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THE LARGE N LIMIT OF HEAVY OPERATOR EXCITATIONS

by

Warren Anthony Carlson

Thesis presented in fulfillment of the requirements for the degree of

Doctor of Philosophy

in the Faculty of Science at the University of the Witwatersrand, Johannesburg

Supervisor: Professor Robert de Mello Koch

July 2023

Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.



Warren Anthony Carlson

13th day of July 2023 in Johannesburg

Abstract

THE LARGE N LIMIT OF

HEAVY OPERATOR EXCITATIONS

W.A. Carlson

School of Physics,

University of the Witwatersrand,

PO Box Wits, Johannesburg, 2050, South Africa

Dissertation: Doctor of Philosophy

July 2023

Operators with bare dimension of order N are studied. These are restricted Schur polynomials labeled by Young diagrams with two long rows or two long columns and are heavy operators in the large N limit. A dramatic simplification of the action of the dilatation operator on these states is found, where the diagonalization of the dilatation operator reduces to solving three-term recursion relations. The solutions to these recursion relations reduce the spectrum of the dilatation operator to that of decoupled harmonic oscillators, showing that these systems are integrable at large N. Then, generating functions for bound states of two giant gravitons are constructed and an extension to more than two giant gravitons is sketched. These generating functions are integrals over auxiliary variables that encode the symmetrization and anti-symmetrization of the fields in the restricted Schur polynomials and give a simple construction of eigenfunctions of the dilatation operator. These generating functions are shown to be eigenfunctions of the dilatation operator in the large N limit. As a byproduct, this construction gives a natural

starting point for systematic $1/N$ expansions of these operators. This includes the prospect to generate asymptotic representations of the symmetric group and its characters via the restricted Schur polynomials. Finally, the asymptotic expansion of the three-point function is computed in three BMN limits by varying one parameter in the large N limit. It is argued that these asymptotic expansions encode non-perturbative effects and are related by a parametric Stokes phenomenon.

Preface

Chapter 3 of this thesis is based on the article

- Warren Carlson, Robert de Mello Koch, and Hai Lin. Nonplanar Integrability. Journal of High Energy Physics, 03:105, 2011. doi: 10.1007/JHEP03(2011)105

and Chapter 4 is based on the article

- Warren Carlson, Robert de Mello Koch, and Minkyoo Kim. Generating functions for giant graviton bound states. Journal of High Energy Physics, 01:104, 2023. doi: 10.1007/JHEP01(2023)104.

The work in Chapter 4 extends that of Chapter 3 to include a formulation for multiple bound giants and generating functions. Chapter 5 contains work that forms the basis for a third publication.

The gap between the research of Chapter 3 and Chapter 4 is twelve years. I would like to explain this gap. After completing my MSc (2009-2010), I worked as a software programmer and continued with Physics in my own time. This resulted in the article 'Nonplanar Integrability' that was published in 2011. At this time, I returned to pursue a PhD in High Energy Phenomenology. This did not succeed and I abandoned its pursuit in 2018. I initiated a new PhD in 2019, resuming the work of 2011.

The two articles on which this work is based have not been submitted for examination or in support of any degree or qualification at any University.

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Chapter 1

Introduction

The AdS/CFT correspondence is a significant step towards understanding the quantum theory of gravity [1–3]. It provides a dictionary that translates between a quantum theory on the boundary of a given spacetime and a theory of gravity in the bulk. For each observable on the field theory side of the conjecture there corresponds an observable on the gravity side. Moreover, this correspondence claims a full dynamical matching between observables on either side of the correspondence. This is the *gauge theory/gravity correspondence*, which is a *holographic duality*, where information from one theory is encoded in another theory of lower dimension. This duality provides a plausible resolution to the tension between quantum theory and relativity theory. While there is currently no proof for this conjecture, evidence in support of the correspondence continues to accumulate (see [4] and references therein). In this thesis, I consider heavy operator excitations as a probe of this correspondence.

We can understand the fundamental building blocks of the AdS/CFT in terms of the three constituent parts: (1) the quantum field theory, (2) the dual gravity theory and (3) the dictionary that connects the two theories. Quantum Field Theory (QFT) is a general framework for describing fundamental processes in nature. Each QFT has a gauge symmetry and is referred to as a *gauge theory*. A gauge theory describes a fundamental process as the excitations of a field, corresponding to point-like particles that interact locally with other point-like particles. QFT has accurately unified the electromagnetic, the weak and

the strong interactions that are encapsulated in the *Standard Model* of particle physics and have shown remarkable agreement with experiments [5–7] [1].

In contrast to the particle-mediated interactions of gauge theories, the *General Theory of Relativity* (GR) uses geometry as the basis of the gravitational interaction. Attempts to include gravity in gauge theories have shown that the size of the contributions due to gravity depend on the energy scale at which the theory is studied. The scale at which gravitational contributions become important is of the order of the Planck scale. Above this scale, divergences arise that cannot be removed. Indeed, until the introduction of the AdS/CFT correspondence, QFT were considered effective theories, at best - valid up to some scale and to be replaced by a more fundamental theory beyond this scale. The AdS/CFT correspondence suggests that QFT is a fundamental theory. GR and QFT make contradictory assumptions about the universe and these contradictions manifest as non-removable divergences in the theories that directly combine them, so these theories are *non-renormalizible* [8, 9]. This is the source of the tension between gauge theory and theories with gravity.

There is a collection of theories that claim to combine gravity and quantum theory, of which gravity appears naturally in *String Theory*. Consistency of the theory requires that the β-function of world-sheet string theory vanishes [10]. Remarkably, the vanishing of the β-function reproduces the Einstein Field Equations of GR. However, string theory has yet to provide a direct description of particle physics as it is understood in the Standard Model [11]. Recent efforts have focused instead on finding a QFT, describing particles and fields, with a dual gravity theory. Such a model for quantum gravity relies on the dual descriptions provided by a QFT and a string theory. The AdS/CFT correspondence is a concrete example of this dual description.

Type IIB string theory on an $AdS_5 \times S^5$ background spacetime with $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory (SYM) as the dual gauge theory is one example of the AdS/CFT correspondence. In this instance, a theory of gravity is related to a theory of

[1] A computation to tenth order in the electric charge (fifth order in the fine structure constant) determined the value of the anomalous magnetic moment of the electron to be $1159652182.78(77) \times 10^{-12}$ [5]. More than twelve thousand Feynman diagrams were evaluated in this calculation, which showed dramatic agreement with experimental measurement of $1159652180.73(0.28) \times 10^{-12}$ [6].

$N \times N$ matrices that interact on a flat background and encode the geometrical and topological properties of the string theory. Quantities in this gauge theory are computed in terms of the parameters $g^2_{\rm YM}$ and $1/N$, but we will often express quantities in terms of the 't Hooft coupling $\lambda = g^2_{\rm YM} N$. Typically, we are interested in computing correlation functions of operators and eigenvalues of the symmetry generators in the CFT. In this thesis, I consider the combinatorial dependence of correlation functions, which generates a $1/N$ expansion, and eigenvalues of the dilatation operator that measure the conformal scaling dimension of the correlator spacetime dependence.

When the coupling $g^2_{\rm YM}$ is small then correlation functions are expanded in powers of the coupling of the theory. This gives rise to the familiar Feynman diagram expansion in perturbation theory, wherein the diagrams in the expansion become increasingly complicated with increasing power of the coupling. In Yang-Mills theory, the diagrammatic expansion can be re-encoded in a topological expansion where higher genus terms are suppressed by powers of $1/N$.

QFT correlation functions generally do not admit convergent series expansions. These series are often asymptotic, so the perturbation theory breaks down [12] and we say that the corresponding quantities are *non-perturbative*. The breakdown of the perturbation theory is a result of the rapid growth in the number of Feynman diagrams that contribute at each order in the theory. It is clear that a non-perturbative understanding is necessary, and group representation theory becomes an indispensable tool in the analysis. Indeed, it is possible to evaluate certain correlation functions exactly using group representation theory [13]. In a CFT with a large amount of supersymmetry these quantities are protected from quantum corrections and are determined exactly in λ as well.

Heavy operators in the CFT have a bare dimension that scales as a power of N making their correlations non-perturbative. These operators probe the dual string theory at short length scales. In particular, heavy operators that are dual to states with mass E allow us to probe effects on a length scale $1/E$. The maximum resolution at which these heavy states probe the theory is bounded. Beyond this limit, the heavy states deform the spacetime, generating new geometries, and the probe cannot resolve quantities in the dual string theory.

These states correspond to operators whose bare dimension scale as $O\left(N^2\right)$, and have been studied in [14]. Below this limit, we study the half-BPS sector of $\mathcal{N}=4$ SYM. This sector admits operators that are protected from quantum corrections. It contains operators dual to *giant graviton* states in the string theory. These are Schur polynomials labeled by Young diagrams with $O(N)$ boxes and correspond to states that carry large angular momentum in the dual string theory. Together, these heavy states and their CFT duals provide a laboratory to study non-perturbative physics in the AdS/CFT correspondence.

The new background geometries dual to operators whose bare dimensions scale as $O\left(N^2\right)$ include AdS black holes [15]. The microscopic states of these black holes can be counted by computing the supersymmetric grand canonical partition function, called the superconformal index, of the dual CFT. Recently, it was shown that this reproduces the Bekenstein-Hawking entropy in the large N limit where the $\mathcal{N}=4$ SYM superconformal index grows as $\exp\left(O\left(N^2\right)\right)$ [16–18]. Indeed, heavy operators provide fertile ground to understand emergent geometries and their CFT duals.

Our interest in studying heavy operators is two-fold: (1) heavy operators are non-perturbative objects with non-trivial dynamics, and (2) heavy operators allow us to probe high energies and short length scales in the dual gravity theory. Additionally, we expect that small non-BPS excitations of heavy half-BPS operators correspond to small deformations of the BPS sector that can be studied perturbatively.

Anomalous dimensions in the CFT correspond to energies in the dual string theory. Interest in CFT anomalous dimensions has been driven by questions on the integrability of certain CFTs. Indeed, integrability theory has played a key role in studying the spectrum of anomalous dimensions of operators in $\mathcal{N}=4$ SYM [4, 19]. The problem of computing anomalous dimensions can be recast as a problem of computing eigenvalues of the dilatation operator. At tree-level, this computation returns the engineering dimension of the operators involved, but gets corrections from quantum interactions at higher loop orders. Later it will be shown that these eigenvalues can be computed exactly for special configurations of giant gravitons in the large N limit, and correspond to the energy spectrum of a collection of harmonic oscillators. The fact that this is possible is a strong

indication that the integrability of the theory extends beyond the planar limit. Hints at integrability of the theory have been found numerically and later by analytic calculations in different sectors and at higher loop orders. The large N limit of the anomalous dimension of excited giant graviton bound state operators is discussed in Chapter 3. These computations can be generalized by constructing generating functions. These generating functions are integrals over auxiliary variables that encode the structure of Young diagrams that label Schur polynomials. This procedure generates eigenstates of the dilatation operator and is outlined in Chapter 4. We can use this construction to give a description of the action of the dilatation operator on these states. As a byproduct, this construction gives a natural starting point for the $1/N$ expansion of these operators by saddle point analysis, thereby allowing us to study perturbations of classical configurations. It also provides an opportunity to identify additional non-perturbative contributions and the relations among them. We recover the partition function of giant graviton bound states from the generating function, whereby each operator contains characters of the symmetric group S_n and suggests a method to generate asymptotic representations of characters at large n, following [20]. Then, Chapter 5 presents the $1/N$ expansion of heavy operators in various BMN limits, providing preliminary results and discussion on the non-perturbative analysis of heavy operator correlation functions. Comments and discussion of these results and analyses are contained in Chapter 6. Technical details of the computations in each chapter are contained in the appendices. A review of the AdS/CFT correspondence, the central theme of this thesis, is presented next, in Chapter 2.

Chapter 2

Background

In this chapter I outline some important details of the AdS/CFT correspondence, following [1–3]. I describe some of the computational tools used to test the correspondence in the context of the type IIB string theory on the $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ SYM with $U(N)$ gauge symmetry in $(3+1)$-dimensions. I start by presenting some background information on $\mathcal{N} = 4$ SYM and AdS spacetime before proceeding to motivate the correspondence. I conclude this chapter with a summary of the progress made toward understanding the spectrum of the dilatation operator in the context of integrability in the AdS/CFT correspondence.

2.1 The AdS/CFT Correspondence

The AdS/CFT correspondence claims a full dynamical equivalence between $\mathcal{N} = 4$ SYM in flat $(3+1)$-dimensional Minkowski spacetime, and Type IIB string theory on an asymptotically $AdS_5 \times S^5$ background spacetime. This is a realization of an equivalence between gauge theories at large N and string theories originally proposed by 't Hooft [21]. The correspondence relates the strong coupling limit of a string theory to the weak coupling limit of a gauge theory, providing *strong/weak* duality between these theories. The AdS/CFT is a profound physical idea that a quantum theory encodes the information of a theory of gravity with higher dimension [22, 23], and provides a non-perturbative

formulation of quantum gravity. In this section, I give a brief motivation for this correspondence and provide introductory details of AdS spacetime and of the correspondence itself.

2.1.1 $\mathcal{N}=4$ Super Yang-Mills Theory

Yang-Mills theory is a general framework that forms the basis for studying fundamental interactions. The Standard Model is an example of a Yang-Mills theory. $\mathcal{N}=4$ SYM is a special example of a Yang-Mills theory that enjoys conformal symmetry. It is a *Conformal Field Theory* (CFT) with non-Abelian gauge symmetry group. In this section I give a brief overview of $\mathcal{N}=4$ SYM theory in $(3+1)$-dimensions.

Start by considering $\mathcal{N}=1$ SYM in $(9+1)$-dimensions. This theory contains the fundamental gauge field A_μ and gauge coupling g_{YM} where the covariant derivative of a generic adjoint field Z is

$$D_\mu Z = \partial_\mu Z - \frac{i}{2} g_{\text{YM}} \big[A_\mu, Z\big]$$

and the field strength is given by

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_{\text{YM}} \big[A_\mu, A_\nu\big].$$

This theory has a Lagrangian

$$\mathcal{L} = -\frac{1}{4} \text{Tr}\big(F_{\mu\nu} F^{\mu\nu}\big) + \text{Tr}\big(\bar{\psi} \Gamma^\mu D_\mu \psi\big) \tag{2.1}$$

where indices μ, ν run over $0, \ldots, 9$ and denote the spacetime dimensions of the theory. ψ is a real chiral spinor in $(9+1)$-dimensions which has 16 real components - this leads to 8 on-shell fermionic degrees of freedom.

Dimensional reduction of $\mathcal{N}=1$ SYM as a theory in $(9+1)$-dimensions to a theory in $(3+1)$-dimensions is achieved by separating out the 6 spatial components in the Lagrangian in (2.1). The resulting theory in $(3+1)$-dimensions comprises the bosonic field A_μ, four Weyl fermions $\Psi_{a\alpha}$, with conjugates $\bar{\Psi}^a{}_{\dot{\alpha}}$, and six real adjoint scalar fields Φ_m. The spacetime indices μ, ν run over $0, 1, 2, 3$ and the spinor indices for two chiralities α, β and $\dot{\alpha}, \dot{\beta}$ run over

1,2. The scalar fields carry indices m,n which run over $1,\ldots,6$, while the spinor fields carry indices a,b that run over $1,\ldots,4$. The $(3+1)$-dimensional effective Lagrangian is

$$\begin{aligned}\mathscr{L} =& -\frac{1}{4}\mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right)-\frac{1}{2}\mathrm{Tr}\left(D^{\mu}\Phi^{m}D_{\mu}\Phi_{m}\right)-\mathrm{Tr}\left(\bar{\Psi}^{a}{}_{\dot{\alpha}}\Gamma_{\mu}{}^{\dot{\alpha}\beta}D^{\mu}\Psi_{a\beta}\right)\\ &+\frac{i}{2}g_{\mathrm{YM}}\,\mathrm{Tr}\left(\Psi_{a\alpha}\Gamma_{m}{}^{ab}\epsilon^{\alpha\beta}\left[\Phi^{m},\Psi_{b\beta}\right]\right)+\frac{i}{2}g_{\mathrm{YM}}\,\mathrm{Tr}\left(\bar{\Psi}^{a}{}_{\dot{\alpha}}\Gamma^{m}{}_{ab}\,\epsilon^{\dot{\alpha}\dot{\beta}}\left[\Phi_{m},\bar{\Psi}^{b}{}_{\dot{\beta}}\right]\right)\\ &+\frac{1}{4}g_{\mathrm{YM}}^{2}\,\mathrm{Tr}([\Phi^{m},\Phi^{n}][\Phi_{m},\Phi_{n}])\end{aligned} \tag{2.2}$$

where $\Gamma_{\mu}{}^{\alpha\dot{\beta}}$ and $\Gamma^{m}{}_{ab}$ denote the generalized Pauli matrices, and $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$ denote totally antisymmetric tensors [24]. The size of each matrix N and the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^{2}N$ appear as parameters in this theory. These parameters appear explicitly in the values of correlation functions of operators in the CFT and the corresponding expectation values of states in the dual gravity theory. Importantly, these correlators are expressed as asymptotic series expansions in the limit that the rank of the gauge group N is large.

Supersymmetry in QFT is an invariance of the predictions of the theory when the constituent bosonic, fermionic and scalar fields are exchanged in a prescribed way. $\mathscr{N}=4$ SYM enjoys four supersymmetries. These symmetries are encoded in four complex generators $\mathfrak{Q}^{a}{}_{\alpha}$ which transform the constituent field in the following way,

$$\begin{aligned}&\mathfrak{Q}^{a}{}_{\alpha}\Phi_{m} = \Gamma_{m}{}^{ab}\Psi_{b\alpha},\\ &\mathfrak{Q}^{a}{}_{\alpha}\Psi_{b\beta} = -\frac{1}{2}\Gamma^{\mu}{}_{\beta\dot{\sigma}}\,\epsilon^{\dot{\sigma}\dot{\gamma}}\Gamma^{\nu}{}_{\alpha\dot{\gamma}}\,F_{\mu\nu}\delta^{a}{}_{b}+\frac{i}{2}\epsilon_{\beta\alpha}\Gamma^{m}{}_{bc}\,\Gamma_{n}{}^{ca}[\Phi_{m},\Phi^{n}],\\ &\mathfrak{Q}^{a}{}_{\alpha}\Psi^{b}{}_{\dot{\gamma}} = i\Gamma_{m}{}^{ab}\Gamma^{\mu}{}_{\alpha\dot{\gamma}}\,D_{\mu}\Phi^{m},\\ &\mathfrak{Q}^{a}{}_{\alpha}A^{\mu} = i\Gamma^{\mu}{}_{\alpha\dot{\gamma}}\,\epsilon^{\dot{\gamma}\dot{\sigma}}\bar{\Psi}^{a}{}_{\dot{\sigma}}.\end{aligned}$$

Conjugate expressions for $\bar{\mathfrak{Q}}^{a}{}_{\alpha}$ hold with $\Psi_{a\alpha}$ and $\bar{\Psi}^{a}{}_{\dot{\alpha}}$ interchanged. These are the supersymmetries of this CFT.

$\mathscr{N}=4$ SYM enjoys a large amount of supersymmetry and there exist quantities that are protected against perturbative corrections in λ. Of these quantities, the half-BPS sector of the theory contains those states that enjoy half the maximal amount of supersymmetry. I shall consider, in this thesis, the half-BPS subsector of $\mathscr{N}=4$ SYM containing only adjoint scalar fields and its relation to the AdS/CFT correspondence.

The AdS/CFT correspondence proposes that there exists a mapping from a string theory in $\mathrm{AdS}_5 \times S^5$ spacetime to $\mathcal{N}=4$ Super Yang-Mills in $(3+1)$-dimensions at the boundary of $\mathrm{AdS}_5 \times S^5$.

2.1.2 AdS Space Summary

The $(d+1)$-dimensional anti-de Sitter space AdS_{d+1} is a solution to the vacuum Einstein field equations with negative cosmological constant

$$\Lambda_{\mathrm{AdS}_{d+1}} = -\frac{d(d-1)}{R^2}.$$

It is a maximally symmetric Lorentzian manifold with constant negative curvature and isometry group $SO(2,d)$, which has $(d+1)(d+2)/2$ symmetry generators. I shall present two different coordinate descriptions of anti-de Sitter space, following [1].

Consider the $(d+1)$-dimensional hyperboloid

$$-(X^{-1})^2 - (X^0)^2 + \sum_{i=1}^{d} (X^i)^2 = -R^2 \tag{2.3}$$

embedded in a flat $(d+2)$-dimensional spacetime, where R is the "radius" of the anti-de Sitter space. The metric induced on the hyperboloid is

$$\mathrm{d}s^2 = -(\mathrm{d}X^{-1})^2 - (\mathrm{d}X^0)^2 + \sum_{i=1}^{d} (\mathrm{d}X^i)^2.$$

A useful reparameterization of this metric follows from the coordinate transformation

$$U = X^{-1} + X^d, \quad V = X^{-1} - X^d = \frac{U}{R^2}x^2 + \frac{R^2}{U} \quad \text{and} \quad x^\mu = \frac{R}{U}X^\mu \quad \mu = 0, 1, \ldots, d-1$$

and the corresponding metric is

$$\mathrm{d}s^2 = \frac{U^2}{R^2}\,\mathrm{d}x^2 + \frac{R^2}{U^2}\,\mathrm{d}U^2.$$

Then, rescaling the coordinate

$$zU = R^2 \qquad \text{and} \qquad z^2\,\mathrm{d}U = -R^2\,\mathrm{d}z$$

gives

$$\mathrm{d}s^2 = \frac{R^2}{z^2}\left(\mathrm{d}z^2 + \mathrm{d}x_\mu\,\mathrm{d}x^\mu\right).$$

This is the metric of AdS_{d+1} space on the Poincaré patch, where the overall factor R^2 is the scale of the metric. Up to a conformal factor, this is just the flat metric of $(d+1)$-dimensional Minkowski space with a warp factor associated with a gravitational potential that depends only on the coordinate z.

Another useful reparameterization of AdS_{d+1} space follows from replacing the embedding coordinates X^i with

$$X^0 = R\cosh(\rho)\cos(\tau), \qquad X^{d+1} = R\cosh(\rho)\sin(\tau) \qquad \text{and} \qquad X^i = R\sinh(\rho)\omega_i$$

where

$$\begin{aligned}
\omega_1 &= \cos(\theta_1) \\
\omega_2 &= \sin(\theta_1)\cos(\theta_2) \\
&\vdots \\
\omega_{d-1} &= \sin(\theta_1)\sin(\theta_2)\cdots\cos(\theta_{d-1}) \\
\omega_d &= \sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{d-1}).
\end{aligned}$$

This recovers the *global* coordinate description for the metric of AdS_{d+1}

$$\mathrm{d}s^2 = -R^2\cosh^2(\rho)\,\mathrm{d}\tau^2 + R^2\,\mathrm{d}\rho^2 + R^2\sinh^2(\rho)\,\mathrm{d}\Omega_{d-1}^2 \tag{2.4}$$

where $\tau \in [0, 2\pi)$, $0 \leq \rho$ and $\mathrm{d}\Omega_{d-1}^2$ is the metric on the $(d-1)$-dimensional unit sphere. The Lorentzian structure of the metric is now manifest. For $\rho \ll 1$ and constant, (2.4) reduces to

$$\mathrm{d}s^2 = R^2\left(-\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\Omega_{d-1}^2\right)$$

which, up to a conformal factor R^2, is the metric on $S^1 \times \mathbb{R}^d$, with τ associated to S^1. Then τ corresponds to a family of S^1s that form closed time-like curves in this space. Unwrapping the S^1 and letting $-\infty < \tau < \infty$ removes the closed time-like curves. Similarly, for $1 \ll \rho$ and constant, Wick rotation transforms (2.4) to

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + \mathrm{d}\Omega_{d-1}^2$$

up to a conformal rescaling and is the metric on $\mathbb{R} \times S^{d-1}$, which we interpret as the boundary of AdS_{d+1}.

More generally, we can embed a d-dimensional space in a space of $(d+1)$-dimensions, and determine the metric induced on the embedded space. A $(d+1)$-dimensional space has $d(d+1)/2$ components in the metric but there are d coordinate transformations one can make that leave the metric invariant. This means that there are $d(d-1)/2$ degrees of freedom to describe the curvature of the space and give rise to different physics.

2.1.3 Motivation for the Correspondence

Initial hints of the correspondence follow from the matching of symmetries and parameters of the resulting theories in a specific low energy limit. Notably, both theories have the same symmetry group $SO(2,d)$, specifically $\mathcal{N}=4$ SYM in $(3+1)$-dimensions with gauge group $U(N)$ and the gravitational theory will be a string theory. This matching serves as motivation for a more general correspondence which claims a matching of quantities on both sides of the correspondence. Moreover, the correspondence claims a complete dynamical matching between these theories, where the large N limit of the CFT includes in its Hilbert space a sector that describes supergravity on $\mathrm{AdS}_5 \times S^5$ [1]. To make the connection concrete, we must identify the corresponding relations between the couplings in each theory and determine the limits of the validity of this identification in the correct low energy limit, and then show that for each state in the gauge theory there exists a corresponding state in the string theory. Following this, we must show that the spacetimes of the corresponding low energy theories of the gravity theory in the bulk and the gauge theory are correspondingly $\mathrm{AdS}_5 \times S^5$ and M_4 - the 4-dimensional Minkowski space boundary of the AdS_5.

We study a string theory where the action

$$S = -\frac{T}{2}\int \mathrm{d}\sigma\,\mathrm{d}\tau\sqrt{-\det(\gamma)}\,\gamma^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

is the area spanned by the moving string that is minimized is due to the tension

$$T = \frac{1}{g_s} = \frac{1}{2\pi\alpha'}$$

where $g_s = g_{\mathrm{YM}}^2$ is the string coupling. Fixing the gauge in the closed string theory reduces the action to that of free 2-dimensional bosons containing left-moving and right-moving

modes. Quantizing these modes generates a particle spectrum, where the massless particles are the graviton, an antisymmetric tensor and a scalar. Since open strings can interact to form closed strings, any theory of open strings is also a theory of closed strings. The supersymmetric string is stable and lives in ten dimensions. To make sense of this, we must first reduce the 10-dimensional superstring theory to a theory in four dimensions, by Kaluza-Klein dimensional reduction.

The AdS/CFT correspondence is obtained by considering a configuration of D3-branes and taking a limit in which theories on the brane decouple from those in the bulk spacetime. Polchinski and others [25, 26] showed that Dp-branes are the supergravity solutions corresponding to dynamical walls to which open strings attach. These Dp-branes set the boundary conditions for open strings. Setting $p = 3$ corresponds to choosing D3-branes, which is appropriate since we live in a $(3+1)$-dimensional space.

Now consider a stack of N parallel D3-branes, separated by a distance r, in type IIB string theory in 10-dimensional spacetime. The low energy limit of the string theory corresponds to the $\alpha' \to 0$ limit. This gives rise to an enormous string tension, and the theory on the D3-branes decouples from that in the bulk. The supergravity solution reduces to an $\mathrm{AdS}_5 \times S^5$. Taking the 't Hooft limit reduces this theory to $\mathcal{N} = 4$ SYM in $(3+1)$-dimensions with $U(N)$ symmetry, which now contains strings from type IIB string theory.

Dp-branes curve space and N D3-branes correspond to the supergravity solution

$$\mathrm{d}s^2 = H^{-\frac{1}{2}}(r)\,\mathrm{d}x_\mu\,\mathrm{d}x^\mu + H^{\frac{1}{2}}(r)\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_5^2\right) \tag{2.5}$$

where

$$H(r) = 1 + \frac{R^4}{r^4} \qquad \text{and} \qquad R^4 = 4\pi\alpha'^2 g_s N$$

where $g_s N$ is the 't Hooft coupling. Far from the brane $R \ll r$, the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}x_\mu\,\mathrm{d}x^\mu + \left(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_5^2\right)$$

corresponding to that of flat 10-dimensional Minkowski spacetime, while near the brane $r \ll R$, the metric becomes

$$\mathrm{d}s^2 = \frac{r^2}{R^2}\,\mathrm{d}x_\mu\,\mathrm{d}x^\mu + \frac{R^2}{r^2}\,\mathrm{d}r^2 + R^2\,\mathrm{d}\Omega_5^2.$$

corresponding to that of $AdS_5 \times S^5$.

The solution (2.5) has a "horizon" that emits "Hawking radiation". There exists a corresponding picture in which this "Hawking radiation" is produced by the collision of two open strings, living on the D-brane, to form a closed string that is not bound to the D-brane and propagates into the bulk spacetime. Following this picture, we expect there to exist a theory relating open strings living on a D3-brane, corresponding to a $\mathcal{N} = 4$ SYM in four dimensions, and a gravity theory living in the space curved by the D3-branes.

We keep track of each string endpoint by assigning indices i and j corresponding to the label of the D-brane on which each string endpoint lies. If there are N D-branes, then $i, j, = 1, \ldots, N$. Then each string state corresponds to a field with colour indices i and j, and each string state corresponds to a field in the adjoint representation of a $U(N)$ gauge theory [27]. Each string now corresponds to an excitation of the D-branes and has a mass energy proportional to the string tension T and the endpoint separation ℓ_s. We now have a theory of open strings in the adjoint of $u(N)$ living on the D-branes. In the $\alpha' \to 0$ limit, corresponding to infinite string tension, the endpoints of each string will be close together, and only the massless modes survive. The low energy theory that lives on the N D-branes, is a Yang-Mills theory with gauge group $U(N)$. We can see this as follows: the string theory has 32 supercharges (32 components $\mathfrak{Q}^i_\alpha$) which form an 11-dimensional spinor, or eight 4-dimensional spinors giving rise to an $\mathcal{N} = 8$ supersymmetry in $d = 4$ dimensions. The D-brane breaks half of the supersymmetry by imposing boundary conditions that select half of the remaining spinors. For $p = 3$ in four dimensions this corresponds to $\mathcal{N} = 4$ SYM with $U(N)$ gauge group. Alternatively, extremal Dp-branes are solutions of supergravity which is the low energy limit ($\alpha' \to 0$) of a string theory. These are black-hole solutions with charge Q and mass M saturating the $|Q| \leq M$ bound. In supersymmetric theories, this bound follows from the supersymmetry algebra and is known as the *BPS bound*. Saturating this bound preserves the maximum number of supersymmetries. Therefore, the extremal Dp-branes also have $\mathcal{N} = 4$ supersymmetry in $d = 4$ dimensions and are solutions of supergravity with horizon at $r = 0$.

2.1.4 The Low Energy Limit

In the low energy limit, the theory decouples into supergravity in the bulk S_{bulk} and an open string theory on a stack of D3-branes. The open strings living on the D3-brane give rise to a theory that reduces to $\mathcal{N}=4$ SYM in the low energy limit with action S_{brane}, while closed strings living in the bulk spacetime give a theory that is supergravity coupled to the massive modes of the string. The interactions of these two ingredients gives rise to the "Hawking radiation", $S_{\text{interaction}}$. Here I give a heuristic motivation for this correspondence by studying a string theory in the presence of D3-branes. We consider two perspectives.

Consider again a stack of D3-branes with open strings attached. We partition the action of an open string theory into

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{interaction}}$$

where S_{bulk} is the action of closed strings propagating in the bulk spacetime, S_{brane} is the action of open strings on the D3-brane and $S_{\text{interaction}}$ defines the open-closed string interactions. The energy of a string is determined by the modes of oscillation of the *center-of-mass* wave-function and of the string itself. In the low energy limit, $\alpha' \to 0$ and the massive string modes decouple and S_{bulk} reduces to 10-dimensional supergravity $S_{\text{supergravity}}$, and S_{brane} reduces to $\mathcal{N}=4$ SYM. In this limit $S_{\text{interactions}} \sim g_s \sim \alpha'$. Then the limit $\alpha' \to 0$ reduces $S_{\text{interaction}}$ to zero and the theory on the brane decouples from that in the bulk spacetime giving rise to two decoupled systems.

As a test of this low energy limit, consider the smallest energy excitation of the brane seen by an observer at a distance r from the D3-brane. Suppose that this energy is E_r. This object will have an apparent energy

$$E_\infty = \frac{1}{1+\frac{R^2}{r^2}} E_r$$

as seen by an observer at infinity. Then, as $r \to 0$, the energy of any mode approaching the branes gets smaller. To this observer, the theory decouples into 10-dimensional supergravity in the bulk and type IIB string theory near the branes where the spacetime is $\text{AdS}_5 \times S^5$. Again we have two decoupled low energy systems (of excitations).

Next, we identify the 10-dimensional supergravity theory in each of the descriptions and are lead to conjecture that $\mathcal{N}=4$ SYM in $(3+1)$-dimensions with $U(N)$ gauge group is equivalent to type IIB string theory on $AdS_5 \times S^5$. This conjecture is the AdS/CFT correspondence.

2.1.5 Validity of the Low Energy Limit Matching Conditions

The validity of this low energy matching is limited to those sectors of the theory where we are able to perform concrete computations. However, it is expected that the correspondence is generally true. Here I present some simple arguments for the validity of this matching in the appropriate limits.

For the supergravity approximation to be valid, the background radius of curvature must be large compared to the string length, that is $R = \sqrt{\alpha'}\left(4\pi g_s N\right)^{1/4} \gg \sqrt{\alpha'} = \ell_s$. This corresponds to the limit where $1 \ll g_s N$ or $1 \ll g_{\mathrm{YM}}^2 N$ and the quantum corrections, given in powers of the string coupling g_s, must be small ($g_s \to 0$). This implies the double scaling limit where $g_s \to 0$ and $N \to \infty$ while keeping $\lambda = g_s N = g_{\mathrm{YM}}^2 N$ large and fixed. However, at large N, the perturbation theory is valid when $\lambda \ll 1$, which is the opposite of the supergravity approximation. The gauge theory perturbation theory, which is valid for $\lambda \ll 1$, and the supergravity theory on $AdS_5 \times S^5$, which is valid for $1 \ll \lambda$ are valid in different coupling regimes. This implies that the AdS/CFT correspondence is a duality between a strongly coupled supergravity theory on $AdS_5 \times S^5$ and weakly coupled $\mathcal{N}=4$ SYM theory with $U(N)$ living on the 4-dimensional boundary of AdS_5.

2.2 Matching Quantities Across the Correspondence

For each quantity in the CFT there should correspond a quantity in the string theory. The quantities include the symmetries of each theory and the corresponding energies and Hilbert spaces. In this section I discuss the matching of these quantities.

2.2.1 Hilbert Spaces

AdS/CFT proposes a matching between the states in the CFT with states in a string theory, and corresponds to matching the Hilbert spaces in each theory. Here I argue for this matching by first matching the time coordinates of each theory, and then matching the Hilbert space of states at each instant. Then the operator/state correspondence in $\mathcal{N}=4$ SYM in $(3+1)$-dimensions carries over to the operator/state correspondence on the boundary of the $\mathrm{AdS}_5 \times S^5$ and each operator in the CFT has a corresponding state in the string theory. We can see this as follows.

A Wick rotation maps the $(3+1)$-dimensional Minkowski spacetime of $\mathcal{N}=4$ SYM to 4-dimensional Euclidean space $\mathbb{R}^4$. By conformal transformation $r = e^{\tau}$ the metric becomes

$$\mathrm{d}s^2 = e^{2\tau}\left(\mathrm{d}\tau^2 + \mathrm{d}\Omega_3^2\right)$$

where conformal rescaling recovers the standard metric

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + \mathrm{d}\Omega_3^2$$

corresponding to the CFT on the cylinder $\mathbb{R}\times S^3$. This maps the origin in $\mathbb{R}^4$ to $\tau \to -\infty$ and spatial infinity corresponds to $\tau \to \infty$ on the cylinder. Similarly, $\tau = 0$ corresponds to the unit circle.

Our task is to find a state $\left|\phi\right\rangle$ on the cylinder that corresponds to the operator $\mathcal{O}(\tau)$ on M_4. Consider an operator $\mathcal{O}(\tau)$, corresponding to a circle in M_4 and the integral

$$F[\phi_b] = \int [D\phi] e^{iS}\mathcal{O}(\tau)\Big|_b$$

where we integrate over all fields ϕ subject to the boundary conditions b. This corresponds is an integral over all fields ϕ on a disc where the fields take values ϕ_b. We interpret $F[\phi_b]$ to mean a state $|F\rangle$ expressed in the basis $\left|\phi_b\right\rangle$. We require that there exists an initial state $\left|\phi\right\rangle$, defined at $\tau \to -\infty$, that evolves into $|F\rangle$ at the time τ,

$$|F\rangle = e^{-i\int_{-\infty}^{\tau}\mathrm{d}\tau'\mathcal{H}(\tau')}\left|\phi\right\rangle.$$

In this way, the operator $\mathcal{O}$ in the CFT on M_4 is mapped to a state in the CFT on the cylinder.

Returning to the metric (2.4) in global coordinates, the limit that $1 \ll \rho$ is large and constant recovers the metric on $\mathbb{R} \times S^3$, up to rescaling, which is the boundary of $AdS_5 \times S^5$ with vanishing boundary S^5. The isometries of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ match those of the $AdS_5 \times S^5$, and we identify the $\mathbb{R}$ subspace of the $\mathbb{R} \times S^3$ at the boundary of the $AdS_5 \times S^5$ with the $\mathbb{R}$ subspace of the $\mathbb{R} \times S^3$ in the CFT. Identifying this boundary with the space on which the CFT lives gives a map between the operators of the CFT in Minkowski space and the states of the string theory.

The boundary of string theory spacetime is matched to the background spacetime of the CFT. Then, at each time t, there exists a Hilbert space in the CFT that is identified with the Hilbert space on the boundary of the $AdS_5 \times S^5$ spacetime in the string theory, and states in the string theory are identified with operators in the CFT.

2.2.2 Scaling Dimensions and Energies

As I have argued, the AdS/CFT matches operators in the CFT to states in the string theory. This matching gives rise to a collection of other matchings across the correspondence. In particular, the scaling dimension of operators in the CFT are matched with the energies of corresponding states in the string theory. Similarly, we identify the $\mathscr{R}$-charge of operators in the CFT with angular momentum of states in the corresponding string theory.

The conformal symmetry of $\mathcal{N} = 4$ SYM is a generalization of scale invariance, where conformal transformations are generalizations of transformations that change distances between points. Conformal transformations are coordinate transformations that act on flat space and a give space-dependent scale factor $\Omega^{-2}(x)$, so conformal invariance is an invariance in flat-space. A scale invariant theory is generally also conformally invariant. In $2 < d$, Minkowski space has conformal symmetry group is $SO(2,d)$, which is the same as the isometry group of AdS_{d+1}. There are $(d+1)(d+2)/2$ components for the parameters of the conformal transformations (we can compare this with the number of symmetries of a maximally symmetric spacetime). This corresponds with the symmetry group $SO(2,d)$ and conformal invariance in flat $(1,d-1)$ dimensions for $2 < d$ corresponds to the symmetry

group $SO(2,d)$ which is the same symmetry group as that for $(d+1)$-dimensional anti-de Sitter space, AdS_{d+1}.

Generators of the conformal symmetry algebra include generators for translations P_μ, generalized rotations $J_{\mu\nu}$, dilatations D and special conformal transformations K_μ. Representations of the conformal group are defined by eigenfunctions of the dilatation operator D with eigenvalues $-i\Delta$, where Δ is the scaling dimension, that is to say that under $x \mapsto \lambda x$, we have

$$\phi(x) \mapsto \phi'(x) = \lambda^\Delta \phi(x)$$

and Δ is increased by P_μ since the $S(2,d)$ conformal algebra acts as

$$[D, P_\mu] = -iP_\mu$$

so

$$D(P_\mu\phi) = P_\mu(D\phi) - iP_\mu\phi = P_\mu(-i\Delta\phi) - iP_\mu\phi = P_\mu\left(-i\Delta\phi - i\phi\right) = -i(\Delta+1)P_\mu\phi$$

and Δ is decreased by K_μ since

$$D(K_\mu\phi) = K_\mu(D\phi) + iP_\mu\phi = K_\mu(-i\Delta\phi) + iK_\mu\phi = K_\mu\left(-i\Delta\phi + i\phi\right) = -i(\Delta-1)K_\mu\phi\,.$$

We can think of K_μ as the annihilation operator $\hat{a}$ and of P_μ as the creation operator $\hat{a}^\dagger$. Since P_μ and K_μ are symmetry generators, we can generate the other states in the theory by successive action of each of these operators. This means that representations are built from the action of creation and annihilation operators P_μ and K_μ.

There will be an operator of lowest dimension, say Φ_0, in the representation of the conformal group so that

$$K_\mu\Phi_0 = 0$$

and Φ_0 is called a *primary operator*. The representations generated by successive operations of P_μ correspond to states with increasing conformal dimension in the gauge theory and increasing energy in the string theory.

Primary fields of dimension Δ scale according to $\phi(\lambda x) = \lambda^\Delta\phi(x)$. In $d = 4$ dimensions, representations of the conformal algebra are obtained by acting with P_μ on the primary

fields. Note that the quantum conformal dimension (scaling dimension) Δ does not need to be the same as the scaling dimension for an operator in $\mathcal{N}=4$ SYM, as some operators receive non-trivial quantum corrections.

In a conformal field theory, with radial coordinate $r = e^{\tau}$, the global rescaling $r \to \lambda r$ gives $e^{\tau} \to e^{\tau \log(\lambda)}$ and corresponds to a time-translation $\tau \to \tau + \log(\lambda)$ in the CFT. The association of coordinate rescaling and time translations suggests an identification between the generators of time translations on the cylinder and dilatations of the metric. By this identification, the Hamiltonian is identified with the dilatation operator, and energies are identified with scaling dimensions. Since the AdS/CFT correspondence identifies the Hilbert space of $\mathcal{N}=4$ SYM on $\mathbb{R} \times S^3$ with that of type IIB string theory on the boundary of $AdS_5 \times S^5$, this suggests that operators of a given conformal dimension in the CFT should map to states of given energy in the string theory.

The dilatation operator can be expanded in powers of the coupling constant g_{YM},

$$D = \sum_{k=0}^{\infty} \left(\frac{g_{\text{YM}}}{4\pi} \right)^{2k} D_{2k}$$

where D_{2k} is the k-loop dilatation operator, which accounts for the corrections to the anomalous scaling dimension of a given operator in powers of $\hbar$. The anomalous dimension is the quantum correction to the energy scaling of the operator $\mathcal{O}$ in $\mathcal{N}=4$ SYM. These corrections are expressed as weighted sums of free theory correlation functions. Interaction energies are measured by the dilatation operator and interaction probabilities are measured by the weighted field averages defined by these correlation functions. Dynamics in the theory are therefore determined by the collection of n-point correlation functions, which give the probabilities of each process, and the eigenvalues of the dilatation operator, which give the energies of interaction in the system [28]. Half-BPS operators receive no quantum corrections to their scaling dimensions. These operators are important in the study of the energy spectrum of subsectors of the theory since calculations involving these operators can be done reliably at weak and strong coupling.

A simple example of the action of the dilatation operator, for a tree-level computation, is given by the action of D_0 on a simple operator $\mathrm{Tr}(Z^n)\mathrm{Tr}(Y^m)$, where

$$D_0 = \mathrm{Tr}(Y\partial_Y) + \mathrm{Tr}(Z\partial_Z) = Y_j^i \frac{\mathrm{d}}{\mathrm{d}\, Y_j^i} + Z_j^i \frac{\mathrm{d}}{\mathrm{d}\, Z_j^i}$$

and

$$\begin{aligned} Z_l^k \frac{\mathrm{d}}{\mathrm{d}\, Z_l^k} \mathrm{Tr}(Z^n) &= Z_l^k \frac{\mathrm{d}}{\mathrm{d}\, Z_l^k} Z_{i_2}^{i_1} Z_{j_3}^{i_2} \cdots Z_{i_1}^{i_n} \\ &= Z_l^k \delta_k^{i_1} \delta_{i_2}^l Z_{j_3}^{i_2} Z_{j_4}^{i_3} \cdots Z_{i_1}^{i_n} + Z_l^k Z_{i_2}^{i_1} \delta_k^{i_2} \delta_{i_3}^l Z_{j_3}^{i_2} \cdots Z_{i_1}^{i_n} + \cdots + Z_l^k Z_{j_2}^{i_1} Z_{j_3}^{i_2} \cdots Z_{j_n}^{i_{n-1}} \delta_k^{i_n} \delta_{i_1}^l \\ &= n\, \mathrm{Tr}(Z^n). \end{aligned}$$

Then

$$D_0 \mathrm{Tr}(Z^n)\mathrm{Tr}(Y^m) = (n+m)\mathrm{Tr}(Z^n)\mathrm{Tr}(Y^m)$$

so the classical, tree-level computation simply counts the number of matrix fields in the trace operator. In the case of $\mathcal{N}=4$ SYM in $(3+1)$-dimensions, each matrix field carries units of inverse length and the tree-level contribution to the dilatation operator D_0 gives the correct classical scaling dimension of the operator. Higher order terms give quantum corrections to this classical scaling dimension as a series in powers of g_{YM}^2. In particular we shall consider the one-loop dilatation operator D_2 that gives the contribution to the dilatation operator at one-loop.

Within the collection of all objects that could exist in the theory, there exit subsectors comprising states that enjoy some given fraction of all supersymmetry in the theory. The vacuum is the only state that is invariant under all supersymmetries while half-BPS states enjoy half of the supersymmetry of the theory. It was shown in [13] that the space of Schur polynomials labeled by Young diagrams with fewer than N rows forms a basis in the space of half-BPS representations corresponding to giant gravitons. This means that the giant graviton dynamics in the dual string theory can be computed using Schur polynomials and the associated group theory.

2.2.3 Limits of the Matching

Objects that occur naturally in the QFT are correlation functions which determine the probability of a specific process occurring in the theory. The subset of all correlation functions that will be of interest in this study have the form

$$\langle \cdots \mathcal{O}_i \mathcal{O}_{i+i} \mathcal{O}_{i+2} \cdots \rangle$$

where each operator $\mathcal{O}_i$ is the trace of some product of fields. Direct computation of these correlators involves the evaluation of formal series expansions in the constituent fields.

The bosonic symmetry group in this matching is $SO(4,2) \times SO(6)$ and by the Noëther Theorem there exist conserved charges associated with this collection of symmetries. These charges are also operators which act on the Hilbert spaces on each side of the correspondence. Each eigenstate with eigenvalue J of an operator corresponding to a conserved charge in the string theory is matched with a corresponding eigenstate in the CFT with the same eigenvalue J. A charge associated with the $SO(6)$ symmetry subgroup in the CFT is called an $\mathscr{R}$*-charge* and corresponds to angular momentum in the string theory.

Each correlator has a series representation in powers of $1/N$ with coefficients that depend on the $\mathscr{R}$-charge. This is well understood and was studied extensively in [13, 29–32]. When N is large, these series give rise to familiar perturbation series expansions. By studying the double scaling limit, where $\lambda = g_{\mathrm{YM}}^2 N$ for fixed λ and infinite N, it was shown that the planar sector of SYM reproduces the free string theory [21]. In this way, the classical limit of the dual string theory is captured by the planar limit of the SYM. The non-planar limit involves operators comprising some number of fields that scale with the rank of the gauge group N. For example, a full computation involving operators comprising N^k fields where $1 \leq k$ will receive non-planar contributions. It is expected that the fully non-planar Yang-Mills theory should correspond to the fully interacting string theory.

2.3 Schur Polynomials and Restricted Schur Polynomials

Schur polynomials have been shown to be powerful computational tools to evaluate correlation functions in $\mathcal{N}=4$ SYM [13, 33]. In this section I give a brief introduction to the Schur polynomial technology and some discussion regarding the evaluation of correlators in the half-BPS sector. A key idea is that the half-BPS sector of $\mathcal{N}=4$ SYM theory is described exactly by the holomorphic sector of a single complex matrix quantum mechanics model which is equivalent to a theory of N free fermions [34, 35]. The half-BPS sector of type IIB string theory on $AdS_5 \times S^5$ contains gravitons, strings and membranes and it is believed that all these objects are contained in the matrix quantum mechanics of $\mathcal{N}=4$ SYM.

We shall occupy ourselves with the scalar field sector of the $\mathcal{N}=4$ SYM theory in four dimensions. In this sector of the theory, the relevant part of the Lagrangian to study is

$$\mathcal{L} = \frac{1}{g_{\rm YM}^2}\,\mathrm{Tr}\left(D_\mu \Phi_i D^\mu \Phi_i + \frac{1}{2}\left[\Phi_i,\Phi_j\right]^2\right)$$

and we do not consider terms involving fermions or gauge bosons. This theory has six real scalar fields that transform in the adjoint representation of $U(N)$. The good angular momentum states in $\mathcal{N}=4$ SYM in (2.2) are constructed from the six scalar fields

$$X = \Phi_1 + i\Phi_2, \qquad Y = \Phi_3 + i\Phi_4 \qquad \text{and} \qquad Z = \Phi_5 + i\Phi_6$$

where fields X, Y, and Z take values in the adjoint representation of $U(N)$. We can see this by acting with the $U(1)$ element $e^{-i\theta}$ on X. This corresponds to a rotation in the $\Phi_1 - \Phi_2$-plane and defines a $U(1)$ conserved charge. Each such transformation corresponds to a rotation in the $\Phi_1 - \Phi_2$-plane of S^5. Similar transformations can be applied to Y and Z fields. In this context, the conserved charge is angular momentum, and each of these fields corresponds to an angular momentum state with $J=1$.

The six scalar fields of $\mathcal{N}=4$ SYM give rise to a $SO(6) \simeq SU(4)$ symmetry, rotating these fields amongst themselves, called *$\mathcal{R}$-symmetry*. The associated conserved $\mathcal{R}$-charge commutes with the bosonic, but not fermionic, spacetime symmetries. This $SO(6)$ symmetry is also present in the corresponding string theory background where the scalar fields Φ_i are the embedding coordinates for the S^5.

Operators built from a product of X, Y or Z fields are half-BPS states. Similarly, operators built from traceless symmetric combinations of the six Hermitian scalars are also half-BPS. This includes single- and multi-trace operators. Within this subset of half-BPS operators is a sector that is invariant under the $SO(4)$ subgroup of $SO(6)$.

Objects with $\mathscr{R}$-charge J have J copies of Z in each corresponding operator. For each partition $\left(p_i, q_i\right)_{i=1}^{m}$ of J satisfying

$$J = \sum_{i=1}^{m} p_i q_i$$

there corresponds a distinct operator

$$\mathscr{O} = \prod_{i=1}^{m} \mathrm{Tr}(Z^{p_i})^{q_i}.$$

This construction puts Schur polynomials of degree J into one-to-one correspondence with half-BPS representations of $\mathscr{R}$-charge J. QFT correlators of thus constructed Schur polynomials and their Hermitian conjugates are protected under the symmetry of the theory and therefore receive no corrections to their scaling dimension. This means that the computations in the weak coupling limit of the theory can be extrapolated to the strong coupling limit.

Half-BPS operators have scaling dimension equal to their $\mathscr{R}$-charge and the large amount of supersymmetry ensures that there are no quantum corrections to this quantity. Then, an operator with $\mathscr{R}$-charge J has $\Delta = J$ and the corresponding string state has energy equal to its angular momentum. For each operator of increasing J there corresponds, following the AdS/CFT conjecture, a state in the dual string theory. Half-BPS giant gravitons in $\mathrm{AdS}_5 \times S^5$ are spherical D3-branes embedded in the S^5 or AdS_5 and are called *sphere giants* and *dual giants*, respectively. These are spatially extended, non-perturbative objects in the string theory that propagate in the bulk spacetime. Giant gravitons carry angular momentum and couple to a background *Ramond-Ramond* 5-form flux. This coupling induces a Lorenz-type force on a giant causing it to expand in a process known as the *Myers Effect* [36]. This expansion is stabilized by the membrane tension that causes the giant to contract. The size of the sphere giant is limited since it cannot expand beyond the embedding S^5 and leads to a cut-off on the maximum angular momentum carried by the

sphere giants. There is no such limit on the angular momentum carried by a dual giant. Later in this chapter I shall explain this cut-off using the Schur polynomial technology.

The radius of the spatial extent of the giant graviton is determined by the angular momentum J carried by the giant graviton following

$$r = R\sqrt{\frac{J}{N}}$$

where R is the radius of the AdS space [29] given in (2.3). We can identify this with the related parameters across the correspondence and find that an object with J units of $\mathscr{R}$-charge has a size

$$r = \left(4\pi g_{\rm YM}^2 N\right)^{1/4} \ell_s \sqrt{\frac{J}{N}}$$

and N is the rank of the gauge group $U(N)$. This follows from the matching of parameters on each side of the correspondence

$$\left(\frac{R}{\ell_s}\right)^4 = 4\pi g_{\rm YM}^2 N = 4\pi\lambda$$

and explicitly demonstrates the strong/weak coupling relation of the duality.

Operators in the CFT are organized according to their $\mathscr{R}$-charge. In this language, operators comprised of $O(1)$ fields are dual to point like objects, like point gravitons, in the gravitational theory, while operators comprised of $O(\sqrt{N})$ fields are dual to objects with fixed size in units of string length, such as strings, and objects comprised of $O(N)$ fields are associated with giant gravitons [29, 37].

2.3.1 Schur Polynomials

Schur polynomials are constructed from traces of $N \times N$ matrices and form a complete basis for the operators in the CFT. Schur polynomials comprised of $O(N)$ fields were proposed as the natural gauge theory dual object of half-BPS giant gravitons in [13]. Following the operator/state correspondence, the AdS/CFT correspondence makes possible the labeling of objects in the string theory by the same parameters J and N that occur in the CFT. The $\mathscr{R}$-charge of the dual gauge theory operators are matched with the degrees of freedom of the string theory states. In this language, Schur polynomials in $\mathscr{N} = 4$ SYM, with scaling

dimension $\sim O(N)$ at large N are dual to giant gravitons and Restricted Schur polynomials are dual to excited giant gravitons in the corresponding string theory.

Operators which are products of traces with the same total number of fields, but different numbers of traces, are orthogonal in the large N limit and are identified with multi-particle graviton states. The number of traces in the operator is identified with the particle number. The orthogonality of operators constructed from products of traces fails when they are comprised of $O(\sqrt{N})$ matrix fields at which point these operators cannot be considered dual to point graviton states. Instead in the near-BPS sector, these operators are dual to strings [35]. Finally, giant gravitons are dual to multi-trace operators comprised of $O(N)$ matrix fields [13, 33] where the interpretation of the objects in the string theory that are dual to operators in the gauge theory is now determined by the $\mathscr{R}$-charge of the operator, corresponding to the number of matrix fields in the operator.

Take $\mathscr{N}=4$ SYM defined in $\mathbb{R}\times S^3$ or R^4 and consider two-point correlators of complex fields

$$\left\langle Z_{ij}(x_1)Z^{\dagger}_{kl}(x_2)\right\rangle \propto \frac{\delta_{il}\delta_{jk}}{|x_1-x_2|^2}$$

where Wick contraction over colour indices give δ-functions and the spacetime dependence is determined by conformal invariance. A general n-point correlator is a sum over all allowed Wick contractions multiplied by the appropriate spacetime factors. I suppress the spacetime dependence in the discussion that follows.

The representation theory of the symmetric and unitary groups is useful in computations of QFT correlators. In particular, the irreducible representations of S_n and $U(N)$ form an important part of the study of gauge theory correlation functions. A matrix representation of a group element is irreducible if it has no invariant subspace. This means that an irreducible matrix representation of a group element cannot be written in block diagonal format. Each irreducible representation R of a group is distinguished from other irreducible representations by the character $\chi_R(\sigma)$ of that representation corresponding to the trace of the matrix representation of the group element σ.

The group theory of the symmetric group and the unitary group are closely connected. This is remarkable considering that the symmetric group S_n is a finite group of order $n!$,

while the unitary group $U(N)$ has an infinite number of group elements. To understand this connection, suppose V_N is an N-dimensional complex vector space, which is also the fundamental representations of $U(N)$, and let

$$V_N^{\otimes n} = V_N \otimes V_N \otimes \cdots \otimes V_N$$

be a N^n-dimensional complex vector space. We can consider the group actions of S_n and $U(N)$ on $V_N^{\otimes n}$. First, consider an element T in the fundamental representation of $U(N)$. Then T is a matrix that acts on vectors $v \in V_N^{\otimes n}$ as follows

$$T(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = T^i_j v_1^j \otimes T^i_j v_2^j \otimes \cdots \otimes T^i_j v_n^j.$$

Now consider an element P in S_n which acts on elements of $V_N^{\otimes n}$ by permuting vectors

$$P(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{P(1)} \otimes v_{P(2)} \otimes \cdots \otimes v_{P(n)}.$$

Note that $PT(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = TP(v_1 \otimes v_2 \otimes \cdots \otimes v_n)$ and actions of P and T commute. Since P and T were arbitrarily chosen elements from S_n and $U(N)$, $V_N^{\otimes n}$ admits commuting actions of the symmetric and unitary groups. Operators in these groups can be simultaneously diagonalized such that $V_N^{\otimes n}$ admits a decomposition

$$V_N^{\otimes n} = \bigoplus_{R \vdash n} V_R^{S_n} \otimes V_R^{U(N)}$$

where $V_R^{S_n}$ and $V_R^{U(N)}$ are the irreducible representations of S_n and $U(N)$, respectively. The dimension of each irreducible representation of S_n organizes the multiplicities of irreducible representations in $U(N)$. Similarly, the irreducible representations of $U(N)$ organize the multiplicities of irreducible representations of S_n. Let $V_R^{S_n}$ have dimension d_R and $V_R^{U(N)}$ have dimension $\dim_R$. Then, $V_R^{U(N)}$ appears d_R times, and $V_R^{S_n}$ appears $\dim_R$ times, in the decomposition of $V_N^{\otimes n}$. Each partition R corresponds to a Young diagram, so representations of S_n and $U(N)$ are labeled by Young diagrams. As a consistency check, we find

$$\sum_{R \vdash n} d_R \dim_R = N^n$$

which is the dimension of the space $V_N^{\otimes n}$. This result is known as the *Schur-Weyl Duality*.

The Schur-Weyl duality allows us to write the characters of the of the unitary group in terms of the characters of the symmetric group. Therefore, the Schur polynomial $\chi_R(Z)$, where $Z \in u(N)$, can be written as

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \mathrm{Tr}\left(\sigma Z^{\otimes n}\right)$$

where $\chi_R(\sigma)$ is the character of an element in S_n in the representation R and

$$\mathrm{Tr}\left(\sigma Z^{\otimes n}\right) = \sum_{i_i, i_2, \ldots, i_n} Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}}.$$

The characters $\chi_R(\sigma)$ assign the same value to group elements from the same conjugacy class. This means that group elements with the same cycle structure in a given representation are assigned the same character value. Similarly, the multi-trace factors associated to group elements with a given cycle structure are identical. The value of each character can be determined by method of strand diagram technology [32].

Each Schur polynomial is labeled by a representation R, corresponding to a partition of n, which has a graphical representation as a *Young diagram*. The Young diagram for R comprises n boxes arranged in left-justified rows, whose lengths form a non-increasing sequence, defined by the partition, from the top of the diagram to the bottom of the diagram. To each diagram R we define

$$\mathrm{hooks}_R = \prod_n (i_n + j_n + 1)$$

where $i_n + j_n + 1$ is called the *hook length* of box n, where j_n and i_n are, respectively, the number of boxes in the column below and in the row to the right of box n in R. We also define

$$f_R = \prod_{i,j} (N - i + j)$$

where $N - i + j$ is the *weight* of a box in the i-th row and j-th column of the Young diagram R. It is clear that $f_R = 0$ for any Young diagram with column heights that are greater than or equal to N. By exploiting the connection between the symmetric and unitary groups, we find the dimension of the irreducible representation R of the unitary group is given by

$$\dim_N(R) = \frac{f_R}{\mathrm{hooks}_R}.$$

The power of the Schur polynomial technology comes from the fact that summation over all Wick contractions in the correlation function value are replaced by a simple product of combinatorial factors associated to a given Young diagram.

The Schur polynomial two-point and three-point functions satisfies the orthogonality relations

$$\left\langle \chi_R(Z)\chi_S(Z^\dagger)\right\rangle = \delta_{RS} f_R \qquad \text{and} \qquad \left\langle \chi_R(Z)\chi_S(Z)\chi_T(Z^\dagger)\right\rangle = g(R,S;T) f_T$$

where δ_{RS} is a Kronecker δ-function, and $g(R,S;T)$ is a Littlewood-Richardson coefficient that counts the multiplicities of the irreducible representation T in the tensor product of irreducible representations R and S [13]. These orthogonality relations can be generalized to arbitrary n-point functions. Correlation functions of operators labeled by a small number of large J_i charges correspond to Schur polynomials labeled by Young diagrams with long columns or long rows, and have been studied in [38].

Schur polynomials comprising $O(1)$ fields are naturally associated with a collection of point particles, while Schur polynomials comprising $O(\sqrt{N})$ fields can be associated with a gas of point particles or with strings. Schur polynomials comprising $O(N)$ fields in representations with a single long column are naturally associated to half-BPS D3-branes that wrap the $S^3 \subset S^5$ in the $AdS_5 \times S^5$ background. Similarly, Schur polynomials comprising $O(N)$ fields in representations with a single long row are naturally associated to half-BPS D3-branes that wrap the $S^3 \subset AdS^5$. A consequence of this association is that the sphere giant $\mathscr{R}$-charge has a maximum upper bound of N, corresponding to the maximum column length for which f_R is zero and the sphere giant correlator vanishes. This is gives a natural cut-off on the maximum angular momentum carried by a sphere giant and hence, the maximum size of the sphere giant [29]. This is known as the *Stringy Exclusion Principle* [39]. There is no such cut-off for the dual giants. This identification of giant gravitons with Schur polynomials labeled by Young diagrams has a natural extension to collections of giant graviton bound states, where a single Young diagram with $O(N)$ boxes and $O(1)$ columns corresponds to a bound state of sphere giants, while a Young diagram with $O(N)$ boxes and

$O(1)$ rows corresponds to a bound state of dual giants. In the limit that the rank of the gauge group N is large, each expectation value can be expanded as a series in $1/N$.

2.3.2 Restricted Schur Polynomials

Restricted Schur polynomials generalize the single matrix Schur polynomials to include matrix words. Matrix words are products of operators, matrix fields, fermions and covariant derivatives. Restricted Schur polynomials provide a natural basis to study open string excitations of giant gravitons in the AdS/CFT correspondence and have been studied extensively [30–32, 40]. A restricted Schur polynomial corresponding to a collection of k open string excitations is given by

$$\chi_{R,(r,s)}(Z^{\otimes n}, Y^{\otimes m}) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \mathrm{Tr}_{(r,s)}(\Gamma_R(\sigma)) Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} \cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}}$$

where R is a Young diagram with $n+m$ boxes that labels an irreducible representation of S_{n+m}, r is a Young diagram with n boxes that labels an irreducible representation of S_n, and s is a Young diagram with m boxes that labels an irreducible representation of S_m. The S_n subgroup acts on $1, 2, \ldots, n$ and therefore permutes indices belonging to the Zs while the S_m subgroup acts on $n+1, n+2, \ldots, n+m$ and permutes indices belonging to the Ys. The sequence (r,s) labels an irreducible representation of $S_n \times S_m$. The trace over group elements $\chi_R(\sigma)$ in the standard Schur polynomials is now replaced with a trace $\mathrm{Tr}_{(r,s)}(\Gamma_R(\sigma))$ over the subspace carrying the irreducible representation (r,s). This trace is obtained by projecting from the carrier space of R to the carrier space of (r,s), and then tracing over R, to give $\mathrm{Tr}(P_{R\to(r,s)}\Gamma_R(\sigma))$. The projectors $P_{R\to(r,s)}$ satisfy the idempotency relation

$$P_{R\to(r,s)} \cdot P_{R\to(r,s)} = P_{R\to(r,s)}$$

and have natural realizations as operators composed of a sum over products of Kronecker δ-functions.

