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Master of Science,
School of Computer Science and Applied Mathematics

Lie Symmetry Analysis of the Two-factor Black-Scholes Equation

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Master of Science.

June 17, 2018

Declaration

I declare that this is my own work except were due references have been made to the literature. It is submitted for the degree of Master of Science by dissertation to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree to any other University.

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June 17, 2018

Abstract

One of the fundamental assumptions of the classical Black-Scholes model is that of constant interest rates through the life of a derivative instrument until expiry. While this assumption greatly simplifies the pricing of derivative instruments it also results in an inherent flaw of the model as interest rates tend to fluctuate over time depending on the economic environment and how market players perceive the future economic environment to be like. This dissertation focuses on a Black-Scholes model in which the constant interest rates assumption is relaxed, i.e. interest rates driven by time-dependent stochastic processes. Lie theory and invariant approaches will be utilized to find solutions of this model, whose derivation shall also presented in this dissertation.

Acknowledgements

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Dedication

To Violet Gumbo nee Siwela, a continuous pillar of strength.
They themselves are makers of themselves ....

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Chapter 1

Introduction

A financial contract whose value depends on the value of another asset (e.g. a share traded on the stock exchange) on which the contract is written on is know as a derivative instrument. Consider a financial contract between two interested participants in which the buyer (B) of the contract, which expires at a predetermined time point in the future τ, obtains the right to buy a specified share with current price S from the seller at a pre-agreed price (known as the strike price, K) when the contract expires. However, the buyer does not have to exercise that right, they are able to allow the contract to expire without acting on it at which point they lose the purchase price of the option. At time τ the the price of the share, S_τ might be greater than the strike price or it might be less. If the price of the share is greater than the strike price then it would be rational for the buyer to exercise their right by purchasing the share from the seller at price K and and immediately sell it in the market for S_τ thereby making a profit of $S_\tau - K$. If $S_\tau < K$ the buyer would choose not to exercise their right to buy, in which case their total loss is capped to the premium paid to the seller to get into the contract. Such a contract is called a call option and the share is the underling asset. The premium paid to the seller should be set high enough such that the seller is compensated for the risk they will taking but not too high such that the buyer finds no benefit in shifting the risk to the buyer. The question which then arises is: how do the two parties determine the price of this derivative?

In their work published in 1973, Fischer Black and Myron Scholes [1] derived a simple and elegant equation which theoretically answered the problem of the valuation of options:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0, \tag{1.1}$$

where V is the value of the option, t is the time to maturity, the stock price is given by S, r is the risk free interest rate and σ is the stock price volatility.

The seller of the option might want to eliminate the risk associated with the call option (theoretically $(S - K) \to \infty$ is possible) which could lead to her financial ruin. To eliminate the risk she can replicate the option value over time by creating a portfolio constituting the underlying stock and a risk free asset, for example a cash in a bank account that bears interest. In a world with no transaction costs she can adjust the weights of the portfolio assets such that the portfolio value is equal to the option value always until the option expires.

This insight provided the basis for deriving the linear Black-Scholes partial differential equation (PDE) (1.1). The following assumptions also provide the building blocks for the derivation:

- Volatility is constant
- Markets are efficient, i.e. all market participants have access to all the information
- Stocks are non-dividend paying
- Interest rates are constant and are known
- Returns on the underlying stock follows a log-normal distribution
- European style options i.e. the option can only be exercised at expiration
- No commissions or transaction costs
- Perfectly liquid markets.

Some of these assumptions are already known to be unrealistic in the real world. The existence of a market phenomenon called 'the volatility smile (or

smirk)' already discredits the assumption of constant volatility. The Black-Scholes model assumes asset prices to be log-normally distributed at all future times. Volatility is unobservable in the market, that is it can only be implied from market prices. The Black-Scholes model assumes that volatility is constant throughout the life of the option. However in practice, it is noted that when asset prices are high the probability that the price will go even higher is quite low and this in turn results in a low volatility of asset prices. However, when asset prices are low, the probability that the asset prices will go even lower is high and this is seen also in the increased volatility of such assets. This essentially means there is a higher probability of extreme negative events occurring than there is of extreme positive events. Hence the classical Black-Scholes understates the downside risk associated with options and this needs to be adjusted.

Interest rates have also been observed to vary with time. They are dependent on prevailing market conditions. Interest rates also give insight into the prevailing market sentiment towards the future state of the economy and the higher the risk or uncertainty participants in the market attach to the future the higher the rate of return or interest rate they would require for their investments in that particular market. In this dissertation the focus is on a modified version of (1.12) in which stochastic interest rates are allowed. Use is made of Lie symmetry and invariant methods to investigate possible solutions to the modified equations.

Research in the application of Lie symmetries in the financial field is still quite a fertile area and has a lot of problems for which it could be applied. Most of the works looked at as part of this research focused on equation (1.1) and on the (2+1) equation is allowed to be stochastic. This dissertation seeks to use Lie symmetry methods to investigate the case where interest rates are driven by a stochastic process. In this research we will limit the analysis to (2+1) model where interest rates follow the Vasicek [10] and Cox-Ingersoll-Ross models [23]. After modelling the two different models, we will try to apply Lie Symmetry theory to find the invariant solutions of the resulting PDEs.

The first chapter will provide an introduction into the basics of stochastic processes and will present important theorems and definitions which shall be

used in deriving the (2+1) Black-Scholes equation. The principles of Lie symmetries will also be introduced to set the foundation for finding solutions to the PDE.

1.1 Deriving the Classical Black-Scholes

Consider a portfolio structured such that it is only made up of a risky asset and an asset earning the risk-free rate which are driven by the following process:

$$dS(t) = \mu S(t)dt + \sigma S(t)dX(t) \tag{1.2}$$

$$dM(t) = rM(t)dt. \tag{1.3}$$

The existence of the stochastic term $dX(t)$ in (1.2) makes the asset risk as it is the source of uncertainty while in (1.3) there is no stochastic term and is risk free.

Also consider a derivative instrument traded in the market maturing at time T whose dynamics follow:

$$df = \mu_f dt + \sigma_f dt dX(t). \tag{1.4}$$

By Itô's Lemma,

$$\mu_f(t) = \mu S(t) f_S + f_t + \frac{1}{2}\sigma^2 S(t)^2 f_{SS} \tag{1.5}$$

$$\sigma_f(t) = \sigma S(t) f_S. \tag{1.6}$$

Now consider a self financing portfolio $V(t)$ made up of one derivative instrument $f(t)$ and a negative holding of $\psi(t)$ units of a risky asset. At each point in time the portfolio is given by

$$V = f - \psi S. \tag{1.7}$$

For a small change in time dt the portfolio value changes as given below

$$\begin{aligned} dV(t) &= df - \psi dS \\ &= (\mu_f dt + \sigma_f(t)dX) - \psi(\mu S(t)dt + \sigma S dX) \\ &= (\mu_f - \psi\mu S)dt + (\sigma_f - \psi\sigma S)dX. \end{aligned}$$

Due to the portfolio $V(t)$ being self financing any change in the portfolio value would purely be as a result of investment gain or loss that is there is no additional funding added into or withdrawn from the portfolio after inception of the portfolio.

The change in portfolio value has randomness in the $dX(t)$ and to eliminate the risk this needs to vanish, that is we need to ensure that the co-efficients of $dX(t)$ equal zero. This can be achieved by making a right choice of $\psi(t)$. Let

$$\psi = \frac{\sigma_f}{\sigma S} = f_S \equiv \frac{\partial f}{\partial S}. \tag{1.8}$$

The dynamics of the value process then becomes

$$dV(t) = (f_t + \frac{1}{2}\sigma^2 S(t)^2 f_{SS})dt, \tag{1.9}$$

that is the portfolio is risk free for small time changes dt. To ensure that the portfolio does not admit arbitrage, the return on the portfolio should be the continuously compounded risk free interest rate of return, that is

$$\begin{aligned} dV(t) &= rV(t) && (1.10)\\ (f_t + \frac{1}{2}\sigma^2 S(t)^2 f_{SS})dt &= r(f(t) - f_S S(t))dt && (1.11)\\ \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} &= rf. && (1.12) \end{aligned}$$

The PDE (1.12) is called the "Black-Scholes equation", the result of which can be used to price call and put options. The intrinsic call option value is given by

$$c(S,t) = (S-K)^+$$

and for the put option

$$p(S,t) = (K-S)^+.$$

The figure that follows shows graphically the payoff of call and put options for a stock with strike price $K = 100$. For stock prices below K the pay-off of the call option is at worst the premium paid for the option. For prices greater

than K the option pays out the differences between the stock price and the strike price that is $S - K$. The solution of (1.12) for a call option is

$$c(S,t) = SN(d_1) - Ke^{-r\tau}N(d_2), \tag{1.13}$$

where

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\tau\right)}{\sigma\sqrt{\tau}}$$

$$d_2 = \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\tau\right)}{\sigma\sqrt{\tau}}$$

$$\tau = T - t, T > t.$$

Given that the pricing of options is in the context of a world in which there is no arbitrage opportunities, the price of the call option and the price of the put option should have a defined relationship as the price of the one should imply the price of the other in this world.
Consider the two portfolios below:

- Portfolio A consists of a call option $c(S,t)$ and a cash amount equivalent to the strike price of the call option. If the option is exercised at maturity the stock is purchased at strike price K using the cash amount equal to K.
- Portfolio B consists of a put option $p(S,t)$ and the underlying asset. At expiration of the option the option seller sells the underlying asset at the strike price if the option is exercised by the holder.

As can be seen from figure 1.1 the two portfolios have the same pay-off structures which implies that at each point in time the value of these two portfolios combined should be the same, that is

$$c(S,t) + K = p(S,t) + S$$

.

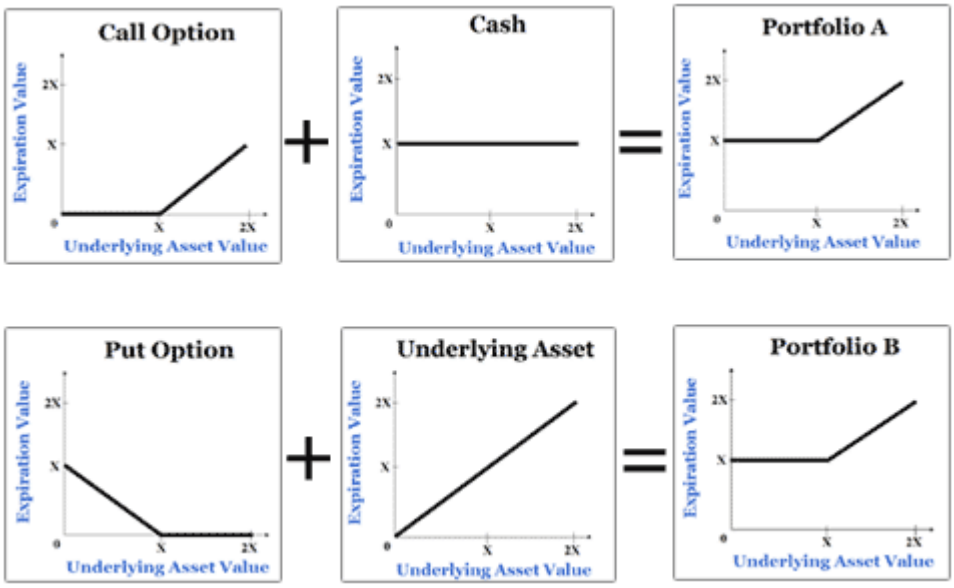


Figure 1.1: Graphical representaion of put-call parity. [28]

1.2 Concluding Remarks

This chapter introduced derivatives as financial instruments whose pricing is based on the Black-Scholes model [1]. The basic theory and derivation of the Black-Scholes model and PDE (1.1) were also introduced. The assumptions of the Black-Scholes model, specifically the assumption of constant interest rates which forms the basis of the problem of interest for this dissertation, were also introduced.

Chapter 2

Literature Review

The application of Lie symmetry theory to finance remains a very fertile field and as such most of the works reviewed so far have mostly been on the classical Black-Scholes model, i.e. the (1+1) parabolic equation (2.2).

The question of how to include stochastic interest rates into the model (1.1) which results in the model having two sources of risk, interest rate risk and asset price volatility has been in literatures and can be traced to back to research by Rabinovitch (1989) [16] in which formulas for call option valuation and bonds when the risk-less interest rate is stochastic are derived. However, in this work numerical analysis is used to price the instruments.

Abudy and Yehuda [7] price stock options with stochastic interest interest rate solution for the Black-Scholes model where the interest rates follow a Gaussian stochastic process. Here they construct a closed-form general solution of the model in which interest rates are allowed to be driven by stochastic processes. They further show that their solution diverges back to (1.12) solution when the constraints on interest rates are reintroduced and also show consistency of their call price solution to the put-call parity conditions. Unlike in Rabinovitch's [16] work Abudy and Yehuda [7] introduce additional risk from rate movements where Rabinovitch's [16] model has only one source of risk. They

find the call option price to be:

$$C(t,T) = S(t)N\left[\frac{\ln\frac{S(t)}{K} + A(t,T) - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau}\right]$$
$$- Ke^{A(t,T)+\frac{1}{2}B^2(t,T)\tau}N\left[\frac{\ln\frac{S(t)}{K} + A(t,T) - \frac{1}{2}\nu^2\tau}{\sigma\sqrt{\tau}} - B(t,T)\sqrt{\tau}\right],$$

where $N()$ is the standard normal cumulative probability distribution, $\tau = T - t$ is the time to maturity, $\nu^2 = \sigma^2 + B^2(t,T) - 2\rho\sigma B(t,T)$, σ the volatility of the underlying asset and ρ is a correlation factor on the asset price and the interest rates. Haowen [13] used martingale theory to propose a European option pricing model with variable interest rates. The result is presented below:

$$V_t + \frac{1}{2}\sigma_1^2X^2V_{XX} + \sigma_1\sigma_2\rho XV_{Xr} + \frac{1}{2}\sigma_2^2 + rXV_X + a(\theta - r)V_r - rV = 0, \quad (2.1)$$

where V is the option price, t is a time variable, X is the asset price, r is the interest rate, σ_1, σ_2 are the volatilities of the asset price and the interest rates respectively. Here ρ is the correlation between interest rates and the asset prices. In this research the derivation will follow Haowen's [13] derivation of the model.

While there are research in literature in which numerical and analytical solutions for the model with stochastic interest rates, there are not as much where solutions are generated through Lie theory for the Black-Scholes equation. Application of Lie theory to financial problems especially of the Black-Scholes type is found mostly in the recent literature but it is largely restricted to the classical model.

Mahomed [3] considered the investigated (1+1) linear parabolic equations using invariant methods. Mahomed [3] investigated linear parabolic equations and showed the effectiveness of using invariant criteria on these equations to reduce them to three Lie canonical forms of a bond pricing model. In this dissertation we shall present the invariants and semi-invariants by the transformation of variables of a (2+1) dimensional equation, using as a foundation work in Mahomed *et. al* [6].

Ibragimov in [5] extended Euler's method to apply on linear parabolic equations and introduced a method to deduce explicit formulae for the general solutions to parabolic equations. Of interest is the derivation of the solution of the Black-Scholes equation which takes the general form of a linear parabolic equation

$$u_t + Au_{xx} + au_x + c = 0, \tag{2.2}$$

with A, a, c are functions of (t, x). By use of the equivalence transformations of the independent variable

$$\tau = \phi(t), \quad y = \psi(t, x)$$

, and linear transformations of the dependent variable

$$v = \sigma(t, x)u, \quad \sigma \neq 0,$$

where $\phi(t), \psi(t, x)$ and $\sigma(t, x)$ are arbitrary functions. Applying these transformations to equation (2.2) and taking derivatives of u after the transformations of the independent variables results in

$$\phi' u_\tau + A\psi_x^2 u_{yy} + (\psi_t + A\psi_{xx} + a\psi_x)\, u_y + cu = 0.$$

Thus equation (2.2) can be rewritten as

$$u_t - u_{xx} + au_x + cu = 0, \tag{2.3}$$

whereby ψ is chosen such that $|A|\psi_x^2 = 1$ and, depending on the sign of A, $\phi = \pm t$. The equivalence transformation is the chosen to be $v = ue^{\rho(t,x)}$ such that

$$u = ve^{-\rho(t,x)}.$$

After differentiating under the transformation and considering the equivalence relation it is found that

$$u_t - u_{xx} + au_x + cu = \left[v_t - v_{xx} + (1 + 2\rho_x)v_x + \left(\rho_{xx} - \rho_x^2 - \rho_t - a\rho_x + c\right) v\right] e^{-\rho(t,x)}. \tag{2.4}$$

The idea is to reduce equation (2.3) to the heat equation

$$v_t - v_{xx} = 0.$$

- $\rho_x = -\frac{1}{2}a$,
- $\rho_{xx} = \frac{1}{2}a_x$ that is $\rho_t = \frac{1}{4}a^2 - \frac{1}{2}a_x + c$.

When these conditions are satisfied then (2.3) can be transformed to (3.44) that is

$$u_t - u_{xx} + au_x + cu = v_t - v_{xx}. \tag{2.5}$$

For the compatibility condition to be satisfied $\rho_{xt} = \rho_{tx}$ should hold true. With the compatibility condition satisfied the semi-invariant of the parabolic equation can be written as

$$K = aa_x - a_{xx} + a_t + 2c_x.$$

The work showed that a parabolic equation of the form (2.3) can be reduced by way of equivalence transformations of the variables to (3.44).

For the equation (2.2) where the semi-invariant K is zero the solution will be:

$$u(t,x) = \frac{1}{2\sqrt{\pi t}} e^{-\rho(t,x)} \int_{-\infty}^{+\infty} u_0(z) e^{\rho(0,x)} e^{\frac{(x-z)^2}{4t}} dz, \tag{2.6}$$

$$t > 0,$$

where $\rho(t,x)$ is a solution to the system (2.7)

$$\begin{aligned} \tfrac{\partial \rho}{\partial x} &= -\tfrac{1}{2}a, \\ \tfrac{\partial \rho}{\partial t} &= \tfrac{1}{4}a^2 - \tfrac{1}{2}a_x + c. \end{aligned} \tag{2.7}$$

These results provide the basis to finding solutions to the equation (1.12) which is of the form (2.2). This work presented an interesting concept with which one can solve parabolic equations of the form (2.2) without the need to calculate the symmetries of the equation by using principles from invariant methods. An extension of this to apply to the (2+1) dimensional equations would be interesting. By extending these results for linear parabolic equations we can also investigate possible solutions to higher-dimensional equations.

Güngör [14] presented necessary conditions for the reduction of the (2+1) dimensional equations of the form(2.8) to be transformed to the heat equation [14].

$$u_t = au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu, \tag{2.8}$$

where $a(x,y), b(x,y), c(x,y), d(x,y), e(x,y), f(x,y)$
where $b^2 - 4ac < 0$. Appropriate transformations are employed to remove the coefficients of u_{xx} and u_{yy} resulting in

$$u_t = u_{xx} + u_{yy} + pu_x + qu_y + Vu. \tag{2.9}$$

where $p(x,y), \quad (x,y), \quad V(x,y,t)$. The reduced equation is then transformed to

$$U_{\overline{t}} = U_{\overline{xx}} + U_{\overline{yy}} + \overline{p}U_{\overline{x}} + \overline{q}U_{\overline{y}} + \overline{V}U \tag{2.10}$$

under the transformations

$$\overline{t} = \tau(t), \overline{x} = \sqrt{\dot{\tau}}(x + \delta(t)), \overline{y} = \sqrt{\dot{\tau}}(y + \rho(t)), u = e^{F(x,y,t)}, \overline{V} = \frac{1}{\dot{\tau}}(V - W)$$

with

$$W = F_t - (F_{xx} + F_{yy}) - (F_x^2 + F_y^2) - \rho F_x - qF_y.$$

If $p_y = p_x$, then F can be defined as

$$F = \frac{\ddot{\tau}}{\dot{\tau}}\left[\frac{1}{8}(x^2 + y^2) + \frac{1}{4}(\delta_x + \rho y)\right] - \frac{1}{2}\phi(x,y) + \log(\nu(t))$$

which enables setting $\overline{p} = \overline{q} = 0$ thus reducing the equation to the two dimensional heat equation.

Li *et. al* [8] focus on finding numerical solutions to (2+1) option pricing models considering the Vasicek and CIR models. Of interest in their work is the modelling of the Black-Scholes equation to include variable interest rates. They consider a portfolio Π with a return equal to the risk-free rate r,

$$d\Pi = r\Pi dt \tag{2.11}$$

and the portfolio Π constituting a stock whose price process is defined the stochastic differential equation

$$dS = rSdt + \sigma dW_1, \tag{2.12}$$

where σ is the stock price volatility and W_1 is a Weiner process. They further assume that the interest rates are driven by

$$dr = (a - \kappa r - \lambda_p \Sigma)dt + \sigma dW_2 \tag{2.13}$$

in which κ is the speed of reversion to the average state of interest rates, $\overline{r}$ is long term average of interest rates, for $a = \kappa\overline{r}$, Σ is the interest rate volatility, λ_p is the price of taking risk in the market and W_2 is a Wiener process. After deriving a stochastic component $x(t)$

$$dx = -\kappa x dt + r^{\beta}\Sigma dW_2 \tag{2.14}$$

and setting limits to β, λ_p to suit the Vasicek and CIR models, they present the two factor model as

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial S} - (r_0 + x) + \frac{\Sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - \kappa x\frac{\partial V}{\partial x} = 0, \tag{2.15}$$

for the Vasicek model and

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial S}(r_0 + x) - (r_0 + x)V \\ + \frac{\Sigma^2(r_0 + x)}{2}\frac{\partial^2 V}{\partial x^2} - \frac{a}{r_0}x\frac{\partial V}{\partial x} = 0,\end{aligned} \tag{2.16}$$

for the CIR model. The Vasicek model was chosen for its mathematical simplicity whilst the CIR is more consistent with the term structure of interest rates which is observed in the markets. In [12], investigations into the two models is conducted in which its noted that the two models fit well with observed market rates and the choice dependent on the instrument being priced.

