UNIVERSITY OF THE WITWATERSRAND

APPROXIMATION THEORY FOR EXPONENTIAL WEIGHTS

By
D.G. Kubayi

A THESIS
submitted to the Faculty of Science
of the University of The Witwatersrand
in partial fulfilment of the requirements for
the degree of Master of Science

Degree awarded with distinction on 10 December 1998

Johannesburg, Republic of South Africa
June 20, 1998.

UNIVERSITY OF THE WITWATERSRAND

Approximation Theory for Exponential Weights

David Giyani Kubayi

Dept of Mathematics

University of The Witwatersrand

PO Wits 2050

Johannesburg, South Africa

20 June 1998

Declaration

I declare that this research report is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

Mubayi

20th day of June 1998

Abstract

Much of weighted polynomial approximation originated with the famous Bernstein qualitative approximation problem of 1910/11. The classical Bernstein approximation problem seeks conditions on the weight functions W such that the set of functions $\{W(x)x^n\}_{n=1}^{\infty}$ is fundamental in the class of suitably weighted continuous functions on $\mathbb{R}$, vanishing at infinity. Many people worked on the problem for at least 40 years. Here we present a short survey of techniques and methods used to prove Markov and Bernstein inequalities as they underlie much of weighted polynomial approximation. Thereafter, we survey classical techniques used to prove Jackson theorems in the unweighted setting. But first we start by reviewing some elementary facts about orthogonal polynomials and the corresponding weight function on the real line. Finally we look at one of the processes of approximation, the Lagrange interpolation and present the most recent results concerning mean convergence of Lagrange interpolation for Freud and Erdös weights.

To

my father, Mbazima Wilson Kubayi

And

my Dear Wife – to – be Agnes Ngobeni

And

my brother and friend, Jimmy Ngoveni.

Acknowledgements

I am greatly indebted to my supervisor, Professor D. S. Lubinsky, for suggesting the research topic, for his expert guidance and teaching in my research, his invaluable assistance in promoting my knowledge and ability, his encouragement, patience, enthusiasm and all support I have received from him since 1995.

I would like to register my sincere gratitude to the Foundation for Research and Development for their generous financial support from 1996 to 1998 during my studying for this MSc. Thanks are also due to the University of The Witwatersrand, Johannesburg, for financial support in the form of Postgraduate Merit Award/Postgraduate Bursary from 1996-1998.

It is my pleasant duty to thank others who have provided support and guidance. I am particularly grateful to my fellow student, John B. Gyamfi Owusu, for helping me with the typing and the final arrangement of the document and my friend, Khayizeni Joas Ngoveni, for all the support he

gave me.

I have tried to give proper credit to various materials used, but I am sure I have benefitted directly and indirectly from countless others. I owe many thanks to all the publishers and authors concerned.

I must thank my brother and friend, Mr Jimmy Ngoveni, for his support and his uncountable words of encouragement which, like streams of water in the desert and the shadow of a great rock in a thirsty land, strengthened me day by day .

Most of all, I am indebted to my parents whose contribution to this effort is truly beyond measure.

D. G. Kubayi

Johannesburg, June 1998.

Contents

Chapter 1

ORTHOGONAL POLYNOMIALS : PRELIMINARIES

In this chapter, we look at some elementary facts about orthogonal polynomials with respect to a general weight function . It is only fitting that before we get into any technical details, we start with a brief historical review of the subject and its relationship with weighted polynomial approximation.

1.1 Brief Historical Review

The general theory of orthogonal polynomials owes much of its "existence" to the great Russian mathematician Chebyshev. Chebyshev published a series of papers in 1855 in which he discussed integrals of the type

$$\int_{-\infty}^{\infty} \frac{p(x)}{x-y} dy, \text{ where } p(x) \geq 0 \text{ in } (-\infty, \infty) \tag{1.1}$$

and the sums of the type

$$\sum_{-\infty}^{\infty} \frac{\theta_i^2}{x - x_i}, \quad \theta_i \neq 0, \tag{1.2}$$

in relation to what was later called the Moment Problem by Stieltjies in his 1894-95 classical paper: "Recherches sur les fractions continues ". Our modern Stieltjies integral comes from this paper.

The Moment Problem: find a bounded non-decreasing function $\psi(x)$ in the interval $[0, \infty)$ such that its "moments"

$$\int_{0}^{\infty} x^n d\psi(x), \quad n = 0, 1, 2, \cdots, \tag{1.3}$$

have a prescibed set of values

$$\int_{0}^{\infty} x^n d\psi(x) = \mu_n, \quad n = 0, 1, 2, \cdots, \tag{1.4}$$

Stieltjes borrowed the terminology "Problem of Moments" from Mechanics. Unlike Stieltjes, Chebyshev was not interested in the existence or construction of a solution of the Moment Problem,

$$\int_0^\infty p(x)\,x^n dx = \mu_n, \quad n = 0, 1, 2, \cdots, \tag{1.5}$$

but mainly in the following two problems:

(I) How far does a given sequence of moments determine the function $p(x)$? In particular, given

$$\int_{-\infty}^\infty p(x)\,x^n dx = \int_{-\infty}^\infty e^{-x^2} x^n dx, \quad n = 0, 1, 2, \cdots, \tag{1.6}$$

can we conclude that

$$p(x) = e^{-x^2}$$

or, in our modern terms, that the distribution characterized by the function

$$\int_{-\infty}^x p(x)\,dt$$

is a normal one? This is a fundamental problem in the theory of probability and in mathematical statistics.

(II) What are the properties of the polynomials $\omega_n(z)$, denominators of successive approximants to the continued fraction:

$$\frac{\lambda_1}{|\, z + c_1} - \frac{\lambda_2 \,|}{|\, z + c_2} - \frac{\lambda_3 \,|}{|\, z + c_3} - \cdots \quad ?$$

In his attempt to find a solution to (II), Chebyshev opened a vast new field, the theory of orthogonal polynomials, of which at the time only the classical polynomials of Legendre, Jacobi, Abel-Laguere and Laplace-Hermite were known before Chebyshev. In his work there are numerous applications of orthogonal polynomials to interpolation, approximate quadratures, expansion of functions in series. Later they have been applied to the general theory of polynomials, theory of best approximants, theory of probability and mathematical stastistics. And most recently they have been applied to the theory of weighted polynomial approximations. This is the area that our investigations will primarily be based. In the sequel we discuss ideas and methods and in between we display some results and their proofs.

1.2 Elementary Facts

Let $W : \mathbb{R} \to [0, \infty)$ be a (Lebesgue) measurable function on the real line. The support of W, denoted by $\mathrm{supp}(W)$, is defined by

$$\mathrm{supp}\,(W) := \overline{\{x \in \mathbb{R} : \forall \varepsilon > 0, \int_{x-\varepsilon}^{x+\varepsilon} W > 0\}}$$

W is said to be a weight function if supp(W) has a positive (Lebesgue) measure, and

$$\int |t|^n W(t)\,dt < \infty, \quad n = 0, 1, 2, \cdots.$$

In the remainder of this chapter, W denotes a fixed weight function. Also we denote the class of all real polynomials of degree at most n by Π_n.

Lemma 1 *Let P be any polynomial which is non-negative on supp(W), and*

$$\int P(x)\,W(x)\,dx = 0.$$

Then

$$P(x) = 0 \;\; \forall x \in \mathbb{R}.$$

Proof. Since $P(x) \geq 0$ for $x \in$supp(W) and $P(x)\,W(x) \geq 0$ almost everywhere, it follows that $P(x)\,W(x) = 0$ almost everywhere. Since the set $\{x \in \mathbb{R} : \forall \varepsilon > 0, \int\limits_{x-\varepsilon}^{x+\varepsilon} W > 0\}$ has positive measure, this implies that $P(x) = 0$ for all $x \in \mathbb{R}$. ■

Let P_n, P_m be polynomials. Then we define an inner product on the class of all polynomials by

$$\langle P_n, P_m \rangle := \int P_n(t)\,P_m(t)\,W(t)\,dt.$$

This definition shows that orthogonal polynomials can be constructed using the Gram-Schmidt orthogonalization procedure. This is illustrated in the following theorem:

Theorem 2 *There exists a unique (infinite) system of orthogonal (orthonormal) polynomials*

$$p_n(x) := p_n(W, x) := \gamma_n x^n + \cdots \in \Pi_n,\ \gamma_n := \gamma_n(W) > 0, n = 0, 1, 2, \cdots, \tag{1.7}$$

such that

$$\int p_n(x)\, p_m(x)\, W(x)\, dx = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m. \end{cases} \tag{1.8}$$

Proof. Existence: we write

$$p_0(x) := \left(\int W(t)\, dt\right)^{-\frac{1}{2}}.$$

Having defined polynomials $p_0, \ldots, p_k$ satisfying (1.7) and (1.8) for some integer $k \geq 0$, we write

$$\begin{aligned} v_{k+1}(x) &:= x^{k+1} \\ u_{k+1}(x) &:= v_{k+1} - \sum_{j=0}^{k} \langle v_{k+1}, p_j \rangle p_j \\ p_{k+1} &:= \langle u_{k+1}, u_{k+1} \rangle^{-\frac{1}{2}}\, u_{k+1}. \end{aligned}$$

Observe that $v_1 = u_1$. Since the system $\{x^k\}$ is linearly independent, u_k is not identically 0 for any k. Hence Lemma 1 shows that $\langle u_k, u_k \rangle > 0$ for all integer $k \geq 0$. Therefore, the process inductively defines an infinite sequence of polynomials $\{p_k\}$ satisfying (1.7) and (1.8). Uniqueness: let

$$\{p_n^* := \gamma_n^* x^n + \cdots \in \Pi_n, \gamma_n^* > 0\}_{n=0}^{\infty}$$

be another system of polynomials satisfying (1.7) and (1.8). We observe that $\{p_k\}_{k=0}^{n-1}$ and $\{p_k^*\}_{k=0}^{n-1}$ are both bases for Π_{n-1} for each integer $n \geq 1$, and hence (1.8) implies that

$$\int P p_n W dx = \int P p_n^* W dx = 0 \text{ for all } P \in \Pi_{n-1}.$$

Since $\gamma_n^{-1} p_n - \gamma_n^{*^{-1}} p_n^* \in \Pi_{n-1}$, this gives

$$\begin{aligned} 0 &= \int \left[\gamma_n^{-1} p_n - \gamma_n^{*^{-1}} p_n^*\right] p_n W dx \\ &= \gamma_n^{-1} \int p_n^2 W dx - \gamma_n^{*^{-1}} \int p_n^* p_n W dx \\ &= \gamma_n^{-1} - \gamma_n^{*^{-1}} \int p_n^* p_n W dx. \end{aligned}$$

Hence,

$$\int p_n^* p_n W dx = \frac{\gamma_n^*}{\gamma_n} \tag{1.9}$$

and switching the roles of p_n and p_n^*, we get

$$\int p_n p_n^* W dx = \frac{\gamma_n}{\gamma_n^*} \tag{1.10}$$

Since γ_n and γ_n^* are both positive, equations (1.9) and (1.10) imply that

$$\gamma_n = \gamma_n^* ,$$

and

$$\int p_n^* p_n W dx = 1.$$

Then

$$\int [p_n - p_n^*]^2 W dx = \int p_n^2 W dx - 2 \int p_n p_n^* W dx + \int p_n^2 W dx$$
$$= 1 - 2(1) + 1 = 0$$

and Lemma 1 implies that

$$\begin{aligned} [p_n - p_n^*]^2 &= 0. \\ \text{i.e., } p_n &= p_n^*. \blacksquare \end{aligned}$$

Remark 1 *Given a sequence of positive numbers $\gamma_n > 0$, we can construct a unique system of polynomials orthogonal with respect to a suitable measure, such that the leading coefficient of the polynomial of degree n is γ_n.*

Next, we prove a recurrence relation which underlies the importance of the sequence $\{\gamma_n\}$ of coefficients.

Theorem 3 *Let $\gamma_{-1} := 0$ and*

$$\beta_n := \int x p_n^2(x) W(x)\,dx, n = 0, 1, 2, \ldots. \tag{1.11}$$

Then

$$(x - \beta_n) p_n(x) = \frac{\gamma_n}{\gamma_{n+1}} p_{n+1}(x) + \frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x), n = 0, 1, 2, \ldots, \tag{1.12}$$

where

$$p_{-1}(x) := 0.$$

Proof. First, consider the case $n = 0$. In view of the equations

$$\gamma_0^2 \int W(x)\,dx = \int p_0^2(x) W(x)\,dx = 1,$$

and

$$\beta_0 = \gamma_0^2 \int x W(x)\,dx,$$

we see that

$$\begin{aligned} \int (x - \beta_0) W(x)\,dx &= \int x W(x)\,dx - \beta_0 \int W(x)\,dx \\ &= \frac{\beta_0}{\gamma_0^2} - \frac{\beta_0}{\gamma_0^2} = 0. \end{aligned}$$

Hence, $x - \beta_0 = \frac{1}{\gamma_1} p_1(x)$, and (1.12) follows for $n = 0$. Next, let $n \geq 1$. Since $x p_n(x) \in \Pi_{n+1}$, there are numbers $a_0, a_1, \ldots, a_{n+1}$ such that

$$x p_n(x) = \sum_{k=0}^{n+1} a_k p_k(x) \tag{1.13}$$

Using the orthogonality relations (1.8), we have

$$a_k = \int x p_n(x)\, p_k(x)\, W(x)\, dx,\ k = 0, 1, \ldots, n+1. \tag{1.14}$$

We calculate the coefficients of a_k one by one. If $k \leq n-2$ then $xp_k(x) \in \Pi_{n-1}$. Since $xp_{n-1}(x) - \frac{\gamma_{n-1}}{\gamma_n} p_n(x) \in \Pi_{n-1}$, we get

$$\begin{aligned} a_{n-1} &= \int x p_{n-1}(x)\, p_n(x)\, W(x)\, dx \\ &= \int \left[x p_{n-1}(x) - \frac{\gamma_{n-1}}{\gamma_n} p_n(x) \in \Pi_{n-1} \right] p_n(x)\, W(x)\, dx + \frac{\gamma_{n-1}}{\gamma_n} \int p_n^2(x)\, W(x)\, dx \\ &= \frac{\gamma_{n-1}}{\gamma_n}. \end{aligned}$$

Similarly, $a_{n+1} = \frac{\gamma_n}{\gamma_{n+1}}$, and finally,

$$\begin{aligned} a_n &= \int x p_n(x)\, p_n(x)\, W(x)\, dx \\ &= \int x p_n^2(x)\, W(x)\, dx \\ &= \beta_n. \text{ (From (1.11))} \ \blacksquare \end{aligned}$$

Remark 2 *Given a polynomial $P \in \Pi_{n-1}$, we can write it in the form*

$$P(x) = \sum_{k=0}^{n-1} a_k p_k(x)$$

where the constants a_k are given by

$$a_k = \int P(t)\, p_k(t)\, W(t)\, dt,\ k = 0, 1, \ldots, n-1. \tag{1.15}$$

Therefore, we get

$$\begin{aligned} P(x) &= \sum_{k=0}^{n-1}\left(\int P(t)\,p_k(t)\,W(t)\,dt\right)p_k(x)\,,\ \forall P\in\Pi_{n-1}. \qquad (1.16)\\ &= \int P(t)\,\{\sum_{k=0}^{n-1}p_k(x)\,p_k(t)\}W(t)\,dt,\ \ \forall P\in\Pi_{n-1}. \end{aligned}$$

Using Theorem 3,we obtain a simple formula for the important Christoffel-Darboux kernel

$$K_n(x,t) := K_n(W;x,t) := \sum_{k=0}^{n-1}p_k(t)\,p_k(x)\,. \qquad (1.17)$$

Theorem 4 (*Christoffel-Darboux Formula*)

For integer $n \geq 1$,

$$K_n(x,t) := \frac{\gamma_{n-1}}{\gamma_n}\,\frac{p_n(x)\,p_{n-1}(t)-p_{n-1}(x)\,p_n(t)}{x-t}. \qquad (1.18)$$

Proof. Using the recurrence equation (1.12), we get

$$\begin{aligned} xK_n(x,t) &= \sum_{k=0}^{n-1}xp_k(x)\,p_k(t)\\ &= \sum_{k=0}^{n-1}\left\{\frac{\gamma_k}{\gamma_{k+1}}p_{k+1}(x)+\beta_k p_k(x)+\frac{\gamma_{k-1}}{\gamma_k}p_{k-1}(x)\right\}p_k(t)\\ &= \sum_{k=1}^{n}\frac{\gamma_{k-1}}{\gamma_k}p_k(x)\,p_{k-1}(t)+\sum_{k=0}^{n-1}\beta_k p_k(x)\,p_k(t)\\ &\quad +\sum_{k=0}^{n-1}\frac{\gamma_{k-1}}{\gamma_k}p_{k-1}(x)\,p_k(t)\,. \qquad (1.19) \end{aligned}$$

Similarly,

$$tK_n(x,t) = \sum_{k=0}^{n-1} \frac{\gamma_{k-1}}{\gamma_k} p_k(x) p_{k-1}(t) + \sum_{k=0}^{n-1} \beta_k p_k(x) p_k(t) + \sum_{k=1}^{n} \frac{\gamma_{k-1}}{\gamma_k} p_{k-1}(x) p_k(t) \tag{1.20}$$

Subtracting (1.20) from (1.19) gives

$$(x-t) K_n(x,t) = \frac{\gamma_{n-1}}{\gamma_n} p_n(x) p_{n-1}(t) - \frac{\gamma_{n-1}}{\gamma_n} p_n(t) p_{n-1}(x).$$

This gives (1.18). ■

Corollary 5 *For integer $n \geq 1$,*

$$K_n(x,x) = \sum_{k=0}^{n-1} p_k^2(x) = \frac{\gamma_{n-1}}{\gamma_n} \left[p_n'(x) p_{n-1}(x) - p_n(x) p_{n-1}'(x) \right] \tag{1.21}$$

The quantity $K_n(x,x)$ and its reciprocal, known as the Christoffel function, denoted by $\lambda_n(x)$, play an important role in the theory of orthogonal polynomials.

Next, we present a specific example to illustrate the general theory discussed thus far.

Example 6 *(The Chebyshev Polynomials)*

These polynomials are defined on $[-1,1]$ by the formula

$$\mathbb{T}_n(\cos t) = \cos nt, \quad n = 0, 1, 2, \ldots. \tag{1.22}$$

It is well known that $\mathbb{T}_n(\cos t)$ is indeed a polynomial of degree n in $\cos t$, $\mathbb{T}_0(x) = 1$ and for integer $n \geq 1$,

$$\mathbb{T}_n(x) = 2^{n-1}x^n + \cdots \in \Pi_n. \tag{1.23}$$

It is also verified easily that

$$\int_{-1}^{1} \mathbb{T}_n(x)\,\mathbb{T}_m(x)\,(1-x^2)^{-\frac{1}{2}}dx = \begin{cases} \pi, \text{if } n = m = 0 \\ \frac{\pi}{2}, \text{ if } n = m \neq 0 \\ 0, \text{ if } n \neq m \end{cases} \tag{1.24}$$

Therefore, the system

$$\left\{\sqrt{\frac{1}{\pi}}, \left(\sqrt{\frac{2}{\pi}}\right)\mathbb{T}_1(x), \sqrt{\frac{2}{\pi}}\mathbb{T}_2(x), \ldots\right\}$$

is a system of orthonormal polynomials with respect to the weight function

$$W(x) = \begin{cases} (1-x^2)^{-\frac{1}{2}}, \text{ if } |x| \leq 1 \\ 0 \qquad\quad, \text{ if } |x| > 1 \end{cases} \tag{1.25}$$

The trigonometric identity

$$2\cos t\ \cos nt = \cos(n+1)t + \cos(n-1)t$$

takes the form of the recurrence relation

$$x\mathbb{T}_n(x) = \left(\frac{1}{2}\right)\mathbb{T}_{n+1}(x) + \left(\frac{1}{2}\right)\mathbb{T}_{n-1}(x)$$

which is consistent with (1.23) and (1.24). The Christoffel function $\lambda_n(x)$ can be calculated explicitly. By definition,

$$\lambda_n^{-1}(x) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^{n-1} \mathbb{T}_k^2(x)$$

so that

$$\begin{aligned}
\frac{\pi}{2}\lambda_n^{-1}(\cos t) &= \frac{1}{2} + \cos^2 t + \cdots + \cos^2(n-1)t \\
&= \frac{1}{2} + (\frac{1+\cos 2t}{2}) + \cdots (\frac{1+\cos 2(n-1)t}{2}) \\
&= \frac{n}{2} + \frac{1}{2}\sum_{k=1}^{n-1} \cos kt.
\end{aligned}$$

Multiplying both sides of the above equation by $2\sin t$, we get

$$\begin{aligned}
\pi \sin t\, \lambda_n^{-1}(\cos t) &= n\sin t + \sum_{k=1}^{n-1} \sin t \, \cos 2kt \\
&= n\sin t + \frac{1}{2}\sum_{k=1}^{n-1} [\sin(2k+1)t - \sin(2k-1)t] \\
&= n\sin t + \frac{1}{2}\sin(2n-1)t - \frac{1}{2}\sin t.
\end{aligned}$$

Hence,

$$\lambda_n^{-1}(\cos t) = \frac{1}{\pi}\{n - \frac{1}{2} + \frac{\sin(2n-1)t}{2\sin t}\}. \tag{1.26}$$

It can be verified directly that

$$\frac{\sin(2n-1)t}{\sin t}$$

is a polynomial in $\cos t$ of degree $2n-2$. This polynomial is ·ed by U_{2n-2}. Thus,

$$\lambda_n^{-1}(x) = \frac{1}{\pi}\{n - \frac{1}{2} + U_{2n-2}\}. \tag{1.27}$$

In the next section, we define the class of weight functions, known as Freud weights, and the corresponding orthogonal polynomials. We also study some simple properties of these weight functions and polynomials.