We can study the two-point correlation functions of restricted Schur polynomials (with the spacetime dependence suppressed for brevity),

$$\left\langle \chi_{R,(r_1,r_2)} \chi_{S,(s_1,s_2)} \right\rangle = \delta_{RS} \delta_{r_1 s_1} \delta_{r_2 s_2} \frac{\mathrm{hooks}_R}{\mathrm{hooks}_{r_1}\, \mathrm{hooks}_{r_2}} f_R.$$

Again, the value of these correlators is determined by the combinatorial factors associated to the corresponding Young diagrams. The two-point function for any restricted Schur polynomial is known, so it will be convenient to work with normalized operators $O_{R,(r,s)}(Z,Y)$, having unit two-point function, where

$$\chi_{R,(r,s)}(Z,Y) = \sqrt{f_R \frac{\text{hooks}_R}{\text{hooks}_r\,\text{hooks}_s}}\; O_{R,(r,s)}(Z,Y).$$

These expressions appear naturally in the construction of Gauss graph operators.

2.3.3 Gauss Graph Operators

Consider the excitations of a p giant graviton state, with m identical strings attached. The Gauss Law constraint on the brane worldvolume forces the number of open strings m_i emanating from brane i to equal the number of strings entering brane i. Since the strings are identical, there is an associated symmetry corresponding to permuting strings such that the configuration of open strings is unchanged. Then the allowed open string configurations are in one-to-one correspondence with the double coset

$$H\backslash S_m/H$$

where

$$H = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_p}$$

is the subgroup of S_m that leaves the open string configuration invariant. The number of double coset elements corresponds to the number of restricted Schur polynomials compatible with this system description. The complete set of functions $O_{R,r}(\sigma)$ constructed on this double coset is given by

$$O_{R,r}(\sigma) = \frac{|H|}{\sqrt{m!}} \sum_{j,k} \sum_{s\vdash m} \sum_{\mu_1,\mu_2} \sqrt{d_s}\; \Gamma^s_{jk}(\sigma) B^{s\to 1_H}_{j\mu_1} B^{s\to 1_H}_{k\mu_2} O_{R,(r,s),\mu_1\mu_2}$$

where $B^{s\to 1_H}_{j\mu_1}$ is a branching coefficient for the irreducible representation s to the μ_1-th copy of the trivial representation of the subgroup H. The index j labels a state in the carrier space of s. The operators $O_{R,r}(\sigma)$ are the *Gauss graph operators*. The structure of these operators has a graphical description where branes correspond to nodes and strings are

directed edges connecting nodes. We call these pictures *Gauss Graphs*. We shall return to these operators again in Chapter 4. Next, I highlight some of the recent progress made in testing the AdS/CFT correspondence and the computation of scaling dimensions of excited giant graviton states.

2.4 The Dilatation Operator and Integrability

Integrability of the theory allows us to compute quantities exactly and finding integrability beyond the planar limit is a strong indicator that the theory is exactly integrable. Indeed, there is considerable evidence supporting the idea that this theory is integrable. Here I summarize some of the progress towards finding integrability beyond the planar limit of the theory.

Numerical computations of the spectrum of anomalous dimensions of restricted Schur polynomials, labeled by two long row or two long column Young diagrams, for small numbers of "impurities" m, were conducted in [41] and [42]. These results suggested that the spectrum of anomalous dimensions is equivalent to a system of decoupled harmonic oscillators. This was demonstrated analytically in the large N limit for operators comprising n fields Z with m impurities Y, with $n, m \sim O(N)$ and $m \ll n$ in [43] . The Young diagrams labeling the operators in this analysis consist of two long rows or columns, where the row or column lengths differ by order N boxes. Evaluating the large N limit of these correlation functions necessarily involves summing all planar and non-planar diagrams. The dilatation operator acting on restricted Schur polynomials was exactly diagonalized in this limit and the corresponding spectrum was again shown to correspond to that of decoupled harmonic oscillators. This analysis showed that the dilatation operator corresponds to the Hamiltonian of an integrable spin-chain, suggesting that integrability of the theory persisted in the non-planar limit.

These results were extended in [44] to Young diagrams with $p \sim O(1)$ rows or columns where each row or column length differed by $O(N)$ boxes. The action of the dilatation operator on Young diagrams R with more than two rows or columns produces a multiplicity

of subduced irreducible representations (r,s) of $S_n \times S_m$. A new version of the Schur-Weyl duality puts states of the $U(p)$ irreducible representation s in one-to-one correspondence with the copies of the irreducible representations (r,s). Using this new technology, the action of the one-loop dilatation operator was evaluated and shown to have a natural connection to the Gauss Law constraint for branes with compact worldvolumes. The action of the dilatation operator on the restricted Schur polynomials splits into a piece that acts on the fields Z and a piece that acts on the "impurity" fields Y. For small values of m, the dilatation operator acting on the Y fields is diagonalized by operators satisfying a version of this Gauss Law. In pictures, this Gauss Law corresponds to a collection of p giant graviton branes with m strings attached. These pictures are Gauss graphs and the corresponding operators, that diagonalize the action of the dilatation acting on the Y fields, are the Gauss graph operators. An explicit construction of Gauss graph operators was later given in [45] where these operators were shown to diagonalize the part of the dilatation operator acting on Y fields for arbitrary values of m. The remaining part of the dilatation operator action is diagonalized by considering the system as a collection of p coupled oscillators that reduces to a system of decoupled harmonic oscillators. Thereafter, extensions to other sectors comprising additional fields [46, 47] show that diagonalizing the dilatation operator again reduces to diagonalizing a set of decoupled oscillators. Similar results were found in different sectors and at higher loop orders [46, 48–50], and eventually to all loop orders in [51]. This strongly suggests that integrability in $\mathcal{N}=4$ SYM theory persists in the large N and non-planar limit.

2.5 Summary and Discussion

In this chapter I have introduced the AdS/CFT correspondence and discussed the key features of a gauge theory and its dual gravity theory. This duality is a promising formulation for non-perturbative quantum gravity. It is motivated by a precise matching of symmetries of the CFT and the dual gravity theory, and by the matching of operators in the CFT with states in the gravity theory in a specific low energy limit, where brane and bulk fields decouple.

While there exists a dictionary that translates quantities from one side of the correspondence into quantities on the other side, it is important to note that this correspondence remains a conjecture and a direct proof of this correspondence is still lacking.

This chapter introduces the powerful Schur polynomial technology that also provides a natural language to compute correlators in the CFT and to describe objects in the dual gravity theory. The subsequent chapters study the excitations of giant graviton bound states in the context of the gauge theory/gravity correspondence using the restricted Schur polynomial technology. The next chapter shows how the dilatation operator acting on the excited two giant systems is diagonalized and that its spectrum is computed exactly in the large N limit.

Chapter 3

Nonplanar Integrability

In this chapter we study operators with a dimension $\Delta \sim O(N)$ and find a simple analytic expressions for the action of the dilatation operator. We study restricted Schur polynomials corresponding to two distinct classes of operators: operators labeled by Young diagrams with two long rows or two long columns. The main complication in working with restricted Schur polynomials is in building a projector from a given S_{n+m} irreducible representation to an $S_n \times S_m$ irreducible representation (both specified by the labels of the restricted Schur polynomial). I give an explicit construction of these projectors by reducing the matter to the simple problem of addition of angular momentum in ordinary non-relativistic quantum mechanics. Diagonalizing the dilatation operator reduces to a problem of solving three-term recursion relations. The fact that the recursion relations have only three-terms is a direct consequence of the weak mixing at one loop of the restricted Schur polynomials. These recursion relations can be solved in terms of symmetric Kravchuk polynomials or Clebsch-Gordan coefficients. This proves that the dilatation operator reduces to a decoupled set of harmonic oscillators and therefore it is integrable.

3.1 Introduction

Ultraviolet divergences inevitably appear in perturbative calculations of various correlation functions in the $\mathcal{N} = 4$ super Yang-Mills theory. These divergences require renormalization,

which induces a mixing among operators with the same bare conformal dimension. The eigenvalues of the new eigenstates under conformal rescalings are a sum of the classical scaling dimension and a loop correction, the *anomalous dimension*. By studying the diagrams which contribute to the renormalization of these operators it is possible to obtain an expression for the dilatation operator. The eigenvalues of this operator are the anomalous dimensions. A key discovery has been the understanding that the dilatation operator can be identified with the Hamiltonian of an integrable spin-chain [19]. This integrability has been found in the planar limit of the theory.

Integrability has proved to be a key ingredient towards finding the exact spectrum of composite operators in $\mathcal{N} = 4$ super Yang-Mills theory. Further, direct perturbative calculations become very cumbersome at high loop orders but can be avoided, by assuming that the observed integrability persists to all loop orders. The dilatation operator can then be determined using general arguments. Clearly, unraveling the integrable structures in the gauge theory is an important problem. In the AdS/CFT correspondence [1–3], these planar results are directly relevant to the problem of determining the exact spectrum of strings in the $AdS_5 \times S^5$ background.

Summing the planar diagrams gives a valid description of the large N limit of those operators in the theory with a bare dimension Δ such that $\Delta^2/N \ll 1$. To correctly construct the large N limit for operators in the theory with an even larger dimension, one must include contributions from non-planar diagrams. There are many good reasons to study operators with a large bare dimension. Operators with a dimension of $O(N)$ are dual to giant gravitons [13, 33, 52], while operators with a dimension of $O\left(N^2\right)$ are dual to new background geometries [13, 14, 52–64]. New methods sparked by the pioneering works [13], employing group representation techniques, probe the theory beyond the planar limit. It is now known how to build a basis for these operators, with the very nice feature that the basis diagonalizes the two-point function [13, 30–32, 40, 65–72]. Techniques to compute correlation functions of these large operators have been developed [59–61, 73] and some investigation into loop corrections have been carried out [30, 31, 41, 42, 69, 74–76]. Formulas for the one loop dilatation operator have been found and these results show that mixing

at one loop is highly constrained. However, much more work is needed. In particular, although formulas for the one loop dilatation operator are known, they are difficult to evaluate explicitly. Ultimately, we would like both explicit formulas for the action of the dilatation operator as well as its eigenvectors and eigenvalues.

Special cases of the problem in the $SU(2)$ sector have been studied numerically in [41, 42]. The computations are rather involved and the expressions for the dilatation operator are complicated. However, upon solving for the spectrum of the dilatation operator numerically, one finds a strikingly simple result: the dilatation operator is equivalent to a sum of harmonic oscillators. Given this simplicity, one might hope that a general analytic approach is possible. In this chapter we study operators with a dimension $\Delta \sim O(N)$ and show that simple analytic expressions for the action of the dilatation operator can be found. Using this construction of the dilatation operator for this class of operators, we can test whether or not it is integrable. I shall argue that the dilatation operator reduces to a decoupled set of harmonic oscillators and hence that it is indeed integrable.

The operators considered here are restricted Schur polynomials built using n complex matrices Z and m complex matrices Y. They are labeled by three Young diagrams. The first Young diagram specifies a representation of the symmetric group S_{n+m} and the second two a representation of $S_n \times S_m$. I consider two distinct classes of operators: those labeled by Young diagrams with two long rows or two long columns. Previous experience with the half-BPS case [13] suggests that these operators are dual to sphere giants (long columns) or AdS giants (long rows). When we say the Young diagrams have two rows (or columns) we mean that all three Young diagram labels have two rows (or columns). By saying the rows (or columns) are large we mean that both of the rows (or columns) of the first Young diagram label contain $O(N)$ boxes.

The main complication in working with restricted Schur polynomials is in building a projector from a given S_{n+m} irreducible representation to an $S_n \times S_m$ irreducible representation (both specified by the labels of the restricted Schur polynomial). One of the main technical advances of this chapter is an explicit and simple construction of these projectors. I shall now outline the logic of this construction. The projectors organize

(partially) labeled Young diagrams into irreducible representations of S_m. Our first step entails showing that these partially labeled Young diagrams can be traded for a (reducible) polynomial representation of the symmetric group. The advantage of the polynomial representation is that it admits the action of an operator d introduced by Dunkl in his study of intertwining functions [77, 78]. This operator is then used to construct a Casimir $d^\dagger d$ whose eigenspaces of definite eigenvalue are precisely the irreducible representations we want. Thus, we are able to substitute the problem of constructing projectors with the eigenproblem of $d^\dagger d$. We then define one more map, which maps the original partially labeled Young diagrams into states of a spin-chain. When acting on the spin-chain, $d^\dagger d$ has a particularly simple form and its eigenproblem is easily solved explicitly. Indeed, we can reduce the problem of computing projectors to the problem of adding angular momentum in ordinary non-relativistic quantum mechanics! This allows us to give rather explicit and simple formulas for the action of the dilatation operator. The problem of diagonalizing the dilatation operator then reduces to a problem solving three-term recursion relations. The fact that the recursion relations have only three-terms is a direct consequence of the weak mixing of the restricted Schur polynomials at one loop. At this point we find that the dilatation operator is very closely related to certain discrete models for the harmonic oscillator [79] and as a consequence the recursion relations can be solved exactly in terms of symmetric Kravchuk polynomials or in terms of Clebsch-Gordan coefficients. If we consider the limit in which the recursion relation can be replaced by a differential equation, which is the large N limit that we have taken, we find a direct connection to the usual harmonic oscillator. In this way we claim that when acting on the class of operators belonging to the $SU(2)$ sector of $\mathcal{N}=4$ super Yang-Mills theory and having a dimension $\Delta \sim O(N)$ the dilatation operator is integrable.

One of the lessons learned from the study of the half-BPS sector of $\mathcal{N}=4$ super Yang-Mills theory is that sphere giants and AdS giants are not independent solutions, but instead are dual to each other. Indeed, the half-BPS states are in one-to-one correspondence with Young diagrams of $SU(N)$. A given Young diagram may be regarded as labeling either a collection of as many sphere giant gravitons as there are columns, or a collection of as

many AdS giant gravitons as there are rows. This correspondence can also be transparently seen from the half-BPS geometries with a white annulus [14]. For example, a single AdS giant of angular momentum k may equally well be thought of as collection of k sphere giant gravitons, each with unit angular momentum. A similar equivalence works for the full spectrum of 1/8-BPS states: the partition function obtained by quantizing the sphere giant graviton is the same as the partition function obtained by quantizing the AdS giant graviton [80, 81], see also [82]. This computation of the dilatation operator at one loop uncovers an incredibly simple relation between the action of the dilatation operator on sphere giants and its action on AdS giants. This result suggests that the duality discovered in the BPS sector may be enlarged to the non-BPS sector of the theory.

In the next section we study the problem in the context of the two sphere giant system. The new method used to construct the symmetric group projectors is most easily developed for AdS giants. Setting-up the problem and constructing the projectors is accomplished in Section 3.3 and Section 3.4, respectively. The explicit expression for the action of the dilatation operator on AdS giants is the last formula obtained in Section 3.4 and Section 3.5 explains the construction for sphere giants and uncovers a very simple relation between the action of the dilatation operator on sphere giants and its action on AdS giants. In Section 3.6, I consider the problem of diagonalizing the dilatation operator and obtain the relevant discrete wave equations. Finally, I shall briefly discuss interesting features of these results in Section 3.7.

3.2 Two Sphere Giant Gravitons

In this section, I shall first review the action of the dilatation operator on a system of two sphere giant gravitons. The two sphere giant graviton system is described by restricted Schur polynomials labeled by Young diagrams that have two long columns. The expression for the dilatation operator has been evaluated explicitly in some special cases; a general expression is not known. This action simplifies considerably in a particular limit that is described in detail. Given this simplicity, the general construction of the dilatation operator

directly in this limit, is considered. I shall then argue that the simplicity of this limit can be exploited when constructing the dilatation operator by employing Young's orthogonal representation for the symmetric group. We are then able to obtain a general expression for the dilatation operator acting on the two sphere giant system by giving a general explicit construction of the projection operators appearing in the definition of the restricted Schur polynomials.

3.2.1 Operators dual to a Two Sphere Giant System

There are six scalar fields ϕ_i in $\mathcal{N}=4$ super Yang Mills theory, which take values in the adjoint of $u(N)$. From these we construct three complex combinations

$$Z=\phi_1+i\phi_2, \qquad Y=\phi_3+i\phi_4 \qquad \text{and} \qquad X=\phi_5+i\phi_6.$$

I shall study operators comprising $O(N)$ fields Z and $O(N)$ fields Y. We always use n to denote the number of Zs, m to denote the number of Ys. I shall refer to the Y fields as "impurities". These operators belong to the $SU(2)$ sector of the theory and have a large $\mathscr{R}$-charge. As a consequence of this large $\mathscr{R}$-charge, non-planar contributions to the correlation functions of these operators are not suppressed at large N. At large $\mathscr{R}$-change, operator mixing is completely unconstrained in the trace basis. This is true even for the free theory. As such, the computation of the anomalous dimensions is a problem of considerable complexity. New methods, that employ group representation theory, allows one to sum all diagrams including planar and non-planar contributions to correlators. The new methods provide bases for the local operators which diagonalize the free two-point function. The mixing of these operators is highly constrained at the quantum level. These methods provide superior bases for our purposes. We employ the basis provided by the restricted Schur polynomials,

$$\chi_{R,(r,s)}\left(Z^{\otimes n},Y^{\otimes m}\right)=\frac{1}{n!\,m!}\sum_{\sigma\in S_{n+m}}\mathrm{Tr}_{(r,s)}(\Gamma_R(\sigma))Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}}Y^{i_{n+1}}_{i_{\sigma(n+1)}}\cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}}$$

where R, r and s are Young diagrams with $n+m$, n and m boxes, labeling irreducible representations of S_{n+m}, S_n and S_m, respectively. As discussed in Chapter 2, the S_n subgroup

permutes indices among the Zs, while the S_m subgroup permutes indices among to the Ys. $\mathrm{Tr}_{(r,s)}$ traces over the subspace carrying the irreducible representation [1] (r,s) of $S_n \times S_m$ inside the carrier space for irreducible representation R of S_{n+m}. This trace is conveniently determined using a projector $P_{R\to(r,s)}$ to give $\mathrm{Tr}\big(P_{R\to(r,s)}\Gamma_R(\sigma)\big)$, which we write as $\mathrm{Tr}_{(r,s)}(\Gamma_R(\sigma))$.

We are interested in operators dual to sphere giant gravitons that wrap a 3-sphere in the S^5 of the $\mathrm{AdS}_5 \times S^5$ background. The two sphere giant system is described by restricted Schur polynomials labeled by Young diagrams with at most two columns. Further, the number of Zs in the operator is αN where $0 < 2-\alpha \equiv \zeta < 1$ and the number of Ys is fixed to be $O(N)$. I have chosen the number of Zs so that both columns of the Young diagram are long, that is, they both always have $O(N)$ boxes. I shall explain the choice for the number of Ys below. It is reasonable to assume that these restricted Schur polynomials are all excitations (including BPS states and non-BPS states) of the two giant system.

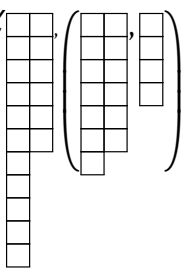
The mixing of these operators with restricted Schur polynomials that have $n \neq 2$ columns is suppressed by, at least, a factor of order $1/\sqrt{N}$ [2]. This factor arises from the normalization of the restricted Schur polynomials. For example, the two-point correlation function of three column restricted Schur polynomials is smaller than that of the two column restricted Schur polynomials by a factor of order $1/N$ [41]. As such, the two column restricted Schur polynomials do not mix with other operators at large N. The two column restricted Schur polynomials form a decoupled sector in this limit. This is a enormous simplification and is the analog of the statement that for operators with a dimension of $O(1)$, different trace structures do not mix in the limit $N \to \infty$. This is expected since these operators correspond to well defined and stable semi-classical objects in spacetime - the two-giant systems. So, for large N and $b_0 \sim O(N)$, the two giant system is decoupled from the three giant system. Similarly, the n column restricted Schur polynomials form a decoupled sector at large N, for the same reason.

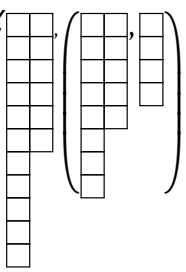
[1] In the two row and two column Young diagrams cases discussed here, each representation (r,s) is subduced exactly once. In general, however, (r,s) is subduced more than once and we need an additional label to track this multiplicity.

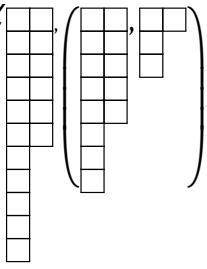
[2] This mixing occurs at the quantum level. There is no mixing in the free theory. If there are a boxes in the second column of the Young diagram, then operator mixing is suppressed by $1/\sqrt{a}$. It is for this reason that we consider Young diagrams where the second column has $O(N)$ boxes.

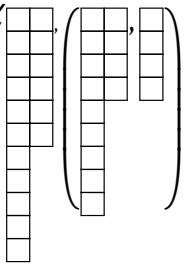
I shall end this section with some comments on notation. The m impurity operators are built using n Zs and m Ys. There are three Young diagrams labeling the restricted Schur polynomial, $\chi_{R,(r,s)}$. The Young diagram R specifies an irreducible representation of S_{n+m}. The Young diagram r (specifying an irreducible representation of S_n) is determined by the number of rows with two boxes b_0 and the number of rows with a single box b_1 while the Young diagram s (specifying an irreducible representation of S_m) with at most two columns, can be specified by stating which boxes in R are to be removed to obtain r.

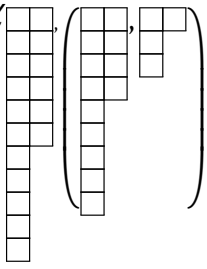
Once R is given, there is a finite number of possible labels $R,(r,s)$, even in the limit $N \to \infty$. To illustrate the problem, it is useful to first study examples with small values for m of order 1. In the next subsection we will be interested in the case $m = 4$. In this case there are nine operators that can be produced once R is given. These operators are given below

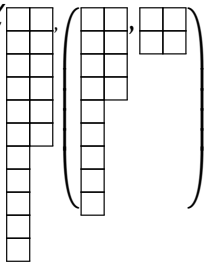
$\chi_A(b_0,b_1) = \chi$  (Z,Y)

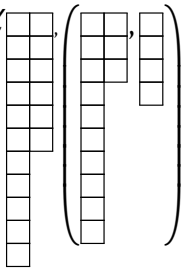
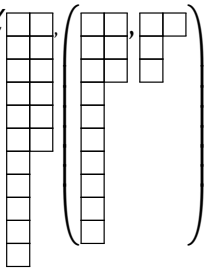
$\chi_B(b_0,b_1) = \chi$  (Z,Y)

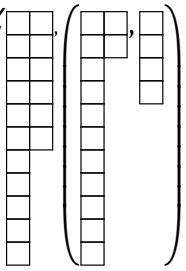
$\chi_C(b_0,b_1) = \chi$  (Z,Y)

$\chi_D(b_0,b_1) = \chi$  (Z,Y)

$\chi_E(b_0,b_1) = \chi$  (Z,Y)

$\chi_F(b_0,b_1) = \chi$  (Z,Y)

$\chi_G(b_0,b_1) = \chi$  (Z,Y) $\chi_H(b_0,b_1) = \chi$  (Z,Y)

$\chi_I(b_0,b_1) = \chi$  (Z,Y)

For the operators we study the number of rows with two boxes in the first and second Young diagram labels scale as N. Thus, the length of the columns in the first two labels are taken to be very large at large N. The third label is exactly as shown for the $m = 4$ case even at large N. The number of boxes removed from each column of the first label to obtain the second label are exactly as shown, even at large N. In what follows we will study the case for which m is of order N. Thus, in general, the column lengths of all the three Young diagrams are of order N so that they are very long and thus are not shown in the figure.

3.2.2 Simplified Action of the Dilatation Operator

The action of the one loop dilatation operator in the $SU(2)$ sector [28]

$$D = -g_{\rm YM}^2 \,\mathrm{Tr}[Y,Z][\partial_Y,\partial_Z]$$

on the restricted Schur polynomial has been studied in [41]. We will find it convenient to work with operators normalized to give a unit two-point function. Towards this end, note that the restricted Schur polynomial two-point correlation function was computed in [67]

$$\left\langle \chi_{R,(r,s)}(Z,Y)\chi_{T,(t,u)}(Z,Y)^\dagger \right\rangle = \delta_{R,(r,s)\,T,(t,u)} f_R \frac{\mathrm{hooks}_R}{\mathrm{hooks}_r\,\mathrm{hooks}_s},$$

where f_R and hooks_R are the product of weights and the product of hook lengths of the Young diagram R, respectively. Using this result, the normalized operators $O_{R,(r,s)}(Z,Y)$ can

be obtained from

$$\chi_{R,(r,s)}(Z,Y)=\sqrt{f_R\frac{\text{hooks}_R}{\text{hooks}_r\,\text{hooks}_s}}\;O_{R,(r,s)}(Z,Y).$$

An important intermediate result is

$$D\chi_{R,(r,s)}\big(Z^{\otimes n},Y^{\otimes m}\big)=\frac{g_{\rm YM}^2}{(n-1)!\,(m-1)!}\sum_{\psi\in S_{n+m}}\text{Tr}_{(r,s)}\big(\Gamma_R((n,n+1)\psi-\psi(n,n+1))\big)$$
$$\times Z^{i_1}_{i_{\psi(1)}}\cdots Z^{i_{n-1}}_{i_{\psi(n-1)}}(YZ-ZY)^{i_n}_{i_{\psi(n)}}\delta^{i_{n+1}}_{i_{\psi(n+1)}}Y^{i_{n+2}}_{i_{\psi(n+2)}}\cdots Y^{i_{n+m}}_{i_{\psi(n+m)}}. \quad (3.1)$$

Notice that one of the indices associated to a Z field and one of the indices associated to a Y field participates. This is reflected in the formula below by the appearance of the group element $\Gamma_R(n,n+1)$ which does not belong to the $S_n\times S_m$ subgroup. In terms of these normalized operators

$$D\,O_{R,(r,s)}(Z,Y)=\sum_{T,(t,u)}N_{R,(r,s);T,(t,u)}O_{T,(t,u)}(Z,Y)$$

and

$$N_{R,(r,s);T,(t,u)}=-g_{\rm YM}^2\sum_{R'}\frac{c_{RR'}d_T\,n\,m}{d_{R'}d_t\,d_u(n+m)}\sqrt{\frac{f_T\,\text{hooks}_T\,\text{hooks}_r\,\text{hooks}_s}{f_R\,\text{hooks}_R\,\text{hooks}_t\,\text{hooks}_u}}$$
$$\times\text{Tr}\Big(\big[\Gamma_R((n,n+1)),P_{R\to(r,s)}\big]I_{R'T'}\big[\Gamma_T((n,n+1)),P_{T\to(t,u)}\big]I_{T'R'}\Big). \quad (3.2)$$

The $c_{RR'}$ is the weight of the box removed from Young diagram R to obtain diagram R', and similarly T' is a Young diagram obtained from T by removing one box. The intertwiner I_{AB} is a map from the carrier space of irreducible representation A to the carrier space of irreducible representation B. Consequently, the Young diagrams A and B must have the same shape. The intertwiner operator relevant for this study will be evaluated in Section 3.2.4 and Section 3.3.3.

This last expression has been evaluated in [41] for the case of two impurities and in [42] for the case of three or four impurities. The results are rather complicated. However, the dynamics simplifies considerably in the limit that $N-b_0=O(N)$, $b_0=O(N)$ and $b_1=O(N)$. For $m=4$ impurities, the action of the dilatation operator is

$$D\,O_A(b_0,b_1)=g_{\rm YM}^2\Delta O_A(b_0,b_1)\times O\left(\frac{1}{b_1}\right)$$

$$
\begin{aligned}
D\,O_B(b_0,b_1) &= -\frac{3}{2}g^2_{\rm YM}\Delta O_B(b_0,b_1)+\frac{\sqrt{3}}{2}g^2_{\rm YM}\Delta O_C(b_0,b_1)\\
D\,O_C(b_0,b_1) &= \frac{\sqrt{3}}{2}g^2_{\rm YM}\Delta O_B(b_0,b_1)-\frac{1}{2}g^2_{\rm YM}\Delta O_C(b_0,b_1)\\
D\,O_D(b_0,b_1) &= -2g^2_{\rm YM}\Delta O_D(b_0,b_1)+\frac{2}{\sqrt{3}}g^2_{\rm YM}\Delta O_E(b_0,b_1)\\
D\,O_E(b_0,b_1) &= -2g^2_{\rm YM}\Delta O_E(b_0,b_1)+\frac{2}{\sqrt{3}}\Delta O_D(b_0,b_1)+\frac{2\sqrt{6}}{3}\Delta O_F(b_0,b_1)\\
D\,O_F(b_0,b_1) &= -2\Delta O_F(b_0,b_1)+\frac{2\sqrt{6}}{3}\Delta O_E(b_0,b_1)\\
D\,O_G(b_0,b_1) &= -\frac{3}{2}g^2_{\rm YM}\Delta O_G(b_0,b_1)+\frac{\sqrt{3}}{2}g^2_{\rm YM}\Delta O_H(b_0,b_1)\\
D\,O_H(b_0,b_1) &= -\frac{1}{2}g^2_{\rm YM}\Delta O_H(b_0,b_1)+\frac{\sqrt{3}}{2}g^2_{\rm YM}\Delta O_G(b_0,b_1)\\
D\,O_I(b_0,b_1) &= g^2_{\rm YM}\Delta O_I(b_0,b_1)\times O\left(\frac{1}{b_1}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta O_X(b_0,b_1) = \sqrt{(N-b_0-b_1)(N-b_0)}\;&\big(O_X(b_0+1,b_1-2)+O_X(b_0-1,b_1+2)\\
&-(2N-2b_0-b_1)O_X(b_0,b_1).
\end{aligned}
$$

with $X=A,B,\ldots,I$. A slightly different limit was considered in [42] - there it was assumed that $b_1\sim O(\sqrt{N})$. Since b_1 sets the difference in angular momentum of the two giants and the angular momentum of the two giants sets their radii [36], this limit set the distance between the two giants to be string size. This is the natural limit to look for (open) stringy excitations of the giants. As we will see, the limit we consider here will allow us to reproduce spectra obtained by numerically diagonalizing the dilatation operator. We can easily identify combinations of operators that are annihilated by D. Apart from $O_A(b_0,b_1)$ and $O_I(b_0,b_1)$ we have $O_B(b_0,b_1)+\sqrt{3}\;O_C(b_0,b_1)$, $O_D(b_0,b_1)+\sqrt{3}\;O_E(b_0,b_1)+\sqrt{2}\;O_F(b_0,b_1)$ and $O_G(b_0,b_1)+\sqrt{3}\;O_H(b_0,b_1)$. If we set

$$
\begin{aligned}
O_{B-C}(b_0,b_1) &\equiv \sqrt{3}\;O_B(b_0,b_1)-O_C(b_0,b_1)\\
O_{D-F}(b_0,b_1) &\equiv \sqrt{2}\;O_D(b_0,b_1)-O_F(b_0,b_1)\\
O_{DF-E}(b_0,b_1) &\equiv O_D(b_0,b_1)-\sqrt{3}\;O_E(b_0,b_1)+\sqrt{2}\;O_F(b_0,b_1)\\
O_{G-H}(b_0,b_1) &\equiv \sqrt{3}\;O_G(b_0,b_1)-O_H(b_0,b_1)
\end{aligned}
$$

then we have

$$D\,O_{B-C}(b_0,b_1) = -2g_{\rm YM}^2 \Delta O_{B-C}(b_0,b_1)$$
$$D\,O_{D-F}(b_0,b_1) = -2g_{\rm YM}^2 \Delta O_{D-F}(b_0,b_1)$$
$$D\,O_{DF-E}(b_0,b_1) = -4g_{\rm YM}^2 \Delta O_{DF-E}(b_0,b_1)$$
$$D\,O_{G-H}(b_0,b_1) = -2g_{\rm YM}^2 \Delta O_{G-H}(b_0,b_1).$$

An exact diagonalization of the dilatation operator gives the spectrum of nine harmonic oscillators. Five of the oscillators have a level spacing $\omega = 0$, three have a level spacing $\omega = 8g_{\rm YM}^2$ and one has a level spacing $\omega = 16g_{\rm YM}^2$. These results can all be obtained from the simplified action of the dilatation operator given above. The generic form of the equation coming from the action of the dilatation operator is

$$D\,O(b_0,b_1) = -\alpha g_{\rm YM}^2 \Big[\sqrt{(N-b_0)(N-b_0-b_1)}\ (O(b_0+1,b_1-2)+O(b_0-1,b_1+2)) \\ -(2N-2b_0-b_1)O(b_0,b_1)\Big] \quad (3.3)$$

which corresponds to an oscillator of level spacing $4\alpha g_{\rm YM}^2$. The derivation of this oscillator level spacing is in Appendix A.2. The goal of this chapter is to derive these simplified equations directly in the $b_1 \sim O(N)$ limit and then to diagonalize them analytically. The main difficulty in deriving these equations is in the explicit construction of the projectors used in defining the restricted Schur polynomials.

3.2.3 Young's Orthogonal Representation

There is a representation for the symmetric group, Young's orthogonal representation, that we will make extensive use of. To define the representation, we will give the rule which determines matrix elements of the matrices representing adjacent permutations, that is, permutations of the form $(i, i+1)$, where i denotes the i-th box. The matrix representing any other element of the group can easily be constructed as some product of the matrices representing the adjacent permutations. I shall use the Young-Yamonouchi basis in which the boxes in the Young diagram are numbered. Thus, for the two Young diagrams shown

below, the diagram on the right corresponds to a valid state; the one on the left does not

$$\begin{array}{|c|c|}\hline 1&4\\\hline 3&2\\\hline 5\\\cline{1-1}\end{array}\qquad\qquad\begin{array}{|c|c|}\hline 5&4\\\hline 3&2\\\hline 1\\\cline{1-1}\end{array}.$$

The dimension of the S_5 irreducible representation labeled by $\begin{array}{|c|c|}\hline&\\\hline&\\\hline\\\cline{1-1}\end{array}$ is 5. The reader can check that there are five possible labels each giving a state in the carrier space of this irreducible representation. Recall that to each labeled box we can associate a weight. Denote the weight of the box labeled i by c_i. If box i is in row b and column a, it has $c_i = N + a - b$. We can now state the rule for the action of adjacent transpositions: If $(i, i+1)$ acts on a given state, it gives back the same state with coefficient $1/(c_i - c_{i+1})$ and it gives the state corresponding to the Young diagram with i and $i+1$ swapped with coefficient [3] $\sqrt{1-1/(c_i-c_{i+1})^2}$. Here are a few examples

$$\Gamma_{\begin{array}{|c|c|}\hline&\\\hline&\\\hline\\\cline{1-1}\end{array}}((12))\left|\begin{array}{|c|c|}\hline 5&3\\\hline 4&2\\\hline 1\\\cline{1-1}\end{array}\right\rangle=-\frac{1}{2}\left|\begin{array}{|c|c|}\hline 5&3\\\hline 4&2\\\hline 1\\\cline{1-1}\end{array}\right\rangle+\frac{\sqrt{3}}{2}\left|\begin{array}{|c|c|}\hline 5&3\\\hline 4&1\\\hline 2\\\cline{1-1}\end{array}\right\rangle$$

$$\Gamma_{\begin{array}{|c|c|}\hline&\\\hline&\\\hline\\\cline{1-1}\end{array}}((12))\left|\begin{array}{|c|c|}\hline 5&3\\\hline 4&1\\\hline 2\\\cline{1-1}\end{array}\right\rangle=\frac{1}{2}\left|\begin{array}{|c|c|}\hline 5&3\\\hline 4&1\\\hline 2\\\cline{1-1}\end{array}\right\rangle+\frac{\sqrt{3}}{2}\left|\begin{array}{|c|c|}\hline 5&3\\\hline 4&2\\\hline 1\\\cline{1-1}\end{array}\right\rangle$$

$$\Gamma_{\begin{array}{|c|c|}\hline&\\\hline&\\\hline\\\cline{1-1}\end{array}}((12))\left|\begin{array}{|c|c|}\hline 5&2\\\hline 4&1\\\hline 3\\\cline{1-1}\end{array}\right\rangle=-\left|\begin{array}{|c|c|}\hline 5&2\\\hline 4&1\\\hline 3\\\cline{1-1}\end{array}\right\rangle.$$

Young's orthogonal representation is particularly useful because it simplifies dramatically when [4] $m/b_1 \ll 1$. Since the generic operator we consider has $b_1 \sim O(N)$, we will consider $m = \gamma N$ impurities with $\gamma \sim O\left(N^0\right) \ll 1$. In this simplification, we can consider the ratio $1 \ll \alpha/\gamma \sim O\left(N^0\right)$ is fixed and large, which does not need to scale as N to some positive power. Indeed, if the boxes i and $i+1$ are in the same column, i must sit below $i+1$ so that

$$\Gamma_R((i, i+1))|\text{same column state}\rangle = -|\text{same column state}\rangle \tag{3.4}$$

If i and $i+1$ are in different columns then $c_i - c_{i+1}$ must be $O(N)$ when $b_1 \sim O(N)$, even in the case where all the impurities are stacked in the longer column. In this case, $1/(c_i - c_{i+1}) =$

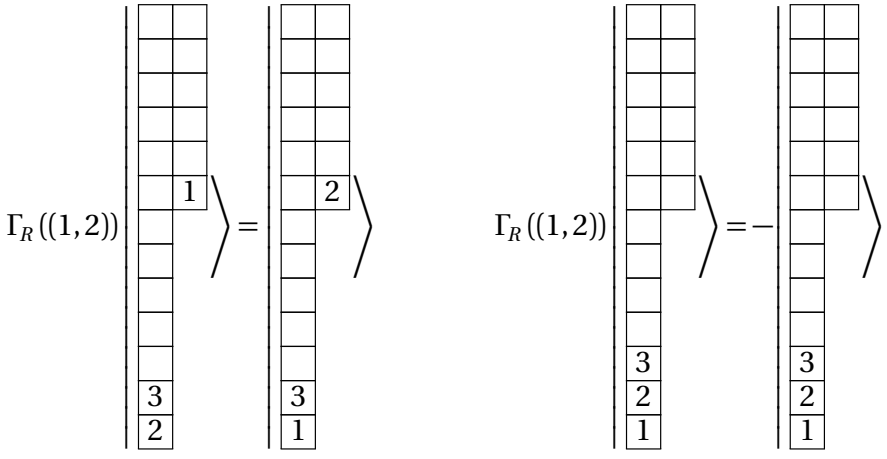
[3] Here to define the representation of S_n we have used the weights c_i which depend on N. The S_n group should know nothing about N. Since only differences between weights appear, this is indeed the case.

[4] Recall that m is the number of Ys.

$O(b_1^{-1})$ which tends to zero, and $\sqrt{1-1/(c_i-c_{i+1})^2} = 1-O(b_1^{-1})$ tends to 1 when N is large. Then,

$$\Gamma_R((i,i+1))|\text{different column state}\rangle = |\text{swapped different column state}\rangle\,. \tag{3.5}$$

The notation in this last equation indicates that i and $i+1$ are in different columns and that the states on either sides of the equation differ by swapping the labels i and $i+1$. All of the states which enter the trace in the restricted Schur polynomial (and hence in the dilatation operator) belong to a particular $S_n \times S_m$ subspace. As I shall now explain, it is possible to directly extract part of this subspace and hence one need not work with the general state in R. Start numbering our states using a Young-Yamonouchi basis. We can obtain r by removing boxes from R. If we only number the first m boxes and further only boxes that are to be removed are numbered, then each partially labeled Young diagram represents a collection of states, all belonging to the correct S_n subspace. Thus, all we need to do now is to take the correct combinations of these states (i.e. of the partially labeled Young diagrams) to get the required S_m subspace. The group element $\Gamma(n,n+1)$ appears in the expression of the dilatation operator, and it acts on one of the m boxes associated to the impurities and one box associated to the Zs. Taken together, these facts imply that all of the operators that we need to consider only have a non-trivial action on the first $m+1$ boxes. Thus we will not label all of the boxes, it is good enough to label the first $m+1$ boxes. Bear in mind that each partially labeled Young diagram corresponds to a collection of states. An example illustrating these rules: (assuming just $m=2$ impurities so that we only label three boxes)



Clearly, there is a dramatic simplification of the representation of the symmetric group in this limit.

3.2.4 Computation of the Dilatation Operator

The dilatation operator includes the coefficient

$$-g_{\rm YM}^2 \frac{c_{RR'}\, d_T\, d_{R^{m+1}}\, nm}{d_{R'}\, d_t\, d_u (n+m)} \sqrt{\frac{f_T\, \text{hooks}_T\, \text{hooks}_r\, \text{hooks}_s}{f_R\, \text{hooks}_R\, \text{hooks}_t\, \text{hooks}_u}}\ .$$

In the above formula $d_{R^{m+1}}$ labels an irreducible representation of S_{n-1} and is obtained from R (which labels an irreducible representation of S_{n+m}) by removing $m+1$ boxes. The dimension factor $d_{R^{m+1}}$ is included in the above coefficient for convenience - it naturally appears when we evaluate the trace

$$\text{Tr}\Big(\big[\Gamma_R((n,n+1)), P_{R\to(r,s)}\big] I_{R'T'} \big[\Gamma_T((n,n+1)), P_{T\to(t,u)}\big] I_{T'R'}\Big). \tag{3.6}$$

There are different choices for the specific R^{m+1}s but, in the limit that we consider, they all give the same contribution. In this limit,

$$\frac{\text{hooks}_T\, \text{hooks}_r}{\text{hooks}_R\, \text{hooks}_t} = 1, \quad \frac{d_T}{d_t} = 2^m, \quad \frac{n}{n+m} = \frac{\alpha}{\alpha+\gamma} \to 1, \quad \frac{d_{R^{m+1}}}{d_{R'}} = \frac{1}{2^m} \quad \text{and} \quad d_u = \frac{m!}{\text{hooks}_u}$$

so that

$$-g_{\rm YM}^2 \frac{c_{RR'}\, d_T\, d_{R^{m+1}}\, nm}{d_{R'}\, d_t\, d_u (n+m)} \sqrt{\frac{f_T\, \text{hooks}_T\, \text{hooks}_r\, \text{hooks}_s}{f_R\, \text{hooks}_R\, \text{hooks}_t\, \text{hooks}_u}} = -\frac{g_{\rm YM}^2 \sqrt{c_{RR'}\, c_{TT'}}\ \sqrt{\text{hooks}_s\, \text{hooks}_u}}{(m-1)!}.$$

All that remains in the evaluation of D is the computation of the trace (3.6). Recall that, because we have only labeled $m+1$ boxes, when we talk about a "state" we actually mean a collection of states. All of the operators we specify below have the same action on the collection of states considered here. It is important to note that, given that we are discussing collections of states, I shall frequently omit an identity matrix factor acting on the collection of states. I shall suppress factors of the identity matrix in all subsequent formulas. It is straight forward to verify that, for m impurities, there are four possible $2^{m+1} \times 2^{m+1}$ intertwiner matrices whose non-zero elements are given by [5]

$$(I_1)_{ij} = 1 \qquad \text{for} \qquad i = j = 1, 2, 3, \ldots, 2^m$$

[5] This next formula nicely illustrates our conventions. The indices i and j do not select a unique state - they select a collection of states. Thus, there is an identity matrix that acts on the collection of states which is suppressed on the right hand side of this equation.

(this intertwiner is used when evaluating terms in $N_{R,(r,s);T,(t,u)}$ for which $R=T$ and the box is removed from the first column of R to obtain R')

$$(I_2)_{ij}=1 \qquad \text{for} \qquad i=j=2^m+1,2^m+2,2^m+3,\ldots,2^{m+1}$$

(again $R=T$ but now the box is removed from the second column of R to obtain R') for the terms with no changes in R and

$$(I_3)_{ij}=1 \qquad \text{for} \qquad j=i+2^m=2^m+1,2^m+2,2^m+3,\ldots,2^{m+1}$$

(now $R\neq T$ and the box is removed from the first column in R to obtain R' and from the second column in T to obtain T')

$$I_4=I_3^{\top}$$

(again $R\neq T$ and the box is removed from first column in T to obtain T' and from the second column in R to obtain R'). The non-zero elements of $\Gamma((n,n+1))$ are

$$\begin{aligned}
&\Gamma((n,n+1))_{ij}=-1, && i=j=1+2p, && p=0,1,\ldots,2^{m-1}-1,\\
&\Gamma((n,n+1))_{ij}=-1, && i=j=2^m+2+2p, && p=0,1,\ldots,2^{m-1}-1,\\
&\Gamma((n,n+1))_{ij}=(-1)^{m-1}, && i=2+2p, && j=2^m+1+2p, && p=0,1,2^m-1,\\
&\Gamma((n,n+1))_{ij}=(-1)^{m-1}, && j=2+2p, && i=2^m+1+2p, && p=0,1,2^m-1.
\end{aligned}$$

Next one needs to compute the projector and then the trace. Without a few new ideas it is not possible to find formulas for general m for the projectors used in constructing restricted Schur polynomials labeled by Young diagrams that have two long columns. These new ideas are most easily developed by studying restricted Schur polynomials labeled by Young diagrams with two long rows. This is the case that we study next.

3.3 Two AdS Giant Gravitons

In this section, I shall first review the action of the dilatation operator on a system of two AdS giant gravitons. This two giant graviton system is described by restricted Schur polynomials labeled by Young diagrams that have two long rows. I again argue that the action of the

dilatation operator simplifies considerably in a particular limit. The general construction of the dilatation operator directly in this limit, is considered. Then, I shall argue that Young's orthogonal representation for the symmetric group in this setting reduces to a representation that is known and has been studied. Using these results I introduce an operator $d^\dagger d$ whose eigenspaces are irreducible representations of the symmetric group. Thus, a general explicit construction of the projection operators appearing in the definition of the restricted Schur polynomials is reduced to solving the eigenvalue problem of $d^\dagger d$. I shall use these results to give an explicit formula for the dilatation operator acting on a general system of two AdS giant gravitons in the next section.

3.3.1 Operators dual to a Two AdS Giant System

I have focused on operators dual to sphere giant gravitons that wrap a 3-sphere in the S^5 of the $AdS_5 \times S^5$ background. In this section, I would like to focus on giant gravitons that are large in the AdS directions but continue to carry angular momentum on the S^5. The two AdS giant system is described by restricted Schur polynomials labeled by Young diagrams with at most two rows. I shall continue to assume that the number of Zs in the operator is αN where $1 < \alpha \sim O\left(N^0\right)$ and the number of Ys is fixed to be γN with $\gamma \sim O\left(N^0\right) \ll 1$. The mixing of these operators with restricted Schur polynomials that have more than two rows, including more general shapes, is again suppressed at least by a factor of order $1/\sqrt{N}$ [6]. The two row restricted Schur polynomials are a decoupled sector at large N. Again, this is expected since these operators correspond to a well defined stable semi-classical object in spacetime (the two AdS giant system).

We only need to make very minor changes in our notation. Each restricted Schur polynomial $\chi_{R,(r,s)}$ is still labeled by three Young diagrams. The representation r (which specifies an irreducible representation of S_n) is specified by stating the number of columns with two boxes b_0 and the number of columns with a single box b_1. The representation s specifies an irreducible representation of S_m that has at most two rows, while R, which specifies an irreducible representation of S_{n+m}, can be specified in terms of r by stating

[6] Again, this is for mixing at the quantum level and there is no mixing in the free theory.

which boxes in R are to be removed to obtain r. Recall that in the sphere giant system, once r has been given there is a finite number of possible labels $R,(r,s)$ even in the limit $N \to \infty$. This is also the case for the AdS giant system. The two AdS giant system and two sphere giant system have an important difference. For the two sphere giant system the parameter b_1 was bounded from above and the parameter b_0 was bounded from below. Indeed, the largest possible value for b_1 and the smallest possible value for b_0 were obtained when the first column contains N boxes. Of course, these bounds on b_0 and b_1 are a consequence of the fact that the number of boxes in any given column is bounded by N. For the AdS giant system, the number of boxes in the first row is not bounded, so that b_0 can range all the way down to 0. For the sphere giant case, the fact that b_0 was bounded from below, and the fact that this bound was $O(N)$, implied a clean decoupling of the two sphere giant system. For the AdS giants, when $b_0 \ll N$ we are transitioning to the state of one giant graviton plus strings, or plus Kaluza-Klein gravitons, or both. The two AdS giant system is still effectively decoupled because it would take a very long time [7] before we decay from the two giant sector. This point does however need to be considered when we study the dilatation operator numerically. In our numerical studies we will only consider Young diagrams R with two rows. This is strictly speaking not always justified because when the second row contains $O(1)$ boxes mixing with Young diagrams with one long row and more than one short row is not suppressed. I shall discuss this point further below (see section 3.7) and argue that it does not affect the accuracy of our results.

3.3.2 Simplified Action of the Dilatation Operator

For the two sphere giant system we have seen that when $1 \ll b_1$ the action of the dilatation operator simplifies dramatically. Is this also the case for the action of the dilatation operator on the two AdS giant system? To compute the action of the dilatation operator we must again evaluate (3.2), but now for the case that R and T have two rows. The result of this computation for the case of two impurities is given in Appendix A.1. There are a number of

[7] What we mean by this is that we need to apply the Hamiltonian $O(N)$ times before we leave the two giant sector.

points worth noting. First, there is a simple relation between the sphere giant and AdS giant results. To obtain the action of the dilatation operator on the AdS giant system from the action of the dilatation operator on the sphere giant system one simply replaces the sphere weights $N-a$ to $N+a$. This is completely explicit if the reader compares the first two expressions in Appendix A.1 to the expressions appearing in [41]. In the last two expressions of Appendix A.1, this is not manifest because I have combined terms. Second, the numerical spectrum for the AdS giant system is surprisingly similar to the spectrum obtained for the sphere giant system [41]. Three quarters of the states are massless while one quarter of the states match those of an oscillator with an energy spacing of $8g^2_{\rm YM}$. This looks identical to what was obtained for the sphere giants. One important difference however, is the fact that for the sphere giant system the levels were of the form $8g^2_{\rm YM}n$ with $n=1,2,3,\ldots$ where as for the AdS giant system the levels are of the form $4g^2_{\rm YM}+8g^2_{\rm YM}n$ with $n=1,2,3,\ldots$. Particularly for the low levels, it seems that these results can be trusted, because the operator with good scale dimension receives no contribution [8] from restricted Schur polynomials with one very long column and one short column. Finally, perhaps the most important result we find is that in the limit that $1\ll b_1/m$ the action of the dilatation operator reduces to a collection of equations of the form

$$D\,O(b_0,b_1)=-\alpha g^2_{\rm YM}\Big[\sqrt{(N+b_0)(N+b_0+b_1)}\;(O(b_0+1,b_1-2)+O(b_0-1,b_1+2)) \\ -(2N+2b_0+b_1)O(b_0,b_1)\Big]. \quad (3.7)$$

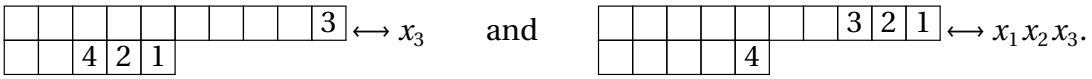
This corresponds to an oscillator of level spacing $4\alpha g^2_{\rm YM}$. More detailed derivations are in Appendix A.2. This result is again related to our previous result for the sphere giant system, by replacing for example $N-b_0\to N+b_0$.

3.3.3 Young's Orthogonal Representation

In the case of AdS giants, Young's orthogonal representation reduces one that has been studied in the mathematics literature [83]. The reader may wish to consult Appendix A.3 where we review a little of the relevant background. The idea is to define a map from a

[8] The contribution is zero to the accuracy of our numerical diagonalization.

labeled Young diagram to a monomial. I shall consider the case of m Y fields. In this case, consider a Young diagram with m labeled boxes [9]; these labels are distributed arbitrarily between the upper and lower rows. Ignore the boxes that appear in the lower row. Include a factor x_i in the monomial for each box in the upper row with label i. A Young diagram with no labeled boxes in the first row maps to 1. Thus, for example, when $m = 4$



$\longleftrightarrow x_3$ and $\longleftrightarrow x_1 x_2 x_3$.

The symmetric group acts by permuting the labels on the monomial factors. For example, $(12)x_1 x_3 = x_2 x_3$. This defines a reducible representation of the symmetric group, S_m. It is clear that the operator

$$d = \sum_{i=1}^{m} \frac{\partial}{\partial x_i}$$

commutes with the action of the symmetric group. Dunkl introduced this operator in the study of intertwining functions [77, 78]. d acts on the monomials by producing the sum of monomials that can be obtained by dropping one factor at a time. For example, $m = 4$ impurities,

$$d\,(x_1 x_2 x_3) = x_2 x_3 + x_1 x_3 + x_1 x_2.$$

The adjoint [10] $d^\dagger$ produces the sum of monomials that can be obtained by appending a factor, without repeating any of the x_is, so

$$d^\dagger (x_1 x_2) = x_1 x_2 x_3 + x_1 x_2 x_4$$

and the generalization to any m is now obvious. The fact that d commutes with all elements of the symmetric group, implies that $d^\dagger$ will too. Indeed, take the dagger of

$$[d, \Gamma(\sigma)] = 0$$

to obtain (use the fact that we are working in an orthogonal representation in the next line)

$$\left[d^\dagger, \Gamma(\sigma)\right] = -\left[d, \Gamma(\sigma^{-1})\right]^\dagger = 0.$$

[9] Up to now we have labeled $m+1$ boxes when considering m impurities. For the present discussion where we want to understand how to decompose into irreducible S_m representations it is more convenient to label only m boxes.

[10] Consult Appendix A.3 for details on the inner product on the space of monomials.

Thus, $d^\dagger d$ commutes with all the elements of the symmetric group. As a consequence, its eigenspaces will furnish irreducible representations of the symmetric group. Additional details and results can be found in [83]. This means that we can substitute the problem of constructing the projectors needed to define the restricted Schur polynomials with the problem of constructing projectors onto eigenspaces of $d^\dagger d$. This amounts to solving for the eigenvectors and eigenvalues of $d^\dagger d$. In the next section I shall argue that this is a surprisingly simple problem.

For the remainder of this section I shall switch back to our previous convention and again label $m+1$ boxes. See Section 3.2.4 for a careful description of our notation. The intertwiners are again $2^{m+1}\times 2^{m+1}$ matrices, and again there are four possible intertwiners. Their non-zero elements are

$$(I_1)_{ij}=1 \qquad \text{for} \qquad i=j=1,2,3,\ldots,2^m$$

($R=T$ and the box is removed from first column)

$$(I_2)_{ij}=1 \qquad \text{for} \qquad i=j=2^m+1,2^m+2,2^m+3,\ldots,2^{m+1}$$

($R=T$ and the box is removed from second column) for the terms with no changes in R and

$$(I_3)_{ij}=1 \qquad \text{for} \qquad j=i+2^m=2^m+1,2^m+2,2^m+3,\ldots,2^{m+1}$$

(the box is removed from first column in R and from the second column in T)

$$I_4={I_3}^\top$$

(the box is removed from first column in T and from second column in R). The non-zero elements of $\Gamma((n,n+1))$ are

$$\begin{aligned}
&\Gamma((n,n+1))_{ij}=1, \quad i=j=1+2p, && p=0,1,\ldots,2^{m-1}-1,\\
&\Gamma((n,n+1))_{ij}=1, \quad i=j=2^m+2+2p, && p=0,1,\ldots,2^{m-1}-1,\\
&\Gamma((n,n+1))_{ij}=1, \quad i=2+2p, && j=2^m+1+2p, \qquad p=0,1,2^m-1,\\
&\Gamma((n,n+1))_{ij}=1, \quad j=2+2p, && i=2^m+1+2p, \qquad p=0,1,2^m-1.
\end{aligned}$$

3.3.4 Computation of the Dilatation Operator

To get to the AdS giant case, we have "flipped" the Young diagrams swapping rows and columns, and mapped the weights $N-b \to N+b$. It is easy to check that the dimension of any Young diagram R is the same as the dimension of the flipped diagram, and (this is the same fact that) the product of hooks is unchanged. The weights $c_{RR'}$ are now for example $N+b_0$ instead of $N-b_0$.

Thus, the coefficient needed to compute the dilatation operator is

$$-g_{\rm YM}^2 \frac{c_{RR'}\, d_T\, d_{R^{m+1}}\, n\, m}{d_{R'}\, d_t\, d_u (n+m)} \sqrt{\frac{f_T\, \text{hooks}_T\, \text{hooks}_r\, \text{hooks}_s}{f_R\, \text{hooks}_R\, \text{hooks}_t\, \text{hooks}_u}} = -\frac{g_{\rm YM}^2 \sqrt{c_{RR'}\, c_{TT'}}\, \sqrt{\text{hooks}_s\, \text{hooks}_u}}{(m-1)!} \tag{3.8}$$

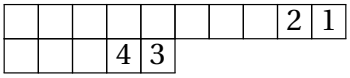
$d_{R^{m+1}}$ again labels an irreducible representation of S_{n-1} and is again obtained from R (which labels an irreducible representation of S_{n+m}) by removing $m+1$ boxes. The dimension factor $d_{R^{m+1}}$ is again included in the above coefficient for convenience - it naturally appears when we evaluate the trace (3.6). All that we need to evaluate now are the traces (3.6). I shall show how to evaluate these for general m in the next section.

3.4 Construction of AdS Giant Projectors and the Dilatation Operator

In this section, I shall define a map from labeled Young diagrams to spin-chain states. The operator $d^\dagger d$ takes a particularly simple form and its eigenvalue problem is solved explicitly. These results then allow a general construction of the projection operators used to define the restricted Schur polynomials and then ultimately of the dilatation operator itself. The last equation in this section gives the final formula for the dilatation operator.

We can map the labeled Young diagrams into states of a spin-chain, for general m. The spin at site i can be in state spin up (1) or state spin down (0). Initially in this section it is again convenient to label only m boxes. In this case, the spin-chain has m sites and the box i tells us the state of site i. If box i appears in the first row, site i is in state 1; if it appears in

the second row site i is in state 0. For example,



$\longleftrightarrow |1100\rangle$.

As I shall now explain, $d^\dagger d$ has a simple action on this spin-chain. Introduce the states

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for the two spins and the operators

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad a^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

which act on these states

$$a^\dagger |0\rangle = |1\rangle \qquad a^\dagger |1\rangle = 0$$

$$a |1\rangle = |0\rangle \qquad a |0\rangle = 0.$$

Consider, for example, the spin-chain state

$$|001011\rangle = |0\rangle \otimes |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle \otimes |1\rangle$$

in a system with six lattice sites. Label the lattice sites, from left to right, starting site 1, then site 2 and so on till the last lattice site 6. The operator a acting at the third site (for example) is

$$a_3 = 1 \otimes 1 \otimes a \otimes 1 \otimes 1 \otimes 1.$$

Then, we can write $d^\dagger d$ as the operator

$$d^\dagger d = \sum_p \sum_n a_n^\dagger a_p \tag{3.9}$$

acting on a long ranged spin-chain. In terms of the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \text{and} \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we can write

$$a = \frac{1}{2}\left(\sigma^1 + i\sigma^2\right), \qquad a^\dagger = \frac{1}{2}\left(\sigma^1 - i\sigma^2\right), \qquad a^\dagger a = \frac{1}{2}\left(1 - \sigma^3\right),$$

and

$$d^\dagger d = \frac{1}{4}\sum_p\sum_n \left(\sigma^1 - i\sigma^2\right)_n \left(\sigma^1 + i\sigma^2\right)_p = -\frac{1}{2}\sum_p \sigma^3_p + \frac{1}{4}\sum_p\sum_n \left(\sigma^1_n\sigma^1_p + \sigma^2_n\sigma^2_p\right).$$

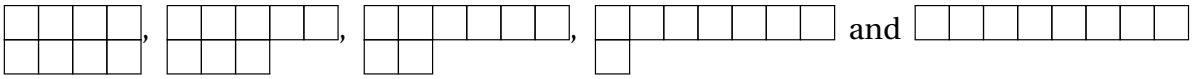
The total spins of the system are

$$J^1 = \frac{1}{2}\sum_p \sigma^1_p, \qquad J^2 = \frac{1}{2}\sum_p \sigma^2_p, \qquad J^3 = \frac{1}{2}\sum_p \sigma^3_p \qquad \text{and} \qquad \mathbf{J}^2 = J^3J^3 + J^1J^1 + J^2J^2.$$

We use upper case letters for operators and lowercase letters for eigenvalues. In terms of these total spins we have

$$d^\dagger d = -J^3 + \left(\mathbf{J}^2 - \left(J^3\right)^2\right) = \mathbf{J}^2 - J^3\left(J^3 + 1\right).$$

Thus, we can label the eigenspaces of $d^\dagger d$ by the eigenvalues of $\mathbf{J}^2$ and J^3. Hence, we can replace the sequences $R, (r,s)$ that label the restricted Schur polynomial with labels (j, j^3). I shall illustrate this connection using a specific example and then state the general rule. Consider the case of eight spins. Then impurities can be organized into any irreducible representation corresponding to a Young diagram with eight blocks and two rows. The possible irreducible representations are



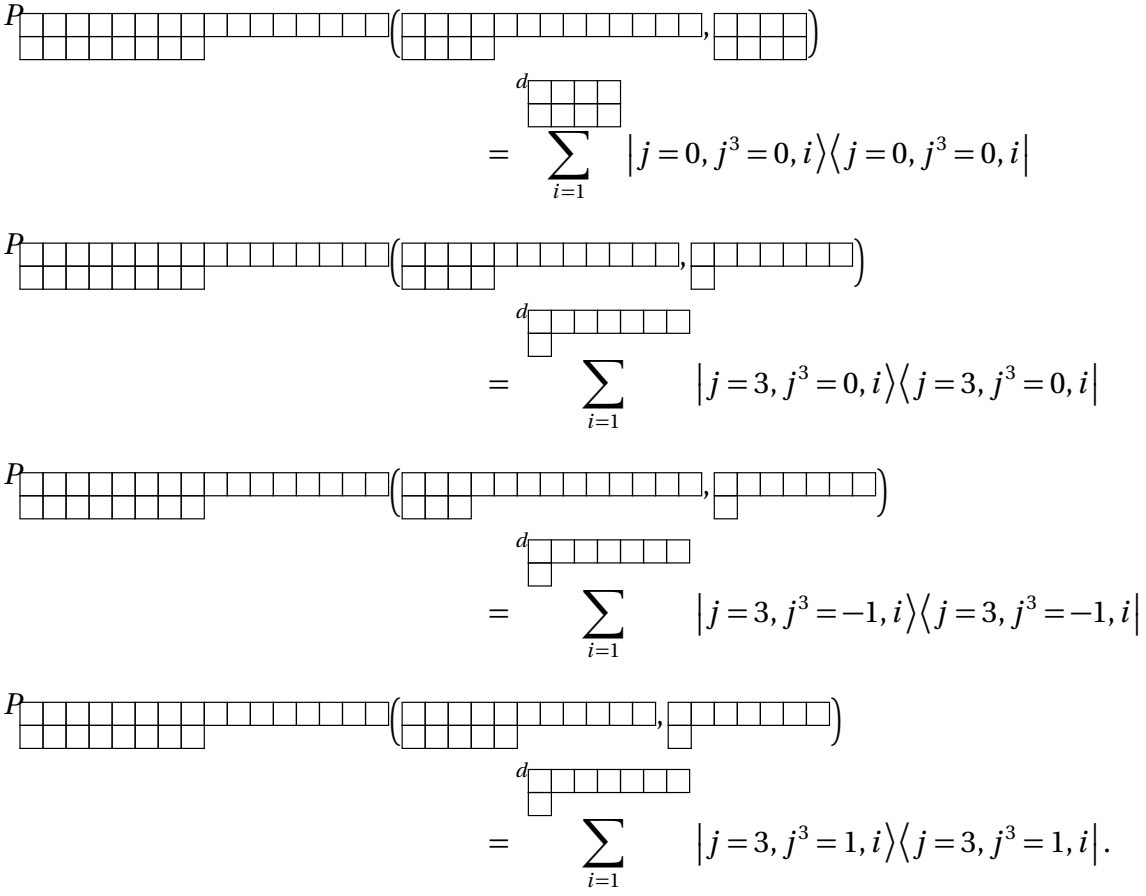
which have dimensions 14, 28, 20, 7 and 1, respectively. Coupling eight spin-$\frac{1}{2}$ particles, using the usual rule for the addition of angular momentum, gives

$$\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = 14\,\mathbf{0} \oplus 28\,\mathbf{1} \oplus 20\,\mathbf{2} \oplus 7\,\mathbf{3} \oplus \mathbf{4}.$$

Notice that the number of angular momentum multiplets matches the number of possible Young diagrams for the impurities and the degeneracy of each multiplet matches the dimension of the S_8 irreducible representation associated to the Young diagram. This is a consequence of the Schur-Weyl duality that connects the representation theory of the symmetric groups and the unitary groups.