Gazizov and Ibragimov [11] also consider the application of symmetry methods to (1+1) Black-Scholes model as well as the two factor Jacobs-Jones model,

$$\begin{aligned}u_t = \frac{1}{2}A^2x^2u_{xx} + ABCxyu_{xy} + \frac{1}{2}B^2y^2u_{yy} + (Dxln\frac{y}{x} - Ex^{\frac{3}{2}})u_x \\ + (Fyln\frac{G}{Y} - Hyx^{\frac{1}{2}})u_y - xu,\end{aligned} \tag{2.17}$$

in which $A, B, C, \ldots, H$ are co-efficients associated with the model. In their work, they classify this model according to its symmetry groups, thus providing a foundation from which one can construct exact (invariant) solutions. They

mention that the number of symmetries of (2.17) is dependent on the choice of co-efficients . The principal Lie algebra (L_ϱ) is spanned by

$$X_1 = \frac{\partial}{\partial t}, \tag{2.18}$$

$$X_2 = u\frac{\partial}{\partial u}, \tag{2.19}$$

$$X_\omega = \omega\frac{\partial}{\partial u}, \tag{2.20}$$

where $\omega = \omega(t, x, y)$ satisfies (2.17). Gazizov and Ibragimov [11] consider and present extensions to L_ϱ for equations satisfying the conditions

$$AB \neq 0,$$

$$C \neq \pm 1$$

and

$$C \neq 0.$$

The extension of L_ϱ were presented in the cases below:

Case 1: $D = 0$

$$X_3 = e^{Ft} y \frac{\partial}{\partial y}.$$

If $AH - BCE = 0$ and $F = 0$,

$$X_4 = 2AB^2(1 - c^2)ty + (2BC \ln x - 2A \ln y + (B - AC)ABt)\, u\frac{\partial}{\partial u}$$

Case 2: $D \neq 0$ **,** $F = -(BD/2AC)$ **,** $H = 0$

$$X_3 = \exp\left(\frac{BD}{2AC}t\right) yy\frac{\partial}{\partial y} + \left(\frac{D}{ABC}\ln\frac{G}{y} + 1\right)\exp\left(\frac{BD}{2AC}t\right) u\frac{\partial}{\partial u}.$$

Case 3: $D \neq 0$ $\quad F$ satisfies $A^2F^2 - A^2D^2 + 2ABCDF + B^2D^2 = 0$ and E, H are related by $BE(ACF + ACD + BD) = AH(AF + AD + BCD)$

$$
\begin{aligned}
X_3 \; &= e^{Dt} y \frac{\partial}{\partial y} - \frac{ACF + ACD + BD}{A^2 B(1 - C^2)} \ln x - \frac{AF + AD + BCD}{AB^2(1 - C^2)} \ln y \\
&\quad + \left(\frac{A^2 CF + A^2 CD - B^2 CD - ABF}{2ABD(1 - C^2)} \right) e^{-Dt} u \frac{\partial}{\partial u} \\
&\quad + \left(\frac{F \ln G(BCD + AF + AD)}{AB^2 D(1 - C^2)} \right) e^{-Dt} u \frac{\partial}{\partial u}.
\end{aligned}
$$

In the case that $B = 2AC$, $F = -D$, $H = 0$, then

$$
X_4 = e^{Dt} y \frac{\partial}{\partial y} + \left(\frac{D}{2A^2 C^2} \ln \frac{G}{y} + 1 \right) e^{Dt} u \frac{\partial}{\partial u}.
$$

These results enable the finding of invariant solutions of equation (2.17). Despite the fact that they mention that equation (2.17) might not be reducible to the heat equation regardless of the choice in $A, B, C, \ldots, H$ it presents interesting results for further investigations.

Of particular interest to this research are solutions to (2+1) heat equation of the type

$$
u_t = u_{xx} + u_{yy}. \tag{2.21}
$$

However, finding the solution to (2.21) is not within the scope of this research and relevant solutions proposed in literature will be considered if necessary.

In [17] the focus was deriving closed form solutions to the Black-Scholes model in which interest rates are allowed to be stochastic. However, in this work the actual derivation of the model was not provided but the derivation of the solution. To find the solution, they used the technique of change in numeraire to derive a closed form solutions. They went further to calculate the option price sensitivities to the model parameters as well as to the price of the zero-coupon bond prices. The price of a stock option was proposed to be

$$
\theta(0) = S(0)N(d_1) - KP(0,T)N(d_2), \tag{2.22}
$$

where

$$
d_2 = \frac{\ln \frac{S(0)}{KP(0,t)} - \frac{1}{2} c \Sigma^2(T)}{\sqrt{\Sigma^2(T)}}
$$

$$d_1 = d_2 + \sqrt{\Sigma^2(T)}$$

and

$$\Sigma^2(T) = T\sigma_1^2 + \frac{\sigma_2^2}{\beta^3}\left(\beta T + \frac{3}{2} - 2e^{\beta T} + \frac{1}{2}e^{2\beta T}\right) + \frac{2\sigma_1\sigma_2\rho}{\beta^2}\left(e^{\beta T} - \beta T - 1\right).$$

In this result Eksi [17] considered the stock price and the interest rates to be correlated given by ρ where σ_1 is the volatility of the stock price, σ_2 is the interest rates volatility and β is a model parameter contributing to the mean reversion of interest rates to their long run average. To make computations easier in this dissertation, only the case where interest rates and stock prices are not correlated is considered that is the assumption $\rho = 0$ was made.

Aligned to the focus of this dissertation is the work of Kovalenko *et. al* [18] in which they apply Lie ymmetry methods to the $2 + 1$ dimensional equation (2.23). In this work they found invariance algebra of fundamental solutions of the linear Fokker-Plank-Kolmogorov equations. The operators found where then used to construct the invariant solution of the equation

$$u_t - u_{xx} + xu_y = 0, \tag{2.23}$$

where $(x, y, t) \in \mathbb{R}^3$ and

$$u = u(t, x, y), u_t = \frac{\partial u}{\partial t}, u_y = \frac{\partial u}{\partial y}, u_{xx} = \frac{\partial^2 u}{\partial x^2}.$$

Defining the general fundamental solution of (2.23) to be

$$u = \delta\left(t - t_0, x - x_0, y - y_0\right)$$

where t_0, x_0, y_0 are model parameters the equation (2.23) becomes

$$u_t - u_{xx} + xu_y = \delta\left(t - t_0, x - x_0, y - y_0\right) \tag{2.24}$$

given that $u|_{t<t_0}$.

It is found that equation (2.23) admits, in addition to the infinite-dimensional Lie algebra of symmetry operations,

$$Y_1 = 2(t - t_0)\partial_t + (x - x_0)\partial_x - (x_0(t - t_0) - 3(y - y_0))\partial_y - 4u\partial_u$$

$$Y_2 = (t^2 - t_0^2)\partial_t + ((tx + 3y) - (t_0x_0 + 3y_0))\partial_x + (3(y - y_0)t - t_0x_0(t - t_0))\partial_y$$
$$-(2(t - t_0) + x^2 - (x_0^2))u\partial_u$$

$$Y_3 = 3(t^2 - t_0^2)\partial_x + (t^3 - 3t_0^2t + 2t_0^3)\partial_y - 3(tx - y - (t_0x_0 - y_o))u\partial_u$$

$$Y_4 = 2(t - t_0)\partial_x + (t - t_0)^2\partial_y - (x - x_0)u\partial_u.$$

Using the two dimensional algebra $\langle Y_1, Y_4\rangle$ they constructed the solution (2.25)

$$u = \frac{C(t,x,y)}{(t-t_0)^2}\exp\left[-\frac{(x-x_0)^2}{4(t-t_0)} - \frac{3}{(t-t_0)^3}\left(y - y_0 - (t-t_0)\frac{x+x_0}{2}\right)^2\right] \tag{2.25}$$

2.1 Concluding Remarks

This chapter focused on previous work in literature in which Lie symmetry analysis was applied in equations of finance.While this dissertation focuses on the (2+1) dimensional Black-Scholes equation, the literature review included work on (1+1) dimensional parabolic equations which, through relevant extensions, can be applied to the higher dimensional equation.

In the next chapter Lie symmetry methods are introduced and an extension of the work done in [3] for (1+1) dimensional parabolic equations is sought for application in solving (2+1) dimensional equations.

Chapter 3

Lie Symmetry Analysis: An Introduction

This chapter aims to provide a high level introduction into key concepts, definitions and theorems of Lie symmetry analysis. A symmetry is a transformation which when applied to an object leaves it unchanged. The theory of transformation groups was discovered by Sophus Lie [2]. A Lie group of transformations map a given differential equation to itself. Under the group of transformations, which are known as the symmetries of the differential equation the differential equation is invariant, i.e. the symmetry groups do not change the structural form of the differential equation.
Symmetries can be used to deduce new solutions from known solutions of a differential equation, reducing the number of independent variables in a partial differential equation (PDE) thereby reducing the PDE, testing numerical algorithms among other uses.

Definition 1. *"A set G of transformations*

$$T_a : \overline{a}^i = f^i(x, u, a), \overline{u}^\alpha = \phi^\alpha(x, u, a), i = 1 \dots n, \alpha = 1 \dots m,$$

where a is a real parameter which continuously ranges in values from a neighbourhood $\mathbb{D} \subset \mathbb{R}$ *of* $a = 0$ *and* f^i, ϕ^α *are differentiable*

functions, is a continuous one-parameter Lie group of transformations in $\mathbb{R}^{n+m}$ *if the following properties are satisfied:*

1. *Closure. If* T_a, T_b *in* G *and* $a, b \in \mathbb{D}' \in \mathbb{D}$
2. *Identity.* $T_0 \in G$ *such that*

$$T_0 T_a = T_a T_0 = T_a$$

3. *Inverses. For* $T_a \in G, a \in \mathbb{D}' \subset \mathbb{D}$*, there exists* $T_a^{-1} = T_{a^{-1}} \in G, a^{-1} \in \mathbb{D}$ *such that* $T_a T_{a^{-1}} = T_0$

"

If the identity transformation occurs at $a = a_0 \neq 0$, i.e, T_{a_0} is the identity, then the shift of the parameter $a = \overline{a} + a_0$ will give T_0 as above.

3.1 Infinitesimal Transformations

Based on Lie's [2] theory the construction of G is equivalent to determining the corresponding infinitesimal transformations:

$$\overline{x}^i \approx x^i + a\xi^i(x,u), \overline{u}^\alpha \approx u^\alpha + a\eta^\alpha(x,u). \tag{3.1}$$

Infinitesimal transformations are obtained through a Taylor series expansion about $a = 0$ with initial conditions $f^i|_{a=0} = x^i, \phi^\alpha|_{a=0} = u^\alpha$. Thus

$$\xi^i(x,u) = \frac{\partial f^i(x,u,a)}{\partial a}|_{a=0},$$

$$\eta^\alpha(x,u) = \frac{\partial \phi^\alpha(x,u,a)}{\partial a}|_{a=0},$$

Let

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

be the infinitesimal operator or generator of G. The infinitesimal transformations can thus be re-written as

$$\overline{x}^i \approx (1 + aX)x^i, \overline{u}^\alpha \approx (1 + aX)u^\alpha. \tag{3.2}$$

Theorem 1. *"Given the infinitesimal transformations* (3.2) or its symbol X, G is then obtained by solution of the Lie equations

$$\frac{d\overline{x}^i}{da} = \xi^i(\overline{x}, \overline{u}), \frac{d\overline{u}^\alpha}{da} = \eta^\alpha(\overline{x}, \overline{u})$$

subject to the conditions

$$\overline{x}^i|_{a=0} = x^i, \overline{u}^\alpha|_{a=0} = u^\alpha.$$

"

One parameter groups are obtained by using this theorem. The transformations (1) form a symmetry group G of the system if the function $\overline{u} = \overline{u}(\overline{x})$ satisfies

$$E^\sigma(\overline{x}, \overline{u}, \overline{u}_{(1)}, \ldots, \overline{u}_{(k)}) = 0$$

whenever $u = u(x)$ satisfies the original differential equation

$$E^\sigma(x, u, u_{(1)}, \ldots, u_{(k)}) = 0$$

By applying the total derivative operator on (3.3)

$$D_i(\phi^\alpha) = D_i(f^j)\overline{D_j}(\overline{u}^\alpha) = D_i(f^j)\overline{u}^\alpha_j. \tag{3.3}$$

Thus

$$\left(\frac{\partial f^j}{\partial x^i} + u^\beta_i \frac{\partial f^j}{\partial u^\beta}\right)\overline{u}^\alpha_j = \frac{\partial \phi^\alpha}{\partial x^i} + u^\beta_i \frac{\partial \phi^\alpha}{\partial u^\beta}$$

The one parameter group, $G^{[1]}$, is the first prolongation that acts in the space (x, u). Here $G^{[1]}$ is formed by transformations in $(x, u, u_{(1)})$ and the transformations $\overline{u}_{(1)} = \psi(x, u, u_{(1)}, a)$. To obtain the prolonged groups $G^{[2]}$ upto $G^{[k]}$ one uses the total derivative transforms.
Let the infinitesimal transformations of the prolonged groups be $G^{[1]}$ to $G^{[k]}$ be (3.2) and

$$\begin{aligned}
\overline{u}^\alpha_i &\approx u^\alpha_i + a\zeta^\alpha_i(x, u, u_{(1)}) \\
\overline{u}^\alpha_{ij} &\approx u^\alpha_i + a\zeta^\alpha_{ij}(x, u, u_{(1)}, u_{(2)}) \\
&\vdots \\
\overline{u}_{i_1 \ldots i^\alpha_k} &\approx u^\alpha_{i_j \ldots i_k} + a\zeta^\alpha_{i_1 \ldots i_k}(x, u, u_{(1)}, \ldots, u_{(k)}),
\end{aligned}$$

with prolongation formulas

$$\begin{aligned}
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j) \\
\zeta_{ij}^\alpha &= D_j(\zeta_i^\alpha) - u_{il}^\alpha D_j(\xi^l) \\
&\vdots \\
\zeta_{i_1\ldots i_k}^\alpha &= D_{i_j}(\zeta_{i_k\ldots i_{k-1}}^\alpha) - u_{i_1\ldots i_{kl}}^\alpha D_j(\xi^l).
\end{aligned}$$

The generators of the prolonged groups are:

$$\begin{aligned}
X^{[1]} = \xi^i(x,u)\frac{\partial}{\partial x^i} &+ \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x,u,u_{(1)}) \\
&\vdots \\
X^{[k]} = \xi^i(x,u)\frac{\partial}{\partial x^i} &+ \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x,u,u_{(1)}) \\
+\ldots &+ \zeta_{i_1\ldots i_k}^\alpha(x,u,\ldots,u_{(k)})\frac{\partial}{\partial u_{i_1\ldots i_k}^\alpha},
\end{aligned}$$

where X is the generator of the group G.

Definition 2. *"A point $(x,u) \in \mathbb{R}^{n+m}$ is an invariant point if it remains unchanged by every transformation of a group G, i.e. $(\overline{x},\overline{u}) = (x,u)$ form all values of $a \in \mathbb{D} \subset \mathbb{D}$"*

Theorem 2. *"A point $(x,u) \in \mathbb{R}^{n+m}$ is an invariant point of a group G with generator $X = \xi^i(x,u)\frac{\partial}{\partial x_i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$ iff $\xi^i(x,u) = 0 = \eta^\alpha(x,u)$"*

Definition 3. *"A function $F(x,u)$ is an invariant of a group G iff*

$$F(\overline{x},\overline{u}) = F(x,u)$$

for all values of x,u and $a \in \mathbb{D}' \subset \mathbb{D}$."

Theorem 3. *"A function $F(x,u)$ is an invariant of a group G with generator $X = \xi^i(x,u)\frac{\partial}{\partial x_i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$ iff*

$$X(F) = \xi^i(x,u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial F}{\partial u^\alpha} = 0$$

"

The characteristic equations to the PDE then are:

$$\frac{dx^1}{\xi^1} = \cdots = \frac{dx^n}{\xi^n} = \frac{du^1}{\eta^1} = \cdots = \frac{du^m}{\eta^m}$$

3.2 Transformations

From the work of Lie [2], Ibragimov [5] and also Mahomed [3] the equivalence transformations of linear (1+1) parabolic PDE's comprise of linear transformations of the dependent variables given by

$$u = \sigma(t, x)\overline{u}, \sigma \neq 0 \tag{3.4}$$

and invertible transformations of the independent variables

$$\begin{aligned} \overline{t} &= \phi(t) \\ \overline{x} &= \psi(t, x) \\ \dot{\phi} &\neq 0 \\ \psi(x) &\neq 0 \end{aligned} \tag{3.5}$$

where ϕ ,ϕ and σ are arbitrary functions and $\overline{u}$ is the new dependent variable. To verify that (3.4) and (3.5) are indeed equivalence transformations we shall consider the (1+1) dimensional canonical linear parabolic equation:

$$u_t = a(t, x)u_{xx} + b(t, x)u_x + c(t, x)u \tag{3.6}$$

Let

$$u(t, x) = \sigma(t, x)\overline{u}$$

be the transformation of the dependent variable. We differentiate with respect to x and t to find the derivatives under the transformation that is:

$$u_x = \sigma_x\overline{u} + \sigma\overline{u}_x \tag{3.7}$$

$$u_{xx} = \sigma_{xx}\overline{u} + 2\sigma_x\overline{u}_x + \sigma\overline{u}_x \tag{3.8}$$

$$u_t = \sigma_t\overline{u} + \sigma\overline{u}_t \tag{3.9}$$

Substitute (3.7) into the equation (3.6) to get:

$$\sigma_t\overline{u} + \sigma\overline{u}_t = \overline{a}\left(\sigma_{xx}\overline{u} + 2\sigma_x\overline{u}_x + \sigma\overline{u}_{xx}\right) + \overline{b}\left(\sigma_x\overline{u} + \sigma\overline{u}_x\right) + \overline{c}\sigma\overline{u}$$

which when re-arranged results in:

$$\overline{u}_t = \overline{a}\left[\left(\frac{\sigma_{xx}}{\sigma}\right)\overline{u} + 2\frac{\sigma_x}{\sigma}\overline{u}_x + \overline{u}_{xx} - \frac{\sigma_t}{\sigma}\right] + \overline{b}\left[\left(\frac{\sigma_x}{\sigma}\right)\overline{u} + \overline{u}_x\right] + \overline{cu}. \tag{3.10}$$

It can be seen that equation (3.6) is invariant under the transformation $u = \sigma\overline{u}$. Since the equation is invariant under the transformation, we can equate the co-efficients of u_{xx},u_x,u. We thus get the following relationships :

$$a = \overline{a}$$

$$b = 2\overline{a}\left(\frac{\sigma_x}{\sigma}\right) + \overline{b}.$$

$$c = \overline{a}\left(\frac{\sigma_{xx}}{\sigma}\right) + \overline{b}\left(\frac{\sigma_x}{\sigma}\right) + \overline{c} - \left(\frac{\sigma_t}{\sigma}\right)$$

For the purposes of this research we seek to extend this to the (2+1) dimensional case with the canonical form

$$u_t = au_{xx} + bu_{yy} + cu_x + du_y + eu \tag{3.11}$$

in which the Black-Scholes equation with stochastic interest rates is contained and a, b, c, d, e are functions of the independent variables (x, y). Let

$$u = \sigma(t, x, y)\overline{u} \tag{3.12}$$

be a transformation of the dependent variable. Differentiating with respect to the independent variables t, x, y:

$$\begin{aligned}
u_t &= \sigma_t\overline{u} + \sigma\overline{u}_t \\
u_x &= \sigma_x\overline{u} + \sigma\overline{u}_x \\
u_y &= \sigma_y\overline{y} + \sigma\overline{u}_y \\
u_{xx} &= \sigma_{xx}\overline{u} + 2\sigma_x\overline{u}_x + \sigma\overline{u}_{xx} \\
u_{yy} &= \sigma_{yy}\overline{u} + 2\sigma_y\overline{u}_y + \sigma\overline{u}_{yy}.
\end{aligned}$$

Substituting back into the original equation (3.11)

$$\begin{aligned}\sigma_t\overline{u} + \sigma\overline{u}_t = &\ \overline{a}\left(\sigma_{xx}\overline{u} + 2\sigma_x\overline{u}_x + \sigma\overline{u}_{xx}\right)\\ &+ \overline{b}\left(\sigma_{yy}\overline{u} + 2\sigma_y\overline{u}_y + \sigma\overline{u}_{yy}\right)\\ &+ \overline{c}\left(\sigma_x\overline{u} + \sigma\overline{u}_x\right)\\ &+ \overline{d}\left(\sigma_y\overline{y} + \sigma\overline{u}_y\right)\\ &+ \overline{e}\sigma\overline{u}\end{aligned}$$

and after grouping to the partial derivatives of $\overline{u}$

$$\begin{aligned}\overline{u}_t = &\ \overline{a}\left[\left(\frac{\sigma_{xx}}{\sigma}\right)\overline{u} + 2\left(\frac{\sigma_x}{\sigma}\right)\overline{u}_x + \overline{u}_{xx}\right]\\ &+ \overline{b}\left[\left(\frac{\sigma_{yy}}{\sigma}\right)\overline{u} + 2\left(\frac{\sigma_y}{\sigma}\right)\overline{u}_y + \overline{u}_{yy}\right]\\ &+ \overline{c}\left[\left(\frac{\sigma_x}{\sigma}\right)\overline{u} + \overline{u}_x\right]\\ &+ \overline{d}\left[\left(\frac{\sigma_y}{\sigma}\right)\overline{u} + \overline{u}_y\right]\\ &+ \overline{eu} - \left(\frac{\sigma_t}{\sigma}\right)\overline{u}.\end{aligned}$$

Comparing co-efficients between equation (3.11) and the transformed equation it can be seen that the following should be hold if (3.11) is invariant under the transformation (3.12), namely

$$\begin{aligned}a &= \overline{a}\\ b &= \overline{b}\\ c &= 2\overline{a}\left(\frac{\sigma_x}{\sigma}\right) + \overline{c}\\ d &= 2\overline{b}\left(\frac{\sigma_y}{\sigma}\right) + \overline{d}\\ e &= \overline{a}\left(\frac{\sigma_{xx}}{\sigma}\right) + \overline{b}\left(\frac{\sigma_{yy}}{\sigma}\right) + \overline{c}\left(\frac{\sigma_x}{\sigma}\right) + \overline{d}\left(\frac{\sigma_y}{\sigma}\right) + \overline{e} - \left(\frac{\sigma_t}{\sigma}\right)\end{aligned}$$

So far we have applied a transformation on the dependent variable (3.4) for the one dimensional equation and (3.12) for the two dimensional case. The next step would be to calculate the semi-invariants and deduce σ in the transformation function which maps u to $\overline{u}$.

