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Difference equations and inverse problems

Moyo, Donald

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Difference equations and inverse problems

Student Name: Donald Moyo

Student Number: 567121

Supervisor: Professor S. Currie

Masters Dissertation



School of Mathematics

Dedication

To my late mom, I know you would be proud of who I have turned out to be.

Declaration

I declare that the work in this dissertation is my own work, that it has not been submitted before to any university for assessment. I further declare that I have acknowledged all sources and have cited them under refecences.

Acknowledgements

I want to thank Professor Currie for all the hard work she put in to make this possible. I'd like to thank all of my family for all the support throughout.

Contents

1 Introduction **5**
1.1 Recurrence relations and orthogonal polynomials 8
1.2 Transformations, factorizations and hierarchies 11
1.3 Inverse problems and recurrence relations 14
1.4 Preliminaries 17

2 Transformations - Cubic λ-dependence at initial end **26**

3 Transformations - Cubic λ-dependence at terminal end **54**

4 Inverse Problems: Reconstructing the potential with cubic λ-dependence at initial end **76**

5 Inverse Problems: Reconstructing the potential with cubic λ-dependence at terminal end **94**

6 Reconstructing *b(n)*: Generalisation of results. **107**
6.1 Cubic dependence at the initial end 107
6.2 Cubic dependence at the terminal end 117

7 Conclusion **127**

8 Appendix **129**

Chapter 1

Introduction

What are difference equations? When do we use them? How do we solve them? What limitations do we have in using them? All the above questions have been asked by yester-mathematicians who have provided us with a plethora of answers, each with great impact on our perception of the difference equations phenomena.

Difference equations, or sometimes called recurrence relations, are of particular importance in science, especially in physics, medicine, and engineering. Even in actuarial work, difference equations play an important role. Finance as well as applied mathematics fields also make use of them. One of the classical problems in the history of difference equations is The Tower of Hanoi (sometimes referred to as the Tower of Brahma or the End of the World Puzzle), invented by the French mathematician, Edouard Lucas, in 1883. He was inspired by a legend that tells of a Hindu temple where the pyramid puzzle might have been used for the mental discipline of young priests. Legend says that at the beginning of time the priests in the temple were given a stack of 64 gold disks, each one a little smaller than the one beneath it. Their assignment was to transfer the 64 disks from one of the three poles to another, with one important proviso – a large disk could never be placed on top of a smaller one. The priests worked very efficiently, day and night. When they finished their work, the myth said, the temple would crumble into dust and the world would vanish [1].

The mathematical puzzle is: "What is the least number of moves required to move n disks from the first pole to the last pole according to the rules given in the last paragraph?" Let t_n represent this number. We desire to obtain some recurrence relation for this sequence. If we have $n+1$ disks, we cannot move the bottom disk off the first pole until we have moved all the others off of it onto the other poles. So if we move the top n disks off to the second pole we need t_n moves. Then we can

move the bottom disk to the third pole, requiring one move. We then move the n disks from the second pole onto the third pole, requiring another t_n moves. Thus

$$t_{n+1} = t_n + 1 + t_n = 2t_n + 1.$$

The change in t_n i.e. $(t_{n+1} - t_n)$ is defined as $\triangle t_n$ and thus subtracting t_n from both sides of this equation gives

$$\Delta t_n = t_n + 1.$$

Hence $t_n = 2^n - 1$.

This can be proven iteratively as follows:

Since

$$t_{n+1} = 2t_n + 1, \tag{1.1}$$

then

$$t_n = 2t_{n-1} + 1. \tag{1.2}$$

Substituting (1.2) into (1.1) we get

$$t_{n+1} = 4t_{n-1} + 2^1 + 2^0. \tag{1.3}$$

Now

$$t_{n-1} = 2t_{n-2} + 1. \tag{1.4}$$

Substituting (1.4) into (1.3) we obtain

$$t_{n+1} = 8t_{n-2} + 2^2 + 2^1 + 2^0.$$

Continuing as above we observe a pattern which we may write in terms of t_1 as

$$t_n = 2^n t_1 + 2^{n-1} + 2^{n-2} + ... + 2^2 + 2^1 + 2^0,$$

which is a geometric progression with first term 1 and common ratio 2.

Using the fact that $t_1 = 1$ and the geometric sum formula we get

$$t_n = 1\left(\frac{2^n - 1}{2 - 1}\right) = 2^n - 1.$$

Legend said that the priests in the temple had 64 disks to work with. So the number of moves required to move these disks from the first pole to the third pole is therefore

$t_{64} = 18,446,744,073,709,551,615.$

There are many other applications of recurrence relations, see Section 1.3. Difference equations are often used as an aid to the study of their continuous counterparts differential equations. This is particularly seen in the examples discussed in Atkinson's book [5]. In this book some real world applications of difference equations are discussed in detail. We touch on two of them here. Consider the three-term recurrence

$$-c(n)y(n+1) + b(n)y(n) - c(n-1)y(n-1) = c(n)\lambda y(n), \tag{1.5}$$

where $y(n)$ is the quantity being modelled e.g growth of a disease, $c(n) > 0$ is a weight function and $b(n)$ a potential function. A weight function is a function that gives some elements in a set more influence on the result of a calculation than other elements in the set while a potential function is a function which when differentiated will give a physical quantity e.g force. There are various applications of (1.5), and we mention a few here. We look firstly at the vibrating string problem. In [5] the string is taken to be weightless and bears m particles $p(0), ..., p(m-1)$ at the points say $x(0), ..., x(m-1)$ with masses $c(0), ..., c(m-1)$ and distances between them given by $x(r+1) - x(r) = 1/c(r)$, $r = 0, ..., m-2$. Beyond $c(m-1)$ the string extends to a length $1/c(m-1)$ and beyond $c(0)$ to a length $1/c(-1)$. The string is stretched to unit tension. If $s(n)$ is the displacement of the particle p_n at time t, the restoring forces on it due to the tension of the string are $c(n-1)(s(n) - s(n-1))$ and -$c(n)(s(n+1) - s(n))$ considering small oscillations only. The solutions must be of the form $s(n) = y(n)\cos(\omega t)$ where $y(n)$ is the amplitude of oscillation of the particle p_n. Solving for $y(n)$ reduces to solving a difference equation of the form (1.5). The second application is in electrical networking. The system of equations is similar to that of the vibrating spring with $c(n)$ representing inductances, $1/c(n)$ capacitances and $s(n)$ the loop currents in the successive meshes. See Section 1.3 for more on applications of difference equations.

1.1 Recurrence relations and orthogonal polynomials

Let $w(x)$ be a nonnegative weight function at infinitely many points in the interval (-1, 1), and is integrable on this interval. Then the inner product of f and g on the semicircle Γ is given by:

$$(f,g) = \int_{\Gamma} f(t)g(t)w(t)(it)^{-1}dt = \int_0^{\pi} f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})d\theta \tag{1.6}$$

where $i = \sqrt{-1}$ and Γ is the upper unit semicircle in the complex plane.

Definition 1.1.1. *The Jacobi polynomials of degree n are defined by*

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\big((1-x)^{n+\alpha}(1+x)^{n+\beta}\big)$$

where $\alpha, \beta > -1$ *and* $-1 < x < 1$.

These univariate Jacobi polynomials defined above satisfy the three-term recurrence:

$$p_{n+1}(x) = (G_n x + H_n)p_n - J_n p_{n-1}(x), \qquad n \geq 0, \qquad p_{-1} = 0.$$

Definition 1.1.2. *The multivariate Jacobi polynomials* $p_{\zeta^\alpha}^{\kappa}$ *are defined by*

$$p_{\zeta^\alpha}^{\kappa} = \frac{(-1)^{|\alpha|}(\kappa+1)_\alpha}{(|\alpha|+|\kappa|+d)_{|\alpha|}} \sum_{\beta \leq \alpha} \frac{(|\alpha|+|\kappa|+d)_{|\beta|}(-\alpha)_\beta \zeta^\beta}{\beta!(\kappa+1)_\beta},$$

where $|\alpha| = n$.

The multivariate Jacobi polynomials satisfy the recurrence relation:

$$p_{\zeta_x^{n+1}} = (\zeta_x + b_{x,n})p_{\zeta_x^n} - c_{x,n}p_{\zeta_x^{n-1}}, \qquad n \geq 1,$$

and $b_{x,n}$ and $c_{x,n}$ are unique.

Given a suitable weight on $\mathbb{R}^d$, there exist many recursive three-term recurrence relations for the corresponding multivariate orthogonal polynomials. In [25], Waldron discusses how to obtain an explicit recursive three-term recurrence for the multivariate Jacobi polynomials on a simplex. He used the idea that some Jacobi polynomials

may be viewed as univariate Jacobi polynomials whose recurrence reduces to the univariate three-term recurrence.

Milovanović and Cvetković, in [38] proved that orthogonal polynomials with respect to an oscillatory weight function $w(x) = xe^{im\pi x}$ on [-1, 1] satisfy the three-term recurrence given by:

$$p_{n+1}(x) = (x - i\alpha_n)p_n(x) - \beta_n p_{n-1}(x),$$

$n = 0, 1, ...,$

with $p_0(x) = 1$ and $p_{-1}(x) = 0$.

On the other hand, Gautschi and Milovanović in [29] introduced and further investigated polynomials on the semicircle $\Gamma = \{z \in \mathbb{C} \,|\, z = e^{i\theta},\ 0 \leq \theta \leq \pi\}$ i.e. orthogonal polynomials relative to the inner product (1.16). Orthogonal polynomials with respect to a complex weight function were considered in [28], while de Bruin provided a generalization of such polynomials on a circular path in [21], with Milovanović and Rajković further investigating these classes of orthogonal polynomials in [39].

Golub and Fischer in [43] looked at the problem of generating the three-term recursion coefficients of orthogonal polynomials for a weight function $v(t) = \tau(t)w(t)$, obtained by modifying a given weight function $w(t)$ by a rational function τ(t). This is interesting because it then implies that any weight function can be modified using a suitable rational function so that a three-term relation may be obtained for any set of orthogonal polynomials. Gautschi in [27] worked on the minimal solution of the three-term recurrence

$$y_{n+1} = (z - a_n)y_n - b_n y_{n-1}, \quad n = 1, 2, ..., b_n > 0.$$

He observed that for any z outside the spectrum of $ds(t)$, where $ds(t)$ is the mass distribution on the real line, the recurrence relation (2.2) has a readily accessible minimal solution whenever the moment problem for $ds(t)$ is determined. There has a been a vast amount of research done on orthogonal polynomials. One such area of research was the work of Zarzo and Godoy in [41].

They considered the elementary procedure of finding recurrence relations between the coefficients $C_m(n)$ when both families of orthogonal polynomials $\{P_n(x)\}_{n\in N}$ and $\{Q_n(x)\}_{n\in N}$ belong to the classical classes of orthogonal polynomials namely, Bessel, Jacobi, Laguerre and Hermite, and are linked by the connection coefficients $C_m(n)$ in the following way:

$$P_n(x) = \sum_{m=0}^{n} C_m(n) Q_m(x). \tag{1.7}$$

The proof of (1.7) is discussed in [41].

These connection coefficients play an important role in many situations of pure and applied mathematics or in mathematical physics, where the nonnegative character of these coefficients has received particular attention in [3], [26] and [42]. The nonnegativity of the coefficients enables us to construct three-term recurrences in many problems. This is important as we study the three term recurrence in our work. An explicit presentation of recurrence relations satisfied by the connection coefficients $C_m(x)$ in (1.7) between two classes of classical orthogonal polynomials of a discrete variable was given in [35]. Lewanowicz in [34] stated two interesting cases for the orthogonal polynomial families $\{P_n\}$ and $\{Q_m\}$ in (1.7), and proposed an alternative way of determining the recurrence relation in (1.8). The two cases were:

1. $\{P_n\}$ is another sequence of classical orthogonal polynomials of a discrete variable;
2. $\{P_n\}$ is a sequence of orthogonal polynomials associated with $\{Q_m\}$

where in each case polynomial $\{P_n\}$ satisfies a linear difference equation in x. In recent work, Ronveaux et al. [40] have discussed the first case and proposed an algorithm for obtaining the recurrence relation in m of the form:

$$\triangle C_{m,k} \equiv \sum_{i=0}^{r} A_i(k) C_{m,k+i} = 0, \tag{1.8}$$

where $\triangle$ is the difference operator.

In [37] Ma and Gao studied the spectrum of discrete second-order difference operators with sign-changing weights and their applications.

Let $T > 1$ be an integer, and let $\mathbb{T} = \{1, 2, ..., T\}$. The authors in [37] consider the spectrum of the second-order linear eigenvalue problem

$$\Delta^2 u(t-1) + \lambda m(t) u(t) = 0, \quad t \in \mathbb{T}, \tag{1.9}$$

with boundary conditions

$$u(0) = u(T+1) = 0, \tag{1.10}$$

where $\lambda \neq 0$ is a parameter, and m changes sign on $\mathbb{T}$; that is m satisfies the following:

$$m(t) > 0, \quad t \in \mathbb{T}_+; \quad m(t) < 0, \quad t \in \mathbb{T} \setminus \mathbb{T}_+.$$

They investigate the existence of sign-changing solutions of discrete nonlinear problems of the form (1.9) - (1.10) by applying spectrum theory and Rabinowitz's bifurcation theorem. According to [37] the equivalent problem to (1.9) is given by:

$$Lu - \lambda m(t) f_\infty u = \lambda m(t) \xi(u) \tag{1.11}$$

where $L : X \mapsto Y$ is defined as

$$Lu(t) = -\Delta^2 u(t-1), \quad u \in X$$

and

$$f_\infty = \lim_{|s| \to \infty} \frac{f(s)}{s}.$$

The results are summarised as follows:

1. For each integer $k \in \{1, 2, ..., n\}$ there exists a continuum $\mathbb{D}^v_{k,+}$ of solutions to (1.11) meeting $(\lambda_{k,+}/rf_\infty, \infty)$.
2. For each integer $l \in \{1, 2, ..., T-n\}$ there exists a continuum $\mathbb{D}^v_{l,-}$ of solutions to (1.11) meeting $(-\lambda_{l,-}/rf_\infty, \infty)$.

For brevity, and since this is not the focus of the dissertation we will not provide the proofs of 1 and 2 above but they may be found in [37].

1.2 Transformations, factorizations and hierarchies

Binding et al. in [8, 11] amongst others discuss transformations and factorisations of continuous Sturm-Liouville boundary value problems. Currie and Love in [19] employed similar ideas to those discussed in [9, 10] to study the transformations of difference equations.

Consider the second-order weighted difference equation (1.5). The authors in [19] transform (1.5) with non-Dirichlet boundary conditions at both ends to an equation of the same form but with Dirichlet boundary conditions at both ends. They also proved that the spectrum of the new boundary value problem is the same as that of the original boundary value problem but with the least eigenvalue lost. Conversely they consider, in Section 3 of [19], an equation of the form (1.5), but with Dirichlet boundary conditions at both ends with an assumption that there exists a strictly positive solution, *z(n)*, to (1.5) for $\lambda = \lambda_0$ with λ_0 less than the least eigenvalue of the given boundary value problem. As a result it was discovered that the given boundary value problem could be transformed to one consisting of an equation of the same type but with specified non-Dirichlet boundary conditions at the ends. The immediate consequence was that the spectrum of the transformed boundary value problem had one extra eigenvalue, namely λ_0.

As a sequel to [19], Currie and Love in [20], show how non-Dirichlet boundary conditions transform to affine λ-dependent boundary conditions. The authors compare the spectra of all feasible combinations (four were discussed) of Dirichlet and non-Dirichlet boundary value problems with their transformed forms. A further result discussed by the authors is of transforming (1.5) with any one of the four combinations of Dirichlet and non-Dirichlet boundary conditions at the end points to end up with a boundary value problem with one extra eigenvalue in each case. Furthermore, the authors showed that we can transform affine λ-dependent boundary conditions back to non-Dirichlet type boundary conditions.

These results are summarised in the following table below:

Original problem		Transformed problem	
Initial endpoint	Terminal Endpoint	Initial Endpoint	Terminal Endpoint
Non-Dirichlet	Affine	Dirichlet	Non-Dirichlet
Affine	Non-Dirichlet	Non-Dirichlet	Dirichlet
Affine	Affine	Non-Dirichlet	Non-Dirichlet

We now define a class of functions called Nevanlinna functions. These will play an important role in the form of our boundary conditions later on. These functions are also called Herglotz or R-functions. The following definition is taken from [36]:

Definition 1.2.1. *A complex valued function f defined on $\mathbb{C}^+$ is called a Nevanlinna function if*

1. *f is analytic in $\mathbb{C}^+$*

2. *Im f(z) $\geq$ 0 for Im z > 0.*
The form of a positive Nevanlinna $N(\lambda)$ function is

$$N(\lambda) = a\lambda + b - \sum_{j=1}^{s} \frac{c_j}{\lambda - d_j} \qquad a, c_j \geq 0 \quad \forall j = 1, ..., s.$$

For the work in this dissertation, we will need the following property of Nevanlinna functions found in [14]:

The inverse of a positive Nevanlinna function is a negative Nevanlinna function, that is

$$\frac{1}{M(\lambda)} = -D(\lambda)$$

where $M(\lambda)$, $D(\lambda)$ are positive Nevanlinna functions as in Definition 1.2.1. In particular, if

$$M(\lambda) = a - \sum_{j=1}^{s} \frac{b_j}{\lambda - c_j}, \quad b_j > 0, \quad a \neq 0,$$

then

$$\frac{1}{M(\lambda)} = \beta - \sum_{j=1}^{s} \frac{\alpha_j}{\lambda - \delta_j}, \quad \alpha_j < 0, \quad \beta \neq 0.$$

In addition, if

$$M(\lambda) = a\lambda + b - \sum_{j=1}^{s} \frac{c_j}{\lambda - d_j}, \quad a_j, c_j > 0,$$

then

$$\frac{1}{M(\lambda)} = -\sum_{j=1}^{s+1} \frac{\sigma_j}{\lambda - \delta_j}, \quad \sigma_j < 0.$$

Currie and Love in [14] discuss some properties of Nevanlinna functions and investigate the effects of the mapping $y \mapsto w$ defined by :

$$w(n) := y(n) - y(n-1)\frac{z(n)}{z(n-1)}, \qquad n = 0, ..., m,$$

where $z(n)$ is a solution to (1.7) for $\lambda = \lambda_0$ such that $z(n) > 0$ for $n = -1, ..., m$ on boundary conditions of the form

$$y(-1) = N(\lambda)y(0), \quad y(m-1) = M(\lambda)y(m),$$

where $N(\lambda)$ and $M(\lambda)$ are positive Nevanlinna functions.

As result, new boundary conditions are explicitly determined. In comparison to the original boundary conditions, the new transformed boundary conditions may have the same dependence on λ, increased dependence on λ or decreased dependence on λ. The authors then go on to compare the spectra of the given boundary value problem with that of the transformed problem, showing as a result the exact conditions under which the transformed problem has one less, one more or the same number of eigenvalues as the given problem.

1.3 Inverse problems and recurrence relations

The most general definition of the inverse problem was formulated by Keller in 1976 in [33]. Often it is cited in literature. It states:

"We call two problems inverses of one another if the formulation of each involves all or part of the solution of the other. Often, for historical reasons, one of the two problems has been studied extensively for some time, while the other is newer and not so well understood. In such cases, the former problem is called the direct problem, while the latter is called the inverse problem."

