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STRONG COMPLETENESS OF A CLASS OF $L^2(T)$-TYPE RIESZ SPACES

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Abstract. The $L^p, 1 \leq p \leq \infty$, spaces have been generalized to the setting of Riesz spaces as $L^p(T)$ spaces, on which there are $R(T)$-valued norms. The strong sequential completeness of the space $L^1(T)$ and the strong completeness of $L^\infty(T)$ with resepct to their respective $R(T)$-valued norms were established by Kuo, Rodda, and Watson. In the current work, the T-strong completeness of $L^2(T)$ is established via the Riesz–Fischer type theorem given by Kalauch, Kuo, and Watson. It is also shown that the conditional expectation operator T is a weak order unit for the T-strong dual.

1. Introduction

Strong convergence was generalized to Dedekind complete Riesz spaces with a conditional expectation operator in [3] as T-strong convergence. Generalized L^p spaces were discussed in the setting of Riesz spaces as $L^p(T)$ spaces with $R(T)$-valued norms in [4, 13, 14]. For T acting on $L^1(T)$ we have that $R(T)$ is a universally complete f-algebra and that the $L^p(T)$ spaces are $R(T)$-modules. We also refer the reader to [20] for an interesting study of sequential order convergence in vector lattices using convergence structures and filters. In [12], the T-strong sequential completeness of $L^1(T)$ was established, i.e., each T-strong Cauchy sequence in $L^1(T)$ converges T-strongly in $L^1(T)$. The term T-strong convergence here means convergence with respect to the $R(T)$-valued norm induced by the conditional expectation operator T in the given space.

Definition 1.1. We say that a net (f_α) in $L^p(T)$, where $p \in [1, \infty]$, is a T-strong Cauchy net if

$$v_\alpha := \sup_{\beta,\gamma \geq \alpha} \|f_\beta - f_\gamma\|_{T,p}$$

is eventually defined (i.e., for $\alpha \geq \alpha^*$ for some α^* in the index set of the net) in $R(T)$ and has order limit zero.

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It should be noted that Definition 1.1 is equivalent to requiring that the net $(\|f_\beta - f_\gamma\|_{T,p})$ as indexed by (β,γ) with componentwise directedness, converges to 0 in order. We are now in a position to give the definition of strong completeness.

Definition 1.2. We say that $L^p(T)$ is strongly complete if each strong Cauchy net (f_α) in $L^p(T)$ is strongly convergent in $L^p(T)$, i.e., there is $f \in L^p(T)$ so that

$$w_\alpha := \sup_{\beta\geq\alpha} \|f_\beta - f\|_{T,p}$$

is eventually defined and has order limit zero.

It was shown in [12] that $L^\infty(T)$ is T-strongly complete. In the current work, the T-strong completeness of $L^2(T)$ is established. This is proved via the Riesz–Fischer type theorem established in [10] where the duality is with respect to the T-strong dual of $L^2(T)$. It is also shown that the conditional expectation operator T is a weak order unit for the T-strong dual of $L^2(T)$. The issue of completeness of $L^2(T)$ is important in the theory of stochastic integrals in Riesz spaces, since these integrals are defined to be limits of Cauchy nets in $L^2(T)$; see, for example, [8]. The results also impact on the study of martingales and strong mixing processes in Riesz spaces; see [18, 19].

2. Preliminaries

Throughout this work, E will denote a Dedekind complete Riesz space with positive cone E_+ and weak order unit. By T being a conditional expectation operator on E, we mean that T is a linear positive order continuous projection on E which maps weak order units to weak order units and has range $R(T)$ closed with respect to order limits in E. This gives us at least one weak order unit, say, e, with $Te = e$, and that $R(T)$ is Dedekind complete, when considered as a subspace of E. By T being strictly positive, we mean that, if $f \in E_+$ and $f \neq 0$, then $Tf \in E_+$ and $Tf \neq 0$. For further details about conditional expectation operators in Riesz spaces, we refer the reader to [11]. Throughout this paper, T will denote a strictly positive conditional expectation operator on E. From [11], we can extend the definition of T from E^+ to its natural domain,

$$\mathrm{dom}(T) := \{f \in E^u_+ | \exists \text{ net } f_\alpha \uparrow f \text{ in } E^u, (f_\alpha) \subset E_+, Tf_\alpha \text{ bounded in } E^u\},$$

where E^u denotes the universal completion of E; see [15, p. 323] or [23]. The extension of T from E^+ to $\mathrm{dom}(T)$ will also be denoted by T and leads to the extension of T to a strictly positive conditional expectation operator T on the Riesz space

