In the light of work already done, reverse transition appears to be a rather complicated process and no simple criterion such as Moller's (1963) can be given. Although Chen and Peube's (1965) criterion stems from theoretical considerations it is rather naive since it is ultimately based on the instability of laminar flows and there is no reason for believing that the conditions defining the onset of instabilities in a laminar flow should also be related to the conditions defining the inception of reverse transition.

1.3 PRESENT STUDY

The purpose of this study will therefore be:

a) To make a critical analysis of the existing theories for radial source flow between parallel disks and hence try and explain some contradictions, e.g. the apparent validity of the uni-directional flow model proposed by McGinn (1955) and Jackson and Symmons (1965a).

b) To investigate the possibility of extending Wilson's (1972) ideas and in so doing try and obtain a uniformly-valid approximation for radial flow between parallel disks.

c) To investigate more thoroughly the inception of reverse transition in radial flows.

2. CRITICAL AND DIMENSIONAL ANALYSES OF RADIAL LAMINAR FLOW

2.1 CRITICAL ANALYSIS OF EXISTING THEORIES FOR RADIAL LAMINAR FLOWS

We consider the steady axially symmetric flow of a Newtonian fluid with constant properties between two 'very large' stationary disks which lie in the plane $z = -h$ and $z = +h$, Figure 1.

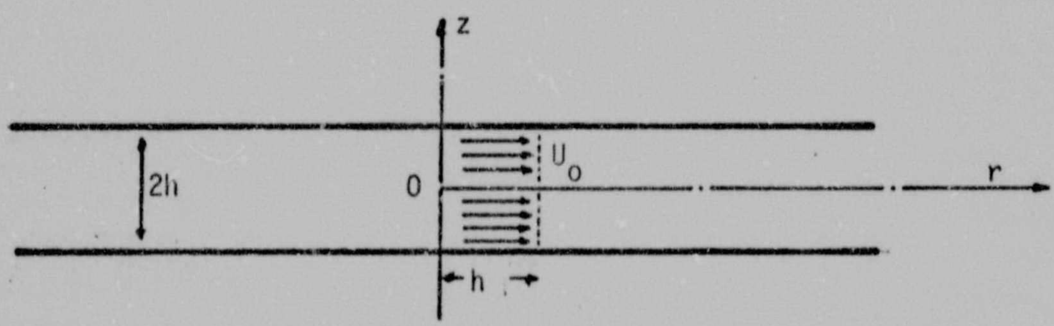


Fig. 1 - Co-ordinate system

The governing equations are

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial z} = 0 \qquad (2\text{-}1)$$

$$\rho(u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z}) = -\frac{\partial p}{\partial r} + \mu(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}) \qquad (2\text{-}2)$$

$$\rho(u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial z}) = -\frac{\partial p}{\partial z} + \mu(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2}) \qquad (2\text{-}3)$$

with boundary conditions

$$u = v = 0 \text{ at } z = \pm h \qquad (2\text{-}4)$$

$$u = U_0, \quad \frac{\partial v}{\partial r} = 0 \text{ at } r = h \qquad (2\text{-}5)$$

A useful relationship is the integral form of continuity

$$Q = 4\pi r \int_0^h u dz \tag{2-6}$$

where Q is the volumetric flow rate.

For convenience we define the following stream function ϕ

$$u = \frac{1}{r}\frac{\partial}{\partial z}(r\phi), \quad v = -\frac{1}{r}\frac{\partial}{\partial r}(r\phi) \tag{2-7}$$

Using ϕ, the governing equations reduce to

$$[\phi_z(\frac{\partial}{\partial r} - \frac{1}{r}) - (\phi_r + \frac{1}{r}\phi)\frac{\partial}{\partial z}]\nabla^2\phi = \nu\nabla^4\phi \tag{2-8}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \tag{2-9}$$

The boundary conditions become

$$\phi_z = -\frac{1}{r}\frac{\partial}{\partial r}(r\phi) = 0 \quad \text{at} \quad z = \pm h \tag{2-10}$$

$$\phi_z = U_o, \quad \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(r\phi)) = 0 \quad \text{at} \quad r = h \tag{2-11}$$

Defining $\phi(r,0) = 0$ and using (2-6) we find

$$\phi(r,h) = \frac{Q}{4\pi r} \tag{2-12}$$

If the following dimensionless variables are introduced into the equation of motion (2-8)

$$\hat{z} = \frac{z}{h}, \quad \hat{r} = \frac{r}{h}, \quad \hat{\phi} = \frac{\phi}{Q/4\pi h}, \quad Re = \frac{Q}{4\pi\nu h} \tag{2-13}$$

we obtain

$$[\hat{\phi}_{\hat{z}}(\frac{\partial}{\partial\hat{r}} - \frac{1}{\hat{r}}) - (\hat{\phi}_{\hat{r}} + \frac{1}{\hat{r}})\frac{\partial}{\partial\hat{z}}]\hat{\nabla}^2\hat{\phi} = \frac{1}{Re}\hat{\nabla}^4\hat{\phi} \tag{2-14}$$