The general rule is now clear: consider the restricted Schur polynomial $\chi_{R,(r,s)}$. The $\mathbf{J}^2 = j(j+1)$ quantum determines the shape of the Young diagram s that organizes the

impurities. Suppose s has N_1 boxes in the first row and N_2 boxes in the second row, then $2j = N_1 - N_2$. The J^3 eigenvalue of the state is always a good quantum number, both in the basis we start in where each spin has a sharp angular momentum or in the basis where the states have a sharp total angular momentum. The j^3 quantum number determines how many impurities sit in the first and second rows of R, that is, it specifies the number of boxes that must be removed from each row of R to obtain r. Let n_1 denote the number of boxes removed from the first row and n_2 the number of boxes removed from the second row, then $2j^3 = n_1 - n_2$. Below are some examples of the projection operators appearing in the restricted Schur polynomials, written in terms of the j and j^3 states:



In each case, i labels all the states with quantum numbers (j, j^3), and takes values from 1 up to the dimension of the irreducible representation organizing the impurities. These results are a general construction of the projection operators used to define the restricted Schur polynomials dual to the two AdS giant system.

To evaluate the traces (3.6) we must consider the action of $\Gamma((n,n+1))$. Towards this end, we again need to switch to our previous convention of labeling $m+1$ boxes in the Young diagram. Equivalently, we must add another site to our chain. $\Gamma((n,n+1))$ can then be taken to act on the first and $(m+1)$-th site of the spin-chain [11]. Including this extra site updates the projectors

$$P_{R,(r,s)} = \sum_i \left|j, j^3, i\right\rangle\left\langle j, j^3, i\right| \otimes 1 = \sum_i \left|j, j^3, i\right\rangle\left\langle j, j^3, i\right| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)$$

where the identity above sits in the $(m+1)$-th slot. I shall also use the notation

$$p_{R,(r,s)} = \sum_i \left|j, j^3, i\right\rangle\left\langle j, j^3, i\right|$$

when the $(m+1)$-th slot is not included.

In terms of the spin-chain language, the intertwiners can be written as

$$I_1 = \left(\frac{1}{2} + \frac{1}{2}\sigma_1^3\right), \qquad I_2 = \left(\frac{1}{2} - \frac{1}{2}\sigma_1^3\right), \qquad I_3 = a_1 \qquad \text{and} \qquad I_4 = a_1^\dagger.$$

When $\Gamma((n,n+1))$ acts it does so by swapping the first and $(m+1)$-th spins, so that

$$\begin{aligned}\Gamma((n,n+1))\, I_1 &= |0\rangle\langle 0| \otimes 1 \otimes \cdots \otimes 1 \otimes |0\rangle\langle 0| + |1\rangle\langle 0| \otimes 1 \otimes \cdots \otimes 1 \otimes |0\rangle\langle 1| \\ &= \left(\frac{1}{2} + \frac{1}{2}\sigma_1^3\right)\left(\frac{1}{2} + \frac{1}{2}\sigma_{m+1}^3\right) + a_1^\dagger a_{m+1}.\end{aligned}$$

In exactly the same way

$$\begin{aligned}I_1\Gamma((n,n+1)) &= \left(\frac{1}{2} + \frac{1}{2}\sigma_1^3\right)\left(\frac{1}{2} + \frac{1}{2}\sigma_{m+1}^3\right) + a_1 a_{m+1}^\dagger \\ \Gamma((n,n+1))\, I_1\Gamma((n,n+1)) &= \left(\frac{1}{2} + \frac{1}{2}\sigma_{m+1}^3\right).\end{aligned}$$

One of the terms contributing to the dilatation operator is

$$\mathrm{Tr}\Big(I_1\big[P_{R,(r,s)}, \Gamma((n,n+1))\big]\, I_1^\top\big[P_{R,(r,s)}, \Gamma((n,n+1))\big]\Big).$$

This is a term that does not change the shape of the Young diagram labels of the restricted Schur polynomial. Using the facts that $I_1 = I_1^\top$, $\Gamma((n,n+1))^\top = \Gamma((n,n+1))$ and $P_{R,(r,s)}^\top = P_{R,(r,s)}$ we can rewrite this term as

[11] Allowing $\Gamma((n,n+1))$ to act on any of the first m sites and the $(m+1)$-th site of the spin-chain does not change the final result.

$$\begin{aligned}
&\mathrm{Tr}\Big(I_1 P_{R,(r,s)}\Gamma((n+1,n))\,I_1 P_{R,(r,s)}\Gamma((n+1,n))\Big)-\mathrm{Tr}\Big(I_1 P_{R,(r,s)}\Gamma((n+1,n))\,I_1\Gamma((n+1,n))\,P_{R,(r,s)}\Big)\\
&=\mathrm{Tr}\bigg(P_{R,(r,s)}\bigg[\Big(\frac{1}{2}+\frac{1}{2}\sigma_1^3\Big)\Big(\frac{1}{2}+\frac{1}{2}\sigma_{m+1}^3\Big)+a_1^\dagger a_{m+1}\bigg]P_{R,(r,s)}\bigg[\Big(\frac{1}{2}+\frac{1}{2}\sigma_1^3\Big)\Big(\frac{1}{2}+\frac{1}{2}\sigma_{m+1}^3\Big)+a_1^\dagger a_{m+1}\bigg]\bigg)\\
&\qquad-\mathrm{Tr}\bigg(\Big(\frac{1}{2}+\frac{1}{2}\sigma_1^3\Big)P_{R,(r,s)}\Big(\frac{1}{2}+\frac{1}{2}\sigma_{m+1}^3\Big)P_{R,(r,s)}\bigg).
\end{aligned}$$

Tracing over the m-th slot we easily find

$$\mathrm{Tr}\bigg(P_{R,(r,s)}\Big(\frac{1}{2}+\frac{1}{2}\sigma_1^3\Big)\Big(\frac{1}{2}+\frac{1}{2}\sigma_{m+1}^3\Big)P_{R,(r,s)}a_1^\dagger a_{m+1}\bigg)=\mathrm{Tr}\bigg(P_{R,(r,s)}\Big(\frac{1}{2}+\frac{1}{2}\sigma_1^3\Big)P_{R,(r,s)}a_1^\dagger\bigg)\langle 0|1\rangle=0$$

and

$$\mathrm{Tr}\big(P_{R,(r,s)}a_1^\dagger a_{m+1}P_{R,(r,s)}a_1^\dagger a_{m+1}\big)=\mathrm{Tr}\big(P_{R,(r,s)}a_1^\dagger P_{R,(r,s)}a_1^\dagger\big)\;\mathrm{Tr}(|0\rangle\langle 1|0\rangle\langle 1|)=0.$$

The expression for the term in the dilatation operator that we are considering becomes

$$\begin{aligned}
&\mathrm{Tr}\big(I_1\big[P_{R,(r,s)},\Gamma((n,n+1))\big]I_1^\top\big[P_{R,(r,s)},\Gamma((n,n+1))\big]\big)\\
&\qquad=2\,\mathrm{Tr}\bigg(P_{R,(r,s)}\Big(\frac{1}{2}+\frac{1}{2}\sigma_1^3\Big)\Big(\frac{1}{2}+\frac{1}{2}\sigma_{m+1}^3\Big)P_{R,(r,s)}\Big(\frac{1}{2}+\frac{1}{2}\sigma_1^3\Big)\Big(\frac{1}{2}+\frac{1}{2}\sigma_{m+1}^3\Big)\bigg)\\
&\qquad\qquad-2\,\mathrm{Tr}\bigg(\Big(\frac{1}{2}+\frac{1}{2}\sigma_1^3\Big)P_{R,(r,s)}\Big(\frac{1}{2}+\frac{1}{2}\sigma_{m+1}^3\Big)P_{R,(r,s)}\bigg).
\end{aligned}$$

Expanding these expressions out and making use of the identities

$$\begin{aligned}
\mathrm{Tr}\big(P_{R,(r,s)}P_{R,(r,s)}\big)&=\mathrm{Tr}\big(P_{R,(r,s)}\sigma_{m+1}^3P_{R,(r,s)}\sigma_{m+1}^3\big)\\
\mathrm{Tr}\big(P_{R,(r,s)}P_{R,(r,s)}\sigma_1^3\big)&=\mathrm{Tr}\big(P_{R,(r,s)}\sigma_{m+1}^3P_{R,(r,s)}\sigma_1^3\sigma_{m+1}^3\big)\\
\mathrm{Tr}\big(P_{R,(r,s)}\sigma_1^3P_{R,(r,s)}\sigma_1^3\big)&=\mathrm{Tr}\big(P_{R,(r,s)}\sigma_1^3\sigma_{m+1}^3P_{R,(r,s)}\sigma_1^3\sigma_{m+1}^3\big)\\
\mathrm{Tr}\big(P_{R,(r,s)}P_{R,(r,s)}\sigma_{m+1}^3\big)&=\mathrm{Tr}\big(P_{R,(r,s)}\sigma_1^3P_{R,(r,s)}\sigma_{m+1}^3\big)=0\\
\mathrm{Tr}\big(P_{R,(r,s)}P_{R,(r,s)}\sigma_1^3\sigma_{m+1}^3\big)&=\mathrm{Tr}\big(P_{R,(r,s)}\sigma_1^3P_{R,(r,s)}\sigma_1^3\sigma_{m+1}^3\big)=0
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathrm{Tr}\big(I_1\big[P_{R,(r,s)},\Gamma((n+1,n))\big]I_1^\top\big[P_{R,(r,s)},\Gamma((n+1,n))\big]\big)&=\frac{1}{4}\,\mathrm{Tr}\big(P_{R,(r,s)}\sigma_1^3P_{R,(r,s)}\sigma_1^3\big)-\frac{1}{4}\,\mathrm{Tr}\big(P_{R,(r,s)}\big)\\
&=\mathrm{Tr}\big(P_{R,(r,s)}I_1P_{R,(r,s)}I_1\big)-\mathrm{Tr}\big(P_{R,(r,s)}I_1\big).
\end{aligned}$$

Using exactly the same types of arguments we find

$$\mathrm{Tr}\big(I_1\big[P_{R,(r,s)},\Gamma((n+1,n))\big]I_1^\top\big[P_{S,(t,u)},\Gamma((n+1,n))\big]\big)=\mathrm{Tr}\big(P_{R,(r,s)}I_1P_{S,(t,u)}I_1\big)$$

$$-\mathrm{Tr}\big(P_{R,(r,s)}I_1\big)\,\delta_{R,(r,s)\,S,(t,u)}$$

$$\mathrm{Tr}\Big(I_2\big[P_{R,(r,s)},\Gamma((n+1,n))\big]\,I_2^{\top}\big[P_{S,(t,u)},\Gamma((n+1,n))\big]\Big)=\mathrm{Tr}\big(P_{R,(r,s)}I_2P_{S,(t,u)}I_2\big)$$

$$-\mathrm{Tr}\big(P_{R,(r,s)}I_2\big)\,\delta_{R,(r,s)\,S,(t,u)}$$

$$\mathrm{Tr}\Big(I_3\big[P_{R,(r,s)},\Gamma((n+1,n))\big]\,I_4^{\top}\big[P_{S,(t,u)},\Gamma((n+1,n))\big]\Big)=\mathrm{Tr}\big(P_{R,(r,s)}I_2P_{S,(t,u)}I_1\big)$$

$$\mathrm{Tr}\Big(I_4\big[P_{R,(r,s)},\Gamma((n+1,n))\big]\,I_3^{\top}\big[P_{S,(t,u)},\Gamma((n+1,n))\big]\Big)=\mathrm{Tr}\big(P_{R,(r,s)}I_1P_{S,(t,u)}I_2\big).$$

To evaluate these traces write each projector as (i is a multiplicity label that runs from 1 to the number of times the irreducible $SU(2)$ representation j_1 appears)

$$p_1=\sum_i\big|j_1,j_1^3,i\big\rangle\big\langle j_1,j_1^3,i\big|. \tag{3.10}$$

Recall that we use a little letter p for the projectors when the $(m+1)$-th site is not included. In this last expression the subscript 1 on the projector stands for $R,(r,s)$ and is a notation which simplifies the equations dramatically. Introduce the following states (the first state on the RHS is the ket describing particle 1 in the lattice; the second ket is a good total angular momentum state obtained by coupling the states of the remaining $m-1$ particles):

$$\begin{aligned}
\big|\phi_1,i\big\rangle&=\Big|\tfrac{1}{2},\ \tfrac{1}{2};j_1-\tfrac{1}{2},j_1^3-\tfrac{1}{2},i\Big\rangle\Big\langle\tfrac{1}{2},\ \tfrac{1}{2};j_1-\tfrac{1}{2},j_1^3-\tfrac{1}{2},i\Big|j_1,j_1^3\Big\rangle\\
\big|\phi_2,i\big\rangle&=\Big|\tfrac{1}{2},\ \tfrac{1}{2};j_1+\tfrac{1}{2},j_1^3-\tfrac{1}{2},i\Big\rangle\Big\langle\tfrac{1}{2},\ \tfrac{1}{2};j_1+\tfrac{1}{2},j_1^3-\tfrac{1}{2},i\Big|j_1,j_1^3\Big\rangle\\
\big|\phi_3,i\big\rangle&=\Big|\tfrac{1}{2},-\tfrac{1}{2};j_1-\tfrac{1}{2},j_1^3+\tfrac{1}{2},i\Big\rangle\Big\langle\tfrac{1}{2},-\tfrac{1}{2};j_1-\tfrac{1}{2},j_1^3+\tfrac{1}{2},i\Big|j_1,j_1^3\Big\rangle\\
\big|\phi_4,i\big\rangle&=\Big|\tfrac{1}{2},-\tfrac{1}{2};j_1+\tfrac{1}{2},j_1^3+\tfrac{1}{2},i\Big\rangle\Big\langle\tfrac{1}{2},-\tfrac{1}{2};j_1+\tfrac{1}{2},j_1^3+\tfrac{1}{2},i\Big|j_1,j_1^3\Big\rangle.
\end{aligned}$$

Using these states the projector can be written as

$$p_1=\sum_i\Big(\big|\phi_1,i\big\rangle+\big|\phi_3,i\big\rangle\Big)\Big(\big\langle\phi_1,i\big|+\big\langle\phi_3,i\big|\Big)+\sum_i\Big(\big|\phi_2,i\big\rangle+\big|\phi_4,i\big\rangle\Big)\Big(\big\langle\phi_2,i\big|+\big\langle\phi_4,i\big|\Big).$$

These states are particularly convenient because the action of the intertwiners I_1 and I_2 on these states is very simple

$$\begin{aligned}
&I_1\big|\phi_1,i\big\rangle=\big|\phi_1,i\big\rangle \qquad I_1\big|\phi_2,i\big\rangle=\big|\phi_2,i\big\rangle, \qquad I_1\big|\phi_3,i\big\rangle=0, \qquad I_1\big|\phi_4,i\big\rangle=0,\\
&I_2\big|\phi_1,i\big\rangle=0, \qquad I_2\big|\phi_2,i\big\rangle=0, \qquad I_2\big|\phi_3,i\big\rangle=\big|\phi_3,i\big\rangle \qquad I_2\big|\phi_4,i\big\rangle=\big|\phi_4,i\big\rangle.
\end{aligned}$$

We can now compute the traces we need

$$\begin{aligned}\mathrm{Tr}(P_1 I_1) &= \sum_i \langle \phi_1, i | \phi_1, i \rangle + \sum_i \langle \phi_2, i | \phi_2, i \rangle \\ &= d_{R^{m+1}} \left(d_{s_1'} \left\langle \frac{1}{2}, \frac{1}{2}; j_1 - \frac{1}{2}, j_1^3 - \frac{1}{2} \middle| j_1, j_1^3 \right\rangle^2 + d_{s_2'} \left\langle \frac{1}{2}, \frac{1}{2}; j_1 + \frac{1}{2}, j_1^3 - \frac{1}{2} \middle| j_1, j_1^3 \right\rangle^2 \right).\end{aligned}$$

s_1' is obtained by dropping a box from the first row of s and $d_{s_1'}$ is the dimension of this irreducible representation. Similarly, s_2' is obtained by dropping a box from the second row of s and $d_{s_2'}$ is the dimension of this irreducible representation. I have summed over i to obtain the factors of $d_{s_1'}$ and $d_{s_2'}$. The overlaps $\langle \phi_1, i | \phi_1, i \rangle$ and $\langle \phi_2, i | \phi_2, i \rangle$ are independent of i so that in the last line above we could sum over i and there is no further need for this index. Recall the general expression for the Clebsch-Gordan coefficient

$$\begin{aligned}\left\langle j_1, j_1^3; j_2, j_2^3 \middle| j, j^3 \right\rangle &= \delta_{j^3, j_1^3 + j_2^3} \sqrt{\frac{(2j+1)(j+j_1-j_2)!\,(j-j_1+j_2)!\,(j_1+j_2-j)!}{(j_1+j_2+j+1)!}} \\ &\times \sqrt{(j+j^3)!\,(j-j^3)!\,(j_1-j_1^3)!\,(j_1+j_1^3)!\,(j_2-j_2^3)!\,(j_2+j_2^3)!} \\ &\times \left(\sum_k \frac{(-1)^k}{k!\,(j_1+j_2-j-k)!\,(j_1-j_1^3-k)!\,(j_2+j_2^3-k)!\,(j-j_2+j_1^3+k)!\,(j-j_1-j_2^3+k)!} \right)^2\end{aligned}$$

where the sum runs over all values of k for which the argument of each factorial is non-negative. It is now straight forward to find

$$\left\langle \frac{1}{2}, \frac{1}{2}; j - \frac{1}{2}, j^3 - \frac{1}{2} \middle| j, j^3 \right\rangle = \sqrt{\frac{1+j+j^3}{2(j+1)}} \quad \text{and} \quad \left\langle \frac{1}{2}, \frac{1}{2}; j + \frac{1}{2}, j^3 - \frac{1}{2} \middle| j, j^3 \right\rangle = \sqrt{\frac{j+j^3}{2j}}$$

so that

$$\mathrm{Tr}(P_1 I_1) = 2 \frac{1+j+j^3}{2(j+1)} d_{s_1'}\, d_{R^{m+1}} + 2 \frac{j+j^3}{2j} d_{s_2'}\, d_{R^{m+1}}.$$

where R^{m+1} is a Young diagram of S_{n-1}, obtained from R by removing $m+1$ boxes, and the factor of 2 comes from tracing over the $(m+1)$-th slot. Now, since $I_1 + I_2$ is the identity we find

$$2 d_s\, d_{R^{m+1}} = \mathrm{Tr}(P_1) = \mathrm{Tr}(P_1 I_1) + \mathrm{Tr}(P_1 I_2) = d_s\, d_{R^{m+1}} + \mathrm{Tr}(P_1 I_2)$$

which implies that

$$\mathrm{Tr}(P_1 I_2) = d_s\, d_{R^{m+1}}.$$

Now consider the second type of term

$$\mathrm{Tr}(P_1 I_1 P_1 I_1) = \mathrm{Tr}(I_1 P_1 I_1 P_1 I_1)$$

where we used the fact that the trace is cyclic and I_1 is a projector so that $(I_1)^2 = I_1$. This is a useful observation because

$$I_1 P_1 I_1 P_1 I_1 = \sum_i \Big(|\phi_1, i\rangle\langle\phi_1, i| + |\phi_2, i\rangle\langle\phi_2, i| \Big) \sum_j \Big(|\phi_1, j\rangle\langle\phi_1, j| + |\phi_2, j\rangle\langle\phi_2, j| \Big).$$

It immediately follows that

$$\mathrm{Tr}(P_1 I_1 P_1 I_1) = 2 \left(d_{s_1'} \left(\frac{1+j+j^3}{2(j+1)} \right)^2 + d_{s_2'} \left(\frac{j+j^3}{2j} \right)^2 \right) d_{R^{m+1}}.$$

Again using the fact that I_1 and I_2 sum to the identity, it is clear that

$$\begin{aligned} \mathrm{Tr}(P_1 I_2 P_1 I_1) &= \mathrm{Tr}(P_1 I_1) - \mathrm{Tr}(P_1 I_1 P_1 I_1) \\ &= 2 d_{R^{m+1}} \left(d_{s_1'} \left(\frac{1+j+j^3}{2(j+1)} \right) + d_{s_2'} \left(\frac{j+j^3}{2j} \right) - d_{s_1'} \left(\frac{1+j+j^3}{2(j+1)} \right)^2 - d_{s_2'} \left(\frac{j+j^3}{2j} \right)^2 \right). \end{aligned}$$

Finally,

$$\begin{aligned} d_s d_{R^{m+1}} &= \mathrm{Tr}(P_1) = \mathrm{Tr}(P_1 P_1) = \mathrm{Tr}(P_1 (I_1 + I_2) P_1 (I_1 + I_2)) \\ &= \mathrm{Tr}(P_1 I_1 P_1 I_1) + 2\,\mathrm{Tr}(P_1 I_2 P_1 I_1) + \mathrm{Tr}(P_1 I_2 P_1 I_2). \end{aligned}$$

We can solve this last equation for $\mathrm{Tr}(P_1 I_2 P_1 I_2)$. This complete the evaluation of the terms we were considering.

We now need to consider the case that the two projectors appearing in the trace have different labels. There is only one term to compute because

$$\begin{aligned} \mathrm{Tr}(P_1 I_1 P_2 I_1) &= \mathrm{Tr}(P_1 I_1 P_2 (I_1 + I_2)) - \mathrm{Tr}(P_1 I_1 P_2 I_2) = -\mathrm{Tr}(P_1 I_1 P_2 I_2) \\ &= -\mathrm{Tr}(P_1 (I_1 + I_2) P_2 I_2) + \mathrm{Tr}(P_1 I_2 P_2 I_2) = \mathrm{Tr}(P_1 I_2 P_2 I_2). \end{aligned}$$

The only time we get a non-zero result for this trace is when $j_2 = j_1 \pm 1$. Without any loss of generality, consider the case that $j_2 = j_1 + 1$. The result for $j_2 = j_1 - 1$ is obtained by swapping $1 \longleftrightarrow 2$. We can write

$$p_2 = \sum_i \Big(|\psi_1, i\rangle + |\psi_3, i\rangle \Big) \Big(\langle\psi_1, i| + \langle\psi_3, i| \Big) + \sum_i \Big(|\psi_2, i\rangle + |\psi_4, i\rangle \Big) \Big(\langle\psi_2, i| + \langle\psi_4, i| \Big)$$

where

$$|\psi_1, i\rangle = \left| \frac{1}{2}, \quad \frac{1}{2}; j_1 + \frac{1}{2}, j_1^3 - \frac{1}{2}, i \right\rangle \left\langle \frac{1}{2}, \quad \frac{1}{2}; j_1 + \frac{1}{2}, j_1^3 - \frac{1}{2}, i \right| j_1 + 1, j_1^3 \Big\rangle$$

$$|\psi_2,i\rangle = \left|\frac{1}{2},\quad \frac{1}{2};j_1+\frac{3}{2},j_1^3-\frac{1}{2},i\right\rangle\left\langle\frac{1}{2},\quad \frac{1}{2};j_1+\frac{3}{2},j_1^3-\frac{1}{2},i\right|j_1+1,j_1^3\Big\rangle$$
$$|\psi_3,i\rangle = \left|\frac{1}{2},-\frac{1}{2};j_1+\frac{1}{2},j_1^3+\frac{1}{2},i\right\rangle\left\langle\frac{1}{2},-\frac{1}{2};j_1+\frac{1}{2},j_1^3+\frac{1}{2},i\right|j_1+1,j_1^3\Big\rangle$$
$$|\psi_4,i\rangle = \left|\frac{1}{2},-\frac{1}{2};j_1+\frac{3}{2},j_1^3+\frac{1}{2},i\right\rangle\left\langle\frac{1}{2},-\frac{1}{2};j_1+\frac{3}{2},j_1^3+\frac{1}{2},i\right|j_1+1,j_1^3\Big\rangle.$$

We now find

$$\begin{aligned}\mathrm{Tr}(P_1I_1P_2I_1) &= 2d_{R^{m+1}}\,\mathrm{Tr}\left[\sum_i\Big(|\psi_1,i\rangle\langle\psi_1,i|+|\psi_2,i\rangle\langle\psi_2,i|\Big)\sum_j\Big(|\phi_1,j\rangle\langle\phi_1,j|+|\phi_2,j\rangle\langle\phi_2,j|\Big)\right]\\ &= 2d_{R^{m+1}}\sum_{i,j}\langle\phi_2,j|\psi_1,i\rangle\langle\psi_1,i|\phi_2,j\rangle\\ &= 2d_{R^{m+1}}\,d_{s_c'}\left\langle\frac{1}{2},\frac{1}{2};j_1+\frac{1}{2},j_1^3-\frac{1}{2},i\right|j_1,j_1^3\Big\rangle^2\left\langle\frac{1}{2},\frac{1}{2};j_1+\frac{1}{2},j_1^3-\frac{1}{2},i\right|j_1+1,j_1^3\Big\rangle^2\\ &= 2d_{R^{m+1}}\,d_{s_c'}\left(\frac{j_1+j_1^3+1}{2(j_1+1)}\right)\left(\frac{j_1-j_1^3+1}{2(j_1+1)}\right).\end{aligned}$$

If s_1 is the third label for p_1 and s_2 is the third label of p_2, then by removing a box from s_1 we can get the same Young diagram as when we remove a box from s_2 - in the last line above I have called this Young diagram, which can be reached from either s_1, s_2 by removing a single box, s_c'.

Putting things together we find (when $j=0$ the term $((m+2)(j^3)^2)/(j(j+1))$ in round braces on the first line below and the last term in the equation must be omitted)

$$\begin{aligned}D\,O_{j,j^3}(b_0,b_1) = g_{\mathrm{YM}}^2\Bigg[&-\frac{1}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right)\Delta O_{j,j^3}(b_0,b_1)\\ &+\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\,\Delta O_{j+1,j^3}(b_0,b_1)\\ &+\frac{(j+j^3)(j-j^3)}{2j}\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\,\Delta O_{j-1,j^3}(b_0,b_1)\Bigg]\end{aligned}\tag{3.11}$$

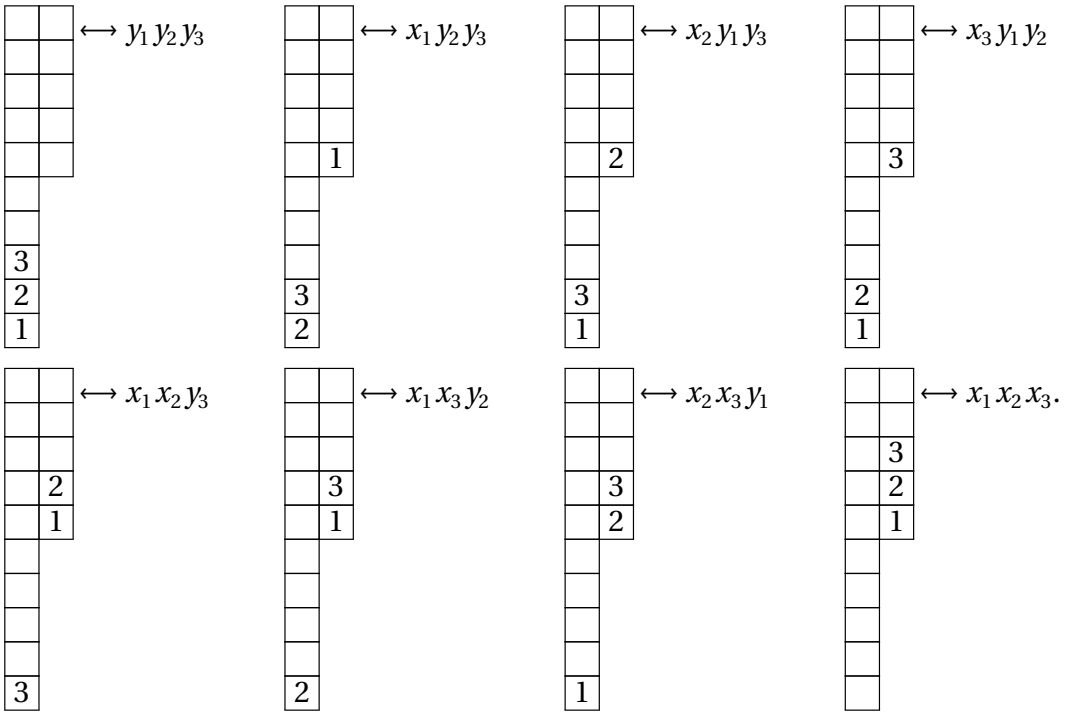
where

$$\begin{aligned}\Delta O(b_0,b_1) = \sqrt{(N+b_0)(N+b_0+b_1)}\,&\big(O(b_0+1,b_1-2)+O(b_0-1,b_1+2)\big)\\ &-(2N+2b_0+b_1)O(b_0,b_1).\end{aligned}\tag{3.12}$$

This concludes our evaluation of the dilatation operator for m impurities.

3.5 Construction of Sphere Giant Projectors and the Dilatation Operator

The starting point of our AdS giant analysis was a polynomial representation which was isomorphic to the Young diagrams. There is a polynomial representation isomorphic to the sphere giants too. For m impurities, introduce the $2m$ variables $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_m$. The x's and y's are Grassmann numbers and all these variables anticommute. The x's and y's are associated to the two columns respectively. Consider three impurities for illustration; in this case the isomorphism is defined by



The generalization to any m is obvious. The polynomials are ordered with (i) x's to the left of the y's and (ii) such that their subscripts (within the x's and y's separately) increase. The group S_n acts on the monomials by exchanging indices without changing the order of the variables.

The natural definition for d is

$$d = \sum_{i=1}^{m} y_i \frac{\partial}{\partial x_i}.$$

It is easy to check that d does indeed commute with the symmetric group. Using the inner product: $\langle y_1 y_2 y_3, y_1 y_2 y_3 \rangle = 1$ (so the inner product of same polynomials is 1 and of different

polynomials is 0) we find

$$d^\dagger = \sum_{i=1}^{m} x_i \frac{\partial}{\partial y_i}.$$

The Casimir we want is

$$d^\dagger d = \sum_i x_i \frac{\partial}{\partial x_i} - \sum_{i,j} x_i y_j \frac{\partial}{\partial y_i}\frac{\partial}{\partial x_j}.$$

This has a nice expression in terms of the following spin model

$$d^\dagger d = \sum_{n,p} (-1)^{n-p} a_n^\dagger a_p = \frac{1}{4}\sum_{p,n} (-1)^{n+p}\left(\sigma^1{}_n \sigma^1{}_p + \sigma^2{}_n \sigma^2{}_p\right) - \frac{1}{2}\sum_p \sigma^3{}_p.$$

If we further define

$$\tilde{\sigma}^i{}_n = \left(\sigma^3{}_n\right)^n \sigma^i{}_n \left(\sigma^3{}_n\right)^n$$

we map this into the AdS giant problem

$$d^\dagger d = \sum_{n,p} (-1)^{n-p} a_n^\dagger a_p = \frac{1}{4}\sum_{p,n}\left(\tilde{\sigma}^1{}_n \tilde{\sigma}^1{}_p + \tilde{\sigma}^2{}_n \tilde{\sigma}^2{}_p\right) - \frac{1}{2}\sum_p \tilde{\sigma}^3{}_p = \tilde{\mathbf{J}}^2 - \tilde{J}^3\left(\tilde{J}^3 + 1\right).$$

To get the sphere giant dilatation operator we will simply need to rewrite the formulas for the intertwiners and $\Gamma((n,n+1))$ in terms of the $\tilde{\sigma}^i$ and then trace. The computations in this case are parallel to those in section 3.4. I_1 and I_2 are unchanged; I_3 and I_4 each pick up a minus sign. Since they always appear together, this change does not affect the answer for the dilatation operator at all. As far as $\Gamma((n,n+1))$, it continues to swap the first and last slots. If m is even, $\Gamma((n,n+1))$ just picks up a minus sign and hence, since $\Gamma((n,n+1))$ always appears twice, the dilatation operator is unaffected. When m is odd only the elements of $\Gamma((n,n+1))$ which are on the diagonal change sign. The intertwiners I_1, I_2 only pick up on the diagonal elements and the intertwiners I_3, I_4 only pick up off the diagonal elements. Thus, we only ever get products of off diagonal elements of $\Gamma((n,n+1))$ with off diagonal elements of $\Gamma((n,n+1))$ or products of on the diagonal elements of $\Gamma((n,n+1))$ with on the diagonal elements of $\Gamma((n,n+1))$. Thus, even for m odd the dilatation operator is unaffected. This proves that

$$\mathrm{Tr}\left(\left[\Gamma_R((n,n+1)), P_{R\to(r,s)}\right] I_{R'T'} \left[\Gamma_T((n,n+1)), P_{T\to(t,u)}\right] I_{T'R'}\right)$$

is the same for the AdS and sphere giant cases and proves our previous observation that to get sphere from AdS we just replace factors like $N + b_0 + b_1 \mapsto N - b_0 - b_1$. We can thus use

exactly the same method as for AdS case. Consider the restricted Schur polynomial $\chi_{R,(r,s)}$ where all Young diagrams have at most two columns. The $\tilde{\mathbf{J}}^2 = j(j+1)$ quantum number again specifies the shape of the Young diagram s that organizes the impurities. Suppose s has N_1 boxes in the first column and N_2 boxes in the second column, then $2j = N_1 - N_2$. The j^3 quantum number again determines how many boxes are removed from the first and second columns of R to produce r. Let n_1 denote the number of boxes removed from the first column and n_2 the number of boxes removed from the second column, then $2j^3 = n_1 - n_2$. The dilatation operator is thus

$$\begin{aligned} D\,O_{j,j^3}(b_0,b_1) = g_{\rm YM}^2\Bigg[&-\frac{1}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right)\Delta O_{j,j^3}(b_0,b_1) \\ &+\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\,\Delta O_{j+1,j^3}(b_0,b_1) \\ &+\frac{(j+j^3)(j-j^3)}{2j}\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\,\Delta O_{j-1,j^3}(b_0,b_1)\Bigg] \end{aligned}$$

where

$$\begin{aligned} \Delta O(b_0,b_1) = \sqrt{(N-b_0-b_1)(N-b_0)}\;&(O(b_0+1,b_1-2)+O(b_0-1,b_1+2)) \\ &-(N-b_0)O(b_0,b_1)-(N-b_0-b_1)O(b_0,b_1). \end{aligned} \quad (3.13)$$

3.6 Diagonalization of the Dilatation Operator

In this section, I introduce a transformation that reduces the dilatation operator to a set of decoupled oscillators. The transformation is constructed by solving a three-term recursion relation. This three-term recursion relation is nothing but the recursion relation of certain Clebsch-Gordan coefficients. This allows the construction of operators with a good scaling dimension in terms of Hahn polynomials. In the limit that the number of impurities is very large, these wave functions become the wave functions of the 2-dimensional radial oscillator.

3.6.1 The $j^3 = 0$ Case and the Corresponding Discrete Wave Equation

For the $j^3 = 0$ case, the action of the dilatation operator is

$$D\,O_{j,0}(b_0,b_1) = g_{\rm YM}^2\left[-\frac{m}{2}\Delta O_{j,0}(b_0,b_1) + \frac{j+1}{2}\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\;\Delta O_{j+1,0}(b_0,b_1)\right.$$
$$\left. + \frac{j}{2}\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\;\Delta O_{j-1,0}(b_0,b_1)\right] \quad (3.14)$$

where

$$\Delta O(b_0,b_1) = \sqrt{(N+b_0)(N+b_0+b_1)}\;\left(O(b_0+1,b_1-2)+O(b_0-1,b_1+2)\right.$$
$$\left. -(2N+2b_0+b_1)O(b_0,b_1).\right.$$

Now make the following ansatz for the operators of good scaling dimension

$$\sum_{b_1} f(b_0,b_1)\,O_p(b_0,b_1) = \sum_{j,b_1} C_p(j)\,f(b_0,b_1)\,O_{j,0}(b_0,b_1) \quad (3.15)$$

and require that

$$-\alpha_p C_p(j) = -\frac{m}{2}C_p(j) + \frac{j+1}{2}\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\;C_p(j+1)$$
$$+\frac{j}{2}\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\;C_p(j-1). \quad (3.16)$$

The dilatation operator changes neither the number of Z fields (n) nor the number of Y fields (m). Since $n = 2b_0 + b_1$ we do not sum over b_0 and b_1 independently in (3.15). Further, b_1 only takes odd or even values. The action of the dilatation operator then reduces to

$$D\,O_p(b_0,b_1) = -\alpha_p g_{\rm YM}^2\left[\sqrt{(N+b_0)(N+b_0+b_1)}\;\left(O_p(b_0+1,b_1-2)+O_p(b_0-1,b_1+2)\right)\right.$$
$$\left. -(2N+2b_0+b_1)O_p(b_0,b_1)\right]. \quad (3.17)$$

Reducing the dilatation operator to a set of decoupled oscillators amounts to determining the coefficients $C_n(j)$ and the values of α_n, by solving the recursion relation (3.16). Now, introduce the Clebsch-Gordan coefficients

$$C^{j_1,j_2,j_T}_{m_1,m_2,m_T} = \left\langle j_T, m_T \middle| j_1, m_1, j_2, m_2 \right\rangle$$

which couples the state with two angular momenta j_1 and j_2 to a state with good total angular momentum j_T. The eigenvalue of the 3-component of angular momentum is denoted in the above using an m. The recursion relation for the Clebsch-Gordan coefficients is

$$\left(2n-\frac{m}{2}\right)C^{\frac{m}{4},\frac{m}{4},j}_{n-\frac{m}{4},\frac{m}{4}-n,0}=\frac{j+1}{2}\sqrt{\frac{(m-2j)(m+2j+4)}{(2j+1)(2j+3)}}\;C^{\frac{m}{4},\frac{m}{4},j+1}_{n-\frac{m}{4},\frac{m}{4}-n,0}$$
$$+\frac{j}{2}\sqrt{\frac{(m-2j+2)(m+2j+2)}{(2j+1)(2j-1)}}\;C^{\frac{m}{4},\frac{m}{4},j-1}_{n-\frac{m}{4},\frac{m}{4}-n,0}.$$

It is now clear that the solution to the recursion relation is $C_p(j)=(-1)^{-j}C^{\frac{m}{4},\frac{m}{4},j}_{p-\frac{m}{4},\frac{m}{4}-p,0}$ with m the number of impurities and

$$-\alpha_p=-2p=0,-2,-4,\ldots,-m\,.$$

From the rules for coupling angular momentum, for the $j^3=0$ case, the range of p is $0\le p\le m/2$. The energy spacing of the oscillators in the $1-2$-directions are $4\alpha_p g^2_{\rm YM}=8pg^2_{\rm YM}$. Thus, in this $j^3=0$ case we obtain $m/2+1$ oscillators with frequencies $0,8g^2_{\rm YM},16g^2_{\rm YM},\ldots,4mg^2_{\rm YM}$.

The "eigenfunctions" $C_p(j)$ are given by Clebsch-Gordan coefficients, or equivalently by (dual) Hahn polynomials, reviewed for example in [79]. They can be written in terms of the hypergeometric function ${}_3F_2$ as

$$C_p(j)=(-1)^{\frac{m}{2}-p}\left(\frac{m}{2}\right)!\sqrt{\frac{2j+1}{\left(\frac{m}{2}-j\right)!\left(\frac{m}{2}+j+1\right)!}}\;{}_3F_2\left(\begin{matrix}-j,\;j+1,\;-p\\ -\frac{m}{2},\;1\end{matrix};1\right)$$

where the range of j and p are $0\le j\le m/2$ and $0\le p\le m/2$, respectively. The lowest energy eigenfunction corresponds to the BPS states. I have fixed the number of Ys to be $m=\gamma N$ with $\gamma\ll 1$ and $b_1=O(N)$ so that

$$\frac{m}{b_1}\sim\gamma\ll 1.$$

This is the condition needed to ensure the dramatic simplifications of Young's orthogonal representation. We can think in terms of a double scaling limit $m\to\infty$ and $b_1\sim N\to\infty$ keeping γ small and fixed. In this limit ${}_3F_2\left(\begin{smallmatrix}-j,\;j+1,\;-p\\ -m/2,\;1\end{smallmatrix};1\right)\to L_p\left(2j^2/m\right)$ with $L_p(\cdot)$ the Laguerre polynomial, so that

$$C_p(j)\to(-1)^{\frac{m}{2}-p}\sqrt{\frac{2}{m}}\;\sqrt{2j+1}\;e^{-\frac{j^2}{m}}L_p\left(\frac{2j^2}{m}\right)\qquad\text{with}\qquad 0\le j\le\frac{m}{2}.$$

These coefficients become the wave function of the 2-dimensional radial Harmonic oscillator for the s-wave, i.e. without a centripetal force

$$\frac{1}{2}\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+r^2\right]C_p(r)=(2p+1)C_p(r) \tag{3.18}$$

where $r=j\sqrt{2/m}$ takes values in $0\leq r\leq\sqrt{m/2}$ and $C_p(r)=C_p(j)/\sqrt{r}$ in the continuum limit. This is a radial oscillator in two dimensions (the $3-4$-directions) with the energy levels labeled by integer p. What is the dual interpretation of these wave functions? To answer this we need to know how r maps into the spacetime description of the dual physics. Since r is a scaled version of j, different values of the r coordinate correspond to different Young diagrams organizing the impurities. These Young diagrams all have two rows. Based on our experience with the half-BPS case, it seems natural to associate each row to one of the giant gravitons. We will simply assume that this is the correct interpretation; this point certainly deserves further study. Recalling that $Y=\phi_3+i\phi_4$ we know that the number of Ys in each operator tells us the angular momentum of the operator in the $3-4$-plane. Denote the two angular momenta of the two giant gravitons J_1^Y and J_2^Y. The relations between j,m,j_1^Y and J_2^Y are

$$j_1^Y+J_2^Y=m,\qquad j_1^Y-J_2^Y=2j,\qquad j_1^Y=\frac{m}{2}+j\qquad\text{and}\qquad J_2^Y=\frac{m}{2}-j.$$

j (and hence r) is directly proportional to the difference in angular momenta of the two states. Giving angular momentum to a graviton causes it to expand as a consequence of [36]. The separation between the two gravitons in the $3-4$-plane will (for separations small compared to the radii of the giants) be directly proportional to the difference in angular momenta of the two giants. It is, therefore, natural to interpret r as a coordinate for the radial separation between the two giants in the $3-4$-plane. How large is this separation? Recall that the radius of a giant with angular momentum J, in units of the AdS scale R is $R\sqrt{J/N}$. The maximum value of the difference between the angular of the giants, j, is bounded by m so that the length between them is always very small, $\sim R\sqrt{\gamma}$. The separation between the giants in $3-4$-direction is thus seen to be small in AdS scale units for these states, due to the γ factor, but it can be large in string scale units. In units of the string length the separation

between the two giants is

$$j\sqrt{\frac{2}{m}}\left(\frac{g_{\rm YM}^2}{N}\right)^{1/4}\ell_s$$

where in the above we used the approximation $\sqrt{m/2+j}-\sqrt{m/2-j}\to j\sqrt{2/m}$. This separation can be an $O(1)$ or very large length. Thus, we propose that r is a coordinate for the radial separation between the two giants in the $3-4$-plane, and the separation naturally ranges from the string length up to small (but non-zero) distances in units of the AdS scale. One still needs to solve the eigenvalue problem of (3.17) that will determine a "wave function" for the $1-2$-plane and we see that $Z=\phi_1+i\phi_2$. Thus, our operators are described by a wave function in four dimensional space. It is rather natural to interpret this space as the 4-dimensional Kähler base appearing in the construction of the 1/4-BPS geometries in [64, 84, 84–90]. We see very concretely the emergence of local physics on the 4-dimensional space from the system of Young diagrams labeling the restricted Schur polynomial. This is strongly reminiscent of the half-BPS case where the Schur polynomials provide wave functions for fermions in a harmonic oscillator potential. Additionally, these wave functions naturally reproduce features of the geometries and the phase space in [14] (for a review see for example [91]).

In the case of the two matrix model we study here it is not true that the matrices Z and Y commute. For this reason, we cannot simultaneously diagonalize these matrices and there is no analog of the eigenvalue basis that is useful for the large N dynamics of single matrix models. For the subsystem describing the BPS states of the Z,Y system however [82] has deduced that we can indeed assume that Z and Y commute in the interacting theory and hence these systems should have a description in terms of eigenvalues. Further, the eigenvalue dynamics is again supposed to be dynamics in an oscillator potential with repulsions preventing the collision of eigenvalues. We have described a part of the BPS sector (as well as non-BPS operators) among the operators we have studied. In the case of a single matrix it is possible to associate the rows of the Young diagram labeling a Schur polynomial with the eigenvalues of the matrix. This connects the eigenvalue description and the Schur polynomial description for single matrix models. Are the oscillators we find

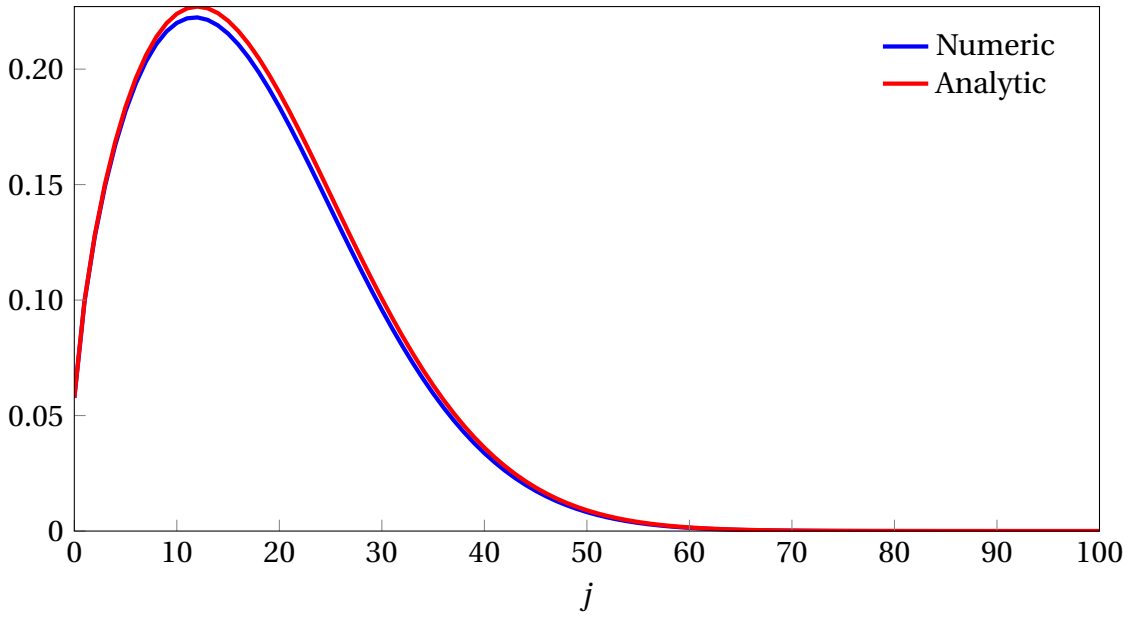


Figure 3.1: A comparison between (3.19) (red) and the numerically computed eigenfunction for $m = 600$ (blue). The agreement is clearly excellent.

here a signal of simple underlying eigenvalue dynamics? Is there a connection between the Young diagram labels and eigenvalues?

I shall now study in detail the state with $p = 0$ so that $\alpha_0 = 0$, i.e. a BPS state. This is the ground state of the wave equation (3.18). In this case, since ${}_3F_2\left(\begin{matrix}-j,\ j+1,\ 0\\ -m/2,\ 1\end{matrix};1\right) = 1$,

$$C_0(j) = (-1)^{\frac{m}{2}} \left(\frac{m}{2}\right)! \sqrt{\frac{2j+1}{\left(\frac{m}{2}-j\right)!\left(\frac{m}{2}+j+1\right)!}} \qquad \text{for} \qquad 0 \le j \le \frac{m}{2}.$$

If we now consider the large m regime,

$$C_0(j) \to (-1)^{\frac{m}{2}} \left(\frac{m}{2}\right)^{-\frac{1}{2}} \sqrt{2j+1}\; e^{-\frac{j^2}{m}} \qquad \text{for} \qquad 1 \ll m \qquad \text{and} \qquad 0 \le j \ll m. \tag{3.19}$$

This formula agrees beautifully with the numerically obtained eigenfunctions - see Figure 3.1 for a comparison.

Another interesting eigenfunction to consider is

$$C_{\frac{m}{2}}(j) = \left(\frac{m}{2}\right)! \sqrt{\frac{2j+1}{\left(\frac{m}{2}-j\right)!\left(\frac{m}{2}+j+1\right)!}}\; {}_2F_1\left(\begin{matrix}-j,\ j+1\\ 1\end{matrix};1\right).$$

Using the identity

$${}_2F_1\left(\begin{matrix}j,\ j+1\\ 1\end{matrix};\frac{1}{2}(1-z)\right) = P_j(z)$$

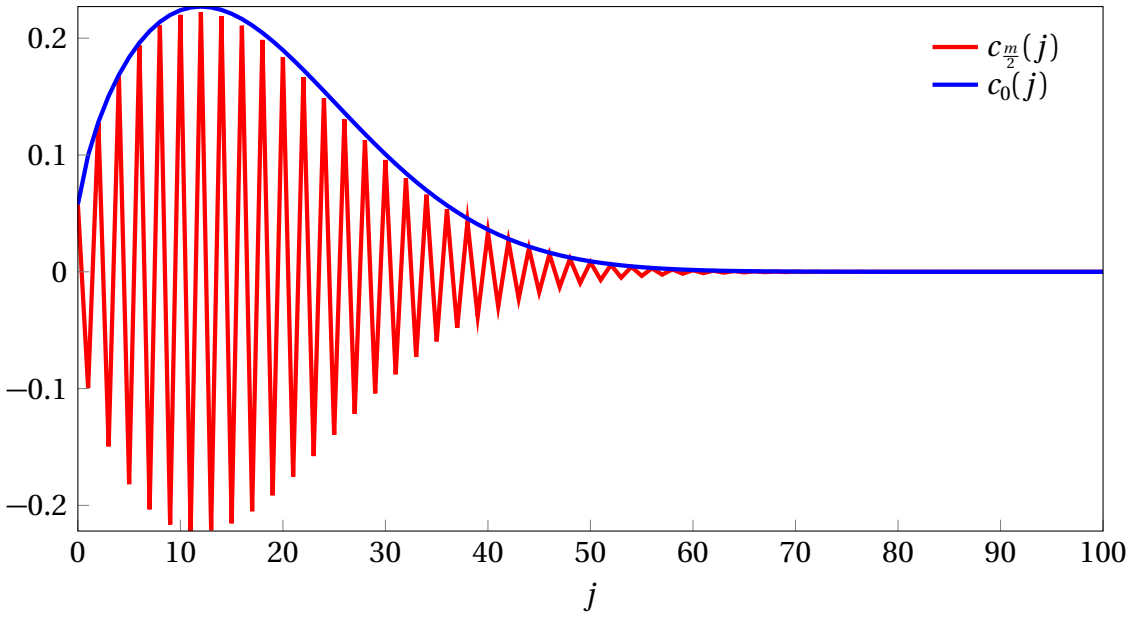


Figure 3.2: A comparison between the numerically computed $C_0(j)$ (blue) and $C_{m/2}(j)$ (red) for $m = 600$. The two solutions are clearly identical up to the alternating signs in $C_{m/2}(j)$, exactly as predicted by the analytic expressions for the eigenfunctions.

where $P_j(z)$ is a Legendre polynomial and $P_j(-1) = (-1)^j$, we find

$$C_{\frac{m}{2}}(j) = (-1)^j \left(\frac{m}{2}\right)! \sqrt{\frac{2j+1}{\left(\frac{m}{2}-j\right)!\left(\frac{m}{2}+j+1\right)!}} \qquad \text{for} \qquad 0 \le j \le \frac{m}{2}.$$

If we now consider the large m regime,

$$C_{\frac{m}{2}}(j) \to (-1)^j \left(\frac{m}{2}\right)^{-\frac{1}{2}} \sqrt{2j+1}\, e^{-\frac{j^2}{m}} \quad \text{for} \quad 1 \ll m \quad \text{and} \quad 0 \le j \ll m. \tag{3.20}$$

This eigenfunction produces an oscillator with a level spacing of $4mg_{\text{YM}}^2$. It is interesting to compare (3.19) and (3.20): there are alternating minus signs, i.e. $(-1)^j$ in (3.20) for superposing j-states, while there are same signs in (3.19) for superposing j-states; these different phases produce vastly different eigenvalues for the two states. This again agrees very well with the numerically computed eigenfunctions. I have compared the numerically computed $C_0(j)$ and $C_{m/2}(j)$ in Figure 3.2.

3.6.2 The Arbitrary j^3 Case and the Corresponding Discrete Wave Equation

For arbitrary j^3, we make the following ansatz for the operators of good scaling dimension

$$\sum_{b_1} f(b_0,b_1)\,O_{p,j^3}(b_0,b_1)=\sum_{j,b_1} C_{p,j^3}(j)\,f(b_0,b_1)\,O_{j,j^3}(b_0,b_1)$$

Repeating the arguments in the Section 3.6.1, we find that the $O_{p,j^3}(b_0,b_1)$'s satisfy the recursion equation (3.17) with the prefactor given by $-\alpha_{p,j^3}$. We also obtain the following recursion relation

$$\begin{aligned}-\alpha_{p,j^3}C_{p,j^3}(j)=&\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\;C_{p,j^3}(j+1)\\&+\frac{(j+j^3)(j-j^3)}{2j}\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\;C_{p,j^3}(j-1)\\&-\frac{1}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right)C_{p,j^3}(j).\end{aligned}$$

This equation has a $j^3\to -j^3$ symmetry, and we will solve for both the $0\le j^3$ case and the $j^3\le$ case. The range for j is

$$0\le\left|j^3\right|\le j\le\frac{m}{2}.$$

The eigenfunctions $C_{p,j^3}(j)$ which solve the recursion relation are easily expressed in terms of the hypergeometric function ${}_3F_2$,

$$C_{p,j^3}(j)=(-1)^{\frac{m}{2}-p}\left(\frac{m}{2}\right)!\sqrt{\frac{2j+1}{\left(\frac{m}{2}-j\right)!\left(\frac{m}{2}+j+1\right)!}}\;{}_3F_2\left(\begin{matrix}|j^3|-j,\;j+|j^3|+1,\;-p\\ \left|j^3\right|-\frac{m}{2},\;1\end{matrix};1\right)\qquad(3.21)$$

where the range of j and p are $\left|j^3\right|\le j\le m/2$, $0\le p\le m/2-\left|j^3\right|$, and the associated eigenvalues are

$$-\alpha_{p,j^3}=-2p=0,-2,-4,\ldots,-\left(m-2\left|j^3\right|\right).$$

This implies an energy spacing for the oscillators in $1-2$-direction of $4\alpha_p g_{\rm YM}^2=8pg_{\rm YM}^2$. Thus, for a given value of j^3 we obtain $m/2-|j^3|+1$ oscillators with frequencies 0, $8g_{\rm YM}^2$, $16g_{\rm YM}^2,\ldots,(4m-8|j^3|)g_{\rm YM}^2$. The lowest energy eigenfunction corresponds to the BPS states. The eigenfunctions (3.21) are related to the Hahn polynomials which are defined by

$$Qn(x;\alpha,\beta,N)={}_3F_2\left(\begin{matrix}-n,\;n+\alpha+\beta+1,\;-x\\ \alpha+1,\;-N\end{matrix};1\right)\qquad\text{for}\qquad n=0,1,2,\ldots,N.$$

The Hahn polynomials are well known to be closely related to the Clebsch-Gordan coefficients of $SU(2)$ [92, 93].

One can take a double scaling limit, so that $m \to \infty$, $b_1 \to \infty$, $m/b_1 \sim \gamma \ll 1$, and $j\sqrt{2/m}$ becomes a continuous variable. In this limit ${}_3F_2\left(\begin{smallmatrix} |j^3|-j,\ 1+j+|j^3|,\ -p \\ |j^3|-m/2,\ 1 \end{smallmatrix};1\right) \to L_p\left(2\left(j^2-|j^3|^2\right)/m\right)$ with $L_p(\cdot)$ the Laguerre polynomial. If we define the continuous variables $u_0 = j^3\sqrt{2/m}$, $r = j\sqrt{2/m}$, $r^2 = u^2+u_0^2$, and $|j^3 \le j|$, then we obtain the following differential equation as the continuous approximation of the difference equation

$$\frac{1}{2}\left[-\frac{1}{u}\frac{\partial}{\partial u}\left(u\frac{\partial}{\partial u}\right)+u^2\right]C_{p,u_0}(u) = (2p+1)C_{p,u_0}(u).$$

In this equation $C_{p,u_0} = C_{p,j^3}(j)/\sqrt{r}$. Note that one can always introduce an overall (normalization) factor in C_{p,u_0}. This limit captures the continuous limit that we took when replacing the hypergeometric function ${}_3F_2$ with the Laguerre polynomial $L_p\left(2(j^2-|j^3|^2)/m\right)$,

$$C_{p,j^3}(j) \to (-1)^{\frac{m}{2}-p}\sqrt{\frac{2}{m}}\,\sqrt{2j+1}\,e^{-\frac{j^2-|j^3|^2}{m}}L_p\left(\frac{2\left(j^2-|j^3|^2\right)}{m}\right)$$

where I have introduced an overall normalization factor.

For fixed m, j^3 ranges from $0 \le |j^3| \le m/2$. For the extreme case $j^3 = \pm m/2$, there is only one oscillator with zero frequency. All states of the corresponding oscillator are BPS states. Another extreme case is $j^3 = 0$. The eigenfunctions obtained in this section nicely recover the results of Section 3.6.1 and we see that both the $j^3 = 0$ and $j^3 \neq 0$ cases are captured by the eigenfunctions described in this section.

For a general j^3, the lowest energy state is given by the $p = 0$ eigenfunction

$$C_{p=0,j^3}(j) = \left(\frac{m}{2}\right)!\,(-1)^{\frac{m}{2}}\sqrt{\frac{2j+1}{\left(\frac{m}{2}-j\right)!\,\left(\frac{m}{2}+j+1\right)!}}\,. \tag{3.22}$$

The lowest energy eigenfunction corresponds to the BPS states. The highest energy state is

$$C_{p=\frac{m}{2}-|j^3|,j^3}(j) = \left(\frac{m}{2}\right)!\,(-1)^j\binom{j+|j^3|}{j-|j^3|}\sqrt{\frac{2j+1}{\left(\frac{m}{2}-j\right)!\,\left(\frac{m}{2}+j+1\right)!}}\,, \tag{3.23}$$

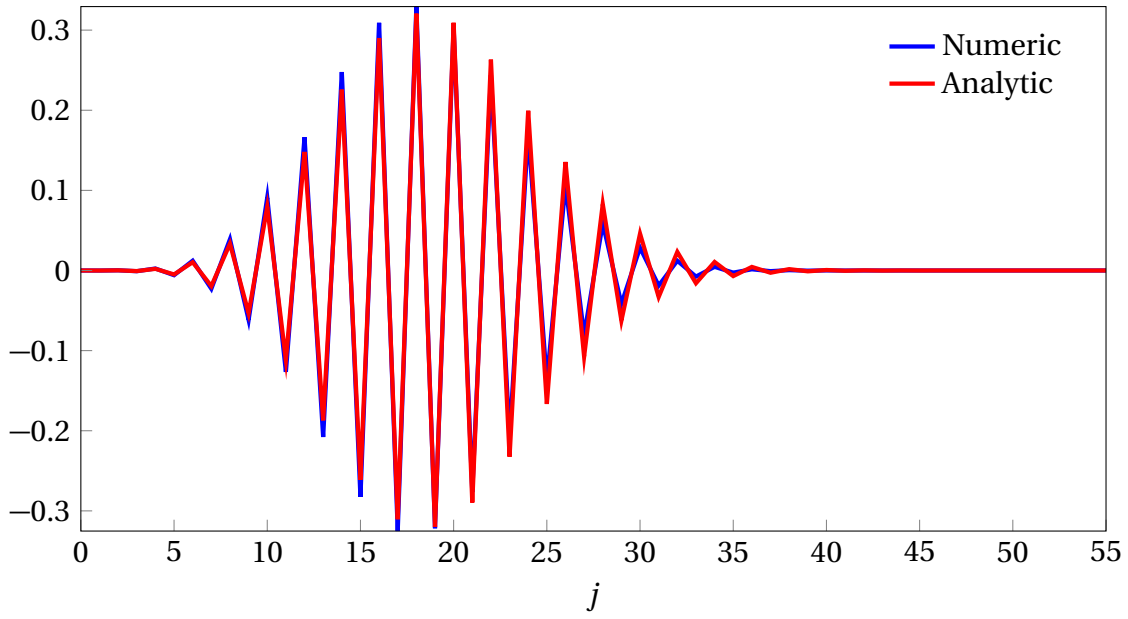


Figure 3.3: A comparison between the numerically computed $C_{p=m/2-|j^3|,j^3}(j)$ (blue) and the analytic formula (3.24) (red) for $m=120$ and $j^3=4$. The agreement is clearly excellent.

where, to obtain this simple result, we have used the formulas relating the hypergeometric function and the Jacobi polynomial $P^{0,2|j^3|}_{j-|j^3|}(z)$:

$$ {}_2F_1\left(\begin{matrix}2\\ |j^3|-j,\ j+|j^3|+1\end{matrix};1\right)\frac{1}{2}(1-z)=P^{0,2|j^3|}_{j-|j^3|}(z),\quad P^{0,2|j^3|}_{j-|j^3|}(-1)=(-1)^{j-|j^3|}\binom{j+|j^3|}{j-|j^3|}. $$

This eigenfunction (3.23) corresponds to an oscillator in the $1-2$-directions with the largest energy spacing of $(4m-8|j^3|)g^2_{\rm YM}$ in this specific j^3 sector.

It is interesting to compare (3.22) and (3.23). There are two main differences. One is the extra alternating signs $(-1)^j$ in (3.23) for superposing j-states. This is familiar from the $j^3=0$ results. Another difference is the extra factor $\binom{j+|j^3|}{j-|j^3|}$ which tends to 1 when $j\to|j^3|$ and tends to $j^{2|j^3|}/(2|j^3|)!$ when $|j^3|\ll j$. Of course, in the special case $j^3=0$ we recover the eigenfunctions discussed in Section 3.6.1.