$$L^1(T) := \mathrm{dom}(T) - \mathrm{dom}(T) = \{f - g | f, g \in \mathrm{dom}(T)\}$$

by taking $Tf = Tf^+ - Tf^-$. We say that E is T-universally complete if $E = L^1(T)$. From the definition of $\mathrm{dom}(T)$, E is T-universally complete if and only if, for each upwards directed net $(f_\alpha)_{\alpha\in\Lambda}$ in E_+ such that $(Tf_\alpha)_{\alpha\in\Lambda}$ is order bounded in E^u, we have that $(f_\alpha)_{\alpha\in\Lambda}$ is order convergent in E. E^u has an f-algebra structure which can be chosen so that e is the multiplicative identity. For T acting on $E = L^1(T)$, $R(T)$ is a universally complete f-algebra and $L^1(T)$ is an $R(T)$-module. From [11, Theorem 5.3], T is an averaging operator, i.e., if $f \in R(T)$ and $g \in E$ then $T(fg) = fT(g)$. This prompts the definition of an $R(T)$ (vector valued) norm $\|\cdot\|_{T,1} := T|\cdot|$ on $L^1(T)$. The homogeneity is with respect to multiplication by elements of $R(T)_+$. The definition of the Riesz space $L^2(T) := \{f \in L^1(T) \mid f^2 \in L^1(T)\}$ was given in [14]. By the averaging property, $L^2(T)$ is an $R(T)$-module, (see [13, Theorem 2.2]) and the map

$$f \mapsto \|f\|_{T,2} := (T(f^2))^{1/2}, \quad f \in L^2(T),$$

is an $R(T)$-valued norm on $L^2(T)$. Aspects of this development for $L^p(T)$ with general $1 < p < \infty$ can be found in [4, 7]. The functional calculus used here is inherited from the f-algebra E^u. Proofs of various Hölder type inequalties in Riesz spaces with conditional expectation operators can be found in [4, 7] and some special cases in [13]. In particular,

$$T|fg| \leq \|f\|_{T,2}\|g\|_{T,2}, \text{ for all } f, g \in L^2(T). \tag{2.1}$$

For $f \in E$, where E is a Dedekind complete Riesz space with weak order unit, we denote by $\mathcal{B}_f$ the band in E generated by f, and by P_f the band projection from E onto $\mathcal{B}_f$. If F is a universally complete Riesz space with weak order unit, say, e, then F is an f-algebra, and e can be taken as the algebraic unit; see [21, Theorem 3.6]. Recall from [10] the following material on partial inverses in Riesz spaces.

Definition 2.1. Let F be a universally complete Riesz space with weak order unit, say e, and take e as the algebraic unit of the associated f-algebra structure. We say that $g \in F$ has a partial inverse if there exists $h \in F$ such that $gh = hg = P_{|g|}e$. We refer to h as the canonical partial inverse of g if, in addition to being a partial inverse to g, we have that $(I - P_{|g|})h = 0$, i.e., $h \in \mathcal{B}_{|g|}$.

The following result from [10], based on [9, Theorem 5] and [17, Remark 3.3], gives existence, uniqueness, and positivity results for partial inverses and canonical partial inverses.

Theorem 2.2. *Let F be a universally complete Riesz space with weak order unit, say, e, which we also take as the algebraic unit of the associated f-algebra structure. Each $g \in F$ has a unique canonical partial inverse, $h \in F$. If $g \in F_+$, then its canonical partial inverse, h, is in F_+.*

3. Dual spaces

Let $E = L^2(T)$. We say that a map $\mathfrak{f}\colon E \to R(T)$ is a T-linear functional on E if it is additive, $R(T)$-homogeneous and order bounded. We denote the space of T-linear functionals on E by E^* and call it the T-dual of E. Further, E^* is a partially ordered vector space with respect to the partial ordering

$$\mathfrak{f} \leq \mathfrak{g} \text{ if and only if } \mathfrak{f}(x) \leq \mathfrak{g}(x) \text{ for all } x \in E_+. \tag{3.1}$$

Let $\mathcal{L}_b(E, R(T))$ denote the space of order bounded linear operators from E into $R(T)$. We note that $E^* \subset \mathcal{L}_b(E, R(T))$, since $R(T)$-homogeneity implies real linearity. Further, as $R(T)$ is Dedekind complete, so is $\mathcal{L}_b(E, R(T))$, (see [2, p. 12]) and $\mathcal{L}_b(E, R(T)) = \mathcal{L}_r(E, R(T))$ where $\mathcal{L}_r(E, R(T))$ denotes the space of regular operators; see [2, page 10]. Following the notation of [2], we denote the space of order continuous operators in $\mathcal{L}_b(E, R(T))$ by $\mathcal{L}_n(E, R(T))$. By [2, p. 44], $\mathcal{L}_n(E, R(T))$ is a band in $\mathcal{L}_b(E, R(T))$, and since $\mathcal{L}_b(E, R(T))$ is Dedekind complete, so is $\mathcal{L}_n(E, R(T))$.