In lubric ion problems and hydrostatic thrust bearings the inertia effects are negligible and it is customary then to set $Re = 0$. The governing equation reduces to

$$\hat{\nabla}^4 \hat{\phi} = 0 \qquad (2\text{-}15)$$

Assuming one-dimensional flow a solution to this problem is

$$\phi = (1{,}5/\hat{r})(\hat{z} - \hat{z}^3/3) \qquad (2\text{-}16)$$

If the inertia effects become important the assumption of a parabolic profile leads to an inconsistency (for details see Appendix A) and to obtain higher order approximations Savage(1964) and Peube (1963) assume the following:

a) the flow field is two-dimensional,

b) the velocity and pressure fields can be represented by a power series in $1/\hat{r}^n$

With these assumptions the solution takes on the following form

$$\hat{r}\hat{\phi} \sim F_0(\hat{z}) + \left(\frac{Re}{\hat{r}^2}\right)F_1(\hat{z}) + 0\left(\frac{1}{\hat{r}^4}\right) \qquad (2\text{-}17)$$

where F_0 and F_1 are defined by the following differential equations

$$F_0^{IV} = 0 \qquad (2\text{-}18)$$

$$F_1^{IV} = -2F_0'F_0'' \qquad (2\text{-}19)$$

with boundary conditions

$$F_0(1) = 1, \quad F_0(0) = F_0'(1) = F_0''(0) = 0 \qquad (2\text{-}20)$$

$$F_1(0) = F_1(1) = F_1'(1) = F_1''(1) = 0 \qquad (2\text{-}21)$$

Wilson (1972) shows by heuristic arguments that (2-17) is actually an asymptotic expansion valid when $(Re/\hat{r}^2)$ tends to zero and hence the error in neglecting the third term is not $0(1/\hat{r}^4)$ but rather $0(Re/\hat{r}^2)^2$. He also points out that the expansion is not uniformly valid for all $\hat{r}$ and becomes singular when $(Re/\hat{r}^2) \sim O(1)$, i.e. $\hat{r} \sim \sqrt{Re}$. He suggests that in this region of non-uniformity the correct dimensionless variables should be

$$z^* = \frac{z}{h}, \quad r^* = \frac{r}{h\sqrt{Re}}, \quad \phi = \frac{\sqrt{Re}\ \phi}{Q/4\pi h} \tag{2-22}$$

and in the limit when Re tends to infinity he obtains the following governing equation

$$[\phi^*_{z^*}(\frac{\partial}{\partial r^*} - \frac{1}{r^*}) - (\phi^*_{r^*} + \frac{1}{r^*}\phi^*)\frac{\partial}{\partial z^*}]\frac{\partial^2\phi^*}{\partial z^{*2}} = \frac{\partial^2\phi^*}{\partial z^{*4}} \tag{2-23}$$

Wilson suggests that a solution of (2-23) can be represented by the following series

$$r^*\phi^* = \sum_0^\infty (\frac{1}{r^{*2}})^N F_N(z^*) + \exp(-\tfrac{1}{2}\alpha r^{*2})f(z^*) \tag{2-24}$$

In a subsequent private communication Dr S. Wilson points out an error in his analysis. Equation (2-24) should have read

$$\phi^* = \frac{1}{r^*}[\sum_0^\infty (\frac{1}{r^{*2}})^N F_N(z^*)] + r^{*\sigma}\exp(-\tfrac{1}{2}\alpha r^{*2})f(z^*) \tag{2-25}$$

This means that not only is it necessary to determine a sequence of eigen values α_1, α_2, α_3 etc. but also a sequence of unknown constants σ_1, σ_2 etc. The problem is therefore intractable.

Wilson's theory implicitly implies that Savage's expansion proceeds in powers of $(Re/\hat{r}^2)^n$. If this is indeed the case it should be possible to generate Savage's expansion by simply assuming an asymptotic expansion of the form

$$\hat{r}\hat{\phi} = F_0(\hat{z}) + (\frac{Re}{\hat{r}^2})F_1(\hat{z}) + (\frac{Re}{\hat{r}^2})^2F_2(\hat{z}) + (\frac{Re}{\hat{r}^2})F_3(\hat{z}) + \ldots \tag{2-26}$$

and then substituting into the equation of motion and solving successively for the functions $F_n(\hat{z})$. If this is done a rather unusual result occurs, the perturbation equations for the functions $F_n(\hat{z})$ turn out to be

$$\left.\begin{aligned}
F_0^{IV} &= 0 \\
F_1^{IV} &= -2F_0'F_0'' \\
16F_1^{IV} &= 0 \\
F_2^{IV} &= -4F_0'F_1'' - 2F_1'F_0'' + 2F_1F_0''
\end{aligned}\right\} \qquad (2\text{-}27)$$

The first two equations are identical to Savage's, however the third equation presents a contradiction since $F_1(\hat{z})$ is governed by two distinct differential equations. Our conclusion therefore, is that the expansion does not proceed in powers of $(Re/r^2)^n$. To explain this apparent anomaly it is necessary to discuss some theory on asymptotic expansions (see Appendix B for a comprehensive survey on asymptotic expansion), i.e. is (2-17) a co-ordinate or parameter-type expansion?

