6)	4	1	3	6	7	0	2	5	7	1
Q.9 (12×18)	8	0	0	3	1	0	1	17	6	0
Q.14 (12×19)	11	0	0	2	2	0	0	11	8	2
Total (180)	***34***	***2***	***11***	***16***	***13***	***1***	***26***	***43***	***29***	***5***

Table 5.11: Strategies used for multiplication word problems in the pre-test

Results in Table 5.11 reveal that unit counting was popular at the beginning of the test, as learners dealt with more multiplication word problems there was a decline in the number of responses using unit counting. Question 1 (19 × 8) had 63,89 % of responses (23/36) using unit counting, whereas Question 14 (12 × 19) had no unit counting responses. Use of unit counting for multiplication word problems mostly had incorrect products. Of the responses which used unit counting, 70,27 % of responses (26/37) had incorrect products and 29,73 % of responses (11/37) had correct products. Common errors with unit counting responses included incorrect model representation of multiplication situations, incomplete working out especially with bigger number ranges and incorrect products while the model representation is correct. Figure 5.6 is an example of incorrect product with correct model representation of problem situation. The learner represented 19 bags of mangoes, with each bag containing 8 mangoes (19 × 8). The model representation was correct, but the solution written was incorrect. The learner obtained a product of 162 instead of 152.

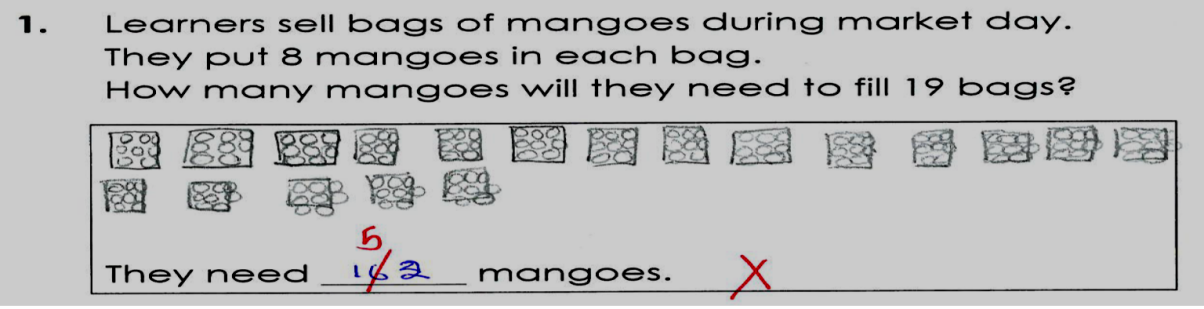


Figure 5.6: Example of incorrect solution using unit counting – pre-test (Learner B)

As mentioned earlier, skip counting and symbolic strategies were most popular strategies used. Over 50 % (101/180 collectively) of multiplication items responses used skip counting and symbolic strategies. Table 5.11 shows that both strategies were mostly used for four multiplication word problems, except Q1. Although skip counting and symbolic strategies were mostly used for responses, data results also reveal that the use of skip counting and symbolic strategies for pre-tests had more incorrect products than correct products.

Of the 59 responses that used skip counting strategies, 27,12 % of responses (16/59) had correct solutions and 72,88 % of responses (43/59) were incorrect solutions. Common errors with skip counting responses included incorrect multiplication facts, counting errors and learners not considering the number itself as a multiple of the number. Figure 5.7 and 5.8 are examples of common errors with skip counting responses.

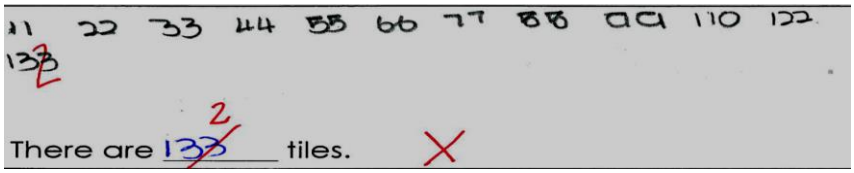


Figure 5.7: Example of counting error using skip counting for Q5 – pre-test (Learner E)

Figure 5.7 shows an example of a learner with counting error of one. First 10 multiples of 11 were correct, the learner made an error with the 11th multiple of 11. There was a counting error by one, which signals that the learner skip counted by 12 instead of 11. This resulted in 11 × 11 as 122 instead of 121. This error was common with other word problems, which can possibly be attributed to a counting error by one. The counting error by one has been identified by Hansen (2011) as a common error in additive calculations. The counting error is usually caused by the learner counting starting number twice (Hansen, 2011).

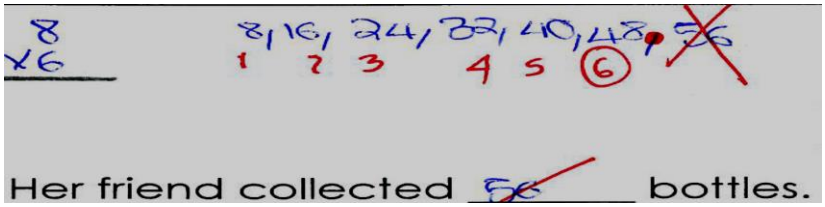


Figure 5.8: Example of misconception of a multiple of a number using skip counting Q6 – pre-test (Learner B)

Figure 5.8 shows that the learner (Learner B) was able to skip count using 8 as all multiples of 8 were correct. However, it appears that the learner does not consider 8 itself as a multiple of 8. It appears that the learner may have considered 16 as the first multiple of 8.

Symbolic strategies were second popular strategy for multiplication word problems in the pre-test, as shown on Figure 5.5 and Table 5.11. Most of the responses that used symbolic strategies, especially column method, had incorrect products. Of 42 responses which used symbolic strategies, 30,95 % of responses (13/42) were correct products and 69,05 % responses (29/42) were incorrect products. Performance was higher when using the symbolic method with lower number ranges, as the number ranges increased there were more incorrect solutions. This was common with responses that used column method. Table 5.11 also shows that two-digits by two-digits multiplication word problems had more incorrect products. Haylock (2010) highlights that an increase in number range when using column method has potential for errors and confusion. Common errors identified in learners' responses included incorrect multiplication facts, place value misconceptions and difficulties with executing column method procedures.

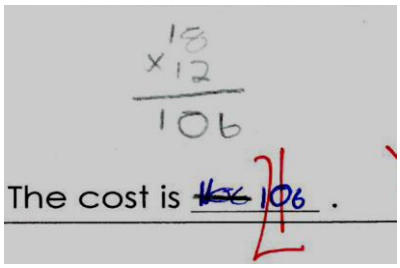


Figure 5.9: Example of incorrect solution using column method

Figure 5.9 response suggest that the learner may possibly not understand the column multiplication algorithm. It appears that the learner understands the purpose of the algorithm, however struggled with performing the algorithm. The error in the learner's response shows that the learner has overgeneralised the rules of column method which may possibly be caused by the learner learning a 'rule' and then applying it inappropriately (Hansen, 2011).

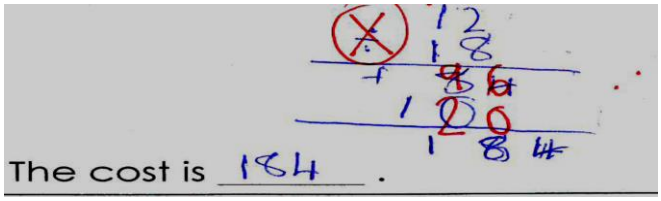


Figure 5.10: Example of partial idea of column method multiplication – pre-test (Learner S)

The response in Figure 5.10 shows that the learner (Learner S) has partial understanding of multiplying two-digits by two-digits. Learner was able to multiply 12 by 8 but had an incorrect product of 84. Then the learner multiplied 10 by 10, instead of 12 by 10. Learner then found the sum of two products. In this case the learner had an idea of zero as a place holder when moving to multiplying tens digits, although it was not written but shown through leaving space in unit column. Ma (2010) also mentions that most learners often leave out the place holder. Leaving out place value holder was also a common error in learners' responses. The error in place value, multiplying 10 by 10 instead of 12 by 10 is caused by failure to understand the position of a digit determines its value (Hansen, 2011). Hansen (2011) argues that this error occurs because learners are introduced to standard algorithms they do not understand, and they do not have a clear understanding of place value.

There was also a common counting error in learners' responses when using symbolic strategies. Figure 5.11 shows a counting error by 1 (Hansen, 2011). In Figure 5.11 the learner used expanded brackets method of multiplication, which was executed correctly, and multiples were also correct. The learner got a sum of 226 instead of 216.

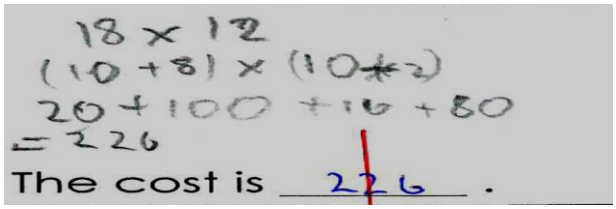


Figure 5.11: Example of a counting error using symbolic method – pre-test (Learner N)

Overall post-test results from multiplication word problems on Figure 5.5 indicated that there was a greater shift towards using formal strategies and decline in responses with incorrect models of multiplication items. None of the responses used unit counting. Symbolic strategies were most common strategies used by learners in the post-test. Symbolic strategies increased to 67,22 % of the responses (121/180) and skip counting strategies responses decreased to 17,78 % responses (32/180). Responses with no evidence of working out declined to 1,11 % (2/180). There was a strong gain of responses with correct products in the post-test, correct

responses in the post-test increased by more than double the pre-test responses. Improved performance and significant shift in using symbolic strategies show that more learners were able to progressively mathematise the multiplication word problems. Table 5.12 below provides detailed strategies for each multiplication word problem in post-test.