In several cases one of the two problems is not well-posed. We define what it means for a problem to be well-posed according to the Hadamard criteria below:

Definition 1.3.1. *A problem is said to be well-posed if*

1. *there exists a solution to the problem (existence),*
2. *there is at most one solution to the problem (uniqueness),*
3. *the solution depends continuously on the data (stability)*

A problem which is not well-posed is called ill-posed. If one of two problems which are inverse to each other is ill-posed, we call it the inverse problem and the other one the direct problem. According to [33] if the data space is defined as a family of solutions to the direct problem, then the existence of the solution to the inverse problem is apparent, but a solution may fail to exist if the data is perturbed by noise. When it comes to the uniqueness property there is a difficulty in showing the solution to the inverse problem. The only way around the non-uniqueness problem is to reformulate the problem as discussed in [7]. Failure to meet the third Hadamard

condition in the above example has proven to be the most difficult to deal with. This is because it arises due to error in measurement and human error, which inevitably occur. These errors are amplified to a point that the resulting solutions are completely meaningless.

Until the beginning of the last century it was generally believed that for natural problems the solution will always depend continuously on the data ("natura non facit saltus") otherwise the mathematical model of the problem was deemed inadequate. This axiom by Darwin Leibnitz provided the impetus for scientists to further investigate and eventually prove that some natural problems were ill-posed and thus laid the foundations for the existence of inverse problems which we study here. See [6] for more on this. Therefore, these problems were called ill-posed problems. Only in the second half of the last century was it noticed that a vast range of problems in science and technology are ill-posed in any reasonable mathematical setting. This initiated a large amount of research in stable and accurate methods for the numerical solution of ill-posed problems. To date, inverse and ill-posed problems are still an active area of research, this is seen in [6, 22, 23, 24].

Inverse problems are important in modern era systems. This includes applications in physics, finance, biochemistry, and engineering. Much work has been done recently on the inverse problems. We note some of the work that has been done here, so as to equip ourselves with the tools that will enable us to achieve our goal in this dissertation.

Consider the following Sturm-Liouville difference equation for $N \in \mathbb{N}$

$$-\Delta^2 y_k + q_k y_{k+1} = \lambda y_{k+1}, \qquad k \in \mathbb{Z} \tag{1.12}$$

with Dirichlet boundary conditions

$$y_0 = y_{N+1} = 0, \tag{1.13}$$

where the sequence $q = q_{k\{k\in\mathbb{Z}\}}$ is the potential.

A well known fact, see [12], is that the problem (1.12), (1.13) has N simple real eigenvalues $\lambda_1 < \lambda_2 < ... < \lambda_N$ with corresponding eigenfunctions. The authors also state the important theorems which we state without proof here.

Theorem 1.3.1. *If two eigenvalue problems of the form (1.12), (1.13) have the same eigenvalues and the same weight numbers, then their potentials are the same on* $\big[0, N-1\big] \cap \mathbb{Z}$.

Theorem 1.3.2. *If two eigenvalue problems of the form (1.12), (1.13) have symmetric potentials and the same eigenvalues, then their potentials are the same on* $\big[0, N-1\big] \cap \mathbb{Z}$.

Bohner and Koyunbakan, in [12], considered inverse problems for Sturm-Liouville difference equations with Dirichlet boundary conditions, particularly of the form (1.12), (1.13). They show that given eigenvalues and weights, we can obtain the potentials uniquely. Moreover, the authors went on to prove that if the potential is symmetric, then it can be uniquely determined using only eigenvalues.

Consider the second order difference equation (1.5) with boundary conditions at the initial and terminal points respectively as follows:

$$y(-1) = \Big[a\lambda + b - \sum_{k=1}^{s} \frac{c_k}{\lambda - d_k}\Big] y(0) := Ay(0), a \geq 0, c_k > 0, \tag{1.14}$$

$$y(m-1) = \Big[\alpha\lambda + \beta - \sum_{j=1}^{p} \frac{\gamma_j}{\lambda - \sigma_j}\Big] y(m) := \Lambda y(m), \alpha \leq 0, \gamma < 0. \tag{1.15}$$

Given the weights and the eigenvalues of the above boundary value problem (1.5), (1.14) and (1.15) with $s \leq 1$ and $p \leq 1$, Currie and Love in [16] obtained a unique reconstruction of the potential and provided an inductive proof. In addition, given the eigenvalues and potential $b(n)$ for a difference boundary value problem of the form (1.5),(1.14),(1.15) with $s \leq 1$ and $p \leq 1$ the authors in [17] discuss the reconstruction of the weights $c(n)$. This reconstruction of the weights $c(n)$ is done inductively on m where $m+1$ is the number of unit intervals. Furthermore, Currie and Love in [18] generalise the results proved in [16] by uniquely reconstructing both the potential and the boundary conditions for given weights and eigenvalues. In the examples in each of the three references above, due to the lack of experimental data the authors solve the forward problem first in order to obtain the given eigenvalues.

In this dissertation we consider the following weighted second-order difference equation of the form

$$c(n)y(n+1) - b(n)y(n) + c(n-1)y(n-1) = -c(n)\lambda y(n),$$

where $c(n) > 0$ represents a weight function and $b(n)$ is the potential function.

Assume the following boundary conditions apply:

$$y(-1) = \Big(f\lambda + k - \sum_{j=1}^{p} \frac{v_j}{\lambda - c_j}\Big) y(0) \tag{1.16}$$

where $f \geq 0$, $v_j > 0$,

$$y(m-1) = \left(a\lambda + b - \sum_{j=1}^{s} \frac{c_j}{\lambda - d_j}\right) \tag{1.17}$$

where $a \leq 0$, $c_j < 0$.

1.4 Preliminaries

This dissertation relies heavily on the work of Currie and Love in [13] and as such we give the definitions and theorems (including the proofs) needed throughout this dissertation in this section.

Consider the equation (1.5). Then the mapping $y \to v$ is defined in [13] by:

$$v(n) := y(n+1) - y(n)\frac{z(n+1)}{z(n)}, \quad n = -1, 0, ..., m-1, \tag{1.18}$$

where $z(n)$ is a solution to (1.5) for $\lambda = 0$ such that $z(n) > 0$ for all $n = -1, ..., m$.

In order to transform (1.5) we require the following theorem.

Theorem 1.4.1. *Let z(n) be a solution of (1.5) corresponding to $\lambda = 0$ and define the difference operators*

$$Sy(n) := y(n) - y(n-1)\frac{z(n-1)c(n-1)}{z(n)c(n)}, \qquad n = 0, ..., m$$

and

$$Q(n) := y(n+1) - y(n)\frac{z(n+1)}{z(n)}, \qquad n = -1, 0, ..., m-1.$$

Then $ly(n) = SQy(n)$ and the "transformed" operator is given by $l_v v(n) = QSv(n), n = 0, ..., m-1$. Hence under the mapping (1.18), equation (1.5) transforms to:

$$c_v(n)v(n+1) - b_v(n)v(n) + c_v(n-1)v(n-1) = -\lambda c_v(n)v(n), \tag{1.19}$$

for $n = 0, ...m-2$, where for $n = -1, 0, ...m-1$,

$$c_v(n) = \frac{c(n)z(n)}{z(n+1)} > 0, \tag{1.20}$$

$$b_v(n) = \left[\frac{c(n)z(n)}{c(n+1)z(n+1)} + \frac{z(n+1)}{z(n)}\right]\frac{c(n)z(n)}{z(n+1)}. \tag{1.21}$$

Proof. By the definition of S and Q we have that

$$\begin{aligned} SQy(n) &= S\left(y(n+1) - y(n)\frac{z(n+1)}{z(n)}\right) \\ &= y(n+1) - y(n)\frac{z(n+1)}{z(n)} - \left(y(n) - y(n-1)\frac{z(n)}{z(n-1)}\right)\frac{z(n-1)c(n-1)}{z(n)c(n)}. \end{aligned}$$

Substituting in for $z(n+1)$ and cancelling terms yields

$$\begin{aligned} SQy(n) &= y(n+1) - \frac{1}{z(n)}\left(-\frac{c(n-1)}{c(n)}z(n-1) + \frac{b(n)}{c(n)}z(n)\right)y(n) \\ &- y(n)\frac{z(n-1)c(n-1)}{z(n)c(n)} + y(n-1)\frac{c(n-1)}{c(n)} \\ &= y(n+1) - \frac{b(n)}{c(n)}y(n) + \frac{c(n-1)}{c(n)}y(n-1) \\ &= -\lambda y(n), \quad n = 0, ..., m-1. \end{aligned}$$

Therefore $l = SQ$.

Now let $Qy(n) = v(n), n = -1, ..., m-1$. This yields

$$\begin{aligned} QSv(n) &= QSQy(n) \\ &= -Q\lambda y(n) \\ &= -\lambda v(n), \quad n = 0, ..., m-1, \end{aligned}$$

which is the required transformed equation.

To find l_v we first need to find $QSv(n)$.

Since

$$Sv(n) = v(n) - v(n-1)\frac{z(n-1)c(n-1)}{z(n)c(n)}, \quad n = 0, ..., m-1,$$

so for $n = 0, ..., m-2$,

$$\begin{aligned} Q(Sv(n)) &= v(n+1) \\ &- v(n)\frac{z(n)c(n)}{z(n+1)c(n+1)} - \left(v(n) - v(n-1)\frac{z(n-1)c(n-1)}{z(n)c(n)}\right)\frac{z(n+1)}{z(n)} \\ &= v(n+1) - v(n)\left(\frac{z(n)c(n)}{z(n+1)c(n+1)} + \frac{z(n+1)}{z(n)}\right) \\ &+ v(n-1)\left(\frac{z(n-1)c(n-1)z(n+1)}{z(n)c(n)z(n)}\right). \end{aligned}$$

Multiplying by $\dfrac{z(n)c(n)}{z(n+1)}$ results in

$$\frac{z(n)c(n)}{z(n+1)}v(n+1) - \left(\frac{z(n)c(n)}{z(n+1)c(n+1)} - \frac{z(n-1)c(n-1)}{z(n)c(n)} + \frac{b(n)}{c(n)}\right)\frac{z(n)c(n)}{z(n+1)}v(n) + \frac{z(n-1)c(n-1)}{z(n)}v(n-1) = -\lambda v(n)\frac{z(n)c(n)}{z(n+1)}.$$

As a result we obtain (1.19). □

Because the number of eigenvalues that a particular boundary value problem has depends on the form of the boundary conditions, the following theorem provided in [13] is fundamental:

Theorem 1.4.2. *Consider the boundary value problem given by (1.5) for n = 0,..., m -1 together with boundary conditions (1.14) and (1.15). The problem has*

1. *s + p + m +1 eigenvalues if $\alpha < 0$,*
2. *s + p + m eigenvalues if $\alpha = 0$ and $\beta \neq 0$,*
3. *s + p + m - 1 eigenvalues if $\alpha = \beta = 0$.*

Proof. From (1.5) with $n = 0$ we get

$$y(1) = -\frac{c(-1)y(-1)}{c(0)} + \Big(\frac{b(0)}{c(0)} - \lambda y(0)\Big).$$

Substituting in for $y(-1)$ from (1.16) yields

$$y(1) = \left(-\frac{c(-1)}{c(0)} \left(a\lambda + b - \sum_{j=1}^{s} \frac{c_j}{\lambda - d_j} \right) + \left(\frac{b(0)}{c(0)} - \lambda \right) \right) y(0),$$

which may be written as follows

$$y(1) := \frac{Q_0^1 + Q_1^1\lambda + ... + Q_{s+1}^1\lambda^{s+1}}{(\lambda - d_1)(\lambda - d_2)...(\lambda - d_s)} y(0), \tag{1.22}$$

where $Q_i^1, i = 0, ..., s+1$ are real constants.

Now (1.5) with $n = 1$ together with (1.22) gives

$$y(2) = \frac{Q_0^2 + Q_1^2\lambda + ... + Q_{s+1}^2\lambda^{s+2}}{(\lambda - d_1)(\lambda - d_2)...(\lambda - d_s)} y(0), \tag{1.23}$$

where $Q_i^2, i = 0, ..., s+2$ are real constants.

By mathematical induction

$$y(r-1) = \frac{Q_0^{r-1} + Q_1^{r-1}\lambda + ... + Q_{s+r-1}^{r-1}\lambda^{s+r-1}}{(\lambda - d_1)(\lambda - d_2)...(\lambda - d_s)} y(0), \tag{1.24}$$

where $Q_i^{r-1}, i = 0, ..., s+r-1$ are real constants. Similarly

$$y(r) = \frac{Q_0^r + Q_1^r\lambda + ... + Q_{s+r}^r\lambda^{s+r}}{(\lambda - d_1)(\lambda - d_2)...(\lambda - d_s)} y(0), \tag{1.25}$$

for real constants $Q_i^r, i = 0, ..., s+r$.

Using (1.17) and the fact that $y \neq 0$ we obtain the following eigencondition

$$\begin{aligned} \frac{Q_0^{r-1} + Q_1^{r-1}\lambda + ... + Q_{s+r-1}^{r-1}\lambda^{s+r-1}}{(\lambda - d_1)(\lambda - d_2)...(\lambda - d_s)} &= \left(a\lambda + b - \sum_{j=1}^{s} \frac{c_j}{\lambda - d_j} \right) \\ &\quad \times \left(\frac{Q_0^r + Q_1^r\lambda + ... + Q_{s+r}^r\lambda^{s+r}}{(\lambda - d_1)(\lambda - d_2)...(\lambda - d_s)} \right). \end{aligned} \tag{1.26}$$

Now with $a = 0$ and $b \neq 0$ in (1.17) the right hand side of (1.26) may be written as

$$\left(\frac{R_0 + R_1\lambda + ... + R_p\lambda^p}{(\lambda - d_1)(\lambda - d_2)...(\lambda - d_p)}\right)\left(\frac{Q_0^r + Q_1^r\lambda + ... + Q_{s+r}^r\lambda^{s+r}}{(\lambda - d_1)(\lambda - d_2)...(\lambda - d_s)}\right), \tag{1.27}$$

where R_i, $i = 0, ..., p$ are real constants.

From this we see that the numerator is a polynomial in λ, of order $p + s + r$ and thus the boundary value problem has $p + s + r$ eigenvalues .

Similarly, with $a = b = 0$ in (1.17) the right hand side of (1.26) may be written as

$$\left(\frac{R_0 + R_1\lambda + ... + R_{p-1}\lambda^p}{(\lambda - d_1)(\lambda - d_2)...(\lambda - d_p)}\right)\left(\frac{Q_0^r + Q_1^r\lambda + ... + Q_{s+r}^r\lambda^{s+r}}{(\lambda - d_1)(\lambda - d_2)...(\lambda - d_s)}\right), \tag{1.28}$$

where R_i, $i = 0, ..., p - 1$ are real constants.

Here the numerator is a polynomial in λ of order $p + r + s - 1$, thus by similar reasoning as above, the boundary value problem has $p + r + s - 1$ eigenvalues .

□

Theorem 1.4.3. *If y(n) obeys the boundary condition*

$$y(m-1) = \left(a\lambda + b - \sum_{j=1}^{s}\frac{c_j}{\lambda - d_j}\right) = F_0(\lambda)y(m) \tag{1.29}$$

then the domain of v(n) may be extended from n = -1,...,m-1 to n = -1,...,m by using

$$\frac{v(m-1)}{v(m)} = \frac{1}{W}$$

where

$$W = \left(\frac{c(m-1)z(m-1)}{c(m)z(m)} - \lambda + \frac{\dfrac{z(m)\lambda}{z(m-1)}}{\dfrac{z(m)}{z(m-1)} - \dfrac{1}{F_0(\lambda)}}\right).$$

Proof. If we transform (1.5) using the transformation (1.18) we get (1.19). Now using (1.19) for $n = m - 1$ together with (1.29) gives

$c_v(m-1)Wv(m-1)-b_v(m-1)v(m-1)+c_v(m-2)v(m-2) = -\lambda c_v(m-1)v(m-1).$

From (1.18) with $n = m-1$ and the boundary condition (1.29), we get

$$v(m-1) = y(m)\left(1 - F_0(\lambda)\frac{z(m)}{z(m-1)}\right). \tag{1.30}$$

Similarly using (1.18) with $n = m-2$ and (1.29) gives

$$v(m-2) = F_0(\lambda)y(m) - y(m-2)\frac{z(m-1)}{z(m-2)}.$$

Rearranging (1.5) with $n = m-1$ yields

$$c(m-2)y(m-2) = b(m-1)y(m-1) - \lambda c(m-1)y(m-1) - c(m-1)y(m).$$

Solving for $y(m-2)$ in the above and substituting we get:

$$v(m-2) = y(m)\left(F_0(\lambda) - \frac{z(m-1)}{c(m-2)z(m-2)}\left(\left(b(m-1) - \lambda c(m-1)\right)F_0(\lambda) - c(m-1)\right)\right). \tag{1.31}$$

Substituting (1.30) and (1.31) into (1.4) and dividing through by $y(m) \neq 0$ and simplifying gives

$$\begin{aligned} W &= \frac{b_v(m-1)}{c_v(m-1)} - \lambda \qquad (1.32)\\ &- \frac{c_v(m-2)}{c_v(m-1)}\left(\frac{1 - \dfrac{z(m-1)}{c(m-2)z(m-2)}\left(b(m-1) - \lambda c(m-1) - \dfrac{c(m-1)}{F_0(\lambda)}\right)}{\dfrac{1}{F_0(\lambda)} - \dfrac{z(m)}{z(m-1)}}\right). \end{aligned}$$

Since $z(n)$ obeys (1.5) for $\lambda = 0$ with $n = m-1$, we get

$$1 - \frac{b(m-1)z(m-1)}{c(m-2)z(m-2)} = -\frac{c(m-1)z(m)}{c(m-2)z(m-2)}. \tag{1.33}$$

Substituting (1.33) into the numerator of the term in the big brackets in (1.32) changes the numerator to

$$\frac{c(m-1)z(m-1)}{c(m-2)z(m-2)}\left(\lambda+\frac{1}{F_0(\lambda)}-\frac{z(m)}{z(m-1)}\right), \tag{1.34}$$

thus the ratio in the big brackets in (1.32) simplifies to

$$\frac{\lambda\dfrac{c(m-1)z(m-1)}{c(m-2)z(m-2)}}{\dfrac{1}{F_0(\lambda)}-\dfrac{z(m)}{z(m-1)}}+\frac{c(m-1)z(m-1)}{c(m-2)z(m-2)}. \tag{1.35}$$

Substituting (1.35) into (1.32) and using the fact that $c_v(n)=\dfrac{c(n)z(n)}{z(n+1)}$ and simplifying gives the following expression for W

$$W=\frac{b_v(m-1)}{c_v(m-1)}-\lambda-\frac{z(m)}{z(m-1)}+\frac{\lambda\dfrac{z(m)}{z(m-1)}}{\dfrac{z(m)}{z(m-1)}-\dfrac{1}{F_0(\lambda)}}. \tag{1.36}$$

Now from (1.21) with $n=m-1$ we get

$$b_v(m-1)=\left(\frac{c(m-1)z(m-1)}{c(m)z(m)}+\frac{z(m)}{z(m-1)}\right)\frac{c(m-1)z(m-1)}{z(m)}, \tag{1.37}$$

and thus

$$W=\frac{c(m-1)z(m-1)}{c(m)z(m)}-\lambda+\frac{\lambda\dfrac{z(m)}{z(m-1)}}{\dfrac{z(m)}{z(m-1)}-\dfrac{1}{F_0(\lambda)}},$$

completing the proof.