1.3 The Weight Function and Orthogonal Polynomials

In this section, we introduce Freud weights and study some of their elementary properties. We will also estimate the order of magnitude of the recurrence coefficients γ_n and the largest zeros of the orthogonal polynomials with respect to these weights.

Definition 7 *Let $W := e^{-Q} : \mathbb{R} \to [0, \infty)$. We say that W is a Freud weight* *function*, *if*

$$Q(x) := \log\left\{\frac{1}{W(x)}\right\} \tag{1.28}$$

is an even, convex function on $\mathbb{R}$, Q *is twice continuously differentiable on* $(0,\infty)$, *and there are constants* c_1 *and* c_2 *such that*

$$0 < c_1 < \frac{xQ''(x)}{Q'(x)} < c_2 < \infty,\ \forall x \in (0,\infty). \tag{1.29}$$

We will often write $W_Q(x) := \exp(-Q(x))$. In the remainder of this section, $W = W_Q$ will denote a fixed Freud weight function. The prototypical Freud weight functions, $\exp(-|x|^{\alpha})$, $\alpha > 1$, will be denoted by W_α. The other cases, $0 < \alpha < 1$ and $\alpha = 1$, will be dealt with in chapter 2.

Notation: $A \sim B$ means that $c_1 A \leq B \leq c_2 A$. Also the integral without limits denotes the integral over the whole real line.

We enumerate some simple properties of Freud weights in the following proposition.

Proposition 8 *(a) If* $0 < x < y < \infty$ *then (with constants* c_1, c_2 *as in* (1.29)*),*

$$\left(\frac{y}{x}\right)^{c_1} Q'(x) \leq Q'(y) \leq \left(\frac{y}{x}\right)^{c_2} Q'(x). \tag{1.30}$$

In particular,

$$\lim_{x\to\infty} Q'(x) = \infty.$$

(*b*) *The weight function* W *satisfies*

$$c_3 \exp\left(-c_4 |x|^{c_2+1}\right) \le W(x) \le c_5 \exp\left(-c_6 |x|^{c_1+1}\right),\ x \in \mathbb{R}. \tag{1.31}$$

In particular,

$$\int |x|^n W(x)\, dx < \infty,\ n = 0, 1, 2, \ldots, \tag{1.32}$$

and

$$\lim_{|x|\to\infty} P(x)\, W(x) = 0 \tag{1.33}$$

for every polynomial P.

(*c*) *For* $x \ge Q'(0+)$, *let* q_x *denote the least positive solution of the equation*

$$q_x Q'(q_x) = x. \tag{1.34}$$

Then

$$2^{\frac{1}{1+c_2}} q_x \le q_{2x} \le 2^{\frac{1}{1+c_1}} q_x.$$

Remark 3 : *The numbers* q_x *introduced in* (1.34) *are called the Freud numbers. We observe that if* W *is a Freud weight, so is* W^{2}. *Proposition* 8 *and* *results in section 1.2 show that there exists a system of polynomials , called Freud polynomials,*

$$p_n(x) := p_n\left(W^2; x\right) := \gamma_n x^n + \cdots \in \Pi_n,\ \gamma_n > 0. \tag{1.35}$$

such that

$$\int p_n(x)\, p_m(x)\, W^2(x)\, dx = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases} \tag{1.36}$$

Moreover, for each $n \geq 1$, p_n has n real and simple zeros,

$$-X_n = x_{n,n} < \cdots < x_{1,n} =: X_n. \tag{1.37}$$

We note that the zeros are symmetric around the origin, since the weight function is even. The recurrence relation given in Theorem 3 takes the form

$$xp_{n-1}(x) = \frac{\gamma_{n-1}}{\gamma_n} p_n(x) + \frac{\gamma_{n-2}}{\gamma_{n-1}} p_{n-2}(x),\ n = 2, 3, \ldots. \tag{1.38}$$

In this section, we estimate the order of magnitude of the recurrence coefficients $\frac{\gamma_{n-1}}{\gamma_n}$ and the largest zero X_n.

First, we make some observations concerning orthogonal polynomials.

Proposition 9 *We have*

$$\frac{\gamma_{n-1}}{\gamma_n} = \int xp_n(x)\, p_{n-1}(x)\, W^2(x)\, dx. \tag{1.39}$$

$$\frac{n}{2}\frac{\gamma_{n-1}}{\gamma_n} = \int Q'(x)\, p_n(x)\, p_{n-1}(x)\, W^2(x)\, dx. \tag{1.40}$$

$$n + \frac{1}{2} = \int xQ'(x)\, p_n^2(x)\, W^2(x)\, dx. \tag{1.41}$$

Proof. Since $xp_{n-1}(x) = \frac{\gamma_{n-1}}{\gamma_n} p_n(x) + P(x)$ for some $P \in \Pi_{n-1}$, then equation (1.39) follows from the orthogonality relations (1.36). Next, we observe that p_n' is a polynomial of degree $n-1$ with leading coefficient $n\gamma_n$. So, $p_n' - n\frac{\gamma_n}{\gamma_{n-1}} p_{n-1} \in \Pi_{n-2}$. Using the orthogonality relations (1.36) again, we get

$$\begin{aligned}\int p_n'(x)\, p_{n-1}(x)\, W^2(x)\, dx &= n\frac{\gamma_n}{\gamma_{n-1}} \int p_{n-1}^2(x)\, W^2(x)\, dx \\ &= n\frac{\gamma_n}{\gamma_{n-1}}.\end{aligned}$$

Integrating by parts and observing that

$$\lim_{|x|\to\infty} p_n'(x)\, p_{n-1}(x)\, W^2(x) = 0,$$

we deduce that

$$n\frac{\gamma_n}{\gamma_{n-1}} = 2\int Q'(x)\, p_n(x)\, p_{n-1}(x)\, W^2(x)\, dx - \int p_n(x)\, p_{n-1}'(x)\, W^2(x)\, dx.$$

Since $p_{n-1}' \in \Pi_{n-2}$, the second integral above is zero, and we get (1.40).

In order to prove (1.41), we observe that $xp_n'(x) - np_n(x) \in \Pi_{n-1}$, and hence,

$$\int xp_n'(x)\, p_n(x)\, W^2(x)\, dx = n\int p_n^2(x)\, W^2(x)\, dx = n.$$

Therefore, an integration by parts as before,

$$n + \frac{1}{2} = \frac{1}{2}\int \left[2xp_n'(x)\, p_n(x) + p_n^2(x)\right] W^2(x)\, dx$$

$$= \frac{1}{2}\int \frac{d}{dx}\left[xp_n^2(x)\right] W^2(x)\,dx$$
$$= \int xp_n^2(x)\,Q'(x)\,W^2(x)\,dx$$

which is (1.41). ■.

We are now in a position to estimate the recurrence coefficients $\frac{\gamma_{n-1}}{\gamma_n}$ and the largest zero X_n. These estimates are not sharp for Freud weight functions.

Theorem 10 *We have*

$$\frac{1}{4}q_n \leq \frac{\gamma_{n-1}}{\gamma_n} \leq 2q_n,\ n = 2, 3, \ldots, \tag{1.42}$$

and

$$\frac{1}{4}q_{n-1} \leq X_n \leq 4q_{n-1},\ n = 2, 3, \ldots. \tag{1.43}$$

In particular,

$$\frac{\gamma_{n-1}}{\gamma_n} \sim X_n \sim q_n. \tag{1.44}$$

Proof. In view of (1.39),

$$\frac{\gamma_{n-1}}{\gamma_n} = \{\int_{|x|\leq q_n} + \int_{|x|\geq q_n}\} xp_n(x)\,p_{n-1}(x)\,W^2(x)\,dx. \tag{1.45}$$

Using Schwarz inequality, we get

$$\left|\int_{|x|\leq q_n} xp_n(x)\,p_{n-1}(x)\,W^2(x)\,dx\right|$$

$$
\begin{aligned}
&\leq q_n \int \left| p_n(x)\, p_{n-1}(x)\, W^2(x) \right| dx \\
&\leq q_n \left(\int p_n^2(x)\, W^2(x)\, dx \right)^{\frac{1}{2}} \left(\int p_{n-1}^2(x)\, W^2(x)\, dx \right)^{\frac{1}{2}} \\
&= q_n. \qquad (1.46)
\end{aligned}
$$

Since W is even , and Q' is increasinr, we have

$$
\begin{aligned}
&\left| \int_{|x|\geq q_n} x p_n(x)\, p_{n-1}(x)\, W^2(x)\, dx \right| \\
&= 2 \left| \int_{q_n}^{\infty} x p_n(x)\, p_{n-1}(x)\, W^2(x)\, dx \right| \\
&\leq \frac{2}{Q'(q_n)} \int_{q_n}^{\infty} \left| x Q'(x)\, p_n(x)\, p_{n-1}(x)\, W^2(x) \right| dx \\
&\leq \frac{2}{Q'(q_n)} \int_{0}^{\infty} \left| x Q'(x)\, p_n(x)\, p_{n-1}(x)\, W^2(x) \right| dx \\
&= \frac{q_n}{n} \int \left| x Q'(x)\, p_n(x)\, p_{n-1}(x)\, W^2(x) \right| dx \qquad (1.47)
\end{aligned}
$$

We observe that $xQ'(x)\, W^2(x) \geq 0$ for all $x \in \mathbb{R}$. Hence, using Schwarz inequality and (1.41), we get

$$
\begin{aligned}
&\int \left| x Q'(x)\, p_n(x)\, p_{n-1}(x)\, W^2(x) \right| dx \\
&\leq \left(\int x Q'(x)\, p_n^2(x)\, W^2(x)\, dx \right)^{\frac{1}{2}} \left(\int x Q'(x)\, p_{n-1}^2(x)\, W^2(x)\, dx \right)^{\frac{1}{2}} \\
&= \sqrt{n + \frac{1}{2}} \sqrt{n - \frac{1}{2}} = \sqrt{n^2 - \frac{1}{4}} \leq n. \qquad (1.48)
\end{aligned}
$$

Substituting from this estimate into (1.47), we get

$$\left| \int_{|x| \geq q_n} x p_n(x) p_{n-1}(x) W^2(x)\, dx \right| \leq q_n \tag{1.49}$$

Using the estimates (1.46) and (1.49) in (1.45), we get the upper estimate for the recurrence coefficients in (1.42). For the lower estimate, we write (1.40) in the form

$$\frac{n}{2} \frac{\gamma_n}{\gamma_{n-1}} = \{ \int_{|x| \leq q_n} + \int_{|x| \geq q_n} \} Q'(x) p_n(x) p_{n-1}(x) W^2(x)\, dx. \tag{1.50}$$

Since W is even, and Q' is increasing, we use the Schwarz inequality to get

$$\left| \int_{|x| \leq q_n} Q'(x) p_n(x) p_{n-1}(x) W^2(x)\, dx \right| \leq Q'(q_n) \int \left| p_n(x) p_{n-1}(x) W^2(x) \right| dx \leq \frac{n}{q_n} \tag{1.51}$$

Similarly, using the fact that W is even and the estimate (1.48), we get

$$\begin{aligned} \left| \int_{|x| \geq q_n} Q'(x) p_n(x) p_{n-1}(x) W^2(x)\, dx \right| &\leq \frac{1}{q_n} \int_{|x| \geq q_n} x Q'(x) \left| p_n(x) p_{n-1}(x) \right| W^2(x)\, dx \\ &\leq \frac{n}{q_n} \end{aligned} \tag{1.52}$$

From (1.50), (1.51) and (1.52), we get

$$\frac{n}{2} \frac{\gamma_n}{\gamma_{n-1}} \leq \frac{2n}{q_n},$$

and hence, the lower estimate in (1.42). To obtain the bounds on the largest zeros X_n, we can use the relation between the recurrence coefficients and the zeros to show that

$$X_n \leq 2 \max_{1\leq k\leq n-1} \frac{\gamma_{k-1}}{\gamma_k} \leq 2 \max_{1\leq k\leq n-1} (2q_n) \leq 4q_{n-1},$$

which is the upper estimate in (1.43) . Similarly,

$$X_n \geq \max_{1\leq k\leq n-1} \frac{\gamma_{k-1}}{\gamma_k} \geq \frac{\gamma_{n-2}}{\gamma_{n-1}} \geq \frac{q_{n-1}}{4}.$$

■

Finally, we obtain an upper bound on the leading coefficient itself.

Proposition 11 *For integer $n \geq 1$,*

$$\gamma_n \leq c \left(\frac{12}{q_n}\right)^n \tag{1.53}$$

Proof. *From* (1.42), *we get*

$$\begin{aligned}
\gamma_n &= \gamma_0 \frac{\gamma_n}{\gamma_{n-1}} \frac{\gamma_{n-1}}{\gamma_{n-2}} \cdots \frac{\gamma_1}{\gamma_0} \\
&\leq 4^n \gamma_0 \left(q_n \cdots q_1\right)^{-1} \\
&= 4^n \gamma_0 \frac{\prod_{k=1}^{n} Q'(q_k)}{n!} \\
&\leq 4^n \gamma_0 \frac{Q'(q_n)^n}{n!} \\
&\leq \frac{\gamma_0}{n!} \left(\frac{4n}{q_n}\right)^n
\end{aligned} \tag{1.54}$$

Since

$$\log n! = \sum_{k=2}^{n} \log k \geq \sum_{k=2}^{n} \int_{k-1}^{k} \log x dx = n \log n - n + 1,$$

we have

$$\frac{n^n}{n!} \leq e^{n-1} \leq 3^{n-1}.$$

Substituting this estimate into (1.54) *we get* (1.53) . ■

Chapter 2

WEIGHTED POLYNOMIAL APPROXIMATION ON $(-\infty, \infty)$

2.1 Introduction

Most of the great work in weighted approximation of this century originated with the famous Bernstein's problem in about 1910/1911. We know that polynomials are unbounded in unbounded sets. So Bernstein realized that there was a need to introduce an extra condition on them so that they can be used to approximate. He introduced a weight function W that will be used to weight these polynomials.

Bernstein's Problem: let $W : \mathbb{R} \to (0,1]$ be continuous. Under what conditions on W is it true that for all continuous $f : \mathbb{R} \to \mathbb{R}$ and for $\varepsilon > 0$, we can find a polynomial P such that

$$\|(f - P)\, W\|_{L_\infty(\mathbb{R})} < \varepsilon\,?$$

We observe that for the problem to make sense, we need $\|PW\|_{L_\infty}$ to be finite for every plynomial P, and hence we need

$$\lim_{|x|\to\infty} x^n W(x) = 0,\ n = 0, 1, 2, \ldots. \tag{2.1}$$

Then PW vanishes at $\pm\infty$ for each polynomial P, so we need

$$\lim_{|x|\to\infty} f(x)\, W(x) = 0. \tag{2.2}$$

We also need f to be continuous , as the locally uniform limit of polynomials is continuous.

With (2.1) and (2.2)as above, we will say that the polynomials are dense for W if the Bernstein problem has a positive solution.

Bernstein's approximation problem was solved in the 1950's independently by Achieser(1954), Pollard(1953) and Mergelyan(1956). All three solutions were different, but all involved "regularizations" or functionals of the weight and are implicit. Mergelyan's solution, for example, involves the function

$$\Omega(x) := \sup\left\{|P(x)| : P \text{ a polynomial with } \left\|\frac{(PW)(t)}{\sqrt{1+t^2}}\right\|_{L_\infty(\mathbb{R})} \leq 1\right\}$$

He showed that Bernstein 's problem has a positive solution if and only if

$$\int_{-\infty}^{\infty} \frac{\ln \Omega(x)}{1+x^2} dx = \infty.$$

In our discussions, we are going to restrict ourselves to the methods which impose regularity conditions on the weight function W that guarantee a simple solution. The following theorem is due to L. Carleson and M.M. Dzrbasjan:

Theorem 12 *Let $W = e^{-Q}$ where $Q : \mathbb{R} \to [0, \infty)$ is even, and $Q(e^x)$ is convex on $(0, \infty)$. Then Bernstein's problem has a positive solution, that is the polynomials are dense for W, iff*

$$\int_{0}^{\infty} \frac{Q(x)}{1+x^2} dx = \infty. \qquad (2.3)$$

Corollary 13 *Let $\alpha > 0$, and*

$$W_\alpha(x) := \exp(-|x|^\alpha) \tag{2.4}$$

Then the polynomials are dense for W iff $\alpha \geq 1$.

With Bernstein's qualitative approximation problem solved, clearly the next step for researchers/approximators is the quantitative approximation: how large must the degree of P be, to achieve a desired accuracy ε ? Dzrbasjan and his co-workers had already started investigations in this direction even before the full solution to Bernstein's problem was published. The search for the quantitative solution led to the beginning of the main theme in weighted polynomial approximation of this century: characterizing the rate of decay of polynomial approximation of a function in terms of its structural /smoothness properties.

Before we get deeper into these results, we present ideas and methods used in proving Markov and Bernstein inequalities.

2.2 Markov and Bernstein Inequalities

In the sequel, P_n denotes an algebraic polynomial of degree $\leq n$,and $C, C_1, C_2, \ldots$ denote positive constants independent of n, P_n and x. The same constant does not necessarily denote the same constant in different occurrences. We write $C \neq C(f)$ to indicate that C is independent of f and $C = C(f)$ to show that it depends on f .

The focus here is on weighted analogues on $(-\infty, \infty)$ of the Markov inequality

$$\left\|P_n'\right\|_{L_p[-1,1]} \leq Cn^2 \left\|P_n\right\|_{L_p[-1,1]} \tag{2.5}$$

and the Bernstein inequality

$$\left\|P_n'(x)\sqrt{1-x^2}\right\|_{L_p[-1,1]} \leq Cn \left\|P_n\right\|_{L_p[-1,1]} \tag{2.6}$$

where $C = C(n,p)$.

For the canonical Freud weights W_α, the Markov inequality has the form

$$\left\|P_n' W_\alpha\right\|_{L_p[\mathbb{R}]} \leq C \left\|P_n W_\alpha\right\|_{L_p[\mathbb{R}]} \times \begin{cases} n^{1-\frac{1}{\alpha}}, \text{ if } \alpha \geq 1 \\ \log n, \text{ if } \alpha = 1 \\ 1, \text{ if } \alpha < 1 \end{cases} \tag{2.7}$$

Here $0 < p \leq \infty$ and $C = C(\alpha, p)$.

We shall dwell at length on the techniques used to prove Markov and Bernstein inequalities, as they underlie much of the weighted approximation in general.

(I) **Dzrbasjan's method** [13]

Let $\varepsilon > 0$ and $x \in \mathbb{R}$. From Cauchy's integral formula for derivatives,

$$\begin{aligned} \left|P_n'(x)\right| &= \left|\frac{1}{2\pi i}\int_{\{t:|t-x|=\varepsilon\}}\frac{P_n(t)}{(t-x)^2}dt\right| \\ &\leq \frac{1}{\varepsilon}\|P_n\|_{L_\infty(\{t:|t-x|=\varepsilon\})} \end{aligned} \tag{2.8}$$

This gives us an estimate for the derivative in terms of values of P_n in the complex plane. To return to the real line, we use an inequality that arises in the theory of functions analytic in the upper-half plane:

$$\log|P_n(x+iy)| \leq \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\log|P_n(s)|}{(x-s)^2+y^2}ds,\ y > 0. \tag{2.9}$$

To estimate the RHS in terms of

$$M := \|P_nW\|_{L_\infty(\mathbb{R})}$$

we split the integral in (2.9) into several pieces. If $x > \delta > 0$, and Q is increasing in $(0,\infty)$, then

$$\frac{y}{\pi}\int_{x-\delta}^{x+\delta}\frac{\log|P_n(s)|}{(x-s)^2+y^2}ds \;\leq\; \frac{y}{\pi}\int_{x-\delta}^{x+\delta}\frac{\log M + Q(s)}{(x-s)^2+y^2}ds$$

$$\leq \ \log M + Q(x+\delta)\,\frac{y}{\pi}\int\limits_{x-\delta}^{x+\delta}\frac{ds}{(x-s)^2+y^2}.$$

The integral over the complementary range is estimated using the Bernstein-type inequality:

$$|P_n(x)| \leq |\mathbb{T}_n(x)|\,\|P_n\|_{L_\infty[-1,1]} \leq (2|x|)^n\,\|P_n\|_{L_\infty[-1,1]}$$

where $\mathbb{T}_n(x)$ denotes the Chebyshev polynomial, P_n is an arbitrary polynomial and $|x| > 1$.