We call $\mathfrak{f} \in E^*$ T-strongly bounded if there is $k \in R(T)_+$ such that

$$|\mathfrak{f}(g)| \leq k\|g\|_{T,2}, \quad \text{for all } g \in E. \tag{3.2}$$

We denote the space of T-strongly bounded T-linear functionals on E by

$$\hat{E} := \{\mathfrak{f} \in E^* \mid \mathfrak{f}\ T\text{-strongly bounded }\}$$

and refer to it as the T-strong dual of E.

The Riesz–Kantorovich theorem [2, p. 12] characterizes suprema and infima in $\mathcal{L}_b(E, R(T))$. We now give a variant of this to characterization of suprema and infima in E^*. However, for this we require Lemma 3.1 (which is certainly known, but included here for completeness).

Lemma 3.1. *Let H be a Dedekind complete Riesz space and F be an order closed Riesz subspace of H, then F is Dedekind complete. Here, by order closed, we mean that if (f_α) is a net in F order convergent in H to f, then (f_α) is order convergent in F with order limit f.*

Proof. In order to show that F is Dedekind complete, we need to show that each upwards directed net (f_α) in F which is bounded above in F, by say g, has a supremum in F. As (f_α) is upwards directed and bounded above by g, considered in H, then (f_α) has supremum f in H, since H is Dedekind complete. Hence (f_α) converges in order in H, to f. We have that the order closedness of F now gives that (f_α) is order convergent to f in F. Hence f is the supremum of (f_α) in F, thus making F Dedekind complete. □

Theorem 3.2 (Riesz–Kantorovich)**.**

(i) *The space E^* is a Riesz space where the lattice operations are given by*

$$(\mathfrak{f} \vee \mathfrak{g})(u) := \sup\{\mathfrak{f}(v) + \mathfrak{g}(w) \mid v, w \in E_+, v + w = u\}, \tag{3.3}$$

$$(\mathfrak{f} \wedge \mathfrak{g})(u) := \inf\{\mathfrak{f}(v) + \mathfrak{g}(w) \mid v, w \in E_+, v + w = u\} \tag{3.4}$$

every $\mathfrak{f}, \mathfrak{g} \in E^*$ and $u \in E_+$.

(ii) $\mathfrak{f}_\alpha \downarrow \mathfrak{o}$ *in E^* if and only if $\mathfrak{f}_\alpha(u) \downarrow 0$ in E for each $u \in E_+$.*

(iii) *E^* is Dedekind complete and an $R(T)$-module.*

Proof.

(i) That E^* is an $R(T)$-module is easily verified from its definition.

Let $\mathfrak{f}, \mathfrak{g} \in E^*$. As $E^* \subset \mathcal{L}_b(E, R(T))$, the Riesz–Kantorovich theorem of [2, p. 12] gives that

$$(\mathfrak{f} \vee_{\mathcal{L}_b(E,R(T))} \mathfrak{g})(u) := \sup\{\mathfrak{f}(v) + \mathfrak{g}(w) \mid v, w \in E_+, v + w = u\}, \tag{3.5}$$

$$(\mathfrak{f} \wedge_{\mathcal{L}_b(E,R(T))} \mathfrak{g})(u) := \inf\{\mathfrak{f}(v) + \mathfrak{g}(w) \mid v, w \in E_+, v + w = u\}. \tag{3.6}$$

Here $\mathfrak{f} \vee_{\mathcal{L}_b(E,R(T))} \mathfrak{g}$ denotes the supremum of $\mathfrak{f}$ and $\mathfrak{g}$ in the space $\mathcal{L}_b(E, R(T))$. As $E^* \subset \mathcal{L}_b(E, R(T))$, to prove the existence of $\mathfrak{f} \vee \mathfrak{g}$, and that $(\mathfrak{f} \vee_{\mathcal{L}_b(E,R(T))} \mathfrak{g}) = \mathfrak{f} \vee \mathfrak{g}$, it suffices to show that $(\mathfrak{f} \vee_{\mathcal{L}_b(E,R(T))} \mathfrak{g})$ is $R(T)$-homogeneous. This in turn gives (3.3). The same applies for the infimum.