Question	**Inappropriate model (#)**	**Omitted (#)**	**Appropriate models with correct answers (#)**				**Appropriate models with incorrect answers (#)**			
			UC	**SC**	**SM**	**NE**	**UC**	**SC**	**SM**	**NE**
Q1 (8×19)	4	1	0	1	16	0	0	3	11	0
Q5 (12×11)	1	2	0	6	13	0	0	4	10	0
Q6 (8×6)	1	5	0	7	11	0	0	5	6	1
Q.9 (12×18)	2	0	0	3	12	0	0	3	16	0
Q.14 (12×19)	8	1	0	0	13	1	0	0	13	0
Total (180)	***16***	***9***	***0***	***17***	***65***	***1***	***0***	***15***	***56***	***1***

Table 5.12: Mathematisation approaches for multiplication items in the post-test

The post-test data in Table 5.12 indicate that there was a decline in use of unit counting and skip counting strategies. Skip counting strategies declined by 15,0 percentage points to 17,78 % (32/180). Of the 32 responses which used skip counting strategies, 53,13 % (17/32) were correct responses and 46,87 % (15/32) were incorrect responses. Skip counting strategies were mostly used for one digit by one-digit multiplication, whereas for 12 × 19 word problem no skip counting strategies were used. Common errors in learners' responses when using skip counting strategies included incorrect multiplication facts.

Post-test showed a significant shift in strategies, learners were more confident to use symbolic strategies such as ratio tables with compressed number relations, lattice methods and column multiplication and division. Symbolic strategies responses increased by about three times the number of responses in comparison to the pre-test. Of the 121 symbolic strategies responses, 53,72 % of the responses (65/121) were correct responses and 46,28 % (56/121) of the responses were incorrect solutions. It appears that number ranges of problem situations may have resulted in variation in success of symbolic strategies, especially when using the ratio tables with compressed number relations. Two digits by two digits multiplication word problems had more incorrect responses than the lower number ranges, as

it can be seen in Table 5.12. Incorrect responses showed evidence of incorrect multiplication facts and errors with mathematical procedures of multiplication.

The emergent use of the lattice method post intervention lessons was because lattice method was introduced and developed during normal mathematics lessons as part of the curriculum coverage. Figure 5.12 below shows post-test response of the learner (Learner N) to Q9. Prior to intervention the learner had an incorrect solution (see Figure 5.11 - page 82), post intervention the use of lattice method enabled the learner to progressively mathematise the word problem.

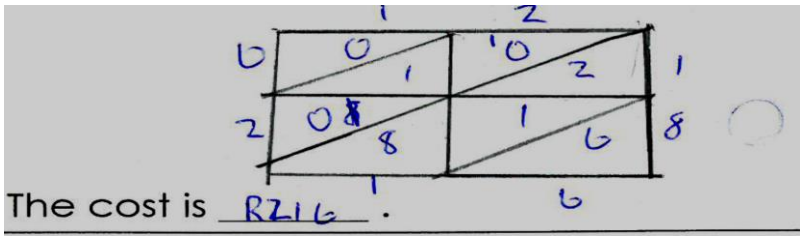


Figure 5.12: Example of lattice method responses in the post-test (Learner N)

Symbolic strategy responses which used column method also improved. Learner S had incorrect solution in the pre-test (see Figure 5.10 – page 82), they used column method as they had partial understanding of column method and incorrect multiplication facts. The learner had overgeneralised the rule of column multiplication (Hansen, 2011). Post-test response of the learner (Figure 5.13) shows the correct response of Q9. The response shows that the learner fully understands the distributive aspect of column method, place value and has correct multiples.

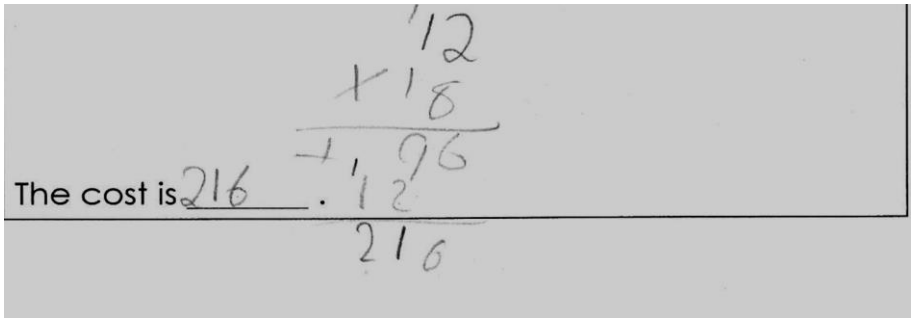


Figure 5.13: Example of improved column response in post-test (Learner S)

With improvements in learners' performance in using symbolic strategies, it is important to note that there were still responses with incorrect multiplication facts and errors with mathematical procedures of multiplication.

Introduction and development of ratio models may have equipped learners with fluency in rapid recall of multiplication facts and work with key representations to help them be more efficient. The use of triple cards at the beginning of the lessons may have assisted learners with rapid recall of multiplication facts, so are rapid recall tasks at the end of the lessons. Figure 5.14 shows the learner's post-test (Learner B) response to Q1. Pre-test the learner had used unit counting (see Figure 5.6 – page 80) which had incorrect product. In the post-test the learner used ratio table and was able to get the correct answer. The learner used doubling, multiplication facts of 10 and 15 to obtain the product of 19 and 8.

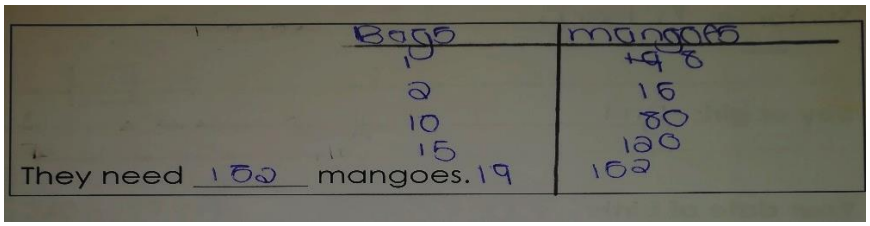


Figure 5.14: Example of ratio table with compressed number relations for Q1 in post-test (Learner B)

Figure 5.15 below is an example of improvement in learner response. The learner (Learner E) used skip counting strategy in the pre-test (see Figure 5.7 – page 80). The response had counting error in one of multiplication facts of 11 (11 × 11), resulting in incorrect product. In the post-test the learner used ratio table, they were able to get a correct product. Learner used multiplication facts of 5 and 10/doubling to help get the product of 12 and 11.

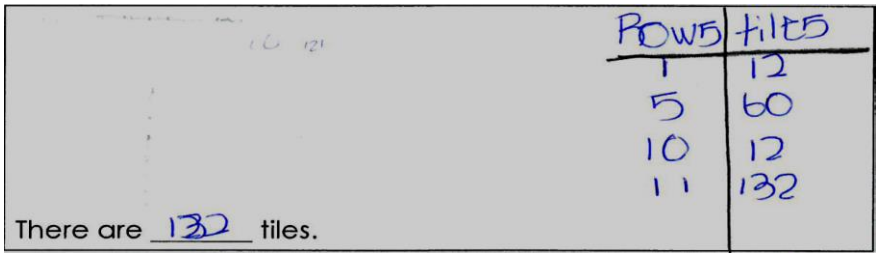


Figure 5.15: Example 2 of using ratio table with compressed number relations for Q5 in post-test (Learner E)

Overall post-test results indicated greater shift in mathematisation approaches learners used as learners were able to better make sense of multiplication items and use more efficient strategies.

5.5.2 Mathematisation approaches for division word problems

Findings and analysis in the previous sections highlighted that learners struggled with making better sense of division items and perform division calculations, in comparison to multiplication items. Unlike multiplication items, for division items the unit counting strategies were the most common strategy learners used in the pre-test. In the post-test for division items there was a greater shift towards using symbolic strategies. Figure 5.16 presents the summary of strategies used for division items in both tests.

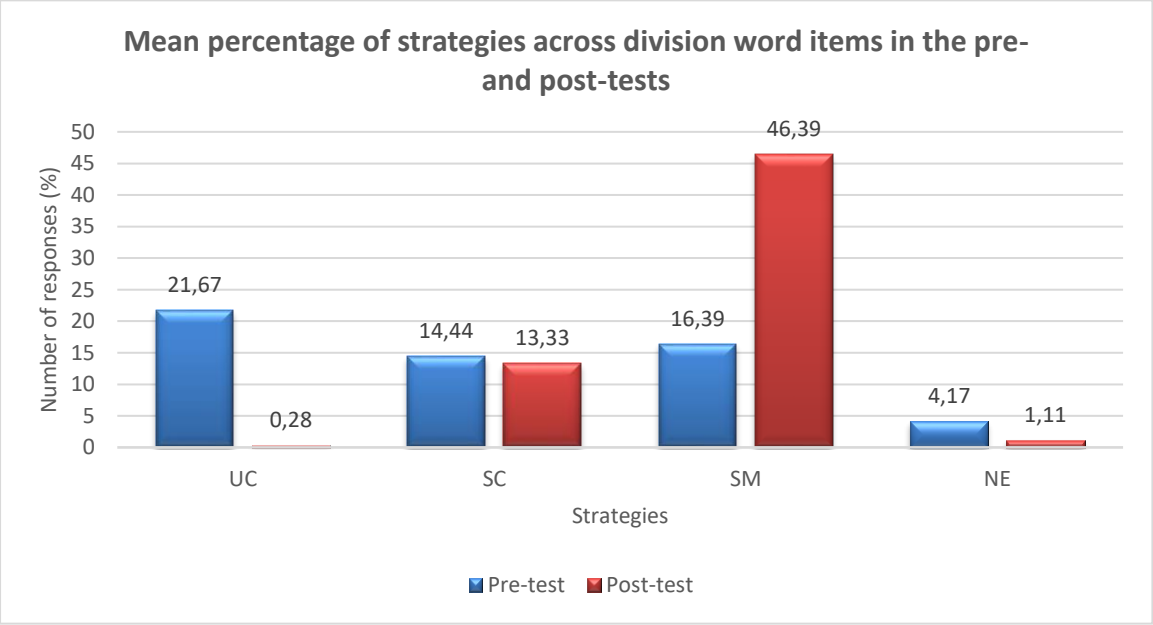


Figure 5.16: Summary of strategies for division items in the pre- and post-tests

Pre-test data reveals that unit counting was most favoured strategies with 21,67 % of responses (78/360) using unit counting. Symbolic method strategies – column method (long division) was the second most favoured strategy with 16,39 % (59/360) of division word problems using the strategies. Skip counting responses were 14,44 % (52/360) and responses with no evidence of working out were 4,17 % (15/360).