□

Theorem 1.4.4. *Suppose y(n) obeys the boundary condition*

$$y(m-1) = \left(a\lambda + b - \sum_{j=1}^{s} \frac{c_j}{\lambda - d_j}\right) = G_0(\lambda)y(m).$$

Then under the mapping given by (1.18), y(n) transforms to v(n) obeying the following:

1. *If z(n) obeys (1.4.4) for $\lambda = 0$ then v(n) obeys a boundary condition of the form*

$$\text{(a) } v(m-1) = \frac{1}{W}v(m) = H_0(\lambda)v(m), \qquad a = -\frac{c(m)}{c(m-1)},$$

$$\text{(b) } v(m-1) = \frac{1}{W}v(m) = J_0(\lambda)v(m), \qquad a \neq -\frac{c(m)}{c(m-1)}.$$

Proof. Here we provide an outline of the proof as it is lengthy. Also, we do not explicitly use the theorem in the remainder of the dissertation as such. It is only used for comparative purposes to show that the workings done in Chapters 2 to 5 are as expected. Firstly it is shown that $\frac{1}{W}$ is a negative Nevanlinna function of the correct form. This involves looking graphically at the Nevanlinna functions involved and using their properties. We then use the fact that $z(n)$ obeys the boundary condition and split into the 2 cases dependent on a and obtain the results. A detailed treatment of the proof can be found in [13]. □

We use the transformation (1.18) to transform the boundary conditions with $a = 0$ and $a < 0$. With $a < 0$ we further examine the case where $b \neq 0$ and $b = 0$. It is important to note that with $a = 0$ and $b = 0$ the interval of the transformed boundary value problem reduces by one.

We use the methods described by Currie and Love in previously cited work, particularly [13] to reconstruct the potential $b(n)$ in (1.5). The work is divided into Chapters 2 to 5. In Chapter 2 we focus on boundary value problems with cubic λ-dependence at the initial end boundary condition. After shifting (using $\lambda = \mu + \lambda_0$) and transforming (see [13]) each boundary value problem, we obtain less λ-dependence in the boundary conditions and hence fewer eigenvalues compared to the initial results before shifting. Chapter 3 deals with the case where the cubic λ-dependence is at the terminal end boundary coondition. The procedure of shifting and transforming is exactly the same as in Chapter 2, resulting in less λ-dependence in the boundary conditions and thus fewer eigenvalues. These are the eigenvalues that we use in the reconstruction of the potential $b(n)$ in Chapters 4 and 5, and would be found

experimentally.

Chapters 4 and 5 deal with reconstruction of the potential $b(n)$ given the eigenvalues and other data calculated in Chapters 2 and 3 respectively. Using Mathematica, we obtain a set of simultaneous equations of various orders, (the orders are contingent of the number of eigenvalues) which are uniquely solved to give the potential $b(n)$.

In Chapter 6 we bring everything together and generalize the work done in Chapters 2 to 5, and build and prove theorems based on the results we obtained from the aforementioned chapters. Finally we mention a few open problems in Chapter 7.

Chapter 2

Transformations - Cubic λ-dependence at initial end

In Chapter 1 we introduced the three-term recurrence and the various properties it possesses. We now consider the three-term recurrence (2.1) which serves as a numerical consideration of the ideas introduced thus far. We will now use the transformation (1.18) given above in order to transform explicit boundary value problems with a cubic dependence in λ at the initial end boundary condition to a boundary value problem with the same form of the equation but with at most quadratic λ-dependence in the initial end boundary condition. Note, the theorems and definitions required to carry out the computations in this chapter are stated in the Introduction and but are still referenced as [13]. Furthermore, all computations are carried out in Mathematica using NSolve and Apart commands throughout this chapter.

The 8 problems considered in this chapter have varying boundary conditions in terms of eigenparameter dependence, up to and including cubic dependence on the eigenparameter. Our ultimate goal is in fact to be able to solve the inverse problem of reconstructing the potential for a boundary value problem with cubic eigenparameter dependence in the boundary conditions. In order to do this we first need to understand the forward problems and their transformations presented here in this Chapter. In essence this Chapter can be seen as a tool which will be used later on in Chapter 4 where inverse problems are considered. Ultimately the results are proven in an abstract general setting in Chapter 6. It should be noted that in the current literature such inverse problems have only been solved for boundary conditions that have at most quadratic dependence on the eigenparameter.

Problem 1

Consider the following boundary value problem on the interval -1, 0, 1, 2, that is for $n = 0, 1,$

$$(n+3)y(n+1) - ny(n) + (n+2)y(n-1) = -\lambda(n+3)y(n), \tag{2.1}$$

$$y(-1) = \left(\lambda + 1 - \frac{1}{\lambda} - \frac{1}{\lambda - 2}\right) y(0), \tag{2.2}$$

$$y(1) = (3 - \lambda)y(2). \tag{2.3}$$

For this problem, the number of unit intervals is 3, $b(n) = n$ and $c(n) = (n + 3)$.

We now solve the forward problem by calculating the eigenvalues of (2.1), (2.2), (2.3) as follows:

For $n = 0$ in (2.1), together with (2.2) we get

$$y(1) = \left(-\frac{2}{3}\left(\lambda + 1 - \frac{1}{\lambda} - \frac{1}{\lambda - 2}\right) - \lambda\right) y(0). \tag{2.4}$$

Similarly, (2.1) with n=1, together with (2.3) gives

$$y(1)\left(\frac{4}{3 - \lambda} + 4\lambda - 1\right) + 3y(0) = 0. \tag{2.5}$$

Substituting (2.4) into (2.5) we get

$$y(0)\left(\left(\frac{4}{3 - \lambda} + 4\lambda - 1\right)\left(\frac{2}{3}\left(\lambda + 1 - \frac{1}{\lambda} - \frac{1}{\lambda - 2}\right) - \lambda\right) + 3\right) = 0. \tag{2.6}$$

We obtain the 5 eigenvalues (see Appendix for code) for the boundary value problem (2.1),(2.2),(2.3), up to 4 decimal places:

λ = -1.2182, λ = -0.0383, λ = 0.5901, λ = 2.1777, λ =3.3386.

By Theorem 2.2 in [13], this is the expected number of eigenvalues.

To ensure that the least eigenvalue of those listed above is equal to zero, we need to shift the boundary value problem (2.1), (2.2), (2.3). To do this we set the least eigenvalue, equal to λ_0, then let $\lambda = \mu + \lambda_0$. Then (2.1) becomes:

$$(n+3)y(n+1) - ny(n) + (n+2)y(n-1) = -(\mu + \lambda_0)(n+3)y(n), \tag{2.7}$$

with the following corresponding boundary conditions

$$y(-1) = \left(\mu + \lambda_0 + 1 - \frac{1}{\mu + \lambda_0} - \frac{1}{\mu + \lambda_0 - 2}\right) y(0), \tag{2.8}$$

$$y(1) = (3 - \mu - \lambda_0)y(2). \tag{2.9}$$

We now transform the shifted boundary value problem (2.7), (2.8), (2.9) using the transformation (1.18). Let $z(n)$ be a solution to (2.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Consequently $z(n)$ satisfies the following equation:

$$(n+3)z(n+1) - nz(n) + (n+2)z(n-1) = -\lambda_0(n+3)z(n). \tag{2.10}$$

Assume that $z(n)$ also satisfies the boundary conditions (2.8) and (2.9) for $\mu = 0$. Then we get from (2.8):

$$\frac{z(-1)}{z(0)} = \left(\lambda_0 + 1 - \frac{1}{\lambda_0} - \frac{1}{\lambda_0 - 2}\right). \tag{2.11}$$

Using (2.10), with $n = 0$, we get

$$\frac{z(1)}{z(0)} = -\lambda_0 - \frac{2}{3}\frac{z(-1)}{z(0)}. \tag{2.12}$$

Similarly, with $n = 1$ in (2.10)

$$\frac{z(2)}{z(1)} = \frac{1}{4} - \frac{3}{4}\frac{z(0)}{z(1)} - \lambda_0. \tag{2.13}$$

For ease of computation define:

$$F(\mu) = \mu + \lambda_0 + 1 - \frac{1}{\mu + \lambda_0} - \frac{1}{\mu + \lambda_0 - 2}, \tag{2.14}$$

$$G(\mu) = 3 - \mu - \lambda_0. \tag{2.15}$$

To find the transformed equation corresponding to (2.7), we calculate the following (rounded to 4 decimal places):

$c_v(-1) = 1.8269,\ \ c_v(0) = 4.9244,\ \ c_v(1) = 16.8727,\ \ b_v(0) = 9.0625,$
$b_v(1) = 60.9377.$

The transformed, shifted equation is given by

$$c_v(n)v(n+1) - b_v(n)v(n) + c_v(n-1)v(n-1) = -\mu c_v(n)v(n), \tag{2.16}$$

with

$$c_v(n) = (n+3)\frac{z(n)}{z(n+1)}, \tag{2.17}$$

$$b_v(n) = \left(\frac{n+3}{n+4}\frac{z(n)}{z(n+1)} + \frac{z(n+1)}{z(n)}\right)(n+3)\frac{z(n)}{z(n+1)}, \tag{2.18}$$

for n = -1, 0, 1.

We will transform the shifted boundary conditions using (1.18) as follows:

Setting $n = -1$ in (1.18) and using (2.8) we get

$$v(-1) = \left(1 - \frac{z(0)}{z(-1)}F(\mu)\right)y(0). \tag{2.19}$$

Also, if $n = 0$ in (1.18) then

$$v(0) = \left(-\mu - \lambda_0 - \frac{2}{3}F(\mu) - \frac{z(1)}{z(0)} \right) y(0). \tag{2.20}$$

Hence

$$\frac{v(-1)}{v(0)} = \frac{1 - \dfrac{z(0)}{z(-1)}F(\mu)}{-\mu - \lambda_0 - \dfrac{2}{3}F(\mu) - \dfrac{z(1)}{z(0)}}. \tag{2.21}$$

As a result we obtain the following Nevanlinna function:

$$\frac{v(-1)}{v(0)} = 0.6568 - \frac{0.1686}{\mu - 3.365} - \frac{0.2774}{\mu - 1.524}. \tag{2.22}$$

This is the expected form of $\dfrac{v(-1)}{v(0)}$ according to Theorem 3.5 in [13], for the case where $g > 0$.

The boundary condition at the terminal end is given by Theorem 3.3 in [13] as

$$\frac{v(1)}{v(2)} = \frac{1}{W}, \tag{2.23}$$

where W is given by

$$W = \frac{c(1)}{c(2)}\frac{z(1)}{z(2)} - \mu + \frac{\mu\dfrac{z(2)}{z(1)}}{\dfrac{z(2)}{z(1)} - \dfrac{1}{G(\mu)}}. \tag{2.24}$$

This results in the following Nevanlinna function

$$\frac{v(1)}{v(2)} = -1.1853. \tag{2.25}$$

This is the expected form of $\frac{v(1)}{v(2)}$ according to Theorem 3.3 in [13].

We now set $n = 0$ in (1.18) and use (2.4) to obtain:

$$v(0) = y(1)\left(1 - \frac{z(1)}{z(0)}\frac{1}{\frac{2}{3}F(\mu) - \mu - \lambda_0}\right). \tag{2.26}$$

By substituting $n = 1$ in (1.18), we get:

$$v(1) = y(2) - y(1)\frac{z(2)}{z(1)}. \tag{2.27}$$

Substituting (2.9) into (2.27) we get

$$v(1) = y(1)\left(\frac{1}{G(\mu)} - \frac{z(2)}{z(1)}\right). \tag{2.28}$$

So

$$\frac{v(0)}{v(1)} = G(\mu)\frac{1 - \frac{z(1)}{z(0)}\frac{1}{\frac{2}{3}F(\mu) - \mu - \lambda_0}}{1 - G(\mu)\frac{z(2)}{z(1)}}. \tag{2.29}$$

The result is the following Nevanlinna function:

$$\frac{v(0)}{v(1)} = 15.2651 - 3.4263\mu. \tag{2.30}$$

We now calculate the 4 eigenvalues of the transformed boundary value problem (2.16), (2.22), (2.25) by substituting (2.22), (2.30) as well as the coefficients $c_v(-1), c_v(0)$, $b_v(0)$ into (2.16) to get (to 4 decimal places):

$$162.5871 - 311.2661\mu + 203.7350\mu^2 - 53.8777\mu^3 + 4.9244\mu^4 = 0. \tag{2.31}$$

Solving (2.31) we obtain the 4 eigenvalues:

$\mu = 1.1799, \quad \mu = 1.8083, \quad \mu = 3.3959, \quad \mu = 4.5568.$

We have obtained four eigenvalues, one less than the number we obtained in the original problem. These new eigenvalues are each less than the original ones by the same amount, namely, λ_0. This shows that we have successfully transformed our original shifted boundary value problem, to the one below:

$$4.9244\frac{v(1)}{v(0)} - 9.0625 + 1.8269\frac{v(-1)}{v(0)} + 4.9244\mu = 0,$$

$$16.8727\frac{v(2)}{v(1)} - 60.9377 + 4.9244\frac{v(0)}{v(1)} + 16.8727\mu = 0,$$

$$\frac{v(-1)}{v(0)} = 0.6568 - \frac{0.1686}{\mu - 3.365} - \frac{0.2774}{\mu - 1.524},$$

$$\frac{v(1)}{v(2)} = -1.1853.$$

Comparing the form (2.2), (2.3) with (2.22), (2.25), we see that (2.22) has quadratic dependence in λ as opposed to cubic dependence in (2.2), and (2.25) is now a non-Dirichlet boundary condition.

Problem 2

Consider the following boundary value problem on the interval -1, 0, 1, 2, that is, $n = 0$, 1, given by (2.1) with (2.3) and

$$y(-1) = \left(2 - \frac{1}{\lambda + 1} - \frac{1}{\lambda + 2}\right) y(0), \tag{2.32}$$

As before, the number of intervals is 3, $b(n) = n$ and $c(n) = n + 3$.

We now solve the forward problem by calculating the eigenvalues of (2.1), (2.32), (2.3) as follows:

For $n = 0$ in (2.1), together with (2.32) we get

$$y(1) = \left(-\frac{2}{3}\left(2 - \frac{1}{\lambda + 1} - \frac{1}{\lambda + 2}\right) - \lambda\right) y(0). \tag{2.33}$$

Recall, (2.1) with $n = 1$, together with (2.3) gives (2.5).

Substituting (2.33) into (2.5) we get

$$y(0)\left(\frac{-2}{3}\left(2 - \frac{1}{\lambda + 1} - \frac{1}{\lambda + 2}\right) - \lambda\right)\left(\left(\frac{4}{3 - \lambda} + 4\lambda - 1\right) + 3\right) = 0. \tag{2.34}$$

Since $y(0) \neq 0$, we obtain the 5 eigenvalues for the boundary value problem (2.1), (2.32), (2.3) (to 4 decimal places)

λ = -2.7990, λ = -1.5483, λ = -0.5875, λ = 0.5092, λ =3.3423.

This is the expected number of eigenvalues by Theorem 2.2 in [13].

To ensure that the least eigenvalue is equal to zero, we need to shift the boundary value problem (2.1), (2.32), (2.3). To do this, as before, we set the least eigenvalue equal to λ_0, then let $\lambda = \mu + \lambda_0$.

Then (2.1) becomes (2.7), with corresponding boundary conditions (2.9) and

$$y(-1) = \left(2 - \frac{1}{\mu + \lambda_0 + 1} - \frac{1}{\mu + \lambda_0 + 2}\right) y(0). \tag{2.35}$$

We now transform the shifted boundary value problem (2.7), (2.35), (2.9) using the transformation in (1.18). Let $z(n)$ be a solution to (2.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Hence $z(n)$ satisfies (2.10). Let $z(n)$ also satisfy the boundary conditions (2.35) and (2.9) for $\mu = 0$ then from (2.35) we get

$$\frac{z(-1)}{z(0)} = \left(2 - \frac{1}{\lambda_0 + 1} - \frac{1}{\lambda_0 + 2}\right). \tag{2.36}$$

If we substitute $n = 0$ in (2.10), recall that we get (2.12), and similarly, with $n = 1$ in (2.10) we get (2.13).

Define the following, for ease of computation:

$$F(\mu) = 2 - \frac{1}{\mu + \lambda_0 + 1} - \frac{1}{\mu + \lambda_0 + 2}, \tag{2.37}$$

$$G(\mu) = 3 - \mu - \lambda_0. \tag{2.38}$$

To find the transformed equation corresponding to (2.7), we calculate the following (rounded to 4 decimal places):

$$\begin{aligned} &c_v(-1) = 7.6148,\ c_v(0) = 11.5063,\ b_v(0) = 36.0985, \\ &c_v(1) = 23.1960,\ b_v(1) = 111.6111. \end{aligned} \tag{2.39}$$

As a result, the transformed, shifted equation is given by (2.16) with the correct coefficients $c_v(n)$ and $b_v(n)$ given by (2.17) and (2.18) respectively, with z relevant to problem 2.

From Theorem 3.5 in [13] we have that $\dfrac{v(-1)}{v(0)}$ is given by

$$\frac{v(-1)}{v(0)} = \frac{z(0)}{z(-1)}\left(\frac{1}{\dfrac{c(-1)}{c(0)} - \dfrac{\mu}{\dfrac{z(-1)}{z(0)} - F(\mu)}}\right). \tag{2.40}$$

Substituting (2.36) and (2.37) into (2.40) and computing we obtain the Nevanlinna function:

$$\frac{v(-1)}{v(0)} = -\frac{0.3950}{\mu - 2.5188} - \frac{0.0797}{\mu - 1.2842}. \tag{2.41}$$

This is the expected form according to Theorem 3.5 in [13], for the case when $2 = h < \dfrac{z(-1)}{z(0)} = 3.8074$ and $g = 0$.

We transform the shifted boundary condition (2.9) using Theorem 3.3 in [13] to get the transformed boundary condition for the terminal end

$$\frac{v(1)}{v(2)} = \frac{1}{W}$$

where W is given by (2.24), with $z(1), z(2)$ and λ_0 corresponding to problem 2.

As a result we obtain

$$\frac{v(1)}{v(2)} = -0.8622. \tag{2.42}$$

This agrees with what we expect to find from Theorem 3.3 in [13].

We now find the eigenvalues of the boundary value problem (2.16), (2.41) and (2.42), with the correct values of $b_v(n)$ and $c_v(n)$ given in (2.39). To do this it helps to find the ratio $\dfrac{v(0)}{v(1)}$ first.

Now using (2.29) where all the quantities are coming from problem 2 we obtain the Nevanlinna function:

$$\frac{v(0)}{v(1)} = 12.0381 - 2.0159\mu. \tag{2.43}$$

Note that for problem 1 and problem 2, the least eigenvalues and the $z(n)'s$ are different, which explains why (2.30) and (2.43) are different.