A careful choice of a small value of y then yields the required result.

(II) **Freud's Method I** [16], [17]

There are two methods to consider here. We begin with the one for the general case. Assume that $W := e^{-Q}$, where Q is even, convex and of smooth polynomial growth at ∞. Let $\{p_j(x)\}_{j=0}^{\infty}$ denote the orthonormal polynomials with respect to W^2 so that

$$\int\limits_{-\infty}^{\infty} p_j(x)\,p_k(x)\,W^2(x)\,dx = \delta_{jk}. \tag{2.10}$$

For $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)\,W^2(x)\,x^j \in L_1(\mathbb{R})$, $j \geq 0$, we can define the formal orthonormal expansion

$$f \sim \sum_{j=0}^{\infty} b_j p_j,\ \ b_j = \int\limits_{-\infty}^{\infty} f p_j W^2\,,\ \ j \geq 0. \tag{2.11}$$

Let

$$S_m[f] := \sum_{j=0}^{m-1} b_j p_j \tag{2.12}$$

denote the mth partial sum of this orthonormal expansion. Using ideas of T. Carleman from Fourier series, and local one-sided approximations to W, Freud proved the $(C, 1)$ –bound

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \left| S_j' [f] \right| W \right\|_{L_\infty(\mathbb{R})} \leq C \, \frac{m}{a_m} \, \|fW\|_{L_\infty(\mathbb{R})} \tag{2.13}$$

with $C \neq C(f, m)$. Here $a_m = a_m(Q)$. The class of weights included W_α, $\alpha \geq 2$, for which $\frac{m}{a_m} = C_1 m^{1-\frac{1}{\alpha}}$. Once we have this, the sup-norm Markov inequality follows. For, introducing the dilated de la Vallee Poussin sum

$$V_m[f] = \frac{1}{m} \sum_{j=m+1}^{2m} S_j[f] \tag{2.14}$$

we see that

$$\left\| V_m' [f] \, W \right\|_{L_\infty(\mathbb{R})} \leq C_2 \frac{m}{a_m} \, \|fW\|_{L_\infty(\mathbb{R})} \tag{2.15}$$

Also V_m has the quasi-projection property

$$V_m[P_m] = P_m.$$

So choosing $f = P_m$ in (2.15) gives a sup-norm Markov inequality. One uses duality and $(C,1)$-bounds such as

$$\left\| \frac{1}{m} \sum_{j=1}^{m} |S_j[f]| W \right\|_{L_\infty(\mathbb{R})} \leq C_3 \|fW\|_{L_\infty(\mathbb{R})} \tag{2.16}$$

to pass to an L_1 analogue of (2.15), and interpolation gives it for $1 < p < \infty$.

(III) Freud's Method II, Levin-Lubinsky's Method [24], [25]

We illustrate the idea for $W := e^{-Q}$, as used by A.L. Levin and D.S. Lubinsky. Suppose that we can find $K > 0$ and polynomials S_n of degree $\leq Kn$ such that for $|x| \leq 2a_n$,

$$C_1 \leq \frac{S_n(x)}{W(x)} \leq C_2 ; \tag{2.17}$$

and

$$\frac{|S_n'(x)|}{W(x)} \leq C_3 \frac{n}{a_n}. \tag{2.18}$$

Then using their infinite-finite range inequality:

$$\|P_n W\|_{Lp(\mathbb{R})} \leq C_3 \|P_n W\|_{L_p\left(-a_n\left(1-Kn^{-\frac{2}{3}}\right) a_n\left(1-Kn^{-\frac{2}{3}}\right)\right)}$$

Levin and Lubinsky showed that

$$\begin{aligned} \|P_n' W\|_{Lp(\mathbb{R})} &\leq \|P_n' W\|_{Lp(-a_n, a_n)} \\ &\leq CC_1^{-1} \|P_n' S_n\|_{Lp(-a_n, a_n)} \\ &\leq CC_1^{-1} \left\|(P_n S_n)' - P_n S_n'\right\|_{Lp(-a_n, a_n)}. \end{aligned} \tag{2.19}$$

Now by the classical Bernstein inequality (2.6) for $[-1,1]$ scaled to $[-2a_n, 2a_n]$,

we have

$$\begin{aligned} \left\|(P_nS_n)'\right\|_{Lp(-a_n,a_n)} &\leq C_4\frac{n}{a_n}\|P_nS_n\|_{L_p(-2a_n,2a_n)} \\ &\leq C_4C_2\frac{n}{a_n}\|P_nW\|_{L_p(-2a_n,2a_n)}. \end{aligned}$$

Using this and (2.18) in (2.19) gives

$$\|P_n'W\|_{L_p(\mathbb{R})} \leq C_5\frac{n}{a_n}\|P_nW\|_{L_p(\mathbb{R})}. \tag{2.20}$$

(IV) **Nevai-Totik's Method** [55]

We assume that $W = e^{-Q}$, where Q is concave and $Q(0) = 0$. Here we make use heavily of the concavity and of fast decreasing polynomials. Let us suppose that for $n \geq 1$, and some $K > 0$, we have polynomials $\overline{S_n}$ of degree $N = N(n)$ say, that decrease rapidly in $[-1,1]$ in the sense that

$$\left|\overline{S}_n(x)\right| \leq Ke^{-Q(a_nx)}, \; x \in [-1,1]. \tag{2.21}$$

but $\overline{S}_n$ peaks at 0:

$$\overline{S}_n(0) = 1; \;\; \overline{S}_n'(0) = 0. \tag{2.22}$$

By a translation, we obtain polynomials S_n of degree $N(n)$, such that

$$|S_n(x)| \leq Ke^{-Q(x)}, \;\; x \in [-a_n, a_n] \tag{2.23}$$

and

$$S_n(0) = 1\ ;\quad S_n'(0) = 0.$$

Then for the Markov inequality at 0,

$$\begin{aligned} |(P_n'W)(0)| &= |P_n'(0)| = \left|(P_nS_n)'(0)\right| \leq \frac{n+N(n)}{a_n}\|P_nS_n\|_{L_\infty[-a_n,a_n]} \\ &\leq K\frac{n+N(n)}{a_n}\|P_nW\|_{L_\infty(\mathbb{R})}. \end{aligned} \tag{2.24}$$

Here we have first used the classical sup-norm Bernstein inequality for $[-1,1]$ scaled to $[-a_n,a_n]$, and then (2.23). For general $x \geq 0$, we use the evenness and concavity of Q to deduce that

$$W(x)W(y) \leq W(x+y).$$

Now fix $x \geq 0$. Applying (2.24) to the polynomial

$$R_n(y) := P_n(x+y)W(x)$$

gives

$$\begin{aligned} |P_n'W|(x) &= |R_n'(0)W(0)| \quad , \quad (W(0)=1) \\ &\leq K\frac{n+N(n)}{a_n}\|R_nW\|_{L_\infty(\mathbb{R})} \\ &= K\frac{n+N(n)}{a_n}\sup_{y\in\mathbb{R}}|P_n(x+y)|W(x)W(y) \\ &\leq K\frac{n+N(n)}{a_n}\|P_nW\|_{L_\infty(\mathbb{R})}. \end{aligned}$$

What are about the size of $N(n)$? For $W = W_1$, so $Q(x) = |x|$, and $a_n = \frac{\pi}{2}n$, Nevai and Totik showed that we can choose $N(n) = O(n \log n)$, so that

$$\|P_n' W_1\|_{L_\infty(\mathbb{R})} \leq C \log n \, \|P_n W_1\|_{L_\infty(\mathbb{R})}.$$

If on the other hand Q is concave, and

$$\int_0^\infty \frac{Q(x)}{1+x^2} dx < \infty$$

then we can choose $N(n) = O(a_n)$, and a_n grows faster than n, so

$$\|P_n' W\|_{L_\infty(\mathbb{R})} \leq C \, \|P_n W\|_{L_\infty(\mathbb{R})}.$$

To obtain the fast decreasing polynomials $\{S_n\}$ of minimal degree, Nevai and Totik used a construction of Marchenko on entire functions of exponential type.

(V) **Levin-Lubinsky's Method** [26], [27]

Initially unaware of Drzbasjan's work, Levin and Lubinsky used Cauchy's integral formula for derivatives to derive (2.7):

$$\|P_n' W_\alpha\|_{L_\infty(\mathbb{R})} \leq C \, \|P_n W\|_{L_\infty(\mathbb{R})} \times \begin{cases} n^{1-\frac{1}{\alpha}}, \text{ if } \alpha \geq 1 \\ \log n, \text{ if } \alpha = 1 \\ 1, \quad \text{if } \alpha < 1 \end{cases}$$

However, rather than using (2.9),they used the weighted Bernstein inequality:

$$|P_n(z)G(z)| \leq \|P_nW\|_{L_\infty(-a,a)}, \quad z \in \mathbb{C}\backslash[-a,a]$$

to estimate $|P_n(t)|$ for $|t-x| = \varepsilon$;i.e.,they used

$$|P_n(t)| \leq |G_{n,a_n}(t)|^{-1} \|P_nW\|_{L_\infty[-a,a]}.$$

Here the explicit representation

$$|G_{n,a_n}(t)|^{-1} = \exp\left(n\int_{-1}^{1}\log\left|\frac{t}{a_n}-s\right|\mu_n(s)\,ds+\sigma_n\right)$$

(where $\mu_n(s)$ is a non-negative "density" function and σ_n is a constant) plays a role. By estimating the density function $\mu_n(s)$ and then the function $|G_{n,a_n}(t)|^{-1}$, and carefully choosing ε, Levin and Lubinsky proved the following theorem:

Theorem 14 *Let $W := e^{-Q}$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, $Q(0) = 0$, Q'' is continuous in $(0,\infty)$, $Q' > 0$ there, and for some $A, B > 0$,*

$$A \leq 1 + \frac{xQ''(x)}{Q'(x)} \leq B, \quad x \in (0,\infty). \tag{2.25}$$

Let $Q^{[-1]}$ denote the inverse of Q on $(0,\infty)$. Then

$$\|P_n'W\|_{L_\infty(\mathbb{R})} \leq C\left(\int_1^n \frac{ds}{Q^{[-1]}(s)}\right)\|P_nW\|_{L_\infty(\mathbb{R})}. \tag{2.26}$$

The condition (2.25) is fairly typical of those used in the theory of Freud weights.

Levin and Lubinsky in [26] were not interested in Markov inequality (2.26), but in its sharper Bernstein cousin. On $[-1,1]$, the Bernstein inequality offers its best estimate for $P_n'(x)$ at $x=0$. For Freud weights, the Bernstein improvement occurs not at 0, but at the effective endpoints $\pm a_n$. To describe this improvement, they used

$$\phi_n(x) := \max\left\{1-\frac{|x|}{a_n}, n^{-\frac{2}{3}}\right\}^{\frac{1}{2}} \leq 1. \tag{2.27}$$

Theorem 15 *Assume the hypotheses of Theorem* 14 *with* $A>1$. *Then*

$$\left|(P_nW)'(x)\right| \leq C\,\frac{n}{a_n}\,\phi_n(x)\,\|P_nW\|_{L_\infty(\mathbb{R})}\ ,\ x\in\mathbb{R}. \tag{2.28}$$

In particular,

$$\left|(P_nW)'(\pm a_n)\right| \leq C\,\frac{n^{\frac{2}{3}}}{a_n}\,\|P_nW\|_{L_\infty(\mathbb{R})}. \tag{2.29}$$

The case $Q(x)=|x|^{\alpha}$, $\alpha\leq 1$, is excluded by the condition $A>1$. When $A\leq 1$, the conclusion of (2.28) holds for $|x|\geq\varepsilon a_n$, for any fixed $\varepsilon>0$. It cannot hold inside $(-\varepsilon a_n,\varepsilon a_n)$ in general, as $\frac{n}{a_n}\to 0$ as $n\to\infty$, whereas the correct Markov factor in (2.26) is bounded below by a positive constant.

(VI) Kroo and Szabados' Method [23]

Kroo and Szabados rediscovered and refined Drzbasjan's method, but they also had the infinite-finite range inequality. They weakened the conditions on Q of Theorem 14 showing that we need something like (2.25), only for large x, and we could replace the lower bound in (2.25) by

$$\frac{xQ'(x)}{Q(x)} \geq A > 0 \text{ for large } x.$$

The first Markov inequality for Erdös weights appeared only in 1990[32]. One of the reasons for this delay was the difficulty in formulating hypothesis: how does one describe smooth but rapid growth of Q ? D.S. Lubinsky [32] found it convenient to use

$$T(x) =: 1 + \frac{xQ''(x)}{Q'(x)}, \tag{2.30}$$

though for most purposes, we can use

$$\overline{T}(x) := \frac{xQ'(x)}{Q(x)}, \tag{2.31}$$

which involves lower derivatives, and typically grows at the same rate as T. Lubinsky then proved the following Markov-type inequality:

Theorem 16 *Let $W := e^{-Q}$ be even, let Q'' be continuous in $(0, \infty)$, $Q' > 0$ there, with positive right limit at 0, and limit ∞ at ∞, and*

$$T(x) = O\left(Q'(x)^{\frac{1}{2}}\right), \quad x \to \infty.$$

Then

$$\|P_n'W\|_{L_\infty(\mathbb{R})} \le CQ'(a_n)\|P_nW\|_{L_\infty(\mathbb{R})}. \tag{2.32}$$

The L_p analogues were subsequently proved by D.S. Lubinsky and T.Z. Mthembu [42], [43], under a few more restrictions on Q. An interesting feature here is that $Q'(a_n)$ grows essentially faster than $\frac{n}{a_n}$, the Markov factor for weights such as W_α, $\alpha > 1$, and may even grow faster than n. For example, if

$$Q(x) := Q_{k,\alpha}(x) = \exp_k(|x|^\alpha)$$

then

$$Q'(x) \sim n\ (\log_k n)^{-\frac{1}{\alpha}}\ \left[\prod_{j=1}^{k} \log_j n\right]^{\frac{1}{2}}.$$

Here $\exp_j := \exp(\exp(\ldots\exp(|x|^\alpha)))$ (the jth exponential), and $\log_j$ is defined in the same way. Note that for $\alpha > 2$, $Q'(a_n)$ does grow faster than n. One form of the Bernstein inequality for these weights is [43] :

$$\left\|(P_n'W)(x)\sqrt{\left|1-\frac{|x|}{n}\right|}\right\|_{L_p(\mathbb{R})} \le C\,\frac{n}{a_n}\|P_nW\|_{L_p(\mathbb{R})}.$$

So we reduce $Q'(a_n)$ to $\frac{n}{a_n}$ by inserting the root factor in the LHS. However, the "true" Bernstein inequality involves $(P_nW)'(x)$ and is quite complicated: There are different phenomena for three ranges of x in $[-a_n, a_n]$, rather than two as in the Freud case or the classical $[-1,1]$ case ; namely near 0 and near ± 1.

2.3 Jackson and Bernstein Theorems For Exponential weights

In this section we shall discuss some of the results concerning the characterization of the rate of decay of polynomial approximation of a function in terms of its structural/smoothness properties. In particular, we shall discuss Jackson type estimates for

$$E_n[f]_p := \|(f-P)\,W\|_{L_p(I)} \tag{2.33}$$

where $I=(-1,1)$ or $\mathbb{R}$, $0<p\leq\infty$ and $W:=e^{-Q}$ is an exponential weight.

As we have seen in the previous section, there are two main classes of weights on $\mathbb{R}$:

• The Frued weights, where Q is even and of polynomial growth at ∞. The prototypical Freud weights are the W_α that we have already seen.

• The Erdös weights, where Q is even and of faster than polynomial growth at ∞. The archetypal examples of Erdös weights are

$$W_{k,\alpha} := e^{-Q_{k,\alpha}} \ ; \ Q_{k,\alpha} := \exp_k\left(|x|^\alpha\right)$$

Here $\alpha > 0$, $k \geq 1$ and $\exp_k := \exp(\exp(\ldots\exp\left(|x|^\alpha\right)$ is the kth iterated exponential. We also set $\exp_0(x) = x$.

One of the main problems in proving Jackson theorems for exponential weights is the rapid decay of the weight at ± 1 or $\pm\infty$. So the techniques for proving Jackson theorems are a key factor. We shall concentrate on the techniques in our discussions.

2.3.1 Techniques For Classical Jackson Theorems

In this section we shall outline five methods that have been, or can be, used to prove Jackson theorems in the classical unweighted setting of $C[-1,1]$ – the continuous real valued functions on $[-1,1]$ with uniform norm; or $C_{2\pi}$–the continuous 2π–periodic functions on $[0,2\pi]$ with uniform norm; or in $L_1[-1,1]$. Our main objective is to review methods that are very popular, or might have some applications to weighted approximation.

(I) Convolution

This is the most popular method of approximation when it comes to Jackson theorems. Even Weierstrass' 1881 proof of his approximation theorem was based on convolution with the scaled heat kernel $\exp\left(-\frac{u}{\delta}\right)^2$, $\delta > 0$. Jackson then modified Fejer's kernel

$$F_n(u) := \frac{1}{2(n+1)}\left[\frac{\sin\left(\frac{n+1}{2}\right)u}{\sin\left(\frac{u}{2}\right)}\right]^2$$

to obtain the Jackson kernel

$$K_n(u) := C_n\left[\frac{\sin\left(\frac{nu}{2}\right)}{\sin\left(\frac{u}{2}\right)}\right]^4, \quad u \in [0, 2\pi]$$

where C_n is chosen so that

$$\int_0^{2\pi} K_n(u)\,du = 1.$$

Hence we obtain the Jackson operator

$$J_n[f](\theta) := \int_0^{2\pi} f(t)\,K_n(\theta - t)\,dt, \quad \theta \in [0, 2\pi], \ f \in C_{2\pi}.$$

Jackson then used $J_n[f]$ to approximate functions satisfying a Lipschitz condition of order 1, and then via a piecewise linear approximation to general functions obtained his celebrated estimate for

$$E_n[f] := \inf_{T_n} \|f - T_n\|_{L_\infty[0,\pi]}$$

(where T_n denotes a trigonometric polynomial of degree $\leq n$), namely

$$E_n[f] \leq C\,\omega\left(f; \frac{1}{n}\right) \tag{2.34}$$

where $C \neq C(f, n)$ and ω denotes the usual modulus of continuity given by

$$\omega(f; \delta) := \inf_{|x-y| \leq \delta} |f(x) - f(y)|.$$

(II) Fourier Series, Multipliers

This is probably the second most popular approach to Jackson theorems.

If $f \in C_{2\pi}$, and

$$f(\theta) \sim \frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j \cos j\theta + b_j \sin j\theta)$$

is its Fourier series, then the partial sums

$$S_n[f](x) := \frac{a_0}{2} + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx)$$

are naturally best approximants in the L_2 norm, but in the L_∞ norm, may incur a projection factor of order $\log n$ when compared to best approximants. The next step then is to modify the partial sums. This idea goes back to Cesaro means of the partial sums, and Lipot Fejer's theorem that the averages of the first n partial sums of the Fourier series of $f \in C_{2\pi}$ converge uniformly to f as $n \to \infty$. However, it was Favard who found in the 1930's the best "multipliers" by solving a dual L_1 approximation problem. Favard formed the linear operator

$$L_n[f](\theta) := \frac{a_0}{2} + \sum_{j=1}^{n} A_j (a_j \cos j\theta + b_j \sin j\theta) \tag{2.35}$$

where the $\{A_j\}$ are chosen according to the rule

$$A_j := \frac{j}{2} (-1)^{j+1} \alpha_j$$

and the $\{\alpha_j\}$ solve a certain dual L_1 mi 'mization problem, namely

$$\int_0^{\pi} \left| x - \sum_{k=1}^{n} \alpha_k \sin kx \right| dx = \min = \frac{\pi^2}{2(n+1)}. \tag{2.36}$$

An integration by parts and some manipulations give the identity

$$(L_n[f] - f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[t - \sum_{k=1}^{n} \alpha_k \sin kt \right] f'(\theta + \pi - t)\, dt \tag{2.37}$$

and hence the 1937 Favard's estimate

$$\begin{aligned} E_n[f] &\leq \|L_n[f] - f\|_{L_\infty[0,2\pi]} \\ &\leq \frac{\pi}{2(n+1)} \|f'\|_{L_\infty[0,2\pi]}. \end{aligned} \tag{2.38}$$

The constant is sharp for each n in the class of functions with continuous derivatives.

(III) The "Unknown Approximator's" Method

The name of t..is method is due to D.S. Lubinsky. The origins of this method are ·t known; though it may have been used by Dzadyk. Ron de Vore used it in 1968. The method involves several steps. Let $f : [-1,1] \to \mathbb{R}$.