Question	**Inappropriate model (#)**	**Omitted (#)**	**Appropriate models with correct answers (#)**				**Appropriate models with incorrect answers (#)**			
			UC	**SC**	**SM**	**NE**	**UC**	**SC**	**SM**	**NE**
Q2 (56÷6)	3	0	11	5	7	1	7	0	1	1
Q3 (63÷7)	12	2	4	1	0	3	7	3	2	2
Q7 (180÷12)	7	1	2	2	2	2	12	3	5	0

Q8 (96÷6)	3	0	6	6	5	1	6	5	3	1
Q10 (128÷8)	9	0	6	2	4	0	4	4	6	1
Q11 (108÷8)	30	1	0	0	0	0	1	1	3	0
Q13 (195÷15)	12	0	5	1	6	1	2	8	1	0
Q16 (84÷14)	26	1	1	0	0	0	2	3	3	0
Q17 (252÷12)	20	1	0	1	0	1	2	4	7	0
Q18 (90÷15)	28	0	0	0	2	1	0	3	2	0
Total (360)	***150***	***6***	***35***	***18***	***26***	***10***	***43***	***34***	***33***	***5***

Table 5.13: Strategies for division word problems in the pre-test

Table 5.13 reveal that unit counting was used throughout for nine division items except Q18, which was misinterpreted by a great number of learners (28/36). Unit counting was regularly used for the first five division items, which were mostly two-digits by one-digit division items. It appears that there was no correlation with the use of unit counting with number ranges. Of 78 responses which used unit counting for division word problems, 44,87 % of responses (35/78) were correct solutions and 55,13 % of responses (43/78) were incorrect solutions. Incorrect solutions usually had incomplete unit counting strategies and incorrect set up of problem situation (misinterpretation of the type of division problem situation). Figure 5.17 shows an example of incorrect set up of unit counting strategy (for Q2). Q2 required learners to divide 56 children into teams of 8 members. The learner (Learner Q) set up 8 teams, instead of creating teams of 8 members. Although the response does not directly model the problem situation presented, the learner represented the division as grouping problem (56 learners divided into 8 teams) instead of division as sharing (56 learners divide into teams of 8 members). Similar findings of incorrect diagrammatic representation model were also found in Hansa (2017)'s study, where learners had inaccurate diagrams drawn for division word problems. The learner's solution in Figure 5.17 also suggest that the learner does not understand the nature of division as equal sharing, as their sharing was not done equally (Freitag, 2014).

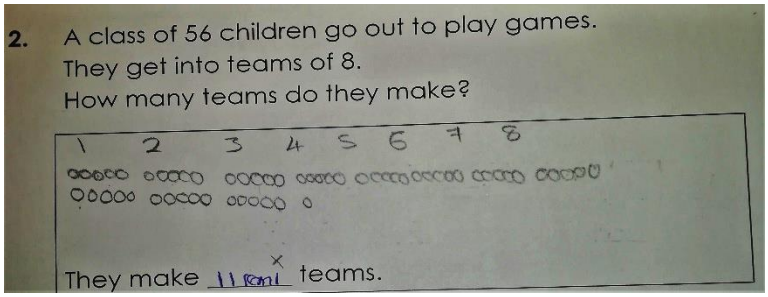


Figure 5.17: Example of incorrect set up of division word problem using unit counting - pre-test (Learner Q)

The pre-test results for symbolic strategies responses show that 44,07 % of responses (26/59) were correct and 55,93 % of responses (33/59) were appropriate models with incorrect solutions. Incorrect symbolic strategies responses had common errors of incorrect multiplication and division facts, and difficulties with executing the standard long division algorithm. The use of standard long division algorithm is prone to errors as it involves several steps which learners may find complex (Freitag, 2014; Haylock, 2010). Errors in standard long division were caused by learners overgeneralising the rule of long division. Hansen (2011) goes on to say that learners often overgeneralise rules because they often want to perform procedural calculations by following poorly understood rules.

Skip counting responses with incorrect solutions were 65,38 % (34/52) and responses with correct solutions were 34,62 % (18/52). Common errors with skip counting strategies included incorrect multiplication and division facts. Responses with no evidence of working out had more correct solutions than incorrect solutions. Of the responses, 66,67 % showed no evidence of working out were correct (10/15) and 33,33 % of responses (5/15) had incorrect solutions.

Moving on to post-test results, there were good shifts in strategies for division items. As shown in Figure 5.16 the use of unit counting declined to 0,28 % of responses (1/360). Skip counting responses also experienced a decline to 13,33 % responses (48/360). Symbolic strategies responses increased to 46,39 % (167/360). Greater shifts to using symbolic strategies and decline in unit counting strategies suggests that learners were better equipped to use efficient strategies. Responses with no evidence of working out were 1,11 % (4/360), which possibly suggest that learners were now confident to perform division calculations rather than just writing answers.

Question	Inappropriate model (#)	Omitted (#)	Appropriate models with correct answers (#)				Appropriate models with incorrect answers (#)			
			UC	SC	SM	NE	UC	SC	SM	NE
Q2 (56÷6)	4	2	0	3	16	0	1	1	9	0
Q3 (63÷7)	4	6	0	4	10	1	0	4	7	0
Q7 (180÷12)	8	0	0	1	19	0	0	2	6	0
Q8 (96÷6)	3	0	0	4	16	1	0	2	10	0
Q10 (128÷8)	8	1	0	0	17	1	0	1	7	1
Q11 (108÷8)	27	3	0	2	1	0	0	0	3	0
Q13 (195÷15)	8	3	0	3	8	0	0	5	9	0
Q16 (84÷14)	12	4	0	9	5	0	0	2	4	0
Q17 (252÷12)	14	1	0	0	15	0	0	3	3	0
Q18 (90÷15)	32	0	0	2	2	0	0	0	0	0
Total (360)	***120***	***20***	***0***	***28***	***109***	***3***	***1***	***20***	***58***	***1***

Table 5.14: Table of strategies for division word problems in post-test

The post-test results in Table 5.14 show that there was a greater shift towards using symbolic strategies and most of the symbolic strategies responses were correct. Of 167 symbolic strategies responses, 65,27 % of responses (109/167) were correct answers and 34,73 % of responses (58/167) were appropriate models with incorrect answers. The response to Q2 in the post-test shown in Figure 5.18 shows correct answer, which the learner (Learner Q) could not represent correctly using unit counting in pre-test (see Figure 5.17 page 88). For the post-test the learner shifted to using ratio table with compressed number relations which was more efficient and showed better sense making of the word problem. The learner was able to use multiples of 3, 5 and 7. Learner's multiples show fluency in rapid recall of multiplication facts and shows progressive mathematisation in learner's modelling of the problem situation.

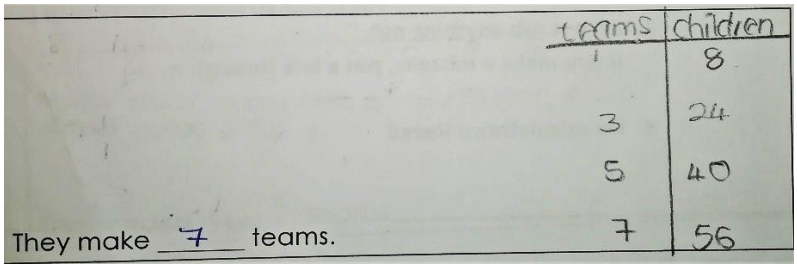


Figure 5.18: Example 3 of correct ratio table with compressed number relations response in post-test (Learner Q)

Figure 5.19 is an example of the learner who used unit counting unsuccessfully for Q2 in pre-test. Post-test the learner shifted to using the standard long division algorithm, which was introduced and developed during normal mathematics curriculum coverage lessons. The procedure was carried out successfully by the learner and both the model and procedure of working out (strategy) were correct. The post-test response also shows better understanding of the problem situation, given that the pre-test response shows that the learner interpreted the word problem as a grouping problem situation instead of a sharing problem situation. That is, in the pre-test the learner created 8 teams with 7 members each, whereas the problem required learners to find the number of teams if there are 8 members in each team. Post test the learner was able to workout that there are 7 teams with 8 members each.