An example

We provide an illustrative example on symmetries which was investigated in [29]. Consider the (2 +1) dimensional PDE

$$e^{-2x}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} \tag{3.13}$$

As shown in [30], the infinitesimal transformations sought for (3.13) are

$$\begin{aligned}
\overline{t} &= t + \varepsilon\xi_1(t,x,y,u) + O(\xi_1)\\
\overline{x} &= x + \varepsilon\xi_2(t,x,y,u) + O(\xi_2)\\
\overline{y} &= x + \varepsilon\xi_3(t,x,y,u) + O(\xi_3)\\
\overline{u} &= u + \varepsilon\eta(t,x,y,u) + O(\eta)
\end{aligned}$$

which are generated by

$$X = \xi_1\frac{\partial}{\partial t} + \xi_2\frac{\partial}{\partial x} + \xi_3 + \eta\frac{\partial}{\partial u} \tag{3.14}$$

For a (2 + 1) dimensional PDE the second prolongation is needed:

$$\begin{aligned}
X^{[2]} = \quad & \xi_1\frac{\partial}{\partial t} + \xi_2\frac{\partial}{\partial x} + \xi_3\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \zeta_t\frac{\partial}{\partial u_t}\\
& + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xx}\frac{\partial}{\partial u_{xx}}\\
& + \zeta_{ty}\frac{\partial}{\partial u_{ty}} + \zeta_{xy}\frac{\partial}{\partial u_{xy}} + \zeta_y\frac{\partial}{\partial u_y} + \zeta_{yy}\frac{\partial}{\partial u_{yy}} + \dots
\end{aligned}$$

The determining equations are found to

$$\begin{gathered}
\xi_x^1 = 0 \\
\xi_y^1 = 0 \\
\eta_{uu} = 0 \\
\xi_u^3 = 0 \\
\xi_u^2 = 0 \\
\xi_u^1 = 0 \\
\xi_y^2 + \xi_x^3 = 0 \\
-2\xi^2 + 2\xi_x^2 - \xi_t^1 = 0 \\
-2\xi^2 + 2\xi_y^3 - \xi_t^1 = 0 \\
e^{2x}\xi_x^2 + 2e^{2x}\eta_{xu} + \xi_t^2 = 0 \\
e^{2x}\eta_{yy} + e^{2x}\eta_x + e^{2x}\eta_{xx} - \eta_t = 0 \\
2e^{2x}\eta_{yu} - e^{2x}\xi_{yy}^3 - e^{2x}\xi_x^3 - e^{2x}\xi_{xx}^3 + \xi_t^3 = 0
\end{gathered} \tag{3.15}$$

When the system of determining equations (3.15) is solved, we find for (3.13) five symmetries

$$X_1 = \frac{\partial}{\partial y}, \tag{3.16}$$

$$X_2 = \frac{\partial}{\partial t}, \tag{3.17}$$

$$X_3 = -\frac{1}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, \tag{3.18}$$

$$X_4 = -t\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - u\left(\frac{e^{-2x}}{4} + \frac{t}{2}\right)\frac{\partial}{\partial u}, \tag{3.19}$$

$$X_5 = u\frac{\partial}{\partial u}. \tag{3.20}$$

Consider the symmetry (3.19). We work out its invariants, thus we have

$$\frac{dx}{-t} = \frac{dt}{t^2} = \frac{du}{u\left(\frac{e^{-2x}}{4} + \frac{t}{2}\right)} = \frac{dy}{0}. \tag{3.21}$$

It immediately follows from (3.21) that

$$I_1 = y$$

From

$$\frac{dx}{-t} = \frac{dt}{t^2}$$

Integrating both sides

$$\ln t = -x + \ln I_2$$

such that

$$I_2 = te^x. \tag{3.22}$$

Again, from (3.21)

$$\frac{dt}{t^2} = \frac{du}{u\left(\frac{e^{-2x}}{4} + \frac{t}{2}\right)}, \tag{3.23}$$

which can be re-written as

$$\frac{du}{u} = -\left(\frac{t}{2} + \frac{e^{-2x}}{4}\right)\frac{dt}{t^2}, \tag{3.24}$$

we can deduce from (3.22) that

$$e^{-x} = \frac{t}{I_2}. \tag{3.25}$$

Substitute (3.25) into equation (3.24) to get

$$\frac{du}{u} = -\left(\frac{1}{2t} + \frac{1}{4I_2^2}\right)dt. \tag{3.26}$$

After integration of both sides of equation(3.26), after substituting back (3.22), we deduce

$$u = F(\alpha, \psi)\frac{1}{\sqrt{t}}\exp\left(-\frac{e^{-2x}}{4t}\right), \tag{3.27}$$

where $\alpha = I_1$, $I_2 = \psi$ and $I_3 = F(\alpha, \psi)$

Taking derivatives of u in terms of the independent variables:

$$\begin{aligned} u_t \quad &= -\tfrac{1}{2}t^{-3/2}\exp\left(\tfrac{-e^{-2x}}{4t}\right)F + t^{-5/2}\tfrac{e^{-2x}}{4}\exp\left(\tfrac{-e^{-2x}}{4t}\right)F \\ &\quad + t^{-1/2}\exp\left(\tfrac{-e^{-2x}}{4t}\right)e^x F_\psi, \end{aligned}$$

$$\begin{aligned} u_x &= \tfrac{e^{-2x}}{2t^{3/2}} \exp\left(\tfrac{-e^{-2x}}{4t}\right) F \\ &\quad + \sqrt{t} e^x \exp\left(\tfrac{-e^{-2x}}{4t}\right) F_\psi, \end{aligned}$$

$$\begin{aligned} u_{xx} &= -\tfrac{e^{-2x}}{t^{3/2}} \exp\left(\tfrac{-e^{-2x}}{4t}\right) F + \tfrac{e^{-4x}}{4t^{5/2}} \exp\left(\tfrac{-e^{-2x}}{4t}\right) F + \tfrac{e^{-x}}{2t^{1/2}} \exp\left(\tfrac{-e^{-2x}}{4t}\right) F_\psi \\ &\quad + \sqrt{t} e^x \exp\left(\tfrac{-e^{-2x}}{4t}\right) F_\psi + \tfrac{e^{-x}}{2\sqrt{t}} \exp\left(\tfrac{-e^{-2x}}{4t}\right) F_\psi \\ &\quad + t^{3/2} e^{2x} \exp\left(\tfrac{-e^{-2x}}{4t}\right) F_{\psi\psi}, \end{aligned}$$

$$u_{yy} = F_{\alpha\alpha} \frac{1}{\sqrt{t}} \exp\left(\frac{-e^{-2x}}{4t}\right). \tag{3.28}$$

Substituting into (3.13) and simplifying (the intermediate steps are not included here to avoid unnecessary cluster), the reduced equation is found as

$$2te^x F_\psi + t^2 e^{2x} F_{\psi\psi} + F_{\alpha\alpha} = 0,$$

which, since $\psi = te^x$,reduces to the simplified reduced equation

$$F_{\alpha\alpha} + \psi^2 F_{\psi\psi} + 2\psi F_\psi = 0. \tag{3.29}$$

By the method of separation of variables a solution for $u(t, x, y)$ is

$$u(t, x, y) = \frac{1}{t} \exp\left(\frac{-e^{-2x}}{4t}\right) (c_1 + yc_2) \frac{c_3}{te^x}, \tag{3.30}$$

where c_1, c_2, c_3 are integration constants.

This example shows that one can use Lie Symmetry methods to reduce an n-dimensional equation to an $(n-1)$-dimensional equation which reduces the original equation to a simpler form.

3.3 Invariant Methods

Equivalence transformations of PDEs are transformations that map the PDE into itself while maintaining the characteristics of the original equation. A lot of research can be found in literature of the use of invariants and semi-invariants with the main focus being on the (1+1)PDE of the form:

$$u_t = a(x)u_{xx} + b(x)u_x + c(x)u, \tag{3.31}$$

where t and x are independent variables, $u = u(t, x)$. Ibragimov [4] calculated invariants for hyperbolic, elliptic and parabolic equations by use of the infinitesimal methods. In his work a new invariant for parabolic equations was provided as K, where

$$K = \left(\frac{1}{2}b^2a_x + a_t + aa_{xx} - a_x^2\right)b + (aa_x - ab)\,b_x - ab_t - a^2b_{xx} + 2a^2c_x, \tag{3.32}$$

where a, b, c are co-efficients of the parabolic equation

$$u_t = a(t, x)u_{xx} + b(t, x)u_x + cu. \tag{3.33}$$

Ibragimov [4] then raises the question of whether the PDE is guaranteed to be reducible to the heat equation iff the invariant K vanishes i.e. when K-invariant is zero. Mahomed [3] further simplified the K invariant of Ibragimov [4] to:

$$k = \frac{\partial}{\partial x}\left(c - a\left(\frac{b}{2a}\right)_x - \frac{b^2}{4a}\right) + \left(\frac{b}{2a}\right)_t, \tag{3.34}$$

in which $K = -2a^2k$.

The k (3.34) invariant for the (1+1) model is not difficult to derive and is shown below. After applying the transformation $u = \sigma(t, x)\overline{u}$ on equation (3.6) the result was (3.10). Since the two equations have the same characteristics under the transformation, corresponding co-efficients of the derivative terms can be equated resulting in:

$$\frac{\sigma_x}{\sigma} = \frac{c}{2a} - \frac{\overline{c}}{2\overline{a}},$$

$$\frac{\sigma_y}{\sigma} = \frac{d}{2b} - \frac{\overline{d}}{2\overline{b}},$$
$$\frac{\sigma_t}{\sigma} = \overline{a}\left(\frac{\sigma_{xx}}{\sigma}\right) + \overline{b}\left(\frac{\sigma_{yy}}{\sigma}\right) + \overline{c}\left(\frac{\sigma_x}{\sigma}\right) + \overline{d}\left(\frac{\sigma_y}{\sigma}\right) + \overline{e} - e.$$

We make use of the compatibility rule

$$\left(\frac{\sigma_x}{\sigma}\right)_y = \left(\frac{\sigma_y}{\sigma}\right)_x,$$

which should hold true. The compatibility condition above will aid in finding the semi-invariants of the equation.

Given

$$\frac{\sigma_x}{\sigma} = \frac{b}{2a} - \frac{\overline{b}}{2\overline{a}},$$

to simplify σ_t/σ we first have to reduce the term σ_{xx}/σ:

$$\frac{\partial}{\partial x}\left(\frac{\sigma_x}{\sigma}\right) = \left(\frac{\sigma_{xx}}{\sigma}\right) - \left(\frac{\sigma_x}{\sigma}\right)^2,$$

$$\frac{\sigma_{xx}}{\sigma} = \left(\frac{\sigma_x}{\sigma}\right)_x + \left(\frac{\sigma_x}{\sigma}\right)^2.$$

Therefore:

$$\frac{\sigma_{xx}}{\sigma} = \left(\frac{b}{2a}\right)_x - \left(\frac{\overline{b}}{2\overline{a}}\right)_x + \left(\frac{b^2}{4a^2}\right) + \left(\frac{\overline{b}^2}{4\overline{a}^2}\right) - \left(\frac{b\overline{b}}{2a\overline{a}}\right).$$

Substituting into $\frac{\sigma_t}{\sigma}$:

$$\frac{\sigma_t}{\sigma} = \overline{c} - a\left(\frac{b}{2a}\right)_x - \overline{a}\left(\frac{\overline{b}}{2\overline{a}}\right)_x + \left(\frac{b^2}{4a}\right) + \left(\frac{\overline{b}^2}{4\overline{a}}\right) - \left(\frac{b}{2\overline{a}}\right) - c,$$

$$\frac{\sigma_x}{\sigma} = \frac{b}{2a} - \frac{\overline{b}}{2\overline{a}}.$$

Since:

$$\left(\frac{\sigma_t}{\sigma}\right)_x = \left(\frac{b}{2a}\right)_t - \left(\frac{\overline{b}}{2\overline{a}}\right)_t,$$

then:

$$\left(\frac{b}{2a}\right)_t - \left(\frac{\overline{b}}{2\overline{a}}\right)_t = \overline{c}_x + \left[a\left(\frac{b}{2a}\right)_x\right]_x - \left[\overline{a}\left(\frac{\overline{b}}{2\overline{a}}\right)_x\right]_x + \left[\left(\frac{b^2}{4a}\right)\right]_x + \left[\left(\frac{\overline{b}^2}{4\overline{a}}\right)\right]_x - \left[\left(\frac{\overline{b}^2}{2\overline{a}}\right)\right]_x - c_x.$$

Re-arranging:

$$c_x - \left[a\left(\frac{b}{2a}\right)_x\right]_x - \left[\left(\frac{b^2}{4a}\right)\right]_x + \left(\frac{b}{2a}\right)_t = \overline{c}_x - \left[\overline{a}\left(\frac{\overline{b}}{2\overline{a}}\right)_x\right]_x - \left(\frac{\overline{b}^2}{4\overline{a}}\right)_x - \left[\left(\frac{\overline{b}^2}{2\overline{a}}\right)\right]_t.$$

Since $k = \overline{k}$ we find the invariant as:

$$k = \frac{\partial}{\partial x}\left[c - a\left(\frac{b}{2a}\right)_x - \left(\frac{b^2}{4a}\right)\right] + \left(\frac{b}{2a}\right)_t. \tag{3.35}$$

For the purposes of this research we need to find the invariants for the (2+1) parabolic equation (3.38) case making use of the same method.

Expanding :

$$\left(\frac{c}{2a}\right)_y - \left(\frac{\overline{c}}{2\overline{a}}\right)_y = \left(\frac{d}{2b}\right)_x - \left(\frac{\overline{d}}{2\overline{b}}\right)_x,$$

$$\left(\frac{c}{2a}\right)_y - \left(\frac{d}{2b}\right)_x = \left(\frac{\overline{c}}{2\overline{a}}\right)_y - \left(\frac{\overline{d}}{2\overline{b}}\right)_x.$$

We find that $h = \overline{h}$ where

$$h = \left(\frac{c}{2a}\right)_y - \left(\frac{d}{2b}\right)_x.$$

Considering now that $\left(\frac{\sigma_x}{\sigma}\right)_t = \left(\frac{\sigma_t}{\sigma}\right)_x$ and expanding $\left(\frac{\sigma_t}{\sigma}\right)_x$:

$$\begin{aligned}\left(\frac{\sigma_t}{\sigma}\right)_x = &\left[a\left(\frac{c}{2a}\right)_x - \overline{a}\left(\frac{\overline{c}}{2\overline{a}}\right)_x + \left(\frac{c^2}{4a}\right) - a\left(\frac{c\overline{c}}{2a\overline{a}}\right) + \left(\frac{\overline{c}^2}{4\overline{a}}\right)\right]_x \\ &+ \left[b\left(\frac{d}{2b}\right)_y - \overline{b}\left(\frac{\overline{d}}{2\overline{b}}\right)_y + \left(\frac{d^2}{4b}\right) - \overline{b}\left(\frac{d\overline{d}}{2b\overline{b}}\right) + \left(\frac{\overline{d}^2}{4\overline{b}}\right)\right]_x \\ &+ \left[\overline{c}\left(\frac{c}{2a}\right)\right]_x - \left[\overline{c}\left(\frac{\overline{c}}{2\overline{a}}\right)\right]_x + \left[\overline{d}\left(\frac{d}{2b}\right)\right]_x - \left[\overline{d}\left(\frac{\overline{d}}{2\overline{b}}\right)\right]_x \\ &+ \overline{e}_x - e_x.\end{aligned}$$

$$\begin{aligned}\left(\frac{c}{2a}\right)_t - \left(\frac{\overline{c}}{2\overline{a}}\right)_t = &\\ &\left[a\left(\frac{c}{2a}\right)_x - \overline{a}\left(\frac{\overline{c}}{2\overline{a}}\right)_x + \left(\frac{c^2}{4a}\right) - \left(\frac{\overline{c}^2}{4\overline{a}}\right)\right]_x \\ &+ \left[b\left(\frac{d}{2b}\right)_y - \overline{b}\left(\frac{\overline{d}}{2\overline{b}}\right)_y + \left(\frac{d^2}{4b}\right) - \left(\frac{\overline{d}^2}{4\overline{b}}\right)\right]_x \\ &+ \overline{e}_x - e_x.\end{aligned}$$

Grouping terms together yields $j = \overline{j}$ where

$$j = \frac{\partial}{\partial x}\left[e - a\left(\frac{c}{2a}\right)_x - \left(\frac{c^2}{4a}\right) - b\left(\frac{d}{2b}\right)_y - \left(\frac{d^2}{4b}\right)\right] + \left(\frac{c}{2a}\right)_t.$$

Again when considering $\left(\frac{\sigma_t}{\sigma}\right)_y = \left(\frac{\sigma_y}{\sigma}\right)_t$ we find again that $m = \overline{m}$ with

$$m = \frac{\partial}{\partial y}\left[e - a\left(\frac{c}{2a}\right)_x - \left(\frac{c^2}{4a}\right) - b\left(\frac{d}{2b}\right)_y - \left(\frac{d^2}{4b}\right)\right] + \left(\frac{d}{2b}\right)_t.$$

To summarise, we state the invariants for the (1+1) D parabolic equation

$$a = \overline{a}, \tag{3.36}$$

$$k = \frac{\partial}{\partial x}\left[c - a\left(\frac{b}{2a}\right)_x - \left(\frac{b^2}{4a}\right)\right] + \left(\frac{b}{2a}\right)_t. \tag{3.37}$$

3.3.1 Application of Invariant Methods to 2 + 1 D Equations

In this section an extension to apply invariant methods to equations with two space variables and one time variable is proposed. Consider the equation of the form

$$u_t + Au_{xx} + Bu_{yy} + Cu_x \quad + Du_y + Eu = 0. \tag{3.38}$$

Using the change of the variables of the equation (3.38)

$$\tau = \phi(t), \alpha = \psi(t, x, y), \beta = \theta(t, x, y),$$

the derivatives of u in (3.38) undergo the transformations

$$\begin{aligned}
u_t &= \phi' u_\tau + \psi_t u_\alpha + \theta_t u_\beta, \\
u_x &= \psi_x u_\alpha + \theta_x u_\beta, \\
u_{xx} &= \psi_{xx} u_\alpha + \psi_{\alpha\alpha}^2 + \theta_{xx} u_\beta + \theta_x^2 u_{\beta\beta}, \\
u_y &= \psi_y u_\alpha + \theta_y u_\beta, \\
u_{yy} &= \psi_{yy} u_\alpha + \psi_y^2 u_{\alpha\alpha} + \theta_{yy} u_\beta + \theta_y^2 u_{\beta\beta}.
\end{aligned}$$

Substituting the transformed derivatives of u into equation (3.38) results in

$$\begin{aligned}
\phi' u_\tau \quad &+ \psi_t u_\alpha + \theta_t u_\beta + A\left(\psi_{xx} u_\alpha + \psi_{\alpha\alpha}^2 + \theta_{xx} u_\beta + \theta_x^2 u_{\beta\beta}\right) \\
&+ B\left(\psi_{yy} u_\alpha + \psi_y^2 u_{\alpha\alpha} + \theta_{yy} u_\beta + \theta_y^2 u_{\beta\beta}\right) \\
&+ C\left(\psi_x u_\alpha + \theta_x u_\beta\right) + D\left(\psi_y u_\alpha + \theta_y u_\beta\right) \\
&+ Eu = 0.
\end{aligned}$$

Grouping like terms in u and derivatives of u yields the equation

$$\begin{aligned}
&\phi' u_\tau + \left(A\psi_x^2 + B\psi_y^2\right) u_{\alpha\alpha} + \left(A\theta_x^2 + B\theta_y^2\right) u_{\beta\beta} \\
&+ \left(\psi_t + A\psi_{xx} + B\psi_{yy} + C\psi_x + D\psi_y\right) u_\alpha \\
&+ \left(\theta_t + A\theta_{xx} + B\theta_{yy} + C\theta_x + D\theta_y\right) u_\beta \\
&+ Eu = 0.
\end{aligned} \tag{3.39}$$

By choosing A, B such that $|A|\psi_x^2 + |B|\psi_y^2 = 1$ and $\theta = \pm t$ equation (3.39) can be written as

$$u_t + u_{\alpha\alpha} + u_{\beta\beta} + a(t,x,y)u_\alpha + b(t,x,y)u_\beta + Eu = 0,$$

where $a(t,x,y), b(t,x,y)$ are continuous variables in the independent variables. Thus equation (3.38) can be transformed to an equation of the form

$$u_t + u_{xx} + u_{yy} + a(t,x,y)u_x + b(t,x,y)u_y + Eu = 0, \tag{3.40}$$

by a suitable choice of transformations of the independent variables y, x, y.