Step 1. Approximate locally, over subintervals of $[-1,1]$.

Choose a partition of $[-1,1]$, not necessarily uniform,

$$-1 = y_0 < y_1 < y_2 < \ldots < y_n < y_{n+1} = 1 \tag{2.39}$$

and on $[y_j, \ldots$ approximate f by a polynomial of low degree, let us say, a cubic polynomial. This gives the approximation

$$\begin{aligned} f &\simeq \sum_{j=0}^{n} p_j \chi_{[y_j, y_{j+1})} \\ &= p_0 + \sum_{j=0}^{n-1} (p_{j+1} - p_j)\, \chi_{[y_j, y_{j+1})} \end{aligned} \tag{2.40}$$

where χ_A denotes the characteristic function of a set A.

Step 2. Replace $\chi_{[y_{j+1},1)}$ *by a polynomial* R_j.

Now approximate each characteristic function $\mathcal{X}_{[y_{j+1},1)}$ in a suitable sense by a polynomial R_j of degree $\leq n$, to give a polynomial approximation

$$f \simeq p_0 + \sum_{j=0}^{n-1} (p_{j+1} - p_j)\, R_{j+1} =: P \tag{2.41}$$

of say degree $\leq n+3$. This method has been extraordinarily useful in solving all sorts of problems of shape preserving (convexity, monotonicity preserving), approximation and also worked well in L_p spaces, $0 < p < 1$, where other methods fail.

(IV) Bojanov's Method

This method was first formulated in full in B. Bojanov's 1995 paper [2]. The method is based on the alternation theorem: if P_n^* is a best polynomial approximation of degree $\leq n$ to f in the uniform norm on $[-1, 1]$, so that

$$\|f - P_n^*\|_{L_\infty[-1,1]} = \min_{P_n} \|f - P_n\|_{L_\infty[-1,1]} =: E_n[f] \tag{2.42}$$

then there are at least $n+2$ alternation points

$$-1 \leq y_0 < y_1 < \ldots < y_{n+1} \leq 1$$

such that

$$(f - P_n^*)(y_j) = \pm(-1)^j \ E_n[f], \ 0 \leq j \leq n+1. \tag{2.43}$$

From this one can derive an explicit formula for $E_n[f]$ in terms of the determinants that arise in studying Chebyshev systems: Define

$$\begin{bmatrix} g_0 & g_1 & g_2 & \cdots & g_k \\ t_0 & t_1 & t_2 & \ldots & t_k \end{bmatrix} := \det\left(g_i(t_j)\right)_{i,j=0}^k.$$

Then

$$E_n[f] = \frac{\begin{bmatrix} 1 & x & \dots & x^n & f \\ y_0 & y_1 & \cdots & y_n & y_{n+1} \end{bmatrix}}{\begin{bmatrix} 1 & x & \dots & x^n & s \\ y_0 & y_1 & \cdots & y_n & y_{n+1} \end{bmatrix}} \tag{2.44}$$

where s is any function with $s(y_j) = \pm(-1)^j$. A little manipulation of the determinants leads to

$$E_n[f] = \sum_{k=0}^{n} (-1)^k c_k \left[f(y_{k+1}) - f(y_k)\right] \tag{2.45}$$

where the coefficients $\{c_k\}$ all have the same sign and satisfy

$$\sum_{k=0}^{n} |c_k| = \frac{1}{2}. \tag{2.46}$$

One readily concludes that

$$E_n[f] \le \frac{1}{2}\omega\left(f; \max_k (y_{k+1} - y_k)\right). \tag{2.47}$$

But we can get much more information by estimating the sum

$$\sum_{k=0}^{n} |c_k| (y_{k+1} - y_k),$$

which turns out to be related to the L_1 norm of a certain piecewise constant function. This method at present yields only

$$E_n[f] \le C\,\omega\left(f; \frac{1}{\sqrt{n}}\right), \tag{2.48}$$

weaker than the Jackson rate. This is promising and one day it will surely lead to the Jackson rate:

$$E_n[f] \leq C\omega\left(f; \frac{1}{n}\right).$$

(V) **The L_1 Christoffel Method.**

This method was first applied by M. Riesz in his classical theorem connecting the poss ty of one-sided L_1 approximation to the uniqueness of the solution of the Harmburger Moment problem. As such it fits more within the framework of weighted approximation on unbounded intervals. Here we present its possibilities for $(-1, 1)$.

The nth Christoffel function associated with a non-negative measure $d\mu$ on the real line is defined by

$$\lambda_n(d\mu, x) := \inf_{P_{n-1}} \int \frac{P_{n-1}^2(t)}{P_{n-1}^2(x)} d\mu(t). \tag{2.49}$$

Let $\Gamma_\xi(x)$ denote the step function

$$\Gamma_\xi(x) := \begin{cases} 1, \; x \leq \xi \\ 0, \; x > \xi \end{cases} \tag{2.50}$$

Then one can construct Φ_n, Ψ_n of degree $\leq 2n-1$, such that in $\mathbb{R}$

$$\Phi_n \leq \Gamma_\xi \leq \Psi_n. \tag{2.51}$$

We obtain Φ_n and Ψ_n by Hermite interpolation of Γ_ξ and its derivatives at Gauss type quadrature points that include ξ. Moreover,

$$\int(\Psi_n - \Phi_n)d\mu_n = \lambda_n(d\mu, \xi).$$

In particular, then

$$\int|\Gamma_\xi - \Psi_n|\,d\mu \leq \lambda_n(d\mu, \xi). \tag{?.52}$$

The point of this method is that for a great deal of "nice" measures μ, one has the correct estimates for the Christoffel functions $\lambda_n(d\mu, x)$ and hence a good estimate for the L_1 error of polynomial approximation of step functions. One can extend this to fairly general functions by first approximating them by splines or one can use Bohr type inequalites and approximate functions of bound[illegible] variation. This idea can be extended to general Mark[illegible]/Chebyshev systems.

Next we present two methods that are used to prove Jackson theorems in the weighted setting. The methods used to prove Jackson theorems in weighted approximation on $\mathbb{R}$ were based on modifying orthogonal expansions and the L_1 Christoffel method. Round about 1993, two sets of authors began to consider alternative methods: on the one hand, Kroó and Szabados developed Bojanov's method and on the other hand, Ditzian and Lubinsky investigated the possibility of applying the unknown approximator's method. We start with Kroó and Szabados method.

2.3.2 Kroó and Szabados' Method

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with fW vanishing at $\pm\infty$. Let P_n^* be a best polynomial approximation of degree $\leq n$ to f in the weighted uniform norm. Then there exist at least n+2 alternation points

$$y_0 < y_1 < y_2 < \ldots < y_{n+1}$$

with

$$\left[\left(f - P_n^*\right) W\right](y_j) = \pm(-1)^j \; E_n\left[f\right]_\infty, \; 0 \leq j \leq n+1. \tag{2.53}$$

Kroó and Szabados [23] showed á lá Bojanov, that

$$E_n\left[f\right]_\infty = \left|\sum_{k=0}^{n} (-1)^k d_k \left[f\left(y_{k+1}\right) - f\left(y_k\right)\right]\right| \tag{2.54}$$

where all $d_k > 0$ and

$$\sum_{k=0}^{n} d_k \left[W^{-\gamma}\left(y_{k+1}\right) + W^{-\gamma}\left(y_k\right)\right] = 1. \tag{2.55}$$

However, the real art lies in their use and estimation of

$$\delta_n := \sum_{k=0}^{n} d_k \left[W^{-\gamma}\left(y_{k+1}\right) + W^{-\gamma}\left(y_k\right)\right]\left(y_{k+1} - y_k\right). \tag{2.56}$$

Here $\gamma \in (0,1)$ is fixed. If we introduce the modulus

$$\omega_\gamma (f;t) := \sup_{|x-t|\leq t} \frac{|f(x) - f(y)|}{W^{-\gamma}(x) + W^{-\gamma}(y)}, \ \ t > 0. \tag{2.57}$$

Kroó and Szabados showed that

$$E_n [f]_\infty \leq 4\, \omega_\gamma (f; \delta_n). \tag{2.58}$$

For a large class of weights, including W_α, $\alpha > 1$, they proved that if all $|y_j| \leq Cq_n$,

$$\delta_n \leq C \frac{q_n \log n}{n} \tag{2.59}$$

and hence deduced the estimate

$$E_n [f]_\infty \leq C_1 \left[\omega_\gamma \left(f; \frac{q_n \log n}{n} \right) + e^{-C_2 n} \|fW^\gamma\|_{L_\infty(\mathbb{R})} \cdot \right] \tag{2.60}$$

The exponentially decaying term reflects the inability of weighted polynomials to approximate outside $[-Cq_n, Cq_n]$.

2.3.3 Some Technicalities

Before we turn to the unknown approximator's method in a weighted setting, we need to present a taste of some fundamental advances made possible by the systematic application of potential theory in the 1980's and 1990's.

We concentrate on the form of Freud's infinite-finite range inequality due to Mhaskar and Saff. Let $I = (-1, 1)$ or $\mathbb{R}$, and $W := e^{-Q}$, where $Q : I \to \mathbb{R}$ is even and convex in I and has limit ∞ at the endpoints of I. Then Mhaskar and Saff showed that if $a_n > 0$ is the root of the equation

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1 - t^2}} dt, \tag{2.61}$$

then

$$\|P_n W\|_{L_\infty(I)} = \|P_n W\|_{L_\infty(-a_n, a_n)}, \tag{2.62}$$

and that asymptotically as $n \to \infty$, a_n is the smallest such number. Later subsequent discussions allowed the convexity of Q to be replaced by $xQ'(x)$ being increasing on $I \cap (0, \infty)$.

In the L_p case, we cannot hope for equality of the L_p norm over I with that over some subinterval, as we are missing out part of the integral. So the L_p analogue of (2.62) has the form

$$\|P_n W\|_{L_p(I)} \leq \left(1 + e^{-Cn}\right) \|P_n W\|_{L_p(-a_n(1+\varepsilon), a_n(1+\varepsilon))}, \tag{2.63}$$

where $0 < p < \infty$, $\varepsilon > 0$ and $C = C(\varepsilon)$. The last inequality in (2.63) follows from the inequality

$$\|P_n W\|_{L_p(I)} \leq C_1 \|P_n W\|_{L_p(-a_n, a_n)}. \tag{2.64}$$

Until recently, almost all the work on weighted approximation on $\mathbb{R}$ concentrated on the so-called Freud case, where Q is of polynomial growth at ∞. This is due to the fact that handling the growth of Q and describing consequent behaviour is more difficult when Q is of faster than polynomial growth at ∞.

In an attempt to find a function that could guarantee the regular growth of Q, D.S. Lubinsky [32] used the function (2.30) :

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}$$

though for most purposes, we can use (2.31) :

$$\overline{T}(x) := \frac{xQ'(x)}{Q(x)} = x\frac{d}{dx}[\log Q(x)].$$

Both quantities typically grow at the same rate and it is not difficult to see that if $T(x)$ is bounded as $x \to \infty$, then $Q(x)$ is of at most polynomial growth at ∞, while if $T(x) \to \infty$ as $x \to \infty$, then $Q(x)$ is of faster than polynomial growth at ∞.

We present three examples to illustrate the growth of T and a_n.

Example 17 . *Some Freud Weights*

Let $W(x) = W_\alpha(x) = e^{-|x|^\alpha}$. We see that

$$T(x) = \frac{x\left[-\alpha x^{\alpha-1}\right]}{-x^\alpha} = \alpha$$

and

$$n = \frac{2}{\pi}\int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}}dt = \frac{2}{\pi}\int_0^1 \frac{a_n t\left[-\alpha(a_n t)^{\alpha-1}\right]}{\sqrt{1-t^2}}dt = -\frac{2\alpha a_n^\alpha}{\pi}\left(\int_0^1 \frac{t^\alpha}{\sqrt{1-t^2}}dt\right)$$

which gives

$$a_n = C_\alpha\, n^{\frac{1}{\alpha}}$$

where

$$C_\alpha = 2\left[\frac{\Gamma\left(\frac{\alpha}{2}\right)}{4\Gamma(\alpha)}\right]^{\frac{1}{\alpha}}.$$

Example 18 . *Some Erdös Weights*

Let $W(x) = W_{k,\alpha}(x) = \exp\left(-e_k^{|x|^\alpha}\right)$, where $\exp_k$ denotes the kth iterated exponential. We see that, after some manipulations,

$$T(x) = \alpha x^\alpha \prod_{j=1}^{k-1} \exp_j(x^\alpha),\ x > 0.$$

An application of the rudiments of Laplace's method to the integral defining a_n gives

$$a_n = \left\{\log_{k-1}\left(\log n - \frac{1}{2}\sum_{j=2}^{k+1}\log_j n + O(1)\right)\right\}^{\frac{1}{\alpha}}$$

and hence

$$a_n = \{\log_k n\}^{\frac{1}{\alpha}} (1 + o(1)).$$

Here $\log_j$ denotes the jth iterated logarithm. One can also show that

$$T(a_n) \sim \prod_{j=1}^{k} \log_j n.$$

Example 19 *Some Exponential Weights on* $(-1, 1)$.

Let $W(x) = W^{0,\alpha}(x) = \exp\left(-(1-x^2)^{-\alpha}\right)$. Then we see that

$$T(x) = \frac{2\alpha x^2 (1-x^2)^{-\alpha-1}}{(1-x^2)^{-\alpha}} = \frac{2\alpha x^2}{(1-x^2)}$$

and

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t \left[2\alpha (a_n t)\left(1-(a_n t)^2\right)^{-\alpha-1}\right]}{\sqrt{1-t^2}} dt$$

giving

$$1 - a_n \sim n^{-\frac{1}{\alpha+\frac{1}{2}}}$$

and hence

$$T(a_n) \sim n^{\frac{1}{\alpha+\frac{1}{2}}}.$$

Taking the limit as $\alpha \to 0^+$, we see that

$$1 - a_n \sim n^{-2} \text{ and } T(a_n) \sim n^2.$$

Of course n^{-2} appears in the classic unweighted estimates for polynomials, such as,

$$\|P_n\|_{L_p[-1,1]} \leq C\,\|P_n\|_{L_p(-1+n^{-2},1+n^{-2})},$$

with $C \neq C(n, P_n)$.

In the sequel, we will use the following notation:

• $\mathcal{F}$ will be used to denote the class of Freud weights; $\mathcal{E}$ will denote the class of Erdös weights and $\mathcal{EXP}$ the class of exponential weights.

Next, we state the Jackson theorem that can be achieved for the classes of weights mentioned above, namely, $\mathcal{F}, \mathcal{E}$ and $\mathcal{EXP}$. Recall that we use the notation

$$E_n[f]_p := \inf_{P\in\prod_n} \|(f-P)\,W\|_{L_p(\mathbb{R})}.$$

Theorem 20 . *Let $0 < p \leq \infty$, $r \geq 1$ and $W \in \mathcal{E}$ or $\mathcal{F}$ or $\mathcal{EXP}$ on the interval I. Let $fW \in L_p(I)$ and if $p = \infty$, let f be continuous with fW vanishing at the endpoints of I. Then*

$$E_n[f]_p \leq C\,\omega_{r,p}\left(f, C_1\frac{a_n}{n}\right). \tag{2.65}$$

Corollary 21 . *If $p \geq 1$, we have*

$$E_n[f]_p \leq C_2\,\left(\frac{a_n}{n}\right)^r \left\|f^{(r)}W\Phi^r_{\frac{a_n}{n}}\right\|_{L_p(I)} \tag{2.66}$$

provided the RHS is meaningful.

For (2.65) and (2.66) to make sense we need to define the modulus of continuity $\omega_{r,p}(f,t)$ and the quantity $\Phi_{\frac{2a}{n}}$. First let us note that

$$\left\|\Phi^r_{\frac{2a}{n}}\right\|_{L_\infty(I)} \leq C, \;\; n \geq 1.$$

Definition 22 . *We define the rth order symmetric difference with increment h by*

$$\Delta^r_h f(x) := \sum_{j=0}^{r} \binom{r}{j} (-1)^j f\left(x + \left(\frac{r}{2} - j\right) h\right) \tag{2.67}$$

One of the fundamental discoveries of Ditzian and Totik was that by letting the size of the increment h depend on x, more specifically by replacing h by $h\sqrt{1-x^2}$, they could achieve elegant characterizations of the condition

$$\inf_{P_n} \|(f - P_n)\|_{L_p[-1,1]} = O\left(n^{-\alpha}\right), \;\; n \to \infty \tag{2.68}$$

in terms of the structural properties of f. Their modulus for $[-1,1]$ has the form

$$\omega^*_{r,p}(f,t) := \sup_{0<h\leq t} \left\|\Delta^r_{h\sqrt{1-x^2}} f(x)\right\|_{L_p[-1,1]}$$

and they showed that if $0 < \alpha < r$, (2.68) is equivalent to

$$\omega^*_{r,p}(f,t) = O\left(t^\alpha\right), \; t \to 0^+.$$

The rth order modulus used by Ditzian and Lubinsky [11] for the weights on

$I = (-1, 1)$ or $\mathbb{R}$ and the L_p norm has the form

$$\omega_{r,p}(f,t) = \underbrace{\sup_{0<h\leq t} \left\| W(x)\, \Delta^r_{h\Phi_t(x)} f(x) \right\|_{L_p(I_h)}}_{\text{main part}} + \underbrace{\inf_{P \in P_{r-1}} \| W(f-P) \|_{L_p(J_t)}}_{\text{tail}}. \tag{2.69}$$

We see that there are 3 undefined quantities: The function Φ_t and the intervals I_h, J_t. Before we define them, we need to account for the "tail" part of (2.69). We know that $P_n W$ is small outside $[-a_n, a_n]$ and so can only approximate inside the interval. The tail part is designed to take account of this. We need an inf over polynomials of degree $\leq r-1$ in the tail to ensure that if f is itself a polynomial of degree $\leq r-1$, we have

$$\omega_{r,p}(f,t) \equiv 0.$$

The function Φ_t is a suitable replacement for the factor $\sqrt{1-x^2}$ in the Ditzian-Totik modulus, and will depend on the class of weights. If $\Phi_t \neq 1$, it reflects an improvement in the degree of approximation towards the endpoints of a certain interval. To make things more explicit, we distinguish between $(-1,1)$ and $\mathbb{R}$:

(I) Freud and Erdös weights on $\mathbb{R}$.

For the classes of weights $\mathcal{E}$, $\mathcal{F}$, $Q(x)$ grows faster than $|x|^\alpha$, $\alpha > 1$, and

a_n grows slower than $n^{\frac{1}{\alpha}}$. So

$$\frac{a_n}{n} \to 0, \ n \to \infty.$$

It turns out that we need something like a_n, where n is evaluated at the inverse of the function $n \to \frac{a_n}{n}$ to describe the interval I_h. Accordingly, we introduce the decreasing function of t,

$$\sigma(t) := \inf\left\{a_n : \ \frac{a_n}{n} \leq t\right\}, \ t > 0.$$

The essential feature of σ here is that

$$\sigma\left(\frac{a_n}{n}\right) \sim a_n.$$

For $\mathcal{F}$, we define

$$\Phi_t(x) \equiv 1; \ I_h := [-\sigma(h), \sigma(h)]; \ J_t := \mathbb{R} \backslash I_h.$$

For $\mathcal{E}$, we define

$$\Phi_t(x) := \sqrt{\left|1 - \frac{|x|}{\sigma(t)}\right|} + T(\sigma(t))^{-\frac{1}{2}};$$

and for $0 < h \leq t$,

$$I_h := [-\sigma(2t), \sigma(2t)]; \quad J_t := \mathbb{R} \backslash [-\sigma(4t), \sigma(4t)].$$

Clearly the choice of $\Phi_t(x) \equiv 1$ for $\mathcal{F}$ does not give any improvement in the degree of approximation, while in the $\mathcal{E}$ case, the choice of Φ_t reflects an improvement in approximation towards the endpoints of the interval $[-\sigma(t), \sigma(t)]$.

(II) Exponential Weights on $(-1,1)$.

An essential simplification in this case is that $a_n \to 1$, $n \to \infty$, so we could actually replace the $\frac{a_n}{n}$ in our Jackson theorem (2.65) by $\frac{1}{n}$. This also means that we do not need the function $\sigma(t)$, and instead can use just $a_{\frac{1}{t}}$. So here we choose

$$\Phi_t(x) := \sqrt{\left|1 - \frac{|x|}{a_{\frac{1}{t}}}\right| + T\left(a_{\frac{1}{t}}\right)};$$

and for $0 < h \leq t$

$$I_h := \left[-a_{\frac{1}{2t}}, a_{\frac{1}{2t}}\right]; \quad J_t := [-1,1] \setminus \left[-a_{\frac{1}{4t}}, a_{\frac{1}{4t}}\right].$$

Remark 4 *For all three classes of weights* $(\mathcal{E}, \mathcal{F}$ *and* $\mathcal{EXP})$, *the modulus does have the usual properties*

$$\omega_{r,p}(f, 2t) \leq C\,\omega_{r,p}(f,t)$$

and for $p \geq 1$,

$$\omega_{r,p}(f,t) \leq C\,t^r \left\| f^{(r)} W \Phi_t^r \right\|_{L_p(I)}.$$

For the Freud weights, Theorem (2.7) is due to Z. Ditzian and D.S. Lubinsky [11]. For the Erdös weights, the result is due to S.B. Damelin [6], and for exponential weights, the result is due to D.S. Lubinsky [38]. The unifying feature here is that the method of proof throughout is the unknown approximator's method and it works simultaneously in all L_p spaces, $0 < p \leq \infty$.