Substituting (2.41) and (2.43) into the shifted, transformed equation gives us the eigencondition:

$$646.58 - 1110.1\mu + 642.03\mu^2 - 148.57\mu^3 + 11.506\mu^4 = 0. \tag{2.44}$$

We solve (2.44) by using the *NSolve* command in Mathematica to give the four eigenvalues (to 4 decimal places):

$\mu = 6.1413, \mu = 3.3082, \mu = 2.2115, \mu = 1.2507.$

We have obtained 4 eigenvalues, one less than the original number of 5. These new eigenvalues differ with the original ones by λ_0. This shows that we have successfully transformed our original shifted boundary value problem to the one below:

$$11.5063\frac{v(1)}{v(0)} - 36.0985 + 7.6148\frac{v(-1)}{v(0)} + 11.5063\mu = 0,$$

$$23.1960\frac{v(2)}{v(1)} - 111.6111 + 11.5063\frac{v(0)}{v(1)} + 23.1960\mu = 0,$$

$$\frac{v(-1)}{v(0)} = -\frac{0.3950}{\mu - 2.5188} - \frac{0.0797}{\mu - 1.2842},$$

$$\frac{v(1)}{v(2)} = -0.8622.$$

Comparing the form of (2.32) and (2.3) with (2.41) and (2.42) we see that both (2.32) and (2.41) have quadratic λ-dependence while (2.3) has linear λ-dependence as opposed to (2.42) which is now non-Dirichlet.

Problem 3

Consider the following boundary value problem on the interval -1, 0, 1, 2, that is, n=0,1, given by (2.1), (2.2) and

$$y(1) = \left(\frac{1}{\lambda + 1}\right) y(2). \tag{2.45}$$

We now solve the forward problem by calculating the eigenvalues of (2.1), (2.2), (2.45) as follows:

For $n = 0$ in (2.1), together with (2.2) we get (2.4), while substituting $n = 1$ in (2.1), together with (2.45) gives

$$y(1)(8\lambda + 3) + 3y(0) = 0. \tag{2.46}$$

Substituting (2.4) into (2.46) we get

$$y(0)\left(\left(-\lambda - \frac{2}{3}\left(\lambda + 1 - \frac{1}{\lambda} - \frac{1}{\lambda - 2}\right)\right)(8\lambda + 3) + 3\right) = 0. \tag{2.47}$$

Since $y(0) \neq 0$, we obtain the 4 eigenvalues (to 4 decimal places) for the boundary value problem (2.1), (2.2), (2.45):

λ = -1.1588, λ = -0.2557, λ = 0.4658, λ = 2.1738.

To ensure that the least eigenvalue is equal to zero, we need to shift the boundary value problem (2.1), (2.2), (2.45) by first selecting the least eigenvalue and calling it λ_0, then letting $\lambda = \mu + \lambda_0$. So the shifted problem is now given by (2.7), (2.8) and

$$y(1) = \frac{1}{\mu + \lambda_0 + 1} y(2). \tag{2.48}$$

Now let $z(n)$ be a solution to (2.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Hence $z(n)$ satisfies (2.10).

Recall, z obeys the boundary conditions (2.8) and (2.48) for μ =0 so we get (2.11). Substituting $n = 0$ in (2.10) gives (2.12). Similarly, (2.10) with $n = 1$ gives (2.13).

For computational ease, we will define the following:

$$F(\mu) = \mu + 1 + \lambda_0 - \frac{1}{\mu + \lambda_0} - \frac{1}{\mu + \lambda_0 - 2}, \tag{2.49}$$

$$G(\mu) = \frac{1}{\mu + \lambda_0 + 1}. \tag{2.50}$$

The transformed shifted equation is given by (2.16), with $c_v(n)$ and $b_v(n)$ given by (2.17) and (2.18) respectively, with z coming from problem 3.

Substituting $n = 0$ in (2.17) and (2.18) we get, respectively:

$$c_v(-1) = 2.0413, \quad c_v(0) = 6.2708, \quad b_v(0) = 12.8306.$$

Recall that $\dfrac{v(-1)}{v(0)}$ is given by (2.21) with all relevant data from problem 3. Substituting (2.11), (2.12) and (2.49) into (2.21) we get the Nevanlinna function (rounded to 4 decimal places):

$$\frac{v(-1)}{v(0)} = 0.5879 - \frac{0.1563}{\mu - 3.3100} - \frac{0.2597}{\mu - 1.4800}. \tag{2.51}$$

This is what we expect to get according to Theorem 3.5 in [13], for $g > 0$.

Note that $n = 0$ in (1.18), together with (2.4) gives $v(0)$ in terms of $y(1)$, that is (2.26), while $n = 1$ in (1.18) gives (2.28).

Hence (2.29) holds, where all the quantities correspond to problem 3 now.

Consequently we obtain to 4 decimal places, the following Nevanlinna function:

$$\frac{v(0)}{v(1)} = 1.2758. \tag{2.52}$$

By Theorem 3.4 in [13], this is the expected form of $\dfrac{v(0)}{v(1)}$.

Substituting (2.51), (2.52) into (2.16) we obtain the third degree polynomial:

$$-30.6631 + 62.0263\mu - 36.7491\mu^2 + 6.2708\mu^3 = 0. \tag{2.53}$$

From (2.53) we obtain the 3 eigenvalues (to 4 decimal places):

$\mu = 0.9032, \mu = 1.6246, \mu = 3.3326.$

We have obtained 3 eigenvalues, one less than the original number of 4. These new eigenvalues differ with the original ones by λ_0. This shows that we have successfully transformed our original shifted boundary value problem to the one below:

$$6.2708\frac{v(1)}{v(0)} - 12.8306 + 2.0413\frac{v(-1)}{v(0)} + 6.2708\mu = 0,$$

$$\frac{v(-1)}{v(0)} = 0.5879 - \frac{0.1563}{\mu - 3.3100} - \frac{0.2597}{\mu - 1.4800},$$

$$\frac{v(0)}{v(1)} = 1.2758.$$

Comparing the form of (2.2) and (2.45) with (2.51) and (2.52) we see that (2.2) has cubic λ-dependence while (2.51) has quadratic λ-dependence. On the other hand, (2.45) depends on λ while (2.52) is now independent of λ, in particular non-Dirichlet.

Problem 4

Consider the following boundary value problem on the interval n = 0,1, given by (2.1) with (2.32) and (2.45).

We calculate the eigenvalues of (2.1), (2.32), (2.45) as follows:

For $n = 0$ in (2.1), together with (2.32) we get (2.33), while substituting $n = 1$ in (2.1), together with (2.45) we get (2.46).

Substituting (2.33) into (2.46) we obtain

$$y(0)\left(\left(-\lambda - \frac{2}{3}\left(2 - \frac{1}{\lambda+1} - \frac{1}{\lambda+2}\right)\right)(8\lambda+3)+3\right) = 0. \tag{2.54}$$

Since $y(0) \neq 0$, we obtain the 4 eigenvalues (to 4 decimal places) of the boundary value problem (2.1), (2.32), (2.45):

λ = -2.7545, λ = -1.5251, λ = -0.6205, λ = 0.1918.

We need to shift the boundary value problem (2.1), (2.32), (2.45) to ensure that the least eigenvalue is equal to zero. Let the least eigenvalue be λ_0, then set $\lambda = \mu + \lambda_0$.

So the shifted boundary value problem is now given by (2.7), (2.35), (2.48).

As usual let $z(n)$ be a solution to (2.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. So $z(n)$ satisfies (2.10). In addition $z(n)$ also satisfies (2.35) and (2.48) for $\mu = 0$, so we get (2.36). Recall $n = 0$ in (2.10) gives (2.12), while substituting $n = 1$ in (2.10) gives (2.13).

To make computations easier, we define the following:

$$F(\mu) = 2 - \frac{1}{\mu + \lambda_0 + 1} - \frac{1}{\mu + \lambda_0 + 2}. \tag{2.55}$$

$$G(\mu) = \frac{1}{\mu + \lambda_0 + 1}. \tag{2.56}$$

The transformed shifted equation is given by (2.16) - (2.18) with z corresponding to problem 4. Here

$$c_v(0) = 19.0360, \quad b_v(0) = 93.5921, \quad c_v(-1) = 7.7907.$$

We now transform the boundary conditions. The form of (2.32) and (2.45) suggest that the boundary conditions will transform according to Theorem 3.5 and Theorem 3.4 in [13], respectively.

Recall from Theorem 3.5 in [13] that $\dfrac{v(-1)}{v(0)}$ is given by (2.21) (where all the quantities correspond to problem 4). Substituting (2.36), (2.12) and (2.55) into (2.21) and solving we obtain:

$$\frac{v(-1)}{v(0)} = -\frac{0.410}{\mu - 2.517} - \frac{0.0764}{\mu - 1.256}. \tag{2.57}$$

Note that this Nevanlinna function is different here from the one in (2.41) due to different values of λ_0, and different $z(n)$.

The other boundary condition $\dfrac{v(0)}{v(1)}$ is given by (2.29), with everything corresponding to problem 4. After computation we obtain the Nevanlinna function:

$$\frac{v(0)}{v(1)} = 0.4203. \tag{2.58}$$

Substituting (2.57), (2.58) and the correct coefficients $c_v(0), b_v(0), c_v(-1)$ into (2.16) we obtain the cubic polynomial in μ:

$$19.036\mu^3 - 120.11\mu^2 + 238.58\mu - 147.14 = 0. \tag{2.59}$$

Solving (2.59) gives (to 4 decimal places):

$$\mu = 2.9463, \mu = 2.1340, \mu = 1.2294.$$

We have obtained 3 eigenvalues, one less than the original number of 4. These new eigenvalues differ with the original ones by λ_0. This shows that we have successfully transformed our original shifted boundary value problem to the one below:

$$19.0359\frac{v(1)}{v(0)} - 93.5921 + 7.7907\frac{v(-1)}{v(0)} + 19.0359\mu = 0,$$

$$\frac{v(-1)}{v(0)} = -\frac{0.410}{\mu - 2.517} - \frac{0.0764}{\mu - 1.256},$$

$$\frac{v(0)}{v(1)} = 0.4203.$$

Comparing the form of (2.32) and (2.45) with (2.57) and (2.58) we see that both (2.32) and (2.57) have quadratic λ-dependence. On the other hand, (2.45) depends on λ whereas (2.58) does not.

Problem 5

Consider the boundary value problem, (2.1), (2.2) and

$$y(1) = 3y(2), \tag{2.60}$$

for $n = 0, 1$.

We now calculate the eigenvalues of (2.1), (2.2), (2.60) as follows:

Recall, $n = 0$ in (2.1), together with (2.2) gives (2.4), while $n = 1$ in (2.1), together with (2.60) gives

$$y(1)\left(4\lambda + \frac{1}{3}\right) + 3y(0) = 0. \tag{2.61}$$

Substituting (2.4) into (2.61) we get:

$$y(0)\left(\left(4\lambda+\frac{1}{3}\right)\left(-\frac{2}{3}\left(\lambda+1-\frac{1}{\lambda}-\frac{1}{\lambda-2}\right)-\lambda\right)+3\right)=0. \tag{2.62}$$

From (2.62), since $y(0) \neq 0$ we obtain 4 eigenvalues for the boundary value problem (2.1), (2.2), (2.60), to 4 decimal places:

λ = -1.2370, λ = -0.0404, λ = 0.6122, λ = 2.1818.

To ensure that the least eigenvalue is equal to zero, let $\lambda = \mu + \lambda_0$ where λ_0 is the least eigenvalue i.e. $\lambda_0 = -1.2370$.

The shifted boundary value problem is now given by (2.7), (2.8) and (2.60).

Now let $z(n)$ be a solution to (2.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. So $z(n)$ satisfies (2.10).

As before, we define terms to make computations easier. These are

$$F(\mu) = \mu + \lambda_0 + 1 - \frac{1}{\mu+\lambda_0} - \frac{1}{\mu+\lambda_0-2}. \tag{2.63}$$

$$G(\mu) = 3. \tag{2.64}$$

Since $z(n)$ satisfies (2.8) and (2.60) for $\mu = 0$ we get (2.11). Again $n = 0$ in (2.10) gives (2.12), and $n = 1$ in (2.10) gives (2.13).

In order to find the transformed equation corresponding to (2.7) we calculate the following:

$c_v(0) = 4.6147, \quad c_v(-1) = 1.7607, \quad c_v(1) = 12.0000,$
$b_v(0) = 8.3234, \quad b_v(1) = 4.6147.$

The shifted, transformed equation is given by (2.16) - (2.18) with z from problem 5. The transformed boundary conditions are obtained using (1.18) as follows:

Recall, $\frac{v(-1)}{v(0)}$ is given by (2.21) (with all relevant quantities $\frac{z(-1)}{z(0)}$, $\frac{z(1)}{z(0)}$ and $F(\mu)$ coming from problem 5 i.e. (2.11), (2.12) and (2.63) respectively), which gives the following Nevanlinna function:

$$\frac{v(-1)}{v(0)} = 0.6816 - \frac{0.1731}{\mu - 3.3825} - \frac{0.2839}{\mu - 1.5384}. \tag{2.65}$$

As before, note that (2.65) is different from (2.57). This is because in each problem, we had a different value for λ_0 and a different $z(n)$.

By Theorem 3.2 in [13] the ratio $\frac{v(1)}{v(2)}$ is given by

$$\frac{v(1)}{v(2)} = \frac{1}{W} = 0. \tag{2.66}$$

We calculate the eigenvalues of the transformed boundary value problem (2.16), (2.65), (2.66) by substituting (2.65), (2.66) as well as the correct values for $c_v(-1)$, $c_v(0)$ and $b_v(0)$ into (2.16) to get:

$$-34.9109 + 58.2645\mu - 29.8321\mu^2 + 4.6147\mu^3 = 0. \tag{2.67}$$

We get the following eigenvalues (to 4 decimal places):

$$\mu = 3.4188, \quad \mu = 1.8492, \quad \mu = 1.1966. \tag{2.68}$$

We have obtained 3 eigenvalues, one less than the original number of 4. These new eigenvalues differ with the original ones by λ_0. We have transformed our original shifted boundary value problem to the one below:

$$4.6147\frac{v(1)}{v(0)} - 8.3234 + 1.7607\frac{v(-1)}{v(0)} + 4.6147\mu = 0,$$

$$12v(2) + 4.6147v(0) = 0,$$

$$\frac{v(-1)}{v(0)} = 0.6816 - \frac{0.1731}{\mu - 3.3825} - \frac{0.2839}{\mu - 1.5384},$$

$$\frac{v(1)}{v(2)} = 0.$$

Comparing the form of (2.2) and (2.60) with (2.65) and (2.66) we see that (2.2) has cubic dependence on λ and (2.65) has quadratic λ-dependence. Meanwhile (2.60) is non-Dirichlet compared with (2.66) which is now Dirichlet.

Problem 6

Consider the following boundary value problem for $n = 0$,1, that is (2.1), (2.2) and

$$y(1) = \left(3 + \frac{1}{\lambda + 1}\right) y(2). \tag{2.69}$$

We now calculate the eigenvalues of (2.1), (2.2), (2.69) as follows:

For $n = 0$ in (2.1), together with (2.2) we get (2.4), while $n = 1$ in (2.1), together with (2.69) gives

$$y(1)\left(\frac{4}{3 + \dfrac{1}{\lambda + 1}} - 1 + 4\lambda\right) + 3y(0) = 0. \tag{2.70}$$

Substituting (2.4) into (2.70) gives

$$y(0)\left(\left(\frac{4}{3 + \dfrac{1}{\lambda + 1}} - 1 + 4\lambda\right)\left(-\frac{2}{3}\left(\lambda + 1 - \frac{1}{\lambda} - \frac{1}{\lambda - 2}\right) - \lambda\right) + 3\right) = 0. \tag{2.71}$$

Because $y(0) \neq 0$, we obtain the 5 eigenvalues for the boundary value problem (2.1), (2.2), (2.69):

$\lambda = 0.0000$, $\lambda =$ -1.4812, $\lambda =$ -1.1470, $\lambda = 0.62944$, $\lambda = 2.1820$.

To ensure that the least eigenvalue is equal to zero, we need to shift the boundary value problem (2.1), (2.2), (2.69) by first selecting the least eigenvalue and calling it λ_0, then letting $\lambda = \mu + \lambda_0$. So the shifted boundary value problem now becomes (2.7), (2.8) and

$$y(1) = \left(3 + \frac{1}{\mu + \lambda_0 + 1}\right) y(2). \tag{2.72}$$

Now let $z(n)$ be a solution to (2.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. So $z(n)$ satisfies (2.10).

Define:

$$F(\mu) = \mu + \lambda_0 + 1 - \frac{1}{\mu + \lambda_0} - \frac{1}{\mu + \lambda_0 - 2}, \tag{2.73}$$

$$G(\mu) = 3 + \frac{1}{\mu + \lambda_0 + 1}. \tag{2.74}$$

To find the transformed equation corresponding to (2.7), we first compute the following:

$c_v(-1) = 0.9624$, $c_v(0) = 2.5854$, $c_v(1) = 3.6872$, $b_v(0) = 4.6710$,
$b_v(1) = 6.7190$.

The shifted, transformed equation is given by (2.16) - (2.18) for $n = 0, 1$.

Now $\dfrac{v(-1)}{v(0)}$ is given by (2.21) (with all quantities corresponding to problem 6).

To get the Nevanlinna function of $\frac{v(-1)}{v(0)}$ we use Mathematica as usual to obtain:

$$\frac{v(-1)}{v(0)} = 1.2469 - \frac{0.2793}{\mu - 3.6128} - \frac{0.4407}{\mu - 1.7346}. \tag{2.75}$$

Note that (2.75) is different from (2.22) because of different values of λ_0 and different $z(n)'s$.

The boundary condition at the terminal end is given by (2.23) with W given by (2.24) using the data coming from problem 6.

Thus

$$\frac{v(1)}{v(2)} = \frac{2.2545}{\mu + 1.1814}. \tag{2.76}$$

This is the form we expect by Theorem 3.2 in [13].

As noted previously, $\frac{v(0)}{v(1)}$ is given by (2.29) (with data from problem 6). This simplifies to the following Nevanlinna function:

$$\frac{v(0)}{v(1)} = 1.8516 - 2.0588\mu. \tag{2.77}$$

We calculate the eigenvalues of the transformed boundary value problem (2.16), (2.75), (2.76) by substituting (2.75), (2.77) as well as the correct values for $c_v(-1)$, $c_v(0)$ and $b_v(0)$ into (2.16) to get the fourth degree polynomial:

$$3.8275 - 16.8944\mu + 18.7086\mu^2 - 7.5893\mu^3 + \mu^4 = 0. \tag{2.78}$$

Consequently we get the following eigenvalues, correct to 4 decimal places:

$$\mu = 3.6632, \quad \mu = 2.1106, \quad \mu = 1.4812, \quad \mu = 0.3342. \tag{2.79}$$

We have obtained 4 eigenvalues, one less than the original number of 5. These new eigenvalues differ with the original ones by λ_0.

This shows that we have successfully transformed our original shifted boundary value problem to the one below:

$$2.5854\frac{v(1)}{v(0)} - 4.6710 + 0.9624\frac{v(-1)}{v(0)} + 2.5854\mu = 0,$$

$$3.6872\frac{v(2)}{v(1)} - 6.7190 + 2.5854\frac{v(0)}{v(1)} + 3.6872\mu = 0,$$

$$\frac{v(-1)}{v(0)} = 1.2469 - \frac{0.2793}{\mu - 3.6128} - \frac{0.4407}{\mu - 1.7346},$$

$$\frac{v(1)}{v(2)} = \frac{2.2545}{\mu + 1.1814}.$$

Comparing the form of (2.2) and (2.69) with (2.75) and (2.76) we see that (2.2) has cubic λ-dependence while (2.75) has quadratic λ- dependence. On the other hand, (2.69) and (2.76) have linear λ-dependence.