For $\alpha \in R(T)_+$ and $x \in E_+$, we have

$$(\mathfrak{f} \vee \mathfrak{g})(\alpha u) := \sup\{\mathfrak{f}(v) + \mathfrak{g}(w) \mid v, w \in E_+, v + w = \alpha u\}.$$

If $v, w \in E_+$ and $v + w = \alpha u$, then $v, w \in B_\alpha$, where B_α is the band in E generated by α.

Now, as $R(T)$ is universally complete, by Theorem 2.2, α has a partial inverse $\beta \in R(T)_+ \cap B_\alpha$ such that

$$\alpha\beta = \beta\alpha = P_\alpha e$$

where P_α denotes the band projection onto the band B_α. Thus $\beta v + \beta w = P_\alpha u$ and $\alpha\beta v = v$ and $\alpha\beta w = w$. Hence

$$\begin{aligned}(\mathfrak{f} \vee \mathfrak{g})(\alpha u) &= \sup\{\alpha(\mathfrak{f}(\beta v) + \mathfrak{g}(\beta w)) \mid v, w \in E_+, \beta v + \beta w = P_\alpha u\}\\ &\le \sup\{\alpha(\mathfrak{f}(v') + \mathfrak{g}(w')) \mid v', w' \in E_+, v' + w' = P_\alpha u\}\\ &\le \sup\{\alpha(\mathfrak{f}(v') + \mathfrak{g}(w') \mid v', w' \in E_+, v' + w' = u\}\\ &= \alpha(\mathfrak{f} \vee \mathfrak{g})(u)\end{aligned}$$

and

$$\begin{aligned}\alpha(\mathfrak{f} \vee \mathfrak{g})(u) &= \sup\{\alpha(\mathfrak{f}(v) + \mathfrak{g}(w)) \mid v, w \in E_+, y + w = u\}\\ &= \sup\{(\mathfrak{f}(\alpha v) + \mathfrak{g}(\alpha w)) \mid v, w \in E_+, v + w = u\}\\ &\le \sup\{(\mathfrak{f}(\alpha v) + \mathfrak{g}(\alpha w)) \mid v, w \in E_+, \alpha v + \alpha w = \alpha u\}\\ &\le \sup\{(\mathfrak{f}(v') + \mathfrak{g}(w')) \mid y', z' \in E_+, v' + w' = \alpha u\}\\ &= (\mathfrak{f} \vee \mathfrak{g})(\alpha u).\end{aligned}$$

Thus $(\mathfrak{f} \vee \mathfrak{g})(\alpha u) = \alpha(\mathfrak{f} \vee \mathfrak{g})(u)$ for $\alpha \in R(T)_+$ and $u \in E_+$. The analogous result holds for $\mathfrak{f} \wedge \mathfrak{g}$ by a similar process. As $(\mathfrak{f} \vee \mathfrak{g})(-u) = -(\mathfrak{f} \wedge \mathfrak{g})(u)$, the homogeneity follows for all $u \in E$ and $\alpha \in R(T)$. Thus E^* is a sublattice of $\mathcal{L}_b(E, R(T))$.

(ii) We note from [2, Theorem 1.13] that $f_\gamma \downarrow 0$ in $\mathcal{L}_b(E, R(T))$ if and only if $f_\gamma(x) \downarrow 0$ in $R(T)$ for each $x \in E_+$.

(iii) Finally, we show that E^* is Dedekind complete. As $\mathcal{L}_b(E, R(T))$ is Dedekind complete, it suffices, by Lemma 3.1, to show that E^* is order closed in $\mathcal{L}_b(E, R(T))$.

If $(\mathfrak{f}_\gamma)$ is a net in E^* with order limit $\mathfrak{f}$ in $\mathcal{L}_b(E, R(T))$ then for each $\alpha \in R(T)$ we have that $\mathfrak{f}_\gamma(\alpha u) = \alpha\mathfrak{f}_\gamma(u)$ for each $u \in E$. Further, by order continuity of multiplication by elements of $R(T)$ it follows that the net $(\alpha\mathfrak{f}_\gamma)$ has order limit $\alpha\mathfrak{f}$. However, for each $u \in E$, $(\mathfrak{f}_\gamma(\alpha u))$ has order limit $\mathfrak{f}(\alpha u)$, and $(\alpha\mathfrak{f}_\gamma(u))$ has order limit $\alpha\mathfrak{f}(u)$ in $R(T) \subset E$. Thus, $\mathfrak{f}(\alpha u) = \alpha\mathfrak{f}(u)$, giving $\mathfrak{f} \in E^*$. $\square$

As in the scalar case, we can define a norm on $\hat{E}$. However, in the case under consideration here, this is an $R(T)$-valued norm or, in the notation of [13], a T-norm.