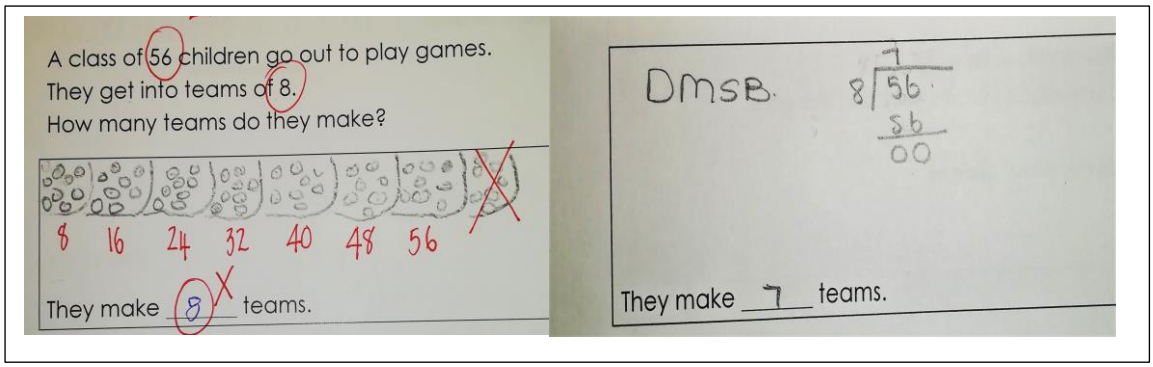


Figure 5.19: Example of a shift in models in pre- and post-tests (Learner T)

Appropriate models with incorrect answers using ratio tables with compressed number relations had evidence of incorrect multiplication and division facts, and difficulties with break down strategy. Learners seemed to struggle with using the breaking down strategy for division as sharing word problems as there were errors with halving and difficulties with breaking down the number if the divisor (number of groups or shares) is an odd number. In a case where learners used column division, common errors remained overgeneralisation of the

rule for long division. However, it must be noted that the use of column division was not as popular as the use ratio table with compressed number relations in the post-test.

With skip counting strategies being second favoured strategy in post-test. Table 5.14 reveals that there were also good improvements in correct responses that used skip counting strategies. Of the 48 skip counting strategies responses, 58,33 % responses (28/48) were correct quotients and 41,67 % responses (20/48) were incorrect quotients. Incorrect quotients were largely caused by incorrect multiplication and division facts.

The overall data from both tests indicate that intervention lessons enabled learners to make better sense of division word problems, improve fluency in rapid recall of multiplication and division facts, and use more efficient strategies to solve division word problems. More learners were able to progressively mathematise the division word problems in the post-test than the pre-test.

5.6. Summary of findings

Findings and analysis of data pointed out that there were substantial gains from multiplicative intervention lessons. Firstly, there were improvements in learners' mathematisation as more learners were able to progressively mathematise multiplicative word problems. For the pre-test, 24,07 % responses were correct solutions to the word problems. In the post-test, it improved to 41,30 % of the responses. Secondly, there were shifts in strategies learners used. In the pre-test, unit counting and skip counting strategies were the most favoured strategies learners used. For the post-test, there was a greater shift towards more efficient strategies. Symbolic and skip counting strategies were most popular strategies, with over 50 % of responses using symbolic strategies. There was an emergent use of ratio tables, which were introduced during intervention lessons. And, learners showed an improved fluency in rapid recall of multiplication and division facts. Data analysis also highlighted that performance for multiplication items differed from division items, which will be the focus of my summary from now on.

For multiplication items the pre-test data revealed that learners' horizontal mathematising skills were stronger than vertical mathematising skills. The difference in mathematising skills was mainly caused by the number ranges of the word problems. Learners performed poorly for multiplication by a two digits multiplier than multiplication by a single digit multiplier. Learners struggled with executing multiplication calculations, especially column method, for higher number ranges. Difficulties with executing column method were mainly caused by

learners overgeneralising the rule of column multiplication as they lack sound understanding of place value and/or not understanding the procedure of column multiplication. Learners' incorrect responses also showed poor recall of multiplication facts, as most multiples were incorrect. Unit counting strategies responses had incorrect representation of multiplication problem situations and incomplete models especially with increased number ranges. Post-test there were significant improvements as the vertical mathematisation skills improved. Learners were more equipped to perform calculations involving multiplication by a two digits multiplier. Correct responses to multiplication by a two digits multiplier increased by three times in the post-test. There was an emergent use of ratio tables with compressed number relations and lattice methods, and learners showed an improved fluency in rapid recall of multiplication facts. None of the responses to multiplication items used unit counting strategy in the post-test.

Division items pre-test data revealed that both horizontal and vertical mathematising skills were poor. Correct responses to division items were less than a quarter of responses, whereas responses with incorrect model representation were 41,67 %. Learners commonly interpreted division items as additive situations and/or multiplication situations. Three factors that seemed to mainly influence performance for division items were; division problem type, language consistency of word problems and the number order consistency of word problems. For the pre-test, division as grouping items had higher success rate than division as sharing items. For division as sharing, learners struggled with deciding on the sequence to use when they used unit counting and skip counting strategies. Learners used trial-and-error to decide on the sequence. It was also noted that both problem types had a high number of inappropriate models for the pre-test. Post intervention lessons there were positive gains, as success rate for both problem types was the same. Learners' responses showed improved sense making of word problem. Language inconsistent division problems had poor performance for both tests. Two division word problems (Q11 and Q18) used the 'times' cue word, which primed learners to use an inappropriate arithmetical operation. For both test learners commonly interpreted the problems as multiplication items. The order of numbers for division items influenced learners' performance. Consistent number order had more appropriate model representation than inconsistent number order items. Learners struggled more with making sense of inconsistent number order word problems as the numbers were in inverted order compared to the symbolic version. Post-test data showed that learners

remained stronger in dealing with consistent number order items, but learners were better equipped to deal with inconsistent number order items.

Strategies for division items improved in the post-test. For the pre-test, unit counting strategies were the most popular strategies learners used. Unit counting strategies responses commonly had incorrect diagrammatic representation of word problems. Learners commonly presented division as sharing problems as division as grouping problems. Symbolic strategies were the second most favoured strategy, but mostly had incorrect solutions. Symbolic strategies responses revealed that learners experienced difficulties with performing standard long division algorithm as learners overgeneralised the rule of long division. There was also evidence of incorrect multiplication and division facts. Post-test there was a decline in use of skip counting and unit counting strategies. Learners were better equipped to use efficient strategies as there was an increase in use of symbolic strategies. However, it was noted that learners experienced difficulties with using the breaking down strategy when working with ratio tables with compressed number relations. Learners seemed to struggle with the concept of halving.

Chapter Six: Conclusion

6.1 Introduction

In this chapter I highlight key findings of my study with respect to the models and strategies used by Grade 6 learners in solving multiplication and division word problems. I also highlight the limitations of the study and offering some recommendations for future research in similar fields of study.

The initial motivation for my study was the poor performance in multiplication and division of Grade 6 learners I have taught. Mathematical knowledge and proficiency studies show that South African learners continuously perform poorly in multiplication and division (DBE, 2013a; 2015; Reddy et al, 2016; Spaull, 2013). Learners I taught often found multiplication and division difficult to understand and were not interested in learning the concepts. Like many South African educators (Barnes, 2005), I used the traditional approach of teaching mathematics as a ready-made system with general applicability to teach multiplication and division concepts. This anti-didactical inversion does not offer learners opportunities to make use of horizontal mathematisation, which later results in learners struggling to find alternative strategies when they forget algorithms. It was due to these issues that I decided to conduct a study on multiplicative reasoning with Grade 6 learners.

Realistic Mathematics Education (RME) was used as a theoretical framework for the study because this approach emphasises that models play a significant role in the process of learning mathematics (Van den Heuvel-Panhuizen & Drijvers, 2014). The RME framework assisted with providing an insight into how learners can be guided to construct models for word problems and the importance of connecting mathematics to learners' everyday world. For learners to understand mathematics there need to be links between mathematics taught and the environment (Gravemeijer & Doorman, 1999). Thus, the use of contextual word problems and models play an important role in sense making process and supporting learners' solution strategies.

The main focus of my study was three-fold, which was: to (i) discover the models and strategies learners are using prior to the intervention lessons, (ii) identify shifts in learners' overall performance in the post-test and (iii) identify models and strategies learners use post intervention lessons.

6.2. Key Findings

The data gathered and analysed from study revealed interesting findings with regards to learners' multiplicative reasoning. Pre-test data revealed that learners used a limited range of strategies to solve multiplicative word problems. Learners' horizontal modelling skills were stronger that vertical modelling skills for multiplication items, but with division items, learners experienced difficulties with horizontally mathematising the items. Over 40 % of division items responses had inappropriate model representation, with learners interpreting the word problems as either additive situations and/or multiplication situations. Skip counting strategies were most favoured strategy for the multiplication items and unit counting strategies were most favoured strategies for the division items. However, the pre-test data revealed a large number of incorrect answers. Symbolic strategies responses for both multiplication and division items had common errors of learners overgeneralising rules of column multiplication or division. Overgeneralisation of rules showed that learners did not understand the importance of place value and the procedural algorithms (Hansen, 2011). An increase in number ranges of multiplication items resulted in learners experiencing difficulties with executing column multiplication. Studies conducted earlier showed that difficulties with executing column multiplication and division as learners overgeneralised the rules of these procedural algorithms (Anghileri, 2006; Juan *et al*, 2016; Ma, 2010). Most learners were also not fluent with recalling multiplication facts, this was evident with responses that used skip counting and symbolic strategies.

Pre-test analysis highlighted that learners experienced more difficulties solving division as sharing problems than division as grouping items. Division as sharing items had a higher number of incorrect calculations because learners struggled with figuring out the counts or sequence to use when using skip counting strategies and/or unit counting strategies. With division items, data also revealed that the order which numbers appear in a word problem influenced learner performance. Learners performed better in consistent number order division items than inconsistent order division items, in both tests. Inconsistent order division items had a high number of inappropriate models in the pre-test. Therefore, the order which numbers appear in the word problems is important (Daroczy *et al*, 2014). Venkat & Mathews' (2018) study with Grade 7 learners also showed that the order in which the numbers appeared had an effect on learners' performance.