One can also look at the large m regime of the eigenfunction (3.23) which behaves as

$$ C_{p=\frac{m}{2}-|j^3|,j^3}(j)\to(-1)^j\left(\frac{m}{2}\right)^{-\frac{1}{2}}\frac{\sqrt{2j+1}}{\Gamma\big(2|j^3|+1\big)}\frac{\Gamma\big(j+|j^3|+1\big)}{\Gamma\big(j-|j^3|+1\big)}e^{-\frac{j^2}{m}} \tag{3.24} $$

for $1\ll m$, $|j^3|\le j\ll\frac{m}{2}$, and where $\Gamma(\cdot)$ is the gamma function. Notice that in both the limit $j\to|j^3|$ and the limit $|j^3|\ll j$ the eigenfunction behaves nicely. The numerically generated eigenfunctions are compared with the exact eigenfunctions in Figure 3.3.

3.7 Discussion

In summary we have found that: For an even number of impurities ($m = 2n$), we obtain a set of oscillators with frequency ω_i and degeneracy d_i where

$$\omega_i = 8ig^2_{\rm YM}, \qquad d_i = 2(n-i)+1 \qquad \text{and} \qquad i = 0,1,\ldots,n.$$

For an odd number of impurities ($m = 2n+1$), we obtain a set of oscillators with frequency ω_i and degeneracy d_i with

$$\omega_i = 8ig^2_{\rm YM}, \qquad d_i = 2(n-i+1) \qquad \text{and} \qquad i = 0,1,\ldots,n.$$

This is exactly the spectrum that was conjectured in [42]. In this chapter, I have analytically explained the degeneracies of the oscillators.

It is useful to review the salient features of our results. The operators I consider, restricted Schur polynomials, are composed of order N Ys and order N Zs. Consequently, the large N spectra are obtained by summing both planar and nonplanar diagrams. The nonplanar diagrams can not be neglected. Recall that the restricted Schur polynomial $\chi_{R,(r,s)}$ is labeled by the three Young diagrams R, r and s, containing $n+m$ boxes, n boxes and m boxes, respectively. We have traded these labels for the integers j (which specifies s), b_0, b_1 (which specify r) and j^3 (which specifies how R and r,s are related). By focusing on the family of operators labeled using two column/row Young diagrams and with fixed numbers of Zs and Ys we have simplified the action of the dilation operator to two recursion relations, one in the b_0, b_1 variables and one in the j variables. A crucial ingredient in this construction is the construction of the projector P used in defining the restricted Schur polynomial. This construction is achieved by mapping each Young diagram s onto a unique state of a spin-chain with m spin variables. The computation of the projector is then reduced to the problem of coupling the individual spins in the spin model to obtain a good total spin.

The operators I have studied include both BPS and non-BPS states. The zero eigenvalue states are the BPS states. We find a discrete wave equation in the $3-4$-directions (corresponding to Y) arising from the recursion relation involving j. The associated "energy levels" are labeled by an integer p and set the parameter α_p which determines the

frequencies of the oscillators described by a second discrete wave equation in the $1-2$-directions (corresponding to Z). This second wave equation arises from the recursion relation involving b_0 and b_1. The associated energy levels are described in detail at the start of this section. We have interpreted these states as the oscillations of the relative positions of the two giant graviton branes. These results give a gauge theory description of the giant graviton brane worldvolume physics.

Thanks to the cut-off on the number of rows in the Young diagram, the sphere giants form a cleanly decoupled sector. The same is not true of the AdS giants. The fact that the AdS giants do not decouple at large N as cleanly as the sphere giants did, has not caused any problems. Indeed, we have checked that all of the low lying energy levels correspond to combinations of operators with all of their support on operators labeled by Young diagrams with two long rows.

The spectra for the AdS giants and sphere giants are closely related. This similarity between the spectra goes even further: we have proved that the action of the dilatation operator on sphere giants is related to its action on AdS giants upon making the substitution $N+b \rightarrow N-b$. It would be nice to study this property in more detail.

There are a number of new features of these results that deserve comment. The spectrum we obtain is that of a set of oscillators with frequency some multiple of $8g^2_{\rm YM}$. It is somewhat unusual to find a dependence on $g^2_{\rm YM}$ itself at large N - we are used to the combination which defines the 't Hooft coupling $\lambda = g^2_{\rm YM}N$. These one loop results suggest that we can hold $g^2_{\rm YM}$ small but fixed. If we take the usual 't Hooft limit, $g^2_{\rm YM} \rightarrow 0$ and we obtain a continuous spectrum. The discrete spectra are obtained in a limit that is not the usual 't Hooft limit. It would be very interesting to study the dilatation operator to two loops and see if this dependence on $g^2_{\rm YM}$, with no dressing by factors of N to some power, persists. Another possibility is that we have an $SU(2)$ gauge symmetry for the sector of the operators with two columns or two rows, and the associated 't Hooft coupling is $2g^2_{\rm YM}$. The factor 2 may be understood as the number that multiplies the effective tension of the two coincident brane system.

The group $SU(2)$ has played a central role. It appears in at least three (apparently) unrelated ways

- The projectors $P_{R\to(r,s)}$ were written in terms of $SU(2)$ Clebsch-Gordan coefficients. There is a natural action of $SU(2)$ defined by a map to the spin-chain (see sections 3.4 and 3.5.

- The eigenvalue problem for the Z oscillator has been solved in terms of symmetric Kravchuk polynomials. The symmetric Kravchuk polynomials satisfy a difference equation that follows from the raising and lowering relations between $SU(2)$ states e.g. [79].

- The eigenvalue problem for the Y oscillator has been solved in terms of Hahn polynomials. Hahn polynomials are closely related to the Clebsch-Gordan coefficients of $SU(2)$.

Since we deal with a two giant graviton system, the worldvolume theory will have an $SU(2)$ gauge symmetry. Could this be related to the $SU(2)$ groups we found above? It would be interesting to study their relation.

The $SU(2)$s found above are not related to the global $SU(2)$ subgroup of the $\mathscr{R}$-symmetry, which rotates Y and Z fields into each other. Indeed, the global $SU(2)$ rotates Y and Z fields into each other so that their action mixes operators with different values of m and n. The $SU(2)$s found above all have an action within a given m and n. It would be very interesting to understand this further. This may well shed further light on the integrability we have uncovered in this chapter.

Subsequent to this work, [46] provided an interesting extension of this analysis to the $SU(3)$ sector by considering restricted polynomials of the form

$$\chi_{R,(r,s,t)}\left(Z^{\otimes n},Y^{\otimes m},X^{\otimes p}\right)=\frac{1}{n!\,m!\,p!}\sum_{\sigma\in S_{n+m+p}}\mathrm{Tr}_{(r,s,t)}(\Gamma_r(\sigma))\,X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}}\,Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}}$$
$$\times\, Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{p+m+n}}_{i_{\sigma(p+m+n)}}$$

comprising n Z fields, m Y fields and p X fields where R is a Young diagram with two rows or two columns. The two point correlation function of restricted Schur polynomials in the $SU(3)$ sector was computed in [67]

$$\left\langle \chi_{R,(r,s,t)}(Z,Y,X)\chi_{R,(r,s,t)}(Z,Y,X)^{\dagger}\right\rangle = \delta_{R,(r,s,t),T,(u,v,w)} f_R \frac{\text{hooks}_R}{\text{hooks}_r\ \text{hooks}_s\ \text{hooks}_t}$$

and the corresponding normalized operators are given by

$$\chi_{R,(r,s,t)}(Z,Y,X) = \sqrt{f_R \frac{\text{hooks}_R}{\text{hooks}_r\ \text{hooks}_s\ \text{hooks}_t}}\ O_{R,(r,s,t)}(Z,Y,X)$$

The one loop dilatation operator acting on the $SU(3)$ sector is now the sum of one loop dilatation operators in each $SU(2)$ subsector of the $SU(3)$

$$D = -g_{\text{YM}}^2(\text{Tr}([Y,Z][\partial_Y,\partial_Z]) + \text{Tr}([X,Z][\partial_X,\partial_Z]) + \text{Tr}([Y,X][\partial_Y,\partial_X])).$$

The action of the dilatation operator on the normalized operators $O_{R,(r,s,t)}(Z,Y,X)$ is given by the sum of actions over each $SU(2)$ subsector

$$D\,O_{R,(r,s,t)}(Z,Y,X) = \sum_{T,(u,v,w)} N_{R,(r,s,t);T,(u,v,w)} O_{T,(u,v,w)}(Z,Y,X)$$

where

$$\begin{aligned}
N_{R,(r,s,t);T,(u,v,w)} = &-\sum_{R'} \frac{g_{\text{YM}}^2 c_{RR'} d_T}{d_{R'}\, d_u\, d_v\, d_w\, (n+m+p)} \sqrt{\frac{f_T\, \text{hooks}_T\, \text{hooks}_r\, \text{hooks}_s\, \text{hooks}_t}{f_R\, \text{hooks}_R\, \text{hooks}_u\, \text{hooks}_v\, \text{hooks}_w}} \\
&\times \Big\{ mp\, \text{Tr}\Big(\Big[\Gamma_R((1,p+1)), P_{R\to(r,s,t)}\Big] I_{R'T'} \Big[\Gamma_T((1,p+1)), P_{T\to(u,v,w)}\Big] I_{T'R'}\Big) \\
&\quad + np\, \text{Tr}\Big(\Big[\Gamma_R((1,m+p+1)), P_{R\to(r,s,t)}\Big] I_{R'T'} \Big[\Gamma_T((1,m+p+1)), P_{T\to(u,v,w)}\Big] I_{T'R'}\Big) \\
&\quad + nm\, \text{Tr}\Big(\Big[\Gamma_R((m+p+1,p+1)), P_{R\to(r,s,t)}\Big] I_{R'T'} \Big[\Gamma_T((m+p+1,p+1)), P_{T\to(u,v,w)}\Big] I_{T'R'}\Big)\Big\}
\end{aligned}$$

and the coefficients $c_{RR'}$, Young diagram labels R, R', and T, T' and intertwiners $I_{R'T'}$ and $I_{T'R'}$ are defined as those in (3.2). The relevant new feature in this expression are the the group elements of S_{n+m+p} that act on matrix indices in each trace. A similar analysis to that provided in this chapter results is a five term recursion relation that is also solved by the Kravchuk and Hahn polynomials.

In the next chapter, I consider a generalization of this construction to generating functions of the two giant graviton bound states and study the action of the dilatation operator on these generating functions.

Chapter 4

Multi-Giants Generating Functions

In this chapter, I shall generalize the arguments from the previous chapter. I shall construct generating functions for operators dual to systems of giant gravitons with open strings attached. These operators have a bare dimension of order N so that the usual methods used to solve the planar limit are not applicable. The generating functions are given as integrals over auxiliary variables, which implement symmetrization and antisymmetrization of the indices of the fields from which the operator is composed. Operators of a good scaling dimension (eigenstates of the dilatation operator) are known as Gauss graph operators. These generating functions give a simple construction of the Gauss graph operators which were previously obtained using a Fourier transform on a double coset. The new description provides a natural starting point for a systematic $1/N$ expansion for these operators as well as the action of the dilatation operator on them, in terms of a saddle point evaluation of their integral representation.

4.1 Introduction

The AdS/CFT correspondence has generated significant progress in understanding both quantum gravity in negatively curved spacetimes, as well as strongly coupled Yang-Mills theory [1–3]. For observables with a dimension that is fixed as we take $N \to \infty$ integrability was discovered [4, 19] giving unprecedented control, at both weak and strong coupling, over

the planar limit of the theory. For these observables, a complete matching with the dual string is evident (see, for example, [94]) and we now know an exquisite matching between string states in the gravity Hilbert space and single trace gauge invariant operators in super Yang-Mills theory.

There are many observables whose bare dimensions scale with a power of N as we take $N \to \infty$, that are of interest in holography. These operators are very heavy states and they probe non-perturbative aspects of the dual string theory. Nice examples include giant gravitons [29, 37, 95] which are dual to operators constructed with order N fields [13, 33, 52], and new spacetime geometries which are dual to operators constructed using order N^2 fields [14]. The construction of the large N expansion for these operators is not straight forward. We do not have a nice analogue of the statement that Feynman diagrams become ribbon graphs and that the genus of the graph determines the dependence on N. The basic problem is that the number of ribbon graphs of a given genus is now dependent on N so that the N dependence is no longer determined by the genus of the ribbon graph being summed. The combinatorics is much more involved and there does not seem to be a nice subset of diagrams that, if summed, produce the leading effect at large N.

Without a better understanding of how to perform a systematic approximation, one could try to sum everything, producing results that are exact to all orders in $1/N$. Remarkably, the article [13] showed that this can actually be done in practice, for the free theory. A key insight of [13] was to show that the combinatorics of ribbon graphs can be reformulated using symmetric groups and their characters. This initiated an active area of research, opening new directions that are still being developed. The original article [13] showed that Schur polynomials provide a basis for the local gauge invariant operators, that they diagonalize the free field two-point function and that the free field two-point function can be explicitly computed, to all orders in $1/N$. In addition, all extremal correlators were also explicitly computed, again to all orders in $1/N$. Schur polynomials are labeled by Young diagrams, which may have columns with a maximum length of N as well as rows that are arbitrarily long. In the same way that there is a beautiful matching between the Hilbert space of strings and single trace operators in the planar limit, [13] argued for a matching

between long columns and sphere giant gravitons, as well as between long rows and AdS giant gravitons, generalizing and completing the observations of [33] which related single sphere giant gravitons to subdeterminant operators. See also [96, 97]. These results were quickly generalized to multi-matrix models in a series of articles [65–67, 73, 98], as well as to other gauge groups [99–107].

Giant gravitons branes can be excited by attaching open strings to them. The gauge theory operators dual to these states are described using decorated Young diagrams [30, 40]. The spin chain language that worked so well for closed strings in the planar limit must be modified when describing the open strings attached to giant gravitons. The new effect that needs to be included is that single fields can hop between the giant graviton and onto the string, so that we are forced to consider the dynamics of a spin chain defined on a dynamical lattice. A creative solution to this problem was put in place in [108, 109] for strings attached to single sphere giant gravitons, and then extended to more general brane bound states in [31, 32]. By now, a compelling physical picture is evident [110–115].

As we have just discussed, combinatorial approaches based on permutation groups and their characters, compute correlation functions exactly to all order in $1/N$. This is a valuable source of information and has stimulated a lot of progress, but it has not given any guidance into how to develop a systematic $1/N$ expansion to the physics of excited giant graviton systems. Since we do not expect any except the most symmetric problems to be exactly solvable, this is an obstacle to progress. One attempt at a simplification has been to use a displaced corners approximation [43, 44]. The idea is that the physics of the open strings should simplify in the limit in which the giant gravitons are well separated in spacetime. This corresponds to a limit in which the lengths of distinct rows (or columns) of the Young diagram differ by an amount $\sim N$. In this limit one can simplify the description of the operators considered, at large N [43, 44], and some mixing problems can be solved exactly [45, 116]. There is a nice harmony between these results and another complimentary approach, based a coherent state description of the giant gravitons [117–120]. However, these small steps do not really improve the situation and a systematic $1/N$ expansion of the

giant plus open string physics, in which, for example, we see the splitting and joining of open strings, remains to be developed.

An important recent development for the case of maximal giants was reported in the series of articles [121–123]. The maximal giant graviton is described by a determinant. The idea of [121–123] is to write the determinant as a fermionic integral. The rewriting replaces the symmetric group projector onto the representation given by a column of length N, written in terms of symmetric group characters, with an integral over some new auxiliary Grassmann fields. By writing the symmetric group projector as an integral one opens the possibility of performing a saddle point analysis which can provide the starting point for a systematic $1/N$ expansion. Similar integrals have been constructed for the projectors onto representations labeled by a single column or row, with an arbitrary length [124, 125]. Another development [126–128], which is closely related, has shown that many of the results from the combinatorial approach can be reproduced by writing suitable generating functions of coherent states as a matrix integral. In some cases these matrix integrals are related to the well known Harish-Chandra-Itzykson-Zuber integral and can be computed using localization. In this approach, non-trivial combinatorial information is repackaged in the manipulations of the coherent state generating function. This is a promising development, relevant to systematic $1/N$ expansions for heavy operators.

The central question motivating us in this chapter, is the systematic $1/N$ expansion for giant graviton systems. Physically this expansion is naturally described in terms of splitting and joining interactions of open strings. Towards this end, we develop integral representations for generating functions of Schur polynomials labeled by Young diagrams that have two rows (corresponding to a bound state of two AdS giants) or two columns (corresponding to a bound state of two sphere giants). Our approach to the $1/N$ expansion entails developing a saddle point expansion of these integral representations [129]. Apart from this application, we will see that the new representations provide a dramatically simpler description of excited giant graviton systems. To demonstrate this, I shall explain how to recover known results in what follows. The class of problems we focus on is rather rich because a bound state of two giant gravitons is described by interesting dynamics. The

worldvolume theory of p branes is a $U(p)$ gauge theory [27], so in this situation with $p=2$ the effective Hamiltonian coming from the dilatation operator should reproduce a $U(2)$ non-Abelian gauge theory [40, 130]. To verify this expectation, we must necessarily correctly describe the splitting and joining of open strings, suggesting that this two giant bound state system is a good testing ground for the methods we are trying to develop.

Constructing the operators dual to a bound state of two giant gravitons requires symmetric group projectors labeled by Young diagrams that have two columns or two rows. The key new progress reported in this chapter starts with the construction of an integral representation for these projectors. To motivate the construction, in Section 4.2 I shall describe how Schur polynomials are constructed using Young projectors. It is the Young projectors that can be given an integral representation. Since there is a direct connection between Schur polynomials and characters of the symmetric group, the same integral representations also provide new formulas for character generating functions. A simple generalization beyond the case of a single matrix produces integral representations for operators that are dual to excited giant graviton bound states. A detailed comparison of the operators constructed show that they correspond to the Gauss graph operators discovered in [45, 116]. We further develop this connection in Section 4.3, where we evaluate the action of the dilatation operator on the excited giant graviton bound states. I discuss these results and suggest how they might be extended in Section 4.4. Appendix B collects technical details needed to perform the computations discussed in the body of this chapter.

4.2 Projectors and Generating Functions

Half-BPS operators in $\mathcal{N}=4$ super Yang-Mills theory, dual to a system of giant graviton branes, are given by Schur polynomials $\chi_R(Z)$. The label R is a Young diagram with at most N rows. Each long column [1] in the Young diagram corresponds to a sphere giant graviton

[1] By a "long column" of a Young diagram R, we mean a column that has order N boxes in it. The number of boxes in each column is bounded by N. Similarly, a "long row" refers to a row with order N boxes. Row lengths are unbounded.

and each long row to an AdS giant graviton [2]. An innovative framework for thinking about the construction of these Schur polynomials and their correlation functions, was developed in [68] where the role of projection operators was stressed and an elegant diagrammatic formalism was established. The relevant projection operators used to construct an operator composed of n Z fields can be written in terms characters of the symmetric group S_n, as well as permutations acting on $V^{\otimes n}$ where V is the vector space that the matrix Z acts on. These characters and permutations are determined by the Young diagram R of the Schur polynomial as well as the number n of fields in the operator. The parameter N does not enter transparently in the construction [3]. Consequently, simplifications as $N \to \infty$ are not immediate and it is not clear how one would pursue a systematic $1/N$ expansion. To go beyond the half-BPS operators, we need to go beyond the single matrix Z sector. The simplest way to do this is by constructing operators which are systematically small deformations of the half-BPS operators. These are nice systems to study as we expect that they will be simpler than the generic system that might be considered. In the dual gravity description, the giant graviton system is excited by attaching open strings to the giant gravitons. These generalizations also have a nice description in terms of projectors which proves useful when evaluating correlation functions and matrix elements of the dilatation operator, but again it is not obvious how a systematic large N expansion can be developed.

In this section we briefly review the projection operator construction and its relation to Young symmetrizers. The language of Young symmetrizers gives a construction of the projectors which employs simple symmetrization and antisymmetrization operations on the indices of the Z fields appearing in the operator. Since these operations can be realized by introducing extra fermionic and bosonic auxiliary fields, we are able to construct an integral representation for the projection operators. More precisely, the integral representation gives a generating function for all Schur polynomials with two rows or two columns. This novel generating function provides a starting point for a systematic saddle point approximation. The generating function can also be interpreted as a generating

[2] Recall that a giant graviton is a spherical S^3 brane that carries a D3-brane dipole moment and the same quantum numbers as a point graviton. A sphere giant graviton is one for which the worldvolume $S^3 \subset S^5$, while an AdS giant graviton is one for which $S^3 \subset \mathrm{AdS}_5$.

[3] N does appear as a cut-off on the number of rows of R.

function for characters, suggesting that one might be able to generate a systematic $1/N$ expansion of the characters themselves. There are natural ways to generalize the generating functions of Schur polynomials, to generating functions describing the Schur polynomials with strings attached. Remarkably, as we will demonstrate, the operators constructed in this way have an immediate and direct connection to other constructions of excited giant graviton operators based on restricted Schur polynomials and Gauss graph operators. I shall develop this connection in the Section 4.3 where we evaluate the action of the dilatation operator on the excited giant graviton states.

4.2.1 Young Symmetrizers

Consider the construction of a Schur polynomial χ_R which is a function of a single complex adjoint scalar Z. The Young diagram R labeling the Schur polynomial specifies the system of D-branes as well as an irreducible representation R of the symmetric group S_n where n is the number of boxes in the diagram R, which is also the number of Z fields used to construct the operator. A simple formula for the Schur polynomial employs a projection operators P_R as follows [68]

$$\chi_R(Z) = \mathrm{Tr}_{V^{\otimes n}}\left(P_R Z^{\otimes n}\right).$$

Here Z acts on the vector space V and both P_R and $Z^{\otimes n}$ are linear operators mapping $V^{\otimes n}$ to itself. The image of the projector corresponds simultaneously to a representation of the symmetric group S_n and a representation of the unitary group $U(N)$, both labeled by the same Young diagram [4] R. A concrete expression for the projection operator is given by [5]

$$P_R = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma \tag{4.1}$$

where the matrix elements of the permutation σ are given by

$$\sigma^{i_1 i_2\cdots i_n}_{j_1 j_2\cdots j_n} = \delta^{i_1}_{j_{\sigma(1)}}\delta^{i_2}_{j_{\sigma(2)}}\cdots\delta^{i_n}_{j_{\sigma(n)}}$$

[4] The projector projects to a subspace of dimension $d_R\mathrm{Dim}_R$, where d_R is the dimension of symmetric group representation R and Dim_R is the dimension of $U(N)$ representation R. States in this subspace carry both an S_n state label and a $U(N)$ state label.

[5] As written P_R is proportional to a projection operator. For a correctly normalized projection operator obeying $P_R P_S = \delta_{RS}P_R$ the right hand side of (4.1) must be multiplied by d_R.

so that

$$\mathrm{Tr}\left(\sigma Z^{\otimes n}\right) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}}. \tag{4.2}$$

Thus, given the characters of S_n a formula for P_R easily follows. These formulas can be extended to study operators constructed using more than one matrix, making it possible to consider perturbations of these half-BPS operators. The characters that are needed are less well known and the combinatorics quickly becomes complicated. Nevertheless, at the level of two-point functions we obtain a complete generalization of the Schur polynomial results i.e. we are able to construct a complete basis for the local operators, this basis takes finite N effects into account and this basis diagonalizes the free field theory two-point function [67].

An alternative (but completely equivalent) approach uses the Young symmetrizer associated to R to construct the projector P_R. A Young symmetrizer is an element of the group algebra of the symmetric group. It is an endomorphism of the vector space $V^{\otimes n}$ which uses the action (4.2) of S_n on $V^{\otimes n}$ given by permuting indices. We illustrate the Young projectors making use of a specific example. Consider a Young diagram R that has two rows, with three boxes in the first row and two in the second. Each box corresponds to a field. The boxes in R are labeled, with a unique integer, as follows

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}. \tag{4.3}$$

There is no rule for the placement of integers in boxes, except that each box should be assigned a unique integer. Each box corresponds to the indices of a field in $Z^{\otimes 5}$ as shown below

$$Z^{i_1}_{j_1} Z^{i_2}_{j_2} Z^{i_3}_{j_3} Z^{i_4}_{j_4} Z^{i_5}_{j_5}. \tag{4.4}$$

For every labeled Young diagram R we can define a "row group" for each row and a "column group" for each column. Each row group permutes the indices in the row, in all possible ways. Thus, for the example given in (4.3) the two row groups are

$$\mathcal{G}_{\mathrm{row}\,1} = \{1,(12),(13),(23),(123),(132)\} \qquad \text{and} \qquad \mathcal{G}_{\mathrm{row}\,2} = \{1,(45)\}$$

where we have written the permutations in cycle notation. The product of these groups is denoted $\mathcal{G}_{\mathrm{row}}$. For the example we consider here $\mathcal{G}_{\mathrm{row}}$ is of order 12. Similarly, the column

groups are given by

$$\mathcal{G}_{\text{col }1} = \{1,(14)\} \qquad \text{and} \qquad \mathcal{G}_{\text{col }2} = \{1,(25)\}$$

and their product, denoted $\mathcal{G}_{\text{col}}$, has order 4. The Young symmetrizer is now given by

$$Y_{\begin{smallmatrix}\square\square\square\\\square\square\end{smallmatrix}} = \sum_{\sigma_1\in\mathcal{G}_{\text{row}}}\sum_{\sigma_2\in\mathcal{G}_{\text{col}}} \text{sgn}(\sigma_2)\sigma_1\sigma_2$$

where $\text{sgn}(\sigma)$ is the parity of σ. A cycle of length 2 is called a transposition. Any permutation can be written as a product of transpositions. A particular permutation is even or odd if it can be expressed using an even or an odd number of transpositions. For an even permutation $\text{sgn}(\sigma)=1$ and for an odd permutation $\text{sgn}(\sigma)=-1$. When applied to the tensor product (4.4) we find the σ_2 factor above antisymmetrizes over the pairs i_1, i_4 and i_2, i_5 of indices (i.e. the indices appearing in the columns of (4.3)) while the σ_1 factor symmetrizes over the i_1, i_2, i_3 indices and over the i_4, i_5 indices (i.e. the indices associated to the rows of R). After a simple sum we find that

$$\begin{aligned}\text{Tr}(Y_{\begin{smallmatrix}\square\square\square\\\square\square\end{smallmatrix}} Z^{\otimes 5}) = {} & \text{Tr}(Z^5) + 2\,\text{Tr}(Z^2)\,\text{Tr}(Z)^3 - 4\,\text{Tr}(Z)^2\,\text{Tr}(Z^3) + 3\,\text{Tr}(Z)\,\text{Tr}(Z^2)^2 \\ & - 6\,\text{Tr}(Z)\,\text{Tr}(Z^4) + 4\,\text{Tr}(Z^2)\,\text{Tr}(Z^3).\end{aligned}$$

The Schur polynomial corresponding to this Young diagram is given by

$$\begin{aligned}\chi_{\begin{smallmatrix}\square\square\square\\\square\square\end{smallmatrix}}(Z) = \frac{1}{5!}\big(& 5\,\text{Tr}(Z^5) + 10\,\text{Tr}(Z^2)\,\text{Tr}(Z)^3 - 20\,\text{Tr}(Z)^2\,\text{Tr}(Z^3) + 15\,\text{Tr}(Z)\,\text{Tr}(Z^2)^2 \\ & - 30\,\text{Tr}(Z)\,\text{Tr}(Z^4) + 20\,\text{Tr}(Z^2)\,\text{Tr}(Z^3)\big)\end{aligned}$$

which illustrates the general formula

$$\chi_R(Z) = \frac{d_R}{n!}\,\text{Tr}\Big(Y_{\begin{smallmatrix}\square\square\square\\\square\square\end{smallmatrix}} Z^{\otimes 5}\Big). \tag{4.5}$$

Thus, up to an unimportant overall constant, we can construct Schur polynomials by using Young projectors. This observation can be used to give an integral representation of the Young projectors needed to construct the two giant graviton bound states.

4.2.2 Bound States of Two AdS Giants

A bound state of two AdS giants, one of momentum p_1 and a second of momentum p_2 with $p_2 \leq p_1$, corresponds to a Schur polynomial labeled by a Young diagram with two rows. There are p_1 boxes in the first row and p_2 boxes in the second. Denote this Young diagram by listing row lengths, i.e. $R = (p_1, p_2)$. To describe a system of two giant gravitons, both p_1 and p_2 are of order N. The Young projector must symmetrize over all indices in row 1 and all indices in row 2, as well as antisymmetrize the pair of indices belonging to each column. The symmetrization to be performed represents the bulk of the work. Fortunately, this symmetrization of indices is easily accomplished with the help of a Gaussian integral. Indeed, a simple application of Wick's theorem implies that

$$\begin{gathered}\frac{1}{\pi^N}\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\; e^{-\bar{\phi}^a\phi_a}\bar{\phi}^{i_1}\bar{\phi}^{i_2}\cdots\bar{\phi}^{i_k}\phi_{j_1}\phi_{j_2}\cdots\phi_{j_k}=\sum_{\sigma\in S_k}\delta^{i_1}_{j_{\sigma(1)}}\delta^{i_2}_{j_{\sigma(2)}}\cdots\delta^{i_k}_{j_{\sigma(k)}}\\ \mathrm{d}^N\phi\equiv\prod_{a=1}^{N}\mathrm{d}\phi_a\qquad\text{and}\qquad \mathrm{d}^N\bar{\phi}\equiv\prod_{a=1}^{N}\mathrm{d}\bar{\phi}^a.\end{gathered}\tag{4.6}$$

To construct the generating function, we perform the antisymmetrization in each column by hand and use $(\varphi_i, \bar{\varphi}^i)$ to symmetrize indices in the first row and $(\phi_i, \bar{\phi}^i)$ to symmetrize indices in the second row. The two giant graviton bound state generating function can then be written as follows (as usual, repeated indices are summed)

$$\begin{aligned}Z_{\mathrm{AdS}}(t_1,t_2)&=\sum_{p_1=p_2}^{\infty}\sum_{p_2=1}^{\infty}t_2^{2p_2}t_1^{p_1-p_2}\frac{(p_1+p_2)!}{d_{(p_1,p_2)}\,p_2!\,(p_1-p_2)!}\chi_{(p_1,p_2)}(Z)\\ &=\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\,\frac{e^{-\bar{\varphi}^i\varphi_i-\bar{\phi}^i\phi_i}}{\pi^{2N}}e^{t_1\bar{\varphi}^iZ_i^j\varphi_j+t_2^2\bar{\varphi}^i\bar{\phi}^j\left(Z_i^kZ_j^l-Z_i^lZ_j^k\right)\varphi_k\phi_l}.\end{aligned}\tag{4.7}$$

The coefficient of the Schur polynomial in the first line above combines factors arising from power expanding the exponential on the second line above (so that $t_2^{2p_2}$ comes with $1/p_2!$ and $t_1^{p_1-p_2}$ comes with $1/(p_1-p_2)!$), as well as from the coefficient in (4.5). We check this result, in Appendix B.1, by performing the integral to second order in t_1 and t_2 and show that we reproduce the correct formulas for the relevant Schur polynomials.

This generating function can also be interpreted as a generating function for characters of the symmetric group $S_{p_1+p_2}$ in the representation labeled by Young diagram $R = (p_1, p_2)$. Permutations with a given cycle structure all belong to the same conjugacy class and the

character is a function on these classes. We follow the standard notation $(1^{n_1}, 2^{n_2}, \ldots, (p_1+p_2)^{n_{p_1+p_2}})$ to denote the cycle structure of the permutation which has n_k k-cycles. In what follows we use the shorthand $\{i^{n_i}\}$ for this cycle structure. As usual we have

$$\sum_{k=1}^{p_1+p_2} k\, n_k = p_1 + p_2$$

for permutations $\sigma \in S_{p_1+p_2}$. Denote the symmetric group $S_{p_1+p_2}$ character, of a permutation with cycle structure $\{i^{n_i}\}$ in irreducible representation $R = (p_1, p_2)$ by $\chi_{(p_1,p_2)}(\{i^{n_i}\})$. The number of permutations with this cycle structure is given by

$$\frac{(p_1+p_2)!}{1^{n_1} n_1!\, 2^{n_2} n_2!\, 3^{n_3} n_3! \cdots (p_1+p_2)^{n_{p_1+p_2}} n_{(p_1+p_2)}!}. \tag{4.8}$$

We can then write the generating function for characters of the irreducible representations $R = (p_1, p_2)$ as follows

$$Z_{\text{AdS}}(t_1, t_2) = \sum_{p_1=p_2}^{\infty} \sum_{p_2=1}^{\infty} \sum_{\{n_i\}} \frac{t_2^{2p_2} t_1^{p_1-p_2}}{1^{n_1} n_1!\, 2^{n_2} n_2! \cdots (p_1+p_2)^{n_{p_1+p_2}} n_{(p_1+p_2)}!} \frac{(p_1+1)!}{(p_1-p_2+1)!}$$
$$\times \chi_{(p_1,p_2)}(\{i^{n_i}\}) \operatorname{Tr}(Z)^{n_1} \operatorname{Tr}(Z^2)^{n_2} \cdots \operatorname{Tr}(Z^{p_1+p_2})^{n_{p_1+p_2}}$$

where $\sum_{\{n_i\}}$ denotes a sum over the conjugacy classes of $S_{p_1+p_2}$. As a simple illustration of this formula, expanding to linear order in t_1 and t_2^2, we find

$$\begin{aligned} Z_{\text{AdS}}(t_1, t_2) &= 1 + t_1 \chi_{\square}((\cdot)) \operatorname{Tr}(Z) + t_2^2 \chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}((\cdot)(\cdot)) \operatorname{Tr}(Z)^2 + t_2^2 \chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}((\cdot\cdot)) \operatorname{Tr}(Z^2) + \cdots \\ &= 1 + t_1 \operatorname{Tr}(Z) + t_2^2 \operatorname{Tr}(Z)^2 - t_2^2 \operatorname{Tr}(Z^2) + \cdots \end{aligned}$$

reproducing the correct characters for S_1 and S_2

$$\chi_{\square}((\cdot)) = 1, \qquad \chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}((\cdot)(\cdot)) = 1 \qquad \text{and} \qquad \chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}((\cdot\cdot)) = -1.$$

This integral representation is a nice repackaging of the computation of characters using Young symmetrizers. Since the computation of the power series expansion of $Z_{\text{AdS}}(t_1, t_2)$ involves evaluating moments of a Gaussian integral, there is no obstacle to computing characters for any reasonable value of p_1 and p_2. Evaluating $Z_{\text{AdS}}(t_1, t_2)$ explicitly is frustrated by the fact that the term with coefficient t_2^2 is quartic in the integration variables. By using

the simple identity

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\, x_1 \int_{-\infty}^{\infty} \mathrm{d}\, x_2\; e^{-(x_1+i\,x_2)(x_1-i\,x_2)+A(x_1+i\,x_2)+B(x_1-i\,x_2)} \equiv \frac{1}{\pi}\int \mathrm{d}\, x \int \mathrm{d}\, \bar{x}\; e^{-x\,\bar{x}+Ax+B\bar{x}} = e^{AB}$$

we find

$$Z_{\mathrm{AdS}}(t_1,t_2) = \frac{1}{\pi^{2N+2}} \int \mathrm{d}^N\, \bar{\varphi} \int \mathrm{d}^N\, \varphi \int \mathrm{d}^N\, \bar{\phi} \int \mathrm{d}^N\, \phi \int \mathrm{d}\, x \int \mathrm{d}\, \bar{x} \int \mathrm{d}\, y \int \mathrm{d}\, \bar{y}$$
$$\times\, e^{-\bar{\varphi}^i\varphi_i - \bar{\phi}^i\phi_i - x\,\bar{x} - y\,\bar{y} + t_1\bar{\varphi}^i Z_i^j \varphi_j + t_2 x \bar{\varphi}^i Z_i^k \varphi_k + t_2 \bar{x} \bar{\phi}^j Z_j^l \phi_l + t_2 y \bar{\varphi}^i Z_i^l \phi_l - t_2 \bar{y} \bar{\phi}^j Z_j^k \varphi_k}.$$

Notice that after this rewriting the integral over both $\varphi_i, \bar{\varphi}^i$ and $\phi_i, \bar{\phi}^i$ are Gaussian so that we can do these integrals exactly. The result is

$$Z_{\mathrm{AdS}}(t_1,t_2) = \frac{1}{\pi^2} \int \mathrm{d}\, x \int \mathrm{d}\, \bar{x} \int \mathrm{d}\, y \int \mathrm{d}\, \bar{y}\, \frac{e^{-x\,\bar{x}-y\,\bar{y}}}{\det(M)}$$

where

$$M = \mathbf{1}_2 \otimes \mathbf{1}_N - m \otimes Z, \qquad m \;=\; \begin{bmatrix} t_1 + t_2 x & t_2 y \\ -t_2 \bar{y} & t_2 \bar{x} \end{bmatrix}$$

and $\mathbf{1}_k$ is the $k \times k$ identity matrix. The eigenvalues λ_a with $a = 1,2$ of m are given by

$$\lambda_{1,2} = \frac{1}{2}\left(t_1 + t_2(x+\bar{x}) \pm \sqrt{(t_1 + t_2(x+\bar{x}))^2 - 4(\bar{x}\, t_1 t_2 + t_2^2(x\,\bar{x} + y\,\bar{y}))} \right). \tag{4.9}$$

Denote the eigenvalues of Z_j^i by z_i. A straight forward computation now shows that

$$\begin{aligned} \frac{1}{\det(M)} &= \prod_{i=1}^{N} \frac{1}{(1-\lambda_1 z_i)(1-\lambda_2 z_i)} \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{\lambda_1^n + \lambda_2^n}{n} \operatorname{Tr}(Z^n) \right) \\ &= \exp\left(\sum_{n=1}^{\infty} f_n \operatorname{Tr}(Z^n) \right) \end{aligned}$$

where f_n is the following polynomial

$$\begin{gathered} f_n = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-\beta)^r}{r!} \frac{\alpha^{n-2r}}{(n-2r)!} (n-r-1)! \\ \alpha = t_1 + t_2(x+\bar{x}) \qquad \text{and} \qquad \beta = t_1 t_2 \bar{x} + t_2^2(x\,\bar{x} + y\,\bar{y}) \end{gathered} \tag{4.10}$$

where $\lfloor n/2 \rfloor$ is an instruction to take the integer part of $n/2$. Thus, we have

$$Z_{\text{AdS}}(t_1,t_2)=\frac{1}{\pi^2}\int \mathrm{d}\,x\int \mathrm{d}\,\bar{x}\int \mathrm{d}\,y\int \mathrm{d}\,\bar{y}\; e^{-x\bar{x}-y\bar{y}+\sum_{n=1}^{\infty} f_n \operatorname{Tr}(Z^n)}. \tag{4.11}$$

This implies the following formula for the character

$$\chi_{(p_1,p_2)}(\{i^{n_i}\})=\left[t_1^{p_1-p_2}t_2^{2p_2}\right]\left(\int \mathrm{d}\,x\int \mathrm{d}\,\bar{x}\int \mathrm{d}\,y\int \mathrm{d}\,\bar{y}\;\frac{e^{-x\bar{x}-y\bar{y}}}{\pi^2}\,\frac{(p_1-p_2+1)!}{(p_1+1)!}\prod_{i=1}^{p_1+p_2}(f_i)^{n_i}\right)$$

where $[x^n]p(x)$ stands for the coefficient of x^n in the power series expansion of the polynomial $p(x)$. This integral formula for characters of the symmetric group is a new result. Lets test this for the case that $p_2=0$, in which we case we know that

$$\chi_{(p_1,0)}(\{i^{n_i}\})=1. \tag{4.12}$$

When $t_2=0$ we easily find $f_n=t_1^n/n$ and it is possible to compute (4.11) explicitly

$$\begin{aligned}
Z_{\text{AdS}}(t_1,0)&=e^{\sum_{n=1}^{\infty}\frac{t_1^n}{n}\operatorname{Tr}(Z^n)}\\
&=\sum_{p_1=0}^{\infty}\sum_{\{n_i\}}t_1^{p_1}\frac{1}{n_1!}\frac{1}{2^{n_2}n_2!}\cdots\frac{1}{2^{n_{p_1}}n_{p_1}!}\operatorname{Tr}(Z)^{n_1}\operatorname{Tr}(Z^2)^{n_2}\cdots\operatorname{Tr}(Z^{p_1})^{n_{p_1}}\\
&=\sum_{p_1=0}^{\infty}\sum_{\{n_i\}}t_1^{p_1}\frac{1}{n_1!}\frac{1}{2^{n_2}n_2!}\cdots\frac{1}{2^{n_{p_1}}n_{p_1}!}\chi_{(p_1,0)}(\{i^{n_i}\})\operatorname{Tr}(Z)^{n_1}\operatorname{Tr}(Z^2)^{n_2}\cdots\operatorname{Tr}(Z^{p_1})^{n_{p_1}}
\end{aligned}$$

which immediately implies (4.12). As a second example, consider $p_1+p_2=6$ and choose a character from the conjugacy class (2^3). We easily find

$$\int \mathrm{d}\,x\int \mathrm{d}\,\bar{x}\int \mathrm{d}\,y\int \mathrm{d}\,\bar{y}\;\frac{e^{-x\bar{x}-y\bar{y}}}{\pi^2}\,(f_2)^3 \;=\; t_1^6-6t_1^4t_2^2+60t_1^2t_2^4-72t_2^6.$$

Thus we find

$$\begin{aligned}
\chi_{\square\square\square\atop\square\square\square}([2^3])&=\left[t_2^6\right]\left(\frac{1}{24}\int \mathrm{d}\,x\int \mathrm{d}\,\bar{x}\int \mathrm{d}\,y\int \mathrm{d}\,\bar{y}\;\frac{e^{-x\bar{x}-y\bar{y}}}{\pi^2}\,(f_2)^3\right)=-3\\
\chi_{\square\square\square\square\atop\square\square}([2^3])&=\left[t_2^4t_1^2\right]\left(\frac{1}{20}\int \mathrm{d}\,x\int \mathrm{d}\,\bar{x}\int \mathrm{d}\,y\int \mathrm{d}\,\bar{y}\;\frac{e^{-x\bar{x}-y\bar{y}}}{\pi^2}\,(f_2)^3\right)=3\\
\chi_{\square\square\square\square\square\atop\square}([2^3])&=\left[t_2^2t_1^4\right]\left(\frac{1}{6}\int \mathrm{d}\,x\int \mathrm{d}\,\bar{x}\int \mathrm{d}\,y\int \mathrm{d}\,\bar{y}\;\frac{e^{-x\bar{x}-y\bar{y}}}{\pi^2}\,(f_2)^3\right)=-1\\
\chi_{\square\square\square\square\square\square}([2^3])&=\left[t_1^6\right]\left(\int \mathrm{d}\,x\int \mathrm{d}\,\bar{x}\int \mathrm{d}\,y\int \mathrm{d}\,\bar{y}\;\frac{e^{-x\bar{x}-y\bar{y}}}{\pi^2}\,(f_2)^3\right)=1
\end{aligned}$$

which are indeed the correct results. This demonstrates that the computation of characters of S_n is reduced to evaluating moments of a Gaussian integral.

An interesting extension of these ideas would be to consider characters for $R=(p_1,p_2)$ with both p_1-p_2 and p_2 of order N. The relevant character can be extracted from $Z_{\rm AdS}(t_1,t_2)$ by performing the following contour integrals

$$\oint_{\mathscr{C}_1}\frac{\mathrm{d}\,t_1}{t_1^{1+p_1-p_2}}\oint_{\mathscr{C}_2}\frac{\mathrm{d}\,t_2}{t_2^{1+2p_2}}\,Z_{\rm AdS}(t_1,t_2).$$

Employing a saddle point evaluation would give a $1/N$ expansion of the characters. The leading term in this expansion should agree with the results of [20].

4.2.3 Excited Bound States of Two AdS Giants

To extend the analysis of the previous subsection we consider excited giant graviton bound states obtained by attaching open strings to the brane system. Open strings are described by words composed of $O(\sqrt{N}\,)$ letters [35]. The letters can be any of the scalar or fermion fields of $\mathcal{N}=4$ super Yang-Mills theory, the field strength tensor or covariant derivatives of them. Attach m_{11} open strings described by the words $W_{A_1},W_{A_2},\ldots,W_{A_{m_{11}}}$ to the first brane (of momentum p_1). Since indices of the first brane are symmetrized by the $\bar\varphi^{\,i}$, φ_i fields attaching these strings is achieved by including the factor

$$\prod_{r=1}^{m_{11}}\bar\varphi^{\,i_r}(W_{A_r})^{j_r}_{i_r}\varphi_{j_r}$$

in the integrand of the generating function. We also attach m_{22} open strings, described by the words $W_{B_1},W_{B_2},\ldots,W_{B_{m_{22}}}$, to the second brane (of momentum p_2). Since these fields are in the second row of the Young diagram describing the brane system, we must antisymmetrize these words with the Z field that lives in the same column but in the first row. Thus, attaching these strings is achieved by including the factor

$$\prod_{s=1}^{m_{22}}\Big(\bar\varphi^{\,i_s}Z^{j_s}_{i_s}\varphi_{j_s}\;\bar\phi^{\,k_s}(W_{B_s})^{l_s}_{k_s}\phi_{l_s}-\bar\varphi^{\,i_s}Z^{j_s}_{i_s}\phi_{j_s}\;\bar\phi^{\,k_s}(W_{B_s})^{l_s}_{k_s}\varphi_{l_s}\Big)$$

in the integrand. We can also attach strings that stretch between the two branes. To do this, we start with the product of a factor which attaches a string to the first brane and a factor

which attaches a string to the second brane. From this we obtain a pair of strings stretched between the branes by permuting column indices, thereby swapping the endpoints of the two strings. There is a non-trivial fact [40] about the Hilbert space of states on a two giant graviton brane bound state that this construction must respect. The worldvolume theory of a system of p branes is a U(p) gauge theory. One consequence of this gauge symmetry is the Gauss Law, which states that for a giant graviton brane, which has a compact worldvolume, the total charge on the brane vanishes. In the open string language, this forces the number of strings leaving any particular brane to be equal to the number of strings terminating on the brane. We implement the Gauss Law constraint by adding open strings in pairs, one stretching from 1 to 2 and one from 2 to 1. Describe the m_{12} open strings stretching from the first to the second brane using the words $W_{C_1}, W_{C_2}, \ldots, W_{C_{m_{12}}}$ and describe the m_{21} $(= m_{12})$ open strings stretching from the second to the first brane using the words $W_{D_1}, W_{D_2}, \ldots, W_{D_{m_{21}}}$. Thus, the generating function for this class of excited bound states, using an obvious dot product notation, is

$$\begin{aligned} Z_{\text{AdS}}(t_1, t_2, \{W\}) = & \int \mathrm{d}^N \bar{\varphi} \int \mathrm{d}^N \varphi \int \mathrm{d}^N \bar{\phi} \int \mathrm{d}^N \phi \; \frac{e^{S_0}}{\pi^{2N}} e^{S_I} \\ & \times \prod_{r=1}^{m_{11}} \left(\bar{\varphi} \cdot W_{A_r} \cdot \varphi \right) \\ & \times \prod_{s=1}^{m_{22}} \left(\bar{\varphi} \cdot Z \cdot \varphi \; \bar{\phi} \cdot W_{B_s} \cdot \phi - \bar{\varphi} \cdot Z \cdot \phi \; \bar{\phi} \cdot W_{B_s} \cdot \varphi \right) \\ & \times \prod_{t=1}^{m_{12}} \left(\bar{\varphi} \cdot Z \cdot \varphi \; \bar{\phi} \cdot W_{D_t} \cdot \varphi \; \bar{\varphi} \cdot W_{C_t} \cdot \phi - \bar{\varphi} \cdot Z \cdot \phi \; \bar{\phi} \cdot W_{D_t} \cdot \varphi \; \bar{\varphi} \cdot W_{C_t} \cdot \varphi \right). \end{aligned} \tag{4.13}$$

where

$$S_0 \equiv -\bar{\varphi} \cdot \varphi - \bar{\phi} \cdot \phi \quad \text{and} \quad S_I \equiv t_1 \bar{\varphi} \cdot Z \cdot \varphi + t_2^2 \left(\bar{\varphi} \cdot Z \cdot \varphi \; \bar{\phi} \cdot Z \cdot \phi - \bar{\varphi} \cdot Z \cdot \phi \; \bar{\phi} \cdot Z \cdot \varphi \right). \tag{4.14}$$

A graphical representation of this generating function is given in Figure 4.1. The dependence on t_1 and t_2 continues to tell us about the brane bound state with no open strings attached, i.e. the coefficient of the term $t_1^{p_1 - p_2} t_2^{2p_2}$ is a two giant bound state for which the first brane has momentum p_1 and the second momentum p_2. To simplify our discussion, we shall focus on the $SU(2)$ sector, in which case the open string words are constructed using two complex adjoint scalars, Y and Z. Further, the last and first letter of the word are both Ys.

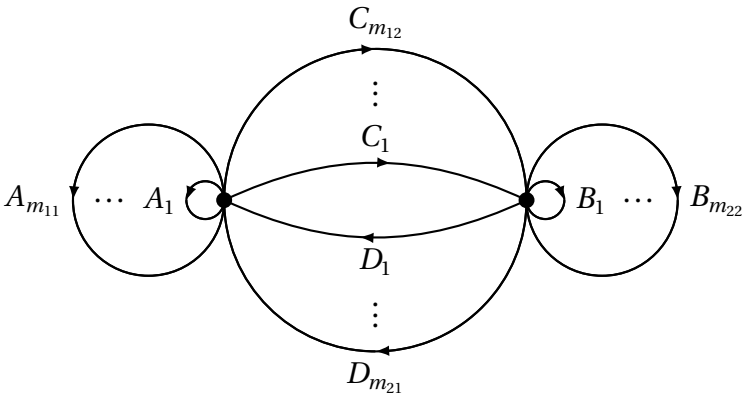


Figure 4.1: This figure gives a graphical representation of the excited brane bound states described by the generating function (4.13). The node on the left represents the brane with momentum p_1 while the node on the right is for the brane of momentum p_2.

By acting on these operators with the dilatation operator, the dynamics of the open string words is described by a Hamiltonian for a spin chain, with non-trivial boundary conditions. In Section 4.3 we consider this Hamiltonian.

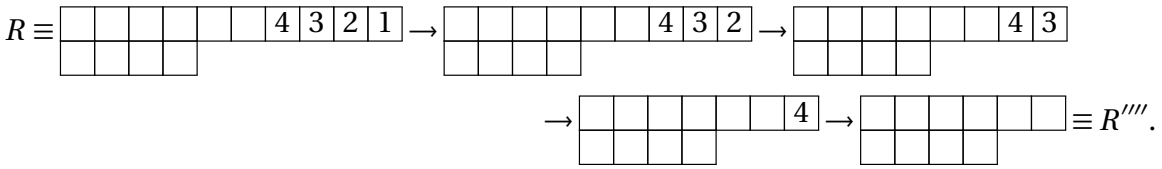
Operators dual to a giant graviton system with strings attached were constructed in [30]. For operators constructed using n Z fields and q open string words, the construction of [30] starts with a chain of subgroups $S_n \subset S_{n+1} \subset S_{n+2} \subset \cdots \subset S_{n+q}$. Using this chain of subgroups, we can define a restriction of a representation R of S_{n+q} to a representation r of S_n, by giving the representation obtained as one successively restricts to each subgroup in the chain. The sequence of representations of the subgroups specify where the open strings are attached, and the representation r defines the giant graviton bound state. All of this information can be summarized by decorating the Young diagram label of the operator. For example, a two AdS giant graviton bound state with four strings attached to the first giant graviton is given by the operator

$$\chi_{\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline &&&&&&&4&3&2&1\\\hline &&&\\\cline{1-4}\end{array}}. \tag{4.15}$$

The chain of representations used to define the operator are obtained by dropping the numbered boxes in order. For the example above, we have the following 5 representations defining the restriction from the original Young diagram R to the Young diagram [6] R''''

[6]The number of primes appearing on R'''' tells us how many open string boxes were dropped from R.

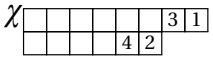
defining the giant bound state



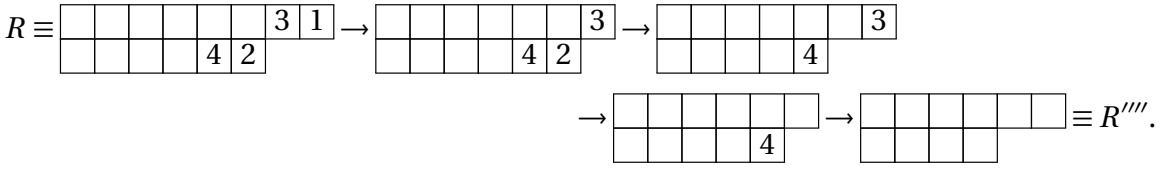
Using this set of restrictions the restricted Schur polynomial of [30] is given by

$$\chi_{R,R''''}(Z,W^{(i)}) = \frac{1}{n!}\sum_{\sigma\in S_{14}} \mathrm{Tr}_{R''''}(\Gamma_R(\sigma)) Z^{i_{14}}_{i_{\sigma(14)}} \cdots Z^{i_5}_{i_{\sigma(5)}} (W_4)^{i_4}_{i_{\sigma(4)}} (W_3)^{i_3}_{i_{\sigma(3)}} (W_2)^{i_2}_{i_{\sigma(2)}} (W_1)^{i_1}_{i_{\sigma(1)}}. \tag{4.16}$$

In [30], $\mathrm{Tr}_{R''''}(\Gamma_R(\sigma))$ is called a restricted character because the trace over R to define the character has been restricted to the subspace R''''. We could also attach two strings to each giant graviton brane



by making use of the following restrictions



To stretch the strings between the branes we use a different restriction for the row and column index of the restricted character. The row restriction specifies the open string start points while the column restriction specifies the endpoints.

What is the relation between the operators of [30] and those constructed above? In Appendix B.1 we explicitly evaluate the form of the operators constructed in [30] and compare them to the operators which follow from the generating function (4.13). Simple examples immediately confirm that

1. In the case that $m_{22} = 0$, $m_{12} = 0 = m_{21}$ with m_{11} arbitrary, the generating function (4.13) precisely reproduces the restricted Schur polynomials constructed in [30].

2. In the case that $m_{11} = 0$, $m_{12} = 0 = m_{21}$ with m_{22} arbitrary, the generating function (4.13) again precisely reproduces the restricted Schur polynomials constructed in [30].

This agreement is not an accident and we can prove that, for the cases described in 1. and 2. above, the two constructions agree. A central result of [30], is a rule for the derivative of a restricted Schur polynomial with respect to an open string. The idea behind the proof is to show that the operators constructed from the generating function (4.13) obey this derivative rule.

To state the derivative rule of [30] we need to review a few facts. When strings are added to the giant system, the order in which strings are added matters [7]. Bearing in mind how the operator was constructed, this is not surprising: changing the order in which open strings are attached changes the set of restrictions used to define the restricted character and the restricted character depends sensitively on which subspace is traced. If we differentiate with respect to the last open string added, the open string together with its box is removed from the restricted Schur polynomial and the remaining polynomial is multiplied by the factor c of the box removed. Recall that the factor of a box in row i and column j is $c = N - i + j$.

The proof of this result, given in [30, 131], uses results from the representation theory of symmetric groups. The derivative with respect to the open string word restricts the sum over S_{n+q} to a sum over S_{n+q-1} and its cosets. A simple rewriting of the sum produces the Jucys-Murphy elements in the group algebra $\mathbb{C}[S_{n+q}]$ of the symmetric group S_{n+q}. They generate a commutative subalgebra of $\mathbb{C}[S_n]$ and they commute with all elements of $\mathbb{C}[S_{n+q-1}]$ with S_{n+q-1} the subgroup we are summing over. The factor that appears multiplying the polynomial reflects the eigenvalues of the Jucys-Murphy elements on the subspace used in the construction of the restricted Schur polynomial. The derivation of this formula is an exercise in the representation theory of finite groups [30].

The operators constructed using the generating function obey exactly the same derivative rule. The proof uses only elementary manipulations. Consider a two giant graviton bound state, with q open strings attached to the first row. An example of the operator we consider is given in (4.15), which has $q = 4$. The box labeled i stands for the

[7] For the specific case that we attach all the open strings to a single giant, as in (4.15) it turns out that attaching the strings in different orders gives the same operator. In this case and only this case the order in which strings are attached does not matter.

open string W_{A_i} for $i = 1,2,3,4$. The rule that we are trying to reproduce says

$$\frac{\partial}{\partial \left(W_{A_1}\right)^a_a} \chi_{\begin{smallmatrix}\square&\square&\square&\square&\square&\square&\square&\square&\square&\square&4&3&2&1\\ \square&\square&\square&\square&\square&\square\end{smallmatrix}} = (N + p_1 + q - 1)\chi_{\begin{smallmatrix}\square&\square&\square&\square&\square&\square&\square&\square&\square&4&3&2\\ \square&\square&\square&\square&\square&\square\end{smallmatrix}}. \tag{4.17}$$

To derive this rule for the operators defined by the generating function, consider

$$\begin{aligned}
&\frac{\partial}{\partial \left(W_{A_1}\right)^a_a} \int \mathrm{d}^N \bar{\varphi} \int \mathrm{d}^N \varphi \int \mathrm{d}^N \bar{\phi} \int \mathrm{d}^N \phi \, \frac{e^{S_0}}{\pi^{2N}} \, e^{S_I} \prod_{r=1}^{q} \left(\bar{\varphi} \cdot W_{A_r} \cdot \varphi\right) \\
&\quad = \int \mathrm{d}^N \bar{\varphi} \int \mathrm{d}^N \varphi \int \mathrm{d}^N \bar{\phi} \int \mathrm{d}^N \phi \, \frac{e^{S_0}}{\pi^{2N}} \, e^{S_I} \prod_{r=2}^{q} \left(\bar{\varphi} \cdot W_{A_r} \cdot \varphi \; \bar{\varphi} \cdot \varphi\right) \\
&\quad = -\int \mathrm{d}^N \bar{\varphi} \int \mathrm{d}^N \varphi \int \mathrm{d}^N \bar{\phi} \int \mathrm{d}^N \phi \; \varphi_a \frac{\partial}{\partial \varphi_a} \left(\frac{e^{S_0}}{\pi^{2N}}\right) e^{S_I} \prod_{r=2}^{q} \left(\bar{\varphi} \cdot W_{A_r} \cdot \varphi\right) \\
&\quad = \int \mathrm{d}^N \bar{\varphi} \int \mathrm{d}^N \varphi \int \mathrm{d}^N \bar{\phi} \int \mathrm{d}^N \phi \, \frac{e^{S_0}}{\pi^{2N}} \left(N + \varphi_a \frac{\partial}{\partial \varphi_a}\right) \left[e^{S_I} \prod_{r=2}^{q} \left(\bar{\varphi} \cdot W_{A_r} \cdot \varphi\right)\right].
\end{aligned}$$

The operator $\varphi_a \frac{\partial}{\partial \varphi_a}$ counts how many boxes there are in the first row, after the box for W_1 is removed. This follows because boxes in the first row represent fields with their upper index contracted with a φ field. Thus, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial \left(W_{A_1}\right)^a_a} \int \mathrm{d}^N \bar{\varphi} \int \mathrm{d}^N \varphi \int \mathrm{d}^N \bar{\phi} \int \mathrm{d}^N \phi \, \frac{e^{S_0}}{\pi^{2N}} \, e^{S_I} \prod_{r=1}^{q} \left(\bar{\varphi} \cdot W_{A_r} \cdot \varphi\right) \\
&\qquad = (N + p_1 + q - 1) \int \mathrm{d}^N \bar{\varphi} \int \mathrm{d}^N \varphi \int \mathrm{d}^N \bar{\phi} \int \mathrm{d}^N \phi \, \frac{e^{S_0}}{\pi^{2N}} \, e^{S_I} \prod_{r=2}^{q} \left(\bar{\varphi} \cdot W_{A_r} \cdot \varphi\right)
\end{aligned}$$

which is indeed the correct rule. Using derivatives which remove all open strings we obtain a Schur polynomial. Given the discussion of Section 4.2.2, we know that the generating function reproduces this operator correctly. Alternatively, since there is no term in the operators considered that is independent of the open string word W_{A_1}, the equality of these derivatives implies the equality of the operators.

We can extend this result to operators corresponding to a bound state of two giant gravitons with all strings attached to the second brane. In this case the derivative rule is

$$\frac{\partial}{\partial \left(W_{B_1}\right)^a_a} \chi_{\begin{smallmatrix}\square&\square&\square&\square&\square&\square&\square&\square&\square&\square&\square&\square\\ \square&\square&\square&\square&\square&\square&4&3&2&1\end{smallmatrix}} = (N + p_2 + q - 2)\chi_{\begin{smallmatrix}\square&\square&\square&\square&\square&\square&\square&\square&\square&\square&\square&\square\\ \square&\square&\square&\square&\square&\square&4&3&2\end{smallmatrix}}.$$

Note that the coefficient on the right hand side is the coefficient of (4.17) with p_1 replaced by p_2 minus one. The extra minus one is because the open string now occupies the second

row. To derive this rule for the operators derived from the generating function, consider

$$\begin{aligned}
&\frac{\partial}{\partial\left(W_{B_1}\right)_a^a}\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\ \frac{e^{S_0}}{\pi^{2N}}e^{S_I}\prod_{s=1}^{q}(\bar{\varphi}\cdot Z\cdot\varphi\ \bar{\phi}\cdot W_{B_s}\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\ \bar{\phi}\cdot W_{B_s}\cdot\varphi)\\
&\quad=\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\ \frac{e^{S_0}}{\pi^{2N}}e^{S_I}\left(\bar{\varphi}\cdot Z\cdot\varphi\ \bar{\phi}\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\ \bar{\phi}\cdot\varphi\right)\\
&\qquad\times\prod_{s=2}^{q}\left(\bar{\varphi}\cdot Z\cdot\varphi\ \bar{\phi}\cdot W_{B_s}\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\ \bar{\phi}\cdot W_{B_s}\cdot\varphi\right)\\
&\quad=(N+p_2+q-1)\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\ \frac{e^{S_0}}{\pi^{2N}}e^{S_I}\\
&\qquad\times\left(\bar{\varphi}\cdot Z\cdot\varphi\right)\prod_{s=2}^{q}\left(\bar{\varphi}\cdot Z\cdot\varphi\ \bar{\phi}\cdot W_{B_s}\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\ \bar{\phi}\cdot W_{B_s}\cdot\varphi\right)\\
&\quad-\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\left(\frac{e^{S_0}}{\pi^{2N}}\right)\\
&\qquad\times\ \varphi_a\frac{\partial}{\partial\phi_a}\left[\left(\bar{\varphi}\cdot Z\cdot\phi\right)\ e^{S_I}\prod_{s=2}^{q}\left(\bar{\varphi}\cdot Z\cdot\varphi\ \bar{\phi}\cdot W_{B_s}\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\ \bar{\phi}\cdot W_{B_s}\cdot\varphi\right)\right].
\end{aligned}$$

Using the obvious facts

$$\begin{aligned}
\varphi_a\frac{\partial}{\partial\phi_a}\left[\bar{\varphi}\cdot Z\cdot\varphi\ \bar{\phi}\cdot W_{B_s}\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\ \bar{\phi}\cdot W_{B_s}\cdot\varphi\right]&=0\\
\varphi_a\frac{\partial}{\partial\phi_a}\left[t_2^2(\bar{\varphi}\cdot Z\cdot\varphi\ \bar{\phi}\cdot Z\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\ \bar{\phi}\cdot Z\cdot\varphi)\right]&=0
\end{aligned}$$

we find

$$\begin{aligned}
&\frac{\partial}{\partial\left(W_{B_1}\right)_a^a}\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\ \frac{e^{S_0}}{\pi^{2N}}e^{S_I}\prod_{s=1}^{q}\left(\bar{\varphi}\cdot Z\cdot\varphi\ \bar{\phi}\cdot W_{B_s}\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\ \bar{\phi}\cdot W_{B_s}\cdot\varphi\right)\\
&\quad=(N+p_2+q-2)\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\ \frac{e^{S_0+S_I}}{\pi^{2N}}\left(\bar{\varphi}\cdot Z\cdot\varphi\right)\\
&\qquad\times\prod_{s=2}^{q}\left(\bar{\varphi}\cdot Z\cdot\varphi\ \bar{\phi}\cdot W_{B_s}\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\ \bar{\phi}\cdot W_{B_s}\cdot\varphi\right).
\end{aligned}$$

The extra factor of $\bar{\varphi}\cdot Z\cdot\varphi$ is simply adding one box to the first row of the Young diagram labeling the operator. The open string W_{B_1} was located in a column of length 2, with the other box in the column occupied by a Z field. After we differentiate with respect to W_{B_1} the second box in the column is left over and this is precisely the extra factor $\bar{\varphi}\cdot Z\cdot\varphi$. This reproduces the correct derivative rule in complete detail. As above, since there is no term in the operators considered that is independent of the open string word W_{B_1}, the equality of

these derivatives implies the equality of the operators. Thus, as long as we attach strings to a specific giant, the generating function we have written down is exactly reproducing the restricted Schur polynomials constructed in [30].