Problem 7

Consider the following boundary value problem for $n = 0,1$, that is (2.1), (2.32) and (2.60).

We now calculate the eigenvalues of (2.1), (2.32), (2.60) as follows:

For $n = 0$ in (2.1), together with (2.32) we get (2.33), while $n = 1$ in (2.1), together with (2.60) gives (2.61).

Substituting (2.33) into (2.61) we get

$$y(0)\left(\left(4\lambda + \frac{1}{3}\right)\left(-\frac{2}{3}\left(2 - \frac{1}{\lambda+1} - \frac{1}{\lambda+2}\right) - \lambda\right) + 3\right) = 0. \tag{2.80}$$

As before since $y(0) \neq 0$, we obtain the 4 eigenvalues (to 4 decimal places):

λ = -2.8060, λ = -1.5540, λ = -0.6050, λ = 0.5477.

We need to shift the boundary value problem (2.1), (2.32), (2.60) as usual by letting $\lambda = \mu + \lambda_0$ where λ_0 is the least eigenvalue i.e. $\lambda_0 = -2.8060$. So the shifted boundary value problem is now given by (2.7), (2.35) and (2.60).

As always, let $z(n)$ be a solution to (2.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. So $z(n)$ satisfies (2.10).

We define the following for computational ease:

$$F(\mu) = 2 - \frac{1}{\mu + \lambda_0 + 1} - \frac{1}{\mu + \lambda_0 + 2}, \tag{2.81}$$

$$G(\mu) = 3. \tag{2.82}$$

The shifted transformed equation is given by (2.16) with

$$c_v(-1) = 7.5903, \quad c_v(0) = 10.8891, \quad c_v(1) = 12.0000, \quad b_v(0) = 32.6430. \tag{2.83}$$

Now, $\frac{v(-1)}{v(0)}$ is given by (2.21) where all the data comes from problem 7. When we apply partial fractions to the expression for $\frac{v(-1)}{v(0)}$ we get the Nevanlinna function:

$$\frac{v(-1)}{v(0)} = -\frac{0.3928}{\mu - 2.5195} - \frac{0.0802}{\mu - 1.2884}. \tag{2.84}$$

This is the expected form according to Theorem 3.5 in [13], for the case when $2 = h < \frac{z(-1)}{z(0)} = 3.7951$ and $g = 0$.

The boundary condition at the terminal end is given by (2.23) with W given by (2.24) using the data for problem 7.

Thus

$$\frac{v(1)}{v(2)} = 0. \tag{2.85}$$

This is the form we expect by Theorem 3.2 in [13].

We now find the eigenvalues of the boundary value problem (2.16), (2.84), (2.85), with the correct values of $b_v(n)$ and $c_v(n)$ as given in (2.83).

Substituting (2.84), (2.85) into the shifted, transformed equation gives us the eigencondition:

$$-9.2379 + 14.3319\mu - 6.8057\mu^2 + \mu^3 = 0. \tag{2.86}$$

We solve (2.86) get the three eigenvalues (to 4 decimal places):

$\mu = 3.3533, \mu = 2.2005, \mu = 1.2519.$

We have, as before, obtained 3 eigenvalues, one less than the original number of 4. These new eigenvalues differ with the original ones by λ_0. Thus we have transformed our original shifted boundary value problem to the one below:

$$10.8891\frac{v(1)}{v(0)} - 32.6430 + 7.5903\frac{v(-1)}{v(0)} + 10.8891\mu = 0,$$

$$12v(2) + 10.8891v(0) = 0,$$

$$\frac{v(-1)}{v(0)} = -\frac{0.3928}{\mu - 2.5195} - \frac{0.0802}{\mu - 1.2884},$$

$$\frac{v(1)}{v(2)} = 0.$$

Comparing the form of (2.32) and (2.60) with (2.84) and (2.85) we see that both (2.32) and (2.84) have quadratic λ-dependence, while (2.60) is non-Dirichlet as opposed to (2.85) which is now Dirichlet.

Problem 8

Consider the following boundary value problem for n = 0,1, that is (2.1), (2.32), (2.69).

The eigenvalues of (2.1), (2.32), (2.69) are calculated as follows:

For $n = 0$ in (2.1), together with (2.32) we get (2.33) while $n = 1$ in (2.1), together with (2.69) gives (2.70).

Substituting (2.33) into (2.70) gives

$$y(0)\left(\left(\frac{-2}{3}\left(2-\frac{1}{\lambda+1}-\frac{1}{\lambda+2}\right)-\lambda\right)\left(\frac{4}{3+\frac{1}{\lambda+1}}-4\lambda-1\right)+3\right)=0. \quad (2.87)$$

Now because $y(0) \neq 0$, we obtain the 5 eigenvalues for the boundary value problem (2.1), (2.32), (2.69):

λ = -2.809, λ = -1.584, λ = -1.378, λ = -0.5606, λ = 0.5818.

We shift the boundary value problem (2.1), (2.69), (2.32) by first selecting the least eigenvalue and calling it λ_0, then letting $\lambda = \mu + \lambda_0$.

The shifted boundary value problem is now given by (2.7), (2.35), (2.72).

Now let $z(n)$ be a solution to (2.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. So $z(n)$ satisfies (2.10).

Define the following for computational ease:

$$F(\mu) = 2 - \frac{1}{\mu + \lambda_0 + 1} - \frac{1}{\mu + \lambda_0 + 2}, \tag{2.88}$$

$$G(\mu) = 3 + \frac{1}{\mu + \lambda_0 + 1}. \tag{2.89}$$

The shifted transformed equation is given by (2.16) with

$c_v(-1) = 7.5779, c_v(0) = 10.6013, c_v(1) = 9.7888,$
$b_v(0) = 31.0969, b_v(1) = 23.1641.$

Also, $\frac{v(-1)}{v(0)}$ is given by (2.21) (with all quantities from problem 8). After computation we obtain the Nevanlinna function:

$$\frac{v(-1)}{v(0)} = -\frac{0.3918}{\mu - 2.5199} - \frac{0.0804}{\mu - 1.2906}. \tag{2.90}$$

Note, this is the form of $\frac{v(-1)}{v(0)}$ that we expect to get according to Theorem 3.5 in [13] for $2 = h < \frac{z(-1)}{z(0)} = 3.7890$ and $g = 0$.

Also, note that (2.90) is different from (2.41) because of different values of λ_0 and different $z(n)'$s, as we would expect.

The boundary condition at the terminal end is given by (2.23), with W calculated from (2.24) using data for problem 8.

We get

$$\frac{v(1)}{v(2)} = \frac{0.2259}{\mu - 1.3667}. \tag{2.91}$$

This is the expected form by Theorem 3.2 in [13].

Also, $\dfrac{v(0)}{v(1)}$ is given by (2.29) with all quantities coming from problem 8. The following Nevanlinna function results:

$$\frac{v(0)}{v(1)} = 7.7716 - 5.0109\mu. \tag{2.92}$$

We now find the eigenvalues of the boundary value problem (2.16), (2.90), (2.91), with the correct values of $b_v(n)$ and $c_v(n)$.

Substituting (2.91) and (2.92) into the shifted, transformed equation (2.16) gives us the eigencondition:

$$13.3615 - 30.1294\mu + 24.3520\mu^2 - 8.2948\mu^3 + \mu^4 = 0. \tag{2.93}$$

We solve (2.93 to give the four eigenvalues (to 4 decimal places):

$\mu = 3.3907, \mu = 2.2484, \mu = 1.4305, \mu = 1.2252.$

We have obtained 4 eigenvalues, one less than the original number of 5. These new eigenvalues differ with the original ones by λ_0. This shows that we have successfully transformed our original shifted boundary value problem to the one below:

$$10.6013\frac{v(1)}{v(0)} - 31.0969 + 7.59779\frac{v(-1)}{v(0)} + 10.6013\mu = 0,$$

$$9.7888\frac{v(1)}{v(2)} - 23.1641 + 10.6013\frac{v(0)}{v(1)} + 9.7888\mu = 0,$$

$$\frac{v(-1)}{v(0)} = -\frac{0.3918}{\mu - 2.5199} - \frac{0.0804}{\mu - 1.2906},$$

$$\frac{v(1)}{v(2)} = \frac{0.2259}{\mu - 1.3667}.$$

Comparing the form of (2.32), (2.69) with (2.90) and (2.91) we see that both (2.32) and (2.90) have quadratic λ-dependence, while both (2.69) and (2.91) have linear dependence on λ.

Chapter 3

Transformations - Cubic λ-dependence at terminal end

We will now use the transformation (1.18) given in Chapter 2 in order to transform explicit boundary value problems with a cubic dependence in λ at the terminal end boundary condition to a boundary value problem with the same form of the equation but with at most quadratic λ-dependence in the terminal end boundary condition.

As in Chapter 2, we consider 8 different problems bearing in mind that these will be used in order to investigate and solve the corresponding inverse problems in Chapter 5. They provide us with a deeper understanding and explicit illustration of what to expect in the abstract setting. As these 8 problems are calculated explicitly there is no need in this Chapter to provide any generalised theory - this is done in Chapter 6.

As in Chapter 2, all computations are carried out in Mathematica using the NSolve and Apart commands. Note that in this chapter we will deviate slightly from the work of Currie and Love, in [13], in the sense that we will allow $c(-1) = 0$. In [13], the assumption is that $c(n) > 0$ for all $n = -1, ..., 2$ so this is in fact a slight generalisation of the work done in [13] and it is interesting to compare the results obtained.

Problem 9

Consider the following boundary value problem for $n = 0,1$:

$$(n+1)^2 y(n+1) - 2ny(n) + n^2 y(n-1) = -(n+1)^2 \lambda y(n), \tag{3.1}$$

$$y(-1) = -2y(0), \tag{3.2}$$

$$y(1) = \left(-2\lambda + 3 + \frac{1}{\lambda - 2} + \frac{1}{\lambda + 1}\right) y(2). \tag{3.3}$$

With $n = 0$ in (3.1) we get

$$y(1) = -\lambda y(0). \tag{3.4}$$

Substituting $n = 1$ into (3.1) together with (3.3) we get:

$$y(1)\left(\frac{4}{-2\lambda + 3 + \dfrac{1}{\lambda - 2} + \dfrac{1}{\lambda + 1}} - 2 + 4\lambda\right) + y(0) = 0. \tag{3.5}$$

Substituting (3.4) into (3.5) and (since $y(0) \neq 0$), simplifying gives

$$1 - \lambda\left(\frac{4}{-2\lambda + 3 + \dfrac{1}{\lambda - 2} + \dfrac{1}{\lambda + 1}} - 2 + 4\lambda\right) = 0. \tag{3.6}$$

We obtain the following eigenvalues (to 4 decimal places):

$\lambda = 2.6498$, $\lambda = 1.4467$, $\lambda =$ -1.2233, $\lambda = 0.5000$, $\lambda = 0.3732$.

To ensure that the least eigenvalue of those listed above is equal to zero, we need to shift the boundary value problem (3.1), (3.2), (3.3). To do this we set the least eigenvalue, equal to λ_0, then let $\lambda = \mu + \lambda_0$. Then (3.1) becomes:

$$(n+1)^2 y(n+1) - 2ny(n) + n^2 y(n-1) = -(n+1)^2(\mu + \lambda_0)y(n), \tag{3.7}$$

with the following boundary conditions:

$$y(-1) = -2y(0), \tag{3.8}$$

$$y(1) = \left(-2\mu - 2\lambda_0 + 3 + \frac{1}{\mu + \lambda_0 - 2} + \frac{1}{\mu + \lambda_0 + 1} \right) y(2). \tag{3.9}$$

We now transform the shifted boundary value problem (3.7), (3.8), (3.9) using the transformation (1.18). Let $z(n)$ be a solution to (3.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Consequently $z(n)$ satisfies the following equation:

$$(n+1)^2 z(n+1) - 2nz(n) + n^2 z(n-1) = -\lambda_0 (n+1)^2 z(n). \tag{3.10}$$

Moreover $z(n)$ satisfies the boundary conditions (3.8), (3.9) for $\mu = 0$ so

$$\frac{z(-1)}{z(0)} = -2. \tag{3.11}$$

With $n = 0$ in (3.10) we get

$$\frac{z(1)}{z(0)} = -\lambda_0. \tag{3.12}$$

Similarly, $n = 1$ in (3.10) gives

$$\frac{z(2)}{z(1)} = \frac{1}{2} - \frac{1}{4}\frac{z(0)}{z(1)} - \lambda_0. \tag{3.13}$$

For ease of computation define:

$$F(\mu) = -2, \tag{3.14}$$

$$G(\mu) = -2\mu - 2\lambda_0 + 3 + \frac{1}{\mu + \lambda_0 - 2} + \frac{1}{\mu + \lambda_0 + 1}. \tag{3.15}$$

Here

$$c_v(-1) = 0,\ c_v(0) = 0.8175,\ c_v(1) = 2.6334,\ b_v(0) = 1.1671,\ b_v(1) = 4.7705. \tag{3.16}$$

The transformed, shifted equation is given by

$$c_v(n)v(n+1) - b_v(n)v(n) + c_v(n-1)v(n-1) = -\mu c_v(n)v(n), \tag{3.17}$$

with

$$c_v(n) = (n+1)^2 \frac{z(n)}{z(n+1)}, \tag{3.18}$$

$$b_v(n) = \left(\frac{(n+1)^2}{(n+2)^2} \frac{z(n)}{z(n+1)} + \frac{z(n+1)}{z(n)} \right) (n+1)^2 \frac{z(n)}{z(n+1)}, \tag{3.19}$$

for n = -1, 0, 1.

The boundary condition at the initial end is given by:

$$\frac{v(-1)}{v(0)} = \frac{z(0)}{z(-1)} \left(\frac{1}{\frac{c(-1)}{c(0)} - \frac{\mu}{\frac{z(-1)}{z(0)} + 2}} \right). \tag{3.20}$$

Simplifying we get

$$\frac{v(-1)}{v(0)} = 0. \tag{3.21}$$

Meanwhile, the boundary condition at the terminal end is given by:

$$\frac{v(1)}{v(2)} = \left(\frac{c(1)}{c(2)} \frac{z(1)}{z(2)} - \mu + \frac{\frac{z(2)}{z(1)}\mu}{\frac{z(2)}{z(1)} - \frac{1}{G(\mu)}} \right)^{-1}. \tag{3.22}$$

Substituting the correct values and simplifying we obtain the Nevanlinna function:

$$W = \frac{0.3367}{\mu - 3.6668} - \frac{0.4514}{\mu - 2.1740}. \tag{3.23}$$

So, by (3.22), $\dfrac{v(1)}{v(2)}$ is given by

$$\frac{v(1)}{v(2)} = -27.3411 + \frac{0.6759}{\mu - 3.0540} + \frac{588.449}{\mu + 18.7604}. \tag{3.24}$$

This agrees with the form of $\dfrac{v(1)}{v(2)}$ in Theorem 3.3 in [13], even under our assumption that $c(-1) = 0$.

We now calculate the 4 eigenvalues of the transformed boundary value problem (3.17), (3.21), (3.24) by substituting (3.21), (3.24) as well as the coefficients $c_v(0), c_v(1)$, $b_v(0)$, $b_v(1)$ into (3.17) to get (to 4 decimal places):

$$15.1505 - 36.1987\mu + 28.6447\mu^2 - 9.1166\mu^3 + \mu^4 = 0. \tag{3.25}$$

Solving (3.25) we obtain the 4 eigenvalues:

$\mu = 3.8731, \quad \mu = 2.6700, \quad \mu = 1.7233, \quad \mu = 0.8501.$

We have obtained 4 eigenvalues, one less than the number we obtained in the original problem. These new eigenvalues are each less than the original ones by the same amount, namely, λ_0. This shows that we have successfully transformed our original shifted boundary value problem, to the one below:

$$0.8175\frac{v(1)}{v(0)} - 1.1671 + 0.8175\mu = 0,$$

$$2.6334\frac{v(2)}{v(1)} - 4.7705 + 0.8175\frac{v(0)}{v(1)} + 2.6334\mu = 0,$$

$$\frac{v(-1)}{v(0)} = 0,$$

$$\frac{v(1)}{v(2)} = -27.3411 + \frac{0.6759}{\mu - 3.0540} + \frac{588.449}{\mu + 18.7604}.$$

Comparing the forms of (3.2) and (3.3) with that of (3.21), (3.24), we see that (3.21) is a Dirichlet boundary condition whereas (3.2) is a non-Dirichlet boundary condition. Also (3.3) has cubic dependence in λ while (3.24) has quadratic dependence on λ.

Problem 10

Consider the boundary value problem (3.1), (3.3) and the following, for $n = 0,1$:

$$y(-1) = \left(\frac{\lambda}{2} - 1\right) y(0). \tag{3.26}$$

With $n = 0$ in (3.1) we get (3.4), while substituting $n = 1$ into (3.1) together with (3.3) and simplifying we get (3.5).

Substituting (3.4) into (3.5) gives (3.6).

After solving, the following eigenvalues (to 4 decimal places) result:

$\lambda = 2.6498$, $\lambda = 1.4467$, $\lambda =$ -1.2233, $\lambda = 0.5000$, $\lambda = 0.3732$.

To ensure that the least eigenvalue of those listed above is equal to zero, we need to shift the boundary value problem (3.1), (3.26), (3.3). To do this we set the least eigenvalue equal to λ_0, then let $\lambda = \mu + \lambda_0$.

We get the shifted boundary value problem (3.7), (3.9) and

$$y(-1) = \left(\frac{\mu + \lambda_0}{2} - 1\right) y(0). \tag{3.27}$$

For convenience we define:

$$F(\mu) = \frac{\mu + \lambda_0}{2} - 1, \tag{3.28}$$

$$G(\mu) = -2\mu - 2\lambda_0 + 3 + \frac{1}{\mu + \lambda_0 - 2} + \frac{1}{\mu + \lambda_0 + 1}. \tag{3.29}$$

As before, we now transform the shifted boundary value problem (3.7), (3.27), (3.9) using the transformation (1.18). Let $z(n)$ be a solution to (3.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Consequently we get that $z(n)$ obeys (3.10) and since $z(n)$ obeys the boundary conditions (3.27) and (3.9) for $\mu = 0$ we get

$$z(-1) = \left(\frac{\lambda_0}{2} - 1\right) z(0), \tag{3.30}$$

$$z(1) = \left(-2\lambda_0 + 3 + \frac{1}{\lambda_0 - 2} + \frac{1}{\lambda_0 + 1}\right) z(2). \tag{3.31}$$

Now by (3.30) we have

$$\frac{z(-1)}{z(0)} = \left(\frac{\lambda_0}{2} - 1\right). \tag{3.32}$$

With $n = 0$ in (3.10) we get (3.12) while $n = 1$ in (3.10) gives (3.13).

To find the transformed equation corresponding to (3.7), we use (3.18) and (3.19) to calculate the following (rounded to 4 decimal places):

$$c_v(0) = 0.8175, \quad c_v(1) = 2.6334, \quad b_v(0) = 1.1671, \quad b_v(1) = 4.7705. \tag{3.33}$$

The transformed, shifted equation is given by (3.17) together with (3.18) and (3.19) for n = -1, 0, 1.