At first glance one may regard (2-17) as a parameter expansion valid for Re tending to zero. This argument is then consistent with previous work on lubrication problems since the expansion reduces to (2-16) when Re is set to zero. However, if one then tries to determine higher order terms valid when Re tends to zero the resulting perturbation equations are different from Savage's, since a parameter expansion valid for small Re is mathematically a separate problem.

Another alternative is to regard Savage's expansion as a co-ordinate expansion valid for large values of the radial co-ordinate but at a fixed Re. To determine whether this is indeed the case we make use of Chang's (1961) artificial parameter method. (An artificial parameter is defined as any parameter in the governing equation that can be eliminated by a suitable co-ordinate transformation). The method can be summarised as follows: we introduce an artificial parameter into the governing equations and then proceed to find an asymptotic expansion in terms of this parameter, i.e. a parameter-type expansion. Once this expansion has been obtained there is a theorem, Chang (1961), that guarantees that the newly found expansion in terms of the artificial parameter has an equivalent co-ordinate-type expansion. The expansion must however satisfy the principle of eliminability, i.e. it must be possible to eliminate the artificial parameter from the resulting expansion which means that only a certain class of functions are allowable.

To determine a co-ordinate expansion valid for large $\hat{r}$ we introduce the following artificial parameter into the equation of motion

$$\sigma = \frac{r}{\varepsilon} \tag{2-28}$$

The governing equation becomes

$$\frac{1}{\varepsilon}\left[\hat{\phi}_{\hat{z}}\left(\frac{\partial}{\partial\sigma} - \frac{1}{\sigma}\right) - \left(\hat{\phi}_{\sigma} + \frac{1}{\sigma}\hat{\phi}\right)\frac{\partial}{\partial\hat{z}}\right]\tilde{\nabla}^2\hat{\phi} = \frac{1}{Re}\tilde{\nabla}^4\hat{\phi} \tag{2-28}$$

where

$$\tilde{\nabla}^2 = \frac{1}{\varepsilon^2}\left(\frac{\partial}{\partial\sigma^2} + \frac{1}{\sigma}\frac{\partial}{\partial\sigma} - \frac{1}{\sigma^2}\right) + \frac{\partial^2}{\partial\hat{z}^2} \tag{2-29}$$

A certain amount of trial and error suggests the following expansion

$$\hat{\phi} \sim \frac{1}{\varepsilon}\hat{\phi}_0 + \frac{1}{\varepsilon^3}\hat{\phi}_1 + \frac{1}{\varepsilon^5}\hat{\phi}_2 + \dots \tag{2-30}$$

Proceeding as in a parameter expansion we define the following limit process

$$\lim_{\varepsilon \to \infty} \hat{\phi}(\sigma, \hat{z}, Re, \varepsilon) \quad \text{with } \sigma, \hat{z} \text{ and } Re \text{ fixed} \tag{2-31}$$

Substituting (2-30) into (2-28) and applying (2-31) we obtain the following perturbation equations

$$\frac{\partial^4\hat{\phi}_0}{\partial\hat{z}^4} = 0 \tag{2-32}$$

$$Re\left\{\left(\hat{\phi}_{0\hat{z}}\left(\frac{\partial}{\partial\sigma} - \frac{1}{\sigma}\right)\frac{\partial^2\hat{\phi}_0}{\partial\hat{z}^2}\right) - \left(\hat{\phi}_{0\sigma}\frac{\partial^3\hat{\phi}_0}{\partial\hat{z}^3} + \frac{1}{\sigma}\hat{\phi}_0\frac{\partial^3\phi_0}{\partial\hat{z}^3}\right)\right\} = \frac{\partial^4\hat{\phi}_1}{\partial\hat{z}^4} +$$

$$+ Re\left\{\hat{\phi}_{0\hat{z}}\left(\frac{\partial}{\partial\sigma} - \frac{1}{\sigma}\right)L\hat{\phi}_0 + \hat{\phi}_{0\hat{z}}\left(\frac{\partial}{\partial\sigma} - \frac{1}{\sigma}\right)\frac{\partial^2\hat{\phi}_1}{\partial\hat{z}^2}\right\} \tag{2-33}$$