In conclusion, we present some of the details of the unknown approximator's method in the weighted setting:

First let us recall that in approximating fW by $P_n W$, we can only work on $[-a_n, a_n]$ as $P_n W$ is very small outside $[-a_n, a_n]$ and so cannot approximate. So we partition this interval as

$$-a_n = y_0 < y_1 < y_2 < \ldots < y_{n+1} = a_n$$

and on each interval $[y_j, y_{j+1})$ approximate f by a polynomial p_j of degree $\leq r$. According to Whitney's theorem, we can choose p_j to approximate f with an error involving a suitable rth order modulus evaluated at $y_{j+1} - y_j$. Thus it is really the size of $(y_{j+1} - y_j)$ that controls the degree of approximation. For Freud weights, we can choose the y_j equally spaced , but for Erdös weights and exponential weights on $[-1, 1]$, one has to choose the y_j in such a way that uniformly in j and n

$$y_{j+1} - y_j \sim \frac{a_n}{n} \Phi_{\frac{a_n}{n}} (y_j).$$

Then one obtains a piecewise polynomial approximation to f

$$f(x) \simeq \sum_{j=0}^{n} p_j \chi_{[y_j, y_{j+1})} = p_0 + \sum_{j=0}^{n-1} (p_{j+1} - p_j)\, \chi_{[y_{j+1}, a_n)}.$$

Next, approximate $\chi_{[y_{j+1}, a_n)}$ by a polynomial R_j of degree $\leq n$. The type of estimate needed here is the following: For $n \geq C$ and $\tau \in [-a_n, a_n]$ there exist polynomials $R_{n,\tau}$ of degree $\leq Ln$ such that for all $x \in I = (-1,1)$ or $\mathbb{R}$,

$$\left|\chi_{[\tau, a_n]} - R_{n,\tau}\right|(x)\, W(x) \leq C_1 W(\tau) \left(1 + \frac{n\,|x-\tau|}{a_n \Phi_{\frac{a_n}{n}}(\tau)}\right)^{-99}$$

where $C, C_1, L \neq C, C_1, L(n, \tau, x)$. Moreover, 99 can be replaced by any positive integer. The estimate is difficult because $W(\tau)$ may be very small for τ close to a_n. In the Freud case , the $\Phi \equiv 1$, but in the other cases, it gets smaller as τ approaches a_n and this is really where one needs the y_j to get closer and closer.

Chapter 3

LAGRANGE INTERPOLATION

3.1 Introduction

In this chapter, we are interested in a study of a certain standard approximation process; namely, the approximation of a function in terms of the Lagrange interpolation polynomials for the function at the zeros of the orthogonal polynomials. In the past twenty years, there has begun to develop a general theory of orthogonal polynomials , and its associated approximation theory, for weights on $\mathbb{R}$ [33] , [54] . In these investigations, it has been helpful to distinguish between Erdös weights and Freud weights. Accordingly our discussions here will be split into two sections: one discussing Lagrange interpolation for Freud weights and the other Lagrange interpolation for Erdös weights.

3.2 Mean Convergence of Lagrange Interpolation for Freud Weights

3.2.1 Introduction and Main Results

The convergence of Lagrange interpolation at zeros of orthogonal polynomials is a classical and widely studied subject. Let us recall the setting.

If $d\alpha$ is a positive Borel measure on a finite or infinite interval (a, b), with support containing infinitely many points, and with all power moments

$$\int_a^b x^j d\alpha\,(x) < \infty\,,\ j = 0, 1, 2, \ldots,$$

then we may define orthonormal polynomials $p_n\,(d\alpha, x)$ or $p_n\,(x)$, $n = 0, 1, 2, \ldots$, which satisfy

$$\int_a^b p_n(d\alpha, x)p_m\,(d\alpha, x)\,d\alpha\,(x) = \delta_{mn}\,,\ m, n = 0, 1, 2, \ldots.$$

We denote the zeros of $p_n(d\alpha, x)$ by

$$a < x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} < b.$$

For functions $f : (a,b) \to \mathbb{R}$, we denote the Lagrange interpolation polynomial of degree $\leq n-1$ to f at the zeros of $p_n(d\alpha, x)$ by $L_n[f](x)$, so that

$$L_n[f](x_{jn}) = f(x_{jn}), \ 1 \leq j \leq n. \tag{3.1}$$

The classical Erdös-Turan theorem asserts that if (a,b) is a finite interval, then for each measurable function $f : (a,b) \to \mathbb{R}$ for which

$$\int_a^b f^2 d\alpha < \infty,$$

we have L_2 convergence :

$$\lim_{n\to\infty} \int_a^b (f - L_n[f])^2 \, d\alpha = 0.$$

In this section, we concentrate on the weights on the whole real line. We shall deal with Freud weights, i.e.; weights of the form $d\alpha(x) = W^2(x)\,dx$ where $W : \mathbb{R} \to \mathbb{R}$ is even of sufficiently smooth and regular decay at infinity The results here apply in particular to

$$W(x) = W_\beta(x) := \exp\left(-\frac{1}{2}|x|^\beta\right), \ x \in \mathbb{R}, \ \beta > 1. \tag{3.2}$$

Many of the methods used to prove these results are taken from a fundamental paper of P. Nevai [53], which dealt with the Hermite weight, the case $\beta = 2$ in (3.2).

Mean convergence of Lagrange interpolation is dependent on suitable estimates for associated orthonormal polynomials. By assuming bounds on orthonormal polynomials that were known to be true for the weights W_β^2, $\beta > 0$ an even integer, Knopfmacher and Lubinsky treated fairly general Freud weights $W^2 = e^{-2Q}$ [20]. For the L_p norm with $1 < p < 2$, the results were sharpened and generalized those in [3], and the functions with integrable singularities at finitely many points were also discussed.

The possibility of treating weights such as W_β^2, $\beta > 1$, was provided by the work of Levin and Lubinsky [27], where the correct bounds were obtained for orthonormal polynomials over $\mathbb{R}$ for a large class of weights. We also use the L_p norms of orthonormal polynomials, and these were estimated above and below in [41], using the results of [27]. The following theorem, due to Lubinsky and Matjila, is a special case to the results in this section:

Theorem 23 *Let* $\beta > 1$, $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\alpha > 0$ *and*

$$\hat{\alpha} := \min\{1, \alpha\}. \tag{3.3}$$

Let

$$\tau = \tau(p) = \begin{cases} \frac{1}{p} - \widehat{\alpha}, & p \leq 4 \\ \frac{1}{p} - \widehat{\alpha} + \frac{\beta}{6}\left(1 - \frac{4}{p}\right), & p > 4 \end{cases}. \tag{3.4}$$

Denote the Lagrange interpolation polynomials at the zeros of $p_n\left(W_\beta^2, .\right)$ *by* $L_n[.]$, $n \geq 1$, *where* W_β *is given by* (3.2). *Then for*

$$\lim_{n\to\infty} \left\|\left(f(x) - L_n[f](x)\right) W_\beta(x)\left(1+|x|\right)^{-\Delta}\right\|_{L_p(\mathbb{R})} = 0 \tag{3.5}$$

to hold for every continuous function $f : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\lim_{|x|\to\infty} |f(x)|\, W_\beta(x)\left(1+|x|\right)^{\alpha} = 0, \tag{3.6}$$

it is necessary and sufficient that

$$\Delta > \tau \quad \text{if } 1 < p \leq 4;$$

$$\Delta > \tau \quad \text{if } p > 4 \text{ and } \alpha = 1;$$

$$\Delta \geq \tau \quad \text{if } p > 4 \text{ and } \alpha \neq 1.$$

Corollary 24 *Let* $\beta > 1$, $\Delta \in \mathbb{R}$, $\alpha > 0$, *and* $\widehat{\alpha} = \min\{1, \alpha\}$. *For the convergence* (3.5) *to hold for every* $1 < p < \infty$ *and every continuous function* $f : \mathbb{R} \to \mathbb{R}$ *satisfying* (3.6), *it is necessary and sufficient that*

$$\Delta \geq -\widehat{\alpha} + \max\left\{1, \frac{\beta}{6}\right\}. \tag{3.7}$$

To formulate the results for more general weights, Lubinsky and Matjila needed the Mhaskar-Rahmanov-Saff number a_n [48], [49]. Let $W = e^{-Q}$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, continuous and $xQ'(x)$ is positive and increasing on $(0, \infty)$, with limits 0 and ∞ at 0 and ∞ respectively. For $n > 0$, the MRS number a_n is the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}} dt. \tag{3.8}$$

The significance of a_n lies partly in the identity

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[-a_n, a_n]} \tag{3.9}$$

which holds for polynomials P of degree $\leq n$ [49].

We now state the main result of this section due to Lubinsky and Matjila [40] :

Theorem 25 *Let $W := e^{-Q}$, where $Q : \mathbb{R} \to \mathbb{R}$ is even and continuous in $\mathbb{R}$, Q'' is continuous in $(0, \infty)$, and $Q' > 0$ in $(0, \infty)$, while for some A, $B > 1$,*

$$A \leq 1 + \frac{xQ''(x)}{Q'(x)} \leq B, \; x \in (0, \infty). \tag{3.10}$$

Let $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\alpha > 0$, and $\hat{\alpha}$ be given by (3.3). *Denote the Lagrange* *interpolation polynomials at the zeros of* $p_n(W^2,.)$ *by* $L_n[.]$, $n \geq 1$. *Then for*

$$\lim_{n\to\infty}\left\|\left(f-L_n[f]\right)(x)\,W(x)\,(1+|x|)^{-\Delta}\right\|_{L_p(\mathbb{R})}=0, \tag{3.11}$$

to hold for every continuous function $f:\mathbb{R}\to\mathbb{R}$ *satisfying*

$$\lim_{|x|\to\infty}|f(x)|\,W(x)\,(1+|x|)^{\alpha}=0, \tag{3.12}$$

it is necessary and sufficient that

$$\Delta > -\widehat{\alpha}+\frac{1}{p} \quad \textit{if } p\leq 4, \tag{3.13}$$

$$a_n^{\frac{1}{p}-(\widehat{\alpha}+\Delta)}n^{\frac{1}{6}\left(1-\frac{4}{p}\right)} = O(1),\ n\to\infty \textit{ if } p>4 \textit{ and } \alpha\neq 1, \tag{3.14}$$

$$a_n^{\frac{1}{p}-(\widehat{\alpha}+\Delta)}n^{\frac{1}{6}\left(1-\frac{4}{p}\right)} = O\left(\frac{1}{\log n}\right),\ n\to\infty \textit{ if } p>4 \textit{ and } \alpha=1. \tag{3.15}$$

Next we present some technical lemmas, such as estimates on orthogonal polynomials and their zeros.

3.2.2 Technical Lemmas

In the sequel , $C, C_1, C_2, \ldots$ denote positive constants independent of n, x and polynomials of degree n. We write $C \neq C(\lambda)$ to indicate that C does not depend on λ. The same symbol does not necessarily represent the same constant in different occurrences. We use $\sim$ in the following sense: If $\{a_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are real sequences, we write

$$b_n \sim c_n$$

if there exist $C_1, C_2 > 0$ such that

$$C_1 \leq \frac{b_n}{c_n} \leq C_2, \ n \geq 1.$$

We write $f^n(x)$ to denote $(f(x))^n$, even for $n = -1$. $\mathbb{P}_n$ denotes the polynomials of degree at most n.

If $W^2 : \mathbb{R} \to [0, \infty)$ is a weight, its nth Christoffel function is given by

$$\lambda_n(x) = \lambda_n\left(W^2, x\right) = \inf_{P \in \mathbb{P}_{n-1}} \int_{-\infty}^{\infty} \frac{(PW)^2(t)}{P^2(x)} dt. \tag{3.16}$$

It is well known [54] that

$$\lambda_n^{-1}(x) = \sum_{j=0}^{n-1} p_j^2(x).$$

We also define the Christoffel numbers

$$\lambda_{jn} := \lambda_n(x_{jn}), \ 1 \leq j \leq n.$$

The leading coefficient of $p_n(x)$ is denoted by γ_n, so that

$$p_n(x) = \gamma_n x^n + \dots.$$

The Lagrange interpolation polynomial $L_n[f]$ admits the representation

$$L_n[f](x) = \sum_{j=1}^{n} f(x_{jn})\, l_{jn}(x),$$

where the fundamental polynomials l_{jn} in turn admit the representation [14, p.23]

$$l_{jn}(x) = \lambda_{jn} \frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x_{jn}) \frac{p_n(x)}{x - x_{jn}}. \tag{3.17}$$

Mean convergence of Lagrange interpolation is closely connected to bounds on orthogonal polynomials and related estimates. Accordingly we recall some results from [27]. Throughout this section, we assume that W is as in Theorem 25.

Theorem 26 *(a) For $n \geq 1$ and $|x| \leq a_n$,*

$$\lambda_n\left(W^2, x\right) \sim \frac{a_n}{n} W^2(x) \max\left\{n^{-\frac{2}{3}}, 1 - \frac{|x|}{a_n}\right\}^{-\frac{1}{2}}. \tag{3.18}$$

Moreover, we replace $\sim$ by $\geq C\times$ for all $x \in \mathbb{R}$.

(b) For $n \geq 1$,

$$\left|\frac{x_{1n}}{a_n} - 1\right| \leq Cn^{-\frac{2}{3}}, \tag{3.19}$$

and uniformly for $n \geq 3$ and $2 \leq j \leq n-1$,

$$x_{j-1,n} - x_{j+1,n} \sim \frac{a_n}{n} \max\left\{n^{-\frac{2}{3}}, 1 - \frac{|x_{jn}|}{a_n}\right\}^{-\frac{1}{2}}. \tag{3.20}$$

(c) For $n \geq 1$,

$$\sup_{x \in \mathbb{R}} |p_n(x)| W(x) \left| 1 - \frac{|x|}{a_n} \right|^{\frac{1}{4}} \sim a_n^{-\frac{1}{2}} \tag{3.21}$$

and

$$\sup_{x \in \mathbb{R}} |p_n(x)| W(x) \sim n^{\frac{1}{6}} a_n^{-\frac{1}{2}}. \tag{3.22}$$

(d) Let $0 < p \leq \infty$. There exists $C > 0$ such that for $n \geq 1$ and $P \in \mathbb{P}_n$,

$$\|PW\|_{L_p(\mathbb{R})} \leq C \|PW\|_{L_p[-a_n, a_n]}. \tag{3.23}$$

(e) For $n \geq 1$,

$$\frac{\gamma_{n-1}}{\gamma_n} \sim a_n. \tag{3.24}$$

(f) Uniformly for $n \geq 2$ and $1 \leq j \leq n-1$,

$$\max\left\{ n^{-\frac{2}{3}}, 1 - \frac{|x_{jn}|}{a_n} \right\} \sim \max\left\{ n^{-\frac{2}{3}}, 1 - \frac{|x_{j+1,n}|}{a_n} \right\}. \tag{3.25}$$

Theorem 27 [41]

(a) Given $0 < p < \infty$, we have for $n \geq 1$,

$$\|p_n W\|_{L_p(\mathbb{R})} \sim a_n^{\frac{1}{p}} \begin{cases} 1, & p < 4 \\ (\log n)^{\frac{1}{4}}, & p = 4 \\ n^{\frac{1}{6}\left(1 - \frac{4}{p}\right)}, & p > 4 \end{cases} \tag{3.26}$$

(*b*) Uniformly for $n \geq 1$, $1 \leq j \leq n$, and $x \in \mathbb{R}$,

$$|l_{jn}(x)| \sim \frac{a_n^{\frac{3}{2}}}{n} W(x_{jn}) \left(\max\left\{ n^{-\frac{2}{3}}, 1 - \frac{|x_{jn}|}{a_n} \right\} \right)^{-\frac{1}{4}} \left| \frac{p_n(x)}{x - x_{jn}} \right|. \tag{3.27}$$

(*c*) Uniformly for $n \geq 1$, $1 \leq j \leq n$, and $x \in \mathbb{R}$,

$$|l_{jn}(x)| W^{-1}(x_{jn}) W(x) \leq C. \tag{3.28}$$

Lemma 28 [27]

(*a*) Given $0 < a < b$, we have uniformly for $n \geq 1$ and $x \in [a, b]$,

$$Q(a_n x) \sim a_n x Q'(a_n x) \sim n. \tag{3.29}$$

(*b*) If A, B are as in (3.10), then

$$u^{\frac{1}{B}} \leq \frac{a_n}{a_1} \leq u^{\frac{1}{A}},\ u \in [1, \infty). \tag{3.30}$$

(*c*) Given $\lambda > 1$, we have for $\nu \in (0, \infty)$, and $u \in \left[\frac{\nu}{\lambda}, \lambda\nu\right]$,

$$\left| \frac{a_u}{a_\nu} - 1 \right| \sim \left| \frac{u}{\nu} - 1 \right|. \tag{3.31}$$

In the remainder of this section, we present three quadrature sum estimates. In the sequel, we set

$$x_{0n} := x_{1n}\left(1 + n^{-\frac{2}{3}}\right);\ \ x_{n+1,n} := x_{nn}\left(1 + n^{-\frac{2}{3}}\right). \tag{3.32}$$

Lemma 29 [40]. *Let $\nu \in \mathbb{R}$. Then uniformly for $n \geq 1$,*

$$\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn})(1+|x_{jn}|)^{\nu} \sim \begin{cases} 1, & \nu < -1 \\ \log n, & \nu = -1 \\ a_n^{1-\nu}, & \nu > -1 \end{cases} \tag{3.33}$$

In several places, we shall need to replace $(1+t^2)^{\nu}$ by an equivalent polynomial on a suitable interval. This is achieved in the following lemma.

Lemma 30 [40]. *Let $\nu \in \mathbb{R}$. There exists $C > 0$ such that for $\lambda \geq 2$, there exist polynomials $\widehat{P_\lambda}$ of degree $\leq C\lambda \log \lambda$ such that*

$$\widehat{P}_\lambda(t) \sim \left(1+t^2\right)^{\nu}, \tag{3.34}$$

uniformly for $t \in [-\lambda, \lambda]$ and $\lambda \geq 2$.

Lemma 31 [40]. *Fix $\sigma \in (0,1)$ and an integer $L \geq 3$, and let $\nu \in \mathbb{R}$. Then for $P \in \mathbb{P}_{Ln}$, and $n \geq 1$,*

$$\sum_{|x_{jn}| \leq \sigma a_n} \lambda_{jn} |P(x_{jn})| W^{-1}(x_{jn}) \left(1+x_{jn}^2\right)^{\nu} \leq C_1 \int_{-a_{Ln}}^{a_{Ln}} |PW|(t) \left(1+t^2\right)^{\nu} dt, \tag{3.35}$$

where $C_1 \neq C_1(P,n)$.

The last quadrature estimate in this section is essentially for part of the Lebesgue function of Lagrange interpolation.

Lemma 32 [40]. *Let* $\beta \in (0, 2)$, $\nu \in \mathbb{R}$, *and*

$$\sum(x) := \sum_{k:|x_{kn}| \geq \beta a_n} |l_{kn}(x)| W^{-1}(x_{kn})(1 + |x_{kn}|)^{-\nu}. \tag{3.36}$$

Then

$$W(x)\sum(x) \leq Ca_n^{-\nu} \begin{cases} 1, & |x| \leq \beta\frac{a_n}{2}; \\ a_n^{-\frac{1}{4}}|p_n W|(x) + \log n, & \beta\frac{a_n}{2} \leq |x| \leq 2a_n; \\ \frac{a_n}{|x|}, & |x| \geq 2a_n. \end{cases} \tag{3.37}$$

3.2.3 Proofs of Theorems in Section 3.2.1

In our proof of Theorem 25, we shall split our function into pieces that vanish inside or outside $[-\beta a_n, \beta a_n]$, some $\beta > 0$. The proof requires several preliminary steps and many of the ideas come from Nevai's fundamental paper [53] . In the sequel, W is as in Theorem 25 and $1 < p < \infty$, $\alpha > 0$, $\Delta \in \mathbb{R}$, and $\hat{\alpha}$ is given by (3.3).