The boundary condition at the initial end is given by:

$$\frac{v(-1)}{v(0)} = \frac{z(0)}{z(-1)} \left(\frac{1}{\dfrac{c(-1)}{c(0)} - \dfrac{\mu}{\dfrac{z(-1)}{z(0)} - \dfrac{\mu + \lambda_0}{2} + 1}} \right). \tag{3.34}$$

Simplifying, we get

$$\frac{v(-1)}{v(0)} = -0.3102. \tag{3.35}$$

Meanwhile, the boundary condition at the terminal end is given by (3.22) where all relevant data corresponds to problem 10. So by (3.22), $\dfrac{v(1)}{v(2)}$ is given by (3.24) owing to the same value of $\dfrac{z(2)}{z(1)}$ in both cases.

This form of $\dfrac{v(1)}{v(2)}$ compares favourably with that in [13], even with $c(-1) = 0$.

We now calculate the 4 eigenvalues of the transformed boundary value problem (3.17), (3.24), (3.35) by substituting (3.35), (3.24) as well as the coefficients $c_v(0), c_v(1)$, $b_v(0)$, $b_v(1)$ into (3.17) to get (to 4 decimal places) the polynomial (3.25).

Solving (3.25) to obtain the 4 eigenvalues:

$\mu = 3.8731, \quad \mu = 2.6700, \quad \mu = 1.7233, \quad \mu = 0.8501.$

We have obtained four eigenvalues, one less than the number we obtained in the original problem. These new eigenvalues are each less than the original ones by the same amount, namely, λ_0. This shows that we have successfully transformed our original shifted boundary value problem, to the one below:

$$0.8175\frac{v(1)}{v(0)} - 1.1671 + 0.8175\mu = 0,$$

$$2.6334\frac{v(2)}{v(1)} - 4.7705 + 0.8175\frac{v(0)}{v(1)} + 2.6334\mu = 0,$$

$$\frac{v(-1)}{v(0)} = -0.3102,$$

$$\frac{v(1)}{v(2)} = -27.3411 + \frac{0.6759}{\mu - 3.0540} + \frac{588.449}{\mu + 18.7604}.$$

Comparing the form of (3.26) and (3.3) with (3.35) and (3.24) respectively, we see that (3.35) is a constant i.e. non-Dirichlet boundary condition whereas (3.26) is linear in λ. Also, (3.3) has cubic dependence in λ while (3.24) has quadratic dependence on λ.

Problem 11

Consider the boundary value problem (3.1), (3.3) and the following, for $n = 0,1$:

$$y(-1) = \left(-\frac{3}{\lambda - 2}\right) y(0). \tag{3.36}$$

Due to the form of (3.1), $y(-1)$ which occurs when $n = 0$ disappears and so (3.36) plays no role in the forward problem. Thus by problems 9 and 10 we have the eigenvalues as before

$\lambda = 2.6498$, $\lambda = 1.4467$, $\lambda =$ -1.2233, $\lambda = 0.5000$, $\lambda = 0.3732$.

The reason why we have 5 eigenvalues and not 6 eigenvalues (as one might expect from [Theorem 2.2, 13]) is that since $c(-1) = 0$ the extra eigenvalue which would come from the λ-dependence in (3.36) is "lost". As $c(-1) = 0$ the initial end boundary condition is in fact inconsequential.

Again let $\lambda_0 =$ -1.2233.

We get the shifted boundary value problem (3.7), (3.9) and

$$y(-1) = \left(-\frac{3}{\mu + \lambda_0 - 2}\right) y(0). \tag{3.37}$$

The only thing remaining to do is to transform boundary condition (3.37) using transformation (1.18). The transformation of (3.7) and (3.9) is identical to problems 9 and 10 due to our specific starting equation (3.1).

Let $z(n)$ be a solution to (3.7) satisfying (3.9) and (3.37) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Consequently we get (3.10), (3.31) together with

$$z(-1) = \left(-\frac{3}{\lambda_0 - 2}\right) z(0). \tag{3.38}$$

Now by (3.38) we have

$$\frac{z(-1)}{z(0)} = \left(-\frac{3}{\lambda_0 - 2}\right). \tag{3.39}$$

With $n = 0$ in (3.10) we get (3.12) while $n = 1$ in (3.10) gives (3.13).

The transformed, shifted equation is given by (3.17) together with (3.18) and (3.19) for n = -1, 0, 1. Note that the actual values for $c_v(n)$, $b_v(n)$ for $n = 0, 1$ are as in (3.33). The boundary condition at the initial end is given by

$$\frac{v(-1)}{v(0)} = \frac{z(0)}{z(-1)} \left(\frac{1}{\dfrac{c(-1)}{c(0)} - \dfrac{\mu}{\dfrac{z(-1)}{z(0)} - \dfrac{3}{\mu + \lambda_0 - 2}}} \right). \tag{3.40}$$

After simplification we obtain

$$\frac{v(-1)}{v(0)} = -\frac{1}{\mu - 3.2230}. \tag{3.41}$$

Define

$$G(\mu) = -2\mu - 2\lambda_0 + 3 + \frac{1}{\mu + \lambda_0 - 2} + \frac{1}{\mu + \lambda_0 + 1}. \tag{3.42}$$

The boundary condition at the terminal end is given by (3.24) because of the same value of $\dfrac{z(2)}{z(1)}$ and λ_0 in both cases.

Since the initial boundary condition plays no role in obtaining the eigencondition we have that (3.25) holds, giving

$\mu = 3.8731, \quad \mu = 2.6700, \quad \mu = 1.7233, \quad \mu = 0.8501.$

We have obtained four eigenvalues, one less than the number we obtained in the original problem. These new eigenvalues are each less than the original ones by the same amount, namely, λ_0. This shows that we have successfully transformed our original shifted boundary value problem, to the one below:

$$0.8175\frac{v(1)}{v(0)} - 1.1671 + 0.8175\mu = 0,$$

$$2.6334\frac{v(2)}{v(1)} - 4.7705 + 0.8175\frac{v(0)}{v(1)} + 2.6334\mu = 0,$$

$$\frac{v(-1)}{v(0)} = -\frac{1}{\mu - 3.2230},$$

$$\frac{v(1)}{v(2)} = -27.3411 + \frac{0.6759}{\mu - 3.0540} + \frac{588.449}{\mu + 18.7604}.$$

We note that the form of (3.36) and (3.41) are the same in terms of eigenparameter dependence.

Problem 12

Consider the boundary value problem (3.1), (3.3) and the following, for $n = 0,1$:

$$y(-1) = \left(1 - \frac{3}{\lambda - 2}\right) y(0). \tag{3.43}$$

The following eigenvalues (to 4 decimal places) result:

$\lambda = 2.6498$, $\lambda = 1.4467$, $\lambda = -1.2233$, $\lambda = 0.5000$, $\lambda = 0.3732$.

Again there are 5 eigenvalues and not 6 eigenvalues due to our value of $c(-1)$, see problem 11. Recall that $\lambda_0 = -1.2233$.

We get the shifted boundary value problem (3.7), (3.9) and

$$y(-1) = \left(1 - \frac{3}{\mu + \lambda_0 - 2}\right) y(0). \tag{3.44}$$

Let $z(n)$ be a solution to (3.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Also $z(n)$ obeys (3.9) and (3.44) for $\mu = 0$. Consequently we get (3.10), (3.31) together with

$$z(-1) = \left(1 - \frac{3}{\lambda_0 - 2}\right) z(0). \tag{3.45}$$

Now by (3.45) we have

$$\frac{z(-1)}{z(0)} = \left(1 - \frac{3}{\lambda_0 - 2}\right). \tag{3.46}$$

With $n = 0$ in (3.10) we get (3.12) while $n = 1$ in (3.10) gives (3.13).

The transformed, shifted equation is still given by (3.17) together with (3.18), (3.19) for $n = -1, 0, 1$ with actual values for $c_v(n)$, $b_v(n)$ for $n = 0, 1$ as in (3.33).

The initial end boundary condition is given by:

$$\frac{v(-1)}{v(0)} = \frac{z(0)}{z(-1)} \left(\frac{1}{\dfrac{c(-1)}{c(0)} - \dfrac{\mu}{\dfrac{z(-1)}{z(0)} - 1 + \dfrac{3}{\mu + \lambda_0 - 2}}} \right). \tag{3.47}$$

Simplification results in

$$\frac{v(-1)}{v(0)} = -\frac{0.4821}{\mu - 3.2233}. \tag{3.48}$$

The boundary condition at the terminal end is as before given by (3.24).

The 4 eigenvalues of the transformed boundary value problem (3.17), (3.48), and (3.24) are given by solving (3.25) i.e.

$\mu = 3.8731, \quad \mu = 2.6700, \quad \mu = 1.7233, \quad \mu = 0.8501.$

We have obtained four eigenvalues, one less than the number we obtained in the original problem. These new eigenvalues are each less than the original ones by the same amount, namely, λ_0. This shows that we have successfully transformed our original shifted boundary value problem, to the one below:

$$0.8175\frac{v(1)}{v(0)} - 1.1671 + 0.8175\mu = 0,$$

$$2.6334\frac{v(2)}{v(1)} - 4.7705 + 0.8175\frac{v(0)}{v(1)} + 2.6334\mu = 0,$$

$$\frac{v(-1)}{v(0)} = -\frac{0.4821}{\mu - 3.2233},$$

$$\frac{v(1)}{v(2)} = -27.3411 + \frac{0.6759}{\mu - 3.0540} + \frac{588.449}{\mu + 18.7604}.$$

Comparing the form of (3.43) with (3.48), we see that we have reduced the λ-dependence after transforming.

Problem 13

Consider the boundary value problem (3.1), (3.2) and

$$y(1) = \left(\frac{1}{2} + \frac{3}{\lambda} + \frac{1}{\lambda - 3}\right) y(2), \tag{3.49}$$

for $n = 0, 1$.

Setting $n = 0$ in (3.1) we get (3.4), while substituting $n = 1$ into (3.1) together with (3.49) gives:

$$y(1)\left(\frac{4}{\left(\frac{1}{2} + \frac{3}{\lambda} + \frac{1}{\lambda - 3}\right)} - 2 + 4\lambda\right) + y(0) = 0. \tag{3.50}$$

Substituting (3.4) in (3.50) and simplifying we get (since $y(0) \neq 0$)

$$1 - \lambda\left(\frac{4}{\left(\frac{1}{2} + \frac{3}{\lambda} + \frac{1}{\lambda - 3}\right)} - 2 + 4\lambda\right) = 0. \tag{3.51}$$

Solving, we obtain the following eigenvalues (to 4 decimal places):

λ = -9.4213, λ = 2.5440, λ = 0.6612, λ = -0.2840.

To ensure that the least eigenvalue of those listed above is equal to zero, we need to shift the boundary value problem (3.1), (3.2), (3.49). To do this we set the least eigenvalue equal to λ_0, then let $\lambda = \mu + \lambda_0$.

We get the shifted boundary value problem (3.7), (3.2) and

$$y(1) = \left(\frac{1}{2} + \frac{3}{\mu + \lambda_0} + \frac{1}{\mu + \lambda_0 - 3}\right) y(2). \tag{3.52}$$

For ease of computation define:

$$F(\mu) = -2, \tag{3.53}$$

$$G(\mu) = \frac{1}{2} + \frac{3}{\mu + \lambda_0} + \frac{1}{\mu + \lambda_0 - 3}. \tag{3.54}$$

We now transform the shifted boundary value problem (3.7), (3.2), (3.52) using the transformation (1.18). Let $z(n)$ be a solution to (3.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Since $z(n)$ obeys (3.2) and (3.52) for $\mu = 0$ we get (3.10), (3.11) and

$$z(1) = \left(\frac{1}{2} + \frac{3}{\lambda_0} + \frac{1}{\lambda_0 - 3}\right) z(2). \tag{3.55}$$

With $n = 0$ in (3.10) we get (3.12) while $n = 1$ in (3.10) gives (3.13).

To find the transformed equation corresponding to (3.7), we use the values (rounded to 4 decimal places);

$$c_v(0) = 0.1061, \quad c_v(1) = 0.4043, \quad b_v(0) = 1.0028, \quad b_v(1) = 4.0182. \tag{3.56}$$

The transformed, shifted equation is given by (3.17) together with (3.18), (3.19) for n = -1, 0, 1 (all relevant data comes from problem 13).

The boundary condition at the initial end is given by (3.20), which when we simplify we obtain (3.21).

The boundary condition at the terminal end is given by (3.22) and thus using the data from problem 13 we obtain:

$$\frac{v(1)}{v(2)} = \frac{0.7253}{\mu - 12.3871} + \frac{3.2221}{\mu - 9.2781}. \tag{3.57}$$

This form of $\dfrac{v(1)}{v(2)}$ is consistent with Theorem 3.2 in [13], even with $c(-1) = 0$.

We now calculate the 3 eigenvalues of the transformed boundary value problem (3.17), (3.21), (3.57) by substituting (3.21), (3.57) as well as the coefficients $c_v(0), c_v(1)$, $b_v(0)$, $b_v(1)$ into (3.17) to get (to 4 decimal places) the polynomial:

$$-1102.3172 + 322.0960\mu - 31.1850\mu^2 + \mu^3 = 0. \tag{3.58}$$

Solving (3.58) we obtain the following 3 eigenvalues:

$\mu = 11.9653, \quad \mu = 10.0824, \quad \mu = 9.1373.$

We have obtained three eigenvalues, one less than the number we obtained in the original problem. These new eigenvalues are each less than the original ones by the same amount, namely, λ_0. This shows that we have successfully transformed our original shifted boundary value problem, to the one below:

$$0.1061\frac{v(1)}{v(0)} - 1.0028 + 0.1061\mu = 0,$$

$$0.4043\frac{v(2)}{v(1)} - 4.0182 + 0.1061\frac{v(0)}{v(1)} + 0.4043\mu = 0,$$

$$\frac{v(-1)}{v(0)} = 0,$$

$$\frac{v(1)}{v(2)} = \frac{0.7253}{\mu - 12.3871} + \frac{3.2221}{\mu - 9.2781}.$$

Comparing the form of (3.2) with (3.21), we see that, as in problem 9, (3.2) is a non-Dirichlet boundary condition and (3.21) is a Dirichlet boundary condition at the

initial end, as expected. Also, in transforming (3.49) to (3.57) we have decreased the dependence on λ in the terminal end boundary condition.

Problem 14

Consider the boundary value problem (3.1), (3.26), (3.49) for $n = 0,1$.

Since the initial end boundary condition (3.26) plays no role in the forward problem we have the eigenvalues as in problem 13, i.e.

λ = -9.4213, λ = 2.5440, λ = 0.6612, λ = -0.2840.

We get the shifted boundary value problem (3.7), (3.27) and (3.52).

Let $z(n)$ be a solution to (3.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Note that as $z(n)$ obeys (3.27) and (3.52) for $\mu = 0$ we get (3.10), (3.30) and (3.55).

From (3.30) we get (3.32), while $n = 0$ in (3.10) gives (3.12) and $n = 1$ in (3.10) gives (3.13).

To find the transformed equation corresponding to (3.7), we use the values in (3.56) (rounded to 4 decimal places).

The transformed, shifted equation is given by (3.17) together with (3.18), (3.19) for n = -1, 0, 1, where all the quantities correspond to problem 14.

The boundary condition at the initial end is given by (3.34), which when we simplify we obtain

$$\frac{v(-1)}{v(0)} = -0.08756. \tag{3.59}$$

By (3.22), $\dfrac{v(1)}{v(2)}$ is given by (3.57).

We now calculate the 3 eigenvalues of the transformed boundary value problem (3.17), (3.59), (3.57) by substituting (3.59), (3.57) as well as the coefficients $c_v(0), c_v(1)$, $b_v(0)$, $b_v(1)$ into (3.17) to get (3.58).

We obtain 3 eigenvalues: (to 4 decimal places)

$\mu = 11.9653, \quad \mu = 10.0824, \quad \mu = 9.1373.$

We have obtained three eigenvalues, one less than the number we obtained in the original problem. These new eigenvalues are each less than the original ones by the same amount, namely, λ_0. This shows that we have successfully transformed our original shifted boundary value problem, to the one below:

$$0.1061\frac{v(1)}{v(0)} - 1.0028 + 0.1061\mu = 0,$$

$$0.4043\frac{v(2)}{v(1)} - 4.0182 + 0.1061\frac{v(0)}{v(1)} + 0.4043\mu = 0,$$

$$\frac{v(-1)}{v(0)} = -0.08776,$$

$$\frac{v(1)}{v(2)} = \frac{0.7253}{\mu - 12.3871} + \frac{3.2221}{\mu - 9.2781}.$$

Comparing the form of (3.26) with (3.59), we see that (3.26) is affine in λ whereas (3.59) has no λ-dependence i.e. is a non-Dirichlet boundary condition.

Problem 15

Consider the boundary value problem (3.1), (3.36), (3.49) for n = 0,1.

Proceeding as before we obtain the following eigenvalues (to 4 decimal places):

λ = -9.4213, λ = 2.5440, λ = 0.6612, λ = -0.2840. Note that as in problems 12 and 13 we get one eigenvalue less than expected from [13, Theorem 2.2] since $c(-1) = 0$.

Let λ_0 = -9.4213 and $\lambda = \mu + \lambda_0$. We obtain the shifted boundary value problem (3.7), (3.37) and (3.52).

Let $z(n)$ be a solution to (3.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. We obtain that z satisfies (3.10), (3.38) and (3.55).

By (3.38) we get (3.39), while $n = 0$ in (3.10) gives (3.12) and $n = 1$ in (3.10) gives (3.13).

To find the transformed equation corresponding to (3.7), we use the values in (3.56) (rounded to 4 decimal places). The transformed, shifted equation is given by (3.17) together with (3.18), (3.19) for n = -1, 0, 1.

The boundary condition at the initial end is given by (3.40), which when we simplify we obtain

$$\frac{v(-1)}{v(0)} = -\frac{1}{\mu - 11.4213}. \tag{3.60}$$

By (3.22), $\dfrac{v(1)}{v(2)}$ is given again by (3.57).

The eigencondition remains as (3.58) and thus we obtain:

$\mu = 11.9653, \quad \mu = 10.0824, \quad \mu = 9.1373.$

We have obtained three eigenvalues, one less than the number we obtained in the original problem. These new eigenvalues differ from the original ones by the same amount, namely, λ_0.

This shows that we have successfully transformed our original shifted boundary value problem, to the one below:

$$0.1061\frac{v(1)}{v(0)} - 1.0028 + 0.1061\mu = 0,$$

$$0.4043\frac{v(2)}{v(1)} - 4.0182 + 0.1061\frac{v(0)}{v(1)} + 0.4043\mu = 0,$$

$$\frac{v(-1)}{v(0)} = -\frac{1}{\mu - 11.4213},$$

$$\frac{v(1)}{v(2)} = \frac{0.7253}{\mu - 12.3871} + \frac{3.2221}{\mu - 9.2781}.$$

We see that in terms of dependence on λ both (3.36) and (3.60) are of the same form.

Problem 16

Consider the boundary value problem (3.1), (3.43), (3.49) for n = 0,1.