$$\mathrm{Re}\{\hat{\phi}_{0\hat{z}}(\frac{\partial}{\partial\sigma} - \frac{1}{\sigma})L\hat{\phi}_0 + \hat{\phi}_{0\hat{z}}(\frac{\partial}{\partial\sigma} - \frac{1}{\sigma})\frac{\partial^2\hat{\phi}_1}{\partial\hat{z}^2}\}$$

$$+ \mathrm{Re}\{\hat{\phi}_{0\sigma} L\frac{\partial\hat{\phi}_0}{\partial\hat{z}} + \hat{\phi}_{0\sigma}\frac{\partial^3\hat{\phi}_1}{\partial\hat{z}_3} + \frac{1}{\sigma}\hat{\phi}_0 L\frac{\partial}{\partial\hat{z}}\ 0\frac{\partial^3}{\partial\hat{z}^3}\hat{\phi}_1$$

$$+ \hat{\phi}_{1\sigma}\frac{\partial^3\hat{\phi}_0}{\partial\hat{z}^3} + \frac{1}{\sigma}\hat{\phi}_1\frac{\partial^3\hat{\phi}_0}{\partial\hat{z}^3}\} = 2L\frac{\partial^2\hat{\phi}_1}{\partial\hat{z}^2} + \frac{\partial^4\hat{\phi}_2}{\partial\hat{z}^4} \qquad (2\text{-}34)$$

where

$$L = \frac{\partial^2}{\partial\sigma^2} + \frac{1}{\sigma}\frac{\partial}{\partial\sigma} - \frac{1}{\sigma^2} \qquad (2\text{-}35)$$

Since the functions $\hat{\phi}_0$, $\hat{\phi}_1$, $\hat{\phi}_2$ etc. must obey the principle of eliminability we must find functions such that when we transform back to $\hat{r}$-space the ε's must disappear from the expansion. The following functions turn out to be suitable

$$\hat{\phi}_0 = \frac{1}{\sigma}F_0(\hat{z})$$

$$\hat{\phi}_1 = \frac{1}{\sigma^3}F_1(\hat{z}) \qquad (2\text{-}36)$$

$$\hat{\phi}_2 = \frac{1}{\sigma^5}F_2(\hat{z})$$

Substituting (2-36) into the perturbation equations we obtain the following system of ordinary differential equations

$$F_0^{IV} = 0$$

$$F_1^{IV} = -2\mathrm{Re}F_0'F_0'' \qquad (2\text{-}37)$$

$$F_2^{IV} = -16F_1 + \mathrm{Re}[2F_1F_0'''' - 4F_0'F_1'' + 2F_1F_0'']$$

with boundary conditions

$$F_0(1) = 1 \quad F_0(0) = 0 \quad F_0'(1) = 0 \quad F_0''(0) = 0$$

$$F_n(0) = 0 \quad F_n(1) = 0 \quad F_n'(1) = 0 \quad F_n''(0) = 0 \qquad (2\text{-}38)$$

for $n > 0$.

The perturbation equations (2-37) are identical to Savage's and this is therefore conclusive evidence that Savage's expansion is indeed a co-ordinate expansion valid for a fixed value of Re. Also the error in neglecting the third term is not $O(Re/r^2)^2$ as Wilson suggested but $O(\sqrt{Re}/r^2)^2$. This is easily seen if we rewrite $F_2(\hat{z})$ as a linear combination of f_{21} and f_{22}, where f_{21} and f_{22} are defined by

$$\begin{aligned} f_{21}{}^{IV} &= -16F_1{}'' \\ f_{22} &= Re[2F_1F_0{}''' - 4F_0{}'F_1{}'' + 2F_1F_0{}''] \end{aligned} \tag{2-39}$$

and rewriting the expansion as

$$\hat{r}\hat{\phi} = F_0(\hat{z}) + \frac{Re}{\hat{r}^2}F_1(\hat{z}) + \frac{Re}{\hat{r}^4}f_{21}(\hat{z}) + \left(\frac{Re}{\hat{r}^2}\right)^2 f_{22}(\hat{z}) \tag{2-40}$$

To explain why McGinn (1955), Jackson and Symmons (1965a) and various other authors mentioned in the literature review have obtained accurate if not identical results to Savage's theory by assuming a uni-directional flow model we rewrite the perturbation equation in terms of velocity and pressure components. These turn out to be

$$\frac{d\hat{P}_0}{d\hat{r}} = \frac{\partial^2\hat{U}_0}{\partial\hat{z}} \tag{2-41}$$

and

$$\hat{U}_0\frac{\partial\hat{U}_0}{\partial\hat{r}} = -\frac{d\hat{P}_1}{d\hat{r}} + \frac{\partial^2\hat{U}_1}{\partial\hat{z}^2} \tag{2-42}$$

We notice that the velocity component of the second order problem $\hat{U}_1$, is only a function of $\hat{U}_0$, i.e. V_0 is still of higher order than $\hat{U}_1$, and thus the assumption of McGinn (1955), Jackson and Symmons (1965a) etc., that the flow field is uni-directional is correct only if the third order effects are neglected.