Lemma 33 *Assume that if* $p \leq 4$, (3.13) *holds and if* $p > 4$, (3.14) *holds. Let* $\varepsilon > 0$, $\beta \in (0, 2)$ *and assume that* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of functions* *from* $\mathbb{R}$ *to* $\mathbb{R}$ *such that*

$$f_n(x) = 0, \quad |x| < \beta a_n, \tag{3.38}$$

and

$$|f_n W|(x) \leq \varepsilon (1 + |x|)^{-\alpha}, \ x \in \mathbb{R}, \ n \geq 1. \tag{3.39}$$

Then

$$\limsup_{n\to\infty} \left\| L_n[f](x) W(x) (1 + |x|)^{-\Delta} \right\|_{L_p(\mathbb{R})} \leq C\epsilon. \tag{3.40}$$

Here $C \neq C(\varepsilon, n, f_n)$.

Proof. Now by Lemma 32, and (3.1) – (3.2),

$$\begin{aligned}
|L_n[f_n](x) W(x)| &= \left| W(x) \sum_{|x_{kn}| \geq \beta a_n} l_{kn}(x) f_n(x_{kn}) \right| \\
&\leq \varepsilon W(x) \sum_{|x_{kn}| \geq \beta a_n} |l_{kn}(x)| W^{-1}(x_{kn}) (1 + |x_{kn}|)^{-\alpha} \\
&\leq C_1 \begin{cases} 1, & |x| \leq \beta a_n; \\ a_n^{\frac{1}{2}} |p_n W|(x) + \log n, & \beta \frac{a_n}{2} \leq |x| \leq 2a_n; \\ \frac{a_n}{|x|}, & |x| \geq 2a_n. \end{cases}
\end{aligned} \tag{3.41}$$

Then

$$\tau_n^{(1)} \ : \ = \left\| L_n[f_n](x) W(x) (1 + |x|)^{-\Delta} \right\|_{L_p\left(|x| \leq \beta \frac{a_n}{2}\right)}$$

$$
\begin{aligned}
&\leq C_1 \varepsilon a_n^{-\alpha} \left\| (1+|x|)^{-\Delta} \right\|_{L_p\left(|x| \leq \beta \frac{a_n}{2}\right)} \\
&\leq C_2 \varepsilon a_n^{-\alpha} \begin{cases} 1, & \Delta p > 1; \\ (\log)^{\frac{1}{p}}, & \Delta p = 1; \\ a_n^{\frac{1}{p}-\Delta}, & \Delta p < 1. \end{cases}
\end{aligned}
$$

Here, if $\Delta p \geq 1$, this term is $o(1)$ as $\alpha > 0$ and a_n grows faster than some positive of n. Suppose now that $\Delta p < 1$. Note that if $p > 4$, then the power of n in (3.14) is positive, so the power of a_n there , namely, $\frac{1}{p} - (\widehat{\alpha} + \Delta)$, must be negative. Hence (3.13) holds for all $p > 1$. So if $\Delta p < 1$, (3.13) shows that

$$
a_n^{-\alpha+\frac{1}{p}-\Delta} \leq a_n^{-\widehat{\alpha}+\frac{1}{p}-\Delta} = o(1). \tag{3.42}
$$

Hence in all cases

$$
\lim_{n\to\infty} \tau_n^{(1)} = 0. \tag{3.43}
$$

Next, from (3.4)

$$
\begin{aligned}
\tau_n^{(2)} \quad : &= \left\| L_n[f_n](x) W(x) (1+|x|)^{-\Delta} \right\|_{L_p\left(\beta \frac{a_n}{2} \leq |x| \leq 2a_n\right)} \\
&\leq C_2 \varepsilon a_n^{-\alpha} \left\{ a_n^{\frac{1}{2}-\Delta} \|p_n W\|_{L_p\left[\beta \frac{a_n}{2}, 2a_n\right]} + (\log n)\, a_n^{\frac{1}{p}-\Delta} \right\} \\
&\leq C_3 \varepsilon a_n^{-\alpha} \left\{ \begin{array}{ll} 1, & p < 4 \\ (\log n)^{\frac{1}{4}}, & p = 4 \\ n^{\frac{1}{6}\left(1-\frac{1}{p}\right)}, & p > 4 \end{array} \right\} + C_3 \varepsilon (\log n)\, a_n^{-(\alpha+\Delta)+\frac{1}{p}}
\end{aligned}
$$

by Theorem 27 (a). Recall that for all $p > 1$, $-(\alpha + \Delta) + \frac{1}{p} < 0$ and a_n is of polynomial growth, so the second term in the last RHS is always $o(1)$. By the same token, for $p \leq 4$, this entire RHS is $o(1)$. When $p > 4$ and $\alpha > 1$, then $\alpha > \widehat{\alpha}$, so (3.14) shows that this whole RHS is $o(1)$. When $p > 4$ and $\alpha \leq 1$, our assumption (3.14) shows that this last term is $O(\varepsilon)$. So in all cases,

$$\lim_{n\to\infty} \sup \tau_n^{(2)} \leq C_4\varepsilon. \tag{3.44}$$

Finally, from (3.4),

$$\begin{aligned} \tau_n^{(3)} \quad : \quad &= \left\| L_n\left[f_n\right](x)\, W(x)\,(1+|x|)^{-\Delta} \right\|_{L_p(|x|\geq 2a_n)} \\ &\leq \quad C_5 a_n^{-\alpha+1} \left\| |x|^{-1}(1+|x|)^{-\Delta} \right\|_{L_p(|x|\geq 2a_n)} \end{aligned}$$

Now from (3.13),

$$\Delta > \frac{1}{p} - \widehat{\alpha} \geq \frac{1}{p} - 1$$

so

$$p\,(\Delta + 1) > 1, \tag{3.45}$$

and hence

$$\tau_n^{(3)} \leq C_6 a_n^{-\alpha+1} a_n^{\frac{1}{p}-(1+\Delta)} = o\,(1)\,, \tag{3.46}$$

by (3.5). Together with (3.6) and (3.7), this yields

$$\lim_{n\to\infty}\sup \left(\tau_n^{(1)} + \tau_n^{(2)} + \tau_n^{(3)}\right) \leq C\varepsilon. \blacksquare$$

Lemma 34 *Assume that if $p \leq 4$, (3.13) holds; if $p > 4$ and $\alpha \neq 1$, (3.14) holds; and if $p > 4$ and $\alpha = 1$, (3.15) holds. Let $\varepsilon > 0$, $\beta \in (0,1)$ and assume that $\{\psi_n\}_{n=1}^{\infty}$ is a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$ such that*

$$\psi_n(x) = 0, \quad |x| \geq \beta a_n, \tag{3.47}$$

and

$$|\psi_n W|(x) \leq \varepsilon (1+|x|)^{-\alpha}, \quad x \in \mathbb{R},\ n \geq 1. \tag{3.48}$$

Then

$$\limsup_{n\to\infty} \left\| L_n[\psi_n](x)\, W(x)\, (1+|x|)^{-\Delta} \right\|_{L_p(|x|\geq 2\beta a_n)} \leq C\varepsilon, \tag{3.49}$$

where $C \neq C(\varepsilon, n, \psi_n)$.

Proof. First note that for $|x| \geq 2\beta a_n$ and $|x_{kn}| \leq \beta a_n$, $|x_{kn} - x| \sim |x|$. Then, from Theorem 27 (b), we have for $|x| \geq 2\beta a_n$,

$$\begin{aligned}
|L_n[\psi_n](x)| &= \left| \sum_{|x_{kn}|<\beta a_n} l_{kn}(x)\, \psi_n(x_{kn}) \right| \\
&\leq \varepsilon \sum_{|x_{kn}|<\beta a_n} |l_{kn}(x)|\, W^{-1}(x_{kn})\, (1+|x_{kn}|)^{-\alpha} \\
&\leq C_1 \varepsilon \frac{a_n^{\frac{3}{2}}}{n} \frac{|p_n(x)|}{|x|} \sum_{|x_{kn}|<\beta a_n} (1+|x_{kn}|)^{-\alpha} \\
&\leq C_2 \varepsilon a_n^{\frac{1}{2}} \frac{|p_n(x)|}{|x|} \sum_{|x|<\beta a_n} (x_{k-1,n} - x_{k+1,n})\, (1+|x_{kn}|)^{-\alpha} \quad \text{(by 3.20)} \\
&\leq C_3 \varepsilon a_n^{\frac{1}{2}} \frac{|p_n(x)|}{|x|} \int_{-2\beta a_n}^{2\beta a_n} (1+|t|)^{-\alpha}\, dt.
\end{aligned}$$

Let

$$(\log n)^* := \begin{cases} \log n, & \alpha = 1 \\ 1, & \text{otherwise} \end{cases}.$$

We see that

$$\int_{-2\beta a_n}^{2\beta a_n} (1+|t|)^{-\alpha}\, dt \le C_4 a_n^{1-\widehat{\alpha}} (\log n)^*, \quad n \ge 2.$$

Then

$$\begin{aligned}
& \left\| L_n[\psi_n](x) W(x) (1+|x|)^{-\Delta} \right\|_{L_p(|x|\ge 2\beta a_n)} \\
\le\ & C_5 \varepsilon a_n^{\frac{3}{2}-\widehat{\alpha}} (\log n)^* \left\| p_n(x) W(x) |x|^{-1-\Delta} \right\|_{L_p(|x|\ge 2\beta a_n)} \\
\le\ & C_5 \varepsilon a_n^{\frac{1}{2}-\widehat{\alpha}-\Delta} (\log n)^* \left\| p_n(x) W(x) \right\|_{L_p(\mathbb{R})} \\
\le\ & C_6 \varepsilon a_n^{\frac{1}{p}-(\widehat{\alpha}+\Delta)} (\log n)^* \times \begin{cases} 1, & p < 4 \\ (\log n)^{\frac{1}{4}}, & p = 4 \\ n^{\frac{1}{6}(1-\frac{4}{p})}, & p > 4 \end{cases}.
\end{aligned}$$

by Theorem 27 (a). Now if $p \le 4$, we know that (3.13) holds and a_n is of polynomial growth, so the RHS is $o(1)$. If $p > 4$ and $\alpha \ne 1$, our hypothesis (3.14) ensures that we obtain $O(\varepsilon)$; if $p > 4$ and $\alpha = 1$, our hypothesis (3.15) ensures that again we obtain $O(\varepsilon)$. ■

Before we proceed to the third lemma on L_p norms of Lagrange interpolants, we need the Hilbert transform $H[f](x)$, and its boundedness in suitable weighted L_p spaces. Recall that

$$H[f](x) := \lim_{\varepsilon\to\infty} \int\limits_{|t-x|\geq\varepsilon} \frac{f(t)}{t-x} dt, \tag{3.50}$$

and if $f \in L_1(\mathbb{R})$, then the limit exists almost everywhere.

Lemma 35 [40] *Let $1 < p < \infty$, $s < 1 - \frac{1}{p}$; $S > -\frac{1}{p}$; $s \leq S$. Then for measurable $f : \mathbb{R} \to \mathbb{R}$, for which the RHS is finite,*

$$\|H[f](x)(1+|x|)^s\|_{L_p(\mathbb{R})} \leq C \left\|f(x)(1+|x|)^S\right\|_{L_p(\mathbb{R})}.$$

Lemma 36 *Assume that (3.13) holds. Let $\varepsilon > 0$, $\beta \in \left(0, \frac{1}{2}\right)$, and assume that $\{\psi_n\}_{n=1}^{\infty}$ is a sequence of functions form $\mathbb{R}$ to $\mathbb{R}$ such that*

$$\psi_n(x) = 0, \quad |x| \geq \beta a_n, \tag{3.51}$$

and

$$|\psi_n W|(x) \leq \varepsilon (1+|x|)^{-\alpha}, \quad x \in \mathbb{R},\ n \geq 1. \tag{3.52}$$

Then

$$\limsup_{n\to\infty} \left\|L_n[\psi_n](x) W(x)(1+|x|)^{-\Delta}\right\|_{L_p(|x|\leq 2\beta a_n)} \leq C\varepsilon, \tag{3.53}$$

where $C \neq C(n, \varepsilon, \psi_n)$.

Proof. Define

$$G(x) := W^{-1}(x)(1+|x|)^{-\alpha}, \quad x \in \mathbb{R}.$$

For $f : \mathbb{R} \to \mathbb{R}$ such that $fW \in L_1(\mathbb{R})$, let $S_n[f](x)$ denote the nth partial sum of the orthonormal expansion of f in $\{p_j\}_0^\infty$. We first show that

$$\begin{aligned} & \left\| L_n[\psi_n](x) W(x) (1+|x|)^{-\Delta} \right\|_{L_p(|x| \le 2\beta a_n)} \\ \le\ & C_2 \varepsilon \sup_{\|h\|_{L_\infty(\mathbb{R})} \le 1} \left\| S_n[hG](x) W(x) (1+|x|)^{-\Delta} \right\|_{L_p(|x| \le 2\beta a_n)}. \end{aligned} \tag{3.54}$$

Let χ_n denote the characteristic function of $[-2\beta a_n, 2\beta a_n]$ and for $n \ge 1$, let

$$g_n(x) := \operatorname{sign}(L_n[\psi_n](x)) |L_n[\psi_n](x)|^{p-1} \chi_n(x) W^{p-2}(x) (1+|x|)^{-\Delta p}.$$

Observe that

$$\begin{aligned} & \left\| L_n[\psi_n](x) W(x) (1+|x|)^{-\Delta} \right\|^p_{L_p(|x| \le 2\beta a_n)} \\ =\ & \int_{-\infty}^{\infty} L_n[\psi_n](x) g_n(x) W^2(x)\, dx \\ =\ & \int_{-\infty}^{\infty} L_n[\psi_n](x) S_n[g_n](x) W^2(x)\, dx \\ =\ & \sum_{k=1}^{n} \lambda_n \psi_n(x_{kn}) S_n[g_n](x_{kn}) \quad \text{(Gauss quadrature)} \\ \le\ & \varepsilon \sum_{|x_{kn}| \le \beta a_n} \lambda_{kn} |S_n[g_n](x_{kn})| W^{-1}(x_{kn}) (1+|x_{kn}|)^{-\alpha} \\ \le\ & C_3 \varepsilon \int_{-\infty}^{\infty} |S_n[g_n](x)| W(x) (1+|x|)^{-\alpha}\, dx \\ \le\ & C_3 \varepsilon \int_{-\infty}^{\infty} S_n[g_n](x) h_n(x) G(x) W^2(x)\, dx \end{aligned}$$

where $h_n(x) :=\mathrm{sign}(S_n[g_n](x))$. Then orthogonality and Hölder's inequality show that this last expression can be continued as

$$\begin{aligned}
& C_3\varepsilon \int_{-\infty}^{\infty} S_n[g_n](x)\, h_n(x)\, G(x)\, W^2(x)\, dx \\
= \;& C_3\varepsilon \int_{|x|\leq 2\beta a_n} g_n(x)\, S_n[h_n G](x)\, W^2(x)\, dx \\
\leq \;& C_3\varepsilon \left\| g_n W (1+|x|)^{\Delta} \right\|_{L_q(|x|\leq 2\beta a_n)} \times \left\| S_n[h_n G]\, W (1+|x|)^{-\Delta} \right\|_{L_p(|x|\leq 2\beta a_n)} \\
= \;& C_3\varepsilon \left\| L_n[\psi_n](x)\, W(x)\, (1+|x|)^{-\Delta} \right\|^{p-1}_{L_q(|x|\leq 2\beta a_n)} \times \left\| S_n[h_n G]\, W (1+|x|)^{-\Delta} \right\|_{L_p(|x|\leq 2\beta a_n)}.
\end{aligned}$$

Thus

$$\begin{aligned}
& \left\| L_n[\psi_n](x)\, W(x)\, (1+|x|)^{-\Delta} \right\|_{L_p(|x|\leq 2\beta a_n)} \\
\leq \;& C_3\varepsilon \sup_{\|h\|_{L_\infty}\leq 1} \left\| S_n[h_n G]\, W (1+|x|)^{-\Delta} \right\|_{L_p(|x|\leq 2\beta a_n)}.
\end{aligned}$$

Next, it is a well known consequence of the Christoffel-Darboux formula that for $f : \mathbb{R} \to \mathbb{R}$ for which $fW \in L_1(\mathbb{R})$,

$$S_n[f](x) = \frac{\gamma_{n-1}}{\gamma_n} \left\{ p_n(x)\, H\left[p_{n-1} f W^2\right](x) - p_{n-1}(x)\, H\left[p_n f W^2\right](x) \right\}.$$

Then using Theorem 26 (c) and (e), we have for $|x| \leq 2\beta a_n$ and $h \in L_\infty(\mathbb{R})$ that

$$\left| S_n[h_n G](x)\, W(x) \right| \leq C_4 a_n^{\frac{1}{2}} \sum_{j=1}^{n} \left| H\left[p_j h G W^2\right](x) \right|. \tag{3.55}$$

Choose $\sigma \in (2\beta, 1)$, and let χ_n^* be the characteristic function of $[-\sigma a_n, \sigma a_n]$, $n \geq 1$. Moreover, let $\|h\|_{L_\infty(\mathbb{R})}$. In estimating the Hilbert transform for

$|x| \leq 2\beta a_n$, we write

$$H\left[p_j hGW^2\right] = H\left[p_j\left(1-\chi_n^*\right)hGW^2\right] + H\left[p_j\chi_n^* hGW^2\right]$$

Now for $|x| \leq 2\beta a_n$, and $j = n-1, n$,

$$\begin{aligned}
& \left|H\left[p_j\left(1-\chi_n^*\right)hGW^2\right](x)\right| \\
= & \left|\int_{|t|\in[\sigma a_n,\infty)} \frac{p_j(t)\left(hGW^2\right)(t)}{x-t}dt\right| \\
\leq & C_5 \int_{\sigma a_n}^{\infty} |p_j W|(t)\, t^{-\alpha-1}dt \\
\leq & C_5 \left(\int_{\sigma a_n}^{\infty} p_j^2(t)\, W^2(t)\, dt\right)^{\frac{1}{2}} \left(\int_{\sigma a_n}^{\infty} t^{-2\alpha-2}dt\right)^{\frac{1}{2}} \\
= & C_5 \left\|p_j W\right\|_{L_2(\sigma a_n,\infty)} \times \left(\frac{1}{1+2\alpha}\left(\sigma a_n\right)^{-(1+2\alpha)}\right)^{\frac{1}{2}} \\
= & C_6 \left\|p_j W\right\|_{L_2(\sigma a_n,\infty)} \sigma^{-\frac{1}{2}-\alpha} a_n^{-\frac{1}{2}-\alpha} \\
= & C_6 a_n^{-\frac{1}{2}-\alpha}.
\end{aligned}$$

This last estimate and (3.55) give for $|x| \leq 2\beta a_n$,

$$\left|S_n[hG](x)\, W(x)\right| \leq C_4 a_n^{-\frac{1}{2}} \sum_{j=n-1}^{n} \left|H\left[p_j\chi_n^* hGW^2\right](x)\right| + C_7 a_n^{-\alpha}. \quad (3.56)$$

Now let us set for small $\delta > 0$,

$$\widehat{\Delta} := \min\left\{\Delta, \frac{1}{p} - \delta\right\}.$$

We now use Lemma 35 with $s = S = -\widehat{\Delta}$. Note that

$$S \cdot \; -\widehat{\Delta} = \max\left\{-\Delta, \delta - \frac{1}{p}\right\} > -\frac{1}{p}.$$

Also, (3.13) gives

$$-\Delta < -\frac{1}{p} + \widehat{\alpha} \leq -\frac{1}{p} + 1.$$

So if $\delta < 1$, then

$$s = -\widehat{\Delta} = \max\left\{-\Delta, \delta - \frac{1}{p}\right\} < 1 - \frac{1}{p}.$$

Thus the requirements of Lemma 35 are met. Then as $\widehat{\Delta} \leq \Delta$, (3.56) and Lemma 35 yield

$$\begin{aligned}
& \left\| S_n [hG](x) W(x) (1+|x|)^{-\Delta} \right\|_{L_p(|x| \leq 2\beta a_n)} \\
\leq \;& C_4 a_n^{\frac{1}{2}} \sum_{j=n-1}^{n} \left\| H\left[p_j \chi_n^* h G W^2\right](x) (1+|x|)^{-\widehat{\Delta}} \right\|_{L_p(\mathbb{R})} \\
& + C_7 a_n^{-\alpha} \left\| (1+|x|)^{-\Delta} \right\|_{L_p(|x| \leq 2\beta a_n)} \\
\leq \;& C_8 a_n^{\frac{1}{2}} \sum_{j=n-1}^{n} \left\| \left(p_j \chi_n^* h G W^2\right)(x) (1+|x|)^{-\widehat{\Delta}} \right\|_{L_p(\mathbb{R})} \\
& + C_8 a_n^{-\alpha} \times \begin{cases} 1, & \Delta p > 1 \\ (\log n)^{\frac{1}{p}}, & \Delta p = 1 \\ a_n^{\frac{1}{p} - \Delta}, & \Delta p < 1 \end{cases} \\
\leq \;& C_8 a_n^{\frac{1}{2}} \sum_{j=n-1}^{n} \left\| (p_j W)(x) (1+|x|)^{-(\alpha + \widehat{\Delta})} \right\|_{L_p(|x| \leq 2\beta a_n)} + o(1), \quad (3.57)
\end{aligned}$$

since $\alpha > 0$, and a_n is of polynomial growth, while also by (3.13),

$$-\alpha + \frac{1}{p} - \Delta \leq -\widehat{\alpha} + \frac{1}{p} - \Delta < 0. \tag{3.58}$$