Lemma 3.3. *Let* $\mathfrak{f} \in \hat{E}$.

(i) *There is an* $R(T)$*-valued norm on* $\hat{E}$ *given by*

$$\|\mathfrak{f}\| := \inf\{k \in R(T)_+ \mid |\mathfrak{f}(g)| \le k\|g\|_{T,2} \text{ for all } g \in E\}. \tag{3.7}$$

(ii) *For each* $g \in E = L^2(T)$,

$$|\mathfrak{f}(g)| \le \|\mathfrak{f}\|\,\|g\|_{T,2}. \tag{3.8}$$

(iii) *The map* $g \mapsto \|g\|_{T,2}$ *is order continuous, and* $\hat{E} \subset E^* \cap \mathcal{L}_n(E, R(T))$.

Proof.

(i) Let $\mathfrak{f} \in \hat{E}$. For each $g \in E$, let

$$Y_g(\mathfrak{f}) := \{k \in R(T)_+ \mid |\mathfrak{f}(g)| \le k\|g\|_{T,2}\}.$$

Then $Y_g(\mathfrak{f})$ is closed with respect to order limits in E, convex, and additive.

$$Y(\mathfrak{f}) := \bigcap_{g \in E} Y_g(\mathfrak{f})$$

is convex, additive and closed with respect to order limits, and infima in $R(T)_+$. Thus $Y(\mathfrak{f})$ has a unique minimal element in $R(T)_+$, which is given by (3.7). As $\|\mathfrak{f}\| \in Y(\mathfrak{f})$, (3.8) holds.

(ii) It is a routine computation to verify that $\mathfrak{f} \mapsto \|\mathfrak{f}\|$ is indeed an $R(T)$-valued norm.

(iii) The domination in (3.2) gives that each $\mathfrak{f} \in \hat{E}$ is order continuous and, hence, it follows that $\hat{E} \subset E^* \cap \mathcal{L}_n(E, R(T))$. □

Lemma 3.4. *The space $\hat{E}$ is an $R(T)$-module and a Dedekind complete Riesz subspace of E^*.*

Proof. From its definition, it is clear that $\hat{E}$ is a vector subspace of E^* and an $R(T)$-module. It remains to prove that $\hat{E}$ is a sublattice of E^* and that it is Dedekind complete.

To show that $\hat{E}$ is a sublattice of E^*, we observe that, if $\mathfrak{f}, \mathfrak{g} \in \hat{E}$, then there exist $k_{\mathfrak{f}}, k_{\mathfrak{g}} \in R(T)_+$ so that $|\mathfrak{f}(u)| \le k_{\mathfrak{f}}\|u\|_{T,2}$ and $|\mathfrak{g}(u)| \le k_{\mathfrak{g}}\|u\|_{T,2}$, for all $u \in E$. From Theorem 3.2,

$$|(\mathfrak{f} \vee \mathfrak{g})(u)| \le \sup\{|\mathfrak{f}(v) + \mathfrak{g}(w)| \mid v, w \in E_+, v + w = u\} \le (k_{\mathfrak{f}} + k_{\mathfrak{g}})\|u\|_{T,2}$$

for all $u \in E$, giving $\mathfrak{f} \vee \mathfrak{g} \in \hat{E}$.

For the Dedekind completeness, let $C \subset \hat{E}_+$ be a nonempty set bounded above by $\mathfrak{g} \in \hat{E}_+$. Then $C \subset E^*_+$, and C is bounded above by $\mathfrak{g} \in E^*$. Thus $\mathfrak{h} := \sup C$ exists in E^*, as E^* is Dedekind complete. Further, $\mathfrak{h} \le \mathfrak{g}$, so there exists $k \in R(T)_+$ so that

$$|\mathfrak{h}(u)| \le |\mathfrak{g}(u)| \le k\|u\|_{T,2}, \text{ for every } u \in E.$$

Thus $\mathfrak{h} \in \hat{E}$. If $\hat{\mathfrak{h}} \in \hat{E}$ were an upper bound of C in $\hat{E}$ then $\hat{\mathfrak{h}}$ is also an upper bound of C in E^* making $\mathfrak{h} \le \hat{\mathfrak{h}}$, thus $\mathfrak{h}$ is the least upper bound of C. □

Combining [10, Proposition 4.1] and [10, Theorem 4.3], we have the following Riesz–Frechet representation theorem.