Now that we have established that Savage's expansion is valid for a fixed Re it is instructive to reanalyse Wilson's hypothesis for obtaining the governing equation for the region of non-uniformity. By introducing new variables defined by (2-22) and applying a limit process whereby Re tends to infinity he is generating perturbation equations for a parameter expansion. To then try and match up a parameter expansion with a co-ordinate expansion is mathematically inappropriate. The reason being that a co-ordinate expansion possesses inherent indeterminacy, i.e. unknown constants appear when one attempts to match up expansions in regions of non-uniformity and these constants can only be determined if the exact solution to the problem is known.

The object now is to try and reformulate the problem as a parameter expansion valid for large values of Re. Intuitively we know that no matter what the value of Re is, the flow will eventually enter a regime, if the radii of the disks are sufficiently large, in which viscous forces will balance pressure forces. In other words we must determine the correct scaling procedure such that in the limit of high values of Re we must obtain a creeping flow solution, i.e. equation (2-16). To do this we make use of dimensional analysis.

2.2 SOME DIMENSIONAL ANALYSIS

The physical parameters that are assumed to characterise the flow of an incompressible fluid between parallel disks are

$$Q, \rho, \mu, h, L \tag{2-43}$$

where Q, ρ, μ and h are the volumetric flowrate, density, dynamic viscosity and half the spacing between the disks respectively. L is an unknown radial length scale which intuitively we would expect to be important since various authors have already shown that the inertia effects are not only determined by the flow rate but also by the radii of the disks (Savage (1964), Peube (1963)).

Making use of the Pi theorem it is possible to reduce the number of variables by forming the following dimensionless groups

$$\pi_1 = f_1\left(\frac{Q}{\nu}h\right) \qquad \pi_2 = f_2\left(\frac{h}{L}\right) \tag{2-44}$$

Thus unlike flow in a pipe or a two-dimensional channel where only one group, i.e. the Reynolds number is needed to characterize the flow, the parallel disk situation is slightly more complicated in that two groups are needed. One is a form of Reynolds number while the other is just a ratio of length scales.

To determine the ratio of inertia to viscous forces we must first define certain velocity and length scales.

We define the following dimensionless variables

$$\bar{r} = r/L, \quad \bar{z} = z/h, \quad \bar{u} = u/U, \quad \bar{v} = v/V \qquad (2\text{-}45)$$

where U, V and L are at the moment unknown velocity and length scales. This procedure ensures that the problem is posed in the most general form possible. To determine U we make use of the integral form of continuity (2-6) which in terms of (2-45) becomes

$$Q = 4\pi\bar{r}L\int_0^1 U\bar{u}h\,d\bar{z} \qquad (2\text{-}46)$$

Defining

$$U = \frac{Q}{4\pi Lh} \qquad (2\text{-}47)$$

(2-46) becomes

$$1 = \bar{r}\int_0^1 \bar{u}\,d\bar{z} \qquad (2\text{-}48)$$

which is the correct dimensionless form of (2-6). An estimation can now be made of the ratio of inertia to viscous forces which is

$$\frac{\text{inertia}}{\text{viscous}} \sim \frac{u\frac{\partial u}{\partial r}}{\nu\frac{\partial^2 u}{\partial z^2}} \sim \frac{Q}{4\pi h\nu}\left(\frac{h}{L}\right)^2 \sim Re\left(\frac{h}{L}\right)^2 \qquad (2\text{-}49)$$

This result is consistent with that given by the Pi theorem. By defining various values of L we notice that the ratio of inertia to viscous forces will vary for different parts of the flow field. Also (2-49) suggests that even if the Reynolds number, Re, is very large the flow will eventually (if the disks are sufficiently large) enter a regime where viscous forces will dominate and the velocity profile will be given by (2-16). At the moment the magnitude of L where (2-16) will apply is unknown but this can be determined by making the equation of motion dimensionless using (2-45). Before this can be done we must first determine V. This can be achieved by rewriting the differential form of the continuity equation (2-1) in terms of (2-45).

$$\frac{\partial}{\partial \bar{r}}(\bar{r}\bar{u}) + \frac{4\pi L^2 V}{Q}\,\bar{r}\,\frac{\partial \bar{v}}{\partial \bar{z}} = 0 \qquad (2\text{-}50)$$