Then using the bound (3.21). we can continue (3.57) as

$$\begin{aligned} &\leq \; C_9 \left\| (1 + |x|)^{-(\alpha + \widehat{\Delta})} \right\|_{L_p(|x| \leq a_{\sigma n})} + o(1) \\ &\leq \; C_{10}. \end{aligned}$$

as again (3.58) implies that

$$p\left(\alpha + \widehat{\Delta}\right) > 1. \tag{3.59}$$

Finally, thi: bound and (3.54) yield the result. ■

Proof of the sufficient conditions of Theorem 25

We remark first that the sufficient conditions in Theorem 25 imply those in Lemmas 33, 34 and 36. Let $\varepsilon > 0$. We can find a polynomial P such that

$$|f - P|(x) W(x) (1 + |x|)^{-\Delta} \leq \varepsilon, \quad x \in \mathbb{R}. \tag{3.60}$$

Then for n large enough,

$$\begin{aligned} &\left\| (f - L_n[f])(x) W(x) (1 + |x|)^{-\Delta} \right\|_{L_p(\mathbb{R})} \\ \leq \; &\left\| (f - P)(x) W(x) (1 + |x|)^{-\Delta} \right\|_{L_p(\mathbb{R})} \end{aligned}$$

$$+\left\|L_n\left[P-f\right](x)\,W(x)\,(1+|x|)^{-\Delta}\right\|_{L_p(\mathbb{R})}$$
$$\leq \varepsilon\left\|(1+|x|)^{-(\alpha+\Delta)}\right\|_{L_p(\mathbb{R})}+\left\|L_n\left[P-f\right](x)\,W(x)\,(1+|x|)^{-\Delta}\right\|_{L_p(\mathbb{R})} \quad (3.61)$$

Here $p(\alpha+\Delta)\geq p(\hat{\alpha}+\Delta)>1$, so the first norm in this RHS is finite. Next, let χ_n denote the characteristic function of $\left[-\frac{a_n}{4},\frac{a_n}{4}\right]$ and write

$$\begin{aligned} P-f &= (P-f)\,\chi_n+(P-f)\,(1-\chi_n) \\ &=: \psi_n+f_n. \end{aligned}$$

We can apply Lemma 33 to $\{f_n\}_{n=1}^{\infty}$ with $\beta=\frac{1}{4}$ to deduce that

$$\limsup_{n\to\infty}\left\|L_n\left[\psi_n\right](x)\,W(x)\,(1+|x|)^{-\alpha}\right\|_{L_p(\mathbb{R})}\leq C_1\varepsilon.$$

Next, in view of (3.60), $\{\psi_n\}_{n=1}^{\infty}$ satisfy (3.47) and (3.48) in Lemma 34 and (3.51), (3.52) in Lemma 36 with $\beta=\frac{1}{4}$, so those lemmas yield

$$\limsup_{n\to\infty}\left\|L_n\left[\psi_n\right](x)\,W(x)\,(1+|x|)^{-\Delta}\right\|_{L_p(\mathbb{R})}\leq C_2\varepsilon.$$

So

$$\limsup_{n\to\infty}\left\|L_n\left[P-f\right](x)\,W(x)\,(1+|x|)^{-\Delta}\right\|_{L_p(\mathbb{R})}\leq C_3\varepsilon. \quad (3.62)$$

Combining (3.61) and (3.62), we have

$$\limsup_{n\to\infty}\left\|(f-L_n[f])(x)\,W(x)\,(1+|x|)^{-\Delta}\right\|_{L_p(\mathbb{R})}\leq C_4\varepsilon.$$

Letting $\varepsilon\to 0^+$ yields the result. ■

In the converse direction, we need the following lemma.

Lemma 37 *Let $\sigma \in (0,1)$, $\eta \in (0, 1-\delta)$, $1 < p < \infty$. Then there exists C such that for $n \geq 1$ and P of degree at most ηn, we have*

$$\|P\|_{L_p[-a_{\delta n}, a_{\delta n}]} \leq C a_n^{\frac{1}{2}} \sum_{j=n-1}^{n} \|p_j W P\|_{L_p[-a_n, a_n]}. \tag{3.63}$$

Proof. Choose $\sigma' \in (\sigma, 1-\eta)$. Let $\delta \in (0,1)$ be such that $\delta\sigma' > \sigma$. For m large enough Theorem 26 (a) shows that

$$\lambda_m^{-1}(W, x) \sim \frac{m}{a_m} W^{-1}(x), \quad |x| \leq a_{2\delta m}.$$

Moreover, we can replace $\sim$ by $\leq C\times$ for all $x \in \mathbb{R}$. Applying this with $m := \left[\sigma' \frac{n}{2}\right]$ and using $2\delta m = 2\delta \left[\sigma' \frac{n}{2}\right] > \delta n$, with n large enough, yields

$$\lambda^{-1}_{\left[\sigma' \frac{n}{2}\right]}(W, x)\, W(x) \begin{cases} \sim \frac{n}{a_n}, & |x| \leq a_{\delta n} \\ \leq C \frac{n}{a_n}, & x \in \mathbb{R} \end{cases}. \tag{3.64}$$

Let P be of degree $\leq \eta n$, and

$$R(x) := \frac{a_n}{n} P(x)\, \lambda^{-1}_{\left[\sigma' \frac{n}{2}\right]}(W, x). \tag{3.65}$$

Then R has degree $\leq \eta n + 2\left[\sigma' \frac{n}{2}\right] \leq (\eta + \delta')\, n < n - 1$, for n large enough, by choice of σ'. Next, with the definition for S_n in Lemma 36,

$$\begin{aligned} R(x) &= S_n[R](x) \\ &= \frac{\gamma_{n-1}}{\gamma_n} \left\{ p_n(x)\, H\left[R p_{n-1} W^2\right](x) - p_{n-1}(x)\, H\left[R p_n W^2\right](x) \right\} \end{aligned}$$

by the Christoffel-Darboux formula. Using the bounds in Theorem 26, we obtain for $|x| \leq \delta a_n$

$$|RW|(x) \leq C_1 a_n^{\frac{1}{2}} \sum_{j=n-1}^{n} \left| H\left[Rp_j W^2\right](x) \right| .$$

Using Riesz's theorem that H is a bounded operator from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$, we obtain

$$\|RW\|_{L_p[-a_{\sigma n}, a_{\sigma n}]} \leq C_2 a_n^{\frac{1}{2}} \sum_{j=n-1}^{n} \left\| Rp_j W^2 \right\|_{L_p(-a_n, a_n)}$$

where we also used the infinite-finite range inequality (3.23) for the weight W^2. Now (3.64) shows that

$$|RW|(x) \begin{cases} \sim |P(x)|, & |x| \leq a_{\sigma n} \\ \leq C|P(x)|, & x \in \mathbb{R} \end{cases} .$$

Then (3.63) follows. ■

Proof of the necessary conditions of Theorem 25

Let $\Delta \in \mathbb{R}$, $\alpha > 0$. We assume the convergence (3.11) for every continuous $f : \mathbb{R} \to \mathbb{R}$ satisfying (3.12). Let $\eta(x) : \mathbb{R} \to (0, \infty)$ be an even continuous function that is decreasing in $[0, \infty)$, with

$$\eta(x) \geq (\log(2 + |x|))^{-\frac{1}{2p}}, \quad x \in [0, \infty), \tag{3.66}$$

but

$$\lim_{x\to\infty} \eta(x) = 0. \tag{3.67}$$

We let X be the space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with

$$\|f\|_X := \max_{x\in\mathbb{R}} |f(x)| W(x) (1+|x|)^{\alpha} \eta(x)^{-1} < \infty.$$

Furthermore, let Y be the space of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ with

$$\|f\|_Y := \left\|(fW)(x)(1+|x|)^{-\Delta}\right\|_{L_p(\mathbb{R})} < \infty.$$

Now each $f \in X$ satisfies (3.12), so our hypothesis ensures that

$$\lim_{n\to\infty} \|f - L_n[f]\|_Y = 0.$$

By the uniform boundedness principle, there exists $C_1 > 0$ such that

$$\|f - L_n[f]\|_Y \le C_1 \|f\|_X \quad \forall n \ge 1,\ \forall f \in X.$$

In particular, as $L_1[f] = f(0)$ we obtain for every continuous $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$, that

$$\|f\|_Y \le C_1 \|f\|_X\ ,$$

provided the norm on the RHS is finite. Hence, for every $n \ge 1$ and every continuous f with $f(0) = 0$ for which the RHS is finite,

$$\left\|L_n[f](x) W(x)(1+|x|)^{-\Delta}\right\|_{L_p(\mathbb{R})} \le C_2 \left\|(fW)(x)(1+|x|)^{\alpha}\eta(x)^{-1}\right\|_{L_\infty(\mathbb{R})}. \tag{3.68}$$

Now choose g_n, $n \geq 1$, such that g_n is continuous in $\mathbb{R}$, $g_n = 0$ in $[0, \infty) \cup \left(-\infty, -\frac{a_n}{2}\right)$;

$$\|g_n\|_X = \left\|(g_n W)(x)(1+|x|)^\alpha \eta(x)^{-1}\right\|_{L_\infty(\mathbb{R})} = 1 \tag{3.69}$$

and for $x_{jn} \in \left[-\frac{a_n}{2}, 0\right)$,

$$(g_n W)(x_{jn})(1+|x_{jn}|)^\alpha \eta(x_{jn})^{-1} \operatorname{sign}(p_n'(x_{jn})) = 1.$$

Then for $x > 0$, our choice of g_n and then (3.25) yield

$$\begin{aligned}
|L_n[g_n](x)| &= \left|\sum_{x_{jn} \in \left[-\frac{a_n}{2}, 0\right)} g_n(x_{jn}) \frac{p_n(x)}{p_n'(x_{jn})(x - x_{jn})}\right| \\
&= \sum_{x_{jn} \in \left[-\frac{a_n}{2}, 0\right)} \left|\frac{p_n(x)}{p_n'(x_{jn})(x - x_{jn})}\right| W^{-1}(x_{jn})(1+|x|)^{-\alpha} \eta(x_{jn}) \\
&\geq C_3 \frac{a_n^{\frac{3}{2}}}{n} \sum_{x_{jn} \in \left[-\frac{a_n}{2}, 0\right)} \frac{|p_n(x)|}{x + |x_{jn}|}(1+|x|)^{-\alpha} \eta(x_{jn}) \\
&\geq C_4 \eta(a_n) a_n^{\frac{1}{2}} |p_n(x)| \sum_{x_{jn} \in \left[-\frac{a_n}{2}, 0\right)} \frac{x_{j-1,n} - x_{j+1,n}}{x + |x_{jn}|}(1+|x_{jn}|)^{-\alpha}
\end{aligned}$$

by (3.20). At least for $x \geq 1$, it is not difficult to see that uniformly for $n \geq 1$ and $1 \leq j \leq n$, we have

$$x + |x_{jn}| \sim x + |t|, \quad t \in (x_{j+1,n}, x_{j-1,n}).$$

So for $x \geq 1$,

$$|L_n[g_n](x)| \geq C_5 \eta(a_n) a_n^{\frac{1}{2}} |p_n(x)| \int_0^{\frac{a_n}{4}} \frac{(1+t)^{-\alpha}}{x+t} dt$$

$$\geq C_5\eta(a_n)\,a_n^{\frac{1}{2}}\frac{|p_n(x)|}{x}\begin{cases}1, & \alpha>1\\ \log\left(1+\min\left\{\frac{a_n}{4},x\right\}\right), & \alpha=1\\ \left(\min\left\{\frac{a_n}{4},x\right\}\right)^{1-\alpha}, & \alpha>1\end{cases}.$$

Let us set

$$(\log x)^* := \begin{cases}\log(1+x), & x=1,\\ 1, & \text{otherwise.}\end{cases}$$

By considering $x\in\left[1,\frac{a_n}{4}\right]$ and $x\in\left[\frac{a_n}{4},2a_n\right]$ separately, we see that we can rewrite the above as

$$|L_n[g_n](x)|\geq C_7\eta(a_n)\,a_n^{\frac{1}{2}}|p_n(x)|\,x^{-\widehat{\alpha}}(\log x)^* \tag{3.70}$$

for $x\in[1,2a_n]$. Then we obtain from (3.66) that for $x\in[1,2a_n]$,

$$|L_n[g_n](x)|\geq C_8(\log n)^{-\frac{1}{2p}}a_n^{\frac{1}{2}}|p_n(x)|\,x^{-2}.$$

Hence, using (3.68) – (3.69),

$$\begin{aligned}C_2 &\geq \left\|L_n[g_n](x)\,W(x)\,(1+|x|)^{-\Delta}\right\|_{L_p[1,2a_n]}\\ &\geq C_9(\log n)^{-\frac{1}{2p}}a_n^{\frac{1}{2}}\left\|(p_nW)(x)\,(1+|x|)^{-\left(\widehat{\alpha}+\Delta\right)}\right\|_{L_p[1,2a_n]}\\ &\geq C_{10}(\log n)^{-\frac{1}{2p}}a_n^{\frac{1}{2}}\left\|(p_nW)(x)\,(1+|x|)^{-\left(\widehat{\alpha}+\Delta\right)}\right\|_{L_p[0,2a_n]}-C_{11}(\log n)^{-\frac{1}{2p}}\end{aligned}$$

where we have used the bound (3.21) for p_n in $[0,1]$. So

$$C_{12}(\log n)^{\frac{1}{2p}}\geq a_n^{\frac{1}{2}}\left\|(p_nW)(x)\,(1+|x|)^{-\left(\widehat{\alpha}+\Delta\right)}\right\|_{L_p[-2a_n,2a_n]}.$$

Now let $\widehat{P}_{2a_n}$ be the polynomial of Lemma 30 of degree $O(a_n \log a_n) = o(n)$ such that for $|x| \leq 2a_n$,

$$\widehat{P}_{2a_n}(x) \sim \left(1+x^2\right)^{-\frac{(\widehat{\alpha}+\Delta)}{2}} \sim (1+|x|)^{-(\widehat{\alpha}+\Delta)}.$$

Then we obtain from Lemma 27,

$$\begin{aligned}
C_{13}(\log n)^{\frac{1}{2p}} &\geq a_n^{\frac{1}{2}} \sum_{j=n-1}^{n} \left\|(p_j W)(x)\, \widehat{P}_{2a_n}(x)\right\|_{L_p\left[-2a_j, 2a_j\right]} \\
&\geq C_{14} \left\|\widehat{P}_{2a_n}(x)\right\|_{L_p\left[-\frac{a_n}{2}, \frac{a_n}{2}\right]} \\
&\geq C_{15} \left\|(1+|x|)^{-(\widehat{\alpha}+\Delta)}\right\|_{L_p\left[-\frac{a_n}{2}, \frac{a_n}{2}\right]} \\
&\geq C_{16} \begin{cases} a_n^{\frac{1}{p}-(\widehat{\alpha}+\Delta)}, & \Delta < \frac{1}{p} - \widehat{\alpha} \\ (\log n)^{\frac{1}{p}}, & \Delta = \frac{1}{p} - \widehat{\alpha} \\ 1, & \Delta > \frac{1}{p} - \widehat{\alpha} \end{cases}
\end{aligned}$$

This is not possible for all n large enough, unless $\Delta > \frac{1}{p} - \widehat{\alpha}$. So, we have proved (3.13) for all $1 < p < \infty$ and in particular, for $p = 4$. This establishes the necessary conditions for $p \leq 4$. To prove the necessary conditions for $p > 4$, we return to (3.70). First, for small enough $\delta \in (0,1)$, it follows from Theorem 26 (c) and (d), that

$$\|p_n W\|_{L_p[\delta a_n, 2a_n]} \sim \|p_n W\|_{L_p(\mathbb{R})}.$$

Then by (3.68) – (3.70)

$$
\begin{aligned}
C_2 &\geq \left\| L_n\left[g_n\right](x) W(x)(1+|x|)^{-\Delta} \right\|_{L_p[\delta a_n, 2a_n]} \\
&\geq C_{17} \eta\left(a_n\right) a_n^{\frac{1}{2}} \left\| p_n W \right\|_{L_p[\delta a_n, 2a_n]} a_n^{-\left(\widehat{\alpha}+\Delta\right)} (\log n)^* \\
&\geq C_{18} \eta\left(a_n\right) a_n^{\frac{1}{p}-\left(\widehat{\alpha}+\Delta\right)} (\log n)^* n^{\frac{1}{6}\left(1-\frac{4}{p}\right)},
\end{aligned}
$$

by (3.25) as $p > 4$. Thus

$$
\limsup_{n \to \infty} \eta\left(a_n\right) a_n^{\frac{1}{p}-\left(\widehat{\alpha}+\Delta\right)} (\log n)^* n^{\frac{1}{6}\left(1-\frac{4}{p}\right)} < \infty, \tag{3.71}
$$

for every function η satisfying (3.66) – (3.67). If

$$
\limsup_{n \to \infty} a_n^{\frac{1}{p}-\left(\widehat{\alpha}+\Delta\right)} (\log n)^* n^{\frac{1}{6}\left(1-\frac{4}{p}\right)} = \infty,
$$

then it is easy to construct $\eta(x)$ decreasing slowly enough to 0 to contradict (3.71). So

$$
\limsup_{n \to \infty} a_n^{\frac{1}{p}-\left(\widehat{\alpha}+\Delta\right)} (\log n)^* n^{\frac{1}{6}\left(1-\frac{4}{p}\right)} < \infty,
$$

and it follows that (3.14) is necessary if $\alpha \neq 1$ and (3.15) is necessary if $\alpha = 1$.

Proof of Theorem 23. For $W = W_\beta$, $a_n = Cn^{\frac{1}{\beta}}$, $n \geq 1$, and the necessary and sufficient conditions in Theorem 25 easily imply those in Theorem 23.

Proof of Corollary 24. Let

$$\tau(p) := \begin{cases} \frac{1}{p} - \widehat{\alpha}, & p \leq 4 \\ \frac{1}{p} - \widehat{\alpha} + \frac{\beta}{6}\left(1 - \frac{4}{p}\right), & p > 4 \end{cases}.$$

For the convergence (3.5) to hold for every $1 < p \leq \infty$ and every continuous function f satisfying (3.6) with a given $\Delta \in \mathbb{R}$, the necessary conditions of Theorem 23 imply that

$$\Delta \geq \tau(p), \quad 1 < p < \infty.$$

Hence

$$\Delta \geq \lim_{p \to 1^+} \tau(p) = 1 - \widehat{\alpha};$$

$$\Delta \geq \lim_{p \to \infty} \tau(p) = \frac{\beta}{6} - \widehat{\alpha}.$$

So it is necessary that (3.7) holds. Conversely, for the convergence (3.5) to hold for every $1 < p < \infty$ and every continuous f satisfying (3.6), the sufficient conditions of Theorem 23 show that it suffices that

$$\Delta > \tau(p), \quad 1 < p < \infty. \tag{3.72}$$

First, (3.7) shows that

$$\Delta \geq 1 - \widehat{\alpha} > \frac{1}{p} - \widehat{\alpha} = \tau(p), \quad 1 < p \leq 4.$$

So for $1 < p \le 4$, (3.72) is fulfilled. Next, we can write for $p > 4$,

$$\tau(p) = \frac{\beta}{6} - \widehat{\alpha} + \frac{1}{p}\left(1 - \frac{2\beta}{3}\right).$$

If $1 - \frac{2\beta}{3} \ge 0$, then for $4 < p < \infty$

$$\tau(p) \le \tau(4) = \frac{\beta}{6} - \widehat{\alpha} + \frac{1}{4}\left(1 - \frac{2\beta}{3}\right) = -\widehat{\alpha} + \frac{1}{4} < 1 - \widehat{\alpha} \le \Delta,$$

by (3.7), so again (3.72) is fulfilled. Finally, if $1 - \frac{2\beta}{3} < 0$, then for $4 < p < \infty$

$$\tau(p) < \tau(\infty) = \frac{\beta}{6} - \widehat{\alpha} \le \Delta,$$

by (3.7). So we have shown that (3.7) implies (3.72) for all $1 < p < \infty$, and this proves that (3.7) is sufficient. ■

3.3 Mean Convergence of Lagrange Interpolation for Erdös Weights

3.3.1 Introduction and Main Results

In this section, we only the state the necessary and sufficient conditions of Lagrange interpolation for Erdös weights without giving emphasis on the proofs of the results. Let us recall that Erdös weights have the form $W^2 = e^{-2Q}$, where $Q : \mathbb{R} \to \mathbb{R}$ is even and of faster than polynomial growth at infinity. In the sequel, we will assume the notation of sections 3.2.1 and 3.2.2. For notational simplicity, we recall Theorem 25 for W_β^2, $\beta > 1$:

Theorem 38 *Let $W_\beta(x) := \exp\left(-\frac{1}{2}|x|^\beta\right)$, $\beta > 1$. Given $f : \mathbb{R} \to \mathbb{R}$, let $L_n[f]$ denote the Lagrange interpolation polynomial to f at the zeros of $p_n(W^2, x)$. Let $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\alpha > 0$, and*

$$\tau := \frac{1}{p} - \min\{1, \alpha\} + \max\left\{0, \frac{\beta}{6}\left(1 - \frac{4}{p}\right)\right\}.$$

Then for

$$\lim_{n\to\infty} \left\|(f - L_n[f])(x)\, W(x)\,(1 + |x|)^{-\Delta}\right\|_{L_p(\mathbb{R})},$$

to hold for every continuous $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} (fW)(x)\,(1 + |x|)^\alpha = 0,$$

it is necessary and sufficient that

$$\begin{aligned} \Delta &> \tau \quad \text{if } 1 < p \le 4; \\ \Delta &> \tau \quad \text{if } p > 4 \text{ and } \alpha = 1; \\ \Delta &\ge \tau \quad \text{if } p > 4 \text{ and } \alpha \ne 1. \end{aligned}$$

To describe analogous results for Erdös weights, we need a class of weights W^2 for which suitable bounds are known for $p_n(W^2, .)$. These were found in [42] and the L_p analogues were found in [39]. For our discussions, the following subclass from [42] is suitable:

Definition 39 *Let $W = e^{-Q}$, where $Q : \mathbb{R} \to \mathbb{R}$ is even. continuous, Q'' exists in $(0, \infty)$, $Q^{(j)} \geq 0$ in $(0, \infty)$, $j = 0, 1, 2$ and the function*

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)} \tag{3.73}$$

is increasing in $(0, \infty)$, with

$$\lim_{x \to \infty} T(x) = \infty;\ T(0+) = \lim_{x \to 0+} T(x) > 1. \tag{3.74}$$

Moreover, we assume that for $C_1, C_2, C_3 > 0$,

$$C_1 \leq \frac{T(x)}{\left(\frac{xQ'(x)}{Q(x)}\right)} \leq C_2,\ x \geq C_3, \tag{3.75}$$

and for every $\varepsilon > 0$,

$$T(x) = O\left(Q(x)^{\varepsilon}\right),\ \ x \to \infty. \tag{3.76}$$

Then we write $W \in \mathfrak{E}_1$.