This agreement between the operators obtained from the generating function (4.13) and those constructed in [30] does not hold in general. Indeed, it is simple to verify that

3. In the case that $m_{12}=0=m_{21}$ with both $m_{11}\neq 0$ and $m_{22}\neq 0$ but otherwise arbitrary, the operators defined by the generating function (4.13) do not agree with the restricted Schur polynomials constructed in [30].

As a simple example, the operator with $p_1=1$ and $p_2=0$, with a single string attached to each row, is given by

$$\int d^N\bar{\varphi}\int d^N\varphi\int d^N\bar{\phi}\int d^N\phi\,\frac{e^{S_0}}{\pi^{2N}}\left(\bar{\varphi}\cdot W_A\cdot\varphi\right)\left(\bar{\varphi}\cdot Z\cdot\varphi\;\bar{\phi}\cdot W_B\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\;\bar{\phi}\cdot W_B\cdot\varphi\right)$$
$$=\mathrm{Tr}(W_A)\mathrm{Tr}(W_B)\mathrm{Tr}(Z)+\mathrm{Tr}(ZW_A)\mathrm{Tr}(W_B)-\mathrm{Tr}(W_A)\mathrm{Tr}(W_BZ)-\mathrm{Tr}(W_AZW_B)$$

while the construction of [30] gives

$$\chi_{\begin{array}{|c|c|}\hline\;&A\\\hline B\\\cline{1-1}\end{array}}=\mathrm{Tr}(W_A)\mathrm{Tr}(W_B)\mathrm{Tr}(Z)+\frac{1}{2}\mathrm{Tr}(W_AW_B)\mathrm{Tr}(Z)-\mathrm{Tr}(W_A)\mathrm{Tr}(W_BZ)$$
$$+\frac{1}{2}\mathrm{Tr}(ZW_A)\mathrm{Tr}(W_B)-\frac{1}{2}\mathrm{Tr}(W_AZW_B)-\frac{1}{2}\mathrm{Tr}(W_AW_BZ).$$

This proves that the restricted Schur construction of [30] and the construction using the generating function (4.13) are not the same. There is a simple and general way to see that these two sets of operators are different. The operators produced using the generating function (4.13) treats all open strings with the same orientation identically. Thus, we can permute any of the strings of the same letter[8] and this is a symmetry of the operator. This symmetry is not present in the restricted Schur polynomial construction of [30].

Describing open string excitations of the giant graviton system in terms of open strings has the advantage that we can make contact with a worldsheet description of the open string [108]. Another possibility is that we excite of the half-BPS giant graviton bound state

[8] By this we mean that we can permute the W_{A_i}'s among themselves, the W_{B_i}'s among themselves and so on.

by including a second field Y in the description of the operator[9]. This description is naturally related to the restricted Schur polynomial construction developed in [67, 73]. Consider operators constructed using n Z fields and m Y fields. The restricted Schur polynomial $\chi_{R,(r,s)\alpha\beta}(Z,Y)$ construction of [67, 73] uses a representation $R \vdash m+n$ to organize the complete set of fields, as well as a representation $r \vdash n$ for the Z fields and a representation $s \vdash m$ for the Y fields. In addition, there are multiplicity labels α,β needed[10] because in general there is more than one way in which the $S_n \times S_m$ representation (r,s) can be obtained after restricting the S_{n+m} representation R to the $S_n \times S_m$ subgroup. This description of the local operators does not immediately have a visible open string interpretation. By diagonalizing the one loop dilatation operator, the studies [43–45, 116] discovered an alternative basis, known as the Gauss graph basis. The operators of this basis, called Gauss graph operators, are eigenstates of the one loop dilatation operator. These eigenstates are labeled by graphs, which provide a diagrammatic picture of the open strings stretching between the different giant graviton branes. The complete basis of operators manifestly obey the Gauss Law of the brane worldvolume theory, motivating the terminology "Gauss graph basis". I shall now argue that it is the Gauss graph operators that correspond to the operators constructed from our generating function. Towards this end, consider the generating function

$$\begin{aligned} Z_{\rm AdS}(t_1,t_2,Y) = \int \mathrm{d}^N\bar{\varphi} \int \mathrm{d}^N\varphi \int \mathrm{d}^N\bar{\phi} \int \mathrm{d}^N\phi \; \frac{e^{S_0}}{\pi^{2N}} e^{S_I} \\ \times \left(\bar{\varphi}\cdot Y\cdot\varphi\right)^{m_{11}} \\ \times \left(\bar{\varphi}\cdot Z\cdot\varphi\;\bar{\phi}\cdot Y\cdot\phi - \bar{\varphi}\cdot Z\cdot\phi\;\bar{\phi}\cdot Y\cdot\varphi\right)^{m_{22}} \\ \times \left(\bar{\varphi}\cdot Z\cdot\varphi\;\bar{\phi}\cdot Y\cdot\varphi\;\bar{\varphi}\cdot Y\cdot\phi - \bar{\varphi}\cdot Z\cdot\phi\;\bar{\phi}\cdot Y\cdot\varphi\;\bar{\varphi}\cdot Y\cdot\varphi\right)^{m_{12}} \end{aligned} \tag{4.18}$$

which is obtained by replacing $W_* \to Y$ in (4.13). The bound states described by this generating function have m_{11} strings attached to the brane corresponding to the first row, m_{22} strings attached to the brane corresponding to the second row, m_{12} strings stretching from the first to the second branes and $m_{21} = m_{12}$ strings stretched from the second to the

[9] We can include the complete set of fields in $\mathcal{N}=4$ super Yang Mills theory. This restriction to a single field Y is for simplicity only.

[10] To define the relevant restricted character, we need one multiplicity label for the row index and one for the column index, that we restrict to.

first. The Gauss Law is clearly manifest in this description since the above expression is only defined for $m_{12} = m_{21}$. In addition, the construction is symmetric under swaps of any strings that have the same orientation. This is a symmetry that the Gauss graph operators enjoy and this agreement of the symmetry enjoyed by the two constructions is a key piece of evidence suggesting that the operators constructed from the generating function (4.18) are the Gauss graph operators. The Gauss graph operators with all strings attached to a single brane agree with the restricted Schur polynomials [44, 45] with all open strings attached in a given row. Consequently, the results we have obtained above already establish that restricted Schur polynomials and Gauss graph operators coincide when the open strings are all attached to a single brane, i.e. when there are no strings stretching between branes.

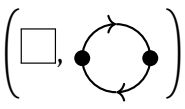
For a detailed comparison of operators that involve stretched strings, study the simpler expression obtained for $m_{11} = 0 = m_{22}$ and $m_{12} = 1 = m_{21}$

$$G_{\mathrm{AdS}}(t_1, t_2) = \int \mathrm{d}^N \bar{\varphi} \int \mathrm{d}^N \varphi \int \mathrm{d}^N \bar{\phi} \int \mathrm{d}^N \phi \, \frac{e^{S_0}}{\pi^{2N}} \, e^{S_I} \times \left(\bar{\varphi} \cdot Z \cdot \varphi \, \bar{\phi} \cdot Y \cdot \varphi \, \bar{\varphi} \cdot Y \cdot \phi - \bar{\varphi} \cdot Z \cdot \phi \, \bar{\phi} \cdot Y \cdot \varphi \, \bar{\varphi} \cdot Y \cdot \varphi \right).$$

The dependence on t_1 and t_2 continues to keep track of the momentum carried by each brane, exactly as above. A simple computation proves that

$$\begin{aligned} G_{\mathrm{AdS}}(t_1, t_2) = {} & \mathrm{Tr}(Z)\,\mathrm{Tr}(Y^2) - \mathrm{Tr}(ZY)\,\mathrm{Tr}(Y) + t_1 \left\{ \mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y^2) + \mathrm{Tr}(Z^2)\,\mathrm{Tr}(Y^2) + \mathrm{Tr}(Z)\,\mathrm{Tr}(ZY^2) \right. \\ & \left. + \mathrm{Tr}(Z^2Y^2) - \mathrm{Tr}(ZY)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y) - \mathrm{Tr}(Z^2Y)\,\mathrm{Tr}(Y) - \mathrm{Tr}(ZY)^2 - \mathrm{Tr}(ZYZY) \right\} \\ & + t_2 \left\{ \mathrm{Tr}(Z)^3\,\mathrm{Tr}(Y^2) + 2\,\mathrm{Tr}(Z^2)\,\mathrm{Tr}(Y^2Z) - \mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y)\,\mathrm{Tr}(YZ) + 2\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y^2Z) \right. \\ & - \mathrm{Tr}(YZ)\,\mathrm{Tr}(YZ^2) - \mathrm{Tr}(Y^2)\,\mathrm{Tr}(Z^3) - 2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(YZ)^2 - 2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y)\,\mathrm{Tr}(YZ^2) \\ & - 2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y^2Z^2) + \mathrm{Tr}(Z)\,\mathrm{Tr}(YZYZ) + 2\,\mathrm{Tr}(Y)\,\mathrm{Tr}(YZ)\,\mathrm{Tr}(Z^2) + \mathrm{Tr}(Y)\,\mathrm{Tr}(YZ^3) \\ & \left. - 2\,\mathrm{Tr}(Y^2Z^3) + 2\,\mathrm{Tr}(YZYZ^2) \right\} + \cdots . \end{aligned} \tag{4.19}$$

In Appendix B we prove that the Gauss graph operators are given by



$$= \mathrm{Tr}(Z)\,\mathrm{Tr}(Y^2) - \mathrm{Tr}(ZY)\,\mathrm{Tr}(Y)$$

$$\left(\begin{array}{c}\square\square\\ \square\end{array}, \bigcirc\!\!\!\!\bigcirc\right) = \mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y^2) + \mathrm{Tr}(Z^2)\,\mathrm{Tr}(Y^2) + \mathrm{Tr}(Z)\,\mathrm{Tr}(ZY^2) + \mathrm{Tr}(Z^2Y^2)$$

$$- \mathrm{Tr}(ZY)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y) - \mathrm{Tr}(Z^2Y)\,\mathrm{Tr}(Y) - \mathrm{Tr}(ZY)^2 - \mathrm{Tr}(ZYZY)$$

$$\left(\begin{array}{c}\square\square\\ \square\end{array}, \bigcirc\!\!\!\!\bigcirc\right) = \mathrm{Tr}(Z)^3\,\mathrm{Tr}(Y^2) + 2\,\mathrm{Tr}(Z^2)\,\mathrm{Tr}(Y^2Z) - \mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y)\,\mathrm{Tr}(YZ) + 2\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y^2Z)$$

$$- \mathrm{Tr}(YZ)\,\mathrm{Tr}(YZ^2) - \mathrm{Tr}(Y^2)\,\mathrm{Tr}(Z^3) - 2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(YZ)^2 - 2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y^2Z^2)$$

$$- 2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y)\,\mathrm{Tr}(YZ^2) + \mathrm{Tr}(Z)\,\mathrm{Tr}(YZYZ) + 2\,\mathrm{Tr}(Y)\,\mathrm{Tr}(YZ)\,\mathrm{Tr}(Z^2)$$

$$+ \mathrm{Tr}(Y)\,\mathrm{Tr}(YZ^3) - 2\,\mathrm{Tr}(Y^2Z^3) + 2\,\mathrm{Tr}(YZYZ^2)$$

in complete agreement with (4.19).

With this encouraging evidence in hand, we would now like to prove that the operators generated by (4.18) are indeed identical to the Gauss graph operators. First in, Appendix B.3, we prove that, at large N, operators derived from the generating function for arbitrary m_{11} and m_{22} agree with the Gauss graph operators. Thus, all that is left is to consider operators with non-zero m_{12}. The proof introduces a graph changing operator: acting on a given Gauss graph, we can easily generate any other Gauss graph operator. Then, given the equality between our operators and the Gauss graph operators when all strings are attached to a definite brane (i.e. there are no stretched string states), and the fact that we can generate any bound state with any number of strings stretched between the giants with the graph changing operator, the proof is complete.

For the proof that follows, it is necessary to briefly review the construction of the Gauss graph operators. The restricted Schur polynomial $\chi_{R,(r,s)\alpha\beta}(Z,Y)$ constructed using n Z fields and m Y fields is labeled by three Young diagrams $R \vdash m+n$, $r \vdash n$ and $s \vdash m$ and two multiplicity labels α, β. The Gauss graph operators $O_{R,r}(\sigma)$ have the same R and r labels, but the s label and the multiplicity labels are replaced by a permutation σ. If we think about each individual Y field as an open string, to specify a state of the Y fields we need to specify a configuration of the open strings. This can be achieved with a permutation, as illustrated in Figure 4.2. This is the permutation σ labeling the Gauss graph operator. From the figure it is clear that swapping endpoints of open strings terminating on the brane and swapping endpoints of open strings originating from the brane is a symmetry of the open

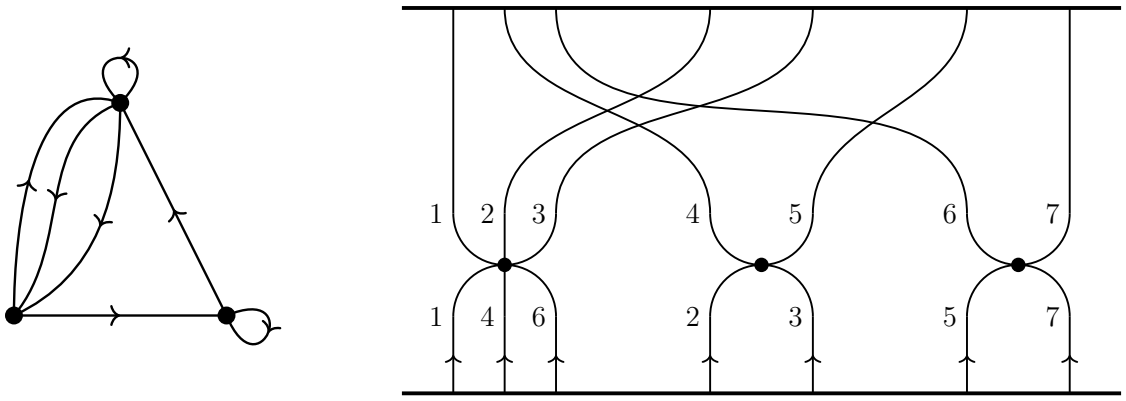


Figure 4.2: Any open string configuration can be mapped to a labeled graph as shown. The two bold horizontal lines in the figure on the right are identified. Rows of R form nodes in the graph. For the graph shown, R has three rows. There are 7 open strings, so the operator is constructed using 7 Y fields. The graph determines a permutation, so each open string configuration maps to a permutation. For the graph shown the permutation, in cycle notation, is $\sigma = (24)(536)$. A nice example is the configuration in which all open strings loop back to the brane they start from, described by the identity permutation.

string configuration. To correctly enumerate distinct physical states we must divide out this symmetry so that each configuration corresponds to an element of a double coset. For the configuration shown in Figure 4.2 the relevant double coset is $H \setminus S_7/H$ where H is the group of symmetries for the endpoints, given by $H = S_3 \times S_2 \times S_2$.

The transformation to the Gauss graph basis is given by

$$O_{R,r}(\sigma) = \sum_{s\vdash m} \sum_{\alpha,\beta} C^s_{\alpha\beta}(\sigma) \chi_{R,(r,s)\alpha\beta} \tag{4.20}$$

where

$$C^s_{\alpha\beta}(\tau) = |H| \sqrt{\frac{d_s}{m!}} \sum_{k,l=1}^{d_s} \Gamma^{(s)}_{kl}(\tau)\, B^{s\to 1_H}_{k\alpha} B^{s\to 1_H}_{l\beta}.$$

This last formula makes use of branching coefficients, which are defined as follows

$$\frac{1}{|H|} \sum_{\gamma\in H} \Gamma^{(s)}_{ik}(\gamma) = \sum_{\mu} B^{s\to 1_H}_{i\mu} B^{s\to 1_H}_{k\mu}$$

where the index μ on the right hand side is a multiplicity label.

As we have just reviewed, the Gauss graph operator is labeled by a permutation σ that specifies how the open strings are stretched between the giant gravitons. We can easily rewrite the formula (4.18) in terms of σ as follows ($m = m_{11} + m_{12} + m_{21} + m_{22}$)

$$Z_{\rm AdS}(t_1,t_2,Y,\sigma)=\int {\rm d}^N\,\bar\varphi\int {\rm d}^N\,\varphi\int {\rm d}^N\,\bar\phi\int {\rm d}^N\,\phi\;\frac{e^{S_0}}{\pi^{2N}}e^{S_I}$$
$$\times\prod_{i=1}^{m_{11}+m_{12}}\left(\bar\varphi^{\,b_i}\,Y^{\,a_i}_{b_i}\,\varphi_{a_{\sigma(i)}}\right)\prod_{j=m_{11}+m_{12}+1}^{m}\left(\bar\varphi\cdot Z\cdot\varphi\;\bar\phi^{\,b_j}\,Y^{\,a_j}_{b_j}\,\phi_{a_{\sigma(j)}}-\bar\varphi\cdot Z\cdot\phi\;\bar\phi^{\,b_j}\,Y^{\,a_j}_{b_j}\,\varphi_{a_{\sigma(j)}}\right).$$

It is clear that σ specifies the row to which the upper indices of the Y fields belong i.e. σ specifies how the Y excitations connect to the giants. The graph changing operator, acting on a system with a total of m Y excitations, has the form

$$\hat G(\sigma)=Y^{\,i_1}_{j_{\sigma(1)}}\cdots Y^{\,i_m}_{j_{\sigma(m)}}\frac{\partial}{\partial\,Y^{\,i_1}_{j_1}}\cdots\frac{\partial}{\partial\,Y^{\,i_m}_{j_m}}.$$

It is simple to verify that

$$\hat G(\sigma_1)Z_{\rm AdS}(t_1,t_2,Y,\sigma_2)=Z_{\rm AdS}(t_1,t_2,Y,\sigma_1\sigma_2).$$

I shall now argue that this graph changing operator has exactly the same action on the Gauss graph operators. When R has only two rows we have no multiplicity labels so that, up to a normalization factor of $|H|/m!$ which we drop, we have[11]

$$\begin{aligned}O(\sigma)&=\sum_{s\vdash m}\sqrt{d_s}\;\Gamma^{(s)}_{jk}(\sigma)B^{s\to 1_H}_j\,B^{s\to 1_H}_k\,O_{R,(r,s)}\\&=\frac{1}{\prod_i m_i!}\sum_{s\vdash m}\sum_{\alpha\in\prod_i S_{m_i}}\sqrt{d_s}\;\Gamma^{(s)}_{jk}(\sigma)\Gamma^{(s)}_{kj}(\alpha)O_{R,(r,s)}\\&=\frac{1}{\prod_i m_i!}\sum_{s\vdash m}\sum_{\alpha\in\prod_i S_{m_i}}\sqrt{d_s}\;\chi_s(\sigma\alpha)O_{R,(r,s)}\end{aligned}\tag{4.21}$$

where $\chi_s(\sigma)$ is the character of $\sigma\in S_m$ in irreducible representation s. Now, again up to a normalization that plays no role in this proof, we have

$$O_{R,(r,s)}=\sum_{\rho\in S_{n+m}}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_m}\frac{1}{\sqrt{d_s}}\chi_r(\sigma_1)\,\chi_s(\sigma_2)\,\chi_R(\sigma_1\circ\sigma_2\,\rho)\,{\rm Tr}(\rho Z^{\otimes n}\,Y^{\otimes m}).$$

In writing this formula we have explicitly used the crucial fact that restricted Schur polynomials with only two rows carry no multiplicity labels[12]. Using this formula, as well as

[11] For a detailed discussion and derivation of this formula, see [45].

[12] In the case that there are no multiplicities we can write the projection operator, which projects to the (r,s) irreducible representation of $S_n\times S_m$, within the carrier space of R as

$$P_{R\to(r,s)}=\frac{d_r\;d_s}{n!\;m!}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_m}\chi_r(\sigma_1)\chi_s(\sigma_2)\Gamma_R(\sigma_1\circ\sigma_2).$$

the character orthogonality identity, which says

$$\sum_{s\vdash m}\chi_s(g)\chi_s(h)=\big|[g]\big|\,\delta_{[g][h]}$$

we have

$$O(\sigma)=\sum_{\rho\in S_{n+m}}\sum_{\sigma_1\in S_n}\sum_{\alpha\in S_{m_1}\times S_{m_2}}\chi_r(\sigma_1)\chi_R(\sigma_1\circ\sigma\alpha\rho)\,\mathrm{Tr}\big(\rho Z^{\otimes n}Y^{\otimes m}\big). \tag{4.22}$$

Using this last formula it is simple to verify that

$$\hat{G}(\sigma_1)O(\sigma_2)=O(\sigma_1\sigma_2).$$

In Appendix B.3 we argue that

$$\sum_{p_1,p_2}t_1^{p_1-p_2}t_2^{p_2}O_{(p_1,p_2)}(\mathbf{1})=Z_{\mathrm{AdS}}(t_1,t_2,Y,\mathbf{1})$$

and since the graph changing operator acts in the same way on both sides of this equation, the proof is complete.

4.2.4 Bound States of Two Sphere Giants

A bound state of two sphere giants, one of momentum p_1 and a second of momentum p_2 with $p_2\le p_1$ corresponds to a Schur polynomial labeled by a Young diagram with two columns. There are p_1 boxes in the first column and p_2 boxes in the second. To describe giant gravitons, both p_1 and p_2 must be order N. The Young symmetrizer must antisymmetrize over all indices in column 1, and all indices in column 2, as well as symmetrize the pair of indices belonging to each row. In contrast to the AdS giants, it is now the antisymmetrization to be performed that represents the bulk of the work. This antisymmetrization of indices can again be accomplished with the help of a Gaussian integral, but now we integrate over Grassmann variables. Again it is a simple application of Wick's theorem to learn that

$$\int \mathrm{d}^N\bar{\xi}\int \mathrm{d}^N\xi\; e^{-\bar{\xi}^a\xi_a}\;\bar{\xi}^{i_1}\xi_{j_1}\bar{\xi}^{i_2}\xi_{j_2}\cdots\bar{\xi}^{i_k}\xi_{j_k}=\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\delta^{i_1}_{j_{\sigma(1)}}\delta^{i_2}_{j_{\sigma(2)}}\cdots\delta^{i_k}_{j_{\sigma(k)}} \tag{4.23}$$

$$\mathrm{d}^N\xi\equiv\mathrm{d}\xi_1\,\mathrm{d}\xi_2\cdots\mathrm{d}\xi_N\qquad\text{and}\qquad\mathrm{d}^N\bar{\xi}\equiv\mathrm{d}\bar{\xi}^N\cdots\mathrm{d}\bar{\xi}^2\,\mathrm{d}\bar{\xi}^1.$$

To write down the generating function for Young diagrams $R=(2^{p_2},1^{p_1-p_2})$ that have two columns, one of length p_1 and one of length p_2, we perform the symmetrization in each

row by hand and use $(\psi_i, \bar{\psi}^i)$ to antisymmetrize indices in the first column and $(\xi_i, \bar{\xi}^i)$ to perform the antisymmetrization of indices in the second column. The two giant graviton bound state generating function can then be written as follows

$$\begin{aligned} Z_{\rm S}(t_1,t_2) &= \sum_{p_1\geq p_2}^{\infty}\sum_{p_2=1}^{\infty} t_2^{p_2}\, t_1^{p_1-p_2}\, \frac{(p_1+p_2)!}{d_{(2^{p_2}1^{p_1-p_2})}p_2!\,(p_1-p_2)!}\,\chi_{(2^{p_2},1^{p_1-p_2})}(Z) \\ &= \int \mathrm{d}^N\bar{\psi}\int \mathrm{d}^N\psi\int \mathrm{d}^N\bar{\xi}\int \mathrm{d}^N\xi\; e^{-\bar{\psi}^i\psi_i-\bar{\xi}^i\xi_i-t_1\bar{\psi}^iZ_i^j\psi_j+t_2^2\bar{\psi}^i\bar{\xi}^j\left(Z_i^kZ_j^l+Z_i^lZ_j^k\right)\psi_k\xi_l}. \end{aligned} \tag{4.24}$$

In Appendix B.1 we check this result by performing the integral to second order in t_1 and t_2 and show that we reproduce the correct formulas for the relevant Schur polynomials.

As was the case as for the AdS giants, this generating function can also be interpreted as a generating function for characters of the symmetric group $S_{p_1+p_2}$ in the representation labeled by Young diagram $R=(2^{p_2},1^{p_1-p_2})$. The generating function is

$$\begin{aligned} Z_{\rm S}(t_1,t_2) = \sum_{p_1=p_2}^{\infty}\sum_{p_2=1}^{\infty}\sum_{\{n_i\}} &\frac{t_2^{2p_2}\,t_1^{p_1-p_2}}{1^{n_1}n_1!\,2^{n_2}n_2!\cdots(p_1+p_2)^{n_{p_1+p_2}}n_{(p_1+p_2)}!}\,\frac{(p_1+1)!}{(p_1-p_2+1)(p_1-p_2)!} \\ &\times\chi_{(2^{p_2},1^{p_1-p_2})}(\{i^{n_i}\})\,\mathrm{Tr}(Z)^{n_1}\,\mathrm{Tr}(Z^2)^{n_2}\cdots\mathrm{Tr}(Z^{p_1+p_2})^{n_{p_1+p_2}} \end{aligned} \tag{4.25}$$

where $\sum_{\{n_i\}}$ denotes a sum over the conjugacy classes of $S_{p_1+p_2}$. To build some intuition for this formula, expand to linear order in t_1 and t_2 to find

$$\begin{aligned} Z_{\rm S}(t_1,t_2) &= 1+t_1\chi_{\square}((\cdot))\,\mathrm{Tr}(Z)+t_2^2\chi_{\square\square}((\cdot)(\cdot))\,\mathrm{Tr}(Z)^2+t_2^2\chi_{\square\square}((\cdot\cdot))\,\mathrm{Tr}(Z^2)+\cdots \\ &= 1+t_1\,\mathrm{Tr}(Z)+t_2^2\,\mathrm{Tr}(Z)^2+t_2^2\,\mathrm{Tr}(Z^2)+\cdots \end{aligned}$$

reproducing the correct characters for S_1 and S_2

$$\chi_{\square}((\cdot))=1,\qquad \chi_{\square\square}((\cdot)(\cdot))=1\qquad\text{and}\qquad \chi_{\square\square}((\cdot\cdot))=1.$$

Following our discussion for characters with long rows, we could easily use the result (4.25) to develop a formula for characters in representations labeled by Young diagrams with two columns.

4.2.5 Excited Bound States of Two Sphere Giants

In this section we extend the results obtained above, constructing generating functions for operators dual to excited sphere giant gravitons. Again consider a bound state of two

sphere giant gravitons, the first with momentum p_1 and the second with momentum p_2. The bound state is excited by attaching open strings. The generating function for this class of bound states is

$$\begin{aligned} Z_S(t_1,t_2,\{W\}) = & \int \mathrm{d}^N\bar{\psi}\int \mathrm{d}^N\psi\int \mathrm{d}^N\bar{\xi}\int \mathrm{d}^N\xi\; e^{\tilde{S}_0+\tilde{S}_1} \\ & \times \prod_{r=1}^{m_{11}}\left(\bar{\psi}\cdot W_{A_r}\cdot\psi\right) \\ & \times \prod_{s=1}^{m_{22}}\left(\bar{\psi}\cdot Z\cdot\psi\;\bar{\xi}\cdot W_{B_s}\cdot\xi-\bar{\psi}\cdot Z\cdot\xi\;\bar{\xi}\cdot W_{B_s}\cdot\psi\right) \\ & \times \prod_{t=1}^{m_{12}}\left(\bar{\psi}\cdot Z\cdot\psi\;\bar{\xi}\cdot W_{D_t}\cdot\psi\;\bar{\psi}\cdot W_{C_t}\cdot\xi+\bar{\psi}\cdot Z\cdot\xi\;\bar{\xi}\cdot W_{D_t}\cdot\psi\;\bar{\psi}\cdot W_{C_t}\cdot\psi\right). \end{aligned} \tag{4.26}$$

where

$$\tilde{S}_0 = -\bar{\psi}\cdot\psi-\bar{\xi}\cdot\xi \qquad \text{and} \qquad \tilde{S}_1 = -t_1\bar{\psi}\cdot Z\cdot\psi+t_2^2\bar{\psi}^i\bar{\xi}^j\left(Z_i^kZ_j^l+Z_i^lZ_j^k\right)\psi_k\xi_l. \tag{4.27}$$

The signs in the final two lines of (4.26) would both have been positive if we had kept all barred variables to the left with $\bar{\psi}$'s before $\bar{\xi}$'s and all unbarred variables to the right with ψ's before ξ's. The dependence on t_1 and t_2 again tells us about the quantum numbers of the brane bound state with no open strings attached: the coefficient of the term $t_1^{p_1-p_2}t_2^{2p_2}$ describes the case that the first brane has momentum p_1 and the second momentum p_2. We again focus mainly on the $SU(2)$ sector in which case the open string words are constructed using two complex adjoint scalars, Z and Y, with the last and first letter of the word both Ys. The Gauss Law is again respected because the open strings are added to the bound state in pairs, with one string W_{C_t} stretching from brane 1 to brane 2 and another W_{D_t} stretching from 2 to 1. In a discussion which closely parallels the discussion for the AdS giant bound state, we can again prove that the operators with strings attached only to brane 1 or only to brane 2 agree with the construction given in [30]. The proof again proceeds by verifying that derivatives with respect to open strings words agree with the rule derived in [30]. I shall sketch the case that strings are attached to the giant of momentum p_1 since it is interesting to see the role of the Grassmann nature of the integration variables. Recall that in the case of the AdS giant (a long row), the factor of the box containing W_{A_1} is $N+(q-1)$, since each time we move along the Young diagram, to the right, we add 1 to the factor. In the case of

the sphere giant (a long column), the factor of the box containing W_{A_1} is $N-(q-1)$, since each time we move down the Young diagram we subtract 1 from the factor. It is the change in the sign of the $q-1$ contribution to the factor that we want to demonstrate. Towards this end, consider

$$\begin{aligned}
&\frac{\partial}{\partial\left(W_{A_1}\right)_a^a}\int \mathrm{d}^N\bar{\psi}\int \mathrm{d}^N\psi\int \mathrm{d}^N\bar{\xi}\int \mathrm{d}^N\xi\, e^{\tilde{S}_0+\tilde{S}_1}\prod_{r=1}^{q}\left(\bar{\psi}\cdot W_{A_r}\cdot\psi\right)\\
&\qquad=\int \mathrm{d}^N\bar{\psi}\int \mathrm{d}^N\psi\int \mathrm{d}^N\bar{\xi}\int \mathrm{d}^N\xi\, e^{\tilde{S}_0+\tilde{S}_1}\,\bar{\psi}\cdot\psi\prod_{r=2}^{q}\left(\bar{\psi}\cdot W_{A_r}\cdot\psi\right)\\
&\qquad=-\int \mathrm{d}^N\bar{\psi}\int \mathrm{d}^N\psi\int \mathrm{d}^N\bar{\xi}\int \mathrm{d}^N\xi\left(\bar{\psi}^i\frac{\partial}{\partial\bar{\psi}^i}e^{\tilde{S}_0}\right)e^{\tilde{S}_1}\prod_{r=2}^{q}\left(\bar{\psi}\cdot W_{A_r}\cdot\psi\right)\\
&\qquad=\left(N-\bar{\psi}^i\frac{\partial}{\partial\bar{\psi}^i}\right)\int \mathrm{d}^N\bar{\psi}\int \mathrm{d}^N\psi\int \mathrm{d}^N\bar{\xi}\int \mathrm{d}^N\xi\, e^{\tilde{S}_0+\tilde{S}_1}\prod_{r=2}^{q}\left(\bar{\psi}\cdot W_{A_r}\cdot\psi\right)
\end{aligned}$$

where the negative sign in the expression in brackets on the last line above is exactly what is required to get the correct factor. This sign comes from the Grassmann nature of $\bar{\psi}^i$ since

$$\frac{\partial}{\partial\bar{\psi}^i}\bar{\psi}^i=N-\bar{\psi}^i\frac{\partial}{\partial\bar{\psi}^i}.$$

Again, exactly as we found for the excited AdS giant bound states, by constructing and comparing operators we learn that the restricted Schur construction of [30] is in general not the same as the construction using the generating function (4.26).

The operators constructed using the generating function (4.26) are again symmetric under swapping strings of the same orientation. This is a strong hint that they are closely related to the Gauss graph operators. This expectation turns out to be correct. The relevant generating function is

$$\begin{aligned}
Z_{\mathrm{S}}(t_1,t_2,Y)=&\int \mathrm{d}^N\bar{\psi}\int \mathrm{d}^N\psi\int \mathrm{d}^N\bar{\xi}\int \mathrm{d}^N\xi\, e^{\tilde{S}_0+\tilde{S}_1}\\
&\times\prod_{r=1}^{m_{11}}\left(\bar{\psi}\cdot Y\cdot\psi\right)\\
&\times\prod_{s=1}^{m_{22}}\left(\bar{\psi}\cdot Z\cdot\psi\;\bar{\xi}\cdot Y\cdot\xi-\bar{\psi}\cdot Z\cdot\xi\;\bar{\xi}\cdot Y\cdot\psi\right)\\
&\times\prod_{t=1}^{m_{12}}\left(\bar{\psi}\cdot Z\cdot\psi\;\bar{\xi}\cdot Y\cdot\psi\;\bar{\psi}\cdot Y\cdot\xi+\bar{\psi}\cdot Z\cdot\xi\;\bar{\xi}\cdot Y\cdot\psi\;\bar{\psi}\cdot Y\cdot\psi\right).
\end{aligned}\tag{4.28}$$

The bound states described by this generating function have m_{11} strings attached to the brane corresponding to the first column, m_{22} strings attached to the brane corresponding to

the second column, m_{12} strings stretching from the first to the second branes and $m_{21} = m_{12}$ strings stretched from the second to the first. The construction is manifestly symmetric under swaps of any strings that have the same orientation. Appendix B.2 contains detailed checks of the construction of operators coming from (4.28), as well as detailed examples demonstrating their equality to the Gauss graph operators. It is possible to generalize the proof given in Appendix B.3 to the case of operators labeled by Young diagrams with long columns, and then to repeat the argument that worked for the AdS giant graviton, making use of the graph changing operator. The details are an obvious generalization of the discussion for the AdS giant graviton.

4.3 Action of the Dilatation Operator

In the previous section we have obtained an integral representation of the operators dual to excited giant graviton bound states. Our goal in this section is study the action of the dilatation operator on these integral representations. I shall restrict to the $SU(2)$ sector, which corresponds to studying open string words constructed using two complex matrices Y and Z. The one loop dilatation operator in the $SU(2)$ sector (up to overall normalization) is [28]

$$D = -g_{\rm YM}^2 \, {\rm Tr}([Z,Y][\partial_Z, \partial_Y]). \tag{4.29}$$

When the open strings are described using words, one can derive a Hamiltonian [108, 109] for particles (representing the Z fields) hopping on a lattice (defined by the Y fields). The Z fields can hop between the open string world sheet and the giant graviton bound state. A key physical question is the description of this hopping, relevant for the dynamics of the string endpoints. We take this up in Section 4.3.1. For the description in which the open strings are described using the Y field, I have argued that the operators we construct are the Gauss graph operators. The Gauss graph operators are eigenstates of the one loop dilatation operator at large N. I shall define an integral representation for the action of the dilatation operator D on the generating function (4.18) in Section 4.3.2. This integral representation is a natural starting point for a systematic $1/N$ expansion.

4.3.1 Open String Boundaries

Our goal is to work out the action of the dilatation operator, in the large N limit. Concretely, this amounts to constructing a lattice Hamiltonian for each of the four types of open strings (attaching to brane 1 or 2, and stretching from 1 to 2 or from 2 to 1) we can consider. What is of most interest is the boundary conditions for the strings. There are two derivatives in D, so it is possible that the two derivatives will act on different open strings. However, these effects are subleading at large N and we ignore them[13]. In this case, we are not losing any information by simplifying the discussion to a single (for W_A and W_B) or pair (for W_C and W_D) of open strings.

The number of fields in the giant graviton is $O(N)$, the planar approximation fails and to obtain the correct large N limit we need to sum all possible contractions of fields in the two giant gravitons. The number of fields in each open string is $\sim L$ with $L \sim O(\sqrt{N})$. If we take $L^2/N \ll 1$, then at large N it is accurate to contract the open string words planarly [30].

We restrict to the $SU(2)$ sector which amounts to constructing all open string words W_* from the pair Z, Y of matrices. To obtain the world sheet description of the open strings, following [109], we interpret the Y fields in W_* as a "lattice" which can be populated by inserting excitations (in this case Z's) into the lattice (gaps between the Y's). I shall, refer to the Z's as impurities. For a word containing L Y fields, there are $L-1$ sites in the lattice

$$W_* = Y Z^{n_1} Y Z^{n_2} Y \cdots Y Z^{n_{L-1}} Y.$$

The $\{n_i\}$ are lattice site occupation numbers. The one loop dilatation operator preserves the number of Y's (the lattice is not dynamical) and allows impurities to hop between neighbouring sites (Z's and Y's can swap their order in W_*). This interpretation maps the problem of determining the anomalous dimensions of operators in the super Yang-Mills theory into the dynamics of a Cuntz oscillator chain [109]. The bulk interactions are described by the Hamiltonian

$$H_{\text{bulk}} = 2\lambda \sum_{l=1}^{l} \hat{a}_l^\dagger \hat{a}_l - \lambda \sum_{l=1}^{L-1} \left(\hat{a}_l^\dagger \hat{a}_{l+1} + \hat{a}_l \hat{a}_{l+1}^\dagger \right), \tag{4.30}$$

[13] Here we are simply alluding to the fact that at large N string interactions are suppressed and hence open string words do not mix. In the closed string sector the analogous statement is that in the planar limit, different trace structures do not mix.

where $\lambda = g_{\rm YM}^2 N$ is the 't Hooft coupling and the Cuntz oscillators obey

$$\hat{a}_i \hat{a}_i^\dagger = I \qquad \text{and} \qquad \hat{a}_i^\dagger \hat{a}_i = I - |0\rangle\langle 0| \,.$$

To obtain the full Hamiltonian, we need to include boundary interactions arising from the string/brane system interaction. This interaction, which introduces sources and sinks for the impurities at the boundaries of the lattice, was first derived in [109] for a string attached to a single sphere giant graviton, and then derived in [31, 32] for strings attached to an arbitrary bound state. One such interaction allows a Z to hop from the first or last site of the string onto the giant, or from the giant into the first or last site of the string. In this process the string exchanges momentum with the giant graviton, so that the giant graviton is exerting a force on the string. There is also a boundary interaction in which a Z belonging to the giant "kisses" the first (or last) Y in the open string word. In this case, no momentum is exchanged, and the string does not feel a force. The logic developed in [31] derives the interaction for the "hop off" process, in which a Z hops off the string and onto the giant. Requiring that the Hamiltonian is hermitian then fixes the "hop on" interaction. The momentum conserving boundary interaction follows by expressing the kiss as a hop on followed by a hop off.

The hop off boundary term for open string W_* is given by [31, 32]

$$H_{\text{hop off}} = \lambda\sqrt{\frac{c}{N}}\ \hat{A}^\dagger \hat{a}_1 + \lambda\sqrt{\frac{c}{N}}\ \hat{A}^\dagger \hat{a}_l \tag{4.31}$$

where c is the factor of the box occupied by the open string word W_*, $\hat{a}_1$ and $\hat{a}_l$ annihilate a Z from the last and first sites of the open string word W_* and $\hat{A}^\dagger$ adds a Z to the giant graviton i.e. it adds a box in the Young diagram label of the operator. The key step in the derivation of this hop off boundary term is an identity obeyed by the restricted Schur polynomials. When a string hops out of the first (or last) site we obtain an open string word of the form $Z W_*$ (or $W_* Z$). The identity expresses the restricted Schur polynomial with words $Z W_*$ or $W_* Z$ in terms of restricted Schur polynomials with word W_*. This identity is derived by writing the sum over σ in (4.16) as a sum over a subgroup and its cosets and again using known results about the eigenvalues of the Jucys-Murphy elements. This is how the factor of the

box appears in (4.31). In terms of the generating function the derivation of this identity is a trivial exercise. Indeed, the identity of [31] is given by

$$\begin{aligned}0 &= \int \mathrm{d}^N\,\bar{\varphi} \int \mathrm{d}^N\,\varphi \int \mathrm{d}^N\,\bar{\phi} \int \mathrm{d}^N\,\phi\ \frac{\partial}{\partial\,\bar{\varphi}^{\,j}} \left(\frac{e^{S_0+S_I}}{\pi^{2N}}\,\bar{\varphi}^{\,i}\,W_i^{\ j} \right) \\ &= \int \mathrm{d}^N\,\bar{\varphi} \int \mathrm{d}^N\,\varphi \int \mathrm{d}^N\,\bar{\phi} \int \mathrm{d}^N\,\phi\ \frac{e^{S_0+S_I}}{\pi^{2N}} \big(\mathrm{Tr}(W) - \bar{\varphi}\cdot W\cdot\varphi + t_1\,\bar{\varphi}\cdot W\cdot Z\cdot\varphi \\ &\qquad + t_2^2(\bar{\varphi}\cdot W\cdot Z\cdot\varphi\ \bar{\phi}\cdot Z\cdot\phi - \bar{\varphi}\cdot W\cdot Z\cdot\phi\ \bar{\phi}\cdot Z\cdot\varphi)\big).\end{aligned}$$

This is identical to the identities given in Appendix C.1 of [32]. The term proportional to $\mathrm{Tr}(W)$ describes gravitational radiation, whereby the open string W is emitted as a closed string $\mathrm{Tr}(W)$ from the excited giant bound state. It represents a subleading contribution at large N. The term proportional to $\bar{\varphi}\cdot W\cdot\varphi$ produces restricted Schur polynomials constructed with open string word W. The remaining terms produce restricted Schur polynomials constructed with open string word $W\cdot Z$. The generating function provides a remarkably simple derivation of this formula.

The identity above describes hopping out of the last site of the open string word. To obtain the identity needed for hopping out of the first site we consider

$$0 = \int \mathrm{d}^N\,\bar{\varphi} \int \mathrm{d}^N\,\varphi \int \mathrm{d}^N\,\bar{\phi} \int \mathrm{d}^N\,\phi\ \frac{\partial}{\partial\,\varphi_i} \left(\frac{e^{S_0+S_I}}{\pi^{2N}}\,W_i^{\ j}\,\varphi_j \right).$$

The identities needed for the sphere giant are equally easy to derive.

4.3.2 Gauss Graph Operators

I have already argued that when each open string word is replaced by a single letter, the operators we generate are the Gauss graph operators. From the results obtained in [45] we know that these are eigenstates of D, in the large N limit. In [44, 45] the simplifications of large N were implemented by exploiting simplifications in the action of the symmetric group. This analysis, known as the distant corners approximation, is clumsy. In particular, it is not obvious exactly what has been discarded in this approximation and, consequently, it is not easy to systematically correct and improve the approximation. It is interesting to reconsider this computation using the generating function where we should have better control over

the large N expansion. Indeed, a systematic large N expansion should follow from a saddle point approximation of the integral representations we have obtained. To carry out this analysis we need to apply D to the operators defined by (4.18). In this subsection we study some illustrative examples. Return to (4.18) which we write as

$$Z_{\rm AdS}(t_1,t_2,Y)=\int {\rm d}^N\,\bar\varphi\int {\rm d}^N\,\varphi\int {\rm d}^N\,\bar\phi\int {\rm d}^N\,\phi\;\frac{e^{S_0+S_I}}{\pi^{2N}}\;e^{t_1\Phi_1+t_2^2\Phi_2}\;\Phi_3^{m_{11}}\Phi_4^{m_{22}}\Phi_5^{m_{12}}$$

where we have introduced the new variables

$$\begin{aligned}
\Phi_1&=\bar\varphi\cdot Z\cdot\varphi\\
\Phi_2&=\bar\varphi\cdot Z\cdot\varphi\;\bar\phi\cdot Z\cdot\phi-\bar\varphi\cdot Z\cdot\phi\;\bar\phi\cdot Z\cdot\varphi\\
\Phi_3&=\bar\varphi\cdot Y\cdot\varphi\\
\Phi_4&=\bar\varphi\cdot Z\cdot\varphi\;\bar\phi\cdot Y\cdot\phi-\bar\varphi\cdot Z\cdot\phi\;\bar\phi\cdot Y\cdot\varphi\\
\Phi_5&=\bar\varphi\cdot Z\cdot\varphi\;\bar\phi\cdot Y\cdot\varphi\;\bar\varphi\cdot Y\cdot\phi-\bar\varphi\cdot Z\cdot\phi\;\bar\phi\cdot Y\cdot\varphi\;\bar\varphi\cdot Y\cdot\varphi.
\end{aligned}$$

Our operators are constructed using a large number of Z fields (order N) and very few Y fields (order 1). It proves helpful to remove the Y fields, as much as possible, from the manipulations that follow. With this in mind, introduce sources for the fields φ, $\bar\varphi$, ϕ and $\bar\phi$ and write

$$\begin{aligned}
Z_{\rm AdS}(t_1,t_2,Y)=\widehat\Phi_3^{m_{11}}\;\widehat\Phi_4^{m_{22}}\;\widehat\Phi_5^{m_{12}}\Bigg(\int {\rm d}^N\,\bar\varphi\int {\rm d}^N\,\varphi\int {\rm d}^N\,\bar\phi\int {\rm d}^N\,\phi\;\frac{e^{S_0+S_I}}{\pi^{2N}}\\
\times\;e^{t_1\Phi_1+t_2^2\Phi_2}\,e^{\bar j^\varphi\cdot\varphi+\bar\varphi\cdot j^\varphi+\bar j^\phi\cdot\phi+\bar\phi\cdot j^\phi}\Bigg)\Big|_{j=0}\\
\equiv\widehat\Phi_3^{m_{11}}\;\widehat\Phi_4^{m_{22}}\;\widehat\Phi_5^{m_{12}}\;Z_{\rm AdS,reduced}(t_1,t_2)\Big|_{j=0}
\end{aligned}\tag{4.32}$$

where $|_{j=0}$ is an instruction to set all sources to zero and

$$\begin{aligned}
\widehat\Phi_3&=\frac{\partial}{\partial j^\varphi}\cdot Y\cdot\frac{\partial}{\partial\bar j^\varphi}\\
\widehat\Phi_4&=\frac{\partial}{\partial j^\varphi}\cdot Z\cdot\frac{\partial}{\partial\bar j^\varphi}\,\frac{\partial}{\partial j^\phi}\cdot Y\cdot\frac{\partial}{\partial\bar j^\phi}-\frac{\partial}{\partial j^\varphi}\cdot Z\cdot\frac{\partial}{\partial\bar j^\phi}\,\frac{\partial}{\partial j^\phi}\cdot Y\cdot\frac{\partial}{\partial\bar j^\varphi}\\
\widehat\Phi_5&=\frac{\partial}{\partial j^\varphi}\cdot Z\cdot\frac{\partial}{\partial\bar j^\varphi}\,\frac{\partial}{\partial j^\phi}\cdot Y\cdot\frac{\partial}{\partial\bar j^\varphi}\,\frac{\partial}{\partial j^\varphi}\cdot Y\cdot\frac{\partial}{\partial\bar j^\phi}\\
&\quad-\frac{\partial}{\partial j^\varphi}\cdot Z\cdot\frac{\partial}{\partial\bar j^\phi}\,\frac{\partial}{\partial j^\phi}\cdot Y\cdot\frac{\partial}{\partial\bar j^\varphi}\,\frac{\partial}{\partial j^\varphi}\cdot Y\cdot\frac{\partial}{\partial\bar j^\varphi}.
\end{aligned}\tag{4.33}$$

The integration is not a Gaussian integral because Φ_2 introduces terms that are quartic in the integration variables. To proceed perform a Hubbard-Stratanovich like transformation, introducing two new complex fields, and rewrite $Z_{\text{AdS,reduced}}(t_1,t_2)$ as follows

$$Z_{\text{AdS,reduced}}(t_1,t_2)=\frac{1}{\pi^{2N+2}}\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\int \mathrm{d}\,x\int \mathrm{d}\,\bar{x}\int \mathrm{d}\,y\int \mathrm{d}\,\bar{y}$$
$$\times\ e^{-\bar{\varphi}\cdot\varphi-\bar{\phi}\cdot\phi-x\bar{x}-y\bar{y}+t_1\bar{\varphi}\cdot Z\cdot\varphi+t_2x\bar{\varphi}\cdot Z\cdot\varphi+t_2\bar{x}\bar{\phi}\cdot Z\cdot\phi+t_2y\bar{\varphi}\cdot Z\cdot\phi-t_2\bar{y}\bar{\phi}\cdot Z\cdot\varphi+\bar{j}^{\varphi}\cdot\varphi+\bar{\varphi}\cdot j^{\varphi}+\bar{j}^{\phi}\cdot\phi+\bar{\phi}\cdot j^{\phi}}.$$

We can streamline this expression as follows

$$Z_{\text{AdS,reduced}}(t_1,t_2)=\frac{1}{\pi^{2N+2}}\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\int \mathrm{d}\,x\int \mathrm{d}\,\bar{x}\int \mathrm{d}\,y\int \mathrm{d}\,\bar{y}$$
$$\times\ e^{-x\bar{x}-y\bar{y}-v\cdot M\cdot\bar{v}+v\cdot\bar{j}+j\cdot\bar{v}}$$

where

$$v_i=\begin{bmatrix}\varphi_i & \phi_i\end{bmatrix},\qquad \bar{v}^i=\begin{bmatrix}\bar{\varphi}^i\\ \bar{\phi}^i\end{bmatrix},\qquad j_i=\begin{bmatrix}j_i^{\varphi} & j_i^{\phi}\end{bmatrix},\qquad \bar{j}^i=\begin{bmatrix}\bar{j}^{\varphi\,i}\\ \bar{j}^{\phi\,i}\end{bmatrix}$$

and
$$M=\begin{bmatrix}\mathbf{1}_N-t_1Z-t_2xZ & -t_2yZ\\ t_2\bar{y}Z & \mathbf{1}_N-t_2\bar{x}Z\end{bmatrix}=\mathbf{1}_2\otimes\mathbf{1}_N-\begin{bmatrix}t_1+t_2x & t_2y\\ -t_2\bar{y} & t_2\bar{x}\end{bmatrix}\otimes Z$$
$$\equiv\mathbf{1}_2\otimes\mathbf{1}_N-\mathscr{M}\otimes Z.$$

The integrals over φ, $\bar{\varphi}$, ϕ and $\bar{\phi}$ are Gaussian and can be done exactly. The result is

$$Z_{\text{AdS,reduced}}(t_1,t_2)=\frac{1}{\pi^2}\int \mathrm{d}\,x\int \mathrm{d}\,\bar{x}\int \mathrm{d}\,y\int \mathrm{d}\,\bar{y}\ \frac{e^{j\cdot(M^{-1})\cdot\bar{j}-x\bar{x}-y\bar{y}}}{\det(M)}. \tag{4.34}$$

The eigenvalues of Z are z_i. The eigenvalues of $\mathscr{M}$ are given in (4.9). In terms of these eigenvalues we can write

$$\frac{1}{\det(M)}=\prod_{a=1}^{2}\prod_{i=1}^{N}\frac{1}{1-\lambda_a z_i}.$$

The Cauchy-Littlewood identity [132] is

$$\prod_{i=1}^{l}\prod_{j=1}^{m}\frac{1}{1-x_iy_j}=\sum_{\Lambda}\chi_\Lambda(x_1,\ldots,x_l)\,\chi_\Lambda\big(y_1,\ldots,y_m\big)$$

where the sum on the right hand side only includes Young diagrams Λ with less than the smaller of L,M rows, and χ_Λ is a Schur polynomial. Applying this formula to our problem

we find

$$\frac{1}{\det(M)} = \sum_{\Lambda} \chi_\Lambda(\mathscr{M})\chi_\Lambda(Z)$$

so that

$$\begin{aligned} Z_{\text{AdS,reduced}}(t_1,t_2) &= \sum_{\Lambda} f_\Lambda(j,\bar{j})\chi_\Lambda(Z) \\ f_\Lambda(j,\bar{j}) &= \frac{1}{\pi^2}\int \mathrm{d}\,x \int \mathrm{d}\,\bar{x} \int \mathrm{d}\,y \int \mathrm{d}\,\bar{y}\; e^{j\cdot(M^{-1})\cdot\bar{j} - x\,\bar{x} - y\,\bar{y}}\;\chi_\Lambda(\mathscr{M}). \end{aligned} \tag{4.35}$$

The sum over Λ is only over Young diagrams with at most two rows.

We can perform a convincing check of (4.35) after we set $j=\bar{j}=0$. When the sources vanish we know that the generating function produces the Schur polynomials with two rows. Further, the Schur polynomial labeled by a Young diagram with r_1 boxes in the first row and r_2 boxes in the second is multiplied by a monomial $t_1^{r_1-r_2}t_2^{2r_2}$. Using the usual formula for the Schur polynomials, we find

$$\begin{aligned} \chi_{\square}(\mathscr{M}) &= t_1 + t_2 x + t_2\bar{x} \\ \chi_{\square\square}(\mathscr{M}) &= t_1^2 + 2t_1t_2x + t_2^2x^2 + t_1t_2\bar{x} + t_2^2x\bar{x} + t_2^2\bar{x}^2 - t_2^2y\bar{y} \\ \chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(\mathscr{M}) &= t_1t_2\bar{x} + t_2^2x\bar{x} + t_2^2y\bar{y} \\ \chi_{\square\square\square}(\mathscr{M}) &= t_1^3 + 3t_1^2t_2x + 3t_1t_2^2x^2 + t_2^3x^3 + t_1^2t_2\bar{x} + 2t_1t_2^2x\bar{x} + t_2^3x^2\bar{x} \\ &\quad + t_1t_2^2\bar{x}^2 + t_2^3x\bar{x}^2 + t_2^3\bar{x}^3 - 2t_1t_2^2y\bar{y} - 2t_2^3xy\bar{y} - 2t_2^3\bar{x}y\bar{y} \\ \chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}(\mathscr{M}) &= t_1^2t_2\bar{x} + 2t_1t_2^2x\bar{x} + t_2^3x^2\bar{x} + t_1t_2^2\bar{x}^2 + t_2^3x\bar{x}^2 + t_1t_2^2y\bar{y} \\ &\quad + t_2^3xy\bar{y} + t_2^3\bar{x}y\bar{y}. \end{aligned}$$

We then easily obtain

$$f_{\square} = t_1, \qquad f_{\square\square} = t_1^2, \qquad f_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = 2t_2^2, \qquad f_{\square\square\square} = t_1^3 \qquad \text{and} \qquad f_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = 3t_1t_2^2$$

which are the correct results. We have also tested that the integer coefficient of each monomial is correct.

If we work in a basis in which Z is diagonal, after a little work we can rewrite (4.34) as

$$Z_{\text{AdS,reduced}}(t_1,t_2) = \frac{1}{\pi^2}\int \mathrm{d}\,x \int \mathrm{d}\,\bar{x} \int \mathrm{d}\,y \int \mathrm{d}\,\bar{y}\;\exp\Big[-x\,\bar{x} - y\,\bar{y} + \sum_{n=1}^{\infty} f_n \operatorname{Tr}(Z^n)$$

$$+ j_\varphi \cdot \left(\sum_{n=0}^{\infty} X^n (1 - t_2 \bar{x} Z) \right) \cdot \bar{j}_\varphi - j_\varphi \cdot \left(\sum_{n=0}^{\infty} X^n t_2 y Z \right) \cdot \bar{j}_\phi$$
$$+ j_\phi \cdot \left(\sum_{n=0}^{\infty} X^n t_2 \bar{y} Z \right) \cdot \bar{j}_\varphi + j_\phi \cdot \left(\sum_{n=0}^{\infty} X^n (1 - t_2 \bar{x} Z - t_1 Z) \right) \cdot \bar{j}_\phi \Bigg] \quad (4.36)$$

where f_n has been defined in (4.10) and

$$X = (t_1 + t_2(x + \bar{x}))Z - \big(t_1 t_2 \bar{x} + t_2^2 (x\bar{x} + y\bar{y})\big) Z^2 .$$

This formula is exact. We are now ready to apply the operators defined in (4.33) to this expression. We evaluate the result at $j = \bar{j} = 0$. One derivative acting produces a term linear in the sources. To get a non-zero result we need to differentiate this again, with respect to the source that appears. Consequently, a pair of derivatives must act before we get a non-zero result. The combinatorics is therefore identical to the combinatorics from Wick's theorem and we only need to know what the basic pairings are. It is easy to read off from the above expression what each possible pairing gives.

It is instructive to consider the case that $t_2 = 0$. This corresponds to a single giant system, which we know is a half-BPS operator. A simple computation shows that

$$Z_{\text{AdS,reduced}}(t_1, 0) = \exp\left[\sum_{n=1}^{\infty} f_n \operatorname{Tr}(Z^n) + j_\varphi \cdot \left(\sum_{n=0}^{\infty} t_1^n Z^n \right) \cdot \bar{j}_\varphi + j_\phi \cdot \bar{j}_\phi \right].$$

For two strings attached to the giant graviton, the generating function is

$$\begin{aligned} Z_{\text{AdS}}(t_1, 0, Y) &= \left(\frac{\partial}{\partial j^\varphi} \cdot Y \cdot \frac{\partial}{\partial \bar{j}^\varphi} \right)^2 Z_{\text{AdS,reduced}}(t_1, 0) \\ &= e^{\sum_{n=1}^{\infty} \frac{t_1^n}{n} \operatorname{Tr}(Z^n)} \sum_{n_1, n_2 = 1}^{\infty} t_1^{n_1 + n_2} \Big(\operatorname{Tr}(Y Z^{n_1}) \operatorname{Tr}(Y Z^{n_2}) + \operatorname{Tr}(Y Z^{n_1} Y Z^{n_2}) \Big) \\ &= \sum_{q=1}^{\infty} \frac{t_1^q}{q!} O_q^{(2)} . \end{aligned}$$

For k strings attached to the giant graviton, the generating function is

$$\begin{aligned} Z_{\text{AdS}}(t_1, 0, Y) &= e^{\sum_{n=1}^{\infty} \frac{t_1^n}{n} \operatorname{Tr}(Z^n)} \sum_{n_1, \ldots, n_k = 1}^{\infty} t_1^{n_1 + n_2 + \cdots + n_k} \sum_{\sigma \in S_k} (Y Z^{n_1})^{i_1}_{i_{\sigma(1)}} (Y Z^{n_2})^{i_2}_{i_{\sigma(2)}} \cdots (Y Z^{n_k})^{i_k}_{i_{\sigma(k)}} \\ &= \sum_{q=1}^{\infty} \frac{t_1^q}{q!} O_q^{(k)} . \end{aligned}$$

This gives a very explicit description of the Gauss graph operator, for a single giant graviton with k strings attached. It is straight forward to verify that

$$\begin{aligned} O_q^{(k)} &= \sum_{\sigma\in S_{k+q}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_q}_{i_{\sigma(q)}} Y^{i_{q+1}}_{i_{\sigma(q+1)}} \cdots Y^{i_{q+k}}_{i_{\sigma(q+k)}} \\ &= k!\, q!\, \chi_{(k+q),(k)(q)}(Z,Y) \end{aligned}$$

where $\chi_{(k+q),(k)(q)}(Z,Y)$ is a restricted Schur polynomial. The corresponding projection operator $P_{(k+q),(k)(q)}$ needed to define the restricted trace is trivial since the carrier space of representation $(k+q)$ is one dimensional. The results of [42] then imply that

$$D\, O_q^{(k)} = 0$$

to all orders in $1/N$.