We obtain (as in the previous 3 problems) the following eigenvalues:

λ = -9.4213, λ = 2.5440, λ = 0.6612, λ = -0.2840,

where again we have 4 eigenvalues instead of the expected 5 eigenvalues since $c(-1) = 0$.

The shifted boundary value problem is given by (3.7), (3.44) and (3.52).

Let $z(n)$ be a solution to (3.7) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. So z obeys (3.10), (3.45), (3.55). In addition (3.12) and (3.13) hold.

Now from (3.45) we get (3.46).

To find the transformed equation corresponding to (3.7), we use the values in (3.56) (rounded to 4 decimal places). The transformed, shifted equation is given by (3.17) together with (3.18), (3.19) for n = -1, 0, 1.

The boundary condition at the initial end is given by (3.47), which when we simplify we obtain

$$\frac{v(-1)}{v(0)} = -\frac{0.2080}{\mu - 11.4213}. \tag{3.61}$$

Furthermore, $\dfrac{v(1)}{v(2)}$ is still given by (3.57).

Solving (3.58) we obtain the following 3 eigenvalues:(to 4 decimal places)

$\mu = 11.9653, \quad \mu = 10.0824, \quad \mu = 9.1373.$

We have obtained three eigenvalues, one less than the number we obtained in the original problem. These new eigenvalues differ from the original ones by the same amount, namely, λ_0. This shows that we have successfully transformed our original shifted boundary value problem, to the one below:

$$0.1061\frac{v(1)}{v(0)} - 1.0028 + 0.1061\mu = 0,$$

$$0.4043\frac{v(2)}{v(1)} - 4.0182 + 0.1061\frac{v(0)}{v(1)} + 0.4043\mu = 0,$$

$$\frac{v(-1)}{v(0)} = \frac{0.2080}{\mu - 11.4213},$$

$$\frac{v(1)}{v(2)} = \frac{0.7253}{\mu - 12.3871} + \frac{3.2221}{\mu - 9.2781}.$$

Comparing the form of (3.43) with (3.61), after transforming the λ-dependence has decreased.

Chapter 4

Inverse Problems: Reconstructing the potential with cubic λ-dependence at initial end

In this chapter, given the eigenvalues for the boundary value problems below we focus on finding $b(0)$ and $b(1)$, i.e., reconstructing the potential. Note that all evaluations are carried out using Mathematica commands Apart and NSolve as done in preceding chapters.

Problem 1

Consider the following boundary value problem on the interval -1, 0, 1, 2, that is for $n = 0, 1,$

$$(n+3)y(n+1) - b(n)y(n) + (n+2)y(n-1) = -\lambda(n+3)y(n), \qquad (4.1)$$

with boundary conditions (2.2) and (2.3).

The given eigenvalues are :

λ_0 = -1.2182, λ_1 = -0.0383, λ_2 = 0.5901, λ_3 = 2.1777, λ_4 =3.3386.

We now shift the boundary value problem so that the least eigenvalue equals zero, that is, set $\lambda = \mu + \lambda_0$ and we get:

$$(n+3)y(n+1) - b(n)y(n) + (n+2)y(n-1) = -(\mu+\lambda_0)(n+3)y(n), \tag{4.2}$$

together with (2.8) and (2.9).

Let $z(n)$ be a solution to (4.2), (satisfying (2.8) and (2.9)) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Consequently $z(n)$ satisfies the following equation:

$$(n+3)z(n+1) - b(n)z(n) + (n+2)z(n-1) = -\lambda_0(n+3)z(n). \tag{4.3}$$

Using (4.3), with $n = 0$, we get

$$\frac{z(1)}{z(0)} = \frac{b(0)}{3} - \lambda_0 - \frac{2}{3}\frac{z(-1)}{z(0)}. \tag{4.4}$$

Similarly, with $n = 1$ in (4.3) we obtain

$$\frac{z(2)}{z(1)} = \frac{b(1)}{4} - \frac{3}{4}\frac{z(0)}{z(1)} - \lambda_0. \tag{4.5}$$

The transformed, shifted equation is given by (2.16), with $c_v(n)$ and $b_v(n)$ given by

$$c_v(-1) = 2\frac{z(-1)}{z(0)}, \tag{4.6}$$

$$c_v(0) = \frac{3}{\dfrac{b(0)}{3} - \lambda_0 - \dfrac{2}{3}\dfrac{z(-1)}{z(0)}}, \tag{4.7}$$

$$c_v(1) = 4\frac{z(1)}{z(2)}, \tag{4.8}$$

$$b_v(0) = 3\frac{z(0)}{z(1)}\left(\frac{3}{4}\frac{z(0)}{z(1)} + \frac{z(1)}{z(0)}\right), \tag{4.9}$$

$$b_v(1) = 4\frac{z(1)}{z(2)}\left(\frac{4}{5}\frac{z(1)}{z(2)} + \frac{z(2)}{z(1)}\right). \tag{4.10}$$

Now we find the transformed boundary condition at the initial end using Theorem 3.5 in [13].

Recall from Chapter 2 that $\dfrac{v(-1)}{v(0)}$ is given by

$$\frac{v(-1)}{v(0)} = \frac{1 - \dfrac{z(0)}{z(1)}F(\mu)}{\dfrac{b(0)}{3} - \mu - \dfrac{2}{3}F(\mu) - \dfrac{z(1)}{z(0)}} \tag{4.11}$$

which can be simplified to (2.40). Thus substituting in (2.11) and (2.14) into (2.40) and evaluating gives the Nevanlinna function (2.22).

Since $z(n)$ obeys (2.9) for $\mu = 0$, we get

$$\frac{z(2)}{z(1)} = \frac{1}{3 - \lambda_0}. \tag{4.12}$$

Equating (4.5) and (4.12) we can solve for $b(1)$ in terms of $b(0)$ to get

$$b(1) = \frac{3}{-\lambda_0 - \dfrac{z(-1)}{z(0)}\dfrac{2}{3} + \dfrac{b(0)}{3}} + 4\lambda_0 + \frac{4}{3 - \lambda_0}. \tag{4.13}$$

Thus we now only have to solve for one unknown namely $b(0)$.

The terminal end boundary condition is given by (2.24) with data from problem 1. This simplifies to:

$$\frac{v(2)}{v(1)} = -0.8436. \tag{4.14}$$

To solve for $b(0)$ we substitute the correct coefficients $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given in (4.6) - (4.10) together with the expression for $\dfrac{v(-1)}{v(0)}$ given in (2.22) into (2.16)

with $n = 0$. In addition we substitute the values for $c_v(0)$, $c_v(1)$ and $b_v(1)$ from (4.7) - (4.10) together with (4.14) into (2.16) with $n = 1$. From the resulting two equations we eliminate the ratio $\dfrac{v(1)}{v(0)}$ to obtain a single equation in μ. Simplification in Mathematica (see Appendix for code) yields the 4th order polynomial in μ:

$$A_0 + A_1\mu + A_2\mu^2 + A_3\mu^3 + A_4\mu^4 = 0, \tag{4.15}$$

where

$A_3 = 99.9790 + 50.3765b(0) + b(0)^2$ and

$A_4 = -9.1381 - 5.0000b(0)^2$.

We only state explicitly the 3rd and 4th order terms as these will be sufficient to solve the inverse problem. For the full polynomial see the Appendix.

Since we are given $\lambda_0, ..., \lambda_4$ we have that:

$$\mu_1 = 1.1799, \mu_2 = 1.8083, \mu_3 = 3.3959, \mu_4 = 4.5568.$$

The eigencondition is then given by the 4th order polynomial in μ:

$$(\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3)(\mu - \mu_4) = 0, \tag{4.16}$$

i.e.

$$33.0164 - 63.2085\mu + 41.3722\mu^2 - 10.9409\mu^3 + \mu^4 = 0. \tag{4.17}$$

To enable us to compare coefficients of powers of μ in (4.15) with those in (4.17) we need to turn (4.15) into a monic polynomial, i.e. the coefficient of μ^4 must be 1. We do this by dividing through the 4th order polynomial by A_4. We then equate powers of μ in the resulting polynomial (see Appendix for full polynomial) with those in (4.17).

We have a set of four simultaneous equations in $b(0)$. Solving $\frac{A_3}{A_4}$ = -10.9409, and $\frac{A_2}{A_4} = 41.3722$ gives $b(0) = 0$ uniquely. The other two equations conform.

Substituting this value of $b(0)$ into (4.13) we obtain $b(1) = 1$ as expected.

Problem 2

Consider the following boundary value problem on the interval -1, 0, 1, 2, that is, n = 0, 1, given by (4.1), (2.32) and (2.3).

Assume that we are given the eigenvalues:

λ_0 = -2.7990, λ_1 = -1.5483, λ_2 = -0.5875, λ_3 = 0.5092, λ_4 =3.3423.

To ensure that the least eigenvalue is equal to zero, we need to shift the boundary value problem as before.

Hence (4.1) becomes (4.2), with corresponding boundary conditions (2.9) and (2.35).

Let $z(n)$ be a solution to (4.2), satisfying (2.9) and (2.35) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Hence $z(n)$ satisfies (4.3).

If we substitute $n = 0$ in (4.3), recall that we get (4.4), and similarly, with $n = 1$ in (4.3) we get (4.5).

As a result, the transformed, shifted equation is given by (2.16) with $c_v(n)$ and $b_v(n)$ given by (4.6) - (4.10) (with values related to problem 2).

Now $\frac{v(-1)}{v(0)}$ is given by (2.40). Thus substituting in (2.36) and (2.37) into (2.40) gives the Nevanlinna function:

$$\frac{v(-1)}{v(0)} = -\frac{0.3950}{\mu - 2.5190} - \frac{0.07970}{\mu - 1.284}. \tag{4.18}$$

Recall that $b(1)$ is given in terms of $b(0)$ by (4.13) where all relevant quantities are from problem 2 and the terminal boundary condition is given by (2.24) with data relevant to problem 2, giving:

$$\frac{v(1)}{v(2)} = -0.8622. \tag{4.19}$$

In order to solve for $b(0)$ firstly we substitute the relevant coefficients $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given in (4.6) - (4.9), together with $\frac{v(-1)}{v(0)}$ given by (4.18) into (2.16) with $n = 0$. Secondly we substitute the values for $c_v(0)$, $c_v(1)$ and $b_v(1)$ obtained from (4.7) - (4.10) together with the reciprocal of (4.19) into (2.16) for $n = 1$. Again, using these two equations we can get rid of the ratio $\frac{v(1)}{v(0)}$ and the resulting expression is the 4th order polynomial:

$$B_0 + B_1\mu + B_2\mu^2 + B_3\mu^3 + B_4\mu^4 = 0, \tag{4.20}$$

where $B_4 = -2.3466 - 3.0000b(0)$ and $B_3 = 30.2980 + 30.8876b(0) + b(0)^2$.

Since we are given $\lambda_0, ..., \lambda_4$ we have that $\mu_1 = 1.2507, \mu_2 = 2.2115, \mu_3 = 3.3082, \mu_4 = 6.1412$.

The eigencondition is then given by the 4th order polynomial in (4.16) with data from problem 2 i.e.

$$56.1941 - 96.4769\mu + 55.7987\mu^2 - 12.9117\mu^3 + \mu^4 = 0. \tag{4.21}$$

To enable us to compare coefficients of powers of μ in (4.20) with those in (4.21) we need to turn (4.20) into a monic polynomial. We do this by dividing through the 4th order polynomial by B_4. We then equate powers of μ in the resulting polynomial with those in (4.21). This gives a set of four simultaneous equations in $b(0)$. Solving two of these equations simultaneously in Mathematica, i.e. solving $\frac{B_3}{B_4}$ = -12.9117 and $\frac{B_2}{B_4} = 55.7987$ gives $b(0) = 0$ uniquely. Substituting this value of $b(0)$ into (4.13) gives $b(1) = 1$ as expected.

Problem 3

Consider the following boundary value problem on the interval -1, 0, 1, 2, that is for $n = 0$, 1, given by (4.1), (2.2) and (2.45).

Assume that we are given the eigenvalues: $\lambda_0 = -1.1588$, $\lambda_1 = -0.2557$, $\lambda_2 = 0.4658$, $\lambda_3 = 2.1738$.

Shifting the boundary value problem in the usual way we get (4.2), (2.8) and (2.48).

Now let $z(n)$ be a solution to (4.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Hence $z(n)$ satisfies (4.3) and thus equations (4.4) and (4.5) hold. Recall $z(n)$ also obeys (2.8) and (2.48) for $\mu = 0$.

The transformed, shifted equation is given by (2.16), with $c_v(n)$ and $b_v(n)$ given by (4.6)-(4.10) with data corresponding to problem 3.

The initial boundary condition $\dfrac{v(-1)}{v(0)}$ is given by (2.40) with $F(\mu)$ as in (2.49). Thus substituting (2.11) and (2.49) into (2.40) and simplifying gives the Nevanlinna function:

$$\frac{v(-1)}{v(0)} = 0.5879 - \frac{0.1563}{\mu - 3.3100} - \frac{0.2597}{\mu - 1.4800}. \tag{4.22}$$

Since $z(n)$ obeys (2.48) for $\mu = 0$, we get

$$\frac{z(2)}{z(1)} = \lambda_0 + 1. \tag{4.23}$$

Equating (4.5) and (4.23) we can solve for $b(1)$ in terms of $b(0)$ to get

$$b(1) = \frac{3}{-\lambda_0 - \dfrac{z(-1)}{z(0)}\dfrac{2}{3} + \dfrac{b(0)}{3}} + 8\lambda_0 + 4. \tag{4.24}$$

Now $\dfrac{v(0)}{v(1)}$ is the terminal boundary condition since our interval reduces by 1. From [13, Theorem 3.4],

$$\frac{v(0)}{v(1)} = \frac{4}{3}\frac{z(1)}{z(0)} - \frac{\dfrac{4}{3}\mu\dfrac{z(1)}{z(0)}}{\dfrac{z(2)}{z(1)} - \dfrac{1}{G(\mu)}} \tag{4.25}$$

where $\dfrac{z(1)}{z(0)}$, $\dfrac{z(2)}{z(1)}$ and $G(\mu)$ are given by (4.4), (4.23) and (2.50) respectively. Thus we obtain:

$$\frac{v(0)}{v(1)} = 1.2758. \tag{4.26}$$

To obtain the eigencondition we substitute the coefficients $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given in (4.6) -(4.9) together with the expressions for $\dfrac{v(-1)}{v(0)}$ in (4.22) and $\dfrac{v(1)}{v(0)}$ (given by the reciprocal of (4.26)) into (2.16), for $n = 0$. Simplifying in Mathematica we obtain the following 3rd order polynomial in μ:

$$D_0 + D_1\mu + D_2\mu^2 + D_3\mu^3 = 0, \tag{4.27}$$

where

$D_2 = 42.0550 + 26.8179b(0) + b(0)^2$ and $D_3 = -7.1762 - 5.0000b(0)$.

Since we are given $\lambda_0, ..., \lambda_3$ we have that: $\mu_1 = 0.9031, \mu_2 = 3.3326, \mu_3 = 1.6246$. Consequently we get the following eigencondition:

$$(\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3) = 0. \tag{4.28}$$

I.e.

$$-4.8898 + 9.8913\mu - 5.8604\mu^2 + \mu^3 = 0. \tag{4.29}$$

In order to compare coefficients of powers of μ in (4.29) with those in (4.27) turn (4.27) into a monic polynomial by dividing through by D_3 then equate powers of μ in the resulting polynomial against the powers of μ in (4.29). This results in 3 simultaneous equations in $b(0)$, two of which are $\frac{D_2}{D_3} = -5.8604$ and $\frac{D_1}{D_3} = 9.8913$, which when solved give $b(0) = 0$. Next, substituting this value of $b(0)$ into (4.24) we obtain $b(1) = 1$ as expected.

Problem 4

Consider the following boundary value problem on the interval n = 0,1, given by (4.1) with boundary conditions (2.32) and (2.45).

Assume that we are given the eigenvalues: λ_0 = -2.7545, λ_1 = -1.5251, λ_2 = -0.6205, λ_3 = 0.1918.

Shifting the boundary value problem we get (4.2), (2.35) and (2.48).

Now let $z(n)$ be a solution to (4.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Hence $z(n)$ satisfies (4.3) and we obtain (4.4) and (4.5). Note that $z(n)$ obeys (2.35) and (2.48) for $\mu = 0$.

The transformed, shifted equation is given by (2.16), with $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given by (4.6), (4.7) and (4.9) with data from problem 4.

Since $\frac{v(-1)}{v(0)}$ is given by (2.40) with $F(\mu)$ given by (2.55), substituting (2.36) and (2.55) into (2.40) gives:

$$\frac{v(-1)}{v(0)} = -\frac{0.4102}{\mu = 2.5166} - \frac{0.0764}{\mu - 1.2559}. \tag{4.30}$$

Since the interval reduces by 1, as in problem 3, $\dfrac{v(1)}{v(0)}$ is given by (6.15) using data relative to problem 4. So the terminal end boundary condition is given by:

$$\frac{v(0)}{v(1)} = 0.4203. \tag{4.31}$$

Recall, $b(1)$ is given in terms of $b(0)$ by (4.24). Furthermore, substituting the coefficients $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given in (4.6), (4.7) and (4.9) respectively, together with the expressions for $\dfrac{v(-1)}{v(0)}$ in (4.30) and the reciprocal of (4.31) into (2.16) for $n = 0$ and simplifying in Mathematica gives:

$$E_0 + E_1\mu + E_2\mu^2 + E_3\mu^3 = 0, \tag{4.32}$$

where $E_2 = 8.9494 + 12.2633b(0) + b(0)^2$ and $E_3 = -1.4184 - 3.0000b(0)$.

Given $\lambda_0, ..., \lambda_3$ we have:

$$\mu_1 = 1.2294, \mu_2 = 2.1340, \mu_3 = 2.9463. \tag{4.33}$$

Thus the eigencondition is given by the 3rd order polynomial in μ namely (4.28). I.e.

$$-7.7295 + 12.5329\mu - 6.3097\mu^2 + \mu^3 = 0. \tag{4.34}$$

To enable us to compare coefficients of powers of μ in (4.32) with those in (4.34) we divide through (4.32) by E_3 and equate powers of μ in the resulting polynomial against the powers of μ in (4.34). This gives a set of 3 simultaneous equations in $b(0)$. We solve two of them, i.e. we solve $\dfrac{E_2}{E_3} = -6.3097$ and $\dfrac{E_1}{E_3} = 12.5329$ to get $b(0) = 0$ uniquely. Substituting this value of $b(0)$ into (4.24) we obtain $b(1) = 1$.

Problem 5

Consider the following boundary value problem for $n = 0$, 1, given by (4.1), (2.2) and (2.60).

Assume that we are given the eigenvalues:

λ_0 = -1.2370, λ_1 = -0.0403, $\lambda_2 = 0.6122$, $\lambda_3 = 2.1818$.

Shifting the boundary value problem (4.1) becomes (4.2), with corresponding boundary conditions (2.8) and (2.60).

Let $z(n)$ be a solution to (4.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Hence $z(n)$ satisfies (4.3), and thus (4.4) and (4.5) hold. For $\mu = 0$, $z(n)$ satisfies (2.8) and (2.60).

The transformed, shifted equation is given by (2.16), with $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given by (4.6), (4.7) and (4.9) with data from problem 5.