As $\bar{r}\dfrac{\partial\bar{v}}{\partial\bar{z}}$ must be of unit order, V must be defined as

$$V = \frac{Q}{4\pi L^2} \tag{2-51}$$

The equation of motion now becomes

$$[\bar{\phi}_{\bar{z}}(\frac{\partial}{\partial\bar{r}} - \frac{1}{\bar{r}}) - (\bar{\phi}_{\bar{r}} + \frac{1}{\bar{r}}\bar{\phi})\frac{\partial}{\partial\bar{z}}]\bar{\nabla}^2\bar{\phi} = \frac{1}{Re(\frac{h}{L})^2}\bar{\nabla}^4\bar{\phi} \tag{2-52}$$

where

$$\bar{\nabla}^2 = (\frac{h}{L})^2(\frac{\partial^2}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}} - \frac{1}{\bar{r}^2}) + \frac{\partial^2}{\partial\bar{z}^2} \tag{2-53}$$

At this stage a comparison between Savage's variables, i.e. (2-13) and those defined by (2-45) is in order. If Savage's variables are used to make the equation of motion dimensionless we find that:

a) the ratio of inertia to viscous forces is proportional to Re which is inconsistent with the Pi Theorem, and

b) all the terms in the equation of motion are of the same order of magnitude which from experience we know is incorrect.

The new variables defined by (2-45) do not have this inherent difficulty and instead it is now quite obvious that various terms will be larger or smaller than others depending on the magnitude of Re and h/L. Also the ratio of inertia to viscous forces has the right form.

By defining different values of L and applying a limit process, in this case $Re \to \infty$, various governing equations can be obtained. We will discuss three possibilities:

a) $(h/L)^2 \sim O(1)$

Inserting $L = h$ into equation (2-52) and taking the limit when $Re \to \infty$, we obtain the well-known Euler equation for inviscid flow

$$[\bar{\phi}_{\bar{z}}(\frac{\partial}{\partial\bar{r}} - \frac{1}{\bar{r}}) - (\bar{\phi}_{\bar{r}} + \frac{1}{\bar{r}}\bar{\phi})\frac{\partial}{\partial\bar{z}}]\bar{\nabla}^2\bar{\phi} = 0 \tag{2-54}$$

b) $(\frac{h}{L})^2 \sim O(\frac{1}{Re})$

The governing equation in this case turns out to be

$$[\bar{\phi}_{\bar{z}}(\frac{\partial}{\partial \bar{r}} - \frac{1}{\bar{r}}) - (\bar{\phi}_{\bar{r}} - \frac{1}{\bar{r}}\bar{\phi})\frac{\partial}{\partial \bar{z}}]\frac{\partial^2 \bar{\phi}}{\partial \bar{z}^2} = \frac{\partial^4 \bar{\phi}}{\partial \bar{z}^4} \qquad (2\text{-}55)$$

This is a boundary layer equation but with an unknown pressure distribution.

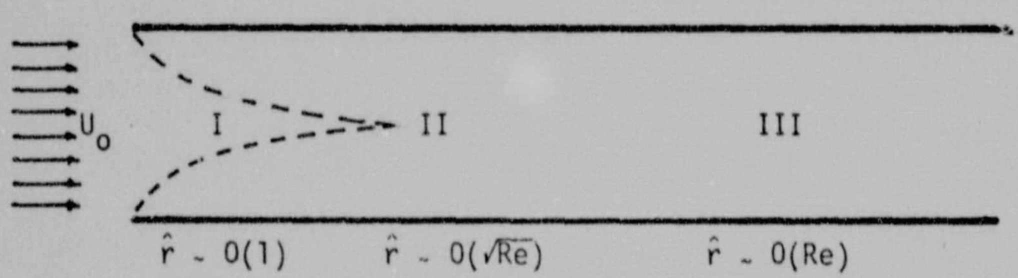
c) $(\frac{h}{L})^2 \sim O(\frac{1}{Re^2})$

In this case the governing equation is simply

$$\frac{\partial^4 \bar{\phi}}{\partial \bar{z}^4} = 0 \qquad (2\text{-}56)$$

which is the well-known equation for radial flow when the inertia effects are negligible.
The flow field is qualitatively sketched in Figure 2.

Figure 2 - Flow Regimes



In region 1, i.e. $L \sim h$ we have essentially inviscid flow and a solution to this problem can be symbolically written down as

$$\phi \sim \phi_0 + Re^{-\frac{1}{2}}\phi_1 + \ldots . \qquad (2\text{-}57)$$

This is just Euler's expansion for inviscid flow and from the dimensional analysis we know that the governing equation for the first term in the expansion will be equivalent to equation (2-54). This equation cannot, however, satisfy the no-slip conditions at the surface of the disks and, therefore, in a small neighbourhood of the surface we will have to match up (in the sense of matched asymptotic expansions) the inviscid core with a typical boundary layer expansion. The second term in the inviscid expansion is the displacement effect of the growing boundary layers which determine why the inviscid core is not uniform near the entrance. Further downstream this effect becomes negligible and when $L \sim \sqrt{Re}$ the boundary layers meet. At this stage fully developed flow has not been achieved and the first term in the asymptotic expansion for this region, i.e. $L \sim \sqrt{Re}$ is governed by a parabolic differential equation (2-55). To determine the initial condition for this equation we simply match the asymptotic expansion for region 2 with the inviscid expansion. When $L \sim Re$, i.e. region 3, the inertia effects become negligible and fully developed flow is achieved.