Post-test data showed shifts in performance; learners showed improved sense making of multiplicative problem situations and substantial shift in more efficient strategies. There was only one response that used unit counting strategy of all post-test responses. Post-test data revealed emergent use of ratio tables, with many responses to multiplication and division items using the model. The emergent use of ratio models was because of the intervention lessons. The study aimed at developing and exploring learners' multiplicative reasoning, which was done through modelling problem situations and using ratio models. Ratio models assist learners to develop mental strategies for solving rate and proportion problems. Ratio models enable learners to represent two quantities involved at the same time (Askew, 2018). Learners were first introduced to modelling multiplicative situations using double number lines, as learners are familiar with using number lines. Orill & Brown (2012) argue that double number lines act as precursors to ratio tables. The advantage of using ratio table is that it can be used to solve both multiplication and division word problems, which assist learners to understand the inverse relation between multiplication and division. In relation to my study, learners were more able to progressively mathematise multiplication word problems using ratio tables than when solving division word problems. Learners experienced difficulties with using the breaking down strategy for division as sharing problems as there were errors in their halving in most cases.

Learners' post-test responses also showed that learners had gained fluency in multiplication facts as it was evident with ratio models (double number line and ratio table), column multiplication and division, and lattice method responses. Learners were also able to perform better with division as sharing calculations. It was also noted that performance with language inconsistent division problems remained poor in the post-test, as learners struggled with making sense of Q11 and Q18.

Improved mathematising of multiplicative situations and dominance of ratio models suggest that the intervention lessons may have assisted learners with sense making of multiplicative situations, modelling problem situations and using efficient strategies. This resulted in positive gains from multiplicative reasoning intervention lessons.

6.3. Limitations

This study has its limitations. Three main limitations which I encountered were data set, time and research instrument.

This study is exploratory and small scale. The data set used a single group of 36 learners in the same class as the sample. Limitations of the data set were discussed in Section 4.7 (page 51).

The study took place over a period of 6 weeks, with 4 weeks allocated to the intervention lessons. One lesson was conducted per week, for an hour. The time allocated to intervention lessons was not enough to explore and develop different multiplicative situations and wide range of strategies. Intervention lessons were limited to array and ratio models. The study was conducted during 'normal' Mathematics lessons and content was aligned to prescribed CAPS Grade 6 curriculum coverage for Mathematics. This required lessons to be worked around allocated time. With more time, learners would have been able to explore and develop multiplicative reasoning using a wide range of multiplicative situations and strategies.

The findings and analysis were only reliant on learners' written responses from pre- and post-tests. In the process of analysing data; I wished to probe further some learners' responses, especially responses with no evidence of working out. Follow up interviews on learners' responses would have been useful for the study. As mentioned in Chapter 4, at the time I had interest in conducting interviews it was too late to make ethical permission application to conduct interviews.

Although the study is exploratory and uses small scale sample, this study provides an insight on the role that different modelling approaches can play in organising contextual problems through RME approach.

6.4 Recommendations

The findings from the study provide strong motivation for future research on how different strategies may be integrated into teaching and learning multiplication and division. The data from the study suggests that using modelling as a tool for solving word problems can assist learners to make sense of multiplicative problem situations and use efficient strategies to solve word problems.

As noted earlier, this study is part of the WMC-P broader study on multiplicative reasoning in Intermediate Phase (MRIP). Future study on multiplicative reasoning should incorporate the use of control group, where traditional approach of teaching multiplication and division is used in order to have comparable data in the broader study. The intervention needs to take place over a longer period. More time will assist to cover a wider range of multiplicative

situations and explore and develop more consistent use of relational models across these situations. This is important in the middle grades as work on rational numbers and other multiplicative topics come to the fore. The findings in this study suggest that it is possible to assist learners with mathematising multiplicative situations in ways that improve their overall performance in these tasks.

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Appendices

Appendix A: The Lessons

Lesson 1

MRIP Grade 6 Lesson 1

Name: _____

1. Lawan is baking chocolate cupcakes.
He puts 8 cupcakes into each bag.
How many cupcakes will Lawan use to fill 6 bags?

He needs _____ cupcakes.

2. Lawan is baking vanilla cupcakes.
He puts 8 cupcakes into each bag.
How many cupcakes will Lawan use to fill 11 bags?

He will use _____ cupcakes.

3. At sports day children are playing in teams of 8.
How many children are in 15 teams?

There are _____ children.

LESSON 1 **Aims:** Identifying equal groups problems
Working t-tables

Rapid recall	**Approx. 5 mins.**

1. Multiplication triples.
Begin with working with multiplication 'triples': write, say, 6 x 4 = 24 on the board and encourage learners to connect multiplication and division facts:

Twenty-four is six times four
Four times six is twenty-four
Twenty-four divided by six is four
Twenty-four divided by four is six
Six times what is twenty-four?
Four times what is twenty-four?

Repeat for a different triple.

Problem solving	**Approx. 40 mins.**

Learners will work on three problems.
After each problem the solution and how it was found is discussed before going on to the next problem.
Read each problem out and check that learners understand what the problem is about but without giving away what mathematics to do.

Problem 1:

Lawan is baking chocolate cupcakes.
He puts 8 cupcakes into each bag.
How many cupcakes will Lawan use to fill 6 bags?

Make it clear that the same number of cupcakes is always put into each bag.
While the learners are working on the problem, walk around the class and select 2 learners who will share their solution method with the class. Tell these learners that you will be asking them to come to the board to share their thinking.
Look out in particular for any learner's working looking like a **t-table line**:

Bags	Cupcakes
1	8
2	16
3	24
4	32
5	40
6	48

Step 1: Agree on the correct answer
When most learners have solved the problem, ask them to **turn-and-talk**; to turn to the person next to them and try to decide which answer is the correct one.
After a short time, invite learners to explain what they think the correct answer is.
Agree on the correct answer.
Step 2: Share solution methods
Invite the learners, one at a time, to come to the board and explain what they did.
Ask the class if anyone can re-explain what they have just heard.

Work on setting up the t-table.

Bags	Cupcakes
1	8
2	16
3	24
4	32
5	40
6	48

Step 3: Refine solution methods

Once the t-table is on the board, ask learners how they can use it to answer questions like:

How many cupcakes in 4 bags?
How many bags are needed for 24 cupcakes?
Suppose the line went up to 10 bags, how many cupcakes would that be?
Can we use the information on the line to work out how many cupcakes in 12 bags?

As you set up Problem 2, ask learners to try and use a t-table to solve it.
Leave the t-table on the board for learners to refer to.

Problem 2:

Lawan is baking vanilla cupcakes.
He puts 8 cupcakes into each bag.
How many cupcakes will Lawan use to fill 11 bags?

As before, walk around and select the 2 learners who will share their solution methods with the class. Look for good use of the double number line.
Look for anyone working out that 10 bags hold 60 cupcakes and so 11 bags hold 66.

Step 1: Agree on the correct answer

As before, ask learners to **turn-and-talk** and try to decide which is the correct answer before reaching agreement on the correct answer.

Step 2: Share solution methods

Invite the learners you selected to come to the board and explain what they did.
Ask the class if anyone can re-explain what they have just heard.
Work on setting up the t-table.

Bags	Cupcakes
1	8
2	16
3	24
4	32
5	40
6	48
7	56
8	64
9	72
10	80
11	88

Step 3: Refine solution methods

Discuss if there is a way to set up the table without writing out all the rows.

Establish that finding the number of cupcakes in 10 bags is easy, so a shorter table can be drawn.

Bags	Cupcakes
1	8
10	80
11	88

As you set up Problem 3, ask learners to try and use a t-table to solve it

Problem 3:

At sports day children are playing in teams of 8.
How many children are in 15 teams?

As before, walk around and select the 2 or 3 learners who will share their solution methods with the class.

Follow Steps 1, 2 and 3 to end up with this t-table:

Teams	Children
1	8
10	80
5	40
15	120

Linking the problems **Approx. 10 mins.**

Ask learners to turn-and-talk about:

What is the same about the three problems?
How are the problems different?
Can they make up a problem that is like these three problems?

Talk about how all three problems are examples of
Equal groups of objects problems.

Practice/Homework

For practice if there is time, or homework, there are more **equal groups of objects** problems for learners to work through.
One of the six problems is NOT an equal groups problem. After working through all six problems, learners should try and identify this problem and say why it is different.
They should also practice their multiplication bonds.

MRIP Grade 6 Problem practice 1

1. Lawan is baking chocolate cupcakes.
He puts 9 cupcakes into each bag.
How many cupcakes will Lawan use to fill 15 bags?

He needs _____ cupcakes.

2. Lawan is baking vanilla cupcakes.
He puts 7 cupcakes into each bag.
How many cupcakes will Lawan use to fill 12 bags?

He will use _____ cupcakes.

3. At sports day children are playing in teams of 7.
How many children are in 15 teams?

There are _____ children.

4. Lawan is baking chocolate cupcakes.
He puts 8 cupcakes into a bag.
He put another 15 cupcakes into the same bag.
How many cupcakes are in the bag?

There are _____ cupcakes.

5. Nomonde is putting out chairs around tables.
She puts 9 chairs at each table.
There are 12 tables.
How many chairs will Nomonde use?

She will use _____ chairs.

6. One of the problems above is NOT an equal groups problem.
Which problem is not an equal groups problem? Explain how you know.

Question _____ is not an equal groups problem.