In the next step the equivalence transformation

$$v = ue^{\rho}, u = ve^{-\rho}, \tag{3.41}$$

is introduced to equation (3.40), where $\rho = \rho(t,x,y)$. Differentiating u under the equivalence transformation results in

$$\begin{aligned}
u &= ve^{-\rho} \\
u_t &= (v_t - \rho_t v)\, e^{-\rho} \\
u_x &= (v_x - \rho_x v)\, e^{-rho} \\
u_y &= (v_y - \rho_y v)\, e^{-rho} \\
u_{xx} &= (v_{xx} - 2\rho_x v_x + (\rho_x^2 - \rho_{xx})\, v)\, e^{-\rho} \\
u_{yy} &= \left(v_{yy} - 2\rho_y v_y + \left(\rho_y^2 - \rho_{yy}\right) v\right) e^{-\rho}
\end{aligned}$$

Substituting the transformations into equation (3.40) yields

$$\begin{aligned}
(v_t - \rho_t)\, e^{-rho} - \left(v_{xx} - 2\rho_x v_x + \left(\rho_x^2 - \rho_{xx}\right) v\right) - \left(v_{yy} - 2\rho_y v_y + \left(\rho_y^2 - \rho_{yy}\right) v\right) e^{-\rho} \\
+ a\,(v_x - \rho_x v)\, e^{-\rho} + b\,(v_y - \rho_y v)\, e^{-\rho} + Eve^{-\rho} = 0.
\end{aligned}$$

Grouping the terms in v and derivatives of v together and dropping the common term $e^{-\rho}$ results in

$$\begin{aligned}
v_t - v_{xx} - v_{yy} + (a + 2\rho_x)\, v_x + (b + 2\rho_y)\, v_y \\
+ \left(\rho_{xx} + \rho_{yy} - \rho_x^2 - \rho_y^2 - a\rho_x - b\rho_y + E - \rho_t\right) v = 0
\end{aligned} \tag{3.42}$$

Due to the equivalence relation equations (3.39) and (3.42) are equal such that

$$\begin{aligned} u_t \quad & +u_{xx} + u_{yy} + a(t,x,y)u_x + b(t,x,y)u_y + Eu \\ & = v_t - v_{xx} - v_{yy} + (a + 2\rho_x)\, v_x + (b + 2\rho_y)\, v_y \\ & + \left(\rho_{xx} + \rho_{yy} - \rho_x^2 - \rho_y^2 - a\rho_x - b\rho_y + E - \rho_t\right) v. \end{aligned} \tag{3.43}$$

Thus equation (3.40) can be reduced to the heat equation

$$v_t - v_{xx} - v_{yy} = 0, \tag{3.44}$$

if the following conditions holds

- $\rho_x = -\frac{1}{2}a$,
- $\rho_y = -\frac{1}{2}b$,
- $\rho_t = \frac{1}{4}a^2 + \frac{1}{4}b^2 - \frac{1}{2}a_x - \frac{1}{2}b_y + e$.

The compatibility conditions $\rho_{xy} = \rho_{yx}, \rho_{xt} = \rho_{tx}, \rho_{yt} = \rho_{ty}$ should hold. By the compatibility condition three semi-invariants for equation (3.40)

- $a_y - b_x = 0$,
- $\frac{1}{2}a_x + \frac{1}{2}b_x - \frac{1}{2}a_{xx} - \frac{1}{2}b_{yx} + \frac{1}{2}a_t + e_x = 0$,
- $\frac{1}{2}a_y + \frac{1}{2}b_y - \frac{1}{2}a_{xy} - \frac{1}{2}b_{yy} + \frac{1}{2}b_t + e_y = 0$.

In this dissertation one of the equations focused on is the Black Scholes model with interest rates that are determined by the Vasicek model. In the following, the idea is to investigate if the equation (6.1) can be transformed to the heat equation.

3.3.2 Application of Invariant Methods 2+1 D Black Scholes model

In this section the idea is to reduce the Black Scholes equation with Vasicek driven interest rates to a form in which the co-efficients of the second partial derivatives V_{SS} and V_{rr} are eliminated i.e. we seek to reduce:

$$u_t = -\frac{1}{2}\sigma^2 S^2 u_{SS} - \frac{1}{2}\Sigma^2 u_{rr} + \kappa r u_r - rSu_S + ru, \tag{3.45}$$

to a form:

$$u_t = u_{cc} + u_{xx} + a(c,x)u_c + b(x,y)u_r + d(c,x,t)u, \tag{3.46}$$

in which $u = u(c,x,t)$.

We shall consider the following transformations on the variables (S,r,t):

$$c = \frac{\sqrt{2}}{\sigma} logS,$$
$$x = \frac{\sqrt{2}x}{\Sigma},$$
$$t = -\bar{t}.$$

Based on these transformations the partial derivatives of u are found as:

$$u_S = \frac{\sqrt{2}}{\sigma S}u_c,$$
$$u_{SS} = \frac{2}{\sigma^2 S^2}u_{cc} - \frac{1}{\sigma S^2}u_c,$$
$$u_r = \frac{\sqrt{2}}{\Sigma}u_x.$$

$$u_{rr} = \frac{2}{\Sigma^2} u_{xx},$$

$$u_t = -u_{\bar{t}}.$$

Substituting the derivatives of u under transformation into (3.45) results in

$$\begin{aligned} -u_{\bar{t}} \; &= \frac{1}{2}\sigma^2 S^2 \left[\frac{2}{\sigma^2 S^2} u_{cc} - \frac{1}{\sigma S^2} u_c \right] - \frac{1}{2}\Sigma^2 \left[\frac{2}{\Sigma^2} u_{xx} \right] \\ &\quad + \kappa r \left[\frac{\sqrt{2}}{\Sigma} u_x \right] - rS \left[\frac{\sqrt{2}}{\Sigma} u_x \right] + ru. \end{aligned}$$

Grouping like terms together into (3.45) results in the transformed equation:

$$u_t = u_{cc} + u_{xx} + \left[\frac{\Sigma x}{\sigma} + \frac{\sigma}{\sqrt{2}} \right] u_c - \kappa x u_x + \frac{\Sigma x}{\sqrt{2}} u. \tag{3.47}$$

From (3.47):

$$\begin{aligned} a &= \frac{\Sigma x}{\sigma} + \frac{\sigma}{\sqrt{2}}, \\ b &= -\kappa x, \\ e &= \frac{\Sigma x}{\sqrt{2}}. \end{aligned}$$

and the compatibility condition is not satisfied since

$$a_c \neq b_x.$$

This raises the question of whether the inability to satisfy the compatibility condition means the (2+1) dimensional model cannot be reduced to the heat equation?

3.4 Concluding Remarks

An introduction into some concepts, definitions and theorems of Lie symmetry methods was provided in this chapter together with an example. Also a possible extension of the work done in [3] for which further investigation are necessary was proposed. The main purpose of this chapter was to provide a basis of mathematical concepts which are relevant for this dissertation.

Chapter 4

Stochastic Calculus in Finance and Interest Rate Models

This chapter provides an introduction to basic stochastic calculus which shall be applied in the derivation of the Black-Scholes model with stochastic interest rates. It will be worth pointing out that this brief introduction of the key definitions and theorems is not meant to be exhaustive of all the principles of stochastic calculus but to provide a high level overview of concepts that apply to the derivation of the model for this dissertation. This chapter is mainly well known concepts and is based on the book by Shreeve [21]. To complete the understanding of the derivations interest rate models will also be introduced, restricted to the Vasicek, C.I.R and simple binomial models.

4.1 Basic Stochastic Calculus

The behaviour of a non-dividend paying stock prices is most generally modelled using Geometric Brownian Motion (GBM). Consider the price of a non-dividend paying stock S_t, a random variable denoting the price of the stock at time $t \geq 0$. Here S_t is regarded as the solution of a stochastic differential equation (SDE)

$$dS_t = \mu S_t + \sigma S_t dt + \sigma S_t dW_t, \tag{4.1}$$

where S_0 is constant , μ is the expected continuous rate of return expected by an investor in the stock, σ is the volatility if the stock price and $W_t(t \geq 0)$ is a Wiener process or Brownian Motion.

Definition 4. *"(Brownian Motion) A stochastic Process $W_t(t \geq 0)$ is called a standard Brownian Motion or Wiener process if*

1. *$W_0 = 0$*
2. *For all $0 \leq t_1 < t_2 \leq t_3 < t_4$, $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ are independent i.e. increments are independent;*
3. *For all $0 \leq t_1 \leq t_2$, $W_{t_2} - W_{t_1} \sim \mathcal{N}(0, t_2 - t_1)$; and*
4. *W has continuous trajectories.*

"

To solve an SDE, one needs to make use of one of the fundamental tools of stochastic calculus which is Itô's formula. We first define Itô's processes.

Definition 5. *"(Itô's Processes). A stochastic process X is called an Itô Process if it is the solution of an SDE*

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \quad X_0 \quad constant \tag{4.2}$$

where W is a Brownian Motion."

Theorem 4. *"(One Dimensional Itô Formula.) Let X be an Itô process determined by* (4.2) and $g \in C^{1,2}([0, \infty)x\mathbb{R})$. Then the

process $g(1, X)$ follows the SDE

$$\begin{aligned} dg(t, X_t) &= \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial d}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}(t, X_t)(dX_t)^2, \\ &= \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial d}{\partial x}(t, X_t)(a(t, X_t)dt + b(t, X_t)dW_t) \\ &+ \frac{1}{2}\frac{\partial^2}{\partial x^2}(t, X_t)(a(t, X_t)dt + b(t, X_t)dW_t), \\ &= \left(\frac{\partial g}{\partial t}(t, X_t) + a(t, X_t)\frac{\partial g}{\partial x}(t, X_t) + \frac{1}{2}b^2(t, X_t)\frac{\partial^2}{\partial x^2}(t, X_t)\right)dt \\ &+ b(t, X_t)\frac{\partial g}{\partial x}(t, X_t)dW_t. \end{aligned}$$

and is also an Itô process following the rules

- $(dt)^2 = 0$,
- $dt \times dW_t = 0$,
- $(dW_t)^2 = 0$.

”

It is generally accepted that prices follow Geometric Brownian Motion (GBM) with the following dynamics

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t). \tag{4.3}$$

Itô's formula is applied in finding the solution to the asset price process (4.3). First we set $g(t, S) := \log S$. By taking derivatives we find

$$\begin{aligned} \frac{\partial g}{\partial t} &= 0, \\ \frac{\partial g}{\partial x} &= \frac{1}{x}, \\ \frac{\partial^2 g}{\partial x^2} &= -\frac{1}{x^2}. \end{aligned}$$

By Itô's formula,

$$\begin{aligned} dg(t, S(t)) &= \frac{1}{S(t)}dS(t) + \frac{1}{2}(-\frac{1}{S(t)^2}(dS(t))^2, \\ &= \frac{1}{S(t)}(\mu S(t)dt + \sigma S(t)dW(t)) - \frac{1}{2S(t)^2}\sigma^2 S(t)^2 dt, \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW(t). \end{aligned}$$

In integral form

$$g(t, S(t)) = g(0, S(0)) + \int_0^t (\mu - \frac{1}{2}\sigma^2)ds + \int_0^t \sigma dW(s).$$

Using Itô's Lemma it can be easily shown that

$$\int_0^t dW(s) = W(t)$$

and

$$\int_0^t W(s)dW(s) = \frac{1}{2}W(t)^2 - \frac{1}{2}t.$$

Applying these results to the Itô integral results in

$$\log S(t) = \log S(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma Wt$$

such that

$$S(t) = S(0) \exp\left[(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\right].$$

Definition 6. *"(Risk free Asset.) There is an asset in the market whose price follows the process*

$$dB(t) = rB(t)dt, r \geq 0 \tag{4.4}$$

where r is the risk free interest rate."

Definition 7. *"(Martingale) A stochastic process* $\{M_t\}_{t\geq 0}$ *on* $(\Omega, \Im, P)$ *is a martingale with respect to the filtration* $\{\Im_t\}_{t\geq o}$ *on* $(\Omega, \Im, P)$ *if the following properties hold:*

1. M_t *is* $\Im_t$*- measurable*
2. $\mathbb{E}[|\mathrm{M_t}|] < \infty$ *for all* t
3. $\mathbb{E}[|\mathrm{M_t}|\Im_{\mathrm{s}}^{\mathrm{M}}] = \mathrm{M_s}$ *for all* $0 \leq s < t < \infty$.

”

Definition 8. *“(Girsanov Theorem) Let P be a probability measure, W^P be a Brownian Motion following the P-dynamics on $(\Omega, \Im, P)$ and let φ be an adapted process. Given a process L defined on $[0,T]$, where T is fixed, by*

$$\begin{aligned} dL_t &= \varphi_t L_t dW_t^P \\ L_0 &= 1 \end{aligned}$$

that is

$$L_t = \exp\left[\int_0^t \varphi_s dW_s^P - \frac{1}{2}\int_0^t ||\varphi_s||^2 ds\right]$$

Assume that

$$\mathbb{E}[\mathrm{L_T}] = 1$$

and define the new probability measure Q on $\Im_T$ to be

$$L_T = \frac{dQ}{dP}.$$

Then

$$dW_t^P = \varphi_t dt + dW_t^Q$$

where W^Q is a dynamic Brownian Motion. This Q is known as the risk neutral probability measure defined such that the asset price and the discounted asset price are equal under this measure. ”

Theorem 5. *“(Feynman-Kac Theorem)” Let $K(x)$ be a non-negative, continuous function,and let $f(x)$ be bounded and continuous. Suppose that $u(t,x)$ is a bounded function that satisfies the heat equation with cooling term*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - K(x)u \tag{4.5}$$

and the initial condition

$$u(0,x) = \lim_{(t,x)\to(0,x)} u(t,x) = f(x)$$

Then

$$u(t,x) = E^x exp\Big\{ -\int_0^t K(W_s)ds \Big\} f(W_t) \qquad (4.6)$$

where, under the probability measure P^x the process $\{W_t\}_{t\geq 0}$ is a Brownian Motion started at x.

The Feynman-Kac theorem provides the basis of the existence of a solution to the heat equation. In this research the conditions necessary for linear (2+1) parabolic equations to be reducible to the heat equation will be proposed.

4.2 Interest Rate Models

The fundamental ideas on which derivative instrument pricing rest on the concepts of replication and arbitrage. To replicate a financial instrument is to construct a portfolio of assets in the economy which, at any given point in time, is equivalent in value to that of the derivative. An arbitrage opportunity is the same as "getting free lunch", i.e. having a profitable trading strategy while taking on no risk. In the absence of arbitrage the derivative value is then also the portfolio value which replicates it. The no arbitrage principle together with the law of one price create the idealized world in which interest rate models are built from and as such, the Black-Scholes models.

The major types of interest rate models are

- Equilibrium models
- Arbitrage free models.

The Vasicek and C.I.R models are short term rate models which fall under equilibrium models. Arbitrage free models, which include the Ho-Lee model, seek to describe the behaviour of the term structure of interest rates based only on variables in the economy which are believed to impact interest rates. Arbitrage free models result in prices which match market prices. The construction of these models is based on a reference set of financial instruments which are assumed to be correctly priced.

4.2.1 Simple Interest Rate Model

Let $P(t,T)$ be the price at time t of a zero coupon bond which matures at time T for $t = 1, 2, 3, \cdots$ and $T = t, t+1, \cdots$. We know that $P(t,t) = 1$ for all t. At time $t = 0$ there is a set of observed prices $P(0,T)$ for $T = 1, 2, \cdots, t$. The risk free rate of interest between t and $t+1$ is defined as

$$r(t+j) = -\ln P(t, t+1)$$

for

$$0 \leq l < 1.$$

The cash account $A(t)$ is then defined as

$$A(0) = 1, \tag{4.7}$$

$$A(t+1) = \frac{A(t)}{P(t,t+1)} = \exp\left[\int_0^{t+1} r(j)dj\right]. \tag{4.8}$$

One factor short-term interest rate models are driven by a process of the form

$$dr_t = (\alpha - \beta r_t)dt + \sigma r_t^{\gamma} dW_t,$$

where r is the interest rate, dW is a standard Brownian Motion, σ is a constant term and α, β, γ are parameters. The Vasicek and C.I.R models are contained in this group of models.

4.2.2 The Vasicek Model

In the Vasicek model, $\gamma = 0$ such that

$$dr_t = a(b - r_t)dt + \sigma dW_t, \tag{4.9}$$

where

- a is the speed of reversion to the mean, $0 \leq a \leq 1$,

- b is the long run average interest rate,
- $\alpha = ab$,
- σ is the volatility,
- W_t is the Brownian Motion at time t.

The Vasicek Model has for the most part been criticized for its admittance of negative interest rates but more recent developments in which countries (Sweden, Switzerland, Japan) have been in negative interest regimes has redeemed it as it has shown that interest can be negative, however the level at which they can be negative might still need to be controlled.

4.2.3 C.I.R model

The C.I.R model (Cox *et. al* [23] [24]) is a modification of the Vasicek model with $\gamma = 0.5$ which results in

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t. \tag{4.10}$$

The square root term ensures that only positive interest rates are admitted by the model as long as a and b are positive.

4.3 Concluding Remarks

This chapter provided an overview of concepts from stochastic calculus that are necessary in the derivation of the (2+1) dimensional Black-Scholes equation. Interest rate theory and models was also introduced since this dissertation is focused on a Black-Scholes model in which interest rates are not assumed to be constant.

Chapter 5

The Black Scholes Equation with Stochastic Interest Rates

In this chapter the derivation of the Black-Scholes models in which the interest rates are stochastic is presented. First, the derivation of the model for the Vasicek driven interest rates Black-Scholes model is shown, followed by the model in which interest rate are driven by the C.I.R model.

5.1 Deriving the Vasicek interest rate dependent Black Scholes model

Considering the standard Black-Scholes model, we consider the case when the assumption of constant interest rates is relaxed. That is in the model to be derived below, interest rates will be allowed to be variable with time. All other assumptions of the standard Black-Scholes model will be taken as they are.

Assume that there is an asset S whose price at a time t is given as S_t. The asset price will follow geometric brownian motion (GBM) as assumed by the Black-Scholes model and satisfies the stochastic differential equation (SDE):

$$dS_t = \mu S_t + \sigma_S S_t dW_t^1, \tag{5.1}$$

where μ is the expected return of the asset S, σ_S is the volatility of S and W_t^1 is a standard Weiner process.

The term structure of interest rates satisfies the stochastic differential equation:

$$dr_t = \beta(r_t, t)dt + \sigma_r(r_t, t)dW_t^2. \tag{5.2}$$

We consider the cases where interest rates will follow the Vasicek Model:

$$dr_t = \kappa(\Theta - r_t)dt + \Sigma_r(r_t, t)dW_t^2. \tag{5.3}$$

Let the price of a European call option be $V_t = V(S_t, r_t, t)$ and $P_t = P(r_t, t : T)$ be the price of a zero-coupon bond with maturity date T where $T > t$. We are now in a position to create a self financing portfolio Π_t which consists of a long position in a unit of the European call option, $\psi(t)$ units of a short position in the asset S and $\alpha(t)$ units of the zero-coupon bond. Mathematically, we have:

$$\Pi_t = V_t - \psi(t)S_t - \alpha(t)P_t. \tag{5.4}$$

For the portfolio to be self financing the change in portfolio value from time t to time $t+dt$, where dt is a small change in time, will come purely from investment returns. This means that the units held on the assets of the portfolio are unchanged over time and there is no further capital injection into the portfolio once it is set up:

$$d\Pi_t = r_t \Pi_t dt, \tag{5.5}$$

$$\Pi_t = r_t[V_t - \psi(t)S_t - \alpha(t)P_t]dt. \tag{5.6}$$

As defined by Haowen [13] and derived by Grace [19], the bond price will follow the general stochastic process:

$$dP = r_t P dt - \sigma_P(\sigma_r, t)P dW_t^2. \tag{5.7}$$

Using Ito's Lemma we find:

$$\begin{aligned} dV =& \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}(dr)^2 + \frac{\partial V}{\partial S}dS, \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{\partial^2 V}{\partial S \partial r}dSdr \end{aligned} \tag{5.8}$$

$$\frac{\partial^2 V}{\partial r^2}(dr)^2 = \sigma_r^2 \frac{\partial^2 V}{\partial r^2}dt, \tag{5.9}$$

$$\frac{\partial^2 V}{\partial S^2}(dS)^2 = \sigma_S^2 \frac{\partial^2 V}{\partial S^2}dt, \tag{5.10}$$

$$\frac{\partial^2 V}{\partial S \partial r}dSdr = \rho \sigma_S \sigma_r S \frac{\partial^2 V}{\partial S \partial r}dt. \tag{5.11}$$

When substituted into (5.6) it results in:

$$\begin{aligned} dV_t =& \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial S^2}(dr)^2 + \rho \sigma_S \sigma_r S \frac{\partial^2 V}{\partial S \partial r}\right] dt \\ &+ \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial S}dS. \end{aligned} \tag{5.12}$$

Substituting (5.12) back into (5.5) yields:

$$\begin{aligned} d\Pi_t =& \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial S^2}(dr)^2 + \rho \sigma_S \sigma_r S \frac{\partial^2 V}{\partial S \partial r}\right] dt \\ &+ \left[\frac{\partial V}{\partial S} - \psi(t)\right] dS + \beta(r_t, t)\frac{\partial V}{\partial r}dt + \sigma_r(r_t, t)\frac{\partial V}{\partial r}dW_t^2 \\ &- \alpha(t)\left[r_t P dt - \sigma_P(\sigma_r, t) P dW_t^2\right] \end{aligned}$$

To ensure that the portfolio is risk free, we choose $\psi(t)$ and $\alpha(t)$ such that stochastic terms are eliminated, i.e. we make $\psi(t) = \frac{\partial V}{\partial S}$ and $\alpha(t) = \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r}$. This results in the simplified equation:

$$\begin{aligned} d\Pi_t =& \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial S^2}(dr)^2 + \rho \sigma_S \sigma_r S \frac{\partial^2 V}{\partial S \partial r}\right] dt \\ &+ \left[\beta(r_t, t)\frac{\partial V}{\partial r} - \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r} r_t P\right] dt \end{aligned} \tag{5.13}$$

Substituting into (5.5)

$$\begin{aligned}&\left[\frac{\partial V}{\partial t}+\frac{1}{2}\sigma_S^2S^2\frac{\partial^2V}{\partial S^2}(dS)^2+\frac{1}{2}\sigma_r^2\frac{\partial^2V}{\partial S^2}(dr)^2+\rho\sigma_S\sigma_rS\frac{\partial^2V}{\partial S\partial r}\right]dt\\&+\left[\beta(r_t,t)\frac{\partial V}{\partial r}-\frac{\partial V}{\partial r}/\frac{\partial P}{\partial r}r_tP\right]dt\\&=r_t\left[V-S\frac{\partial V}{\partial S}-\frac{\partial V}{\partial r}/\frac{\partial P}{\partial r}r_tP\right]dt.\end{aligned}\tag{5.14}$$

Simplifying the above equation leads us to the general model with stochastic interest rates:

$$\begin{aligned}&\frac{\partial V}{\partial t}+\frac{1}{2}\sigma_S^2S^2\frac{\partial^2V}{\partial S^2}(dS)^2+\frac{1}{2}\sigma_r^2\frac{\partial^2V}{\partial S^2}(dr)^2\\&+\rho\sigma_S\sigma_rS\frac{\partial^2V}{\partial S\partial r}+\beta(r_t,t)\frac{\partial V}{\partial r}+r_tS\frac{\partial V}{\partial S}-r_tV=0.\end{aligned}\tag{5.15}$$

Assuming that the interest rate and asset price stochastic processes have zero correlation in the short term, we find for the Vasicek model:

$$\begin{aligned}&\frac{\partial V}{\partial t}+\frac{1}{2}\sigma_S^2S^2\frac{\partial^2V}{\partial S^2}(dS)^2+\frac{1}{2}\Sigma_r^2\frac{\partial^2V}{\partial S^2}(dr)^2\\&+\rho\sigma_S\sigma_rS\frac{\partial^2V}{\partial S\partial r}+\kappa r\frac{\partial V}{\partial r}+r_tS\frac{\partial V}{\partial S}-r_tV=0.\end{aligned}\tag{5.16}$$

Here we differ from Li *et.al* [8] in which they have $r=r_0+x(t)$ where $x(t)$ follows a decaying Ornstein-Uhlenbeck [20] process. To ensure that the partial differential equation is simpler to use mathematically ,without losing its properties, the assumption made was the $r_0=0$ for the Vasicek dependent model.