Theorem 3.5 (Riesz–Frechet representation theorem in Riesz space)**.** *For $E = L^2(T)$ and its T-strong dual, $\hat{E}$, the map*

$$\Psi\colon E \to \hat{E} \quad \textit{with} \quad \Psi(f) = T_f, \textit{ for all } f \in E, \tag{3.9}$$

is a bijection. Here, $T_f(g) := T(fg)$ for $f, g \in E$. The map Ψ is additive, $R(T)$-homogeneous, and $R(T)$-valued norm preserving in the sense that $\|\Psi(f)\| = \|f\|_{T,2}$ for all $f \in E$.

Theorem 3.6. *$T = \Psi(e)$ is a weak order unit for $\hat{E}$.*

Proof. Let $\mathfrak{f} \in \hat{E}_+$. For $g \in L^2(T)$ with $g \ge 0$, from Theorem 3.2, we have

$$\begin{aligned}
\sup_{n\in\mathbb{N}}(\mathfrak{f} \wedge nT)(g) &= \bigvee_{n\in\mathbb{N}} \inf\{\mathfrak{f}(u) + nT(v) | u + v = g,\, u, v \ge 0\}\\
&= \bigvee_{n\in\mathbb{N}} \inf\{\mathfrak{f}(g) - \mathfrak{f}(v) + nT(v) | 0 \le v \le g\}\\
&= \mathfrak{f}(g) + \bigvee_{n\in\mathbb{N}} \inf\{nT(v) - \mathfrak{f}(v) | 0 \le v \le g\}.
\end{aligned}$$

From Theorem 3.5, $\mathfrak{f}(v) = T(v\Psi^{-1}(\mathfrak{f}))$, from which it follows that

$$nT(v) - \mathfrak{f}(v) = T((ne - \Psi^{-1}(\mathfrak{f}))v).$$

Thus

$$\sup_{n\in\mathbb{N}}(\mathfrak{f} \wedge nT)(g) = \mathfrak{f}(g) + \bigvee_{n\in\mathbb{N}} \inf \left\{T((ne - \Psi^{-1}(\mathfrak{f}))v)|0 \le v \le g\right\}.$$

However,

$$-(ne - \Psi^{-1}(\mathfrak{f}))^- \le ne - \Psi^{-1}(\mathfrak{f})$$

and $-(ne - \Psi^{-1}(\mathfrak{f}))^- \le 0$ so, for $0 \le v \le g$, we have that

$$-g(ne - \Psi^{-1}(\mathfrak{f}))^- \le -v(ne - \Psi^{-1}(\mathfrak{f}))^- \le v(ne - \Psi^{-1}(\mathfrak{f})).$$

Thus

$$-T(g(ne - \Psi^{-1}(\mathfrak{f}))^-) \le \inf_{g\ge v\ge 0} T(v(ne - \Psi^{-1}(\mathfrak{f}))) \le 0.$$

As e is a weak order unit, so $(ne - \Psi^{-1}(\mathfrak{f}))^- \downarrow 0$ in order as $n \to \infty$, giving that $-T(g(ne - \Psi^{-1}(\mathfrak{f}))^-) \uparrow 0$ in order as $n \to \infty$. Hence

$$\bigvee_{n\in\mathbb{N}} \inf \left\{T((ne - \Psi^{-1}(\mathfrak{f}))v)|0 \le v \le g\right\} = 0$$

and

$$\sup_{n\in\mathbb{N}}(\mathfrak{f} \wedge nT)(g) = \mathfrak{f}(g).$$

Thus, T is a weak order unit for $\hat{E}$. □

Theorem 3.7. *Ψ, as defined in* (3.9), *is an order continuous vector lattice isomorphism between E and $\hat{E}$ with Ψ^{-1} order continuous.*

Proof. From Theorem 3.5, the map $\Psi\colon E \to \hat{E}$ is a linear bijection.