I know this because

MRIP Grade 6 Rapid practice 1

8 x 5 =	80 ÷ 8 =
8 x 10 =	80 ÷ 10 =
8 x 11 =	40 ÷ 8 =
8 x 9 =	40 ÷ 5 =
9 x 8 =	48 ÷ 8 =
10 x 8 =	48 ÷ 6 =
5 x 8 =	40 ÷ 2 =
2 x 8 =	16 ÷ 8 =
4 x 8 =	32 ÷ 8 =
8 x 8 =	64 ÷ 8 =
12 x 8 =	40 ÷ 8 =
8 x 12 =	80 ÷ 8 =
8 x 3 =	120 ÷ 8 =
7 x 8 =	32 ÷ 8 =
8 x 10 =	32 ÷ 4 =

MRIP Grade 6 Lesson 2
Name: _____

1. Hamsa is putting out chairs.
She puts 7 chairs around each table.
How many chairs will she need for 8 tables?

She needs _____ chairs.

2. Hamsa is putting plates on tables.
She puts 7 plates on each table.
She has 56 plates.
How many tables can Hamsa put plates on?

She can put plates on _____ tables.

3. Sameera is putting stickers onto pages in a book.
She puts 5 stickers onto each page.
She has 120 stickers.
How many pages can Sameera fill with stickers?

She can fill _____ pages.

LESSON 2 **Aims:** Identifying equal groups problems
Working with the t-table

Rapid recall **Approx. 5 mins.**

1. Multiplication triples.
Remind the learners about **triples and the relationship between the numbers**.
Write up 6, 7, 42 and cover up the product (42). Ask learners to say the **TWO** number sentence linking the two numbers that they can see, and the answer ('six times seven is 24' and 'seven times six is 24')
Cover up one of the factors (6, say). Ask learners to say **TWO** number sentences linking the two numbers they can see, and the answer ('Forty-two divided by seven is six' and 'seven times what is 42? Six' Repeat for the other factor and different triples.

Problem solving **Approx. 40 mins.**

Learners work on three problems. After each problem the solution and how it was found is discussed before going on to the next problem.
Read each problem out and check that learners understand what the problem is about.
Problem 1:

Hamsa is putting out chairs.
She puts 7 chairs around each table.
How many chairs will she need for 8 tables?

Make it clear that the same number of chairs is always put around each table.
Remind the learners of the t-table and set this blank one up on the board:

Tables	Chairs

Ask 'Where do we put the 7 from the problem? Under "Tables" or under "Chairs"?'
Answer: 'Under chairs.'
Ask 'Why do we put 7 under "Chairs"? Ans: 'Because there are 7 chairs at each table.'
Write the 7 under "Chairs".
Ask 'What do we put under "Tables"?' Answer: 'One.'
Ask 'Why do we put 1 under "Tables"?' Answer: 'Because one table has 7 chairs.'
Complete the first row of the t-table:

Tables	Chairs
1	7

Ask 'Where do we put the 8 from the problem? Under "Tables" or under "Chairs"?'
Answer: 'Under tables.'
Ask 'Why do we put 8 under "Tables"?' Ans: 'Because there are eight tables.'
Write the 8 under "Tables", leaving space between it and the first row.
Ask 'So what do we need to find out?' Ans: 'How many chairs are needed for 8 tables.'
Write a question mark alongside the 8 and under "Chairs".

Tables	Chairs
1	7
8	?

Set the learners off to find the answer. While the learners are working on the problem, walk around the class and select 2 learners to share their solution method with the

class. Tell these learners that they will come to the board to share their thinking.
Look out in particular for any learner using doubling to find the answer:

Tables	Chairs
1	7
2	14
4	28
8	56

Step 1: Agree on the correct answer

When most learners have solved the problem, ask learners to **turn-and-talk**; to turn to the person next to them and try to decide on the correct answer. As a class agree on the correct answer.

Step 2: Share solution methods

Invite the learners, one at a time, to come to the board and explain what they did.
Ask the class if anyone can re-explain what they have just heard.
Work on setting up the t-table using doubles as shown above.

Step 3: Refine solution methods

Once the t-table is on the board, ask how they can use it to answer questions like:

Suppose the line went up to 10 tables, how many chairs would that be?
Can we use the t-table to work out how many chairs for 16 tables?

As you set up Problem 2, leave the t-table on the board for learners to refer to.

Problem 2:

Hamsa is putting plates on tables.
She puts 7 plates on each table.
She has 48 plates.
How many tables can Hamsa put plates on?

NOTE: Although this is very similar to Problem 1, it is a division problem so learners may get confused in setting up the t-table.

Ask the learners to draw the t-table and set this blank one up on the board:

Tables	Plates

Ask 'Where do we put the 7 from the problem? Under "Tables" or under "Plates"?'
Answer: 'Under Plates.'
Ask 'Why do we put 7 under "Plates"? Ans: 'Because there are 7 plates on each table.'
Write the 7 under "Plates".
Ask 'What do we put under "Tables"?' Answer: 'One.'
Ask 'Why do we put 1 under "Tables"?' Answer: 'Because one table has 7 Plates.'
Complete the first row of the t-table:
Ask 'Where do we put the 56 from the problem? Under "Tables" or under "Plates"?'
Answer: 'Under "Plates".'
Ask 'Why do we put 56 under "Plates"?' Ans: 'Because there are 56 plates altogether.'
Write the 56 under "Plates", leaving space between it and the first row.
Ask 'So what do we need to find out?' Ans: 'How many tables are needed for 56 plates.'
Write a question mark alongside the 56 and under "Tables".

Tables	Plates
1	4
?	56

Set the learners off to find the answer.

As before, walk around and select the 2 learners who will share their solution methods with the class. Look out for any learner using, as before, doubling to find the answer and also for any learners who use the answer to the first problem.

Step 1: Agree on the correct answer

As before, ask learners to **turn-and-talk** and try to decide on the correct answer.

Step 2: Share solution methods

Invite the learners you selected to come to the board and explain what they did.
Ask the class if anyone can re-explain what they have just heard.
Work on setting up the t-table.

Tables	Plates
1	7
2	14
4	28
8	56

Step 3: Refine solution methods

Discuss what is similar to question 1 and what is different.
As you set up Problem 3, ask learners to try and use a t-table to solve it

Problem 3:

Sameera is putting stickers onto pages in a book.
She puts 5 stickers onto each page.
She has 70 stickers.
How many pages can Sameera fill with stickers?

As before, set up the initial t-table:
Walk around and select the 2 learners to share their solutions with the class.
Follow Steps 1, 2 and 3 to end up with this t-table:

Pages	Sticker
1	5
10	50
20	100
4	20
24	120

Linking the problems **Approx. 10 mins.**

Ask learners to turn-and-talk about:

What is the same about the three problems?
How are the problems different?
Can they make up a problem that is like these three problems?

Talk about how all three problems are examples of

Equal groups problems

Talk about how 1 was a multiplication problem, and 2 and 3 division problems.

Practice/Homework

For practice or homework, there are more **equal groups** problems to work through. One of the five problems is NOT an **equal groups** problem. After working through all five problems, learners should try and identify this problem and say why it is different. They should also practice their multiplication bonds.

MRIP Grade 6 Problem practice 2

1. Hamsa is putting out cups on tables.
She puts 8 cups on each table.
How many cups will she need for 9 tables?

She needs _____ cups.

2. Corin is putting biscuits on plates.
He puts 8 biscuits on each plate.
He has 88 biscuits.
How many plates can Corin put biscuits on?

He can put biscuits on _____ plates.

3. Val is packing pencils into a box.
She puts 8 pencils into one box.
She puts another 80 pencils into the box.
How many pencils are in the box?

There are _____ pencils in the box.

4. Viren is packing books into boxes.
He puts 10 books onto each box.
He has 120 books.
How many boxes can Viren fill with books?

He can fill _____ boxes.

5. Herman is putting out rows of chairs.
He puts 8 chairs in each row.
There are 12 rows.
How many chairs will Herman use?

He will use _____ chairs.

6. One of the problems above is NOT an equal groups problem.
Which problem is not an equal groups problem? Explain how you know.

Question _____ is not an equal groups problem.
I know this because

MRIP Grade 6 Rapid practice 2

9 x 10 =	90 ÷ 9 =
9 x 5 =	90 ÷ 10 =
9 x 11 =	45 ÷ 9 =
8 x 9 =	45 ÷ 5 =
9 x 8 =	54 ÷ 9 =
10 x 9 =	54 ÷ 6 =
5 x 9 =	50 ÷ 2 =
2 x 9 =	18 ÷ 9 =
4 x 9 =	36 ÷ 9 =
8 x 9 =	72 ÷ 9 =
12 x 9 =	45 ÷ 9 =
9 x 12 =	90 ÷ 9 =
9 x 3 =	135 ÷ 9 =
9 x 6 =	81 ÷ 9 =
9 x 7 =	36 ÷ 9 =

MRIP Grade 6 Lesson 3
Name: _____

1. One mango costs R 9.
Mike buys 8 mangoes.
How much does Mike spend?

He spends R _____ .

2. One cool drink costs R 9.
Constance spends R 72 buying cool drinks.
How many cool drinks did Constance buy?

She bought _____ cool drinks.

3. Emmanuel spends R 96 buying bananas.
He buys 8 bananas.
How many Rand does one banana cost?

One banana costs R _____.

LESSON 3 **Aims:** Identifying equal size measures problems
Working with the t-table

Rapid recall	**Approx. 5 mins.**
Multiplication and division triples. Work on learners rapidly recalling the three numbers in various triples.	
Problem solving	**Approx. 40 mins.**

Learners work on three problems. After each problem the solution and how it was found is discussed before going on to the next problem.
Read each problem out and check that learners understand what the problem is about.

Problem 1:

One mango costs R 9.
Mike buys 8 mangoes.
How much does Mike spend?