In the next section the derivation for the C.I.R model is provided.

5.2 Deriving the C.I.R interest rate dependent Black-Scholes model

The derivation of the C.I.R dependent model is similar to the derivation of the Vasicek dependent model. The only difference is in the stochastic process which drives the short term interest rates. All other assumptions of the standard Black-Scholes model will be taken as they are.

Assume that there is an asset S whose price at a time t is given as S_t. The asset price will follow GBM as assumed by the Black Scholes model and satisfies the SDE:

$$dS_t = \mu S_t + \sigma_S S_t dW_t^1 \tag{5.17}$$

where μ is the expected return of the asset S,
σ_S is the volatility of S and
W_t^1 is a standard Weiner process.

The term structure of interest rates satisfies the stochastic differential equation:

$$dr_t = \beta(r_t, t)dt + \sigma_r(r_t, t)dW_t^2. \tag{5.18}$$

We consider the case where interest rates will follow the C.I.R model:

$$dr_t = \kappa(\Theta - r_t)dt + \Sigma_r(r_t, t)dW_t^2. \tag{5.19}$$

Let the price of a European call option be $V_t = V(S_t, r_t, t)$ and $P_t = P(r_t, t : T)$ be the price of a zero-coupon bond with maturity date T where $T > t$. We are now in a position to create a self financing portfolio Π_t which consists of a long position in a unit of the European call option, $\psi(t)$ units of a short position in the asset S and $\alpha(t)$ units of the zero-coupon bond. Mathematically, we have:

$$\Pi_t = V_t - \psi(t)S_t - \alpha(t)P_t. \tag{5.20}$$

For the portfolio to be self financing the change in portfolio value from time t to

time $t+dt$, where dt is a small change in time, will come purely from investment return. This means that the units held on the assets of the portfolio are unchanged over time and there is no further capital injection into the portfolio once it is set up:

$$d\Pi_t = r_t \Pi_t dt, \tag{5.21}$$

$$\Pi_t = r_t[V_t - \psi(t)S_t - \alpha(t)P_t]dt. \tag{5.22}$$

As defined by Haowen [13] and derived by Grace [19], the bond price will follow the general stochastic process:

$$dP = r_t P dt - \sigma_P(\sigma_r, t) P dW_t^2. \tag{5.23}$$

Using Ito's Lemma we find:

$$\begin{aligned} dV =& \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}(dr)^2 + \frac{\partial V}{\partial S}dS \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{\partial^2 V}{\partial S \partial r}dSdr \end{aligned} \tag{5.24}$$

$$\frac{\partial^2 V}{\partial r^2}(dr)^2 = \sigma_r^2 \frac{\partial^2 V}{\partial r^2}dt \tag{5.25}$$

$$\frac{\partial^2 V}{\partial S^2}(dS)^2 = \sigma_S^2 \frac{\partial^2 V}{\partial S^2}dt \tag{5.26}$$

$$\frac{\partial^2 V}{\partial S \partial r}dSdr = \rho \sigma_S \sigma_r S \frac{\partial^2 V}{\partial S \partial r}dt. \tag{5.27}$$

When substituted into (5.22) results in:

$$\begin{aligned} dV_t =& \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial S^2}(dr)^2 + \rho\sigma_S\sigma_r S \frac{\partial^2 V}{\partial S \partial r}\right] dt \\ &+ \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial S}dS. \end{aligned} \tag{5.28}$$

Substituting (5.28) back into (5.21) yields:

$$\begin{aligned} d\Pi_t = &\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial S^2}(dr)^2 + \rho\sigma_S\sigma_r S\frac{\partial^2 V}{\partial S \partial r}\right] dt \\ &+ \left[\frac{\partial V}{\partial S} - \psi(t)\right] dS + \beta(r_t, t)\frac{\partial V}{\partial r}dt \\ &+ \sigma_r(r_t, t)\frac{\partial V}{\partial r}dW_t^2 \\ &- \alpha(t)\left[r_t P dt - \sigma_P(\sigma_r, t) P dW_t^2\right]. \end{aligned} \tag{5.29}$$

To ensure that the portfolio is risk free, we choose $\psi(t)$ and $\alpha(t)$ such that stochastic terms are eliminated that is we make

$$\psi(t) = \frac{\partial V}{\partial S}$$

and

$$\alpha(t) = \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r}.$$

This results in the simplified equation:

$$\begin{aligned} d\Pi_t = &\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial S^2}(dr)^2 + \rho\sigma_S\sigma_r S\frac{\partial^2 V}{\partial S \partial r}\right] dt \\ &+ \left[\beta(r_t, t)\frac{\partial V}{\partial r} - \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r} r_t P\right] dt. \end{aligned} \tag{5.30}$$

Substituting into (5.21)

$$\begin{aligned} &\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial S^2}(dr)^2 + \rho\sigma_S\sigma_r S\frac{\partial^2 V}{\partial S \partial r}\right] dt \\ &+ \left[\beta(r_t, t)\frac{\partial V}{\partial r} - \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r} r_t P\right] dt \\ &= r_t \left[V - S\frac{\partial V}{\partial S} - \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r} r_t P\right] dt. \end{aligned} \tag{5.31}$$

Simplifying the above equation leads us to the general model with stochastic interest rates:

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial S^2}(dr)^2 \\ &+ \rho\sigma_S\sigma_r S\frac{\partial^2 V}{\partial S \partial r} + \beta(r_t, t)\frac{\partial V}{\partial r} + r_t S\frac{\partial V}{\partial S} - r_t V = 0. \end{aligned} \tag{5.32}$$

Assuming that the interest rate and asset price stochastic processes have zero correlation in the short term, we find for the C.I.R model:

$$V_t + \frac{\sigma^2 S^2}{2} V_{SS} + S(r_0 + x)V_S - (r_0 + x)V + \frac{\Sigma^2 (r_0 + x)}{2} V_{xx} - \frac{a}{r_0} x V_x = 0. \quad (5.33)$$

5.3 Concluding Remarks

The modified Black-Scholes partial differential equations (5.16) and (5.33) have been derived in this chapter making use of the concepts in Stochastic processes that where introduced earlier. In the next chapter Lie Symmetry methods are applied to the equations to try to find the solutions to the partial differential equations.

Chapter 6

Lie symmetry analyis

In this section we apply Lie symmetry methods on the (5.16). Conditions necessary to reduce the (2+1) partial differential equations (PDEs) to the heat equation were proposed previously. In chapter three we applied these conditions to determine if equation (5.16) can be reduced to the heat equation and preliminary result was that it cannot be transformed to the heat equation. However, more research needs to be done to determine the complete conditions necessary to use to test reducibility to the heat heat equation. We consider the equation of the Black-Scholes with interest rates driven by the Vasicek model:

$$u_t + \frac{\sigma^2 S^2}{2} u_{SS} + Sxu_S - xu + \frac{\Sigma^2}{2} u_{xx} + \kappa u_x = 0, \tag{6.1}$$

where t is the time to maturity, S is the stock price, x is the time dependant interest rate driven by a stochastic process and the price of the option given by u.

6.1 The Black Scholes Equation with Vasicek Interest Rates

We first consider (6.1) in the symmetry analysis. We look for the infinitesimal symmetry operator of the form

$$X = \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial S} + \xi_3 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} \tag{6.2}$$

which generates the infinitesimal transformations

$$\begin{aligned} \overline{t} &= t + \varepsilon \xi_1(t, x, S, u) + O(\xi_1) \\ \overline{S} &= S + \varepsilon \xi_2(t, x, S, u) + O(\xi_2) \\ \overline{x} &= x + \varepsilon \xi_3(t, x, S, u) + O(\xi_3) \\ \overline{u} &= u + \varepsilon \eta(t, x, S, u) + O(\eta). \end{aligned}$$

Since we are concerned with the second order partial differential equation, the second prolongation is

$$\begin{aligned} X^{[2]} = \quad & \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial x} + \xi_3 \frac{\partial}{\partial S} + \eta \frac{\partial}{\partial u} + \zeta_t \frac{\partial}{\partial u_t} \\ & + \zeta_{tt} \frac{\partial}{\partial u_{tt}} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} \\ & + \zeta_{tS} \frac{\partial}{\partial u_{tS}} + \zeta_{xS} \frac{\partial}{\partial u_{xS}} + \zeta_S \frac{\partial}{\partial u_S} + \zeta_{SS} \frac{\partial}{\partial u_{SS}} + \dots \end{aligned}$$

where

$$\begin{aligned} \zeta_t &= D_t \eta - u_t D_t \xi_1 - u_x D_t \xi_2 - u_S D_t \xi_3 \\ \zeta_x &= D_x \eta - u_x D_x \xi_2 - u_t D_x \xi_1 - u_S D_x \xi_3 \\ \zeta_S &= D_S \eta - u_S D_S \xi_3 - u_x D_S \xi_2 - u_t D_S \xi_1 \end{aligned}$$

and

$$\begin{aligned}
\zeta_{tt} &= D_t\zeta_t - u_{tt}D_t\xi_1 - u_{tx}D_t\xi_2 - u_{tS}D_t\xi_3 \\
\zeta_{tx} &= D_x\zeta_t - u_{tt}D_t\xi_1 - u_{tx}D_x\xi_2 - u_{tS}D_x\xi_3 \\
\zeta_{tS} &= D_S\zeta_t - u_{tx}D_S\xi_1 - u_{xx}D_S\xi_2 - u_{xS}D_x\xi_3 \\
\zeta_{xx} &= D_x\zeta_x - u_{tx}D_x\xi_1 - u_{xx}D_x\xi_2 - u_{Sx}D_x\xi_3 \\
\zeta_{xS} &= D_x\zeta_S - u_{tS}D_x\xi_1 - u_{xy}D_x\xi_2 - u_{SS}D_x\xi_3 \\
\zeta_{SS} &= D_S\zeta_S - u_{St}D_S\xi_1 - u_{xy}D_S\xi_2 - u_{SS}D_y\xi_3.
\end{aligned}$$

The equation (6.1) has the following determining equations

$$\sigma\xi_S^1 = 0$$
$$\Sigma\xi_x^1 = 0$$
$$\Sigma\xi_{uu}^1 = 0$$
$$\sigma\xi_{uu}^1 = 0$$
$$\Sigma\xi_u^1 = 0$$
$$\sigma\xi_u^1 = 0$$
$$\sigma\Sigma\xi_{Su}^1 = 0$$
$$\sigma\Sigma\xi_S^1 = 0$$
$$\sigma\Sigma\xi_{xu}^1 = 0$$
$$\sigma\Sigma\xi_x^1 = 0$$
$$\sigma\Sigma\xi_{uu}^1 = 0$$
$$\sigma\Sigma\xi_u^1 = 0$$
$$\Sigma\left(x\kappa\xi_{uu}^1 + \xi_{uu}^3\right) = 0$$
$$\sigma\left(x\kappa\xi_{uu}^1 + \xi_{uu}^3\right) = 0$$
$$\sigma\left(x\kappa\xi_u^1 + \xi_u^3\right) = 0$$
$$\Sigma\left(Sx\xi_{uu}^1 - \xi_{uu}^2\right) = 0$$
$$\sigma\left(Sx\xi_{uu}^1 - \xi_{uu}^2\right) = 0$$
$$\Sigma\left(Sx\xi_u^1 - \xi_u^2\right) = 0$$
$$\Sigma\left(-2x\kappa\xi_u^1 - 2\xi_u^3 + \Sigma^2\xi_{xu}^1\right) = 0$$
$$\sigma\left(2S\left(x+\sigma^2\right)\xi_u^1 - 2\xi_u^2 + S^2\sigma^2\xi_{Su}^1\right) = 0$$
$$Sx\Sigma^2\xi_x^1 - \Sigma^2\xi_x^2 - S^2\sigma^2\left(x\kappa\xi_S^1 + \xi_S^3\right) = 0$$
$$\sigma\left(2(-1+S)x\xi_u^1 - \eta_{uu} + Sux\xi_{uu}^1 - 2Sx\xi_{Su}^1 + 2\xi_{Su}^2\right) = 0$$
$$\Sigma\left(2(Sx+\kappa)\xi_u^1 - \eta_{uu} + Sux\xi_{uu}^1 + 2x\kappa\xi_{xu}^1 + 2\xi_{xu}^3\right) = 0$$
$$S\Sigma^2\xi_u^1 + Sx\Sigma^2\xi_{xu}^1 - \Sigma^2\xi_{xu}^2 - S^2x\kappa\sigma^2\xi_{Su}^1 - S^2\sigma^2\xi_{Su}^3 = 0$$
$$\Sigma\left(2Sux\xi_u^1 - 6x\kappa\xi_x^1 - 4\xi_x^3 + \Sigma^2\xi_{xx}^1 + 2Sx\xi_S^1 + S^2\sigma^2\xi_{SS}^1 + 2\xi_t^1\right) = 0$$
$$\sigma\left(4\xi^2 + S\left(2Sux\xi_u^1 - 2x\kappa\xi_x^1 + \Sigma^2\xi_{xx}^1 + 6Sx\xi_S^1 + 4S\sigma^2\xi_S^1 - 4\xi_S^2 + S^2\sigma^2\xi_{SS}^1 + 2\xi_t^1\right)\right) = 0$$
$$-2Sx\eta - 2ux\xi^2 - 2Su\xi^3 + 2Sux\eta_u - 2S^2u^2x^2\xi_u^1 - 2x\kappa\eta_x + 2Sux^2\kappa\xi_x^1$$
$$-2Su\Sigma^2\xi_x^1 + \Sigma^2\eta_{xx} - Sux\Sigma^2\xi_{xx}^1 + 2Sx\eta_S$$
$$-2S^2ux^2\xi_S^1 - 2S^2ux\sigma^2\xi_S^1 + S^2\sigma^2\eta_{SS} - S^3ux\sigma^2\xi_{SS}^1 + 2\eta_t - 2Sux\xi_t^1 = 0$$

$$-2\kappa\xi^3 - 2Su\left(x^2\kappa + \Sigma^2\right)\xi_u^1 - 2Sux\xi_u^3 + 2x^2\kappa^2\xi_x^1 - 2Sx\Sigma^2\xi_x^1 - 2\kappa\Sigma^2\xi_x^1 + 2x\kappa\xi_x^3$$
$$+2\Sigma^2\eta_{xu} - 2Sux\Sigma^2\xi_{xu}^1 - x\kappa\Sigma^2\xi_{xx}^1 - \Sigma^2\xi_{xx}^3 - 2Sx^2\kappa\xi_S^1 - 2Sx\xi_S^3$$
$$-S^2x\kappa\sigma^2\xi_{SS}^1 - S^2\sigma^2\xi_{SS}^3 - 2x\kappa\xi_t^1 - 2\xi_t^3 = 0$$
$$2x\xi^2 + 2S\xi^3(t,S,x,u) + 2S^2ux^2\xi_u^1 - 2S^2ux\sigma^2\xi_u^1 - 2Sux\xi_u^2 - 2Sx^2\kappa\xi_x^1 + 2S\Sigma^2\xi_x^1$$
$$+2x\kappa\xi_x^2 + Sx\Sigma^2\xi_{xx}^1 - \Sigma^2\xi_{xx}^2 + 2S^2x^2\xi_S^1 + 2S^2x\sigma^2\xi_S^1$$
$$-2S^3x\sigma^2\xi_S^1 - 2Sx\xi_S^2 + 2S^2\sigma^2\eta_{Su} - 2S^3ux\sigma^2\xi_{Su}^1$$
$$+S^3x\sigma^2\xi_{SS}^1 - S^2\sigma^2\xi_{SS}^2 + 2Sx\xi_t^1 - 2\xi_t^2 = 0$$

After solving the determining equations the equation is found to have two symmetries:

$$X_1 = \frac{\partial}{\partial t} \tag{6.3}$$

$$X_2 = u\frac{\partial}{\partial u} \tag{6.4}$$

Consider the linear combination of the symmetries below

$$X = c_1\frac{\partial}{\partial t} + c_2u\frac{\partial}{\partial u} \tag{6.5}$$

where c_1 and c_2 are constants. The Lagrangian system is

$$\frac{dt}{c_1} = \frac{du}{c_2u} = \frac{dx}{0} = \frac{dS}{0} \tag{6.6}$$

From the system we easily deduce that

$$I_1 = x$$

and

$$I_2 = S$$

Consider

$$\frac{du}{c_2u} = \frac{dt}{c_1}$$
$$\ln u = \frac{c_2}{c_1}t + \ln I_3$$

Thus we find the find the general solution to the equation as

$$u = I_3 e^{-\frac{c_2}{c_1}t}$$

Assuming that $I_3 = F(x, S)$ then,

$$u = F(x, S)e^{-\frac{c_2}{c_1}t} \tag{6.7}$$

Now the general solution is substituted into the original equation. First we take the derivatives of u to t, S, x

$$u_t = \frac{c_2}{c_1}F(x, S)e^{\frac{c_2}{c_1}t} \tag{6.8}$$

$$u_x = F_x e^{\frac{c_2}{c_1}t} \tag{6.9}$$

$$u_{xx} = F_{xx} e^{\frac{c_2}{c_1}t} \tag{6.10}$$

$$u_S = F_S e^{\frac{c_2}{c_1}t} \tag{6.11}$$

$$u_{SS} = F_{SS} e^{\frac{c_2}{c_1}t} \tag{6.12}$$

Substitute into the original equation

$$\frac{c_2}{c_1}e^{\frac{c_2}{c_1}t}F + \frac{\sigma^2 S^2}{2}e^{\frac{c_2}{c_1}t}F_{SS} + Sxe^{\frac{c_2}{c_1}t}F_S - xe^{\frac{c_2}{c_1}t} + \frac{\Sigma^2}{2}e^{\frac{c_2}{c_1}t}F_{xx} - \kappa x e^{\frac{c_2}{c_1}t} = 0$$

$$\frac{\sigma^2 S^2}{2}F_{SS} + SxF_S + \left(\frac{c_2}{c_1} - x\right)F + \frac{\Sigma^2}{2}F_{xx} - \kappa x F_x = 0 \tag{6.13}$$

To solve the equation we use the method of separation of variables. Let $F = G(x)H(S)$, the dependent variables are dropped.Then

$$\frac{\sigma^2 S^2}{2}GH'' + SxGH' + \left(\frac{c_2}{c_1} - x\right)GH + \frac{\Sigma^2}{2}X''S - \kappa x G'H = 0 \tag{6.14}$$

Dividing throughout by GH

$$\frac{\sigma^2 S^2}{2}\frac{H''}{H} + Sx\frac{H'}{H} + \left(\frac{c_2}{c_1} - x\right) + \frac{\Sigma^2}{2}\frac{X''}{X} - \kappa x\frac{G'}{G} = 0 \tag{6.15}$$

Separating the equations in $G(x)$ and $H(S)$ as

$$\frac{\sigma^2 S^2}{2}\frac{H''}{H} + Sx\frac{H'}{H} + \left(\frac{c_2}{c_1}\right) = -\lambda = \frac{\Sigma^2}{2}\frac{X''}{X} - \kappa x\frac{G'}{G} - x \tag{6.16}$$

in two equations

$$\frac{\sigma^2 S^2}{2} H'' + SxH' + \left(\frac{c_2}{c_1} + \lambda\right) H = 0 \tag{6.17}$$

and

$$\frac{\Sigma^2}{2} X'' - \kappa x G' + (\lambda - x) G = 0 \tag{6.18}$$

Case 1. $\lambda \neq 0$

The solution of equation, confirmed by Mathematica, is

$$\begin{aligned} G(x) = \quad & e^{-\frac{x}{\kappa}} C_1 H\left(\frac{2\kappa^2\lambda + \Sigma^2}{2\kappa^3}, \frac{x\sqrt{\kappa}}{\Sigma} + \frac{\Sigma}{\kappa^{3/2}}\right) \\ & + e^{-\frac{x}{\kappa}} C_2 F_1\left(-\frac{2\kappa^2\lambda + \Sigma^2}{4\kappa^3}, \frac{1}{2}, \left(\frac{x\sqrt{\kappa}}{\Sigma} + \frac{\Sigma}{\kappa^{3/2}}\right)^2\right) \end{aligned}$$

where F_1 is a hypergeometric function and

$$\begin{aligned} H(S) \quad = S^{\frac{\sqrt{\lambda c_1 + c_2}\,(-\Phi - \Psi)}{\sqrt{2}\sigma\sqrt{c_1}}} C_3 \\ + S^{\frac{\sqrt{\lambda c_1 + c_2}\,(\Phi - \Psi)}{\sqrt{2}\sigma\sqrt{c_1}}} C_4. \end{aligned}$$

where

$$\Phi = \frac{1}{2}\sqrt{-16 - \frac{8xc_1}{\lambda c_1 + c_2} + \frac{8x^2 c_1}{\sigma^2(\lambda c_1 + c_2)} + \frac{2\sigma^2 c_1}{\lambda c_1 + c_2}}$$

and

$$\Psi = \frac{2\sqrt{2}x\sqrt{c_1}\sqrt{\lambda c_1 + c_2} - \sqrt{2}\sigma^2\sqrt{c_1}\sqrt{\lambda c_1 + c_2}}{2\sigma(\lambda c_1 + c_2)}.$$

When $\lambda \neq 0$ the solution for $G(x)$ is not mathematically friendly and makes it difficult to use in practical environments due to the presence of Hermit and Hypergeometric functions in the solution. Next the case when $\lambda = 0$ is considered.