As an example of a non-trivial eigenstate of the dilatation operator D, consider a pair of stretched strings, i.e. $m_{11} = 0 = m_{22}$ and $m_{12} = 1 = m_{21}$. To obtain a manageable problem, consider the coefficient of t_1^n in the expansion of (4.32), which is given by

$$\begin{aligned} \mathcal{O} &= \sum_{\sigma\in S_{n+2}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n+1}}_{i_{\sigma(n+1)}} (Y^2)^{i_{n+2}}_{i_{\sigma(n+2)}} - \sum_{\sigma\in S_{n+2}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} (ZY)^{i_{n+2}}_{i_{\sigma(n+2)}} \\ &= \mathcal{O}_1 - \mathcal{O}_2. \end{aligned}$$

We work at the leading order in a large N expansion and we take n to be of order N. In this limit $\mathcal{O}$ simplifies. Recall that in any conformal field theory, we have the operator/state correspondence, which associates a state to every operator. Thus associated to the two operators above, we have the states $|\mathcal{O}_1\rangle$ and $|\mathcal{O}_2\rangle$. Norms of states map to correlation functions of operators

$$\langle \mathcal{O}_1|\mathcal{O}_1\rangle = \left\langle \mathcal{O}_1 \mathcal{O}_1^\dagger \right\rangle \qquad \text{and} \qquad \langle \mathcal{O}_2|\mathcal{O}_2\rangle = \left\langle \mathcal{O}_2 \mathcal{O}_2^\dagger \right\rangle.$$

In Appendix B.4 we show that

$$\frac{\langle \mathcal{O}_2|\mathcal{O}_2\rangle}{\langle \mathcal{O}_1|\mathcal{O}_1\rangle} = O(N^{-1})$$

so that at large N we can simplify

$$\mathcal{O} = \mathcal{O}_1 + O\left(N^{-1}\right) = \sum_{\sigma\in S_{n+2}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n+1}}_{i_{\sigma(n+1)}} (Y^2)^{i_{n+2}}_{i_{\sigma(n+2)}}.$$

It is interesting to trace this simplification back to the generating function $Z_{\rm AdS}(t_1,t_2,Y)$. This simplification amounts to dropping the second term in

$$\Phi_5 = \bar\varphi\cdot Z\cdot\varphi\ \bar\phi\cdot Y\cdot\varphi\ \bar\varphi\cdot Y\cdot\phi - \bar\varphi\cdot Z\cdot\phi\ \bar\phi\cdot Y\cdot\varphi\ \bar\varphi\cdot Y\cdot\varphi$$

which is intuitively appealing. Indeed, the factor $\bar\phi\cdot Y\cdot\varphi\ \bar\varphi\cdot Y\cdot\phi$ has the very natural interpretation that the two Y fields have endpoints on different branes. Acting on $\mathcal{O}$ with D given in (4.29) we obtain

$$\begin{aligned} D\mathcal{O} = -g_{\rm YM}^2(n+1)\sum_{\sigma\in S_{n+2}} Z^{i_2}_{i_{\sigma(2)}}\cdots Z^{i_{n+1}}_{i_{\sigma(n+1)}}\Big(& Y^{i_1}_{i_{\sigma(n+2)}}[Z,Y]^{i_{n+2}}_{i_{\sigma(1)}} + Y_i^{\,i_{n+2}}[Z,Y]^{i}_{i_{\sigma(1)}}\delta^{i_1}_{i_{\sigma(n+2)}} \\ & -[Z,Y]^{i_1}_{j}\,Y^{j}_{i_{\sigma(n+2)}}\delta^{i_{n+2}}_{i_{\sigma(1)}} - Y^{i_{n+2}}_{i_{\sigma(1)}}[Z,Y]^{i_1}_{i_{\sigma(n+2)}}\Big). \end{aligned}$$

To simplify this expression we need a few identities. The most basic of these follow by simply changing variables in the sum. For example

$$\begin{aligned} \sum_{\sigma\in S_{n+2}} Z^{i_2}_{i_{\sigma(2)}}\cdots Z^{i_{n+1}}_{i_{\sigma(n+1)}}Y^{i_1}_{i_{\sigma(n+2)}}[Z,Y]^{i_{n+2}}_{i_{\sigma(1)}} &= \sum_{\rho\in S_{n+2}} Z^{i_2}_{i_{\rho(2)}}\cdots Z^{i_{n+1}}_{i_{\rho(n+1)}}Y^{i_1}_{i_{\rho(1)}}[Z,Y]^{i_{n+2}}_{i_{\rho(n+2)}} \\ &= \sum_{\psi\in S_{n+2}} Z^{i_1}_{i_{\psi(1)}}\cdots Z^{i_n}_{i_{\psi(n)}}Y^{i_{n+1}}_{i_{\psi(n+1)}}[Z,Y]^{i_{n+2}}_{i_{\psi(n+2)}} \end{aligned}$$

where for the first equality $\sigma=\rho(1,n+2)$ and for the second $\rho=(1,n+1)\psi(1,n+1)$. Another useful trick is to write sums in terms of cosets of a subgroup. For example, examine the sum

$$\mathcal{S} = \sum_{\sigma\in S_{n+3}} Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n+1}}_{i_{\sigma(n+1)}}Y^{i_{n+2}}_{i_{\sigma(n+2)}}Y^{i_{n+3}}_{i_{\sigma(n+3)}}.$$

Consider the subgroup S_{n+2} defined by $\rho(n+3)=n+3$ for $\rho\in S_{n+2}$, i.e. the subgroup leaves $n+3$ inert. Now rewrite the above sum in terms of the right cosets, labeled by the integer $i=1,2,\ldots,n+3$ defined by

$$\sigma=\rho(i,n+3)\qquad\text{where}\qquad\sigma\in S_{n+3}\qquad\text{and}\qquad\rho\in S_{n+2}.$$

The sum over σ is equivalent to separately summing over ρ and i. After rewriting we find

$$\begin{aligned} \mathcal{S} = \sum_{\rho\in S_{n+2}} Z^{i_1}_{i_{\rho(1)}}\cdots Z^{i_{n+1}}_{i_{\rho(n+1)}}Y^{i_{n+2}}_{i_{\rho(n+2)}}\,{\rm Tr}(Y) + \sum_{\rho\in S_{n+2}} Z^{i_1}_{i_{\rho(1)}}\cdots Z^{i_{n+1}}_{i_{\rho(n+1)}}(Y^2)^{i_{n+2}}_{i_{\sigma(n+2)}} \\ +(n+1)\sum_{\rho\in S_{n+2}} Z^{i_1}_{i_{\sigma(1)}}\cdots Y^{i_{n+1}}_{i_{\sigma(n+1)}}(ZY)^{i_{n+2}}_{i_{\sigma(n+2)}}. \end{aligned} \tag{4.37}$$

If we use left cosets we find

$$\mathscr{S} = \sum_{\rho\in S_{n+2}} Z^{i_1}_{i_{\rho(1)}} \cdots Z^{i_{n+1}}_{i_{\rho(n+1)}} Y^{i_{n+2}}_{i_{\rho(n+2)}} \mathrm{Tr}(Y) + \sum_{\rho\in S_{n+2}} Z^{i_1}_{i_{\rho(1)}} \cdots Z^{i_{n+1}}_{i_{\rho(n+1)}} (Y^2)^{i_{n+2}}_{i_{\sigma(n+2)}}$$
$$+(n+1)\sum_{\rho\in S_{n+2}} Z^{i_1}_{i_{\sigma(1)}} \cdots Y^{i_{n+1}}_{i_{\sigma(n+1)}} (YZ)^{i_{n+2}}_{i_{\sigma(n+2)}}. \quad (4.38)$$

Comparing (4.37) and (4.38) we have

$$\sum_{\rho\in S_{n+2}} Z^{i_1}_{i_{\sigma(1)}} \cdots Y^{i_{n+1}}_{i_{\sigma(n+1)}} (ZY)^{i_{n+2}}_{i_{\sigma(n+2)}} = \sum_{\rho\in S_{n+2}} Z^{i_1}_{i_{\sigma(1)}} \cdots Y^{i_{n+1}}_{i_{\sigma(n+1)}} (YZ)^{i_{n+2}}_{i_{\sigma(n+2)}}.$$

Making use of these identities we find

$$D\mathscr{O} = -g^2_{\mathrm{YM}}(n+1)\sum_{\sigma\in S_{n+2}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} (2YZY - Y^2Z - ZY^2)^{i_{n+1}}_{i_{\sigma(n+1)}} \delta^{i_{n+2}}_{i_{\sigma(n+2)}}.$$

Next, introduce the S_{n+1} subgroup defined by $\rho(n+2) = n+2$. By making use of right cosets exactly as above, we find

$$D\mathscr{O} = -g^2_{\mathrm{YM}}(N+n+1)(n+1)\sum_{\sigma\in S_{n+1}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} (2YZY - Y^2Z - ZY^2)^{i_{n+1}}_{i_{\sigma(n+1)}}.$$

The next identity we need is obtained by starting from the sum which defines $\mathscr{O}$

$$\mathscr{O} = \sum_{\sigma\in S_{n+2}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Z^{i_{n+1}}_{i_{\sigma(n+1)}} (Y^2)^{i_{n+2}}_{i_{\sigma(n+2)}}$$

and writing in terms of cosets of the subgroup S_{n+1} which leaves $n+2$ inert. We learn that

$$\mathscr{O} = \sum_{\sigma\in S_{n+1}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Z^{i_{n+1}}_{i_{\sigma(n+1)}} \mathrm{Tr}(Y^2) + (n+1)\sum_{\sigma\in S_{n+1}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} (ZY^2)^{i_{n+1}}_{i_{\sigma(n+1)}}$$

when using right cosets, and

$$\mathscr{O} = \sum_{\sigma\in S_{n+1}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Z^{i_{n+1}}_{i_{\sigma(n+1)}} \mathrm{Tr}(Y^2) + (n+1)\sum_{\sigma\in S_{n+1}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} (Y^2Z)^{i_{n+1}}_{i_{\sigma(n+1)}}$$

when using left cosets. Recalling that n is of order N, it is clear that the first term on the right hand side of these last two formulas can be dropped at large N. We then find

$$D\mathscr{O} = 2g^2_{\mathrm{YM}}(N+n+1)\mathscr{O} + 2\mathscr{O}'$$

where

$$\mathscr{O}' = \sum_{\sigma\in S_{n+1}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} (YZY)^{i_{n+1}}_{i_{\sigma(n+1)}}.$$

In Appendix B.4 we show that

$$\frac{\langle \mathcal{O}'|\mathcal{O}'\rangle}{\langle \mathcal{O}|\mathcal{O}\rangle} = O\left(N^{-1}\right)$$

so that the $\mathcal{O}'$ contribution can be dropped at large N. Thus, in the end we have

$$D\,\mathcal{O} = 2g_{\rm YM}^2(N+n+1)\mathcal{O}$$

proving that our operators are indeed eigenstates of D at large N. There are a few points worth stressing:

1. The above calculation has summed much more than just the planar diagrams.

2. The operator $\mathcal{O}$ is a sum of many different trace structures.

3. It is not the usual 't Hooft coupling $g_{\rm YM}^2 N$ that appears, but rather $g_{\rm YM}^2(N+n+1)$. This is also fixed in the large N limit so that we have a finite renormalization of the 't Hooft coupling. Notice that $N+n+1$ is the factor of the last box in the first row of the R label of the Gauss graph operator. The same renormalization is evident in the results of [42, 130], obtained using very different methods.

4. The above result is another nice check of the equivalence of the operators defined by the generating function and the Gauss graph operators.

4.4 Discussion and Conclusions

The spectrum of type IIB string theory on $AdS_5 \times S^5$ includes giant graviton branes. These are half-BPS solutions. Superposing a collection of giant graviton branes gives a half-BPS bound state of giant gravitons. The CFT operators dual to these bound states are given by Schur polynomials [13]. Excitations of these bound states, which are in general not BPS states, are obtained by attaching open strings to the branes. CFT operators describing these excited giant graviton bound states, known as Gauss graph operators, were described in [45, 116]. Their construction employs group representation theory and a Fourier transform on a double coset. The construction is technical which obscures many details. For example, it is not easy to see how the description simplifies at large N or how systematic $1/N$ corrections

can be computed. The discussion is general in the sense that it applies to arbitrary bound states of giant gravitons. An important assumption is that the number of open strings that are attached is limited so that we are considering a small deformation of the original half-BPS operator.

The question of developing a systematic $1/N$ expansion for excited giant graviton systems is the central motivation for our study. A standard way of generating a $1/N$ expansion is through a saddle point evaluation of the path integral. Consequently, a promising starting point is an integral representations for the relevant operators. This philosophy was recently employed, in a very similar setting, in [126]. In this chapter we have obtained these integral representations, for operators dual to a bound state of two sphere giant gravitons or two AdS giant gravitons. These representations provide a simple description of the relevant operators.

The starting point for the construction is the observation that Schur polynomials can be written using Young symmetrizers. These Young symmetrizers symmetrize and antisymmetrize indices of matrix fields from which our operator is composed. Motivated by the results of [121–125] these symmetrization and antisymmetrizations can be implemented as integrals over ordinary (for symmetrization) and Grassmann (for antisymmetrization) variables which carry a single $U(N)$ index. In this way, one obtains an integral representation for Schur polynomials with two rows or two columns. One consequence of this representation is a simple integral formula for characters of the symmetric group, in irreducible representations labeled by Young diagrams with two rows, or two columns. There is a natural proposal for operators dual to excited giant graviton bound states. One of our most important results has been the demonstration that this gives an integral representation for the Gauss graph operators themselves.

We have considered the action of the dilatation operator on these novel generating functions. In the case that the open strings are described using an open string word, a key physical question is the boundary condition imposed on the string endpoints. The dilatation operator translates into a world sheet Hamiltonian as usual. The open string endpoints are represented by sources and sinks in this Hamiltonian, allowing the brane and

open string to exchange momentum, i.e. to exert a force on each other. The form of this interaction was first obtained in [108, 109] for a single brane, and in [31, 32] for a general brane bound state. In the case of the general brane bound state, the discussion makes use of identities obeyed by restricted characters and these identities were derived using sums over cosets of the group and properties of Jucys-Murphy elements. We have recovered these known results, but with much less effort: the restricted character identity follows from the statement that the integral of a total derivative vanishes! In the case that each open string is described by a single Y field, we are considering the Gauss graph operators which are eigenstates of D [45, 116]. By introducing a set of sources we have written down a set of differential operators (4.33) which create the open strings when applied to the generating function (4.36). In (4.36) there are only four integrals left to do and consequently, this is a promising starting point to generate a systematic $1/N$ expansion through a saddle point evaluation.

Our analysis has been specialized to the case of a bound state of two giant gravitons. A natural extension is to bound states of more giant gravitons. The extension is straight forward, if a little tedious. To sketch what is involved, I shall give some of the details relevant to describe bound states of three AdS giant gravitons. Since there are three rows, and the indices belonging to each row must be symmetrized, the description will involve three sets of fields, φ_i, $\bar{\varphi}^i$, ϕ_i, $\bar{\phi}^i$ as well as η_i, $\bar{\eta}^i$. There is also a new antisymmetrization, associated to the new columns of length three. As usual, we list row lengths so that $R=(p_1,p_2,p_3)$ denotes a Young diagram with p_1 boxes in the first row, p_2 in the second and p_3 in the third. The generating function for Schur polynomials with three rows is given by

$$\begin{aligned} Z_{\rm AdS}(t_1,t_2,t_3) &= \sum_{p_1=p_2+p_3}^{\infty}\sum_{p_2=p_3}^{\infty}\sum_{p_3=1}^{\infty}\frac{t_3^{p_3}\,t_2^{p_2-p_3}\,t_1^{p_1-p_2}(p_1+p_2+p_3)!}{d_{(p_1,p_2,p_3)}\,p_3!\,(p_2-p_3)!\,(p_1-p_2)!}\chi_{(p_1,p_2,p_3)}(Z) \\ &= \int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\phi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\eta\int \mathrm{d}^N\bar{\eta}\,\frac{e^{S_0}}{\pi^{3N}}\,e^{t_1S_1+t_2^2S_2+t_3^3S_3} \end{aligned}$$

where

$$\begin{aligned} S_0 &= -\bar{\varphi}^i\varphi_i-\bar{\phi}^i\phi_i-\bar{\eta}^i\eta_i \\ S_1 &= \bar{\varphi}^i Z_i^{\ j}\varphi_j \end{aligned}$$

$$S_2 = \bar{\varphi}^i \bar{\phi}^j \left(Z_i^k Z_j^l - Z_i^l Z_j^k \right) \varphi_k \phi_l$$
$$S_3 = \bar{\varphi}^i \bar{\phi}^j \bar{\eta}^k \Big(Z_i^l Z_j^m Z_k^n + Z_i^m Z_j^n Z_k^l + Z_i^n Z_j^l Z_k^m - Z_i^l Z_j^n Z_k^m - Z_i^m Z_j^l Z_k^n - Z_i^n Z_j^m Z_k^l \Big) \varphi_l \phi_m \eta_n$$

When considering excited bound states, we now have many different possibilities for the open strings that can be attached, since the open string can start on any brane and terminate on any brane. Factors that attach strings to brane 1, brane 2 and brane 3 respectively, are given by

$$\bar{\varphi}^i Y_i^j \varphi_j, \qquad \bar{\varphi}^i \bar{\phi}^j \left(Z_i^k Y_j^l - Z_i^l Y_j^k \right) \varphi_k \phi_l \qquad \text{and}$$
$$\bar{\varphi}^i \bar{\phi}^j \bar{\eta}^k \left(Z_i^l Z_j^m Y_k^n + Z_i^m Z_j^n Y_k^l + Z_i^n Z_j^l Y_k^m - Z_i^l Z_j^n Y_k^m - Z_i^m Z_j^l Y_k^n - Z_i^n Z_j^m Y_k^l \right) \varphi_l \phi_m \eta_n .$$

To describe strings that stretch between branes, we again use a product of the above factors and simply swap column indices. It is no longer necessary to attach pairs of strings. We can for example define a factor with three Y fields, corresponding to a configuration in which a Y stretches from the first to the second brane, another from the second to the third and the last from the third to the first.

We can also attach strings that stretch between the two branes. To do this, we start with the product of a factor which attaches a string to the first brane and a factor which attaches a string to the second brane. From this we obtain a pair of strings stretched between the branes by permuting column indices, thereby swapping the endpoints of the two strings. The resulting generating function is again consistent with the Gauss Law arising from the worldvolume theory of the giant gravitons. Further, it is manifestly invariant under swapping strings of the same orientation. This again suggests that these operators are identical to the corresponding Gauss graph operators, something that we have checked in some examples. It is equally easy to consider bound states with even more AdS giant gravitons, or to consider bound states of more than two sphere giant gravitons.

The worldvolume dynamics of a bound state of p giant gravitons is described in terms of the open string excitations of the branes [25]. At low energy this should become a non-Abelian Yang-Mills theory with gauge group $U(p)$ [27]. Attempts to construct this emergent Yang-Mills theory starting from a description of the giant graviton bound state in

conformal field theory have been given in [40, 130, 133, 134]. Although these initial results are encouraging, they are frustrated by the technical difficulty of the combinatorics. Perhaps the simpler description of giant graviton bound states obtained in this chapter will allow progress. Specifically, a systematic $1/N$ expansion, through a saddle point evaluation of (4.36), should lead to $1/N$ interactions which allow open strings to split and join. These interactions should then match those of the expected emergent Yang-Mills theory.

The next chapter studies some examples of $1/N$ expansions of correlation functions. These expansions are given by saddle point approximations for large N, where interesting non-perturbative effects are found. It is expected that similar expansions can be made using the generating functions described in this chapter.

Chapter 5

Asymptotic Expansions

The previous chapters studied excitations of giant graviton bound states in the limit of large N. In each case, there are significant simplifications to the corresponding quantities. We have considered quantities that can be computed exactly, however, to gain a better understanding, we want to generate systematic approximations to these quantities. The saddle point analysis is a standard method to generate a systematic $1/N$ expansion, given by a non-perturbative leading factor multiplying an asymptotic series in $1/N$. These expansions generally do not converge and are determined up to a non-perturbative term.

The discussion in this chapter follows from the analysis in [135] and [136], where saddle point analysis of complex contour integrals gives a relations between expansions at other saddles. I shall consider the saddle point analysis of a single integral, however, there are multidimensional versions of this treatment, for example [137] and [138].

The analysis in [139] suggests that extremal three-point correlators exhibit a parametric Stokes phenomenon. This was shown by mapping the exact extremal three-point correlator to a semi-classical expansion of the Schrödinger equation. In this chapter I generalize these arguments by considering the asymptotic expansion of extremal multi-point correlators which depend only on the rank of the gauge group N and the $\mathscr{R}$-charge J of each operator. These quantities have integral representations, with known exact values given by ratios of gamma functions. Since the gamma function has a nice integral representation, we can generate asymptotic expansions by saddle point analysis. I discuss aspects of these

expansions corresponding to three BMN limits. Finally, I summarize the results from the different asymptotic expansions of three-point functions corresponding to different classical configurations in the dual string theory. Further discussion and illustrative examples of the basic logic motivating this study is contained in Appendix C.

5.1 Introduction

A rich mathematical structure underpins the theory of asymptotic analysis. This structure suggests novel and interesting effects in the asymptotic analysis of QFT observables that should be fully described by the theory of resurgence. Resurgence theory will allow us to resolve non-perturbative effects using information about the asymptotic expansions of the theory.

The standard procedure of summing diagrams in the topological expansion defines the construction and summation of an asymptotic series. An equivalent construction follows from the saddle point analysis of an integral representation. In these series expansions, a non-perturbative leading contribution, given by the value of the integrand at a saddle point, multiplies a formal power series in the expansion parameter. In the case of QFT correlators, we may choose to expand in the coupling constant g or in $1/N$. A very general argument by Dyson [12] suggests that these perturbation series are asymptotic with a vanishing radius of convergence. This means that there exists no region where the perturbation series converges. The asymptotic nature of these series can be explained by the rapid growth in the number of Feynman diagrams [140, 141] that contribute to each correlation function. Following an argument Lipatov [142] it follows that the large order behaviour of these perturbation series follows a factorial growth and an example of such a growth is presented in detail in Appendix C. This factorial growth generates a divergence in the perturbation series which is now, at best, understood as a formal series. While the divergence of perturbation series might otherwise lead us to believe that perturbation theory is inconsistent and, perhaps, not very useful, it is exactly the divergence of the perturbation theory that gives rise to the non-perturbative physics that we want to study.

The saddle point approximation has a zero radius of convergence corresponding to expanding about a singular point. Additional singular points in the Borel plane generate Stokes phenomena whereby a single function has different asymptotic series representations in different regions of the complex plane. Moreover, these expansions will admit parametric Stokes phenomena where varying a parameter about a critical value in the theory causes the asymptotic expansion to change discontinuously. Then, given an asymptotic series, the procedure of Borel resummation recovers a function associated with a given asymptotic expansion and make sense of the corresponding Stokes phenomena in terms of the contour deformation of complex integrals. We associate the Stokes phenomenon with moving from a region in the complex plane where one saddle dominates, to a region where a different saddle dominates. The divergence of the perturbation series is associated to the presence other saddles, associated to singularities in the Borel plane. The Stokes phenomenon has an intuitive explanation in terms of contour integrals where the integration contour is caught on a singularity in the Borel plane and the integral picks-up a non-perturbative contribution corresponding to the residue of the singularity.

5.2 Extremal Three-Point Correlators

Consider the extremal three-point correlation function. This is in the half-BPS sector of $\mathcal{N} = 4$ SYM. While these quantities are protected against quantum correction and their values determined exactly, their perturbation series expansions are asymptotic. This means that the perturbation series diverge and are determined only up to a collection of non-perturbative factors. In general, these correlation functions have more than one saddle point and each saddle point corresponds to a classical configuration. Since these correlators have more than one saddle point, they will exhibit a Stokes phenomenon and the value of each correlator is fixed by determining the contributing saddles.

We study correlators that depend on at least one additional parameter J. Borel resummation of the asymptotic series about one saddle point recovers information about the function at another saddle point. Therefore, the corresponding Borel sum also depends

on J, and varying J introduces a parametric Stokes phenomenon to the correlator. This parametric Stokes phenomenon corresponds to moving between saddle points in the asymptotic expansion. The ultimate goal is to make sense of the perturbation series by relating the asymptotic series generated about each saddle point by varying J.

Suppose $\mathcal{O}$ is a half-BPS operator that is built from traces of products of $N \times N$ complex matrices Z and $Z^\dagger$. The standard the path integral representation for an unnormalized correlation function

$$\langle \mathcal{O} \rangle = \int \left[\mathrm{d}Z \, \mathrm{d}Z^\dagger \right] e^{-S} \mathcal{O} \qquad \text{with} \qquad S = \mathrm{Tr}\left(Z Z^\dagger \right) \tag{5.1}$$

gives a starting point for the saddle point analysis of correlators. The integral in (5.1) defines the expectation value of $\mathcal{O}$ as a function of J and N. While (5.1) give an integral representation for the correlator, the number of integration variables increases with N, so we will use a different integral representation instead.

Let χ_{1^J} be a Schur polynomial associated with a Young diagram comprising a single column with J boxes. Let $J = J_1 + J_2$. These correspond to sphere giants, but a similar analysis is possible for dual giants. The exact three-point correlation function is

$$\left\langle \chi_{1^{J_1}} \chi_{1^{J_2}} \chi^\dagger_{1^J} \right\rangle = \frac{\Gamma(N+1)}{\Gamma(N-J+1)} \delta_{J_1+J_2,J} = \delta_{J_1+J_2,J} N(N-1)(N-2)\cdots(N-J+1). \tag{5.2}$$

Changing the value of J corresponds to assigning a different $\mathscr{R}$-charge value to one of the fields in the correlator and to a different classical configuration in the dual string theory. When $J \sim O(1)$, each operator in (5.2) corresponds to an order 1 collection of Kaluza-Klein gravitons. Then, when $J \sim O(\sqrt{N})$, these operators correspond collections of strings. Finally, for $J \sim O(N)$, these operators are giant graviton states. Each of these limits corresponds to a different classical limits in the dual gravity, and varying J interpolates from one classical configuration to the next. We would like to reproduce this picture in the asymptotic expansion, by identifying the corresponding saddle contributions associated with each classical limit. Then, we would like to relate the series coefficients in the asymptotic expansions, thereby generating the resurgence relations among the different saddle points.

We can expand the product of gamma functions in (5.2) as a product of asymptotic series by saddle point expansion of the following integrals

$$\Gamma(s)=\int_0^\infty \frac{\mathrm{d}\,x}{x}e^{-x}x^s \qquad \text{and} \qquad \frac{1}{\Gamma(s)}=-\frac{s\,e^{-\pi i s}}{2\pi i}\int_{\mathscr{H}}\frac{\mathrm{d}\,x}{x}e^{-x}x^{-s} \tag{5.3}$$

where $\mathscr{H}$ is the doubly infinite *Hankel contour*. Then the asymptotic expansion of the ratio is given by the product of the asymptotic expansions of (5.3). For generic values of s we find

$$\Gamma(s)=\sqrt{2\pi s}\;s^{s-1}e^{-s}\sum_{m=0}^{\infty}\frac{\gamma_m}{s^m} \qquad \text{and} \qquad \frac{1}{\Gamma(s)}=\frac{1}{\sqrt{2\pi s}\;s^{s-1}e^{-s}}\sum_{m=0}^{\infty}(-1)^m\frac{\gamma_m}{s^m}$$

where the Stirling coefficients γ_m satisfy

$$\sum_{j=0}^{m}(-1)^m\gamma_j\gamma_{m-j}=0.$$

Then, using these expressions, we can generate asymptotic expansions for (5.2).

First consider the case $J=j\sim O(1)$, corresponding to a gas of point gravitons, then

$$\frac{\Gamma(N+1)}{\Gamma(N-J+1)}\sim e^{j\log(N)}\sum_{n=0}^{\infty}\frac{B_n(j)}{n!}q_n(j)\frac{1}{N^n} \tag{5.4}$$

where

$$q_0(j)=1 \qquad \text{and} \qquad q_n(j)=\prod_{i=1}^{n}\big(j-i\big)$$

and $B_n(j)$ is the n-th Nörlund polynomial. These polynomials are defined by the generating function

$$\left(\frac{z}{e^z-1}\right)^x=\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!}$$

where $B_n(1)=B_n$ is the n-th Bernoulli number. Next, consider the case $J=j\sqrt{N}$, then (5.2) expands to

$$\begin{aligned}\frac{\Gamma(N+1)}{\Gamma(N-J+1)}\sim e^{j\sqrt{N}\log(N)}e^{-\frac{j^2}{2}}\Bigg\{1&-\left(\frac{j^3-3j}{6}\right)\frac{1}{N^{1/2}}+\left(\frac{j^6-12j^4+27j^2}{72}\right)\frac{1}{N}\\&-\left(\frac{5j^9-135j^7+999j^5-2025j^3+540j}{6480}\right)\frac{1}{N^{3/2}}\\&+\left(\frac{5j^{12}-240j^{10}+3726j^8-22032j^6+44685j^4-19440j^2}{155520}\right)\frac{1}{N^2}\\&-\left(\frac{7j^{15}-525j^{13}+13986j^{11}-164430j^9+862191j^7-1788885j^5+1020600j^3}{6531840}\right)\frac{1}{N^{5/2}}+\cdots\Bigg\}\end{aligned} \tag{5.5}$$

Finally, let $J = jN$, for $0 < j < 1$, then,

$$\begin{aligned}\frac{\Gamma(N+1)}{\Gamma(N-J+1)} \sim e^{jN\log(N)} e^{-jN} \left(1-j\right)^{-N(1-j)-1/2} \Bigg\{ 1 &+ \left(\frac{1}{12} - \frac{1}{12(1-j)}\right)\frac{1}{N} \\ &+ \left(\frac{1}{288} - \frac{1}{144(1-j)} + \frac{1}{288(1-j)^2}\right)\frac{1}{N^2} \\ &- \left(\frac{139}{51840} + \frac{1}{3456(1-j)} - \frac{1}{3456(1-j)^2} - \frac{139}{51840(1-j)^3}\right)\frac{1}{N^3} \\ &- \left(\frac{571}{2488320} - \frac{139}{622080(1-j)} - \frac{1}{82944(1-j)^2} - \frac{139}{622080(1-j)^3} + \frac{571}{2488320(1-j)^4}\right)\frac{1}{N^4} + \cdots \Bigg\}.\end{aligned} \tag{5.6}$$

Let $x = 1/(1-j)$, then

$$\frac{\Gamma(N+1)}{\Gamma(N-J+1)} \sim e^{\frac{x-1}{x}N\log(N)} e^{\frac{1-x}{x}N} x^{\frac{N}{x}+1/2} \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}(-1)^n \gamma_n \gamma_{k-n} x^n\right)\frac{1}{N^k} \tag{5.7}$$

where $\gamma_0 = 1$, $\gamma_1 = 1/12$, $\gamma_2 = 1/288$, $\gamma_3 = -139/51840$, and so on, are the Stirling numbers from the expansion of the gamma function.

Notice that the coefficient of the term with the largest power of j terms in these polynomials grows as a power, while those with the smallest power of j grow like a Bernoulli number. Indeed, at leading order in j, the n-th Nörlund polynomial has coefficient $2^{-(n+1)}$, while at the lowest order in j, the $2n$-th Nörlund polynomial has coefficient $B_{2n}/2n$. The constant term in the coefficient of N^{-n} is the n-th Stirling coefficient γ_n. Clearly, the series (5.4), (5.5) and (5.6) are asymptotic, having approximately factorial growth.

We can check if the expansion in (5.5) is recovered from (5.4) by rescaling $j \mapsto j\sqrt{N}$ and collecting terms and resumming over j. Indeed, this is the case. It suggests that (5.4) and (5.5) are expansions about a common saddle, corresponding to a common classical limit. Similarly, (5.6) is recovered from (5.4) and (5.5) after appropriate rescaling and summing over j and with an additional summation over N. The additional summation over N recovers the additional non-perturbative leading factors in $J \sim O(N)$ and demonstrates that simple rescaling and rearranging of terms is insufficient to relate the different expansions since an additional non-perturbative factor must be included. This corresponds to a parametric Stokes phenomenon associated with moving between saddle points and expanding about different classical limits of the theory, by varying a single parameter J.

Each of these expansions was checked to 8-th order in $1/N$. Therefore, (5.4) and (5.5) are expansions about the same saddle point of the correlator (5.2), while (5.7) is an expansion about a different saddle point of (5.2).

Similarly, we can consider expansions of the normalized Schur polynomials

$$O_R = \frac{1}{\sqrt{f(R)}}\chi_R \qquad \text{where} \qquad \left\langle O_R O_S^\dagger \right\rangle = \delta_{R,S}.$$

The three-point correlation function of the rescaled operators, labeled by a single column of boxes, is

$$\left\langle O_{1^{J_1}} O_{1^{J_2}} O_{1^{J_3}}^\dagger \right\rangle = \delta_{J_1+J_2,J_3} \sqrt{\frac{\Gamma(N-J_1+1)\,\Gamma(N-J_2+1)}{\Gamma(N-J_3+1)\,\Gamma(N+1)}}\,.$$

Again, we expect that this correlator to have an asymptotic expansion in $1/N$. Similarly, the three-point correlation function of the rescaled operators, labeled by a single row of boxes, is

$$\left\langle O_{J_1} O_{J_2} O_{J_3}^\dagger \right\rangle = \delta_{J_1+J_2,J_3} \sqrt{\frac{\Gamma(N+J_3+1)\,\Gamma(N)}{\Gamma(N+J_1+1)\,\Gamma(N+J_2+1)}}\,.$$

Now let $J_3 = J_1 + J_2$. We can now study the asymptotic expansions of these normalized correlators. Let $j_1, j_2 \sim O(1)$, then for $J_1 = j_1$, $J_2 = j_2$, we find

$$\begin{aligned}\left\langle O_{1^{j_1}} O_{1^{j_2}} O_{1^{j_3}}^\dagger \right\rangle \sim \Bigg\{ 1 &- \left(\frac{j_1 j_2}{2}\right)\frac{1}{N} + \left(\frac{j_1^2 j_2^2 - 2j_1^2 j_2 - 2j_1 j_2^2 + 2j_1 j_2}{8}\right)\frac{1}{N^2} \\ &- \left(\frac{j_1^3 j_2^3 - 6j_1^3 j_2^2 - 6j_1^2 j_2^3 + 8j_1^3 j_2 + 8j_1 j_2^3 + 18 j_1^2 j_2^2 - 12 j_1^2 j_2 - 12 j_1 j_2^2 + 4 j_1 j_2}{48}\right)\frac{1}{N^3} + \cdots \Bigg\}.\end{aligned} \tag{5.8}$$

Similarly, for $J_1 = j_1\sqrt{N}$ and $J_2 = j_2\sqrt{N}$, we find

$$\begin{aligned}\left\langle O_{1^{J_1}} O_{1^{J_2}} O_{1^{J_3}}^\dagger \right\rangle \sim e^{-j_1 j_2/2} \Bigg\{ 1 &- \left(\frac{j_1^2 j_2 + j_1 j_2^2}{4}\right)\frac{1}{\sqrt{N}} \\ &+ \left(\frac{3j_1^4 j_2^2 + 3j_1^2 j_2^4 + 6j_1^3 j_2^3 - 16 j_1^3 j_2 - 16 j_1 j_2^3 - 24 j_1^2 j_2^2 + 24 j_1 j_2}{96}\right)\frac{1}{N} + \cdots \Bigg\}.\end{aligned} \tag{5.9}$$

Finally, for $J_1 = j_1 N$ and $J_2 = j_2 N$, we find

$$\begin{aligned}\left\langle O_{1^{J_1}} O_{1^{J_2}} O_{1^{J_3}}^\dagger \right\rangle \sim &\left(1-j_1\right)^{(1-j_1)N/2} \left(1-j_2\right)^{(1-j_2)N/2} \left(1-j_1-j_2\right)^{-(1-j_1-j_2)N/2} \\ &\times \left\{ 1 + \left(\frac{j_1 j_2}{4}\right)\frac{1}{N^2} + \left(\frac{j_1^2 j_2 + j_1 j_2^2}{4}\right)\frac{1}{N^3} + \cdots \right\}.\end{aligned} \tag{5.10}$$

Similarly, the normalized correlator labeled by Young diagrams with a single row of boxes, for $J_1, J_2 \sim O(1)$, gives

$$\left\langle O_{J_1} O_{J_2} O_{J_3}^{\dagger} \right\rangle \sim \frac{1}{\sqrt{N}} \left\{ 1 + \left(\frac{j_1 j_2}{2} \right) \frac{1}{N} + \left(\frac{j_1^2 j_2^2 - 2 j_1^2 j_2 - 2 j_1 j_2^2 - 2 j_1 j_2}{8} \right) \frac{1}{N^2} + \left(\frac{j_1^3 j_2^3 - 6 j_1^3 j_2^2 - 6 j_1^2 j_2^3 + 6 j_1^2 j_2^2 + 8 j_1^3 j_2 + 8 j_1 j_2^3 + 12 j_1^2 j_2 + 12 j_1 j_2^2 + 4 j_1 j_2}{48} \right) \frac{1}{N^3} + \cdots \right\}. \tag{5.11}$$

Similarly, choosing $J_1, J_2 \sim O(\sqrt{N})$, gives

$$\left\langle O_{J_1} O_{J_2} O_{J_3}^{\dagger} \right\rangle \sim e^{j_1 j_2/2} \frac{1}{\sqrt{N}} \left\{ 1 - \left(\frac{j_1^2 j_2 + j_1 j_2^2}{4} \right) \frac{1}{\sqrt{N}} + \left(\frac{3 j_1^4 j_2^2 + 3 j_1^2 j_2^4 + 6 j_1^3 j_2^3 + 14 j_1^3 j_2 + 24 j_1^2 j_2^2 + 16 j_1 j_2^3 - 24 j_1 j_2}{96} \right) \frac{1}{N} + \cdots \right\} \tag{5.12}$$

and choosing $J_1, J_2 \sim O(N)$ gives

$$\left\langle O_{J_1} O_{J_2} O_{J_3}^{\dagger} \right\rangle \sim \left(1 + j_1\right)^{-\left(1+j_1\right)N/2} \left(1 + j_2\right)^{-\left(1+j_2\right)N/2} \left(1 + j_1 + j_2\right)^{\left(1+j_1+j_2\right)N/2} \frac{1}{N} \times \left\{ 1 - \left(\frac{j_1 j_2}{4} \right) \frac{1}{N^2} + \left(\frac{j_1^2 j_2 + j_1 j_2^2}{4} \right) \frac{1}{N^3} + \cdots \right\}. \tag{5.13}$$

Again, these asymptotic expansions carry different non-perturbative factors multiplying perturbative series expansions in $1/N$. The appearance of additional parameters j_1 and j_2, in (5.8), (5.9) and (5.10), and again in (5.11), (5.12) and (5.13) indicate the presence of multiple parametric Stokes phenomena in this class of correlators. The choice of normalization makes different physical effects apparent in each expansion. A judicious choice of normalization makes the different physical processes manifest in each correlator. Again, the fact that non-perturbative leading factors are not related by simple rescalings of the parameters indicates that each asymptotic series is characterized by a parametric Stokes phenomenon associated to the appearance, or disappearance, of different saddle contributions. In this case, the appearance of j_1 and j_2 in the series expansions suggests the presence of two independent parametric Stokes phenomena associated with varying each parameter independently. Unfortunately, the polynomial dependence on these two parameters makes the recovery of the resurgence relations among the different saddle points significantly more difficult than in the non-normalized case.

5.3 Discussion

It has been shown that extremal three-point giant graviton correlators exhibit a parametric Stokes phenomenon. This means that the exact value of the three-point function is associated to different classical configurations depending on the value of the $\mathscr{R}$-charge J relative to the rank of the gauge group N. We associated to each classical configuration a different pole in the Borel plane of the corresponding saddle-point integral. Moreover, the different normalizations of the correlators introduce additional parameters in the asymptotic series, to which there must correspond a different parametric Stokes phenomenon. By varying each parameter in the integral representation, we expect we can interpolate between different classical configurations where the amplitude of each transition to a new classical configuration of the system is weighted by a distinct non-perturbative factor. By smoothly varying the $\mathscr{R}$-charge J, the saddle-point integral encounters different singularities in the Borel plane associated to different classical configurations. Traversing a singularity of the contour integral in the Borel plane corresponds to transitioning from one classical configuration to another. We see this in the discontinuous jump in the value of the asymptotic expansion of the correlator about one classical configuration to a new classical configuration with a new asymptotic expansion. Since the same correlation function describes different classical systems, we expect that this parametric Stokes phenomenon is a generator for a duality between theories of particles, strings, branes and other heavier objects with $\mathscr{R}$-charge J in the string theory. That this parametric Stokes phenomenon gives a plausible explanation for the duality between systems of particles, strings and branes adds to the significance of the contribution made in the previous previous chapters to the construction of asymptotic expansions of correlation functions. To make this duality concrete requires that we demonstrate that the series coefficients of one asymptotic series about one saddle can be recovered from the series coefficients of the asymptotic series about another saddle. This forms the basis of the work in [143].

Chapter 6

Conclusion

I conclude here with a summary of the results, highlighting key points and give a general commentary on the contents of this work. I also comment below on similar work, open problems and future prospects.

6.1 Summary

CFT operators whose bare dimensions scale as a power of N correspond to heavy states. These operators probe non-perturbative physics in the dual gravity theory. Some of these operators correspond to massive, half-BPS D3-brane solutions, that carry angular momentum. These are giant gravitons which are dual to operators built from $n \sim O(N)$ complex scalar fields corresponding to Schur polynomials and labeled by Young diagrams with n boxes, in the CFT. While superpositions of giant gravitons, corresponding to giant graviton bound states, are also half-BPS states, their excitations are not. These excitations are obtained by attaching open strings to the giant graviton branes and are dual to restricted Schur polynomials in the CFT. Allowed configurations of open string attachments satisfy a Gauss Law condition on the brane worldvolume. Operators describing these excitations are the Gauss graph operators. These excitations correspond to perturbations of the unexcited states. With this in mind, we would like to generate systematic $1/N$ expansions of these observables.

Methods from group representation theory allow us to compute quantities exactly. These methods were used in Chapter 3 to compute of the exact spectrum of the one-loop dilatation operator on excitations of bound states of two giant gravitons. This spectrum was shown to match that of decoupled harmonic oscillators in the large N limit. The analytic results of this analysis agree beautifully with the numerically determined solutions and shows a dramatic simplification of the dynamics of these systems, at large N. This has given one of the earliest indications that integrability of the theory persists beyond the planar limit. However, these exact solutions give no insight into how we might construct systematic approximations.

While correlation functions of operators whose bare dimension remains constant as $N \to \infty$ are well approximated by the planar limit of the Feynman diagrams expansion, no such analogue is available for heavy operators. This follows since the genus of the Feynman diagrams fail to capture the additional N dependence of heavy operators in the non-planar limit. The saddle point analysis is a standard method of generating $1/N$ expansions. However, this requires an integral representation of the relevant operators. This is a challenging problem when considering matrix fields, since the number of integration variables grows with N. In contrast, the generating function technology discussed in Chapter 4 has only four integration variables, which is a promising starting point for saddle point expansions. This was used to construct the partition function for bound states of two giant gravitons, and a heuristic argument was given on how this could be extended to more giants. Moreover, the generating function of giant graviton bound states was shown to be an eigenfunction of the one-loop dilatation operators at large N. This is a significant new result.

The asymptotic expansions of the three-point correlator were computed in three different BMN limits in Chapter 5. Surprisingly, the non-perturbative contribution to the $J \sim O(1)$ and $J \sim O(\sqrt{N})$ expansions suggest that these limits correspond to expansions about a common saddle and therefore are perturbations about a common classical configuration. Moreover, the $J \sim O(N)$ expansion is a function of the Stirling numbers while the $J \sim O(1)$ and $J \sim O(\sqrt{N})$ expansions are functions of generalized Bernoulli

numbers. This is interesting since the combinatorial data for these correlation functions is given by a ratio of gamma functions, which have a simple saddle point structure. However, varying the parameters in the correlation function introduces a parametric Stokes phenomenon brought by the interaction between saddles. It is expected that a complete description of the $1/N$ expansion of these quantities will demonstrate a parametric Stokes phenomenon associated to the exchange of dominant saddles at different values of J, in the large N limit. Importantly, the different finite saddle contributions multiplying each expansion suggests that these are indeed perturbations around different classical configurations.

6.2 Discussion

A common feature of the constructions in this thesis is the use of ordinary variables (for symmetrization) and Grassmann variables (for antisymmetrization). This reflects the symmetric and antisymmetric representations of each of the operators in terms of the Young diagrams. These were used to write down the dual monomial functions in the spin-chain construction in Chapter 3 and again in Chapter 4 where symmetrization and antisymmetrization was implemented as integrals over ordinary and Grassmann variables. A new feature of the restricted Schur construction in Chapter 4 is the use of Young symmetrizers that provide an efficient construction of projectors needed to build the restricted Schur polynomials. A key idea in the analysis in Chapter 3 involved a mapping to a spin-chain, which appeared again in Chapter 4 where it described the dynamical lattice allowing interactions between fields on the strings and fields on the branes.

A byproduct of the generating function developed in Chapter 4 is the ability to generate characters of the symmetric group. In particular, there is the possibility to generate the asymptotic characters where n is large. However, the saddle point analysis does not easily disentangle the information regarding these asymptotic characters.

Furthermore, we would like to demonstrate a direct re-encoding of the asymptotic series coefficients of the three-point correlator expansions about different saddles. It is possible

to generate a $1/N$ expansion of the three-point correlator in different BMN limits. These different expansions correspond to choosing different values of J proportional to a power of N. Then, following the insights from resurgence theory, we should expect we can relate these different expansions and in turn relate the different classical limits of the theory. However, finding this relationship is not straight forward, and while it might be possible to do so directly from the expansions, it is likely that such a matching will require information from a single integral representation.

6.3 Outlook

Given the previous discussion, some comments are now in order. A next step could be to use the integral representations in Chapter 4 to study the asymptotic expansion of correlators and symmetric group characters. In principle this can be done by direct saddle point computation of the generating function and the characters can be read off the coefficient of the corresponding term in the partition function at large n corresponding to late order terms. However, this poses an additional question - Is it meaningful to expect that a generic integral representation of the character of the symmetric group is directly accessible from this expansion and if so, what would it mean? The reason for this question is that for each choice of parameters t_i in the generating function, there will be a corresponding representation, however, at large n, these characters will appear later in the series expansion.

The stability of these series expansions requires the computation of many terms. For a given number of boxes n there will correspond many sequences t_i labeling states and corresponding representations. However, it is not yet clear that the asymptotic representation will isolate a single representation, rather the asymptotic representation will generate a series with a polynomial dependence on the powers of t_i and it is yet to be shown that these can be disentangled at large n to give a specific character. Moreover, the appearance of late order terms in the expansion in the large n limit means that the for a given choice of n and powers of t_i in the expansion, many terms must be computed before a pattern is found and the stability of later order terms is computationally intensive to

obtain. Until the large N and large n computation of these character representations is carried out, the limits of this representation remain unknown.

We have considered operators labeled by only two long rows/columns. More general computations are necessary for more general configurations, however, the extensions to more rows and columns are possible in principle. It is an interesting prospect to consider extensions to the generating function that would allow for operators with $O(N)$ rows or columns. Such a construction would allow us to interpolate between the operators of bare dimension $O(N)$ and operators of bare dimension $O(N^2)$. These operators are sufficiently heavy so as to back-react on the spacetime corresponding to new classical configurations. The inclusion of these back-reacted spacetimes requires computational technology associated with LLM geometries [14]. For now, the direct construction of multi-row/column generating functions can be carried out, in principle, for only finitely many rows and columns. It is similarly expected that the large N limit will again be a crucial element in the simplification of the generating function representation that will make the construction of the operators a tractable exercise.

There is some scope to extend the analysis and thereby interpolate between different classical configurations. In the simple case of the three-point functions, correlators depending on parameters N and J have well defined asymptotic series and there is, in principle, no limit to the scaling of N and J or the number of boxes that appear in each operator representation that would allow $O(N^2)$ physics to be accessible via asymptotic series expansions and by relating the asymptotic series. However, this requires more work to relate the different asymptotic series for different values of N and J and thereby relate the different operators with different scaling dimensions.

Following this study, several open problems remain. Computations of the dilatation operator have shown the theory to be integrable in certain large N limits, and it is an interesting prospect to consider if integrability can be proven in general. A more interesting open problem is to determine the explicit $1/N$ expansion of heavy operators. The asymptotic expansions of the three-point functions suggests a rich mathematical structure the underlying their correlation functions. It will be interesting to find the exact resurgence

relations among the different asymptotic expansions connecting the theory that interpolate between the different classical limits. Additionally, it remains to be shown whether the asymptotic characters of the symmetric group corresponding to Young diagrams, where the number of boxes $n \to \infty$, can be determined from a saddle point expansion of the giant graviton bound state generating function.

A related analysis was performed in [144] wherein the generating function of a specific $U(N)$ matrix model admits a power series representation with integer coefficients. These integer coefficients count the superconformal index of BPS states in a 4-dimensional gauge theory with a given charge. In the context of the AdS/CFT correspondence, these coefficients also count the index of BPS states of the same charge in the dual gravity in theory in 5-dimensions. In that analysis, the integral over unitary matrices is also a generating function of graviton states represented as a sum over products of traces of powers of the unitary matrices, with coefficients given by the characters of the symmetric group S_N.

In addition to the large N analysis presented here, the finite N dynamics have been studied in the context of Spin Matrix Theory (SMT) describing the near-BPS limits of $\mathcal{N}=4$ SYM [145–147]. SMT describes the decoupling limit of $\mathcal{N}=4$ SYM, giving rise to an effective field theory described by the one-loop dilatation operator of $\mathcal{N}=4$ SYM. This theory breaks the exact BPS conditions and suggests a path to understanding near-BPS black holes. Importantly, SMT provides an avenue to study non-planar integrability in different sectors of $\mathcal{N}=4$ SYM.

Approaches, similar to those presented here, using coherent state generating functions have been shown to encode combinatorial data of correlators [126–128]. These approaches are also relevant to the systematic generation of $1/N$ expansions. The $1/N$ expansion of extremal three-point correlators was studied in [139] in the context of Exact WKB Analysis of a dual Schrödinger equation. The work of Chapter 5 extends this study to general integral representations and forms the basis of another article [143].

Appendices

Appendix A

Nonplanar Integrability

This appendix contains additional details and discussion the dilatation operator spectrum for two impurities, Young's orthogonal representation and inner products on the space of monomials, following Chapter 3.

A.1 Dilatation Operator for AdS Giants with Two Impurities

We are interested in the two AdS giant system for the case that our operators have two impurities. In this case there are four operators that can be produced once r is given. These operators are

$$\chi_A(b_0, b_1) = \chi_{\begin{smallmatrix}\square\square\square\square\square\square\square\square\square\square\\ \square\square\square\square\square\end{smallmatrix}}\left(\begin{smallmatrix}\square\square\square\square\square\square\square\square\\ \square\square\square\square\square\end{smallmatrix}, \begin{smallmatrix}\square\square\end{smallmatrix}\right)(Z, Y)$$

$$\chi_B(b_0, b_1) = \chi_{\begin{smallmatrix}\square\square\square\square\square\square\square\square\square\square\\ \square\square\square\square\square\end{smallmatrix}}\left(\begin{smallmatrix}\square\square\square\square\square\square\square\square\square\\ \square\square\square\square\end{smallmatrix}, \begin{smallmatrix}\square\square\end{smallmatrix}\right)(Z, Y)$$

$$\chi_C(b_0, b_1) = \chi_{\begin{smallmatrix}\square\square\square\square\square\square\square\square\square\square\\ \square\square\square\square\square\end{smallmatrix}}\left(\begin{smallmatrix}\square\square\square\square\square\square\square\square\square\\ \square\square\square\square\end{smallmatrix}, \begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)(Z, Y)$$

$$\chi_D(b_0, b_1) = \chi_{\begin{smallmatrix}\square\square\square\square\square\square\square\square\square\square\\ \square\square\square\square\square\end{smallmatrix}}\left(\begin{smallmatrix}\square\square\square\square\square\square\square\square\square\square\\ \square\square\square\end{smallmatrix}, \begin{smallmatrix}\square\square\end{smallmatrix}\right)(Z, Y).$$

The length of the rows in the first two labels are taken to infinity at large N. The third label is exactly as shown even at large N. The number of boxes removed from each row of the first label to obtain the second label are exactly as shown even at large N. The dilatation

operator is

$$
\begin{aligned}
D\,O_A(b_0,b_1) =& 2g_{\rm YM}^2(N+b_0+b_1+1)\left[\frac{1}{(b_1+2)}\sqrt{\frac{b_1+1}{b_1+3}}\;O_C(b_0-1,b_1+2)\right.\\
&\left.+\frac{2}{(b_1+2)^2}O_A(b_0,b_1)-\frac{b_1+4}{(b_1+2)^2}\sqrt{\frac{b_1+1}{b_1+3}}\;O_B(b_0-1,b_1+2)\right]\\
&+2g_{\rm YM}^2\sqrt{(N+b_0+b_1+1)(N+b_0-1)}\left[\frac{b_1}{(b_1+2)^2}\sqrt{\frac{b_1+3}{b_1+1}}\;O_B(b_0,b_1)\right.\\
&\left.-\frac{1}{b_1+2}\sqrt{\frac{b_1+3}{b_1+1}}\;O_C(b_0,b_1)+2\frac{1}{(b_1+2)^2}O_D(b_0-1,b_1+2)\right]\\
D\,O_D(b_0,b_1) =& 2g_{\rm YM}^2(N+b_0)\left[\frac{b_1-2}{b_1^2}\sqrt{\frac{b_1+1}{b_1-1}}\;O_B(b_0+1,b_1-2)\right.\\
&\left.-\frac{1}{b_1}\sqrt{\frac{b_1+1}{b_1-1}}\;O_C(b_0+1,b_1-2)+2\frac{1}{b_1^2}O_D(b_0,b_1)\right]\\
&+2g_{\rm YM}^2\sqrt{(N+b_0)(N+b_0+b_1)}\left[\frac{2}{b_1^2}O_A(b_0+1,b_1-2)\right.\\
&\left.-\frac{b_1+2}{b_1^2}\sqrt{\frac{b_1-1}{b_1+1}}\;O_B(b_0,b_1)+\frac{1}{b_1}\sqrt{\frac{b_1-1}{b_1+1}}\;O_C(b_0,b_1)\right]\\
D\,O_C(b_0,b_1) =& g_{\rm YM}^2\sqrt{(N+b_0+1)(N+b_0+b_1+1)}\left[-\frac{2}{b_1+2}\sqrt{\frac{b_1+3}{b_1+1}}\;O_A(b_0,b_1)\right.\\
&\left.-O_C(b_0-1,b_1+2)+\frac{b_1+4}{b_1+2}O_B(b_0-1,b_1+2)\right]\\
&+g_{\rm YM}^2\sqrt{(N+b_0+b_1)(N+b_0)}\left[\frac{2}{b_1}\sqrt{\frac{b_1-1}{b_1+1}}\;O_D(b_0,b_1)\right.\\
&\left.-O_C(b_0+1,b_1-2)+\frac{b_1-2}{b_1}O_B(b_0+1,b_1-2)\right]\\
&+g_{\rm YM}^2(2N+2b_0+b_1-3)\,O_C(b_0,b_1)\\
&+g_{\rm YM}^2\frac{(N+b_0)(4-4b_1-2b_1^2)-b_1^3-b_1^2+4b_1}{b_1(b_1+2)}O_B(b_0,b_1)\\
&-2g_{\rm YM}^2\frac{(N+b_0-1)}{b_1+2}\sqrt{\frac{b_1+3}{b_1+1}}\;O_D(b_0-1,b_1+2)\\
&+2g_{\rm YM}^2\frac{(N+b_0+b_1)}{b_1}\sqrt{\frac{b_1-1}{b_1+1}}\;O_A(b_0+1,b_1-2)\\
D\,O_B(b_0,b_1) =& g_{\rm YM}^2\sqrt{(N+b_0-1)(N+b_0+b_1+1)}\left[2\frac{b_1}{(b_1+2)^2}\sqrt{\frac{b_1+3}{b_1+1}}\;O_A(b_0,b_1)\right.\\
&\left.+\frac{b_1}{b_1+2}O_C(b_0-1,b_1+2)-\frac{b_1(b_1+4)}{(b_1+2)^2}O_B(b_0-1,b_1+2)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+g_{\rm YM}^2\sqrt{(N+b_0+b_1)(N+b_0)}\left[-2\frac{b_1+2}{b_1^2}\sqrt{\frac{b_1-1}{b_1+1}}\,O_D(b_0,b_1)\right.\\
&\qquad\left.+\frac{b_1+2}{b_1}O_C(b_0+1,b_1-2)-\frac{(b_1+2)(b_1-2)}{b_1^2}O_B(b_0+1,b_1-2)\right]\\
&+g_{\rm YM}^2\frac{(N+b_0)(4-2b_1^2-4b_1)-b_1^3-b_1^2+4b_1}{b_1(b_1+2)}O_C(b_0,b_1)\\
&+g_{\rm YM}^2\frac{2(N+b_0)(b_1^4+4b_1^3+4b_1^2-8)+b_1^5+5b_1^4+8b_1^3-16b_1}{b_1^2(b_1+2)^2}O_B(b_0,b_1)\\
&+2g_{\rm YM}^2\frac{(N+b_0-1)b_1}{(b_1+2)^2}\sqrt{\frac{b_1+3}{b_1+1}}\,O_D(b_0-1,b_1+2)\\
&-2g_{\rm YM}^2\frac{(N+b_0+b_1)(b_1+2)}{b_1^2}\sqrt{\frac{b_1-1}{b_1+1}}\,O_A(b_0+1,b_1-2).
\end{aligned}
$$

A.2 Oscillators

We look for operators with good scaling dimension that diagonalize the equation (3.7). These take the form of superpositions

$$\sum_{b_1} f(b_0,b_1)O(b_0,b_1).$$

There are no independent sums over b_0 and b_1 because $n=2b_0+b_1$ is fixed. Since

$$
\begin{aligned}
\sum_{b_1} f(b_0,b_1)D\,O(b_0,b_1) &= -\alpha g_{\rm YM}^2\left(\sum_{b_1} O(b_0,b_1)\Big[\sqrt{(N+b_0)(N+b_0+b_1)}\,\big(f(b_0-1,b_1+2)\right.\\
&\qquad\left.+f(b_0+1,b_1-2)\big)\Big]\right.\\
&\qquad\left.-\sum_{b_1} O(b_0,b_1)(2N+2b_0+b_1)f(b_0,b_1)\right)\\
&=\sum_{b_1}\kappa f(b_0,b_1)O(b_0,b_1)
\end{aligned}
$$

where we have assumed $N+b_0,b_1\gg 1$, the $f(b_0,b_1)$ satisfy the recursion relation

$$
\begin{aligned}
-\alpha g_{\rm YM}^2[\sqrt{(N+b_0)(N+b_0+b_1)}\,(f(b_0-1,b_1+2)+f(b_0+1,b_1-2))&\\
-(2N+2b_0+b_1)f(b_0,b_1)] = \kappa f(b_0,b_1).&\quad\text{(A.1)}
\end{aligned}
$$

In the large $N+b_0,b_1$ regime it is accurate to take a continuum limit of this recursion relation. This gives a particularly simple description of the coefficients $f(b_0,b_1)$. Towards

this end, introduce the continuous variable $\rho = 2b_1/\sqrt{N+b_0}$, replace $f(b_0,b_1)$ with $f(\rho)$ and expand

$$\sqrt{(N+b_0+b_1)(N+b_0)} = (N+b_0)\left(1-\frac{1}{2}\frac{b_1}{N+b_0}-\frac{1}{8}\frac{b_1^2}{(N+b_0)^2}+\cdots\right)$$
$$f\left(\rho-\frac{1}{\sqrt{N+b_0}}\right) = f(\rho)-\frac{1}{\sqrt{N+b_0}}\frac{\partial f}{\partial\rho}+\frac{1}{2(N+b_0)}\frac{\partial^2 f}{\partial\rho^2}+\cdots.$$

These expansions are only valid if $b_1 \ll N+b_0$, which is certainly not always the case. However, for eigenfunctions with all of their support in the small ρ region we do expect the continuum limit of the recursion relation to give accurate answers. We find the recursion relation becomes

$$\left(2\alpha g_{\rm YM}^2\right)\frac{1}{2}\left[-\frac{\partial^2}{\partial\rho^2}+\rho^2\right]f(\rho) = \kappa f(\rho) \tag{A.2}$$

which is half of the harmonic oscillator with frequency $2\alpha g_{\rm YM}^2$. Since the length of the rows (or columns) of the Young diagram are non-increasing, this selects only half the oscillator. This implies that $0 \le \rho$, so that only half of the wave functions are selected (those that vanish at $\rho = 0$) and the energy spacing of the remaining oscillator states is $4\alpha g_{\rm YM}^2$. Clearly the description of the coefficients $f(b_0,b_1)$ obtained by solving (A.2) will be accurate for the operators corresponding to the low lying oscillator eigenstates.

It is also possible to solve the recursion relation (A.1) directly. Since we work in the large $N+b_0, b_1$ regime, we can replace (A.1) by

$$\kappa f(b_0,b_1) = -\alpha g_{\rm YM}^2\Big[\sqrt{(N+b_0)(N+b_0+b_1+1)}\, f(b_0-1,b_1+2) + \sqrt{(N+b_0+1)(N+b_0+b_1)}\, f(b_0+1,b_1-2) - (2N+2b_0+b_1)f(b_1,b_1)\Big].$$

This recursion relation is solved by [79]

$$f(b_0,b_1) = (-1)^n\left(\frac{1}{2}\right)^{N+b_0+\frac{b_1}{2}}\sqrt{\binom{2N+2b_0+b_1}{N+b_0+b_1}\binom{2N+2b_0+b_1}{n}} \times {}_2F_1\left(\begin{matrix}-n,\ -(N+b_0+b_1)\\ -(2N+2b_0+b_1)\end{matrix};2\right) \tag{A.3}$$

where the hypergeometric function ${}_2F_1$ which defines the symmetric Kravchuk polynomial $K_n\left(x,\frac{1}{q},p\right)$,

$${}_2F_1\left(\begin{matrix}-n,\ -x\\ -p\end{matrix};q\right) = K_n\left(x,\frac{1}{q},p\right).$$

The corresponding eigenvalue is $\kappa = 2n\alpha g_{\rm YM}^2$. The restriction $0 \leq b_1$ selects half of the wave functions for the continuous solutions and the energy level spacing is again $4\alpha g_{\rm YM}^2$. The solutions (A.3) are accurate for all levels of the oscillator. We have checked that the low lying solutions of (A.3) are in excellent agreement with the harmonic oscillator wave functions that vanish at the origin. Note that for any finite value of N the spectrum is bounded. It is only in the $N \to \infty$ limit that the tower of levels is infinite. In this limit, the harmonic oscillator wave functions are an excellent description for any arbitrarily high but finite energy level.

The analysis for the equation (3.3) is similar once we make the change $N + b_0 \to N - b_0$. We have a similar recursion relation for $f(b_0, b_1)$ in this second case. By taking a continuum limit, and using a continuous variable $\tilde{\rho} = 2b_1/\sqrt{N - b_0}$, we have

$$\left(2\alpha g_{\rm YM}^2\right)\frac{1}{2}\left[-\frac{\partial^2}{\partial \tilde{\rho}^2} + \tilde{\rho}^2\right] f(\tilde{\rho}) = \kappa f(\tilde{\rho}) \tag{A.4}$$

which is again an oscillator with frequency $2\alpha g_{\rm YM}^2$, but since $0 \leq \tilde{\rho}$, again, only half of the wave functions are selected and the energy spacing is $4\alpha g_{\rm YM}^2$. Finally, the original recursion relation can again be solved using symmetric Kravchuk polynomials.

A.3 The Space $L\left(\Omega_p\right)$

In this Appendix we discuss the representation relevant for the problem of multiple AdS giants. We highly recommend the article [83] for background material. Consider S_m the symmetric group on m-objects. Then

$$\Omega_p = S_m / S_{m-p} \times S_p$$

is the space of all p subsets of $\{1, 2, \ldots, m\}$. If $m = 4$ then $\Omega_1 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$ and $\Omega_2 = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$, etc. You can identify a p subset with a monomial. For example, we identify $\{1,3\}$ with $x_1 x_3$ and $\{1,2,4\}$ with $x_1 x_2 x_4$. Thus, we can consider Ω_p to be the space of distinct monomials with p factors and no factor repeats. Ordering of factors is not important, so that $x_1 x_2 x_4$ and $x_4 x_1 x_2$ are the element of Ω_3. We are interested in the space $L\left(\Omega_p\right)$ of complex valued functions on Ω_p. The symmetric group has a natural action

on $L\left(\Omega_p\right)$ defined by its action on each monomial and acts by permuting the labels on the factors in the monomial. Thus, for example,

$$(12)x_1x_2x_3 = x_1x_2x_3 \qquad \text{and} \qquad (24)x_1x_2x_3 = x_1x_4x_3.$$

There is a natural inner product under which the monomials are orthonormal, so that, for example

$$\langle x_1x_2x_3, x_1x_2x_3\rangle = 1, \qquad \text{and} \qquad \langle x_1x_2x_3, x_1x_2x_4\rangle = \langle x_1x_2x_3, x_1x_3x_4\rangle = 0.$$

$L\left(\Omega_p\right)$ furnishes a reducible representation of the symmetric group S_m. The relevance of $L\left(\Omega_p\right)$ for us is that the projectors acting in $L\left(\Omega_p\right)$ projecting onto an irreducible representation of S_m are precisely the projectors we must define the restricted Schur polynomials. Then, the operator

$$d = \sum_{i=1}^{m} \frac{\partial}{\partial x_i}$$

maps from $L\left(\Omega_p\right)$ to $L(\Omega_{p-1})$. Further, it commutes with the action of S_m so that elements of the kernel of d form an invariant S_m subspace. The intersection of the kernel of d and $L\left(\Omega_p\right)$ is called $S^{m-p,p}$ in [83] and it is proved that $S^{m-p,p}$ is an irreducible representation of S_m.