Make it clear that each mango costs the same.
Remind the learners of the t-table and set this blank one up on the board:

Mangoes	Cost (Rand)

Ask 'Where do we put the 9 from the problem? Under "Mangoes" or under "Cost"?'
Answer: 'Under "Cost".'
Ask 'Why do we put 9 under "Cost"? Ans: 'Because each mango costs R 9.'
Write the 9 under "Cost".
Ask 'What do we put under "Mangoes"?' Answer: 'One.'
Ask 'Why do we put 1 under "Mangoes"?' Answer: 'Because one mango costs 9 Rand.'
Complete the first row of the t-table:

Mangoes	Cost (Rand)
1	9

Ask 'Where do we put the 8 from the problem? Under "Mangoes" or under "Cost"?'
Answer: 'Under "Mangoes".'
Ask 'Why do we put 8 under "Mangoes"?' Ans: 'Because Mike buys eight mangoes.'
Write the 8 under "Tables", leaving space between it and the first row.
Ask 'So what do we need to find out?' Ans: 'How many Rand Mike paid for 8 mangoes.'
Write a question mark alongside the 8 and under "Cost".

Mangoes	Cost (Rand)
1	9
8	?

Set the learners off to find the answer. Walk around and select 2 learners to share their solutions with the class. Alert them that they will to come to the board.
Look out in particular for any learner using doubling to find the answer:

Mangoes	Cost (Rand)
1	9
2	18
4	36
8	72

Step 1: Agree on the correct answer

Ask learners to **turn-and-talk**; to turn to their neighbor agree on an answer.

After a short time, invite learners to explain what they think the correct answer is.

Step 2: Share solution methods

Invite the learners, one at a time, to come to the board and explain what they did.

Ask the class if anyone can re-explain what they have just heard.

Work on setting up the t-table using doubles as shown above.

Step 3: Refine solution methods

Once the t-table is on the board, ask how they can use it to answer questions like:

How many mangoes can Mike buy with 36 Rand?

Suppose the line went up to 12 mangoes, how much would that cost?

Can we use the t-table to work out how much 16 mangoes would cost?

As you set up Problem 2, leave the t-table on the board for learners to refer to.

Problem 2:

One cool drink costs R97.

Constance spends R 72 buying cool drinks.

How many cool drinks did Constance buy?

Ask the learners to draw the t-table and set this blank one up on the board:

Drinks	Cost (Rand)

Ask 'Where do we put the 9 from the problem? Under "Drinks" or under "Cost"?'

Answer: 'Under "Cost".'

Ask 'Why do we put 9 under "Cost"? Ans: 'Because each drink costs 9 rand.'

Write the 9 under "Cost".

Ask 'What do we put under "Drinks"?' Answer: 'One.'

Ask 'Why do we put 1 under "Drinks"?' Answer: 'Because one drink costs 9 rand.'

Complete the first row of the t-table:

Drinks	Cost (Rand)
1	9

Ask 'Where do we put the 73 from the problem? Under "Drinks" or under "Cost"?'

Answer: 'Under "Cost".'

Ask 'Why do we put 72 under "Cost"?' Ans: 'Because she spends 72 rand altogether.'

Write the 72 under "Cost", leaving space between it and the first row.

Ask 'So what do we need to find out?' Ans: 'How many drinks she bought for 72 rand.'

Write a question mark alongside the 72 and under "Drinks".

Drinks	Cost (Rand)
1	9
?	72

Set the learners off to find the answer.

As before, walk around and select the 2 learners who will share their solution methods with the class. Look out in particular for any learner using, as before, doubling to find the answer. Also look out for any learners who see that they can use the answer to the first problem.

Step 1: Agree on the correct answer

Ask learners to **turn-and-talk** and to decide on the correct answer.

Step 2: Share solution methods

Invite the learners you selected to come to the board and explain what they did.
Ask the class if anyone can re-explain what they have just heard.
Work on setting up the t-table.
Look out for any learners who see that they can uses the answer to the first problem

Drinks	Cost (Rand)
1	9
2	18
4	36
8	72

Step 3: Refine solution methods

Discuss what is similar to question 1 and what is different.
As you set up Problem 3, ask learners to try and use a t-table to solve it

Problem 3:

Emmanuel spends R 96 buying bananas.
He buys 8 bananas.
How many Rand does one banana cost?

NOTE: Although this is very similar to Problem 1, it is a division problem so learners may get confused in setting up the t-table.

As before, set up the initial t-table but this time the 8 does NOT go under the cost. It is the number of bananas – this is the critical point that learners may need help with.

Bananas	Cost (Rand)
1	?
8	96

Walk around and select the 2 learners who will share their solution methods with the class. Look out for any learners who use halving to find the cost of 4 bananas (28 rand), then 2 bananas (14 rand) and 1 banana (7 rand)
Follow Steps 1, 2 and 3 to end up with this t-table:

Bananas	Cost (Rand)
1	12
2	24
4	48
8	96

Linking the problems **Approx. 10 mins.**

Ask learners to turn-and-talk about:
What is the same about the three problems?
How are the problems different?
Can they make up a problem that is like these three problems?
Talk about how all three problems are examples of
Equal size measures problems
Talk about how 1 was a multiplication problem, and 2 and 3 division problems.

Practice/Homework

For practice, or homework, there are more **equal groups or equal size measures** problems. One problem is NOT an **equal groups or equal size measures** problem: learners should say why it is different.
They should also practice their multiplication bonds.

MRIP Grade 6 Problem practice 3

1. One sack of potatoes weighs 7 kg.
Tuhlella buys 12 sacks of potatoes.
How heavy are the sacks altogether?

The weight of all the sacks together is _____ kg.

2. One can of cola costs R 9.
Val spends R 108 buying cans of cola.
How many cans of cola did Val buy?

She bought _____ cans.

3. Trust cycles 112 kilometres.
He cycles for 8 hours.
How many kilometres does Trust cycle in one hour?

He cycles _____ kilometres in one hour

4. Sonny spends R 56 buying bananas.
He spends R 7 on a mango.
How much does Sonny spend altogether?

He spends R _____.

5. The total weight of 12 same-sized bags of onions is 108 kg.
What is the weight of one bag of onions?

The weight of one bag is _____ kg.

6. One of the problems above is NOT an equal groups or equal measures problem.
Which problem is not an equal groups or equal measures problem?
Explain how you know.

Question _____ is not an equal groups problem.

I know this because

MRIP Grade 6 Rapid practice 3

9 x 9 =	99 ÷ 9 =
8 x 5 =	90 ÷ 11 =
9 x 12 =	48 ÷ 8 =
8 x 11 =	48 ÷ 6 =
11 x 8 =	54 ÷ 9 =
10 x 12 =	54 ÷ 6 =
5 x 12 =	120 ÷ 2 =
2 x 12 =	24 ÷ 12 =
4 x 12 =	48 ÷ 12 =
8 x 12 =	96 ÷ 12 =
12 x 8 =	36 ÷ 6 =
9 x 12 =	72 ÷ 6 =
12 x 3 =	144 ÷ 6 =
12 x 6 =	49 ÷ 7 =
12 x 7 =	98 ÷ 7 =

MRIP Grade 6 Lesson 4

Name: _____

1. Sam is planting rows of cabbages.
She plants 9 cabbages in each row.
Sam plants 12 rows of cabbages.
How many cabbages does Sam plant?

Sam plants _____ cabbages.

2. Viren is putting out rows of chairs in the hall.
He puts 13 chairs in each row.
He puts out 156 chairs altogether.
How many rows of chairs did Viren put out?

Viren put out _____ rows of chairs.

3. Patrick is putting out tiles in rows.
He puts out 8 equal rows of tiles on the floor.
He puts out 96 tiles altogether.
How many tiles does Patrick put in each row?

There are _____ tiles in each row

LESSON 4 **Aims:** Identifying equal groups in arrays problems
Working with the t-table

Rapid recall	**Approx. 5 mins.**
Multiplication and division triples. Remind the learners about **triples and the relationship between the three numbers.** Work on learners rapidly recalling the three numbers in a triple.	
Problem solving	**Approx. 40 mins.**

Learners work on three problems.

As before, after each problem the solution and how it was found is discussed before going on to the next problem.

Problem 1:

Sam is planting rows of cabbages.
She plants 9 cabbages in each row.
Sam plants 12 rows of cabbages.
How many cabbages does Sam plant?

Make it clear that each each row has the same number of cabbages in it.

Remind the learners of the t-table and set this initial one up on the board:

Rows	Cabbages
1	9
12	[?]

Step 1: Agree on the correct answer

Step 2: Share solution methods

Step 3: Refine solution methods

As you set up Problem 2, leave the t-table on the board for learners to refer to.

Problem 2:

Viren is putting out rows of chairs in the hall.
He puts 13 chairs in each row.
He puts out 156 chairs altogether.
How many rows of chairs did Viren put out?

Make it clear that each each row has the same chairs in it.

Working with the learners set this blank t-table on the board:

Rows	Chairs
1	13
[?]	156

Step 1: Agree on the correct answer

Step 2: Share solution methods

Step 3: Refine solution methods

As you set up Problem 3, leave the t-table on the board for learners to refer to.

Problem 3:

Patrick is putting out tiles in rows.
He puts out 8 equal rows of tiles on the floor.
He puts out 96 tiles altogether.
How many tiles does Patrick put in each row?

A quick sketch of some tiles in rows may help learners understand the situation.
As before, set up the initial t-table:

Rows	Tiles
1	?
8	96

Step 1: Agree on the correct answer

Step 2: Share solution methods

Step 3: Refine solution methods

Linking the problems	**Approx. 10 mins.**
Ask learners to turn-and-talk about: What is the same about the three problems? How are the problems different? Can they make up a problem that is like these three problems? Talk about how all three problems are examples of **Equal groups in arrays problems** Talk about how Problem 1 was a multiplication problem, and 2 and 3 division problems.	
Practice/Homework	
For practice if there is time, or homework, there are more **equal groups and arrays** problems for learners to work through. One of the five problems is NOT an **equal group or arrays** problem. After working through all five problems, learners should try and identify this problem and say why it is different. They should also practice their multiplication bonds.	