Case 2. $\lambda = 0$

The solution of equation, confirmed in Mathematica, is

$$G(x) = e^{-2\left(\frac{x}{2\kappa} + \frac{\Sigma^2 \text{Log}\left[2x\kappa - \Sigma^2\right]}{4\kappa^2}\right)} C_1 \tag{6.19}$$

and

$$H(S) = S^{\frac{(-2\gamma - \vartheta)\sqrt{c_2}}{2\sqrt{2}\sigma\sqrt{c_1}}} C_2 + S^{\frac{(2\gamma - \vartheta)\sqrt{c_2}}{2\sqrt{2}\sigma\sqrt{c_1}}} C_3.$$

where

$$\gamma = \sqrt{-4 + \frac{-4\sigma^2 x c_1 + 4x^2 c_1 + 2\sigma^4 c_1}{2\sigma^2 c_2}}$$

and

$$\vartheta = \frac{2\sqrt{2}x\sqrt{c_1} + \sqrt{2}\sigma^2\sqrt{c_1}}{\sigma\sqrt{c_2}}$$

The solution in this case is much simpler than when $\lambda \neq 0$. From the general solution (6.7), the solution to equation (6.1) can be written as

$$u(t, S, x) = G(x)H(S)e^{-\frac{c_2}{c_1}t}, \tag{6.20}$$

where $G(x)$ is given by (6.19) and $H(S)$ is given by (6.20).

When the result is plotted it is observed that the relationship between the call option price and the stock price is either positive or negative depending on the highly unstable relationship between the option price and interest rates (figure 6.1).

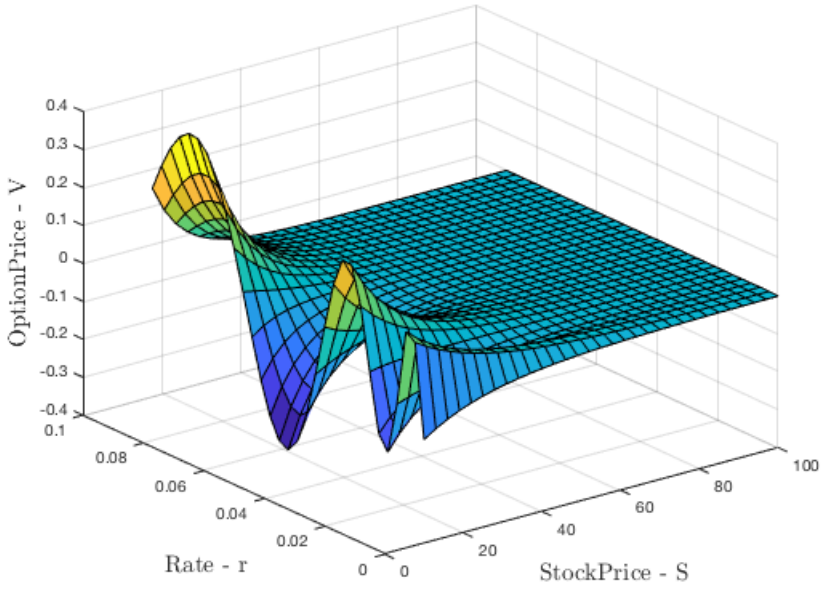


Figure 6.1: The solution $u(t, S, r)$ shows an unstable oscillatory behaviour in relation to the interest rate .

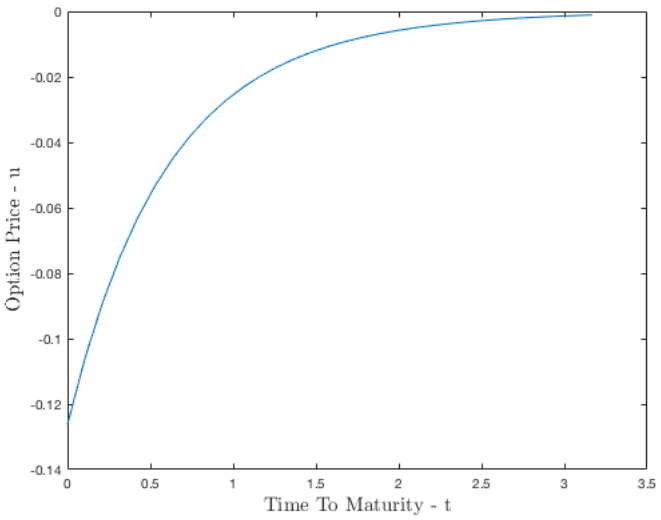


Figure 6.2: The solution $u(t, S, r)$ shows a positive relation to time to maturity.

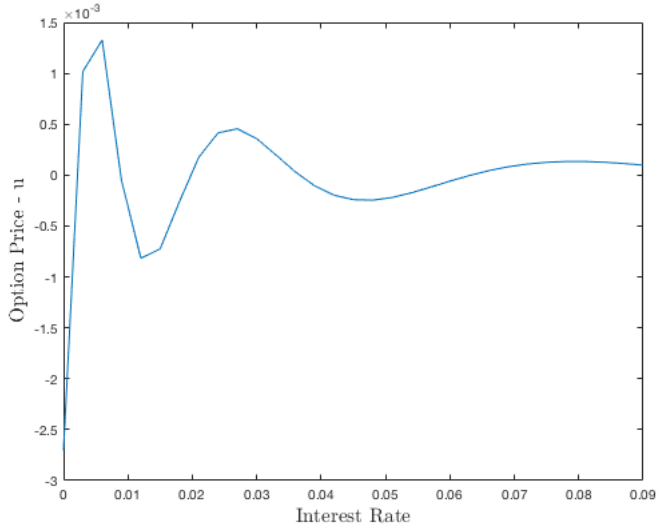


Figure 6.3: The solution $u(t, S, r)$ relation to interest rates.

6.2 The Transformed Black Scholes with Vasicek Interest Rates

We define new variables c and r as

$$c = \frac{1}{\sigma}\log S,$$
$$r = \frac{1}{\Sigma}x,$$

and

$$\tau = \frac{1}{2}t.$$

We differentiate u and have the transformation variables

$$u_t = \frac{1}{2}u_\tau$$
$$u_S = \frac{1}{\sigma S}u_c$$
$$u_{SS} = -\frac{1}{\sigma S^2}u_c + \frac{1}{\sigma^2 S^2}u_{cc}$$
$$u_x = \frac{1}{\Sigma}u_x$$
$$u_{xx} = \frac{1}{\Sigma^2}u_{xx}.$$

Substituting into the original equation

$$\begin{aligned}\frac{1}{2}u_\tau + \frac{\sigma^2 S^2}{2}\left(-\frac{1}{\sigma S^2}u_c + \frac{1}{\sigma^2 S^2}u_{cc}\right) + Sx\left(\frac{1}{\sigma S}u_c\right)\\ - xu + \frac{\Sigma^2}{2}\left(\frac{1}{\Sigma^2}u_{rr}\right) - \kappa x\left(\frac{1}{\Sigma}u_r\right) = 0.\end{aligned} \tag{6.21}$$

Expanding the equation results in

$$\frac{1}{2}u_\tau - \frac{\sigma}{2}u_c + \frac{1}{2}u_{cc} + \frac{x}{\sigma}u_c - xu + \frac{1}{2}u_{rr} - \frac{\kappa x}{\Sigma} = 0, \tag{6.22}$$

which, after simplification and substituting in the transformation variables, becomes,

$$u_\tau + u_{cc} + u_{rr} + \left(\frac{2\Sigma r}{\sigma} - \sigma\right)u_c - 2\Sigma r u - 2\kappa r u_r = 0. \tag{6.23}$$

We again seek the infinitesimal symmetry operator of the form:

$$X = \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial c} + \xi_3 + \eta \frac{\partial}{\partial r} \tag{6.24}$$

which generates the infinitesimal transformations

$$\begin{aligned} \overline{t} &= t + \varepsilon \xi_1(t, c, r, u) + O(\xi_1) \\ \overline{c} &= c + \varepsilon \xi_2(t, c, r, u) + O(\xi_2) \\ \overline{r} &= r + \varepsilon \xi_3(t, c, r, u) + O(\xi_3) \\ \overline{u} &= u + \varepsilon \eta(t, c, r, u) + O(\eta) \end{aligned}$$

Since we are concerned with the second order partial differential equation, the second prolongation is

$$\begin{aligned} X^{[2]} = \quad & \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial c} + \xi_3 \frac{\partial}{\partial r} + \eta \frac{\partial}{\partial u} + \zeta_t \frac{\partial}{\partial u_t} \\ & + \zeta_{tt} \frac{\partial}{\partial u_{tt}} + \zeta_{tc} \frac{\partial}{\partial u_{tc}} + \zeta_c \frac{\partial}{\partial u_c} + \zeta_{cc} \frac{\partial}{\partial u_{cc}} \\ & + \zeta_{tr} \frac{\partial}{\partial u_{tr}} + \zeta_{xr} \frac{\partial}{\partial u_{xr}} + \zeta_r \frac{\partial}{\partial u_r} + \zeta_{rr} \frac{\partial}{\partial u_{rr}} + \dots \end{aligned}$$

By use of the SYM package in Mathematica we find the determining equations

to be

$$\xi_c^1 = 0 \tag{6.25}$$

$$\xi_r^1 = 0 \tag{6.26}$$

$$\eta_{uu} = 0 \tag{6.27}$$

$$\xi_u^3 = 0 \tag{6.28}$$

$$\xi_u^1 = 0 \tag{6.29}$$

$$\sigma\xi_{cu}^2 = 0 \tag{6.30}$$

$$\sigma\xi_{ru}^2 = 0 \tag{6.31}$$

$$\sigma\xi_{uu}^2 = 0 \tag{6.32}$$

$$\sigma\xi_u^2 = 0 \tag{6.33}$$

$$\sigma\left(\xi_r^2 + \xi_c^3\right) = 0 \tag{6.34}$$

$$\sigma\left(-2\xi_c^2 + \xi_t^1\right) = 0 \tag{6.35}$$

$$\sigma\left(-2\xi_r^3 + \xi_t^1\right) = 0 \tag{6.36}$$

$$-2\sqrt{2}\kappa\sigma\xi^3(t,c,r,u) + 2\sqrt{2}r\kappa\sigma\xi_r^3 + 4\sigma\eta_{ru} - 2\sigma\xi_{rr}^3 + \sqrt{2}\sigma^2\xi_c^3 - 2r\Sigma\xi_c^3 \tag{6.37}$$

$$-2\sigma\xi_{cc}^3 - 2\sqrt{2}r\kappa\sigma\xi_t^1 - 2\sigma\xi_t^3 = 0 \tag{6.38}$$

$$-\sqrt{2}r\sigma\Sigma\eta(t,c,r,u) - \sqrt{2}u\sigma\Sigma\xi^3(t,c,r,u) + \sqrt{2}ru\sigma\Sigma\eta_u - 2\sqrt{2}r\kappa\sigma\eta_r + 2\sigma\eta_{rr} \tag{6.39}$$

$$-\sqrt{2}\sigma^2\eta_c + 2r\Sigma\eta_c + 2\sigma\eta_{cc} + 2\sigma\eta_t - \sqrt{2}ru\sigma\Sigma\xi_t^1 = 0 \tag{6.40}$$

$$\sigma\left(2\Sigma\xi^3(t,c,r,U) - \sqrt{2}ru\sigma\Sigma\xi_u^2 + 2\sqrt{2}r\kappa\sigma\xi_r^2 - 2\sigma\xi_{rr}^2 + \sqrt{2}\sigma^2\xi_c^2\right) \tag{6.41}$$

$$-\sigma\left(2r\Sigma\xi_c^2 + 4\sigma\eta_{cu} - 2\sigma\xi_{cc}^2 - \sqrt{2}\sigma^2\xi_t^1 + 2r\Sigma\xi_t^1 - 2\sigma\xi_t^2\right) = 0. \tag{6.42}$$

Solving the determining equations results in six symmetries

$$X_1 = \frac{\partial}{\partial \tau},$$

$$X_2 = \frac{e^{2\tau\kappa}}{2\kappa}\frac{\partial}{\partial r} + \frac{e^{2\tau\kappa}}{2\kappa^2\sigma}\frac{\partial}{\partial c} - u\left(\frac{e^{2\tau\kappa}\left(2r\kappa^2+\Sigma\right)}{2\kappa^2}\right)\frac{\partial}{\partial u},$$

$$X_3 = -\frac{e^{-2\tau\kappa}}{2\kappa}\frac{\partial}{\partial r} + \frac{e^{-2\tau\kappa}\Sigma}{2\kappa^2\sigma}\frac{\partial}{\partial c} + u\left(\frac{e^{-2\tau\kappa}\Sigma}{2\kappa^2}\right)\frac{\partial}{\partial u},$$

$$X_4 = \frac{\partial}{\partial r} + \frac{2\tau(\kappa^2\sigma^2+\Sigma^2)}{\sigma\Sigma}\frac{\partial}{\partial c} + u\left(r\kappa + \frac{c\kappa^2\sigma}{\Sigma} + \frac{\tau\kappa^2\sigma^2}{\Sigma} + 2\tau\Sigma\right)\frac{\partial}{\partial u},$$

$$X_5 = u\frac{\partial}{\partial u},$$

$$X_6 = \frac{\partial}{\partial c}.$$

This is apart from the infinte solution symmetries.

Table 6.1: Commutator table for symmetries of the reduced Vasicek driven Black-Scholes model

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	$-2\kappa X_4$	$2\kappa X_5$	$2\beta X_2 + \theta X_3$
X_2	0	0	0	0	0	$\frac{\kappa^2\sigma}{\Sigma}X_3$
X_3	0	0	0	0	0	0
X_4	$2\kappa X_4$	0	0	0	$-\frac{2}{\kappa}X_3$	0
X_5	$-2\kappa X_5$	0	0	$\frac{2}{\kappa}X_3$	0	0
X_6	$-(\beta X2+ \theta X_3)$	$-\frac{\kappa^2\sigma}{\Sigma}X_3$	0	0	0	0

The commutator table is shown in table (6.1) where

$$\beta = \frac{2(\kappa^2\sigma^2 + \Sigma^2)}{\sigma\Sigma},$$

and

$$\theta = \frac{(\kappa^2\sigma^2 + 2\Sigma^2)}{\Sigma}.$$

We can use any of the symmetries or combination of symmetries to reduce the equation. For the reduction of the equation symmetry X_3 is chosen. The corresponding Lagrangian system of X_3 is

$$\frac{dr}{\left(-\frac{e^{-2\tau\kappa}}{2\kappa}\right)} = \frac{dc}{\left(\frac{e^{-2\tau\kappa}\Sigma}{2\kappa^2\sigma}\right)} = \frac{du}{\left(\frac{e^{-2\tau\kappa}\Sigma}{2\kappa^2}\right)} = \frac{d\tau}{0}. \tag{6.43}$$

It is easily deduced from the system that

$$I_1 = \tau$$

Consider

$$\frac{dr}{\left(-\frac{e^{-2\tau\kappa}}{2\kappa}\right)} = \frac{dc}{\left(\frac{e^{-2\tau\kappa}\Sigma}{2\kappa^2\sigma}\right)}.$$

Thus

$$r = I_2 - \frac{\kappa\sigma c}{\Sigma}$$

such that

$$I_2 = \frac{r\Sigma + \kappa\sigma c}{\Sigma}$$

To get a solution to the equation we consider

$$\frac{dc}{\left(\frac{e^{-2\tau\kappa}\Sigma}{2\kappa^2\sigma}\right)} = \frac{du}{\left(\frac{e^{-2t\kappa}\Sigma}{2\kappa^2}\right)}$$

which gives the solution

$$u = I_3 e^{\sigma c}.$$

Let $I_3 = F(I_1, I_2) = F(\alpha, \psi)$ where

$$\alpha = \tau$$

and

$$\psi = \frac{r\Sigma + \kappa\sigma c}{\Sigma}.$$

The solution of equation can thus be written as

$$u = F(\alpha, \psi)e^{\sigma c}. \tag{6.44}$$

The derivatives of u in terms of α and ψ are

$$u_t = F_\alpha e^{\sigma c}$$

$$u_c = \left(\frac{\kappa\sigma}{\Sigma}\right) F_\psi e^{\sigma c} + \sigma F e^{\sigma c}$$

$$u_{cc} = \left(\frac{\kappa\sigma}{\Sigma}\right)^2 F_{\psi\psi} e^{\sigma c} + 2\left(\frac{\kappa\sigma^2}{\Sigma}\right) e^{\sigma c} + \sigma^2 F e^{\sigma c}$$

$$u_r = F_\psi e^{\sigma c}$$

$$u_{rr} = F_{\psi\psi} e^{\sigma c}.$$

Substitute into the original equation

$$F_\alpha + \left(F_{\psi\psi}\left(\frac{\kappa\sigma}{\Sigma}\right)^2 + 2F_\psi\left(\frac{\kappa\sigma^2}{\Sigma}\right) + \sigma^2 F\right) + F_{\psi\psi} + \left(\frac{2\Sigma r}{\sigma} - \sigma\right)\left(F_\psi\left(\frac{\kappa\sigma}{\Sigma}\right) + F\sigma\right) - 2\kappa r F_\psi - 2\Sigma r F = 0$$

Grouping terms together and simplifying results in the partial differential equation we have

$$F_\alpha + (1 + \frac{\kappa^2\sigma^2}{\Sigma^2})F_{\psi\psi} - (\frac{2\kappa\sigma^2}{\Sigma})F_\psi = 0 \tag{6.45}$$

We use the method of separation of variables. Let

$$F = G(\alpha)H(\psi)$$

then dropping the dependent variable, we find

$$G'H + (1 + \frac{\kappa^2\Sigma^2}{\Sigma^2})GH'' - (\frac{2\kappa\sigma^2}{\Sigma})GH' = 0. \tag{6.46}$$

Divide through out by GH

$$\frac{G'}{G} + \left(1 + \frac{\kappa^2\Sigma^2}{\Sigma^2}\right)\frac{H''}{H} - \left(\frac{2\kappa\sigma^2}{\Sigma}\right)\frac{H'}{H} = 0$$

The equation can then be separated into two equations in $G(\alpha)$ and $H(\psi)$

$$\frac{G'}{G} = -\lambda = \left(1 + \frac{\kappa^2\Sigma^2}{\Sigma^2}\right)\frac{H''}{H} - \left(\frac{2\kappa\sigma^2}{\Sigma}\right)\frac{H'}{H}$$

We therefore seek the solutions to the equations

$$G'(\alpha) + \lambda G(\alpha) = 0 \tag{6.47}$$

and

$$(1 + \frac{\kappa^2\Sigma^2}{\Sigma^2})H'' - (\frac{2\kappa\sigma^2}{\Sigma})H' + \lambda H = 0. \tag{6.48}$$

Case 3. $\lambda \neq 0$

We solve the two differential equations assuming that $\lambda \neq 0$. We find the solutions to be

$$G(\alpha) = C_0 e^{-\alpha\lambda} \tag{6.49}$$

and

$$H(\psi) = e^{\frac{1}{2}\Gamma - \Lambda}C_1 + e^{\frac{1}{2}\Gamma + \Lambda}C_1$$

where

$$\Gamma = -\frac{\kappa\sigma^2\Sigma}{\kappa^2\sigma^2 + \Sigma^2}$$

and

$$\Lambda = \sqrt{\frac{(\kappa^2\sigma^4\Sigma^2)}{(\kappa^2\sigma^2 + \Sigma^2)^2} - \frac{(4\lambda\Sigma^2)}{(\kappa^2\sigma^2 + \Sigma^2)}}\psi$$

The solution of the equation is the product of $G(\alpha)$ and $H(\psi)$ which is

$$u(t, c, r) = e^{\sigma c - \alpha\lambda}\left[e^{\frac{1}{2}\Gamma - \Lambda}C_1 + e^{\frac{1}{2}\Gamma + \Lambda}C_1\right]$$

where

$$\alpha = \tau$$
$$\psi = \frac{c\kappa\sigma + \Sigma r}{\Sigma}.$$

One reverts to the solution $u(t, S, x)$ by reversing the initial transformations performed initially, that is substitute

$$\tau = \frac{1}{2}t,$$
$$c = \frac{1}{\sigma}\log S,$$

and

$$r = \frac{1}{\Sigma}x.$$

into the solution.