3. ANALYSIS

3.1 ASYMPTOTIC EXPANSIONS

The dimensional analysis has shown that it is possible to demarcate the problem into three distinct regions. These have been sketched schematically in Figure 2. We will discuss each region separately.

Region 1 (The upstream region)

In this region we define the following upstream variables

$$\hat{r} = \frac{r}{h} \qquad \hat{z} = \frac{z}{h} \qquad \hat{\phi} = \frac{\phi}{Q/4\pi h} \tag{3-1}$$

In terms of these variables the equation of motion becomes

$$[\hat{\phi}_{\hat{z}} (\frac{\partial}{\partial \hat{r}} - \frac{1}{\hat{r}}) - (\hat{\phi}_{\hat{r}} + \frac{1}{\hat{r}} \hat{\phi}) \frac{\partial}{\partial \hat{z}}] \hat{\nabla}^2 \hat{\phi} = \frac{1}{Re} \hat{\nabla}^4 \hat{\phi} \tag{3-2}$$

where $\hat{\nabla}^2 = \frac{\partial}{\partial \hat{r}^2} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} - \frac{1}{\hat{r}^2} + \frac{\partial^2}{\partial \hat{z}^2}$

The disk walls are at $\hat{z} = \pm 1$ and the boundary conditions there are

$$\hat{\phi}_{\hat{z}} (\hat{r}, \pm 1) = \frac{1}{\hat{r}} (\frac{\partial}{\partial \hat{r}} (\hat{r}\hat{\phi}(\hat{r}, \pm 1)) = 0 \tag{3-3}$$

while the conditions at the inlet are

$$\hat{\phi} (1, \hat{z}) = \hat{z}, \qquad \frac{\partial}{\partial \hat{r}} (\frac{1}{\hat{r}} (\frac{\partial}{\partial \hat{r}} (\hat{r}\hat{\phi}(1, \hat{z})))) = 0 \tag{3-4}$$

Now associated with the upstream variables must be some limit process and in our case it is just

$$\lim_{Re \to \infty} \hat{\phi} (\hat{r}, \hat{z}) \text{ with } \hat{r} \text{ and } \hat{z} \text{ fixed} \tag{3-5}$$

As mentioned earlier, in this region, i.e. $L \sim h$ the flow is essentially inviscid and, therefore, the asymptotic expansion must be similar to the Euler expansion for inviscid flow, i.e.

$$\hat{\phi} \sim \hat{\phi}_0 + Re^{-\frac{1}{2}} \hat{\phi}_1 + \ldots . \tag{3-6}$$

To determine the functions $\hat{\phi}_o$ and $\hat{\phi}_1$, we merely substitute (3.6) into equation (3-2) and successively apply (3-5). The perturbation equations turn out to be

$$[\hat{\phi}_{o\hat{z}} (\frac{\partial}{\partial \hat{r}} - \frac{1}{\hat{r}}) - (\hat{\phi}_{o\hat{r}} + \frac{1}{\hat{r}}\hat{\phi}_o) \frac{\partial}{\partial \hat{z}}] \hat{\nabla}^2 \hat{\phi}_o = 0 \qquad (3\text{-}7)$$

and

$$[\hat{\phi}_{oz} (\frac{\partial}{\partial \hat{r}} - \frac{1}{\hat{r}}) - (\hat{\phi}_{o\hat{r}} + \frac{1}{\hat{r}} \hat{\phi}_o) \frac{\partial}{\partial \hat{z}}] \hat{\nabla}^2 \hat{\phi}_1 = 0 \qquad (3\text{-}8)$$

$$+ [\hat{\phi}_{1\hat{z}} (\frac{\partial}{\partial \hat{r}} - \frac{1}{\hat{r}}) - (\hat{\phi}_{1\hat{r}} + \frac{1}{\hat{r}} \hat{\phi}_1) \frac{\partial}{\partial \hat{z}}] \hat{\nabla}^2 \hat{\phi}_o = 0$$

Since vorticity is defined by

$$\omega = \hat{\nabla}^2 \hat{\phi} \qquad (3\text{-}9)$$

it is possible to give a physical interpretation for equations (3-7) and (3-8). The first order perturbation equation for $\hat{\phi}_o$ implies that for a first approximation vorticity is simply convected, its viscous diffusion being negligible, while in the second approximation, equation (3-8) the first term represents second order vorticity being convected along first order streamlines and the second term infers that first order vorticity is convected along second order corrections to the streamlines. As in the first approximation viscous diffusion is negligible.