MRIP Grade 6 Problem practice 4

1. One bag of apples contains 13 apples.
Mark buys 15 bags of apples.
How many apples is that altogether?

Mark buys _____ apples.

2. Karin is putting rows of pies on a tray.
Each row has 12 pies in it.
Karin puts out 25 rows of pies.
How many pies did Karin put out?

She put out _____ pies.

3. Trust drives 348 kilometres.
Sonny drives 356 kilometres.
How much further does Sonny cycle?

Sonny drives _____ kilometres more than Trust.

4. A farmer is planting rows of carrots.
He plants 15 carrots in each row.
He plants 180 carrots altogether.
How many rows of carrots does the farmer plant?

He plants _____ rows of carrots.

5. The total weight of 16 same-sized bags of potatoes is 192 kg.
What is the weight of one bag of potatoes?

One bag of potatoes will weigh _____ kilograms.

6. One of the problems above is NOT an equal groups problem.
Which problem is not an equal groups problem?
Explain how you know.

Question _____ is not an equal groups problem.

I know this because

MRIP Grade 6 Rapid practice 4

7 x 9 =	88 ÷ 8 =
8 x 7 =	88 ÷ 11 =
11 x 12 =	42 ÷ 6 =
9 x 11 =	42 ÷ 7 =
11 x 9 =	56 ÷ 8 =
10 x 14 =	56 ÷ 7 =
5 x 14 =	160 ÷ 2 =
2 x 16 =	36 ÷ 12 =
4 x 16 =	72 ÷ 12 =
8 x 16 =	108 ÷ 12 =
12 x 9 =	48 ÷ 6 =
9 x 12 =	96 ÷ 6 =
12 x 4 =	144 ÷ 6 =
12 x 8 =	49 ÷ 7 =
12 x 16 =	98 ÷ 7 =

Appendix B: Ethics Clearance Letter

Wits School of Education

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28 August 2017

Student Number: 481433

Protocol Number: 2017ECE033M

Dear Tumelo Ramaoka

Application for Ethics Clearance: Master of Science

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate, has considered your application for ethics clearance for your proposal entitled:

Solving multiplication and division word problems - strategies used by grade 6 learners

The committee recently met and I am pleased to inform you that clearance was granted.

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

All the best with your research project.

Yours sincerely,

M Mabeta

Wits School of Education

011 717-3416

Appendix C: Information Letters

UNIVERSITY OF THE
WITWATERSRAND,
JOHANNESBURG

LETTER TO THE PRINCIPAL

June 2017

Dear Mr Mashau

I am Tumelo Ramaoka undertaking a research study as a M.Sc. (Science Education) student in the School of Education at the University of the Witwatersrand. I am doing research on *Solving multiplication and division word problems – models and strategies used by Grade 6 learners*.

My research study aims to investigate the effects of a multiplicative reasoning intervention. In this study I will explore the models and strategies Grade 6 learners use to solve multiplication and division word problems prior to, and following, a sequence of intervention lessons.

My data collection process will take place over a period of six lessons during scheduled mathematics periods. One lesson will be conducted each week for six weeks. A single Grade 6 class will be used as participants for the study. The study will be carried out as follows:

i. Pre-test - for the first lesson learners will be given a pre-test on multiplication and division word problems;
ii. Intervention lessons – lesson two to five will be based on interpreting word problems and learning how to use t-tables, double number lines and arrays to solve multiplication and division problems; and
iii. Post-test – the last lesson will be a post-test, which will be the same as the pre-test.

The post-test and pre-test results will be used to identify the models and strategies learners use to approach the word problems and whether the models and strategies learners use differ between the pre-test and post-test.

The reason I chose your school is because of accessibility, convenience and experience working with learners as an educator at the school.

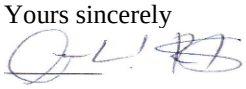
With reference to the above kindly consider my request to collect data at your school whilst engaging learners in this study. Learners' participation will be completely voluntary, and their parents will receive a similar letter seeking their permission.

The research participants will not be advantaged or disadvantaged in any way. They will be reassured that they can withdraw from participating at any stage of data collection. There are no foreseeable risks in participating in this study. The participants will not be paid for this study.

The names of the research participants and identity of the school will always be kept confidential and in all academic writing about the study. The school's privacy will be maintained in all writing resulting from the study.

All research data will be kept safe behind lock and key and destroyed between 3-5 years after completion of the project.
Please let me know if you require any further information. I look forward to your response as soon as is convenient.

Yours sincerely



NAME: Tumelo Ramaoka
EMAIL: 481433@students.wits.ac.za

UNIVERSITY OF THE
WITWATERSRAND,
JOHANNESBURG

LETTER TO LEARNERS

June 2017

Dear Learner

I am Mr. Tumelo Ramaoka studying Master of Science (Science Education) at the University of the Witwatersrand. I am doing research on *Solving multiplication and division word problems – models and strategies used by Grade 6 learners*.

My research study aims to collect information on methods Grade 6 learners use to solve word problems before a series of lessons on word problems, and after a series of lessons.

My data collection process will take place over a period of six lessons during mathematics lessons. One lesson will take place each week for six weeks. For the first lesson you will write a test on word problems. Then the next four lessons you will learn how to interpret word problems, and how to use t-tables, double-number lines and arrays to solve multiplication and division word problems. For the last lesson you will write another test on word problems. Please note that the two tests will not count towards your marks for mathematics.

I was wondering whether you would mind participating in the research study. I need your help by participating in the process of data collection, which will use tests as a method of collecting data.

Note, that taking part in the study is voluntary, which means that you don't have to do it. Also, if you decide halfway through that you prefer to stop, this is completely your choice and will not affect you negatively in any way. You will not be paid for participating in the study.

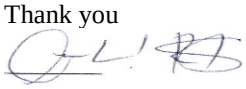
All information about you will be kept secretive in all my writing about the study. And, all the collected information will be stored safely in a locker and destroyed between 3-5 years after I have completed my study.

Your parents/guardians have also been given an information sheet and consent form, but at the end of the day it is your decision to join us in the study. Please discuss this with your parents or guardians before you sign the permission form.

I look forward to working with you!

Please feel free to contact me if you have any questions.

Thank you



NAME: Tumelo Ramaoka
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CELLPHONE NUMBERS: 074 380 0878

UNIVERSITY OF THE
WITWATERSRAND,
JOHANNESBURG

LETTER TO PARENTS/GUARDIANS

June 2017

Dear Parent/Guardian

I am Tumelo Ramaoka undertaking a research study as a M.Sc. (Science Education) student in the School of Education at the University of the Witwatersrand. I am doing research on *Solving multiplication and division word problems– models and strategies used by Grade 6 learners*.

My research study aims to investigate the effects of multiplicative reasoning intervention. In this study I will explore the models and strategies Grade 6 learners use to solve multiplication and division word problems prior to, and following, a sequence of intervention lessons.

The reason I chose your school is because of accessibility, convenience and experience working with learners as an educator at the school. I was wondering whether you would mind if your child participated in the research study. The study will use tests as a method of collecting data. The data will be collected during mathematics lessons, for a period of six lessons. One lesson will be conducted each week; therefore, this will take six weeks to complete data collection.

I do not anticipate any risk greater than normal life. The content of the intervention lessons are aligned with the curriculum and your child will not therefore not forfeit any mathematics lesson time by participating in this research. Your child may enjoy this research while learning more about interpreting multiplication and division word problems, and how to use different models and strategies for multiplication and division.

Your child's participation in this project is completely voluntary. In addition to your permission, your child will also be asked if he or she would like to take part in this study. Any child may withdraw from the study at any time. The choice to participate or not will not impact your child's grades or status at school. Participants will not be paid.

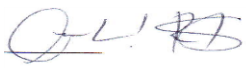
Documented information that will be obtained during this research project will be kept strictly confidential and secure and will not become a part of your child's school record.

Your child's name and identity will always be kept confidential and in all academic writing about the study. Your child will be given pseudonym to ensure confidentiality. His/her individual privacy will be maintained in all published and written data resulting from the study.

All research data will be destroyed between 3-5 years after completion of the study.

Please let me know if you require any further information.
Thank you very much for your help.

Yours sincerely,



NAME: Tumelo Ramaoka
EMAIL: 481433@students.wits.ac.za
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Appendix D: Consent Forms

Learner Consent Form

Please fill in and return the reply slip below indicating your willingness to participate in the research project called *Solving multiplication and division word problems – models and strategies used by Grade 6 learners*.

My name is: _____

I agree to write the tests for this study. YES/NO

I understand that:

- my name and information will not be made public,
- I do not have to answer every question and can withdraw from the study at any time,
- the test results from the study will not count towards my marks for mathematics, and
- all the data collected during this study will be destroyed within 3-5 years after completion of the project.

Sign_____ Date_____

UNIVERSITY OF THE
WITWATERSRAND,
JOHANNESBURG

Parent/Guardian Consent Form

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called *Solving multiplication and division word problems – models and strategies used by Grade 6 learners*:

I, _____ the parent of _____

I agree that my child may write the tests for this study. YES/NO

I understand that:

- my child's name and information will be kept confidential and safe and that my name and the name of the school will not be revealed.
- he/she does not have to answer every question and can withdraw from the study at any time
- all the data collected during this study will be destroyed within 3-5 years after completion of the project.

Sign_____ Date_____