Analysis of Results: $\lambda \neq 0$

Market expectations are that the price of a call option will go up as the value of the underlying asset goes up. Also it is expected that the price of the option be an increasing function relative to interest rates. From figure 6.4 it is observed that the price of the option is positive related to both the underlying asset price and slightly related positively with the interest rates. It is necessary to understand how the solution found for the option price behaves in relation to time left to maturity of the option.

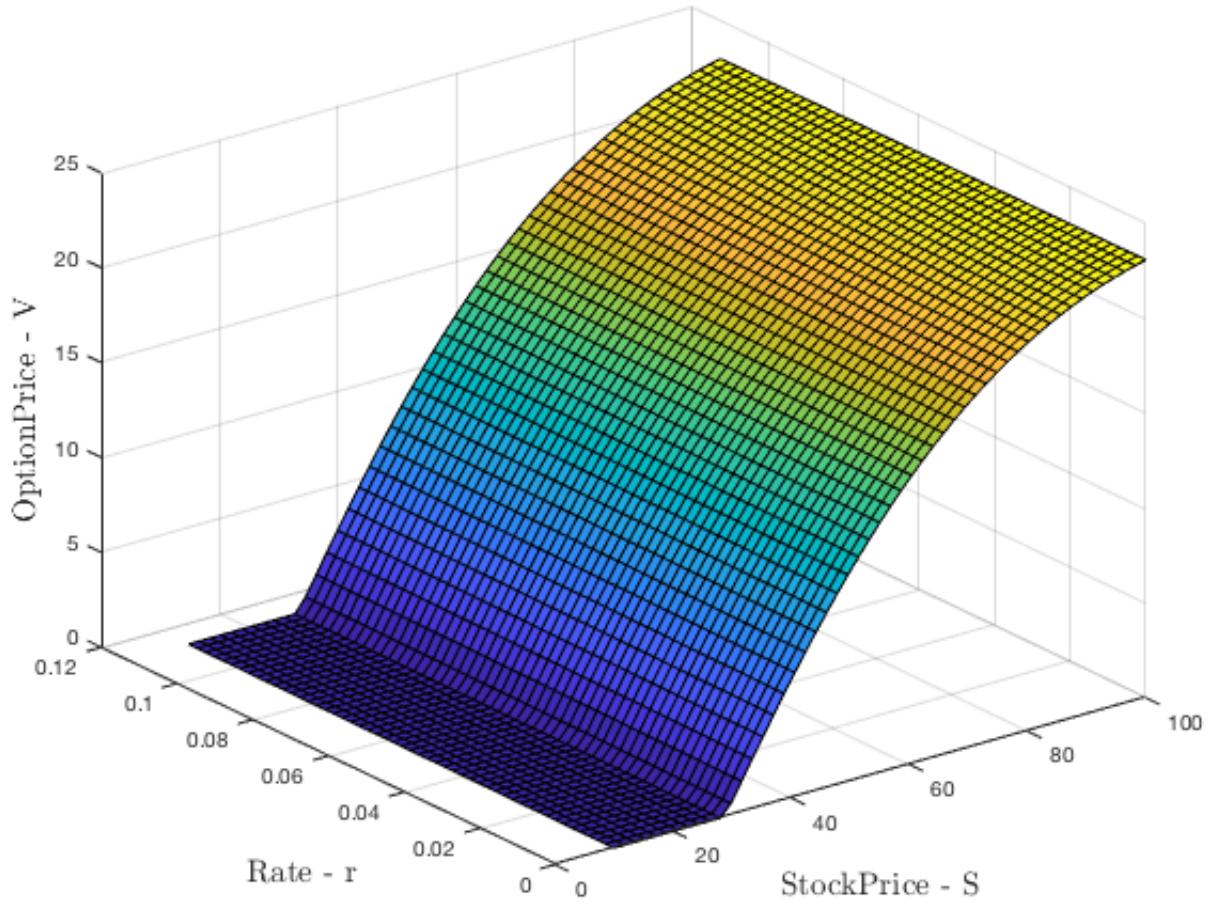


Figure 6.4: Surf plot of option price relative to interest rates and underlying asset price

In the figure 6.5 the price of the call option rises as the time to maturity increases, that is the longer the time the option is away from the exercise date the higher the price of the option. In the market as the time to maturity increases the likelihood of an adverse event affecting the underlying asset value increases, thus the price of the option rises to compensate for the increase in perceived risk in the underlying.

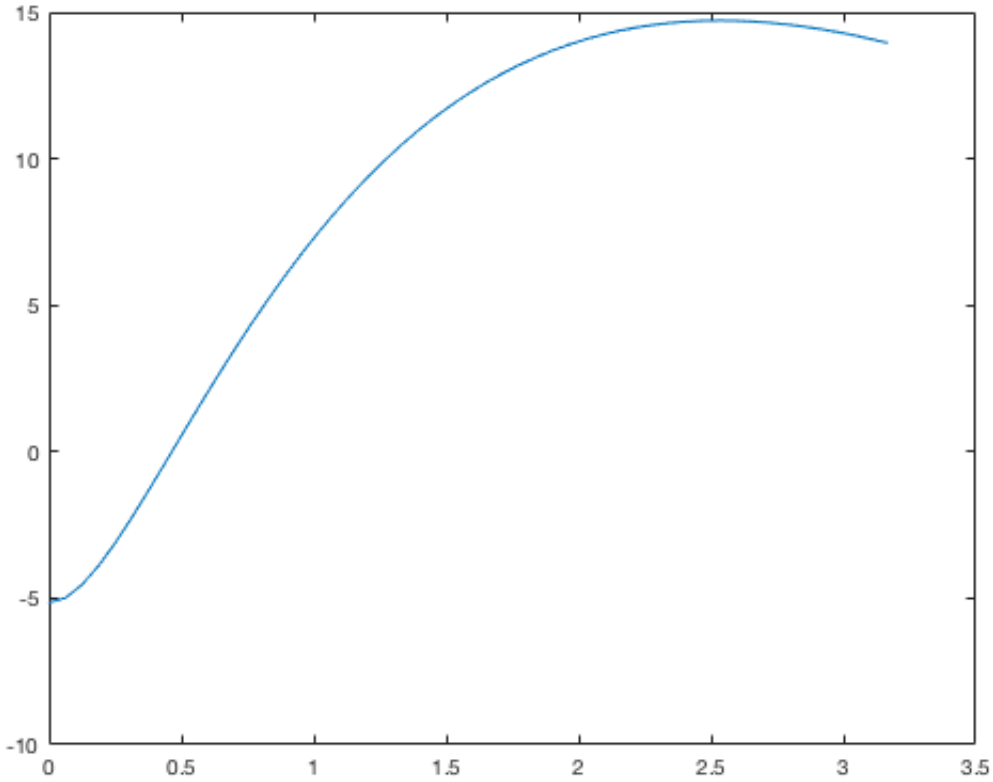


Figure 6.5: 2D plot of option price relative to time to maturity

With regards to the relationship between option price and interest rates the solution plot also show a positive relation between a call option and the interest rates, that is the price of option is an increasing function of interest rates. However from figure 6.6 the option price is not very sensitive to changes in interest rate. As the interest rate increased from 0% to 9% the option priced changed marginally from $R11.8316$ to $R11.83176$ which represents just a 0.000845% increase in option price. It would be fair for one to conclude based on this observation that the option price is negligibly dependent on interests

giving some credibility to the assumption of constant interest rates. However, one would have to to consider that once the transaction value goes up the error in pricing caused by not accounting for rates also goes up.

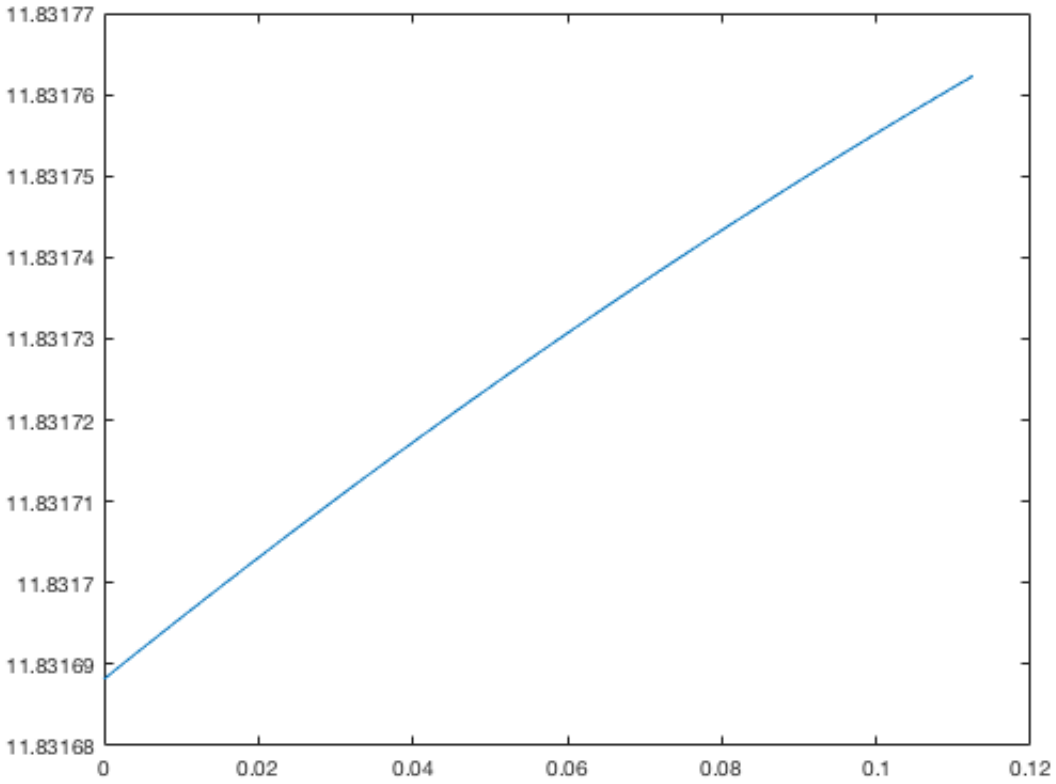


Figure 6.6: 2D plot of option price relative to interest rates

An interesting observation was in the sensitivity of the solution to the parameter λ. For the tested values of varying λ the solution appears to only behave in the range of $\lambda = |0.1|$ which is shown in figure 6.4. As soon λ is out of the range the solution exhibits oscillatory behaviour as in Figure 6.5. The more the parameter diverges from the range of stability the more of the oscillatory behaviour is observed and also the option value becomes very small.

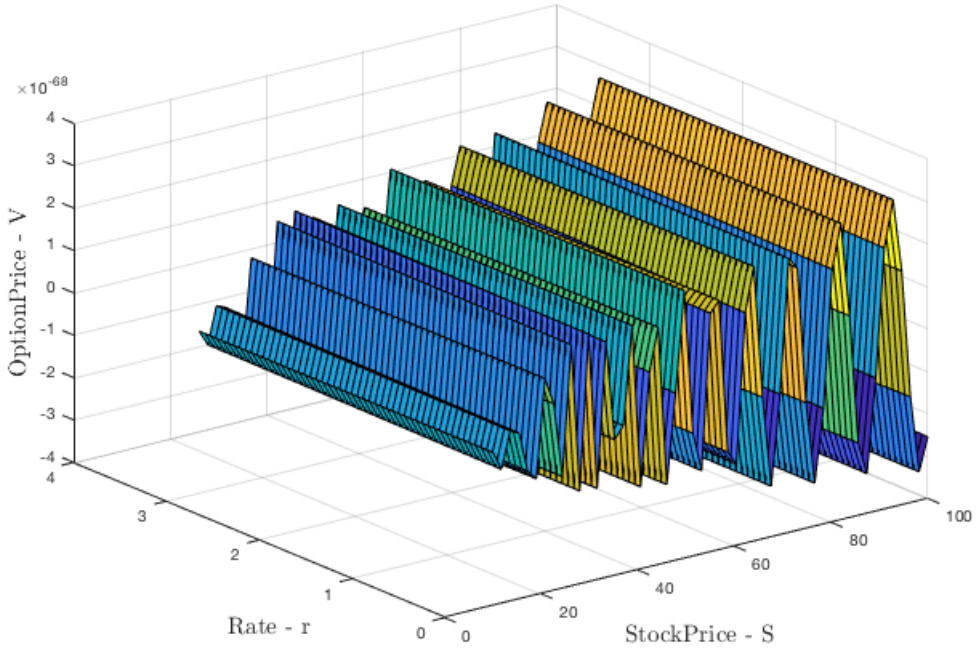


Figure 6.7: The solution $u(t, S, r)$ shows an unstable oscillatory behaviour

Case 4. $\lambda = 0$

When $\lambda = 0$ the solution of the equations changes to

$$G(\alpha) = C_0 \tag{6.50}$$

where C_0 is a constant and

$$H(\psi) = \frac{e^{\frac{-\kappa^2\sigma^2\Sigma\psi}{\kappa^2\sigma^2+\Sigma^2}}(-\kappa^2\sigma^2 - \Sigma^2)C_1}{\kappa\sigma^2\Sigma} + C_2. \tag{6.51}$$

The solution is then

$$u(\tau, c, r) = C_0 \left(\frac{e^{\frac{-\kappa^2\sigma^2\Sigma\psi}{\kappa^2\sigma^2+\Sigma^2}}(-\kappa^2\sigma^2 - \Sigma^2)C_1}{\kappa\sigma^2\Sigma} + C_2 \right), \tag{6.52}$$

where

$$\alpha = \tau$$

$$\psi = c + \frac{\Sigma r}{\kappa\sigma}.$$

One finds the solution $u(t, S, x)$ by reversing the initial transformations done initially, that is substitute

$$\tau = \frac{1}{2}t$$

$$c = \frac{1}{\sigma}\log S$$

and

$$r = \frac{1}{\Sigma}x$$

into the solution.

Analysis of Results : $\lambda = 0$

The solution of the equation when $\lambda = 0$ comes with some interesting observations. The existence of the term $-\kappa^2\sigma^2 - \Sigma^2$ results in the constants C_0, C_1 and C_2 being the variables that will determine if the value of $u(t, S, r)$ will be negative or positive. Consider when $C_0, C_1, C_2 > 0$, then $u(t, S, r)$ is a decreasing function of both the interest rates and underlying asset price (figure 6.8). This is an impractical relation since call options are expected to have a positive relation with the underlying stock price. Now consider when either C_0

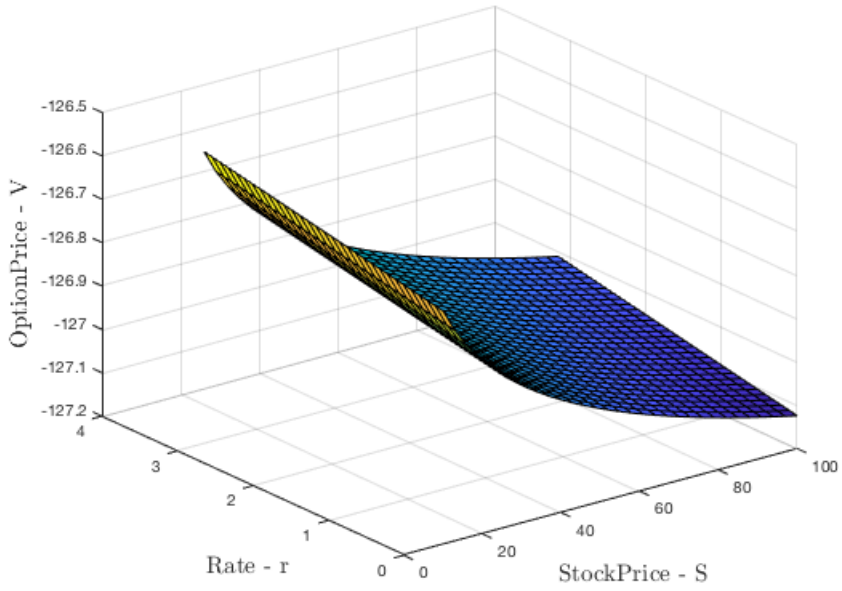


Figure 6.8: Surf plot of option price to interest rates and underlying price-negative relationship

or C_1 is negative. In this case the price of the option rises as the price of the underlying asset increases. The option price is still observed to be negatively related to interest rates. As in the case for the case when $\lambda \neq 0$ option price are affected much less by a change in interest rates compared to changes in the underlying asset price.

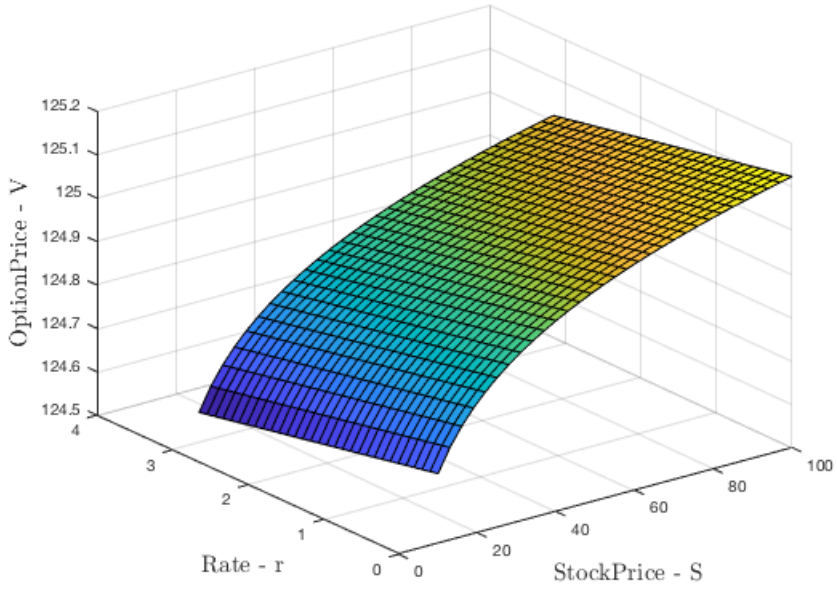


Figure 6.9: Surf plot of option price to interest rates and underlying price-positive relationship

6.3 Black Scholes Equation with C.I.R Interest Rates

In this section we look at the Black-Scholes model where the interest rates are driven by the C.I.R model

$$V_t + \frac{\sigma^2 S^2}{2} V_{SS} + S(r_0 + x)V_S - (r_0 + x)V + \frac{\Sigma^2 (r_0 + x)}{2} V_{xx} - \frac{a}{r_0} x V_x = 0 \quad (6.53)$$

For this equation we seek the infinitesimal symmetry operator of the form

$$X = \xi_1 \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial S} + \xi_3 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial V} \quad (6.54)$$

which generates the infinitesimal transformations

$$\begin{aligned} \overline{t} &= t + \varepsilon \xi_1(t, x, S, V) + O(\xi_1) \\ \overline{S} &= S + \varepsilon \xi_2(t, x, S, V) + O(\xi_2) \\ \overline{x} &= x + \varepsilon \xi_3(t, x, S, V) + O(\xi_3) \\ \overline{V} &= V + \varepsilon \eta(t, x, S, V) + O(\eta) \end{aligned}$$

Since we are concerned with the second order partial differential equation, the second prolongation is

$$\begin{aligned} X^{[2]} = \quad & \xi_1\frac{\partial}{\partial t} + \xi_2\frac{\partial}{\partial x} + \xi_3\frac{\partial}{\partial S} + \eta\frac{\partial}{\partial V} + \zeta_t\frac{\partial}{\partial V_t} \\ & + \zeta_{tt}\frac{\partial}{\partial V_{tt}} + \zeta_{tx}\frac{\partial}{\partial V_{tx}} + \zeta_x\frac{\partial}{\partial V_x} + \zeta_{xx}\frac{\partial}{\partial V_{xx}} \\ & + \zeta_{tS}\frac{\partial}{\partial V_{tS}} + \zeta_{xS}\frac{\partial}{\partial V_{xS}} + \zeta_S\frac{\partial}{\partial V_S} + \zeta_{SS}\frac{\partial}{\partial V_{SS}} + \dots \end{aligned}$$