Let f_α be a net in E with order limit f and let $g \in E$, then

$$\Psi(f_\alpha)(g) = T(f_\alpha g) \to T(fg)$$

as pairwise multiplication is an order continuous map from $L^2(T) \times L^2(T) \to L^1(T)$, and T is order continuous on $L^1(T)$, see the Riesz space Hölder inequality in [4, Theorem 3.7] and [7, Corollary 6.4]. Hence Ψ is order continuous.

Suppose that $(\mathfrak{f}_\alpha) \subset \hat{E}_+$ is a downwards directed net with $\mathfrak{f}_\alpha \downarrow 0$. Then $(\Psi^{-1}(\mathfrak{f}_\alpha)) \subset E_+$ is a downwards directed net with $\Psi^{-1}(\mathfrak{f}_\alpha) \downarrow h \ge 0$. So, for $g \in E_+$, we have

$$T(\Psi^{-1}(\mathfrak{f}_\alpha)g) \downarrow T(hg)$$

from the order continuity of the multiplication and of T. However,

$$T(\Psi^{-1}(\mathfrak{f}_\alpha)g) = \mathfrak{f}_\alpha(g) \downarrow 0.$$

Thus $T(hg) = 0$ for all $g \in E_+$, in particular for $g = h$. We obtain $T(h^2) = 0$, which with the strict positivity of T gives $h^2 = 0$, and thus $h = 0$. Hence, Ψ^{-1} is order continuous.

It remains to show that Ψ preserves lattice operations. We prove preservation of pairwise supremum. Let $f, g \in E$, then for $u \in E_+$ we have, from Theorem 3.2, that

$$\begin{aligned}(\Psi(f) \vee \Psi(g))(u) &= \sup\{T((fv + g(u - v)) \mid 0 \le v \le u\} \\ &= T(gu) + \sup\{T((f - g)v) \mid 0 \le v \le u\} \\ &= T(gu) + T((f - g)^+ u) \\ &= T((g + (f - g)^+)u).\end{aligned}$$

However $g + (f - g)^+ = f \vee g$, see [24, page 18, equation (1)], so

$$(\Psi(f) \vee \Psi(g))(u) = T((f \vee g)u) = \Psi(f \vee g)(u)$$

giving $\Psi(f) \vee \Psi(g) = \Psi(f \vee g)$. □

Note 3.8. Combining the facts that Ψ is an order preserving map that maps e to T it follows directly that Ψ is a lattice isomorphism between the components of e in E and the components of T in $\hat{E}$.

The above results, when combined with the ideas of abstract integration in Riesz spaces (with scalar field replaced by the commutative ring $R(T)$) of Nakano, [16, pp. 445-453], and Luxemburg and Zaanen, [15, pp. 257-260], give the following Riesz representation of Radon measures.

Theorem 3.9. *Each $\mathfrak{f} \in \hat{E}$ is an $R(T)$-valued Radon measure $\mu_\mathfrak{f}$ defined on the σ-algebra, C_e, of components of e in E by $\mu_\mathfrak{f}(p) := \mathfrak{f}(p)$. The Radon–Nikodým derivative of $\mu_\mathfrak{f}$ with respect to μ_T is $\Psi^{-1}(\mathfrak{f})$.*

Proof. Let $f = \Psi^{-1}(\mathfrak{f})$. We begin with case of

$$f = \sum_{k=1}^{n} g_k p_k,$$

where $p_k \in C_e$ with $p_k p_j = 0$ for $k \neq j$ and

$$e = \sum_{k=1}^{n} p_k.$$

Then, for each $p \in C_e$,

$$\mu_\mathfrak{f}(p) = \mathfrak{f}(p) = T(fp) = \sum_{k=1}^{n} g_k T(p_k p) = \sum_{k=1}^{n} g_k \mu_T(p_k p) = \int_p f \, d\mu_T.$$

This can be extended via order limits to general $f \in E$. □

4. Completeness

We are now in a position to prove the strong completeness of $\hat{E}$ and, from this deduce the strong completeness of E.

Theorem 4.1. *The dual space $\hat{E}$ is strongly complete.*

Proof. Let $(\mathfrak{f}_\alpha)$ be a strong Cauchy net in $\hat{E}$ and $x \in E = L^2(T)$, then

$$\|\mathfrak{f}_\alpha - \mathfrak{f}_\beta\| \to 0, \tag{4.1}$$

in order, in $R(T)$. Thus, eventually,

$$v_\alpha := \sup_{\beta,\gamma\geq\alpha} \|\mathfrak{f}_\beta - \mathfrak{f}_\gamma\| \tag{4.2}$$

exists as an element of $R(T)$, and, by (4.1), $v_\alpha \downarrow 0$.