Since the flow is assumed to be irrotational (see inlet boundary conditions) a solution of the first order problem is just a solution of

$$\hat{\nabla}^2 \hat{\phi}_o = 0 \qquad (3\text{-}10)$$

and must satisfy the boundary conditions given by (3.4). This turns out to be

$$\hat{\phi}_o = \frac{\hat{z}}{\hat{r}} \quad \text{for } \hat{r} \geq 1 \qquad (3\text{-}11)$$

(Note the no-slip conditions (3-3) have been dropped since they are unenforceable.) In a small neighbourhood of the wall we must supplement the Euler expansion with a boundary layer expansion in order to satisfy the no-slip conditions there.

The boundary layer expansion may be found by redefining the variables near the wall and then applying a special limit to the governing equation. The new variables associated with the boundary layer are

$$x = Re^{\frac{1}{2}} (\hat{z} - 1), \qquad \hat{r} = \frac{r}{h} \tag{3-12}$$

i.e. we have magnified the transverse direction as customarily done in boundary layer analysis. Following van Dyke (1962) we take the boundary layer expansion to be

$$\hat{\phi} \sim Re^{-\frac{1}{2}} \phi_o + Re^{-1} \phi_1 + \ldots . \tag{3-13}$$

and the limit process to be

$$\begin{matrix} \text{Limit} \\ Re \to \infty \end{matrix} \; \hat{\phi} (x, \hat{r}) \text{ with } x \text{ and } \hat{r} \text{ fixed} \tag{3-14}$$

The first order equation for ϕ_o then turns out to be

$$[\phi_{ox} (\frac{\partial}{\partial \hat{r}} - \frac{1}{\hat{r}}) - (\phi_{o\hat{r}} + \frac{1}{\hat{r}}\phi_o) \frac{\partial}{\partial x}] \frac{\partial^2 \phi_o}{\partial x^2} = \frac{\partial^4 \phi_o}{\partial x^4} \tag{3-15}$$

with boundary conditions

$$\phi_o (\hat{r}, o) = \frac{1}{\hat{r}}, \quad \phi_{ox} (\hat{r}, o) = 0 \tag{3-16}$$

$$\phi_{ox}(\hat{r}, \infty) = \hat{\phi}_{o\hat{z}}(\hat{r}, 1)$$

The last condition comes from applying van Dyke's (1964) matching principle (for details of the matching procedure see Appendix C) which physically means that the tangential velocity at the outer edge of the boundary must approach the inviscid speed, $\hat{\phi}_{o\hat{z}}(\hat{r},1)$.

Equation (3-15) may be simplified if we recognise that it is a perfect differential and can be integrated to give

$$\Phi_{oxxx} - \Phi_{ox}\,\Phi_{ox\hat{r}} + \Phi_{o\hat{r}}\,\Phi_{oxx} + \frac{1}{\hat{r}}\Phi_o\,\Phi_{oxx} = \frac{d\hat{p}}{d\hat{r}} \qquad (3\text{-}17)$$

where
$$\frac{d\hat{p}}{d\hat{r}} = \phi_{o\hat{z}}\,\phi_{o\hat{z}\hat{r}} = -\frac{1}{\hat{r}^3} \qquad (3\text{-}18)$$

To solve equation (3-17) we define a modified Mangler transformation such that the resulting equation reduces to Görtler's boundary layer equation. The new variables are

$$\Phi_o = (2\xi)^{\frac{1}{2}}\,F(\xi,\eta)/\hat{r}$$

$$\xi = \int_o^{\hat{r}} \hat{\phi}_{o\hat{z}}\,\hat{r}^2\,d\hat{r} \qquad (3\text{-}19)$$

$$\eta = \frac{\hat{\phi}_{o\hat{z}}\,\hat{r}x}{(2\xi)^{\frac{1}{2}}}$$

In terms of the above variables (3-17) becomes

$$F\eta\eta\eta + F\eta\eta F + \beta(\xi)\,[1 - F^2\eta] = 2\xi\,[F\eta F\eta\xi - F\xi F\eta\eta] \qquad (3\text{-}20)$$

where
$$\beta(\xi) = \frac{2\xi}{\hat{\phi}_{o\hat{z}}}\,\frac{d}{d\xi}\,\hat{\phi}_{o\hat{z}}$$

while the boundary conditions transform to

$$F(\xi,o) = F\eta(\xi,o) = 0$$

$$\lim_{\eta\to\infty} F\eta\,(\xi,\eta) = 1 \qquad (3\text{-}21)$$