Example 40 (a) *The principal example of $W = e^{-Q} \in \mathfrak{E}_1$ is*

$$W_{k,\alpha}(x) = \exp\left(-Q_{k,\alpha}(x)\right),\ \alpha > 1,$$

where

$$Q_{k,\alpha}(x) =: \exp_k\left(|x|^{\alpha}\right),\ k \geq 1.$$

For this W,

$$T(x) = T_{k,\alpha}(x) = \alpha \left[1 + x^{\alpha} \sum_{l=1}^{k} \prod_{j=1}^{l-1} \exp_j (x^{\alpha}) \right], \; x \geq 0.$$

Here (3.75) *holds in the stronger form*

$$\lim_{x \to \infty} \frac{T(x)}{\left[\frac{xQ'(x)}{Q(x)}\right]} = 1. \tag{3.77}$$

and (3.76) *holds in the stronger form*

$$\lim_{x \to \infty} \frac{T(x)}{\left[\prod_{j=1}^{k} \log_j Q(x)\right]} = \alpha. \tag{3.78}$$

(*b*) Another example of $W = e^{-Q} \in \mathcal{E}_1$ is given by

$$Q(x) := \exp\left[\left(\log\left(A + x^2\right)\right)^{\beta}\right], \tag{3.79}$$

$\beta > 1$, A large enough; for which

$$T(x) = \frac{2x^2}{A + x^2} \left[\frac{\beta - 1}{\log(A + x^2)} + \beta \left\{ \log\left(A + x^2\right)^{\beta - 1} \right\} \right] + \frac{2A}{A + x^2}. \tag{3.80}$$

Again (3.75) holds in the stronger form (3.77), while (3.76) holds in the stronger form

$$\lim_{x \to \infty} \frac{T(x) \log x}{\log Q(x)} = \beta. \tag{3.81}$$

Next we present the main results of this section. The following theorem is due to Damelin and Lubinsky [9]:

Theorem 41 *Let $W = e^{-Q} \in \mathfrak{E}_1$. Let $L_n[.]$ denote the Lagrange interpolation polynomial to f at the zeros of $p_n(W^2, .)$. Let $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\kappa > 0$. Then for*

$$\lim_{n\to\infty} \left\| (f - L_n[f])(x) W(x) (1 + Q(x))^{-\Delta} \right\|_{L_p(\mathbb{R})}, \tag{3.82}$$

to hold for every continuous $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} |fW|(x) (\log|x|)^{1+\kappa} = 0,$$

it is necessary and sufficient that

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)\right\}. \tag{3.83}$$

At first, the choice of the extra weighting factor $(1 + Q)$ in (3.82) may seem rather severe. Recall that Q grows faster than any polynomial. However, even if f vanishes outside a fixed interval, the following theorem shows why we need such a factor if $p > 4$:

Theorem 42 *Let W, L_n be as in Theorem 41 and $p > 4$. Suppose that measurable $U : \mathbb{R} \to \mathbb{R}$ satisfies*

$$\liminf_{x\to\infty} U(x) x^{-\left(\frac{3}{2} - \frac{1}{p}\right)} Q(x)^{\frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)} > 0. \tag{3.84}$$

Then there exists continuous $f : \mathbb{R} \to \mathbb{R}$, vanishing outside $[-2, 2]$ such that

$$\limsup_{n\to\infty} \|L_n[f]\,WU\|_{L_p(\mathbb{R})} = \infty. \tag{3.85}$$

Remark 5 *(a) For $p > 4$, there is no growth restriction on f, no matter how severe it may be , that can allow us a weighting factor weaker than a power of $(1+Q)$. One can formulate versions of Theorem 41 for $p > 4$ that involve $\Delta = \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)$, and then one has to introduce extra factors in (3.82), such as negative powers of $(1+|x|)$ and negative powers of T or $\log(2+Q)$. Unfortunately one then needs extra conditions on T to avoid very complicated formulations.*

(b) For $p \leq 4$, the weighting factor $(1+Q)$ is unnecessarily strong. Let us recall the Erdös-Turan theorem, as extended by Shohat [56]. If $f : \mathbb{R} \to \mathbb{R}$ is Riemann integrable in each finite interval, and there exists an even entire function G with all non-negative Maclaurin series coefficients such that

$$\lim_{|x|\to\infty} \frac{f^2(x)}{G(x)} = 0,$$

and

$$\int_{-\infty}^{\infty} G(x)\,W^2(x)\,dx < \infty,$$

then

$$\lim_{n\to\infty} \|(f - L_n[f])\, W\|_{L_2(\mathbb{R})} = 0. \tag{3.86}$$

For the "nice" weights in $\mathfrak{E}_1$, a result of Clunie and Kovari [5] allows one to choose G with

$$G(x) \sim W^{-2}(x)\,(1+|x|)^{-1-\kappa},\ x\in\mathbb{R},\ \kappa>0$$

or

$$G(x) \sim W^{-2}(x)\,(1+|x|)^{-1}\,(\log(2+|x|))^{-1-\kappa},\ x\in\mathbb{R},\ \kappa>0.$$

Thus we can ensure that (3.86) holds provided

$$\lim_{|x|\to\infty} (fW)(x)\,(1+|x|)^{\frac{1}{2}+\frac{\kappa}{2}} = 0,$$

or

$$\lim_{|x|\to\infty} (fW)(x)\,(1+|x|)^{\frac{1}{2}}\,(\log(2+|x|))^{\frac{1}{2}+\frac{\kappa}{2}} = 0.$$

Thus the result here does not extend the classical result for $p=2$.

Next we present a result of Damelin and Lubinsky [9] which does essentially constitute an extension of the classical Erdös-Turan result.

Theorem 43 *Let $W = e^{-Q} \in \mathfrak{E}_1$. Let $1 < p < 4$, and $\alpha \in \mathbb{R}$. Let $L_n[f]$ denote the Lagrange interpolation polynomial to f at the zeros of $p_n(W^2,.)$. Then the following are equivalent.*

(a) For every continuous $f : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{|x|\to\infty} |f(x)| W(x) (1+|x|)^{\alpha} = 0, \tag{3.87}$$

we have

$$\lim_{n\to\infty} \|(f - L_n[f]) W\|_{L_p(\mathbb{R})} = 0. \tag{3.88}$$

(b) $\alpha > \frac{1}{p}$.

The proof of this result is based on the following converse quadrature sum estimate which is of independent interest:

Theorem 44 *Let $W := e^{-Q} \in \mathfrak{E}_1$ and $1 < p < 4$. There exists $C > 0$ such that for $n \geq 1$ and $P \in \mathbb{P}_{n-1}$,*

$$\|PW\|_{L_p(\mathbb{R})} \leq C \left\{ \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) |PW|^p (x_{jn}) \right\}^{\frac{1}{p}}. \tag{3.89}$$

Next, we state a result which shows that we cannot insert any positive power of $(1+|x|)$ inside the L_p norm in (3.88) at least when $\alpha > \frac{1}{p}$:

Theorem 45 *Let $W = e^{-Q} \in \mathfrak{E}_1$. Let $1 < p < 4$ and $\Delta \in \mathbb{R}$. Then the following are equivalent:*

(a) For every $\alpha > \frac{1}{p}$ and every continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying (3.87), we have

$$\lim_{n\to\infty} \left\| (f - L_n[f])(x) W(x) (1+|x|)^{\Delta} \right\|_{L_p(\mathbb{R})} = 0. \tag{3.90}$$

(b)

$$\Delta \leq 0. \tag{3.91}$$

Remark 6 *We note that with more effort, we can replace the continuity of* *.n Theorems* 43 *and* 45 *by Riemann integrability, and we can replace* $(1+|x|)^{\alpha}$, $\alpha > \frac{1}{p}$, *by* $(1+|x|)^{\frac{1}{p}} (\log(2+|x|))^{\frac{1}{p}}$, *some* $\varepsilon > 0$.

Finally, in Theorem 42 it was shown that even for f vanishing outside $[-2, 2]$ and $p > 4$, we needed $(1+Q)^{-\Delta}$ in (3.82) with

$$\Delta \geq \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right).$$

We now present an analogous result for $p = 4$:

Theorem 46 *Let* $W := e^{-Q} \in \mathfrak{E}_1$. *Suppose that a measurable function* $U : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$\lim_{x\to\infty} U(x) x^{-\frac{3}{4}} (\log Q(x))^{\frac{1}{4}} = \infty. \tag{3.92}$$

Then there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ vanishing outside $[-2, 2]$ such that

$$\limsup_{n \to \infty} \|L_n[f]\, WU\|_{[illegible](\mathbb{R})} = \infty.$$

Remark 7 *If $Q(x)$ grows faster than* $\exp(x^{3+\varepsilon})$*, for some $\varepsilon > 0$, then Theorem* 46 *shows that we cannot choose $U \equiv 1$ and hope for convergence. So there is no analogue of Theorem* 44 *for $p =$ [illegible] [illegible]wever, it seems that a negative power of* $\log Q$*, rather than the* $(1 +$ [illegible] [illegible]quired for $p > 4$*, will allow some analogue of Theorem* 41 *for $p = 4$.*

Bibliography

[1] N.I. Achieser, Theory of Approximation, Dover, New York, 1992.

[2] B. Bojanov, A Jackson Type Theorem for Chebyshev systems, Math. Balkania.

[3] S.S. Bonan, Weighted mean convergence of Lagrange Interpolation, Ph.D. thesis, Ohio State University, Columbus, OH, 1982.

[4] P. Borwein and T. Erdelyi, Polynomials and Polynomial Inequalities, Springer, Berlin, 1995.

[5] J. Clunie and T. Kovari, On integral functions having prescribed asymptotic growth II, Can. J. Math. 20(1968), 7-20

[6] S.B. Damelin, Converse and Smoothness Theorems for Erdös Weights in L_p $(0 < p \leq \infty)$, to appear in J. Approx. Theory.

[7] S.B. Damelin and D.S. Lubinsky, Jackson Theorems for Erdös Weights in L_p $(0 < p \leq \infty)$, to appear in J. Approx. Theory.

[8] S.B. Damelin and D.S. Lubinsky, Necessary and Sufficient conditions for Mean Convergence of Lagrange Interpolation for Erdös Weights, Can. J. Math. Vol. 48(4), (1996), 710-736.

[9] S.B. Damelin and D.S. Lubinsky, Necessary and Sufficient conditions for Mean Convergence of Lagrange Interpolation for Erdös Weights II, Can. J. Math. Vol. 48(4), (1996), 737-757.

[10] R. DeVore, D. Leviatan and X.M. Yu, Polynomial Approximation in L_p, $0 < p < 1$, Constructive Approximation 8(1992), 187-201.

[11] Z. Ditzian and D.S. Lubinsky, Jackson and Smoothness Theorems for Freud Weights in L_p $(0 < p \leq \infty)$, Constructive Approximation, 13(1992), 99-152.

[12] Z. Ditzian and V. Totik, Moduli of Smoothness, Springer Series in Computational Mathematics, Vol. 9, Springer, Berlin, 1987.

[13] M.M. Dzrbasjan, Some questions in the theory of weighted approximation on the entire real axis, Mat. Sb. 36(1955), 353-440.

[14] G. Freud, Orthogonal Polynomials, Pergamon Press/Academia Kiado, Budapest, 1970.

[15] G. Freud, Weighted polynomial approximation and K-functionals, in Theory of Approximation with Applications, A.G. Law et al. (eds.), Academic Press, New York, 1976, pp9-23.

[16] G. Freud, Markov-Bernstein Type Inequalities in $L_p(-\infty,\infty)$, in Approximation Theory II, G.G. Lorentz et al.(eds.), Academic Press, New York,1976, 369-377.

[17] G. Freud, On Markov-Bernstein Type Inequalities and Their Applications, J. Approx. Theory 19(1977), 22-37.

[18] G. Freud, On the greatest zero of an orthogonal polynomial, J. Approx. Theory 46(1986), 16-24.

[19] D. Jackson, The theory of Approximation, AMS Colloq. Publns. Vol. 11, American Mathematical Society, Providence, 1930.

[20] A. Knopfmacher and D.S. Lubinsky, Mean Convergence of Lagrange Interpolation for Freud's weights with Applications to product integration rules, J. Comput. Appl. Math. 17(1987), pp79-103.

[21] P. Koosis, The Logarithmic Integral I, Cambridge Advanced Studies in Mathematics, Vol. **12**, Cambridge University Press, Cambridge, 1992.

[22] P. Koosis, The Logarithmic Integral II, Cambridge Advanced Studies in Mathematics, Vol. **21**, Cambridge Uinversity Press, Cambridge, 1992.

[23] A. Kroó and J. Szabados, Weighted Polynomial Approximation on the Real Line, J. Approx. Theory **83** (1995), 41-64.

[24] A.L. Levin and D.S. Lubinsky, Canonical products and the weights exp(-$|x|^{\alpha}$), $\alpha > 1$, J. Approx. Theory **49** (1987), 149-169.

[25] A.L. Levin and D.S. Lubinsky, Weights on the real line that admit good relative polynomial approximation, With Applications, J. Approx. Theory 49 (1987), 170-195.

[26] A.L. Levin and D.S. Lubinsky, L_{∞} Markov and Bernstein Inequalities for Freud weights, SIAM J. Math. Anal. **21** (1990), 1065-1082.

[27] A.L. Levin and D.S. Lubinsky, Christoffel functions, Orthogonal polynomials, and Nevai's Conjecture for Freud Weights, Constructive Approximation 8 (1992), 463-535.

[28] A.L. Levin and D.S. Lubinsky, L_p Markov-Bernstein Inequalities for Freud Weights, J. Approx. Theory 77 (1994), 229-248.

[29] A.L. Levin and D.S. Lubinsky, Orthogonal Polynomials and Christoffel Functions for Exp($-|x|^{\alpha}$), $\alpha \leq 1$, J. Approx. Theory Vol. 80(2), (1995), 219-252.

[30] A.L. Levin, D.S. Lubinsky and T.Z. Mthembu, Christoffel functions and orthogonal polynomials for Erdös Weights on $(-\infty, \infty)$, Rendiconti di Mathematicae e delle sue Applicazioni (di Roma) **14** (1994), 199-289.

[31] G.G. Lorentz, Approximation of Functions, 2nd Edition, Chelsea, New York, 1986.

[32] D.S. Lubinsky, L_∞ Markov and Bernstein Inequalities for Erdös Weights, J. Approx. Theory 60 (1990), 188-230.

[33] D.S. Lubinsky, An Update on orthogonal polynomials and weighted Approximation on the Real line, Acta Applicandae Mathematicae 33 (1993), 121-164.

[34] D.S. Lubinsky, An Extension of the Erdös-Turan inequality for the sum of successive fundamental polynomials, Ann. Num. Math. 2 (1995), 305-309.

[35] D.S. Lubinsky, Ideas of Weighted Polynomial Approximation on $(-\infty, \infty)$, in Approximation Theory VIII, Vol. 1: Approximation and Interpolation, C.K. Chui and L L. Schumacher (eds.), World Scientific, Singapore, 1995, 371-396.

[36] D.S. Lubinsky, Jackson and Bernstein Theorems for Exponential Weights, in Proceedings of the Budapest Conference: Approximation and Function Series, Budapest, Hungary, 1995, 85-115.

[37] D.S. Lubinsky, Converse quadrature sum estimate inequalities for polynomials with Freud Weights, Acta Sci. Math. (Szeged) 60 (1995), 527-557.

[38] D.S. Lubinsky, Forward and Converse Theorems of Polynomial Approximation form Exponential Weights on [-1,1], I, J. Approx. Theory Vol. 91(1), (1997), 1-47.

[39] D.S. Lubinsky, The Weighted L_p norms of orthonormal polynomials for Erdös Weights, Comput. Math. Appl.

[40] D.S. Lubinsky and D.M. Matjila, Necessary and Sufficient conditions for Mean Convergence of Lagrange interpolation for Freud Weights, SIAM J. Math. Anal. Vol. **26** (1), (1995), 238-262.

[41] D.S. Lubinsky and F. Moricz, The Weighted L_p -norms of orthogonal polynomials for Freud Weights, J. Approx. Theory Vol. **77**(1), (1994), 42-50.

[42] D.S. Lubinsky and T.Z. Mthembu, L_p Markov-Bernstein inequalities for Erdös Weights, J. Approx. Theory **65** (1991), 301-321.

[43] D.S. Lubinsky and T.Z. Mthembu, Orthogonal expansions and the error of weighted polynomial approximation for Erdös Weights, Numer. Func. Anal. and Optimiz. **13** (1992), 327-347.

[44] D.S. Lubinsky and T.Z. Mthembu, Mean Convergence of Lagrange Interpolation for Erdös Weights, J. Comput. Appl. Math. **47** (1993), 369-390.

[45] D.S. Lubinsky and P. Nevai, Markov-Bernstein Inequalities revisited, Chinese J. Approx. Theory and its Applications **3** (1987), 98-119.

[46] D.S. Lubinsky and V.Totik, Best Weighted Polynomial Approximation via Jacobi Expansions, SIAM J. Math. Anal. **25** (1994), 555-570.

[47] H.N. Mhaskar, Introduction to the Theory of Weighted Polynomial Approximation, Series in Approximations and Decompositions Vol. 7, World Scientific, Singapore, 1996.

[48] H.N. Mhaskar and E.B. Saff, Extremal Problems in Exponential Weights, Trans. Amer. Soc. 285 (1984), 203-234.

[49] H.N. Mhaskar and E.B. Saff, Where does the sup-norm of a Weighted polynomial live? Constructive Approximation 1 (1985), 71-91.

[50] H.N. Mhaskar and E.B. Saff, Where does the L_p norm of a Weighted Polynomial live? Trans. Amer. Math. Soc. 303 (1987), 109-124.

[51] T.Z. Mthembu, Bernstein and Nikolskii Inequalities for Erdös Weights, J. Approx. Theory 75 (1993), 214-235.

[52] P. Nevai, Lagrange Interpolation at zeros of orthogonal polynomials, in Approximation Theory II, G.G. Lorentz et al. (eds.), Academic Press, New York, 1976, 163-201.

[53] P. Nevai, Mean Convergence of Lagrange Interpolation II, J. Approx. Theory 30 (1980), 263-276.

[54] P. Nevai, Geza Freud, Orthogonal Polynomials and Christoffel Functions: A Case Study, J. Approx. Theory 48 (1986), 3-167.

[55] P. Nevai and V. Totik, Weighted Polynomial Inequalities, Constructive Approximation 2 (1986), 113-127.

[56] J.A. Shohat and J.D. Tamarkin, The Problem of Moments, Mathematical Surveys, No.1, AMS, New York, 1943.