From (3.8) and (4.2), we have

$$|\mathfrak{f}_\beta(x) - \mathfrak{f}_\gamma(x)| \leq \|\mathfrak{f}_\alpha - \mathfrak{f}_\beta\| \, \|x\|_{T,2} \leq v_\alpha \, \|x\|_{T,2}, \tag{4.3}$$

for $\beta, \gamma \geq \alpha$. Hence $(\mathfrak{f}_\alpha(x))$ is a Cauchy net in $R(T)$, for each $x \in E$.

Taking $\beta = \alpha$ in (4.3) gives

$$|\mathfrak{f}_\beta(x)| \leq |\mathfrak{f}_\beta(x) - \mathfrak{f}_\alpha(x)| + |\mathfrak{f}_\alpha(x)| \leq (\|\mathfrak{f}_\alpha\| + v_\alpha) \, \|x\|_{T,2}, \tag{4.4}$$

for each $\beta \geq \alpha$. Thus, for each x, $(\mathfrak{f}_\alpha(x))$ is a Cauchy net and is eventually bounded in $R(T)$. We may thus define $\underline{\mathfrak{f}}(x) := \liminf_\alpha \mathfrak{f}_\alpha(x)$, $\bar{\mathfrak{f}}(x) := \limsup_\alpha \mathfrak{f}_\alpha(x)$ in $R(T)$. Here

$$0 \leq \bar{\mathfrak{f}}(x) - \underline{\mathfrak{f}}(x) = \lim_\alpha (\sup_{\beta\geq\alpha} \mathfrak{f}_\beta(x) - \inf_{\gamma\geq\alpha} \mathfrak{f}_\gamma(x)) = \lim_\alpha \sup_{\beta,\gamma\geq\alpha} (\mathfrak{f}_\beta(x) - \mathfrak{f}_\gamma(x)). \tag{4.5}$$

Now, applying (4.3) to (4.5) gives

$$0 \leq \bar{\mathfrak{f}}(x) - \underline{\mathfrak{f}}(x) \leq \lim_\alpha v_\alpha \|x\|_{T,2} = 0. \tag{4.6}$$

So, we can set $\mathfrak{f}(x) := \bar{\mathfrak{f}}(x) = \underline{\mathfrak{f}}(x)$, with $(\mathfrak{f}_\alpha(x))$ converging in order to $\mathfrak{f}(x) \in R(T)$, for each $x \in E$; see [3].

From the order continuity of addition and multiplication by elements of $R(T)$ and the $R(T)$-homogeneity and additivity of each $\mathfrak{f}_\alpha$ it follows that $\mathfrak{f}\colon L^2(T) \to R(T)$ is an $R(T)$-linear map. Taking the order limit in (4.4) with respect to β gives

$$|\mathfrak{f}(x)| \leq (\|\mathfrak{f}_\alpha\| + v_\alpha)\|x\|_{T,2}$$

for some α, and all $x \in L^2(T)$, from which we have that $\mathfrak{f} \in \hat{E}$. Taking the order limit with respect to γ in (4.3) yields

$$|\mathfrak{f}_\beta(x) - \mathfrak{f}(x)| \leq v_\alpha \, \|x\|_{T,2}, \tag{4.7}$$

for all $\beta \geq \alpha$, and $x \in E$. Hence

$$\|\mathfrak{f}_\beta - \mathfrak{f}\| \leq v_\alpha, \qquad \text{for all } \beta \geq \alpha,$$

giving that $(\mathfrak{f}_\alpha)$ strongly converges to $\mathfrak{f}$. □

Theorem 4.2. *$L^2(T)$ is strongly complete.*

Proof. The map $\Psi\colon y \mapsto T_y$, as defined in Theorem 3.5, is an $R(T)$-linear $R(T)$-valued vector norm preserving bijection between $L^2(T)$ and $\hat{E}$. Given a strong Cauchy net (y_α) in $L^2(T)$, $(\Psi(y_\alpha))$ is a strong Cauchy net in $\hat{E}$ and is thus strongly convergent to some $\mathfrak{f} \in \hat{E}$. As Ψ is a bijection, there is $y \in L^2(T)$ so that $\mathfrak{f} = \Psi(y) = T_y$. Due to the $R(T)$-linearity of Ψ and this map being $R(T)$-valued vector norm preserving, the strong convergence of $(\Psi(y_\alpha))$ to $\mathfrak{f}$ in $\hat{E}$ yields the strong convergence of (y_α) to y in $L^2(T)$. □

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