We again use MATHEMATICA specifically the SYM package to find the symmetries of the equation. The determining equations are found to be

$$\sigma\xi^1_S = 0$$
$$\sigma\xi^1_{VV} = 0$$
$$\sigma\xi^1_V = 0$$
$$\Sigma\left(x + r_0\right)\xi^1_x = 0$$
$$\Sigma\left(x + r_0\right)\xi^1_{VV} = 0$$
$$\Sigma\left(x + r_0\right)\xi^1_V = 0$$
$$\sigma\Sigma\left(x + r_0\right)\xi^1_{SV} = 0$$
$$\sigma\Sigma\left(x + r_0\right)\xi^1_S = 0$$
$$\sigma\Sigma\left(x + r_0\right)\xi^1_{xV} = 0$$
$$\sigma\Sigma\left(x + r_0\right)\xi^1_x = 0$$
$$\sigma\Sigma\left(x + r_0\right)\xi^1_{VV} = 0$$
$$\sigma\Sigma\left(x + r_0\right)\xi^1_V = 0$$
$$\sigma\left(ax\xi^1_{VV} + r_0\xi^3_{VV}\right) = 0$$
$$\sigma\left(ax\xi^1_V + r_0\xi^3_V\right) = 0$$
$$\sigma\left(S\left(x + r_0\right)\xi^1_{VV} - \xi^2_{VV}\right) = 0$$
$$\Sigma\left(x + r_0\right)\left(ax\xi^1_{VV} + r_0{\xi^3}_{VV}\right) = 0$$
$$\Sigma\left(x + r_0\right)\left(S\left(x + r_0\right)\xi^1_{VV} - \xi^2_{VV}\right) = 0$$
$$\Sigma\left(x + r_0\right)\left(S\left(x + r_0\right)\xi^1_V - \xi^2_V\right) = 0$$
$$\sigma\left(2S\left(x + \sigma^2 + r_0\right)\xi^1_V - 2\xi^2_V + S^2\sigma^2\xi^1_{SV}\right) = 0$$

$$\sigma\left(\eta_{VV}-V\left(x+r_0\right)\xi_{VV}^1+2\left(S\left(x+r_0\right)\xi_{SV}^1-\xi_{SV}^2\right)\right)=0$$
$$\Sigma\left(x+r_0\right)\left(\left(-2ax+\Sigma^2r_0\right)\xi_V^1+r_0\left(-2\xi_V^3+\Sigma^2\left(x+r_0\right)\xi_{xV}^1\right)\right)=0$$
$$\Sigma\left(x+r_0\right)$$
$$*\left(2\left(a+xr_0+r_0^2\right)\xi_V^1+Vr_0^2\xi_{VV}^1+2ax\xi_{xV}^1+r_0\left(-\eta_{VV}+Vx\xi_{VV}^1+2\xi_{xV}^3\right)\right)=0$$
$$S\Sigma^2r_0^3\xi_x^1+\Sigma^2r_0^2\left(2Sx\xi_x^1-\xi_x^2\right)-aS^2x\sigma^2\xi_S^1$$
$$-r_0\left(-Sx^2\Sigma^2\xi_x^1+x\Sigma^2\xi_x^2+S^2\sigma^2\xi_S^3\right)=0$$
$$S\Sigma^2r_0^3\xi_{xV}^1+\Sigma^2r_0^2\left(S\xi_V^1+2Sx\xi_{xV}^1-\xi_{xV}^2\right)-aS^2x\sigma^2\xi_{SV}^1$$
$$+r_0\left(Sx\Sigma^2\xi_V^1+Sx^2\Sigma^2\xi_{xV}^1-x\Sigma^2\xi_{xV}^2-S^2\sigma^2\xi_{SV}^3\right)=0$$
$$\sigma\left(-2aSx\xi_x^1+Sr_0^2\left(2V\xi_V^1+\Sigma^2\xi_{xx}^1+6S\xi_S^1\right)\right)$$
$$+\sigma\left(r_0\left(4\xi^2+S\left(2Vx\xi_V^1+x\Sigma^2\xi_{xx}^1+6Sx\xi_S^1+4S\sigma^2\xi_S^1-4\xi_S^2+S^2\sigma^2\xi_{SS}^1+2\xi_t^1\right)\right)\right)=0$$
$$\Sigma\left(-6ax^2\xi_x^1+r_0^3\left(2V\xi_V^1+\Sigma^2\xi_{xx}^1+2S\xi_S^1\right)\right)$$
$$+\left(\Sigma r_0^2\left(4Vx\xi_V^1+2\Sigma^2\xi_x^1-4\xi_x^3+2x\Sigma^2\xi_{xx}^1+4Sx\xi_S^1+S^2\sigma^2\xi_{SS}^1+2\xi_t^1\right)\right)$$
$$+\left(\Sigma r_0\left(2\xi^3+x\left(2Vx\xi_V^1+\left(-6a+2\Sigma^2\right)\xi_x^1-4\xi_x^3+x\Sigma^2\xi_{xx}^1+2Sx\xi_S^1+S^2\sigma^2\xi_{SS}^1+2\xi_t^1\right)\right)\right)=0$$
$$2ax\left(-\eta_x+Vx\xi_x^1\right)-Vr_0^3\left(2V\xi_V^1+\Sigma^2\xi_{xx}^1+2S\xi_S^1\right)$$
$$-r_0^2\left(2\eta-2V\eta_V+4V^2x\xi_V^1+2V\Sigma^2\xi_x^1-\Sigma^2\eta_{xx}\right)$$
$$-r_0\left(2Vx\Sigma^2\xi_{xx}^1-2S\eta S+4SVx\xi_S^1+S^2V\sigma^2\xi_{SS}^1+2V\xi_t^1\right)$$
$$-r_0\left(2x\eta+2V\xi^3-2Vx\eta_V+2V^2x^2\xi_V^1-2aVx\xi_x^1+2Vx\Sigma^2\xi_x^1-x\Sigma^2\eta_{xx}\right)$$
$$-r_0\left(Vx^2\Sigma^2\xi_{xx}^1-2Sx\eta_S+2SVx^2\xi_S^1-S^2\sigma^2\eta_{SS}+S^2Vx\sigma^2\xi_{SS}^1-2\eta_t+2Vx\xi_t^1\right)=0$$
$$2a^2x^2\xi_x^1-2\Sigma^2r_0^4\left(\xi_x^1+V\xi_{xV}^1\right)$$
$$-r_0^3\left(2V\Sigma^2\xi_V^1+2V\xi_V^3+4x\Sigma^2\xi_x^1-2\Sigma^2\eta_{xV}+4Vx\Sigma^2\xi_{xV}^1+\Sigma^2\xi_{xx}^3+2S\xi_S^3\right)$$
$$-ar_0\left(2\xi^3+x\left(2Vx\xi_V^1+2\Sigma^2\xi_x^1-2\xi_x^3+x\Sigma^2\xi_{xx}^1+2Sx\xi_S^1+S^2\sigma^2\xi_{SS}^1+2\xi_t^1\right)\right)$$
$$-r_0^2\left(2Vx\left(a+\Sigma^2\right)\xi_V^1+2Vx\xi_V^3+2a\Sigma^2\xi_x^1+2x^2\Sigma^2\xi_x^1-2x\Sigma^2\eta_{xV}+2Vx^2\Sigma^2\xi_{xV}^1\right)$$
$$-r_0\left(ax\Sigma^2\xi_{xx}^1+x\Sigma^2\xi_{xx}^3+2aSx\xi_S^1+2Sx\xi_S^3+S^2\sigma^2\xi_{SS}^3+2\xi_t^3\right)=0$$
$$2ax\left(-Sx\xi_x^1+\xi_x^2\right)+Sr_0^3\left(2V\xi_V^1+\Sigma^2\xi_{xx}^1+2S\xi_S^1\right)$$
$$+r_0^2\left(2\xi^2+4SVx\xi_V^1-2V\xi_V^2+2S\Sigma^2\xi_x^1+2Sx\Sigma^2\xi_{xx}^1-\Sigma^2\xi_{xx}^2\right)$$
$$+r_0\left(4S^2x\xi_S^1-2S\xi_S^2-2S^2V\sigma^2\xi_{SV}^1+S^3\sigma^2\xi_{SS}^1+2S\xi_t^1\right)$$
$$+r_0\left(2x\xi^2+2S\xi^3+2SVx^2\xi_V^1-2Vx\xi_V^2-2aSx\xi_x^1+2Sx\Sigma^2\xi_x^1+Sx^2\Sigma^2\xi_{xx}^1-x\Sigma^2\xi_{xx}^2\right)$$
$$+r_0\left(2S^2x^2\xi_S^1-2Sx\xi_S^2+2S^2\sigma^2\eta_{SV}-2S^2Vx\sigma^2\xi_{SV}^1+S^3x\sigma^2\xi_{SS}^1-S^2\sigma^2\xi_{SS}^2+2Sx\xi_t^1-2\xi_t^2\right)$$
$$=0$$

Solving the the determining equations yields three symmetries:

$$X_1 = \frac{\partial}{\partial t}, \tag{6.55}$$

$$X_2 = S\frac{\partial}{\partial S}, \tag{6.56}$$

$$X_3 = V\frac{\partial}{\partial V}. \tag{6.57}$$

We take the linear combination of the three to arrive at

$$X = c_1\frac{\partial}{\partial t} + c_2 S\frac{\partial}{\partial S} + c_3 V\frac{\partial}{\partial V} \tag{6.58}$$

where c_1, c_2, c_3 are constants. This results in the system

$$\frac{dt}{c_1} = \frac{dS}{c_2 S} = \frac{dV}{c_3 V} = \frac{dx}{0}. \tag{6.59}$$

It is quickly deduced that

$$I_1 = x = \alpha.$$

Now we consider:

$$\frac{dt}{c_1} = \frac{dS}{c_2 S},$$

$$\frac{c_2}{c_1}t + \ln I_2 = \ln S,$$

which solves to

$$S = I_2 e^{\frac{c_2}{c_1}t}$$

and it can be observed that

$$\begin{aligned} I_2 = \quad & Se^{-\frac{c_2}{c_1}t}, \\ & = \psi. \end{aligned}$$

Now we consider

$$\frac{dt}{c_1} = \frac{dV}{c_3 V},$$

$$\frac{c_3}{c_1}t + \ln I_3 = \ln V$$

where $c_1 \neq 0$, which results in

$$V = I_3 e^{\frac{c_3}{c_1}t}. \tag{6.60}$$

Let us assume that I_3 is a function of I_1 and I_2, that is

$$I_3 = F(\alpha, \psi).$$

The general solution of the equation is then

$$V = F(\alpha, \psi) e^{\frac{c_3}{c_1}t}. \tag{6.61}$$

Taking derivatives of the original equation we get

$$V_t = F_\psi((-\frac{c_2}{c_1}t)S) e^{-\frac{c_2}{c_1}t} e^{\frac{c_3}{c_1}t} + F(\frac{c_3}{c_1}) + e^{\frac{c_3}{c_1}t} \tag{6.62}$$

$$V_S = F_\psi e^{-\frac{c_2}{c_1}t} e^{\frac{c_3}{c_1}t} \tag{6.63}$$

$$F_{\psi\psi} = \left(e^{-\frac{c_2}{c_1}t} \right)^2 e^{\frac{c_3}{c_1}t} \tag{6.64}$$

$$V_x = F_\alpha e^{\frac{c_3}{c_1}t} \tag{6.65}$$

$$V_{xx} = F_{\alpha\alpha} e^{\frac{c_3}{c_1}t} \tag{6.66}$$

Substituting into (6.53) results in

$$\begin{aligned} F_\psi \left(S - \frac{c_2}{c_1} \right) e^{-\frac{c_2}{c_1}t} e^{\frac{c_3}{c_1}t} + F \left(\frac{c_3}{c_1} \right) e^{\frac{c_3}{c_1}t} + \frac{\sigma^2 S^2}{2} F_{\psi\psi} e^{-\frac{2c_2}{c_1}} e^{\frac{c_3}{c_1}t} \\ + S(r_0 + x) F_\psi e^{-\frac{c_2}{c_1}t} e^{\frac{c_3}{c_1}t} - (r_0 + x) F e^{\frac{c_3}{c_1}t} \\ + \frac{\Sigma^2 (r_0 + x)}{2} F_{\alpha\alpha} e^{\frac{c_3}{c_1}t} - \frac{a}{r_0} x F_\alpha e^{\frac{c_3}{c_1}t} = 0 \end{aligned}$$

We can drop the common term $e^{\frac{c_3}{c_1}t}$ by dividing through by the term to result

in:

$$F_\psi \left(S - \frac{c_2}{c_1}\right) e^{-\frac{c_2}{c_1}t} e^{\frac{c_3}{c_1}t} + F\left(\frac{c_3}{c_1}\right) + \frac{\sigma^2 S^2}{2} F_{\psi\psi} \left(e^{-\frac{c_2}{c_1}}\right)^2$$
$$+ S(r_0 + x) F_\psi e^{-\frac{c_2}{c_1}t} - (r_0 + x)F$$
$$+ \frac{\Sigma^2(r_0 + x)}{2} F_{\alpha\alpha} - \frac{a}{r_0} x F_\alpha e^{\frac{c_3}{c_1}t} = 0$$

We also note that

$$\alpha = x$$

and

$$\psi = S e^{-\frac{c_2}{c_1}t}.$$

We can now write equation (6.53) in term of only α and ψ and is simplified to

$$F_\psi \left(r_0 + \alpha - \frac{c_2}{c_1}\right)\psi + \frac{\sigma^2\psi^2}{2} F_{\psi\psi} - \left(r_0 + \alpha - \frac{c_3}{c_1}\right) F + \frac{\Sigma^2(r_0 + \alpha)}{2} F_{\alpha\alpha} - \frac{a}{r_0}\alpha F_\alpha = 0 \tag{6.67}$$

We use the method of separation of variables. Assume

$$F(\alpha, \psi) = G(\alpha)H(\psi)$$

Substituting $F(\alpha, \psi)$ in the equation

$$H'(\psi)G(\alpha) \left(r_0 + \alpha - \frac{c_2}{c_1}\right)\psi + \frac{\sigma^2\psi^2}{2} H''(\psi)G(\alpha) - \left(r_0 + \alpha - \frac{c_3}{c_1}\right) H(\psi)G(\alpha)$$
$$+ \tfrac{\Sigma^2(r_0+\alpha)}{2} H(\psi)G''(\alpha) - \frac{a}{r_0}\alpha H(\psi)G'(\alpha) = 0$$

Divide both sides of the equation by $H(\psi)G(\alpha)$

$$\left(r_0 + \alpha - \frac{c_2}{c_1}\right)\psi \frac{H'(\psi)}{H(\psi)} + \frac{\sigma^2\psi^2}{2}\frac{H''(\psi)}{H(\psi)} - \left(r_0 + \alpha - \frac{c_3}{c_1}\right) + \frac{\Sigma^2(r_0 + \alpha)}{2}\frac{G''(\alpha)}{G(\alpha)}$$
$$- \frac{a}{r_0}\alpha\frac{G'(\alpha)}{G(\alpha)} = 0$$

Separating into two ordinary differential equations, we have

$$\left(r_0 + \alpha - \frac{c_2}{c_1}\right)\psi H'(\psi) + \frac{\sigma^2\psi^2}{2} H''(\psi) + \left(\frac{c_3}{c_1} + \lambda\right) H(\psi) = 0 \tag{6.68}$$

and

$$\frac{\Sigma^2(r_0+\alpha)}{2}G''(\alpha)-\frac{a}{r_0}\alpha G'(\alpha)-(r_0+\alpha-\lambda)G(\alpha)=0. \tag{6.69}$$

Case 5. $\lambda \neq 0$

Here we have

$$G(\alpha)=\beta_1 C_1 \mathrm{U}\left[-\gamma_1, 1+\frac{-2a+\Sigma^2}{\Sigma^2}, \frac{2\sqrt{a^2+2\Sigma^2 r_0^2}}{\Sigma^2}+\frac{2\alpha\sqrt{a^2+2\Sigma^2 r_0^2}}{\Sigma^2 r_0}\right] + \beta_1 C_2 \mathrm{L}\left[\gamma_1, \frac{-2a+\Sigma^2}{\Sigma^2}, \frac{2\sqrt{a^2+2\Sigma^2 r_0^2}}{\Sigma^2}+\frac{2\alpha\sqrt{a^2+2\Sigma^2 r_0^2}}{\Sigma^2 r_0}\right]$$

where

$$\beta_1 = e^{\frac{\left(-2a+\Sigma^2\right)\mathrm{Log}\left[\alpha+r_0\right]r_0+\alpha\left(a-\sqrt{a^2+2\Sigma^2 r_0^2}\right)}{\Sigma^2 r_0}}$$

and

$$\gamma_1=\frac{a^2+\lambda\Sigma^2 r_0+a\sqrt{a^2+2\Sigma^2 r_0^2}-\Sigma^2\sqrt{a^2+2\Sigma^2 r_0^2}}{\Sigma^2\sqrt{a^2+2\Sigma^2 r_0^2}}.$$

Case 6. $\lambda = 0$

Solving the equations when $\lambda = 0$ in MATHEMATICA produces the results

$$G(\alpha)=\beta_2 C_1 \mathrm{U}\left[-\gamma_2, 1+\frac{-2a+\Sigma^2}{\Sigma^2}, \frac{2\sqrt{a^2+2\Sigma^2 r_0^2}}{\Sigma^2}+\frac{2\alpha\sqrt{a^2+2\Sigma^2 r_0^2}}{\Sigma^2 r_0}\right] + \beta_2 C_2 \mathrm{L}\left[\gamma_2, \frac{-2a+\Sigma^2}{\Sigma^2}, \frac{2\sqrt{a^2+2\Sigma^2 r_0^2}}{\Sigma^2}+\frac{2\alpha\sqrt{a^2+2\Sigma^2 r_0^2}}{\Sigma^2 r_0}\right],$$

where

$$\beta_2 = e^{\frac{\left(-2a+\Sigma^2\right)\mathrm{Log}\left[\alpha+r_0\right]r_0+\alpha\left(a-\sqrt{a^2+2\Sigma^2 r_0^2}\right)}{\Sigma^2 r_0}}$$

$$\gamma_2=\frac{a^2+a\sqrt{a^2+2\Sigma^2 r_0^2}-\Sigma^2\sqrt{a^2+2\Sigma^2 r_0^2}}{\Sigma^2\sqrt{a^2+2\Sigma^2 r_0^2}}$$

and

$$H(\psi) = \psi^{\frac{\sqrt{c_3}\,(-\beta_3-\gamma_3)}{\sqrt{2}\sigma\sqrt{c_1}}} C_1 + \psi^{\frac{\sqrt{c_3}\,(-\beta_3+\gamma_3)}{\sqrt{2}\sigma\sqrt{c_1}}} C_2,$$

where

$$\beta_3 = \frac{2\sqrt{2}\alpha c_1 - \sqrt{2}\sigma^2 c_1 - 2\sqrt{2}c_2 + 2\sqrt{2}c_1 r_0}{2\sigma\sqrt{c_1}\sqrt{c_3}}$$

and

$$\gamma_3 = \sqrt{-4 + \frac{\left(2\sqrt{2}\alpha c_1 - \sqrt{2}\sigma^2 c_1 - 2\sqrt{2}c_2 + 2\sqrt{2}c_1 r_0\right)^2}{4\sigma^2 c_1 c_3}}.$$

In both cases the solutions are complicated involving Hypergeometric function U and Laugauerre function L which, as in the case of the solution of equation (6.1), makes them unlikely to be used in practical market environments.

6.4 Concluding Remarks

In this chapter, Lie symmetry methods have been applied on both the untransformed Vasicek driven Black-Scholes equation (6.1) and the transformed Vasicek equation (6.23), and also the CIR driven model (6.53) was considered. The solutions were provided and analysis of results for the transformed equation was also given. The software package SYM for Mathematica was used.

Chapter 7

Conclusion

The purpose of this dissertation was to apply Lie symmetry and invariant methods in solving of the (2 + 1) Black-Scholes equations where the interest rates are driven by the Vasicek model and also when the variable interest rate structure is determined by the C.I.R model. It was also intended to provide an introduction into key mathematical concepts which are used in the field of mathematical finance, which are especially important in the derivation of the various variations of the Black Scholes model.

In chapter 1 the classical Black-Scholes model was introduced, the derivation of which was also provided. The understanding of the classical model provides the foundation for the derivation of the more complicated model in which interest rates are relaxed and are allowed to be time-variable in line with market behaviour. Chapter 2 looked at previous work done on (2 + 1) dimensional models in finance. However, as not much work was achieved on the the equation of focus,the literature review was not limited to the application of symmetry methods on the the (2 + 1) dimensional equations but also looked at interesting research on (1 + 1) dimensional equations which was extended to (2 + 1) dimensional equations where we proposed conditions for which (2+1) parabolic equations can be reduced to the heat equation (3.44) .

Chapter 3 and chapter 4 provided high level introductions to key concepts that were used in this dissertation. Chapter 3 introduced Lie symmetry meth-

ods and slightly expanded the work of Mahomed [6] in the application of invariant methods to equations of finance. This minor expansion considered in this dissertation opens up the need for more research to be performed in the application of invariant methods to higher dimensional equations of finance as this will likely result in more simpler and easily derivable solutions to financial equations.
Chapter 4 provided a basic introduction to stochastic calculus. In working with equations of finance it is important to understand Stochastic Calculus as the concepts are used extensively in equations of finance in their derivations as well as in the solving thereafter.

In chapter 5, the derivation of the (2+1) dimensional Black-Scholes model was provided, following the work of Haowen [13]. This dissertation differed from [13] in that while Haowen [13] allowed stochastic processes to be correlated for simplicity an assumption was made that the processes for the underlying asset price and the interest rate process were independent of each other that is $\rho_{W_1,W_2} = 0$. For the Vasicek model it was also assumed that the initial interest rate be zero ($r_0 = 0$). This assumption also further simplified the equation used in this dissertation.

In chapter 6 three symmetries of the Vasicek rate dependent Black-Scholes model were found. The solution was also derived from these and provided in this chapter. However due to the complexity of the results and their probable inability to provide a practical use the equation independent variables were transformed to give a more simplified equation of the model. Six symmetries were found and the solution was found with the application of one of the symmetries in reducing the dimension of the equation. The solution is observed to be in line with market expectations with regards to the relationship between call option prices to underlying asset prices and interest rates. However, it also points to a lack of sufficient dependence of the price to varying interest rates which likely gives credibility to the assumption of constant interest rates.
Chapter 6 also provided an Lie symmetry analysis of the C.I.R dependent model. As was the case for the untransformed Vasicek equation, three symmetries were found. The solution found was also too complicated to provide any relief in a practical option pricing environment. Further research is necessary to find a more simpler solution.

As far as known at the time of this research not much work had been done in the application of symmetry methods to (2+1) equations of finance specifically the Black-Scholes equation with variable interest rates. This dissertation found the symmetries for the two models based on Vasicek and C.I.R model dependent interest rates. The application of invariant methods was slightly touched on as well in this dissertation and provided a basis for further research with focus on higher dimensional equations in finance.

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