We can get a concrete sense of these ideas by considering the following example. Consider $S^{3,1}$ which is spanned by the polynomials (this basis was found by writing the obvious polynomials linear in the x_is that are annihilated by d and then using the Gram-Schmidt algorithm to get an orthonormal basis)

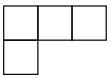
$$\phi_1 = \frac{x_1 - x_2}{\sqrt{2}}, \qquad \phi_2 = \frac{x_3 - x_4}{\sqrt{2}} \qquad \text{and} \qquad \phi_3 = \frac{x_1 + x_2 - x_3 - x_4}{2}.$$

It is easy to check that

$$\begin{aligned}
&(12)\phi_1 = -\phi_1, && (12)\phi_2 = \phi_2, && (12)\phi_3 = \phi_3,\\
&(23)\phi_1 = \frac{1}{2}\phi_1 - \frac{1}{2}\phi_2 + \frac{1}{\sqrt{2}}\phi_3, && (23)\phi_2 = -\frac{1}{2}\phi_1 + \frac{1}{2}\phi_2 + \frac{1}{\sqrt{2}}\phi_3, && (23)\phi_3 = \frac{1}{\sqrt{2}}\phi_1 + \frac{1}{\sqrt{2}}\phi_2,\\
&(34)\phi_1 = \phi_1, && (34)\phi_2 = -\phi_2, && (34)\phi_3 = \phi_3.
\end{aligned}$$

Thus, we have the following group elements

$$\Gamma((12)) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Gamma((23)) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad \text{and} \quad \Gamma((34)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is now possible to compute all elements of the group and then also the characters, using these matrices. In this way, it is a simple matter to identify this as the  irreducible representation.

Appendix B

Multi-Giants Generating Functions

This appendix contains explicit checks and proofs supporting the discussion in Chapter 4.

B.1 Check of the Generating Function for a Two AdS Giant Bound State

In this appendix we expand the generating function for the bound states of two AdS giant gravitons, in a power series in t_1 and t_2. This allows us to test their correctness since, for a small number of fields, the construction of the relevant Schur polynomials is entirely straightforward. Here, it will be conventient to use S_0 and S_1 following (4.14) to simplify the notation.

B.1.1 Operators Constructed from the Generating Function

No open strings attached: To start, consider the two AdS giant graviton bound state generating function, which is given by

$$\begin{aligned} Z_{\text{AdS}}(t_1,t_2) &= \sum_{p_1=p_2}^{\infty}\sum_{p_2=1}^{\infty} t_2^{2p_2}\, t_1^{p_1-p_2}\, \frac{(p_1+p_2)!}{d_{(p_1,p_2)}\; p_2!\,(p_1-p_2)!}\,\chi_{(p_1,p_2)}(Z) \\ &= \int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\;\frac{e^{S_0}}{\pi^{2N}}e^{S_I} \end{aligned} \tag{B.1}$$

where $d_{(p_1,p_2)}$ is the dimension of the symmetric group $S_{p_1+p_2}$ representation labeled by a Young diagram with p_1 boxes in the first row and p_2 boxes in the second row, $p_2 \leq p_1$. After a simple computation we have

$$d_{(p_1,p_2)} = \frac{(p_1+p_2)!\,(p_1-p_2+1)}{(p_1+1)!\,p_2!}$$
$$\mathcal{M}_{p_1,p_2} = \frac{(p_1+p_2)!}{d_{(p_1,p_2)}\,p_2!\,(p_1-p_2)!} = \frac{(p_1+1)!}{(p_1-p_2+1)!}.$$

Power series expanding the integrand of (B.1) in the parameters t_1 and t_2, we generate a sequence of polynomials in the variables $\bar{\varphi}^i$, φ_i, $\bar{\phi}^i$ and ϕ_i. The resulting integrals are all easily performed using (4.6) and the obvious analogous formula for integrals over $\bar{\varphi}^i$ and φ_i. The result is

$$\begin{aligned} Z(t_1,t_2) = 1 &+ t_1\,\mathrm{Tr}(Z) + t_2^2\left(\mathrm{Tr}(Z)^2 - \mathrm{Tr}(Z^2)\right) + t_1 t_2^2\left(\mathrm{Tr}(Z)^3 - \mathrm{Tr}(Z^3)\right) \\ &+ \frac{t_1^2}{2}\left(\mathrm{Tr}(Z)^2 + \mathrm{Tr}(Z^2)\right) + \frac{t_2^4}{2}\left(\mathrm{Tr}(Z)^4 + 3\,\mathrm{Tr}(Z^2)^2 - 4\,\mathrm{Tr}\,Z\,\mathrm{Tr}(Z^3)\right) + \cdots . \end{aligned} \tag{B.2}$$

In the first column below we show the value of $\mathcal{M}_{p_1,p_2}$, in the second we show the Schur polynomial $\chi_{(p_1,p_2)}(Z)$ and in the third we show the coefficient of $t_1^{p_1-p_2}\,t_2^{2p_2}$. The fact that the product of the first two entries on any row reproduces the third entry shows that the result (B.2) has indeed confirmed the formula (B.1).

$$\begin{array}{lll} \mathcal{M}_{1,0} = 1 & \chi_{\ydiagram{1}} = \mathrm{Tr}(Z) & \mathrm{Tr}(Z) \\ \mathcal{M}_{1,1} = 2 & \chi_{\ydiagram{1,1}} = \dfrac{\mathrm{Tr}(Z)^2 - \mathrm{Tr}(Z^2)}{2} & \mathrm{Tr}(Z)^2 - \mathrm{Tr}(Z^2) \\ \mathcal{M}_{2,1} = 3 & \chi_{\ydiagram{2,1}} = \dfrac{\mathrm{Tr}(Z)^3 - \mathrm{Tr}(Z^3)}{3} & \mathrm{Tr}(Z)^3 - \mathrm{Tr}(Z^3) \\ \mathcal{M}_{2,0} = 1 & \chi_{\ydiagram{2}} = \dfrac{\mathrm{Tr}(Z)^2 + \mathrm{Tr}(Z^2)}{2} & \dfrac{\mathrm{Tr}(Z)^2 + \mathrm{Tr}(Z^2)}{2} \\ \mathcal{M}_{2,2} = 6 & \chi_{\ydiagram{2,2}} = \dfrac{\mathrm{Tr}(Z)^4 - 4\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Z^3) + 3\,\mathrm{Tr}(Z^2)^2}{12} & \dfrac{\mathrm{Tr}(Z)^4 + 3\,\mathrm{Tr}(Z^2)^2 - 4\,\mathrm{Tr}\,Z\,\mathrm{Tr}(Z^3)}{2}. \end{array}$$

This example nicely illustrates how the checks that follow will be performed. We will not however quote the explicit formulas for all of the operators we produce, since these become rather lengthy expressions that are not very illuminating.

Two open strings attached, to the giant of momentum p_1: Next consider the generating function obtained when we attach two open strings to the giant graviton with momentum

p_1. We employ the description in which each open string is described with a word. The relevant generating function is

$$Z(t_1,t_2)=\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\ \frac{e^{S_0}}{\pi^{2N}}e^{S_I}\left(\bar{\varphi}\cdot W_{A_1}\cdot\varphi\ \bar{\varphi}\cdot W_{A_2}\cdot\varphi\right).$$

Expanding to second order in t_1 and t_2^2 we obtain

$$Z(t_1,t_2)=a_0+a_1t_1+a_2t_2^2+t_1^2a_3+t_1t_2^2a_4+\frac{t_2^4}{2}a_5+\cdots \tag{B.3}$$

where [1], for example

$$\begin{aligned}a_0&=\mathrm{Tr}(W_{A_1})\mathrm{Tr}(W_{A_2})+\mathrm{Tr}(W_{A_1}W_{A_2})\\ a_1&=\mathrm{Tr}(Z)\mathrm{Tr}(W_{A_1})\mathrm{Tr}(W_{A_2})+\mathrm{Tr}(Z)\mathrm{Tr}(W_{A_1}W_{A_2})+\mathrm{Tr}(ZW_{A_1})\mathrm{Tr}(W_{A_2})\\ &\quad+\mathrm{Tr}(W_{A_1})\mathrm{Tr}(ZW_{A_2})+\mathrm{Tr}(W_{A_1}W_{A_2}Z)+\mathrm{Tr}(W_{A_1}ZW_{A_2}).\end{aligned} \tag{B.4}$$

Two open strings attached, to the giant of momentum p_2: Next consider the generating function obtained when we attach two open strings to the giant graviton with momentum p_2. The relevant generating function is

$$\begin{aligned}Z(t_1,t_2)=\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\ \frac{e^{S_0+S_I}}{\pi^{2N}}&\left(\bar{\varphi}\cdot Z\cdot\varphi\ \bar{\phi}\cdot W_{B_1}\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\ \bar{\phi}\cdot W_{B_2}\cdot\varphi\right)\\ &\times\left(\bar{\varphi}\cdot Z\cdot\varphi\ \bar{\phi}\cdot W_{B_1}\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\ \bar{\phi}\cdot W_{B_2}\cdot\varphi\right).\end{aligned}$$

Expanding to second order in t_1 and t_2^2 we obtain

$$Z(t_1,t_2)=b_0+b_1t_1+b_2t_2^2+t_1^2b_3+t_1t_2^2b_4+\frac{t_2^4}{2}b_5+\cdots \tag{B.5}$$

where, for example

$$\begin{aligned}b_0=&\mathrm{Tr}(W_{B_1})\mathrm{Tr}(W_{B_2})\mathrm{Tr}(Z)^2+\mathrm{Tr}(Z)^2\mathrm{Tr}(W_{B_1}W_{B_2})-\mathrm{Tr}(W_{B_2})\mathrm{Tr}(Z)\mathrm{Tr}(W_{B_1}Z)\\ &-\mathrm{Tr}(W_{B_1})\mathrm{Tr}(Z)\mathrm{Tr}(W_{B_2}Z)+2\,\mathrm{Tr}(W_{B_1}Z)\mathrm{Tr}(W_{B_2}Z)+\mathrm{Tr}(W_{B_1})\mathrm{Tr}(W_{B_2})\mathrm{Tr}(Z^2)\\ &+\mathrm{Tr}(W_{B_1}W_{B_2})\mathrm{Tr}(Z^2)-\mathrm{Tr}(Z)\mathrm{Tr}(W_{B_1}W_{B_2}Z)-\mathrm{Tr}(W_{B_2})\mathrm{Tr}(W_{B_1}Z^2)-\mathrm{Tr}(W_{B_2}W_{B_1}Z^2)\\ &-\mathrm{Tr}(Z)\mathrm{Tr}(W_{B_2}W_{B_1}Z)-\mathrm{Tr}(W_{B_1})\mathrm{Tr}(W_{B_2}Z^2)-\mathrm{Tr}(W_{B_1}W_{B_2}Z^2)+2\,\mathrm{Tr}(W_{B_1}ZW_{B_2}Z)\\ b_1=&\mathrm{Tr}(W_{B_1})\mathrm{Tr}(W_{B_2})\mathrm{Tr}(Z)^3+\mathrm{Tr}(Z)^3\mathrm{Tr}(W_{B_1}W_{B_2})-\mathrm{Tr}(W_{B_2})\mathrm{Tr}(Z)^2\mathrm{Tr}(W_{B_1}Z)\end{aligned}$$

[1] When writing Z^i_j the upper i index is a row index and the lower j index is a column index.

$$
\begin{aligned}
&-\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}(W_{B_2}Z)+2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_1}Z)\,\mathrm{Tr}(W_{B_2}Z)+3\,\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Z^2)\\
&+3\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_1}W_{B_2})\,\mathrm{Tr}(Z^2)-\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(W_{B_1}Z)\,\mathrm{Tr}(Z^2)-\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(W_{B_2}Z)\,\mathrm{Tr}(Z^2)\\
&-\mathrm{Tr}(Z)^2\,\mathrm{Tr}(W_{B_1}W_{B_2}Z)-\mathrm{Tr}(Z^2)\,\mathrm{Tr}(W_{B_1}W_{B_2}Z)-2\,\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_1}Z^2)\\
&+2\,\mathrm{Tr}(W_{B_2}Z)\,\mathrm{Tr}(W_{B_1}Z^2)-\mathrm{Tr}(Z)^2\,\mathrm{Tr}(W_{B_2}W_{B_1}Z)-\mathrm{Tr}(Z^2)\,\mathrm{Tr}(W_{B_2}W_{B_1}Z)\\
&-2\,\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_2}Z^2)+2\,\mathrm{Tr}(W_{B_1}Z)\,\mathrm{Tr}(W_{B_2}Z^2)+2\,\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(Z^3)\\
&+2\,\mathrm{Tr}(W_{B_1}W_{B_2})\,\mathrm{Tr}(Z^3)-2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_1}W_{B_2}Z^2)+2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_1}ZW_{B_2}Z)\\
&-2\,\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(W_{B_1}Z^3)-2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_2}W_{B_1}Z^2)-2\,\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(W_{B_2}Z^3)\\
&-2\,\mathrm{Tr}(W_{B_1}W_{B_2}Z^3)+2\,\mathrm{Tr}(W_{B_1}ZW_{B_2}Z^2)-2\,\mathrm{Tr}(W_{B_2}W_{B_1}Z^3)\\
&+2\,\mathrm{Tr}(W_{B_2}ZW_{B_1}Z^2).
\end{aligned}
$$

A pair of strings attached, one to each giant: The generating function for two strings attached, one to each of the two giants in the bound state, again employing a description in which each open string is described with a word, is given by

$$
\begin{aligned}
Z(t_1,t_2)=\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\;\frac{e^{S_0}}{\pi^{2N}}e^{S_I}\left(\bar{\varphi}\cdot W_A\cdot\varphi\right)\\
\times\left(\bar{\varphi}\cdot Z\cdot\varphi\;\bar{\phi}\cdot W_B\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\;\bar{\phi}\cdot W_B\cdot\varphi\right).
\end{aligned}
$$

Again expanding to second order in t_1 and t_2 we obtain

$$
Z(t_1,t_2)=c_0+t_1c_1+t_2^2c_2+t_1^2c_3+t_1t_2^2c_4+t_2^4c_5+\cdots \tag{B.6}
$$

where, for example

$$
\begin{aligned}
c_0&=\mathrm{Tr}(W_A)\,\mathrm{Tr}(W_B)\,\mathrm{Tr}(Z)-\mathrm{Tr}(W_B)\,\mathrm{Tr}(W_AZ)+\mathrm{Tr}(W_A)\,\mathrm{Tr}(W_BZ)-\mathrm{Tr}(W_AW_BZ)\\
c_1&=\mathrm{Tr}(W_A)\,\mathrm{Tr}(W_B)\,\mathrm{Tr}(Z)^2-\mathrm{Tr}(W_B)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_AZ)+2\,\mathrm{Tr}(W_A)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_BZ)\\
&\quad-\mathrm{Tr}(W_AZ)\,\mathrm{Tr}(W_BZ)+\mathrm{Tr}(W_A)\,\mathrm{Tr}(W_B)\,\mathrm{Tr}(Z^2)-\mathrm{Tr}(Z)\,\mathrm{Tr}(W_AW_BZ)\\
&\quad-\mathrm{Tr}(W_B)\,\mathrm{Tr}(W_AZ^2)+2\,\mathrm{Tr}(W_A)\,\mathrm{Tr}(W_BZ^2)-\mathrm{Tr}(W_AW_BZ^2)-\mathrm{Tr}(W_AZW_BZ).
\end{aligned} \tag{B.7}
$$

A pair of strings stretched between the giants: Finally, consider the generating function for two strings stretched between the two giants in the bound state, again employing a description in which each open string is described with a word

$$Z(t_1,t_2)=\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\ \frac{e^{S_0}}{\pi^{2N}}\ e^{S_I}$$
$$\times\left(\bar{\varphi}\cdot Z\cdot\varphi\ \bar{\phi}\cdot W_C\cdot\varphi\ \bar{\varphi}\cdot W_D\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\ \bar{\phi}\cdot W_C\cdot\varphi\ \bar{\varphi}\cdot W_D\cdot\varphi\right).$$

Again expanding to second order in t_1 and t_2 we obtain

$$Z(t_1,t_2)=s_0+t_1s_1+t_2^2s_2+t_1^2s_3+t_1t_2^2s_4+t_2^4s_5+\cdots$$

where, for example

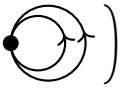
$$\begin{aligned}
s_0 &= \mathrm{Tr}(Z)\,\mathrm{Tr}(W_C W_D)-\mathrm{Tr}(W_D)\,\mathrm{Tr}(W_C Z)-\mathrm{Tr}(W_C W_D Z)+\mathrm{Tr}(W_D W_C Z)\\
s_1 &= \mathrm{Tr}(Z)^2\,\mathrm{Tr}(W_C W_D)-\mathrm{Tr}(W_D)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_C Z)-\mathrm{Tr}(W_C Z)\,\mathrm{Tr}(W_D Z)+\mathrm{Tr}(W_C W_D)\,\mathrm{Tr}(Z^2)\\
&\quad-\mathrm{Tr}(Z)\,\mathrm{Tr}(W_C W_D Z)-\mathrm{Tr}(W_D)\,\mathrm{Tr}(W_C Z^2)+2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_D W_C Z)-\mathrm{Tr}(W_C W_D Z^2)\\
&\quad-\mathrm{Tr}(W_C Z W_D Z)+2\,\mathrm{Tr}(W_D W_C Z^2).
\end{aligned}\tag{B.8}$$

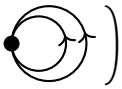
B.1.2 Restricted Schur Polynomial and Gauss Graph Operators

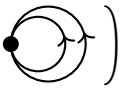
The formulas derived above, in terms of open string words W_*, are extremely useful. As written, they describe the operators obtained when each open string excitation is described using a word. These are the operators we use in Section 4.3 to determine the lattice Hamiltonians describing the dynamics of open strings attached to giant gravitons. Alternatively, we can set all open string words equal to a single letter $W_*=Y$ and compare to the Gauss graph operators. It is this second exercise that we will carry out in this Appendix.

To construct the Gauss graph operators, we need to construct the restricted Schur polynomials. There are a number of simplifications that arise when we consider restricted Schur polynomial operators labeled by a single row (this is the case for the coefficients a_0, a_1 and a_3 in (B.3)). In this case the Gauss graph operators and the restricted Schur polynomial bases coincide. Further, since the representation R is one dimensional, the subspace we restrict to when defining the restricted characters is not a proper subspace and the restricted characters simply reduce to ordinary characters. In this case it is simple

to check the following equalities

$$a_0 = 2\chi_{\square\square,(\cdot,\square\square)}(Z,Y) = \Big(\cdot,$$  $$\Big)$$

$$a_1 = 2\chi_{\square\square\square,(\square,\square\square)}(Z,Y) = \Big(\square,$$  $$\Big)$$

$$a_3 = 2\chi_{\square\square\square\square,(\square\square,\square\square)}(Z,Y) = \Big(\square\square,$$  $$\Big).$$

The equality above is between (1) the coefficient a_* appearing in (B.4)), (2) the restricted Schur polynomial with three Young diagram labels and (3) the Gauss graph operator which has both a Young diagram label and a graph label. The graph is the same in all three cases because in all three we have a single brane with two strings attached to it. The equalities above are explicit checks of the equality between Gauss graph operators with all strings attached to a single brane, and our generating function for excited bound states with all excitations associated to a given row. Here we are, of course, using the fact that these equalities are meant to be exact and not just large N identities.

The remaining terms in (B.3) should be compared to Gauss graph operators constructed from restricted Schur polynomials that are labeled by Young diagrams that have two rows. The two row restricted Schur polynomial is still simpler than the general case, since with only two rows we do not need to introduce multiplicity labels. However, the representation R is no longer one dimensional and the subspace we restrict to when defining the restricted characters is usually a proper subspace. Thus, the restricted characters are not equal to ordinary characters, and they must be computed. Since we are attaching both strings to a single brane, it is again true that, for these operators, the Gauss graph operators and the restricted Schur polynomial bases coincide. This is again an exact statement. The restricted Schur polynomials that we need are given by

$$\begin{aligned}\chi_{\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix},\left(\begin{smallmatrix}\square\\\square\end{smallmatrix},\square\square\right)}(Z,Y) = \frac{1}{4}\Big(&\mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y)^2 + \mathrm{Tr}(Y^2)\,\mathrm{Tr}(Z)^2 + 2\,\mathrm{Tr}(YZ)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y)\\ &-\mathrm{Tr}(Z^2)\,\mathrm{Tr}(Y)^2 - \mathrm{Tr}(Y^2)\,\mathrm{Tr}(Z^2) - 2\,\mathrm{Tr}(YZ^2)\,\mathrm{Tr}(Y)\\ &+2\,\mathrm{Tr}(ZY^2)\,\mathrm{Tr}(Z) - 2\,\mathrm{Tr}(Y^2Z^2)\Big)\end{aligned}$$

$$\chi_{\begin{smallmatrix}\square\square\square\square\\\square\end{smallmatrix},\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square\right)}(Z,Y) = \frac{1}{6}\Big(\mathrm{Tr}(Z)^3\,\mathrm{Tr}(Y)^2 + 4\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y)\,\mathrm{Tr}(ZY) + 4\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}(ZY^2)$$

$$
\begin{aligned}
&+2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y)\,\mathrm{Tr}(Z^2Y)+\mathrm{Tr}(Z)^3\,\mathrm{Tr}(Y^2)+2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(ZY)^2\\
&+2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Z^2Y^2)+2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(ZYZY)-2\,\mathrm{Tr}(YZ)\,\mathrm{Tr}(Y)\,\mathrm{Tr}(Z^2)\\
&-\mathrm{Tr}(Y)^2\,\mathrm{Tr}(Z^3)-\mathrm{Tr}(Y^2)\,\mathrm{Tr}(Z^3)-2\,\mathrm{Tr}(ZY)\,\mathrm{Tr}(Z^2Y)\\
&-2\,\mathrm{Tr}(Z^2)\,\mathrm{Tr}(ZY^2)-4\,\mathrm{Tr}(Z^3Y)\,\mathrm{Tr}(Y)-4\,\mathrm{Tr}(Y^2Z^3)\\
&-2\,\mathrm{Tr}(YZYZ^2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\chi_{\ydiagram{4,2}}\Big(\ydiagram{3,2},\ydiagram{2,1}\Big)(Z,Y)=\frac{1}{24}\Big(&\mathrm{Tr}(Y)^2\,\mathrm{Tr}(Z)^4+\mathrm{Tr}(Z)^4\,\mathrm{Tr}(Y^2)+4\,\mathrm{Tr}(Y)\,\mathrm{Tr}(Z)^3\,\mathrm{Tr}(YZ)\\
&+2\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}(YZ)^2+2\,\mathrm{Tr}(YZ)^2\,\mathrm{Tr}(Z^2)+3\,\mathrm{Tr}(Y)^2\,\mathrm{Tr}(Z^2)^2\\
&+3\,\mathrm{Tr}(Y^2)\,\mathrm{Tr}(Z^2)^2+4\,\mathrm{Tr}(Z)^3\,\mathrm{Tr}(Y^2Z)-4\,\mathrm{Tr}(Z)\,\mathrm{Tr}(YZ)\,\mathrm{Tr}(YZ^2)\\
&+12\,\mathrm{Tr}(Y)\,\mathrm{Tr}(Z^2)\,\mathrm{Tr}(YZ^2)+4\,\mathrm{Tr}(YZ^2)^2-4\,\mathrm{Tr}(Y)^2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Z^3)\\
&-4\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y^2)\,\mathrm{Tr}(Z^3)-4\,\mathrm{Tr}(Y)\,\mathrm{Tr}(YZ)\,\mathrm{Tr}(Z^3)-4\,\mathrm{Tr}(Y^2Z)\,\mathrm{Tr}(Z^3)\\
&+12\,\mathrm{Tr}(Z^2)\,\mathrm{Tr}(Y^2Z^2)+2\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}(YZYZ)+2\,\mathrm{Tr}(Z^2)\,\mathrm{Tr}(YZYZ)\\
&-12\,\mathrm{Tr}(Y)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(YZ^3)-4\,\mathrm{Tr}(YZ)\,\mathrm{Tr}(YZ^3)-12\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y^2Z^3)\\
&-4\,\mathrm{Tr}(Z)\,\mathrm{Tr}(YZYZ^2)-4\,\mathrm{Tr}(YZYZ^3)+4\,\mathrm{Tr}(YZ^2YZ^2)\Big).
\end{aligned}
$$

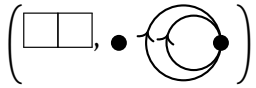
It is now a simple exercise to check the following equalities

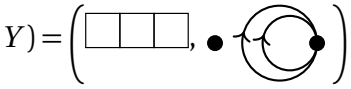
$$
\begin{aligned}
a_2&=4\chi_{\ydiagram{3,1}}\Big(\ydiagram{1,1},\ydiagram{2}\Big)(Z,Y)=\Big(\ydiagram{1,1},\ \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\Big)\\
a_4&=6\chi_{\ydiagram{4,1}}\Big(\ydiagram{2,1},\ydiagram{2}\Big)(Z,Y)=\Big(\ydiagram{2,1},\ \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\Big)\\
a_5&=12\chi_{\ydiagram{4,2}}\Big(\ydiagram{2,2},\ydiagram{2}\Big)(Z,Y)=\Big(\ydiagram{2,2},\ \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\Big)
\end{aligned}
\tag{B.9}
$$

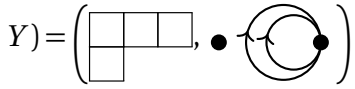
after setting $W_1=W_2=Y$ in a_2, a_4 and a_5.

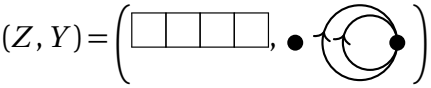
It is equally easy to test the results for strings attached to the second giant graviton, given in (B.5). In this case, again because all strings are attached to a single giant, there is an exact equality between the restricted Schur polynomials and the Gauss graph operators. We have

tested the equalities

$b_0 = 4\chi_{\square\square,\square}(\square\square,\square\square)(Z,Y) =$ 

$b_1 = 12\chi_{\square\square\square,\square}(\square\square\square,\square\square)(Z,Y) =$ 

$b_2 = 16\chi_{\square\square\square,\square\square}(\square\square\square,\square\square)(Z,Y) =$ 

$b_3 = 24\chi_{\square\square\square\square,\square\square}(\square\square\square\square,\square\square)(Z,Y) =$ 

after setting $W_1 = W_2 = Y$ in b_0, b_1, b_2 and b_3.

Testing the results when there is a string attached to each giant graviton is also interesting. In this case we gave a proof that the two sets of operators agree at large N, but have not proved an exact statement. These results are given in (B.6). For this class of operators it is no longer true that the Gauss graph operators and the restricted Schur polynomials coincide. Recall that from (4.21) we have

$$O(\sigma) = \frac{1}{\prod_i m_i!} \sum_{s\vdash m} \sum_{\alpha\in\prod_i S_{m_i}} \sqrt{d_s}\ \chi_s(\sigma\alpha) O_{R,(r,s)}$$

where $\chi_s(\sigma)$ is the character of $\sigma \in S_m$ in irreducible representation s. Since we only have a single Y box in each row, the groups S_{m_i} are trivial we have the lovely simple formula

$$O(\sigma) = \sum_{s\vdash m} \sqrt{d_s}\ \chi_s(\sigma) O_{R,(r,s)}. \tag{B.10}$$

To obtain the state with a string attached to each of the two branes we need to choose σ equals the identity in (B.10). We then have $\chi_s(\mathbf{1}) = d_s$ and hence

$$O(\mathbf{1}) = O_{R,}\Big(r,\square\square\Big) + O_{R,}\Big(r,\begin{smallmatrix}\square\\\square\end{smallmatrix}\Big). \tag{B.11}$$

Using this formula, we can test the polynomials defined in (B.6). A subset of the restricted Schur polynomials that we need are

$$\chi_{\square\square,\square}(\square,\square\square) = \frac{1}{2}\Big(\mathrm{Tr}(Y)^2\,\mathrm{Tr}(Z) + \mathrm{Tr}(Y^2)\mathrm{Tr}(Z) - \mathrm{Tr}(Y)\,\mathrm{Tr}(YZ) - \mathrm{Tr}(Y^2Z)\Big)$$

$$\chi_{\square\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}}\Big(\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}\Big) = \frac{1}{2}\Big(\mathrm{Tr}(Y)^2\,\mathrm{Tr}(Z) - \mathrm{Tr}(Y^2)\mathrm{Tr}(Z) + \mathrm{Tr}(Y)\,\mathrm{Tr}(YZ) - \mathrm{Tr}(Y^2Z)\Big)$$

$$\chi_{\square\square\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}}(\square\square,\square\square) = \frac{1}{4}\Big(\mathrm{Tr}(Z^2)\mathrm{Tr}(Y^2) + \mathrm{Tr}(Z^2)\mathrm{Tr}(Y)^2 + \mathrm{Tr}(Y^2)\mathrm{Tr}(Z)^2$$

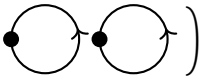
$$-2\,\mathrm{Tr}(YZYZ)+\mathrm{Tr}(Y^2)\mathrm{Tr}(Z^2)-2\,\mathrm{Tr}(YZ)^2\Big)$$

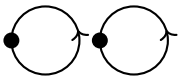
$$\chi_{\square\square\square,\left(\square\square,\square\right)}=\frac{1}{4}\Big(\mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y)^2+\mathrm{Tr}(Z^2)\mathrm{Tr}(Y)^2-\mathrm{Tr}(Y^2)\mathrm{Tr}(Z)^2$$

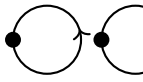
$$-2\,\mathrm{Tr}(Y^2Z^2)-\mathrm{Tr}(Y^2)\mathrm{Tr}(Z^2)+2\,\mathrm{Tr}(YZ^2)\mathrm{Tr}(Y)$$

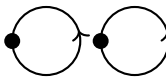
$$-2\,\mathrm{Tr}(ZY^2)\mathrm{Tr}(Z)+2\,\mathrm{Tr}(YZ)\mathrm{Tr}(Z)\mathrm{Tr}(Y)\Big).$$

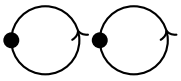
We have tested the equalities

$$c_0=\chi_{\square\square,\left(\square,\square\square\right)}(Z,Y)+\chi_{\square\square,\left(\square,\square\right)}(Z,Y)=\Big(\square,$$  $$\Big)$$

$$c_1=\chi_{\square\square\square,\left(\square\square,\square\square\right)}(Z,Y)+\chi_{\square\square\square,\left(\square\square,\square\right)}(Z,Y)=\Big(\square\square,$$  $$\Big)$$

$$c_2=3\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\left(\begin{smallmatrix}\square\\\square\end{smallmatrix},\square\square\right)}(Z,Y)+3\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\left(\begin{smallmatrix}\square\\\square\end{smallmatrix},\square\right)}(Z,Y)=\Big(\begin{smallmatrix}\square\square\\\square\end{smallmatrix},$$  $$\Big)$$

$$c_3=\frac{3}{2}\chi_{\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix},\left(\square\square\square,\square\square\right)}(Z,Y)+\frac{3}{2}\chi_{\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix},\left(\square\square\square,\square\right)}(Z,Y)=\Big(\square\square\square,$$  $$\Big)$$

$$c_4=8\chi_{\begin{smallmatrix}\square\square\square\\\square\square\end{smallmatrix},\left(\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix},\square\square\right)}(Z,Y)+8\chi_{\begin{smallmatrix}\square\square\square\\\square\square\end{smallmatrix},\left(\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix},\square\right)}(Z,Y)=\Big(\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix},$$  $$\Big)$$

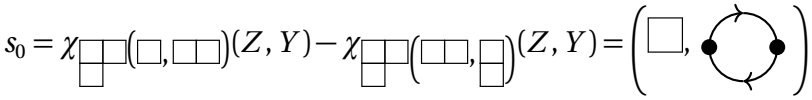
(B.12)

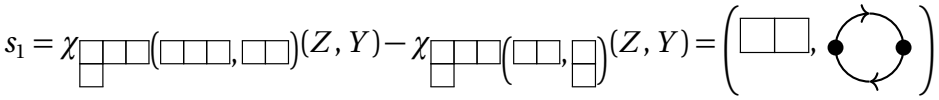
after setting $W_1=W_2=Y$ in c_0, c_1, c_2, c_3 and c_4. The fact that we obtain exact agreement is not expected, suggesting we should attempt to extend the proof given in Appendix B.3 to finite N.

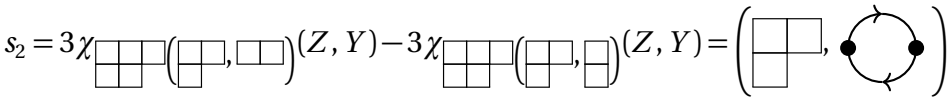
We would now like to test the results for the stretched string operators, summarized in (B.8) above. Since the operators with a string on each brane are related to the stretched string operators by the action of the graph changing operator, the results (B.12) suggest we should again find exact agreement. To obtain the state with strings stretching between the two branes we need to choose $\sigma=(1,2)$ in (B.10). Thus, in the end we have

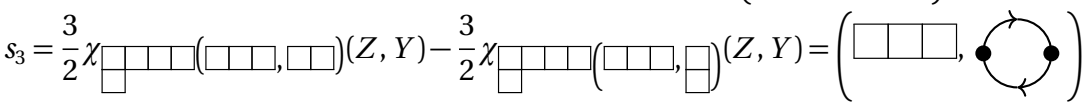
$$O((1,2))=O_{R,\left(r,\square\square\right)}-O_{R,\left(r,\square\right)}. \tag{B.13}$$

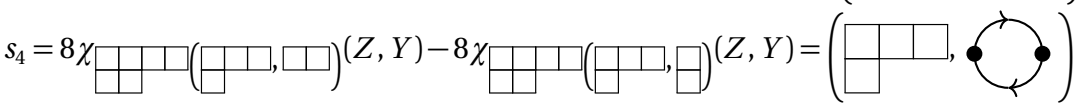
Using this formula, we have argued for and tested the equalities

$s_0 = \chi_{\square\square/\square}(\square,\square\square)(Z,Y) - \chi_{\square\square/\square}(\square\square,\square)(Z,Y) =$ 

$s_1 = \chi_{\square\square\square/\square}(\square\square\square,\square\square)(Z,Y) - \chi_{\square\square\square/\square}(\square\square,\square)(Z,Y) =$ 

$s_2 = 3\chi_{\square\square\square/\square\square}(\square\square,\square\square)(Z,Y) - 3\chi_{\square\square\square/\square\square}(\square\square,\square)(Z,Y) =$ 

$s_3 = \frac{3}{2}\chi_{\square\square\square\square/\square}(\square\square\square\square,\square\square)(Z,Y) - \frac{3}{2}\chi_{\square\square\square\square/\square}(\square\square\square\square,\square)(Z,Y) =$ 

$s_4 = 8\chi_{\square\square\square\square/\square\square}(\square\square\square\square,\square\square)(Z,Y) - 8\chi_{\square\square\square\square/\square\square}(\square\square\square\square,\square)(Z,Y) =$ 

after setting $W_1 = W_2 = Y$ in s_0, s_1, s_2, s_3 and s_4.

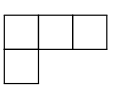
Our final comparison is between operators for which the open string excitations are described using open string words. A simple example is given by attaching two strings to the first brane and setting $p_1 = p_2 = 1$. As above, each row of R is, of course, not a giant graviton, but this simple example allows us to compare the construction of [30] with the construction developed above. From the above construction this operator is given as the coefficient of t_2^2 in the power series expansion of the expression

$$\int \mathrm{d}^N\bar{\varphi}\int \mathrm{d}^N\varphi\int \mathrm{d}^N\bar{\phi}\int \mathrm{d}^N\phi\,\frac{e^{S_0}}{\pi^{2N}}e^{S_I}\prod_{r=1}^{2}\big(\bar{\varphi}\cdot W_{A_r}\cdot\varphi\big).$$

The result is

$$\begin{aligned}O_{11}(Z,W_{A_1},W_{A_2}) = {}& \mathrm{Tr}(Z)^2\,\mathrm{Tr}(W_{A_1})\,\mathrm{Tr}(W_{A_2}) - \mathrm{Tr}(Z^2)\,\mathrm{Tr}(W_{A_1})\,\mathrm{Tr}(W_{A_2}) + \mathrm{Tr}(Z\,W_{A_1})\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{A_2})\\ &+ \mathrm{Tr}(Z\,W_{A_2})\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{A_1}) + \mathrm{Tr}(Z)^2\,\mathrm{Tr}(W_{A_1}W_{A_2}) - \mathrm{Tr}(Z^2)\,\mathrm{Tr}(W_{A_1}W_{A_2})\\ &- \mathrm{Tr}(Z^2W_{A_1})\,\mathrm{Tr}(W_{A_2}) - \mathrm{Tr}(Z^2W_{A_2})\,\mathrm{Tr}(W_{A_1}) + \mathrm{Tr}(Z\,W_{A_1}W^{A_2})\,\mathrm{Tr}(Z)\\ &+ \mathrm{Tr}(Z\,W_{A_2}W^{A_1})\,\mathrm{Tr}(Z) - \mathrm{Tr}(Z^2W^{A_1}W^{A_2}) - \mathrm{Tr}(Z^2W^{A_2}W_{A_1}).\end{aligned} \tag{B.14}$$

We want to compare this to the construction given in [30], which makes use of projection operators. For the operator that we are studying, we define a representation R_1'' by removing two boxes from representation R, as follows

$R =$  $\longrightarrow$  $\longrightarrow$ $R_1'' =$  .

We use the notation R_1'' to match the notation of Appendix C1 of [30]. The projection operator that acts within the carrier space of R and projects from R to R_1'' is denoted by $P_{R,R_1''}$. In terms of this projector, the giant bound state with two strings attached is given by

$$\chi_{R,R_1''}(Z,W_{A_1},W_{A_2})=\frac{1}{2}\sum_{\sigma\in S_4}Z^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}\left(W_{A_1}\right)^{i_3}_{i_{\sigma(3)}}\left(W_{A_2}\right)^{i_4}_{i_{\sigma(4)}}\mathrm{Tr}\left(P_{R,R_1''}\Gamma_R(\sigma)\right).$$

Using the results of Appendix C1 of [30] and the explicit expression (B.14) it is simple to verify that

$$O\left(Z,W_{A_1},W_{A_2}\right)=2\chi_{R,R_1''}\left(Z,W_{A_1},W_{A_2}\right)$$

so that our generating function has correctly reproduced the restricted polynomial constructed in [30]. In Section 4.2.3 our arguments for the equality of restricted Schur polynomials and the operators obtained from the generating function made extensive use of the derivative rule proved in [30]. It is worth reviewing this rule with the formula (B.14) in hand. Towards this end, we employ a convenient graphical notation (for this example, when defining the restrictions, string W_{A_1} is removed first and then W_{A_2} is removed) as follows

$$\chi_{R,R_1''}\left(Z,W_{A_1},W_{A_2}\right)=\chi_{\begin{array}{|c|c|c|}\hline\ &2&1\\\hline\ \\\cline{1-1}\end{array}}.$$

More details about this notation are available in [30]. For this example, the derivative rule of [30] states that

$$\frac{\partial}{\partial\left(W_{A_1}\right)^a_a}\chi_{\begin{array}{|c|c|c|}\hline\ &2&1\\\hline\ \\\cline{1-1}\end{array}}=(N+2)\chi_{\begin{array}{|c|c|}\hline\ &2\\\hline\ \\\cline{1-1}\end{array}}\qquad\text{and}\qquad\frac{\partial}{\partial\left(W_{A_2}\right)^a_a}\chi_{\begin{array}{|c|c|}\hline\ &2\\\hline\ \\\cline{1-1}\end{array}}=(N+1)\chi_{\begin{array}{|c|}\hline\ \\\hline\ \\\cline{1-1}\end{array}}.$$

It is a simple matter to verify these formulas using (B.14).

Finally, consider the case that a pair of strings are stretched between the two giant gravitons. Giant graviton bound states with a single pair of stretched strings are generated by the integral

$$\int \mathrm{d}^N\bar{\varphi}\int\mathrm{d}^N\varphi\int\mathrm{d}^N\bar{\phi}\int\mathrm{d}^N\phi\;\frac{e^{S_0}}{\pi^{2N}}e^{S_I}$$
$$\times\left(\bar{\varphi}\cdot Z\cdot\varphi\;\bar{\phi}\cdot W_D\cdot\varphi\;\bar{\varphi}\cdot W_C\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\;\bar{\phi}\cdot W_D\cdot\varphi\;\bar{\varphi}\cdot W_C\cdot\varphi\right).$$

The term independent of t_1 and t_2 in the power series expansion of integral is

$$O_{ss}=\mathrm{Tr}(Z)\mathrm{Tr}(W_C\,W_D)+\mathrm{Tr}(Z\,W_C\,W_D)-\mathrm{Tr}(Z\,W_D)\mathrm{Tr}(W_C)-\mathrm{Tr}(Z\,W_D\,W_C)\tag{B.15}$$

which is in precise agreement with the construction of [30]. To see this, note that the restricted character needed to construct the restricted Schur polynomial is given by a single matrix element (we use the Young's orthonormal basis)

$$\chi_{ss}(\sigma)=\left\langle \begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}\right|\Gamma_{\begin{array}{|c|c|}\hline\ &\ \\\hline\ \\\cline{1-1}\end{array}}(\sigma)\left|\begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}\right\rangle$$

The complete set of restricted characters are now easily evaluated

$$\chi_{ss}((1)(2)(3))=0 \qquad \chi_{ss}((12))=\frac{\sqrt{3}}{2} \qquad \chi_{ss}((13))=-\frac{\sqrt{3}}{2}$$
$$\chi_{ss}((23))=0 \qquad \chi_{ss}((123))=-\frac{\sqrt{3}}{2} \qquad \chi_{ss}((132))=\frac{\sqrt{3}}{2}.$$

It is now trivial to verify that

$$\chi_{ss}(Z,W_C,W_D)\equiv\sum_{\sigma\in S_3}\chi_{ss}(\sigma)(W_D)^{i_1}_{i_{\sigma(1)}}(W_C)^{i_2}_{i_{\sigma(2)}}Z^{i_3}_{i_{\sigma(3)}}=\frac{\sqrt{3}}{2}O_{ss}.$$

It is again interesting to explore the derivative of this operator. The derivative rule of [30] states that the derivative with respect to string D, stretching between the two branes, vanishes [2]. It is a simple exercise to verify that this is indeed the case

$$\frac{\partial}{\partial\,(W_D)^a_a}O_{ss}=0.$$

B.2 Check of the Generating Function for a Two Sphere Giant Bound State

In this Appendix we expand the generating functions for the bound states of two sphere giant gravitons, in a power series in t_1 and t_2, in order to test their correctness. Again, it will be conventient to use $\tilde{S}_0$ and $\tilde{S}_1$ following (4.27) to simplify the notation.

[2] The thing that is special about string D is that it occupies slot 1, i.e. its upper index is i_1.

B.2.1 Operators Constructed from the Generating Function

No open strings attached: The two sphere giant graviton bound state generating function is given by

$$\begin{aligned} Z(t_1,t_2) &= \sum_{p_1\geq p_2}^{\infty}\sum_{p_2=1}^{\infty} t_2^{2p_2}\, t_1^{p_1-p_2}\, \frac{(p_1+p_2)!}{d_{(2^{p_2}1^{p_1-p_2})}\; p_2!\,(p_1-p_2)!}\,\chi_{(2^{p_2},1^{p_1-p_2})}(Z) \\ &= \int \mathrm{d}^N\,\bar{\psi}\int \mathrm{d}^N\,\psi\int \mathrm{d}^N\,\bar{\xi}\int \mathrm{d}^N\,\xi\; e^{\tilde{S}_0+\tilde{S}_1} \end{aligned} \tag{B.16}$$

where $d_{(2^{p_2}1^{p_1-p_2})}$ is the dimension of the symmetric group $S_{p_1+p_2}$ representation labeled by a Young diagram with p_1 boxes in the first column and $p_2\leq p_1$ boxes in the second column. After a simple computation we have

$$\begin{aligned} d_{(2^{p_2}1^{p_1-p_2})} &= \frac{(p_1+p_2)!\,(p_1-p_2+1)}{(p_1+1)!p_2!} \\ \mathscr{M}_{p_1,p_2} &= \frac{(p_1+p_2)!}{d_{(2^{p_2}1^{p_1-p_2})}\; p_2!\,(p_1-p_2)!} \;=\; \frac{(p_1+1)!}{(p_1-p_2+1)!}. \end{aligned}$$

Power series expanding this integral in the parameters t_1 and t_2^2 we find a sequence of polynomials in the variables $\bar{\psi}^i$, ψ_i, $\bar{\xi}^i$ and ξ_i. These are all easily performed using (4.23) and the obvious analogous formula for integrals over $\bar{\xi}^i$ and ξ_i. The result is

$$Z(t_1,t_2,\{W\}) = 1+t_1d_1+t_2^2d_2+t_1^2d_3+t_1t_2^2d_4+t_2^4d_5+\cdots$$

where

$$\begin{aligned} d_1 &= \mathrm{Tr}(Z) = \mathscr{M}_{1,0}\chi_{\square} \\ d_2 &= \mathrm{Tr}(Z)^2+\mathrm{Tr}(Z^2) = \mathscr{M}_{2,0}\chi_{\square\square} \\ d_3 &= \mathrm{Tr}(Z)^2-\mathrm{Tr}(Z^2) = \mathscr{M}_{1,1}\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \\ d_4 &= \mathrm{Tr}(Z)^3-\mathrm{Tr}(Z^3) = \mathscr{M}_{2,1}\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} \\ d_5 &= \mathrm{Tr}(Z)^4+3\,\mathrm{Tr}(Z^2)^2-4\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Z^3) = \mathscr{M}_{2,2}\chi_{\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}} \end{aligned}$$

confirming (B.16).

Two open strings attached, to the giant of momentum p_1: Next consider the generating function describing bound states of two sphere giant gravitons, with two strings attached

to the first giant graviton. The generating function is given by

$$\begin{aligned} Z(t_1,t_2,\{W\}) &= \int \mathrm{d}^N\bar{\psi}\int \mathrm{d}^N\psi\int \mathrm{d}^N\bar{\xi}\int \mathrm{d}^N\xi\; e^{\bar{S}_0+\bar{S}_1}\left(\psi\cdot W_{A_1}\cdot\bar{\psi}\;\psi\cdot W_{A_2}\cdot\bar{\psi}\right) \\ &= f_0+t_1f_1+t_2^2f_2+t_1^2f_3+t_1t_2^2f_4+t_2^4f_5+\cdots \end{aligned} \tag{B.17}$$

where, for example

$$\begin{aligned} f_0 &= \mathrm{Tr}(W_{A_1})\,\mathrm{Tr}(W_{A_2})-\mathrm{Tr}(W_{A_1}W_{A_2}) \\ f_1 &= \mathrm{Tr}(W_{A_1})\,\mathrm{Tr}(W_{A_2})\,\mathrm{Tr}(Z)-\mathrm{Tr}(Z\,W_{A_1})\,\mathrm{Tr}(W_{A_2})-\mathrm{Tr}(Z\,W_{A_2})\,\mathrm{Tr}(W_{A_1}) \\ &\quad -\mathrm{Tr}(W_{A_1}W_{A_2})\,\mathrm{Tr}(Z)+\mathrm{Tr}(W_{A_1}W_{A_2}Z)+\mathrm{Tr}(W_{A_2}W_{A_1}Z). \end{aligned}$$

Two open strings attached, to the giant of momentum p_2: Next consider the generating function describing bound states of two sphere giant gravitons, with two strings attached to the first giant graviton. The generating function is given by

$$\begin{aligned} Z(t_1,t_2,\{W\}) &= \int \mathrm{d}^N\bar{\psi}\int \mathrm{d}^N\psi\int \mathrm{d}^N\bar{\xi}\int \mathrm{d}^N\xi\; e^{\bar{S}_0+\bar{S}_1}\left(\bar{\psi}\cdot Z\cdot\psi\;\bar{\xi}\cdot W_{B_1}\cdot\xi-\bar{\psi}\cdot Z\cdot\xi\;\bar{\xi}\cdot W_{B_1}\cdot\psi\right) \\ &\qquad\qquad \times\left(\bar{\psi}\cdot Z\cdot\psi\;\bar{\xi}\cdot W_{B_2}\cdot\xi-\bar{\psi}\cdot Z\cdot\xi\;\bar{\xi}\cdot W_{B_2}\cdot\psi\right) \\ &= g_0+t_1g_1+t_2^2g_2+t_1^2g_3+t_1t_2^2g_4+t_2^4g_5+\cdots \end{aligned} \tag{B.18}$$

where, for example

$$\begin{aligned} g_0 &= \mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(Z)^2-\mathrm{Tr}(Z)^2\,\mathrm{Tr}(W_{B_1}W_{B_2})+\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_1}Z) \\ &\quad +\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_2}Z)+2\,\mathrm{Tr}(W_{B_1}Z)\,\mathrm{Tr}(W_{B_2}Z)-\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(Z^2) \\ &\quad +\mathrm{Tr}(W_{B_1}W_{B_2})\,\mathrm{Tr}(Z^2)-\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_1}W_{B_2}Z)-\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(W_{B_1}Z^2)-\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_2}W_{B_1}Z) \\ &\quad -\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(W_{B_2}Z^2)+\mathrm{Tr}(W_{B_1}W_{B_2}Z^2)-2\,\mathrm{Tr}(W_{B_1}Z\,W_{B_2}Z)+\mathrm{Tr}(W_{B_2}W_{B_1}Z^2) \\ g_1 &= \mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(Z)^3-\mathrm{Tr}(Z)^3\,\mathrm{Tr}(W_{B_1}W_{B_2})+\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}(W_{B_1}Z) \\ &\quad +\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}(W_{B_2}Z)+2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_1}Z)\,\mathrm{Tr}(W_{B_2}Z)-3\,\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Z^2) \\ &\quad +3\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_1}W_{B_2})\,\mathrm{Tr}(Z^2)-\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(W_{B_1}Z)\,\mathrm{Tr}(Z^2)-\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(W_{B_2}Z)\,\mathrm{Tr}(Z^2) \\ &\quad -\mathrm{Tr}(Z)^2\,\mathrm{Tr}(W_{B_1}W_{B_2}Z)+\mathrm{Tr}(Z^2)\,\mathrm{Tr}(W_{B_1}W_{B_2}Z)-2\,\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_1}Z^2) \\ &\quad -2\,\mathrm{Tr}(W_{B_2}Z)\,\mathrm{Tr}(W_{B_1}Z^2)-\mathrm{Tr}(Z)^2\,\mathrm{Tr}(W_{B_2}W_{B_1}Z)+\mathrm{Tr}(Z^2)\,\mathrm{Tr}(W_{B_2}W_{B_1}Z) \\ &\quad -2\,\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_2}Z^2)-2\,\mathrm{Tr}(W_{B_1}Z)\,\mathrm{Tr}(W_{B_2}Z^2)+2\,\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(Z^3) \end{aligned}$$

$$
\begin{aligned}
&-2\,\mathrm{Tr}(W_{B_1}W_{B_2})\,\mathrm{Tr}(Z^3)+2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_1}W_{B_2}Z^2)-2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_1}ZW_{B_2}Z)\\
&+2\,\mathrm{Tr}(W_{B_2})\,\mathrm{Tr}(W_{B_1}Z^3)+2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_{B_2}W_{B_1}Z^2)+2\,\mathrm{Tr}(W_{B_1})\,\mathrm{Tr}(W_{B_2}Z^3)\\
&-2\,\mathrm{Tr}(W_{B_1}W_{B_2}Z^3)+2\,\mathrm{Tr}(W_{B_1}ZW_{B_2}Z^2)-2\,\mathrm{Tr}(W_{B_2}W_{B_1}Z^3)+2\,\mathrm{Tr}(W_{B_2}ZW_{B_1}Z^2).
\end{aligned}
$$

A pair of strings, one attached to each giant: The generating function describing bound states of two sphere giant gravitons, with a string attached to each of the two giant gravitons, is given by

$$
\begin{aligned}
Z(t_1,t_2,\{W\}) &= \int \mathrm{d}^N\bar{\psi}\int \mathrm{d}^N\psi\int \mathrm{d}^N\bar{\xi}\int \mathrm{d}^N\xi\; e^{\tilde{S}_0+\tilde{S}_1}\left(\bar{\psi}\cdot W_A\cdot\psi\right)\\
&\qquad\times\left(\bar{\psi}\cdot Z\cdot\psi\;\bar{\xi}\cdot W_B\cdot\xi-\bar{\psi}\cdot Z\cdot\xi\;\bar{\xi}\cdot W_B\cdot\psi\right)\\
&= h_0+t_1h_1+t_2^2h_2+t_1^2h_3+t_1t_2^2h_4+t_2^4h_5+\cdots
\end{aligned}
$$

where, for example

$$
\begin{aligned}
h_0 &= \mathrm{Tr}(W_A)\,\mathrm{Tr}(W_B)\,\mathrm{Tr}(Z)+\mathrm{Tr}(W_B)\,\mathrm{Tr}(W_AZ)-\mathrm{Tr}(W_A)\,\mathrm{Tr}(W_BZ)-\mathrm{Tr}(W_AW_BZ)\\
h_1 &= \mathrm{Tr}(W_A)\,\mathrm{Tr}(W_B)\,\mathrm{Tr}(Z)^2+\mathrm{Tr}(W_B)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_AZ)-2\,\mathrm{Tr}(W_A)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_BZ)\\
&\quad-\mathrm{Tr}(W_AZ)\,\mathrm{Tr}(W_BZ)-\mathrm{Tr}(W_A)\,\mathrm{Tr}(W_B)\,\mathrm{Tr}(Z^2)-\mathrm{Tr}(Z)\,\mathrm{Tr}(W_AW_BZ)+\mathrm{Tr}(W_AW_BZ^2)\\
&\quad-\mathrm{Tr}(W_B)\,\mathrm{Tr}(W_AZ^2)+2\,\mathrm{Tr}(W_A)\,\mathrm{Tr}(W_BZ^2)+\mathrm{Tr}(W_AZW_BZ).
\end{aligned}
$$

A pair of strings stretched between the giants: Finally consider the generating function describing bound states of two sphere giant gravitons, with two strings stretched between the two giant gravitons. The generating function is given by

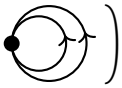
$$
\begin{aligned}
Z(t_1,t_2,\{W\}) &= \int \mathrm{d}^N\bar{\psi}\int \mathrm{d}^N\psi\int \mathrm{d}^N\bar{\xi}\int \mathrm{d}^N\xi\; e^{\tilde{S}_0+\tilde{S}_1}\\
&\qquad\times\left(\bar{\psi}\cdot Z\cdot\psi\;\bar{\xi}\cdot W_C\cdot\psi\;\bar{\psi}\cdot W_D\cdot\xi-\bar{\psi}\cdot Z\cdot\xi\;\bar{\xi}\cdot W_C\cdot\psi\;\bar{\psi}\cdot W_D\cdot\psi\right)\\
&= j_0+t_1j_1+t_2^2j_2+t_1^2j_3+t_1t_2^2j_4+t_2^4j_5+\cdots
\end{aligned}
$$

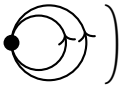
where, for example

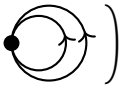
$$
\begin{aligned}
j_0 &= \mathrm{Tr}(Z)\,\mathrm{Tr}(W_CW_D)-\mathrm{Tr}(W_D)\,\mathrm{Tr}(W_CZ)+\mathrm{Tr}(W_CW_DZ)-\mathrm{Tr}(W_DW_CZ)\\
j_1 &= \mathrm{Tr}(Z)^2\,\mathrm{Tr}(W_CW_D)-\mathrm{Tr}(W_D)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_CZ)+\mathrm{Tr}(Z)\,\mathrm{Tr}(W_CW_DZ)-2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(W_DW_CZ)\\
&\quad+\mathrm{Tr}(W_CZ)\,\mathrm{Tr}(W_DZ)-\mathrm{Tr}(Z^2)\,\mathrm{Tr}(W_CW_D)+\mathrm{Tr}(W_D)\,\mathrm{Tr}(W_CZ^2)-\mathrm{Tr}(W_CW_DZ^2)\\
&\quad-\mathrm{Tr}(W_CZW_DZ)+2\,\mathrm{Tr}(W_DW_CZ^2).
\end{aligned}
$$

B.2.2 Restricted Schur Polynomial and Gauss Graph Operators

Our goal is again to compare the operators we have obtained above to the Gauss graph operators, after setting the open string words equal to Y. As for the case of the AdS giants, the simplest possibility to consider is when the Young diagram labeling the operator has a single column, corresponding to a single giant graviton. In this case, the restricted Schur polynomial basis and the Gauss graph basis coincide, and restricted characters reduce to ordinary characters. It is simple to verify that

$$f_0 = 2\chi_{\square,(\cdot,\square)}(Z,Y) = \Big(\cdot,$$  $$\Big)$$

$$f_1 = 2\chi_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},(\square,\square)}(Z,Y) = \Big(\square,$$  $$\Big)$$

$$f_3 = 2\chi_{\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix},(\square,\square)}(Z,Y) = \Big(\begin{smallmatrix}\square\\\square\end{smallmatrix},$$  $$\Big).$$

The remaining terms in (B.17) correspond to restricted Schur polynomials that are labeled by Young diagrams with two columns. Since we are attaching both strings to a single brane, it is still true that the Gauss graph operators and the restricted Schur polynomial bases coincide. The restricted Schur polynomials that we need to evaluate are given by

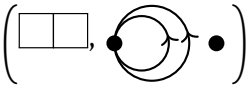
$$\chi_{\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix},(\square\square,\square)} = \frac{1}{4}\Big(\mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y)^2 + 2\,\mathrm{Tr}(Y^2Z^2) - \mathrm{Tr}(Y^2)\,\mathrm{Tr}(Z^2) - 2\,\mathrm{Tr}(YZ^2)\,\mathrm{Tr}(Y)$$
$$+ 2\,\mathrm{Tr}(ZY^2)\,\mathrm{Tr}(Z) - \mathrm{Tr}(Y^2)\,\mathrm{Tr}(Z)^2 - 2\,\mathrm{Tr}(YZ)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y) + \mathrm{Tr}(Z^2)\,\mathrm{Tr}(Y)^2\Big)$$

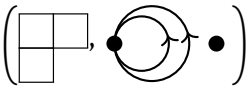
$$\chi_{\begin{smallmatrix}\square\square\\\square\\\square\\\square\end{smallmatrix},(\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square)} = \frac{1}{6}\Big(\mathrm{Tr}(Z)^3\,\mathrm{Tr}(Y)^2 - 4\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y)\,\mathrm{Tr}(ZY) - \mathrm{Tr}(Z)^3\,\mathrm{Tr}(Y^2)$$
$$- 2\,\mathrm{Tr}(Z^2)\,\mathrm{Tr}(ZY)\,\mathrm{Tr}(Y) + 2\,\mathrm{Tr}(Z^2Y)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y) - \mathrm{Tr}(Z^3)\,\mathrm{Tr}(Y)^2$$
$$+ 4\,\mathrm{Tr}(Y^2Z)\,\mathrm{Tr}(Z)^2 + 2\,\mathrm{Tr}(ZY)^2\,\mathrm{Tr}(Z) + \mathrm{Tr}(Y^2)\,\mathrm{Tr}(Z^3)$$
$$+ 2\,\mathrm{Tr}(ZY)\,\mathrm{Tr}(Z^2Y) - 2\,\mathrm{Tr}(ZYZY)\,\mathrm{Tr}(Z) + 2\,\mathrm{Tr}(Z^2)\,\mathrm{Tr}(ZY^2)$$
$$- 2\,\mathrm{Tr}(Z^2Y^2)\,\mathrm{Tr}(Z) + 4\,\mathrm{Tr}(Z^3Y)\,\mathrm{Tr}(Y) - 2\,\mathrm{Tr}(Z^2YZY) - 4\,\mathrm{Tr}(Z^3Y^2)\Big)$$

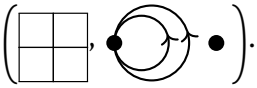
$$\chi_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix},(\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix},\square)} = \frac{1}{24}\,\mathrm{Tr}(Y)^2\,\mathrm{Tr}(Z)^4 - \frac{1}{24}\,\mathrm{Tr}(Z)^4\,\mathrm{Tr}(Y^2) - \frac{1}{6}\,\mathrm{Tr}(Y)\,\mathrm{Tr}(Z)^3\,\mathrm{Tr}(YZ) + \frac{1}{12}\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}(YZ)^2$$

$$
\begin{aligned}
&-\frac{1}{12}\mathrm{Tr}(YZ)^2\,\mathrm{Tr}(Z^2)+\frac{1}{8}\mathrm{Tr}(Y)^2\,\mathrm{Tr}(Z^2)^2-\frac{1}{8}\mathrm{Tr}(Y^2)\,\mathrm{Tr}(Z^2)^2+\frac{1}{6}\mathrm{Tr}(Z)^3\,\mathrm{Tr}(Y^2Z)\\
&+\frac{1}{6}\mathrm{Tr}(Z)\,\mathrm{Tr}(YZ)\,\mathrm{Tr}(YZ^2)-\frac{1}{2}\mathrm{Tr}(Y)\,\mathrm{Tr}(Z^2)\,\mathrm{Tr}(YZ^2)+\frac{1}{6}\mathrm{Tr}(YZ^2)^2\\
&-\frac{1}{6}\mathrm{Tr}(Y)^2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Z^3)+\frac{1}{6}\mathrm{Tr}(Z)\,\mathrm{Tr}(Y^2)\,\mathrm{Tr}(Z^3)+\frac{1}{6}\mathrm{Tr}(Y)\,\mathrm{Tr}(YZ)\,\mathrm{Tr}(Z^3)\\
&-\frac{1}{6}\mathrm{Tr}(Y^2Z)\,\mathrm{Tr}(Z^3)+\frac{1}{2}\mathrm{Tr}(Z^2)\,\mathrm{Tr}(Y^2Z^2)-\frac{1}{12}\mathrm{Tr}(Z)^2\,\mathrm{Tr}(YZYZ)\\
&+\frac{1}{12}\mathrm{Tr}(Z^2)\,\mathrm{Tr}(YZYZ)+\frac{1}{2}\mathrm{Tr}(Y)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(YZ^3)-\frac{1}{6}\mathrm{Tr}(YZ)\,\mathrm{Tr}(YZ^3)\\
&-\frac{1}{2}\mathrm{Tr}(Z)\,\mathrm{Tr}(Y^2Z^3)-\frac{1}{6}\mathrm{Tr}(Z)\,\mathrm{Tr}(YZYZ^2)+\frac{1}{6}\mathrm{Tr}(YZYZ^3)-\frac{1}{6}\mathrm{Tr}(YZ^2YZ^2).
\end{aligned}
$$

It is now a simple exercise to check the following equalities

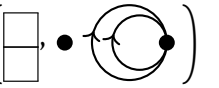
$f_2 = 4\chi_{\square\square,\square}\left(\square\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)(Z,Y) =$ 

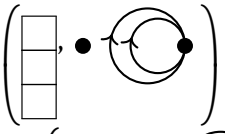
$f_4 = 6\chi_{\square\square,\square}\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)(Z,Y) =$ 

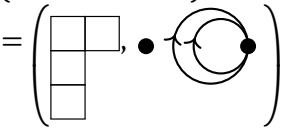
$f_5 = 12\chi_{\square\square,\square}\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)(Z,Y) =$  .

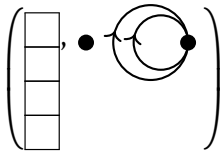
after setting $W_1 = W_2 = Y$ in f_2, f_4 and f_5.

Using the results for strings attached to the second giant graviton, given in (B.18) we have checked that the identities

$g_0 = 4\chi_{\square\square,\square}\left(\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)(Z,Y) =$ 

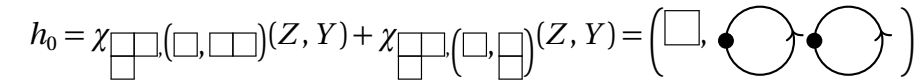
$g_1 = 2\chi_{\square\square,\square}\left(\begin{smallmatrix}\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)(Z,Y) =$ 

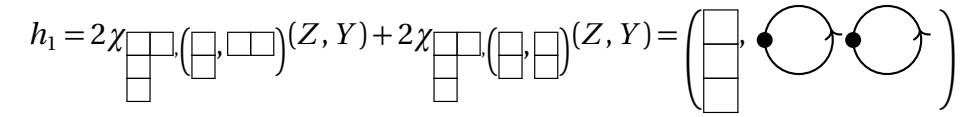
$g_2 = 16\chi_{\square\square,\square}\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)(Z,Y) =$ 

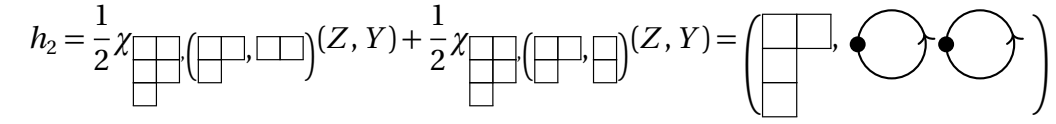
$g_3 = 24\chi_{\square\square,\square}\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)(Z,Y) =$ 

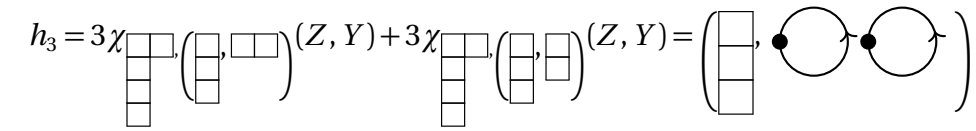
are obeyed. To test the results when there is one string attached to each giant graviton, we must again compare to a linear combination of restricted Schur polynomials. The discussion is parallel to the discussion for the AdS giants, so we simply quote the results. We have

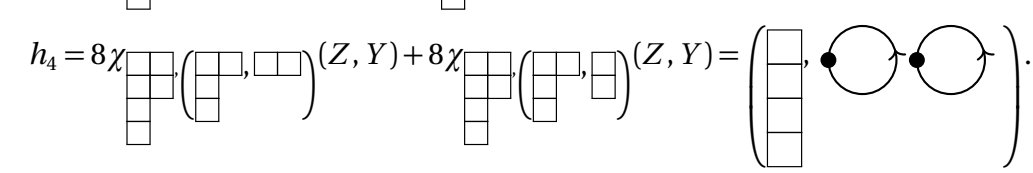
verified the identities

$h_0 = \chi_{\square\square,\square}(\square,\square\square)(Z,Y) + \chi_{\square\square,\square}(\square,\square)(Z,Y) =$ 

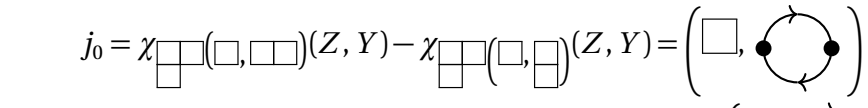
$h_1 = 2\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square}(\square,\square\square)(Z,Y) + 2\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square}(\square,\square)(Z,Y) =$ 

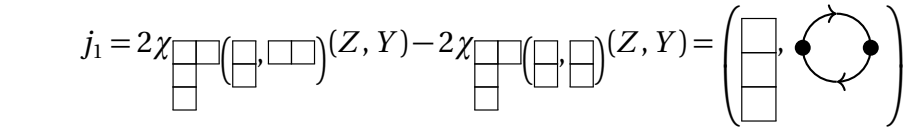
$h_2 = \frac{1}{2}\chi_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix},\square}(\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square)(Z,Y) + \frac{1}{2}\chi_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix},\square}(\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square)(Z,Y) =$ 

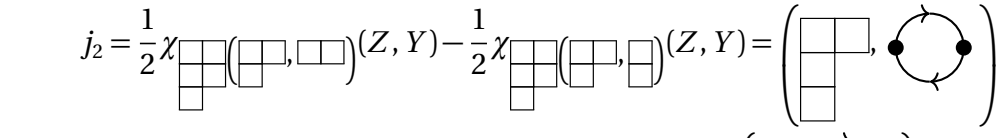
$h_3 = 3\chi_{\begin{smallmatrix}\square\square\\\square\\\square\\\square\end{smallmatrix},\square}(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\square\square)(Z,Y) + 3\chi_{\begin{smallmatrix}\square\square\\\square\\\square\\\square\end{smallmatrix},\square}(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\square)(Z,Y) =$ 

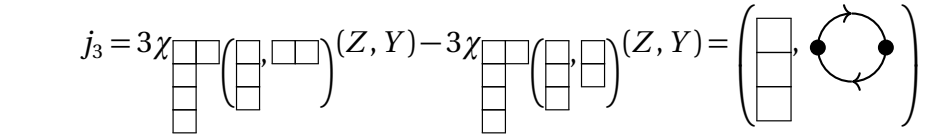
$h_4 = 8\chi_{\begin{smallmatrix}\square\square\\\square\square\\\square\\\square\end{smallmatrix},\square}(\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix},\square\square)(Z,Y) + 8\chi_{\begin{smallmatrix}\square\square\\\square\square\\\square\\\square\end{smallmatrix},\square}(\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix},\square)(Z,Y) =$  .

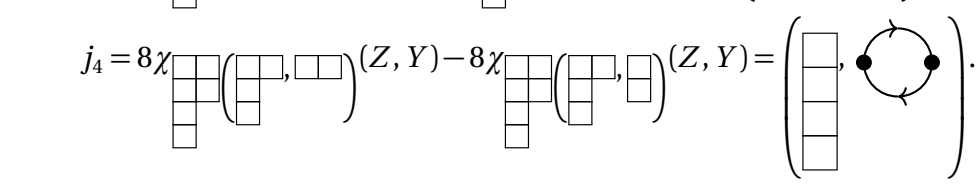
Note once again that we have a complete match, even at finite N. Finally, for the operators corresponding to the stretched string states we have verified the identities

$j_0 = \chi_{\square\square,\square}(\square,\square\square)(Z,Y) - \chi_{\square\square,\square}(\square,\square)(Z,Y) =$ 

$j_1 = 2\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square}(\square,\square\square)(Z,Y) - 2\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square}(\square,\square)(Z,Y) =$ 

$j_2 = \frac{1}{2}\chi_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix},\square}(\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square)(Z,Y) - \frac{1}{2}\chi_{\begin{smallmatrix}\square\square\\\square\square\\\square\end{smallmatrix},\square}(\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square)(Z,Y) =$ 

$j_3 = 3\chi_{\begin{smallmatrix}\square\square\\\square\\\square\\\square\end{smallmatrix},\square}(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\square\square)(Z,Y) - 3\chi_{\begin{smallmatrix}\square\square\\\square\\\square\\\square\end{smallmatrix},\square}(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\square)(Z,Y) =$ 

$j_4 = 8\chi_{\begin{smallmatrix}\square\square\\\square\square\\\square\\\square\end{smallmatrix},\square}(\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix},\square\square)(Z,Y) - 8\chi_{\begin{smallmatrix}\square\square\\\square\square\\\square\\\square\end{smallmatrix},\square}(\begin{smallmatrix}\square\square\\\square\\\square\end{smallmatrix},\square)(Z,Y) =$  .

B.3 A Gauss Graph Identity

In this Appendix we will prove an identity for Gauss graph operators which do not involve any stretched strings. This corresponds to setting $\sigma = \mathbf{1}$ in $O_{R,r}(\sigma)$. We choose r to have p_1 boxes in the first row and p_2 boxes in the second and take both p_1 and p_2 to be $O(N)$. R has

$p_1 + m_1$ boxes in the first row and $m_2 + p_2$ boxes in the second row so that m_1 of the boxes in the first row and m_2 in the second row correspond to Y's. We take m_1 and m_2 to be $O(1)$. Recall formula (4.22) which, after setting σ to the identity, says

$$O_{R,r}(\mathbf{1}) = \sum_{\rho \in S_{n+m}} \sum_{\sigma_1 \in S_n} \sum_{\alpha \in S_{m_1} \times S_{m_2}} \chi_r(\sigma_1) \chi_R(\sigma_1 \circ \alpha \rho) \mathrm{Tr}\big(\rho Z^{\otimes n} Y^{\otimes m}\big).$$

There is a nice interpretation of this formula that follows after recalling that the projection operator onto representation q, within a vector space V, is given by

$$P_q = \frac{d_q}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \chi_q(\sigma) \sigma.$$

Here σ is the action of group $\mathcal{G}$ on the space V, $\chi_q(\sigma)$ is the character of group element σ in representation q, d_q is the dimension of representation q and $|\mathcal{G}|$ is the order of the group $\mathcal{G}$. If there is more than one subspace of V that carries representation q, P_q will project to their direct sum. Using (m) to denote the trivial representation of S_m, labeled by a Young diagram with a single row of length m, we can write (up to an unimportant constant which we drop)

$$O_{R,r}(\mathbf{1}) = \sum_{\rho \in S_{n+m}} \chi_R\big(P_r P_{(m_1)} P_{(m_2)} \rho\big) \mathrm{Tr}\big(\rho Z^{\otimes n} \otimes Y^{\otimes m}\big).$$

We are going to work out a formula for the derivative of this operator with respect to Y^a_a. This analysis is motivated by Appendix C of [30] and we recommend that the reader consult this source for further details that we do not spell out here. It is useful to generalize the operators we consider by including a third field

$$O^{(3)}_{R,r}(\mathbf{1}) = \sum_{\rho \in S_{n+m}} \chi_R\big(P_r P_{(m_1)} P_{(m_2)} \rho\big) \mathrm{Tr}\big(\rho Z^{\otimes n} \otimes X^{\otimes m_2} \otimes Y^{\otimes m_1}\big).$$

Describing the states in the carrier space of $R \vdash m+n$ using Young-Yammanouchi patterns, the indices of the fields Y correspond to boxes with labels from 1 to m_1, the indices of the fields X to boxes with labels from m_1+1 to $m_1+m_2=m$ and Z to labels from $m+1$ to $m+n$. The Y fields are in representation (m_1) and the X fields in representation (m_2). The Y fields populate boxes in the first row, while the X fields populate boxes in the second row. Now, consider the derivative

$$\frac{\mathrm{d}}{\mathrm{d}\, Y^a_a} O^{(3)}_{R,r}(\mathbf{1}) = m_1 \sum_{\rho \in S_{n+m}} \chi_R\big(P_r P_{(m_1)} P_{(m_2)} \rho\big) \mathrm{Tr}\big(\rho Z^{\otimes n} \otimes X^{\otimes m_2} \otimes Y^{\otimes m_1 - 1} \otimes \mathbf{1}\big)$$

where a is summed on the left hand side. Here we have chosen the index of $\mathbf{1}$ to correspond to the box in R labeled 1 and we have used the fact that we can differentiate any of the Y fields and then cycle the identity $\mathbf{1}$ into slot 1, possible because the Y fields are in representation (m_1). Writing the indices more explicitly we have

$$\frac{\mathrm{d}}{\mathrm{d}\,Y^a_a} O^{(3)}_{R,r}(\mathbf{1}) = m_1 \sum_{\rho\in S_{n+m}} \chi_R\left(P_r P_{(m_1)} P_{(m_2)} \rho\right) Z^{i_{n+m}}_{i_{\rho(n+m)}} \cdots Z^{i_{n+1}}_{i_{\rho(n+1)}} X^{i_{m_1+m_2}}_{i_{\rho(m_1+m_2)}} \cdots X^{i_{m_1+1}}_{i_{\rho(m_1+1)}} \times Y^{i_{m_1}}_{i_{\rho(m_1)}} \cdots Y^{i_2}_{i_{\rho(2)}} \delta^{i_1}_{i_{\rho(1)}}.$$

To proceed, write the sum over S_{n+m} as a sum over the subgroup S_{n+m-1} which permutes $2,3,\cdots,m+n$ but leaves 1 inert. The result is

$$\frac{\mathrm{d}}{\mathrm{d}\,Y^a_a} O^{(3)}_{R,r}(\mathbf{1}) = m_1 \sum_{\rho\in S_{n+m-1}} \chi_R\left(P_r P_{(m_1)} P_{(m_2)} \rho\, C\right) Z^{i_{n+m}}_{i_{\rho(n+m)}} \cdots Z^{i_{n+1}}_{i_{\rho(n+1)}} X^{i_{m_1+m_2}}_{i_{\rho(m_1+m_2)}} \cdots X^{i_{m_1+1}}_{i_{\rho(m_1+1)}} \times Y^{i_{m_1}}_{i_{\rho(m_1)}} \cdots Y^{i_2}_{i_{\rho(2)}}$$

where

$$C = N + \sum_{i=2}^{n+m} (1,i) = N + \hat{j}.$$

In this formula, $\hat{j}$ is a Jucys-Murphy element. Denote the representation obtained after dropping the box labeled 1 from R by R'. The box that is dropped is from the first row of R. Acting on this representation $\hat{j}$ gives the content of the dropped box and we have

$$\frac{\mathrm{d}}{\mathrm{d}\,Y^a_a} O^{(3)}_{R,r}(\mathbf{1}) = m_1\, c_{RR'}\, O_{R',r}(\mathbf{1}).$$

This is an exact result. Now consider the derivative with respect to X. A simple calculation gives

$$\frac{\mathrm{d}}{\mathrm{d}\,X^a_a} O^{(3)}_{R,r}(\mathbf{1}) = m_2 \sum_{\rho\in S_{n+m}} \chi_R\left(P_r P_{(m_1)} P_{(m_2)} \rho\right) Z^{i_{n+m}}_{i_{\rho(n+m)}} \cdots Z^{i_{n+1}}_{i_{\rho(n+1)}} X^{i_{m_1+m_2}}_{i_{\rho(m_1+m_2)}} \cdots \delta^{i_{m_1+1}}_{i_{\rho(m_1+1)}} \times Y^{i_{m_1}}_{i_{\rho(m_1)}} \cdots Y^{i_2}_{i_{\rho(2)}} Y^{i_1}_{i_{\rho(1)}}.$$

To proceed, we write the sum over S_{n+m} as a sum over the subgroup S_{n+m-1} which permutes $1,2\cdots,m_1,m_1+2,\cdots m+n$ but leaves m_1+1 inert. The result is

$$\frac{\mathrm{d}}{\mathrm{d}X^a_a}O^{(3)}_{R,r}(\mathbf{1}) = m_2 \sum_{\rho\in S_{n+m-1}} \chi_R\left(P_r P_{(m_1)} P_{(m_2)} \rho\, C\right) Z^{i_{n+m}}_{i_{\rho(n+m)}} \cdots Z^{i_{n+1}}_{i_{\rho(n+1)}} X^{i_{m_1+m_2}}_{i_{\rho(m_1+m_2)}} \cdots X^{i_{m_1+2}}_{i_{\rho(m_1+2)}}$$
$$\times\, Y^{i_{m_1}}_{i_{\rho(m_1)}} \cdots Y^{i_1}_{i_{\rho(1)}}$$

where

$$C = \kappa_1 + \kappa_2$$
$$\kappa_1 = N + \sum_{i=m_1+2}^{n+m} (m_1+1, i) \qquad \text{and} \qquad \kappa_2 = \sum_{i=1}^{m_1} (m_1+1, i).$$

We will now argue that at large N, the terms collected in κ_2 can be ignored. Above we are summing transpositions (2-cycles). Transpositions square to the identity matrix, so that their eigenvalues square to 1. Consequently, the matrix elements of the transpositions are bounded, independent of N. In the term κ_2 we sum only $m_1 = O(1)$ terms, while in κ_1 we sum $m_2 + n = O(N)$ terms. In addition, the first term in κ_1 is N. This certainly bounds the eigenvalues of κ_2 to be $O(1)$ at most. One might worry that there are cancellations between terms summed in κ_1, which might in the end imply that κ_1 itself has eigenvalues of order 1. This is not the case: it is clear $\kappa_1 = N + \hat{j}$. For the case that we consider here, $\hat{j}$ gives the content of the last box in the second row, which is $p_2 - 2$. Thus, κ_1 has all eigenvalues of order N and we can safely ignore κ_2 at large N. In this case we have

$$\frac{\mathrm{d}}{\mathrm{d}X^a_a}O^{(3)}_{R,r}(\mathbf{1}) = m_2\, c_{RR'} O_{R',r}(\mathbf{1}).$$

In this formula the box dropped to obtain R' from R is the right most box in the second row. The formula we want to derive, is for the derivative of $O_{R,r}(\mathbf{1})$. This can be written as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}Y^a_a}O_{R,r}(\mathbf{1}) &= \frac{\mathrm{d}}{\mathrm{d}Y^a_a}O^{(3)}_{R,r}(\mathbf{1})\bigg|_{X=Y} + \frac{\mathrm{d}}{\mathrm{d}X^a_a}O^{(3)}_{R,r}(\mathbf{1})\bigg|_{X=Y} \\ &= m_1\, c_{RR'_1} O_{R'_1,r}(\mathbf{1}) + m_2\, c_{RR'_2} O_{R'_2,r}(\mathbf{1}) \end{aligned} \tag{B.19}$$

where we have used the notation R'_i to denote the representation obtained by dropping a box from row i of R.

It is a much simpler exercise to show that this is the rule obeyed by operators obtained from generating function. The relevant generating function is

$$Z_{\mathrm{AdS}}(t_1, t_2, Y) = \int \mathrm{d}^N \bar{\varphi} \int \mathrm{d}^N \varphi \int \mathrm{d}^N \bar{\phi} \int \mathrm{d}^N \phi\, \frac{e^{S_0}}{\pi^{2N}} e^{S_I}$$

$$\times\left(\bar{\varphi}\cdot Y\cdot\varphi\right)^{m_{11}}\left(\bar{\varphi}\cdot Z\cdot\varphi\;\bar{\phi}\cdot Y\cdot\phi-\bar{\varphi}\cdot Z\cdot\phi\;\bar{\phi}\cdot Y\cdot\varphi\right)^{m_{22}}.$$

Setting m_{11} equal to m_1 and m_{22} equal to m_2, and extracting the coefficient of $t_2^{2p_2}t_1^{p_1-p_2}$ gives the operator $O_{R,r}(\mathbf{1})$ with m_1 Y boxes in the first row, m_2 Y boxes in the second and r a Young diagram with p_1 boxes in the first row and p_2 boxes in the second row. Differentiating with respect to Y_a^a and using the arguments of Section 4.2.3, we easily recover (B.19).

This proves that the derivative with respect to Y of the Gauss graph operator matches the derivative with respect to Y of the operators defined by our generating function. Since there are no contributions to either operator that is independent of Y, this proves their equality. There is an obvious extension of these arguments to the case of Gauss graph operators for sphere giant gravitons.

B.4 Correlation Functions

In this section we evaluate the correlation functions that were used in Section 4.3.2. Consider

$$\mathcal{O}_1=\sum_{\sigma\in S_{n+2}}Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_{n+1}}_{i_{\sigma(n+1)}}(Y^2)^{i_{n+2}}_{i_{\sigma(n+2)}}.$$

We would like to evaluate the correlator $\langle\mathcal{O}_1\mathcal{O}_1^\dagger\rangle$. The Y fields only contract with $Y^\dagger$ fields. Note that

$$\left\langle(Y^2)^i_j(Y^{\dagger 2})^k_l\right\rangle=\delta^k_l\delta^i_j+N\delta^i_l\delta^k_j.$$

At large N we need only keep the second term on the right hand side. This gives

$$\left\langle(Y^2)^i_j(Y^{\dagger 2})^k_l\right\rangle=N\left\langle Y_j^{\,i}\,Y_l^{\,k}\right\rangle\left(1+O\left(N^{-1}\right)\right).$$

We will not display the $O\left(N^{-1}\right)$ correction in what follows. Using this formula we have (the operator $P_{(n+2)}$ is defined in (4.1) above)

$$\begin{aligned}\left\langle\mathcal{O}_1\mathcal{O}_1^\dagger\right\rangle&=N\sum_{\sigma_1,\sigma_2\in S_{n+2}}\left\langle Z^{i_1}_{i_{\sigma_1(1)}}\cdots Z^{i_{n+1}}_{i_{\sigma_1(n+1)}}Y^{i_{n+2}}_{i_{\sigma_1(n+2)}}Z^{\dagger\,j_1}_{j_{\sigma_2(1)}}\cdots Z^{\dagger\,j_{n+1}}_{j_{\sigma_2(n+1)}}Y^{\dagger\,j_{n+2}}_{j_{\sigma_2(n+2)}}\right\rangle\\&=N\sum_{\sigma_1,\sigma_2\in S_{n+2}}\sum_{\rho\in S_{n+1}}\mathrm{Tr}\left(\sigma_1\rho\sigma_2\rho^{-1}\right)\\&=N(n+1)!\sum_{\sigma_1,\sigma_2\in S_{n+2}}\mathrm{Tr}(\sigma_1\sigma_2)\end{aligned}$$

$$= N(n+1)!(n+2)! \sum_{\sigma_1 \in S_{n+2}} \mathrm{Tr}(\sigma_1)$$

$$= N(n+1)!((n+2)!)^2 \,\mathrm{Tr}\left(P_{(n+2)}\right)$$

$$= N(n+1)!(n+2)! \prod_{i=0}^{n+1} (N+i).$$

Next, consider

$$\mathcal{O}_2 = \sum_{\sigma \in S_{n+2}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} (ZY)^{i_{n+2}}_{i_{\sigma(n+2)}}.$$

Our previous strategy of performing the Y contractions by hand is not very useful when evaluating the $\langle \mathcal{O}_2 \mathcal{O}_2^\dagger \rangle$ correlator. A better approach is to start from the identity

$$\sum_{\sigma \in S_{n+3}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n+1}}_{i_{\sigma(n+1)}} Y^{i_{n+2}}_{i_{\sigma(n+2)}} Y^{i_{n+3}}_{i_{\sigma(n+3)}} = \sum_{\sigma \in S_{n+2}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n+1}}_{i_{\sigma(n+1)}} Y^{i_{n+2}}_{i_{\sigma(n+2)}} \mathrm{Tr}(Y)$$
$$+ \sum_{\sigma \in S_{n+2}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n+1}}_{i_{\sigma(n+1)}} (Y^2)^{i_{n+2}}_{i_{\sigma(n+2)}} + (n+1) \sum_{\sigma \in S_{n+2}} Z^{i_1}_{i_{\sigma(1)}} \cdots Y^{i_{n+1}}_{i_{\sigma(n+1)}} (ZY)^{i_{n+2}}_{i_{\sigma(n+2)}}$$

which follows after summing, on the right hand side, over the subgroup and its cosets. The first two terms on the right hand side can be neglected at large N so that

$$\mathcal{O}_2 = \frac{1}{n+1} \sum_{\sigma \in S_{n+3}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_{n+1}}_{i_{\sigma(n+1)}} Y^{i_{n+2}}_{i_{\sigma(n+2)}} Y^{i_{n+3}}_{i_{\sigma(n+3)}}.$$

A simple computation now gives

$$\left\langle \mathcal{O}_2 \mathcal{O}_2^\dagger \right\rangle = 2(n!)^2 (n+2)(n+3) \prod_{i=0}^{n+2} (N+i)$$

and hence we have

$$\frac{\left\langle \mathcal{O}_2 \mathcal{O}_2^\dagger \right\rangle}{\left\langle \mathcal{O}_1 \mathcal{O}_1^\dagger \right\rangle} = \frac{2(n+3)(N+n+2)}{(n+1)^2 N} = O(N^{-1}).$$

Finally, consider

$$\mathcal{O}' = \sum_{\sigma \in S_{n+1}} Z^{i_1}_{i_{\sigma(1)}} \cdots Z^{i_n}_{i_{\sigma(n)}} (YZY)^{i_{n+1}}_{i_{\sigma(n+1)}}.$$

A very similar argument to the one just given establishes that

$$\frac{\left\langle \mathcal{O}' \mathcal{O}'^\dagger \right\rangle}{\left\langle \mathcal{O}_1 \mathcal{O}_1^\dagger \right\rangle} = O\left(N^{-2}\right).$$

Appendix C

Asymptotic Expansions

This appendix contains an outline of the basic logic, with illustrative examples, of Borel resummation and divergent series, supporting the discussion in Chapter 5.

C.1 Basic Logic

Correlation functions are typically studied using perturbation series expansion in powers of some large parameter. These series are asymptotic, meaning that they diverge and this divergence is naturally explained by the combinatorial growth of the number of Feynman diagrams in the series. Different functions can have the same asymptotic expansion, since adding a non-perturbative part to a function does not change the asymptotic series of that function. Therefore, non-perturbative contributions make the asymptotic expansion non-unique. Each asymptotic expansion of a function is defined in wedge in the complex plane about a *saddle point* of the function, and the boundary of each wedge is called a *Stokes Line*. Crossing a Stokes line corresponds to moving from a region with one expansion to a region with a different expansion. The value of the asymptotic expansion changes discontinuously across a Stokes line and this discontinuous change is called the *Stokes Phenomenon*. Therefore, a function that is defined on the entire complex plane will have a different asymptotic expansion in each wedge about each of its saddle points. A *parametric Stokes phenomenon* corresponds moving between different saddles of the

function, where different asymptotic series dominate, by varying a parameter in the series. Each saddle point corresponds to a different classical configuration, therefore, moving between different saddle points corresponds to moving from one classical configuration to another. Resurgence theory states that the information about asymptotic expansion at one saddle point is encoded into the asymptotic expansion at another saddle point.

I shall briefly outline some of the details of asymptotic series representations for Gaussian integrals. First, consider the identities

$$1=\int_0^\infty \mathrm{d}\,s\; e^{-s}\frac{s^n}{n!} \qquad \text{and} \qquad f(t)=\sum_{n=0}^\infty a_n t^n.$$

Then, given f, we find

$$f(t)=\sum_{n=0}^\infty 1\cdot a_n t^n=\sum_{n=0}^\infty \int_0^\infty \mathrm{d}\,s\; e^{-s}\frac{a_n}{n!}(st)^n=\int_0^\infty \mathrm{d}\,s\; e^{-s}\sum_{n=0}^\infty \frac{a_n}{n!}(st)^n=\mathscr{L}\left\{\mathscr{B}[f]\right\}(t) \tag{C.1}$$

where

$$\mathscr{B}[f](s)=\sum_{n=0}^\infty \frac{a_n}{n!}s^n \qquad \text{and} \qquad \mathscr{L}\left\{\mathscr{B}[f]\right\}(t)=\int_0^\infty \mathrm{d}\,s\; e^{-s}\mathscr{B}[f](st)$$

are the Borel transform of f and the inverse Borel transform, respectively. The inverse Borel transform is the Laplace transform of $\mathscr{B}[f]$. The exchange of the sum and integral in (C.1) is allowed when the sum converges. Now, as an illustrative example, consider the asymptotic formal series in t

$$f(t)=\sum_{n=0}^\infty a\,n!\,(bt)^n.$$

Then

$$f(t)=\int_0^\infty \mathrm{d}\,s\; e^{-s}\frac{a}{1-bts}=a\frac{1}{bt}e^{-\frac{1}{bt}}\,\mathrm{Ei}\left(\frac{1}{bt}\right) \tag{C.2}$$

where

$$\mathrm{Ei}(x)=-\int_{-x}^\infty \frac{\mathrm{d}\,z}{z}\; e^{-z}.$$

The function $\mathrm{Ei}(x)$ has a branch cut running from $-\infty$ to 0 and a simple pole at $x=0$. Expanding (C.2) as a series, for small values of t, recovers

$$f(t)=a\left\{1+bt+2(bt)^2+6(bt)^3+24(bt)^4+120(bt)^5+720(bt)^6+\cdots\right\}$$

$$= a \sum_{n=0}^{\infty} n! \, (b\,t)^n \tag{C.3}$$

as expected. Expanding (C.2) as a series for large t gives

$$\begin{aligned} f(t) = & \frac{a}{b\,t}\left[\left(\gamma - \log(b\,t)\right)\left\{1 - \frac{1}{b\,t} + \frac{1}{2}\frac{1}{(b\,t)^2} - \frac{1}{6}\frac{1}{(b\,t)^3} + \frac{1}{24}\frac{1}{(b\,t)^4} - \frac{1}{120}\frac{1}{(b\,t)^5} + \cdots\right\}\right. \\ & \left. + \frac{1}{b\,t}\left\{1 - \frac{3}{4}\frac{1}{(b\,t)^1} + \frac{11}{36}\frac{1}{(b\,t)^2} - \frac{25}{288}\frac{1}{(b\,t)^3} + \frac{137}{7200}\frac{1}{(b\,t)^4} - \frac{49}{14400}\frac{1}{(b\,t)^5} + \cdots\right\}\right] \\ = & \left(\gamma - \log(b\,t)\right)\frac{a}{b\,t}\sum_{n=0}\frac{1}{n!}(b\,t)^{-n} + \frac{a}{(b\,t)^2}\sum_{n=0}\sigma_n (b\,t)^{-n} \end{aligned}$$

where $\gamma = 0.577216$ is the *Euler-Mascheroni* constant and

$$\sigma_n = \left.\frac{\mathrm{d}^n \log(x)}{\mathrm{d}\,x^n}\right|_{x=1}$$

such that $|\sigma_n|$ grows without bound as $n \to \infty$. It is now clear that (C.2) has different asymptotic expansion for different values of t.

C.2 Gaussian Moment Integral

We study the moments of the Gaussian integral

$$\mathscr{I}_k = \int_{-\infty}^{\infty} \mathrm{d}\,x \; e^{-x^2} x^{2k} = \sqrt{\pi}\,\frac{(2k-1)!!}{2^k}, \tag{C.4}$$

where double factorial in the Gaussian moment matches the coefficients associated with Wick contraction of fields in Feynman diagram expansion in QFT. Note that we could have studied the Gaussian integral

$$\mathscr{I}_k^{(m)} = \int_{-\infty}^{\infty} \mathrm{d}\,x \; e^{-k^m x^2} x^{2k} = \frac{\sqrt{\pi}}{k^{m(k+\frac{1}{2})}}\frac{(2k-1)!!}{2^k} = k^{-m(k+1/2)}\mathscr{I}_k$$

for any non-negative integer m. It is clear that the rescaling of the Gaussian rescales the leading factor of (C.4) without altering the form of the series in (C.5). Each moment of the Gaussian integral can be expanded as a series, for large k

$$\mathscr{I}_k = \sqrt{2\pi}\; k^k e^{-k}\left\{1 - \frac{1}{24k} + \frac{1}{1152k^2} + \frac{1003}{414720k^3} - \frac{4027}{39813120k^4} - \frac{5128423}{6688604160k^5} + \cdots\right\}. \tag{C.5}$$

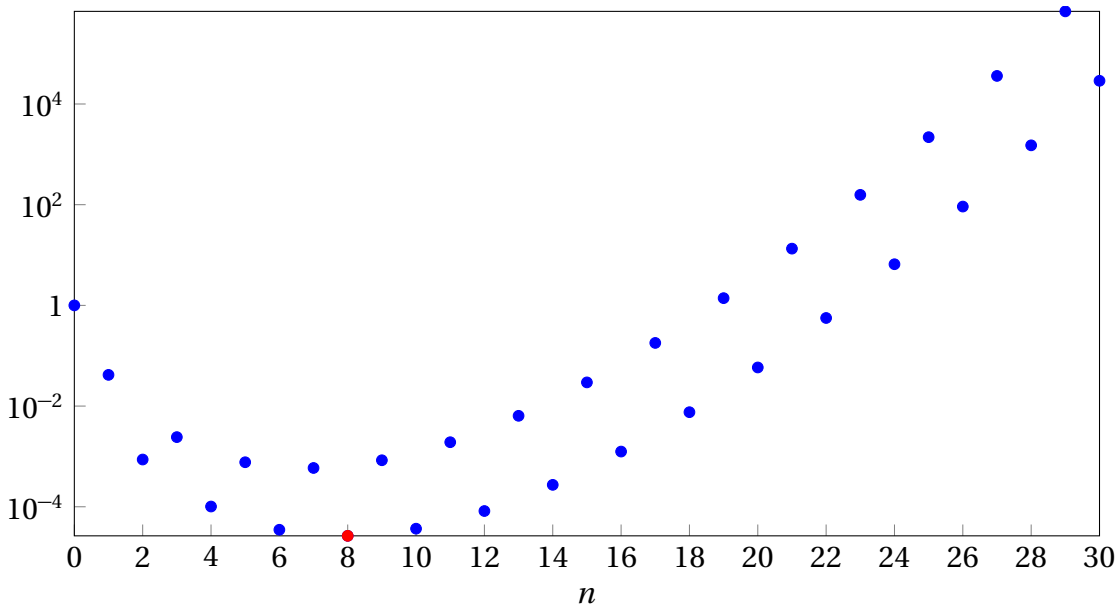


Figure C.1: A plot of the coefficients $|c_n|$, of k^{-n}, in (C.5), with smallest coefficient $|c_8|$ (red). The growth of the coefficients indicates that the series in (C.5) is asymptotic.

Let c_n denote the coefficient of k^{-n} in the asymptotic series in (C.5), with $c_0 = 1$, and $c_1 = -1/24$ and so on. Then, for increasing n, $|c_n|$ decreases until $n = 8$, where $c_8 = -0.0000264647$, whereafter $|c_n|$ increases without bound. Figure C.1 shows the growth of $|c_n|$. We write

$$\mathscr{I}_k = \sqrt{2\pi}\; k^k e^{-k} \sum_{n=0}^{\infty} \frac{c_n}{k^n} \tag{C.6}$$

where growth of the series coefficients indicates that the series in (C.5) is asymptotic.

Similarly, expanding (C.4) for small k yields

$$\mathscr{I}_k = \sum_{n=0}^{\infty} b_n k^n = 1 - 1.96k + 4.39k^2 - 8.91k^3 + 17.98k^4 - 36.05k^5 + 72.18k^6 - 144.41k^7 + 288.85k^8 - 577.72k^9 + 1155.45k^{10} + \cdots \tag{C.7}$$

where each coefficient is expressed in terms of polygamma function $\psi^{(n)}(z) = \frac{\mathrm{d}^n \log(z)}{\mathrm{d}^n z}$. The series coefficients b_n in (C.7) take the asymptotic form

$$b_n \sim (-1)^n 2^{n+1} \tag{C.8}$$

for large n. Figure C.2 shows the growth of $|b_n|$.

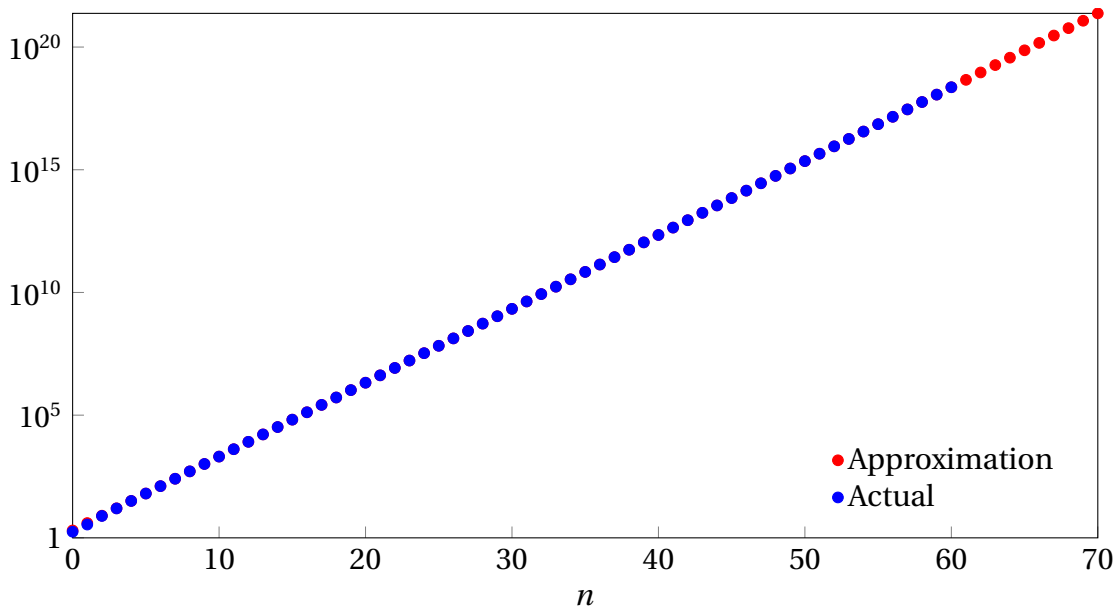


Figure C.2: A plot of the coefficients $|b_n|$, of k^n, in (C.7). The growth of the coefficients indicates that the series in (C.7) is asymptotic.

We can rewrite the Gaussian integral in (C.4) as

$$\mathscr{I}_k = \int_{-\infty}^{\infty} \mathrm{d}\,x\; e^{-x^2} x^{2k} = \int_{-\infty}^{\infty} \mathrm{d}\,x\; e^{-x^2+k\log\left(x^2\right)} \tag{C.9}$$

which has saddle points x_k such that

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(-x^2+k\log\left(x^2\right)\right)_{x=x_k} = -2x_k + \frac{2k}{x_k} = 0$$

then $x_k^{(\pm)} = \pm\sqrt{k}$. There are two saddle points on the contour of integration which accounts for the asymptotic behaviour of the series in (C.5) depicted in Figure C.1. To help generate the asymptotic series expansions of (C.9) associated with each saddle point $x_k^{(\pm)}$, we define the phase function

$$f(x) = x^2 - k\log\left(x^2\right). \tag{C.10}$$

Then expanding the integral about the saddle point at $x_k^{(\pm)}$ gives

$$\mathscr{I}_k^{(\pm)} = e^{-f\left(x_k^{(\pm)}\right)} \int_{-\infty}^{\infty} \mathrm{d}\,x\; e^{-\left(f(x)-f\left(x_k^{(\pm)}\right)\right)}$$

where

$$f\left(x_k^{(\pm)}\right) = k - k\log(k) \qquad \text{and} \qquad e^{-f\left(x_k^{(\pm)}\right)} = k^k e^{-k}.$$

Next, we expand $f(x)-f(x_k^{\pm})$ in a Taylor series about $x_k^{(\pm)}$ and use

$$\delta x = x - x_k^{(\pm)} \qquad \text{and} \qquad \mathrm{d}x = \mathrm{d}\delta x.$$

Expanding about $x_k^{(-)}$ yields

$$f(x)-f\left(x_k^{(-)}\right) = 2\delta x^2+\frac{2}{3k^{1/2}}\delta x^3+\frac{1}{2k^{2/2}}\delta x^4+\frac{2}{5k^{3/2}}\delta x^5+\frac{1}{3k^{4/2}}\delta x^6+\frac{2}{7k^{5/2}}\delta x^7+\frac{1}{4k^{6/2}}\delta x^8+\cdots \tag{C.11}$$

and expanding about $x_k^{(+)}$ yields

$$f(x)-f\left(x_k^{(+)}\right) = 2\delta x^2-\frac{2}{3k^{1/2}}\delta x^3+\frac{1}{2k^{2/2}}\delta x^4-\frac{2}{5k^{3/2}}\delta x^5+\frac{1}{3k^{4/2}}\delta x^6-\frac{2}{7k^{5/2}}\delta x^7+\frac{1}{4k^{6/2}}\delta x^8-\cdots. \tag{C.12}$$

Equations (C.11) and (C.12) differ only in the sign of the odd powers of δx. Then, expanding the non-quadratic part of the phase functions as series in δx yields

$$e^{-\left(f(x)-f\left(x_k^{(-)}\right)\right)} = e^{-2\delta x^2}\left\{1-\frac{2}{3k^{1/2}}\delta x^3-\frac{1}{2k^{2/2}}\delta x^4-\frac{2}{5k^{3/2}}\delta x^5+\frac{2k-3}{9k^{4/2}}\delta x^6+\frac{7k-6}{21k^{5/2}}\delta x^7+\cdots\right\} \tag{C.13}$$

and

$$e^{-\left(f(x)-f\left(x_k^{(+)}\right)\right)} = e^{-2\delta x^2}\left\{1+\frac{2}{3k^{1/2}}\delta x^3-\frac{1}{2k^{2/2}}\delta x^4+\frac{2}{5k^{3/2}}\delta x^5+\frac{2k-3}{9k^{4/2}}\delta x^6-\frac{7k-6}{21k^{5/2}}\delta x^7+\cdots\right\}. \tag{C.14}$$

Similarly to (C.11) and (C.12), (C.13) and (C.14) differ only in the sign of the odd powers of δx, which will integrate to zero. Then, integrating over the even powers of δx yields

$$\begin{aligned}\mathscr{I}_k^{(-)} &= e^{-k}k^k\int_{-\infty}^{\infty}\mathrm{d}\delta x\; e^{-2\delta x^2}\left\{1-\frac{1}{2k}\delta x^4+\frac{2k-3}{9k^2}\delta x^6+\frac{47k-30}{120k^3}\delta x^8\right.\\ &\qquad\left.-\frac{350k^2-1377k+630}{3150k^4}\delta x^{10}+\cdots\right\}\\ &= e^{-k}k^k\sqrt{\frac{\pi}{2}}\left\{1-\frac{1}{24k}+\frac{1}{1152k^2}+\frac{1003}{414720k^3}+\cdots\right\}\end{aligned} \tag{C.15}$$

and

$$\mathscr{I}_k^{(+)} = e^{-k}k^k\int_{-\infty}^{\infty}\mathrm{d}\delta x\; e^{-2\delta x^2}\left\{1-\frac{1}{2k}\delta x^4+\frac{2k-3}{9k^2}\delta x^6+\frac{47k-30}{120k^3}\delta x^8\right.$$

$$-\frac{350k^2-1377k+630}{3150k^4}\delta x^{10}+\cdots\Bigg\}$$
$$=e^{-k}k^k\sqrt{\frac{\pi}{2}}\left\{1-\frac{1}{24k}+\frac{1}{1152k^2}+\frac{1003}{414720k^3}+\cdots\right\}. \tag{C.16}$$

Summing the expansions about each saddle gives

$$\mathscr{I}_k=\mathscr{I}_k^{(-)}+\mathscr{I}_k^{(+)}=\sqrt{2\pi}\;e^{-k}k^k\left\{1-\frac{1}{24k}+\frac{1}{1152k^2}+\frac{1003}{414720k^3}+\cdots\right\}, \tag{C.17}$$

which matches the expansion in (C.5).

Each correlation function has an associated Feynman diagram expansion, where the action generates the corresponding Feynman rules. Rewrite the action as a series

$$S_\pm=\sum_{m=2}^{\infty}a_m^{(\pm)}\delta x^m$$

and define

$$a_m^{(-)}=\hat{a}_m \qquad \text{and} \qquad a_m^{(+)}=(-1)^m\hat{a}_m$$

where

$$\hat{a}_m=\frac{2}{m+2}k^{-m/2}.$$

To get the Feynman rules, I shall first determine the propagator using

$$\frac{\langle\delta x^2\rangle}{\langle 1\rangle}=\frac{\int_{-\infty}^{\infty}\mathrm{d}\delta x\;e^{-2\delta x^2}\delta x^2}{\int_{-\infty}^{\infty}\mathrm{d}\delta x\;e^{-2\delta x^2}}=\frac{\frac{1}{4}\sqrt{\frac{\pi}{2}}}{\sqrt{\frac{\pi}{2}}}=\frac{1}{4}$$

then the Feynman rules are

$$\longleftrightarrow\frac{1}{4} \qquad \text{(propagator)}$$


$$\longleftrightarrow -a_n^{(\pm)} \qquad (n\text{-point vertex}).$$

Each Feynman diagram constructed by the Feynman rules has a corresponding symmetry factor that fixes the value of the diagram. Symmetry factors are assigned according to the following rules:

1. Edges that start and end at the same vertex form closed loops. Divide by 2 for each closed loop.

2. Edges without orientation and identical endpoints are equivalent. Divide by $n!$ for n equivalent edges.

3. A vertex with n edges attached to it has degree n. Multiply by $n!$ for each vertex of degree n.

4. Vertices that can be swapped without changing the topology of the diagram are equivalent. Divide by $n!$ for n equivalent vertices.

5. Connected diagrams with identical topology are equivalent. Divide by $n!$ for n equivalent diagrams.

Reorganizing the sums (C.15) and (C.16) in we recover the following Feynman diagram expansion

$$\mathscr{I}_k^{(\pm)} = \sqrt{\frac{\pi}{2}}\, k^k e^{-k} \left\{ 1 + \quad + \quad + \quad + O\left(k^{-2}\right) \right\}$$

where

$$= \left(-a_3^{(\pm)}\right)^2 \left(\frac{1}{4}\right)^3 \left(\frac{3!3!}{2\cdot 2\cdot 2}\right) = \frac{1}{32k}$$

$$= \left(-a_3^{(\pm)}\right)^2 \left(\frac{1}{4}\right)^3 \left(\frac{3!3!}{2\cdot 3!}\right) = \frac{1}{48k}$$

$$= \left(-a_4^{(\pm)}\right) \left(\frac{1}{4}\right)^2 \left(\frac{4!}{2\cdot 2\cdot 2}\right) = -\frac{3}{32k}$$

and

$$+ \quad + \quad = -\frac{1}{24k},$$

which matches each order k^{-1} term in (C.15) and (C.16). Then (C.17) becomes

$$\mathscr{I}_k = \sqrt{2\pi}\, k^k e^{-k} \left\{ 1 + \quad + \quad + \quad + O\left(k^{-2}\right) \right\}.$$

The coefficients c_n in (C.6) can be generated by recursion, following Example 4.2 in [148], and the series [149, 150]

$$c_0 = 1 \qquad \text{and} \qquad c_n = \frac{1}{n} \sum_{j=2}^{n+1} \frac{c_{n-j+1}}{j} B_j\left(\frac{1}{2}\right)$$

where $B_j(x)$ is the j-th Bernoulli polynomial defined by

$$B_j(x) = \sum_{i=0}^{j} \binom{j}{i} B_i x^{j-i}$$

and B_j is the j-th Bernoulli number [151]. It then follows that

$$B_{2j+1}\left(\frac{1}{2}\right) = B_{2j+1} = 0 \qquad \text{and} \qquad \frac{B_{2j}\left(\frac{1}{2}\right)}{B_{2j}} = -\frac{2^{2j-1}-1}{2^{2j-1}}$$

for any non-negative integer j. Then,

$$c_0 = 1 \qquad \text{and} \qquad c_n = -\frac{1}{n}\sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{2^{2k}-2}{2^{2k}} \frac{B_{2k}}{2k} c_{n+1-2k}.$$

Similarly, using a well known formula [151] we express the Bernoulli numbers B_{2k} in terms of the Riemann zeta function $\zeta(x)$,

$$B_{2k} = (-1)^{k-1} 2 \frac{\zeta(2k)}{(2\pi)^{2k}} (2k)!.$$

This allow us to write

$$c_n = \frac{2}{n}\sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k \left(\frac{2^{2k-1}-1}{2^{2k-1}}\right) \frac{\zeta(2k)}{(2\pi)^{2k}} (2k-1)! c_{n+1-2k}.$$

Suppose that $n = 2m+1$ for $m = 0, 1, 2, 3, \ldots$, then we can write

$$c_{2m+1} = -\frac{2}{2m+1}\sum_{k=1}^{m+1} (-1)^k \left(\frac{2^{2k-1}-1}{2^{2k-1}}\right) \frac{\zeta(2k)}{(2\pi)^{2k}} (2k-1)! c_{2(m+1-k)}.$$

Let $k = p+1$, then we have the associations $k = 1 \longleftrightarrow p = 0$ and $k = m+1 \longleftrightarrow p = m$, which allows us to write

$$c_{2m+1} = -\left(\frac{2}{2m+1}\right)\sum_{p=0}^{m} (-1)^p \left(\frac{2^{2p+1}-1}{2^{2p+1}}\right) \frac{\zeta(2(p+1))}{(2\pi)^{2(p+1)}} (2p+1)! c_{2(m-p)}.$$

We find a similar expression for $n = 2(m+1)$ for $m = 0, 1, 2, 3, \ldots$. Now define the convenience functions

$$r_p = \frac{2^{2p+1}-1}{2^{2p+1}} \qquad \text{and} \qquad f_p = (2p+1)!$$

and

$$q_p = \frac{\zeta(2(p+1))}{(2\pi)^{2(p+1)}} \qquad \text{and} \qquad \check{q}_p = \frac{2^{2(p+1)}+1}{2^{2(p+1)}} (2\pi)^{2(p+1)}$$

where $\check{q}(p) \to q(p)$ as $p \to \infty$ and

$$\alpha_0 = \frac{1}{48} \qquad \text{and} \qquad \alpha_p = (-1)^p r_p \check{q}_p f_p.$$

Then, we have

$$c_0 = 1, \qquad c_{2m+1} = -\left(\frac{2}{2m+1}\right)\sum_{p=0}^{m} \alpha_p c_{2(m-p)} \qquad \text{and} \qquad c_{2(m+1)} = -\left(\frac{1}{m+1}\right)\sum_{p=0}^{m} \alpha_p c_{2(m-p)+1}$$

where coefficients c_{2m+1} are now functions of $c_{2(m+1)}$ only and, similarly, coefficients $c_{2(m+1)}$ are functions of c_{2m+1} only. The magnitude of each c_m grows like some combinatorial factor. The dominant contribution to the value of c_{2m+1} comes from the term with $p = m$. Similarly, the dominant contributions to $c_{2(m+1)}$ comes from the terms with $p = 0$ and $p = m$, which are of comparable size. Each coefficient carries a correction factor

$$\check{h}_m = 1, \qquad h_0 = 1 \qquad \text{and} \qquad h_m = \frac{2m^2+1}{2m^2}$$

where $h(m) \to \check{h}(m) = 1$ as $m \to \infty$. The function h_m corrects under-estimation of $c_{2(m+1)}$ by $\check{c}_{2(m+1)}$ at large m. We approximate the coefficients c_n as $\check{c}_n$,

$$\check{c}_{2m+1} = -\frac{2}{2m+1}\alpha_m \check{c}_0 \qquad \text{and} \qquad \check{c}_{2(m+1)} = -\frac{h_m}{m+1}(\alpha_m \check{c}_1 + \alpha_0 \check{c}_{2m+1}) \tag{C.18}$$

where

$$\check{c}_0 = 1, \qquad \check{c}_1 = -\frac{1}{24} \qquad \text{and} \qquad \check{c}_2 = -\frac{1}{1152}. \tag{C.19}$$

Note that using $\check{h}_m$ in place of h_m in (C.18) simplifies the analysis of the Borel sum to follow, at the expense of a larger relative error in the coefficient approximations. Combining (C.6) and (C.18), we find

$$\mathscr{I}_k \sim \sqrt{2\pi}\; k^k e^{-k} \sum_{n=0}^{\infty} \frac{\check{c}_n}{k^n}.$$

Figure C.3 shows an overlay of the coefficients c_n and $\check{c}_n$. Figure C.4 demonstrates the exponential decay in the relative errors in the approximation of the coefficients c_n. Next, we study a variation on this model.

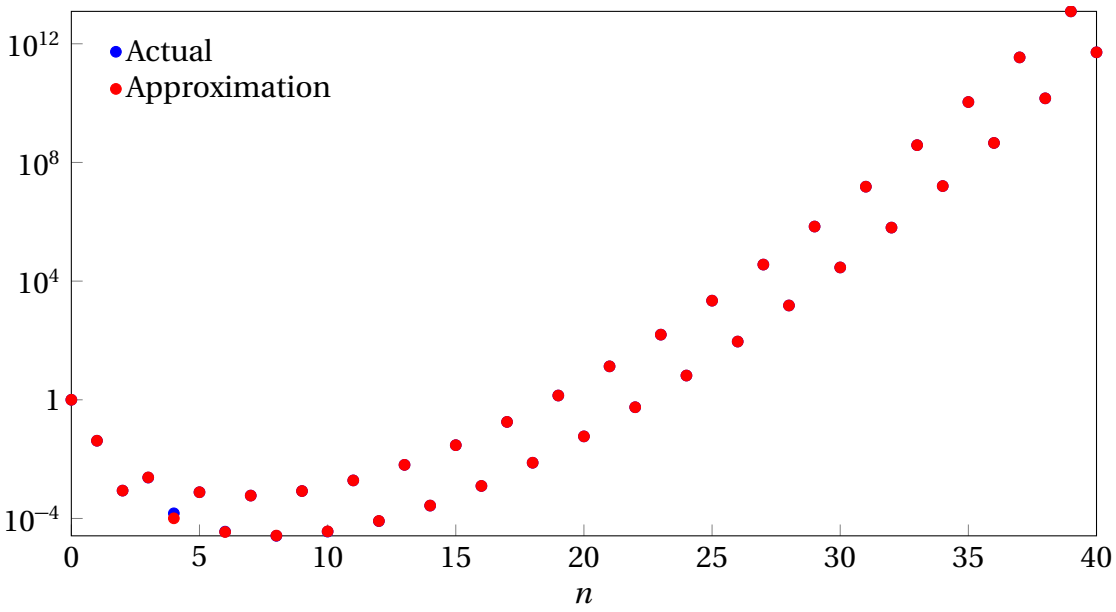


Figure C.3: An overlay of the coefficients c_n (blue) and the coefficient approximations $\check{c}_n$ (red). This plot shows the good agreement between the c_n and the $\check{c}_n$.

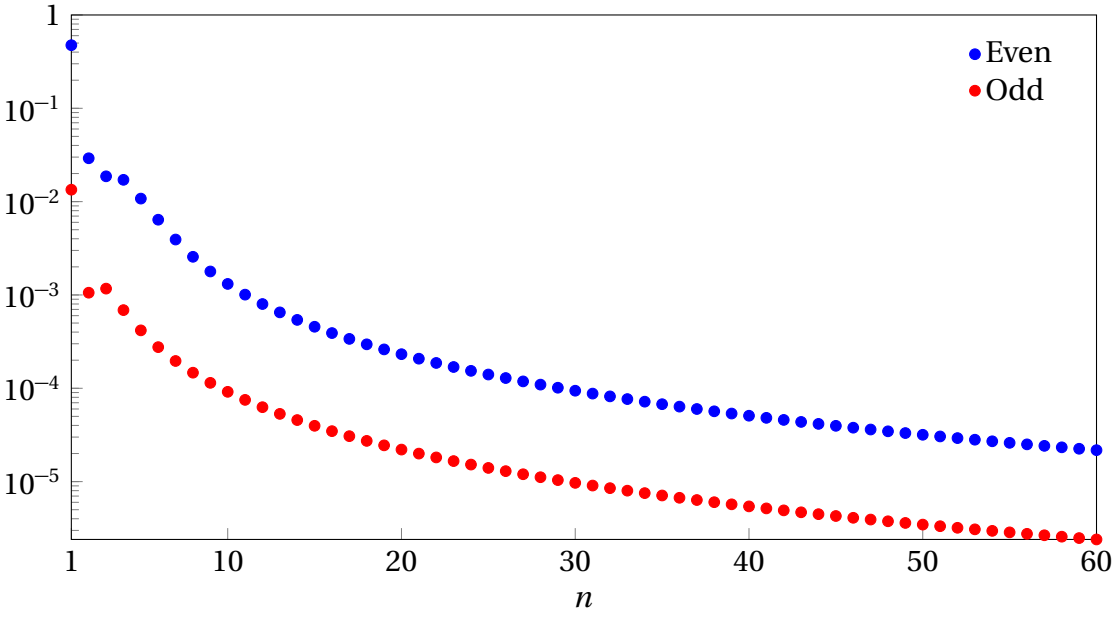


Figure C.4: A plot of the relative error for the even (blue) and odd (red) coefficient approximation $\check{c}_n$. The relative errors in the approximation in (C.18) tend to zero as $n \to \infty$.

C.3 Quartic Gaussian Model

We start by considering the Gaussian model with quartic interaction term $\frac{g}{6}x^4$,

$$\mathscr{I}(g) = \sqrt{\frac{1}{\pi}} \int_{-\infty}^{\infty} \mathrm{d}\, x\; e^{-x^2 - \frac{g}{6}x^4} = \sqrt{\frac{1}{\pi}\left(\frac{3}{2g}\right)}\, e^{\frac{1}{2}\left(\frac{3}{2g}\right)} K_{\frac{1}{4}}\left(\frac{1}{2}\left(\frac{3}{2g}\right)\right) \tag{C.20}$$

where

$$K_\nu(z)=\frac{\sqrt{\pi}\,\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_1^\infty \mathrm{d}\,t\;e^{-zt}(t^2-1)^{\nu-\frac{1}{2}}\qquad\text{for}\qquad \Re(\nu)>-\frac{1}{2}\text{ and }\mathrm{ph}(z)<\frac{\pi}{2}\tag{C.21}$$

is the ν-th modified Bessel function of the second kind [152]. The factor of $1/6$ multiplying g in the integral in (C.20) helps to establish the Borel sum to come. Indeed, dividing g by any number greater that 6 will help the Borel sum to converge. The integral in (C.20) has saddle points at $x_0=0$ and $x_\pm=\pm i\sqrt{3/g}$. Evaluating the exponential in (C.20) at the saddle points x_0 and $x_\pm$ yields

$$e^{-x_0^2-\frac{g}{6}x_0^4}=1\qquad\text{and}\qquad e^{-x_\pm^2-\frac{g}{6}x_\pm^4}=e^{\frac{3}{2g}}\tag{C.22}$$

and expanding (C.21) about $x_\pm$, using $x=\delta x-x_\pm$ yields

$$\mathscr{I}^{(\pm)}(g)=\sqrt{\frac{1}{\pi}}\;e^{\frac{3}{2g}}\int_{-\infty+x_\pm}^{\infty+x_\pm}\mathrm{d}\,\delta x\;e^{2\delta x^2}\left\{1\mp\frac{2i\sqrt{g}}{\sqrt{3}}\delta\mathrm{x}^3+\frac{g}{6}\delta\mathrm{x}^4-\frac{2g}{3}\delta\mathrm{x}^6\mp\frac{ig^{3/2}}{3\sqrt{3}}\delta\mathrm{x}^7+\frac{g^2}{72}\delta\mathrm{x}^8\right.$$
$$\left.\pm\frac{4ig^{3/2}}{9\sqrt{3}}\delta\mathrm{x}^9+\cdots\right\}.$$

Notice that expanding about x_+ gives the same contribution to the integral as expanding about x_- which is associated with the non-perturbative factor $e^{\frac{3}{2}g}$. The integral over δx is divergent for real values of δx and a new contour of integration is needed for this integral to converge. Next, expanding (C.21) about x_0, using $x=\delta x-x_0$ yields

$$\mathscr{I}^{(0)}(g)=\sqrt{\frac{1}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}\,\delta x\;e^{-\delta x^2}\left\{1-\frac{g}{6}\delta x^4+\frac{g^2}{72}\delta x^8-\frac{g^3}{1296}\delta x^{12}+\frac{g^4}{31104}\delta x^{16}+\cdots\right\}$$

and there is no non-perturbative factor associated with expanding about x_0. Then integrating over δx yields

$$\mathscr{I}^{(0)}(g)=\left\{1-\frac{1}{8}g+\frac{35}{384}g^2-\frac{385}{3072}g^3+\cdots\right\}.$$

Expanding (C.20) as a series for small g yields

$$\mathscr{I}(g)=1-\frac{1}{8}g+\frac{35}{384}g^2-\frac{385}{3072}g^3+\frac{25025}{98304}g^4-\frac{1616615}{2359296}g^5+\frac{260275015}{113246208}g^6-\frac{929553625}{100663296}g^7+\cdots\tag{C.23}$$

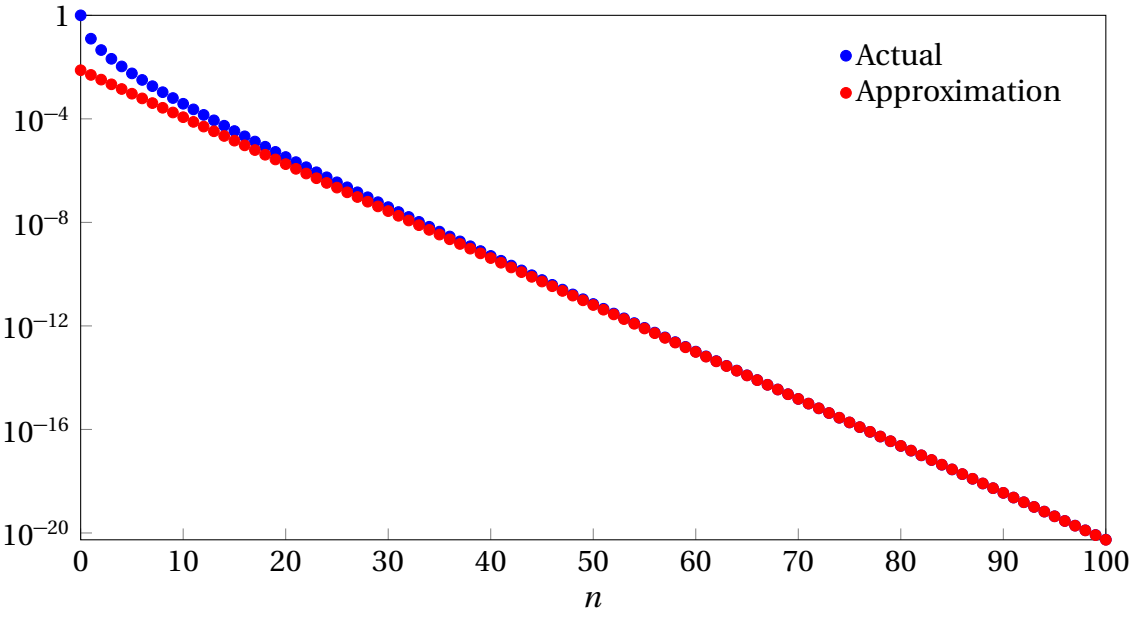


Figure C.5: An overlay of the coefficients $\log(|c_n/n!|)$ (blue) and the approximation $\log(|(-1)^n AB^n|)$ (red).

The coefficients c_n are given by

$$c_n = \frac{1}{24^n}\frac{(4n-1)!!}{n!}.$$

We approximate the large order behaviour of $c_n/n!$ using

$$\frac{c_n}{n!} \sim (-1)^n AB^n \tag{C.24}$$

where

$$A = \frac{653111}{86288845} \approx 0.00756889 \qquad \text{and} \qquad B = \frac{8431419}{12803560} \approx 0.658521.$$

Figure C.5 shows that $c_n/n!$ is asymptotic to $(-1)^n AB^n$ as $n \to \infty$. Similarly, by Stirling approximation, we find

$$\frac{1}{24^n}\frac{(4n-1)!!}{n!} \sim \left(\frac{2n}{3}\right)^n$$

then (C.23) converges only for $2ng/3 < 1$. However, for each choice of g there is an n such that this bound is not satisfied and the series diverges.

The Borel transform of (C.23) can be summed exactly to give

$$\sum_{n=0}^{\infty} \frac{c_n}{n!} s^n = \frac{2\sqrt{3}\, K\left(\frac{2\sqrt{6s}}{\sqrt{6s}+3}\right)}{\pi\sqrt{\sqrt{6s}+3}} = \frac{2}{\pi}\sqrt{\frac{3}{\sqrt{6s}+3}}\, K\left(\frac{2\sqrt{6s}}{\sqrt{6s}+3}\right)$$

where $K(x)$ is the complete elliptic integral of the first kind. The inverse Borel transform of this function is not easily done, so instead we Borel sum and then inverse Borel transform the modeled series instead. This corresponds to an alternating series with coefficients

$$a_n = (-B)^n n! \qquad \text{and} \qquad \frac{1}{B} = 1.51 \approx \frac{3}{2}.$$

Notice that the non-perturbative contribution to the integral in (C.20), corresponding to the value of the integral at the saddle point $x_\pm$ in (C.22) is recovered from the large order behaviour of the coefficients is the asymptotic series in (C.23).

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