By (2.60):

$$\frac{z(2)}{z(1)} = \frac{1}{3}. \tag{4.35}$$

Equating (4.5) and (4.35) we can solve for $b(1)$ in terms of $b(0)$ to get

$$b(1) = \frac{3}{-\lambda_0 - \dfrac{z(-1)}{z(0)}\dfrac{2}{3} + \dfrac{b(0)}{3}} + 4\lambda_0 + \frac{4}{3}. \tag{4.36}$$

The initial end boundary condition $\dfrac{v(-1)}{v(0)}$ is given by (2.40) with $F(\mu)$ given by (2.63). Now substituting (2.11) and (2.63) into (2.40) and solving gives the Nevanlinna function:

$$\frac{v(-1)}{v(0)} = 0.6816 - \frac{0.1731}{\mu - 3.3825} - \frac{0.2839}{\mu - 1.5384}. \tag{4.37}$$

The terminal end boundary condition is given by (2.24) with all quantities corresponding to problem 5, i.e.

$$\frac{v(1)}{v(2)} = 0. \tag{4.38}$$

By (4.38), $v(1) = 0$. Substituting $v(1) = 0$, the coefficients $c_v(-1)$, $c_v(0)$ and $b_v(0)$ (given in (4.6), (4.7) and (4.9) respectively), together with the expression for $\frac{v(-1)}{v(0)}$ (in (4.35)) into (2.16) for $n = 0$ and simplifying yields:

$$F_0 + F_1\mu + F_2\mu^2 + F_3\mu^3 = 0, \tag{4.39}$$

where $F_3 = -9.7515 - 5.0000b(0)$ and $F_2 = 63.0404 + 28.5053b(0) + b(0)^2$.

Recall that we are given $\lambda_0, ..., \lambda_3$, so we have:

$$\mu_1 = 1.1966, \mu_2 = 1.8492, \mu_3 = 3.4188. \tag{4.40}$$

The eigencondition is then given by the polynomial (4.28).

I.e.

$$-7.5652 + 12.6260\mu - 6.4647\mu^2 + \mu^3 = 0. \tag{4.41}$$

By comparing the coefficients of powers of μ in (4.39) with those in (4.41) we are able to solve for $b(0)$ uniquely. To do this we turn (4.39) into a monic polynomial by dividing through by F_3 and equating powers of μ in the resulting polynomial with those in (4.41) to get $\frac{F_2}{F_3} = -6.4647$ and $\frac{F_1}{F_3} = 12.6260$. From these two equations we get $b(0) = 0$. Substituting this value of $b(0)$ into (4.36) we obtain $b(1) = 1$ as expected.

Problem 6

Consider the following boundary value problem (4.1), (2.2) and (2.69), with $n = 0, 1$.

Let the eigenvalues:

λ_0 = -1.4812, λ_1 = -1.1470, λ_2 = 0.0000, λ_3 = 0.6294, λ_4 = 2.1820, be given.

We shift the boundary value problem (4.1), (2.2), (2.69) so that (4.1) becomes (4.2), with corresponding boundary conditions (2.8) and (2.72).

As always $z(n)$ is a solution to (4.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Hence $z(n)$ satisfies (4.3). In addition $z(n)$ also satisfies the shifted boundary conditions (2.8) and (2.72) for $\mu = 0$.

Again, the transformed, shifted equation is given by (2.16), with $c_v(-1)$, $c_v(0)$, $b_v(0)$, $c_v(1)$, $b_v(1)$, given by (4.6) - (4.10) with data from problem 6.

The initial end boundary condition $\dfrac{v(-1)}{v(0)}$ is given by (2.40) with $F(\mu)$ given by (2.73). Substituting (2.11) and (2.73) into (2.40) and evaluating gives the Nevanlinna function:

$$\frac{v(-1)}{v(0)} = 1.2469 - \frac{0.2793}{\mu - 3.6130} - \frac{0.4407}{\mu - 1.7350}. \tag{4.42}$$

The terminal end boundary condition is given by (2.24) using data from problem 6, i.e.

$$\frac{v(1)}{v(2)} = \frac{2.2545}{\mu + 1.1814}. \tag{4.43}$$

Now since $z(n)$ obeys (2.72) for $\mu = 0$, we have

$$\frac{z(2)}{z(1)} = \frac{1}{\left(3 + \dfrac{1}{\lambda_0 + 1}\right)}. \tag{4.44}$$

Equating (4.5) and (4.44) we can find $b(1)$ in terms of $b(0)$ as follows:

$$b(1) = \frac{3}{-\lambda_0 - \dfrac{z(-1)}{z(0)}\dfrac{2}{3} + \dfrac{b(0)}{3}} + 4\lambda_0 + \frac{4}{3 + \dfrac{1}{\lambda_0 + 1}}. \tag{4.45}$$

By substituting the correct coefficients $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given in (4.6), (4.7), (4.9) together with $\dfrac{v(-1)}{v(0)}$ in (4.42) into (2.16) with $n = 0$ we get an expression for $\dfrac{v(0)}{v(1)}$. Consequently, substituting this expression for $\dfrac{v(0)}{v(1)}$, together with the values of $c_v(0)$, $c_v(1)$, $b_v(1)$ (given in (4.7), (4.8), (4.10)) and the reciprocal of (4.43) into (2.16) for $n = 1$ and using Mathematica to simplify we obtain the following 4th order polynomial:

$$G_0 + G_1\mu + G_2\mu^2 + G_3\mu^3 + G_4\mu^4 = 0, \tag{4.46}$$

where $G_3 = 132.0959 + 38.1957b(0) + b(0)^2$ and $G_4 = -17.4057 - 5.0000b(0)$.

As $\lambda_0, ..., \lambda_3$ we have that:

$$\mu_1 = 0.3342, \mu_2 = 1.4812, \mu_3 = 2.1106, \mu_4 = 3.6632. \tag{4.47}$$

The eigencondition is then given by:

$$(\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3)(\mu - \mu_4) = 0. \tag{4.48}$$

i.e.

$$3.8275 - 16.8944\mu + 18.7086\mu^2 - 7.5893\mu^3 + \mu^4 = 0. \tag{4.49}$$

As before, to enable us to compare coefficients of powers of μ in (4.46) with those in (4.48) we turn (4.46) into a monic polynomial, by dividing through (4.46) by G_4. Equating powers of μ in the resulting polynomial with those in (4.49) gives $\dfrac{G_3}{G_4} = -7.5893$ and $\dfrac{G_2}{G_4} = 18.7086$. Solving these equations simultaneously gives $b(0) = 0$ uniquely.

Substituting this value of $b(0)$ into (4.45) gives $b(1) = 1$ as anticipated.

Problem 7

Consider the boundary value problem given by (4.1), (2.32) and (2.60), on the interval -1,..,2.

The eigenvalues

λ_0 = -2.8056, λ_1 = -1.5537, λ_2 = -0.6050, λ_3 = 0.5477 are given.

Shifting the boundary value problem (4.1), (2.32), (2.60) results in (4.1) becoming (4.2), with corresponding boundary conditions (2.35) and (2.60).

We have $z(n)$ a solution of (4.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Hence $z(n)$ satisfies (4.3) as well as (2.35) and (2.60) for $\mu = 0$.

The transformed, shifted equation is given by (2.16), with $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given by (4.6), (4.7) and (4.9) with the information corresponding to problem 7.

Equation (2.40) with $F(\mu)$ given by (2.81), gives an expression for $\dfrac{v(-1)}{v(0)}$. Substituting (2.36) and (2.81) into (2.40) and simplifying gives the Nevanlinna function:

$$\frac{v(-1)}{v(0)} = -\frac{0.3928}{\mu - 2.5195} - \frac{0.0802}{\mu - 1.2884}. \tag{4.50}$$

To find the eigencondition we require the terminal end boundary condition given by (2.24), i.e.

$$\frac{v(1)}{v(2)} = 0. \tag{4.51}$$

Since $z(n)$ obeys (2.60) for $\mu = 0$, $\dfrac{z(2)}{z(1)}$ is given by (4.35). Hence $b(1)$ can be written in terms of $b(0)$ as (4.36).

As before, using (4.51), we get $v(1) = 0$. Substituting this value and the coefficients $c_v(-1)$, $c_v(0)$ and $b_v(0)$ (related to problem 7, given in (4.6), (4.7) and (4.9) respectively), together with the expression for $\dfrac{v(-1)}{v(0)}$ given in (4.55) into (2.16) for the case when $n = 0$ and simplifying using Mathematica results in:

$$H_0 + H_1\mu + H_2\mu^2 + H_3\mu^3 = 0. \tag{4.52}$$

where $\quad H_3 = -2.4796 - 3.0000b(0)$ and $H_2 = 16.8752 + 13.0769b(0) + b(0)^2$.

Note that:

$$\mu_1 = 1.2519, \mu_2 = 2.2005, \mu_3 = 3.3533, \tag{4.53}$$

i.e. the eigencondition (4.28) becomes (for this problem):

$$-9.2379 + 14.3319\mu - 6.8057\mu^2 + \mu^3 = 0. \tag{4.54}$$

Dividing through (4.52) by H_3 enables us to solve for $b(0)$ by comparing coefficients of μ with those in (4.54). This results in three simultaneous equations in $b(0)$. Solving two of these i.e. solving $\dfrac{H_2}{H_3} = -6.8057$ and $\dfrac{H_1}{H_3} = 14.3319$ gives $b(0) = 0$. The other equation conforms.

Now substituting this value of $b(0)$ into (4.36) gives $b(1) = 1$.

Problem 8

Consider the boundary value problem given by (4.1), (2.32) and (2.69), on the interval -1,..,2.

Assume the following eigenvalues

λ_0 = -2.8090, λ_1 = -1.5837, λ_2 = -1.3785, λ_3 = -0.5606, λ_4 = 0.5818

are given.

As a result of shifting the boundary value problem (4.1), (2.32), (2.69), (4.1) becomes (4.2), with corresponding boundary conditions (2.35) and (2.72).

With $z(n)$ a solution of (4.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$, $z(n)$ satisfies (4.3) as well as (2.35) and (2.72) for $\mu = 0$.

The shifted, transformed equation is given by (2.16), with $c_v(-1)$, $c_v(0)$, $c_v(1)$, $b_v(0)$, $b_v(1)$ given by (4.6) - (4.10) with data relevant to problem 8.

The initial end boundary condition $\frac{v(-1)}{v(0)}$ is obtained by substituting (2.88) and other data relevant to problem 8 into (2.40) giving:

$$\frac{v(-1)}{v(0)} = -\frac{0.3918}{\mu - 2.5199} - \frac{0.0804}{\mu - 1.2906}. \tag{4.55}$$

The terminal end boundary condition given by (2.24), i.e.

$$\frac{v(1)}{v(2)} = \frac{0.2259}{\mu - 1.3667}. \tag{4.56}$$

Since $z(n)$ obeys (2.72) for $\mu = 0$, $\frac{z(2)}{z(1)}$ is given by (4.44), so $b(1)$ can be written in terms of $b(0)$ as (4.45).

Substituting the coefficients $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given in (4.6) - (4.9) together with the expression for $\dfrac{v(-1)}{v(0)}$ in (4.55) into (2.16) for the case when $n = 0$ gives an expression for $\dfrac{v(0)}{v(1)}$. Using the resulting expression for $\dfrac{v(0)}{v(1)}$ together with the values for $c_v(0)$, $b_v(0)$, $c_v(1)$ (obtained from (4.7) - (4.10)) and the reciprocal of (4.56) gives a 4th order polynomial in μ. Simplifying using Mathematica results in the following polynomial:

$$J_0 + J_1\mu + J_2\mu^2 + J_3\mu^3 + J_4\mu^4 = 0, \tag{4.57}$$

where

$J_3 = 21.1256 + 17.7823b(0) + b(0)^2$ and $J_4 = -2.5469 - 3.0000b(0)$.

Since

$$\mu_1 = 1.2252, \mu_2 = 1.4305, \mu_3 = 3.3907, \mu_4 = 2.2484, \tag{4.58}$$

the eigencondition given by (4.28) for this particular problem becomes:

$$13.3615 - 30.1294\mu + 24.3520\mu^2 + 8.2948\mu^3 + \mu^4 = 0. \tag{4.59}$$

First we turn (4.57) into a monic polynomial by dividing through by J_4 then we equate powers of μ in the resulting polynomial with those in (4.59). We get four simultaneous equations in $b(0)$. Solving two of these i.e. solving $\dfrac{J_3}{J_4} = -8.2948$ and $\dfrac{J_2}{J_4} = 24.3520$ gives $b(0) = 0$ uniquely.

To obtain $b(1)$ we substitute this value of $b(0)$ into (4.45) to get $b(1) = 1$ as expected.

Chapter 5

Inverse Problems: Reconstructing the potential with cubic λ-dependence at terminal end

Problem 9

Consider the following boundary value problem for $n = 0, 1,$

$$(n+1)^2 y(n+1) - b(n)y(n) + n^2 y(n-1) = -(n+1)^2 \lambda y(n), \tag{5.1}$$

with (3.2) and (3.3).

From the eigenvalues we wish to reconstruct $b(0)$ and $b(1)$. Assume we are given the eigenvalues:

λ_0 = -1.2233, λ_1 = 0.3732, λ_2 = 0.5000, λ_3 = 1.4467, λ_4 = 2.6498.

The boundary value problem is then shifted to ensure that the least eigenvalue is equal to 0, i.e. we set $\lambda = \mu + \lambda_0$. Then we get the shifted boundary value problem:

$$(n+1)^2 y(n+1) - b(n)y(n) + n^2 y(n-1) = -(n+1)^2 (\mu + \lambda_0) y(n) \tag{5.2}$$

together with the boundary conditions (3.8) and (3.9).

If $z(n)$ is a solution to (5.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = 0, 1$, then $z(n)$ satisfies

$$(n+1)^2 z(n+1) - b(n)z(n) + n^2 z(n-1) = -\lambda_0 (n+1)^2 z(n). \tag{5.3}$$

Note that $z(n)$ also obeys (3.8) and (3.9) for $\mu = 0$.

Setting $n = 0$ in (5.3) we get

$$\frac{z(1)}{z(0)} = b(0) - \lambda_0. \tag{5.4}$$

Substituting $n = 1$ in (5.3) we get

$$\frac{z(2)}{z(1)} = \frac{b(1)}{4} - \frac{1}{4}\frac{z(0)}{z(1)} - \lambda_0. \tag{5.5}$$

The transformed, shifted equation is given by (3.17), with $c_v(n)$ and $b_v(n)$ given by

$$c_v(-1) = 0, \tag{5.6}$$

$$c_v(0) = \frac{z(0)}{z(1)}, \tag{5.7}$$

$$c_v(1) = 4\frac{z(1)}{z(2)}, \tag{5.8}$$

$$b_v(0) = \frac{z(0)}{z(1)}\left(\frac{1}{4}\frac{z(0)}{z(1)} + \frac{z(2)}{z(1)}\right). \tag{5.9}$$

$$b_v(1) = \frac{4z(1)}{z(2)}\left(\frac{4}{9}\frac{z(1)}{z(2)} + \frac{z(2)}{z(1)}\right). \tag{5.10}$$

Now $\dfrac{v(-1)}{v(0)}$ is given by:

$$\frac{v(-1)}{v(0)} = \frac{z(0)}{z(-1)}\left(\frac{1}{\dfrac{c(-1)}{c(0)} - \dfrac{\mu}{\dfrac{z(-1)}{z(0)} - F(\mu)}}\right) \tag{5.11}$$

where $F(\mu)$ is given by (3.14). Using (5.11) and simplifying gives the Nevanlinna function:

$$\frac{v(-1)}{v(0)} = 0. \tag{5.12}$$

The terminal end boundary condition is given by (3.22) (with data from problem 9). Simplifying in Mathematica gives:

$$\frac{v(1)}{v(2)} = -27.3411 + \frac{0.6759}{\mu - 3.0540} + \frac{588.4488}{\mu + 18.7600}. \tag{5.13}$$

Since z obeys (3.9) for $\mu = 0$ we get

$$\frac{z(2)}{z(1)} = \frac{1}{\left(-2\lambda_0 + 3 + \dfrac{1}{\lambda_0 - 2} + \dfrac{1}{\lambda_0 + 1}\right)}. \tag{5.14}$$

Equating (5.5) and (5.14) we obtain $b(1)$ in terms of $b(0)$ i.e.

$$b(1) = \frac{z(0)}{z(1)} + 4\lambda_0 + \frac{4}{-2\lambda_0 + 3 + \dfrac{1}{\lambda_0 - 2} + \dfrac{1}{\lambda_0 + 1}}. \tag{5.15}$$

In order to solve for $b(0)$ we substitute $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given by formulae (5.6), (5.7) and (5.9), together with the expression for $\dfrac{v(-1)}{v(0)}$ from (5.12) into (3.17) with $n = 0$. We also substitute $c_v(0), c_v(1), b_v(1)$ (given in (5.7), (5.8) and (5.9)) and the reciprocal of (5.13) into (3.17) with $n = 1$.

Consequently we now have two equations which we use to eliminate the ratio $\frac{v(1)}{v(0)}$ to obtain a single equation in μ which we simplify in Mathematica to get:

$$H_0 + H_1\mu + H_2\mu^2 + H_3\mu^3 + H_4\mu^4 = 0, \tag{5.16}$$

where

$H_3 = 11.152 + 10.1360b(0) + b(0)^2$ and $H_4 = -1.2233 - b(0)$.

Given $\lambda_0, ...\lambda_4$ we have:

$$\mu_1 = 0.8501, \mu_2 = 1.7233, \mu_3 = 2.6700, \mu_4 = 3.8731. \tag{5.17}$$

As a result, the eigencondition is then given by the 4th order polynomial in μ:

$$(\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3)(\mu - \mu_4) = 0, \tag{5.18}$$

i.e.

$$15.1505 - 36.1987\mu + 28.6447\mu^2 - 9.1166\mu^3 + \mu^4 = 0. \tag{5.19}$$

To compare coefficients of powers of μ in (5.16) with those in (5.19) we need to make (5.16) a monic polynomial by dividing through the 4th order polynomial by H_4. As before, we next equate powers of μ in the resulting polynomial to those in (5.19). Thus we will have a set of four simultaneous equations in $b(0)$. Solving two of these equations i.e. solving $\frac{H_3}{H_4}$ = -9.1166 and $\frac{H_2}{H_4}$ = 28.6447 gives $b(0) = 0$. The other equations concur.

Next, substituting this value of $b(0)$ into (5.15) we obtain $b(1) = 2$ as expected.

Problem 10

Consider the boundary value problem for $n = 0$, 1 given by (5.1), (3.3) and (3.26).

Assume we are given the following eigenvalues:

λ_0 = -1.2233, λ_1 = 0.3732, λ_2 = 0.5000, λ = 1.4467, λ = 2.6498.

As before, to ensure that the least eigenvalue is equal to zero, the boundary value problem is shifted by setting $\lambda = \mu + \lambda_0$ resulting in the shifted boundary value problem (5.2), (3.27), (3.9).

Since $z(n)$ is a solution to (5.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = 0, 1$, then as before, $z(n)$ satisfies (5.3).

Indeed, $z(n)$ also obeys (3.27) and (3.9) for $\mu = 0$.

Now setting $n = 0$ and $n = 1$ in (5.3) gives (5.4) and (5.5) respectively.

As determined before, the transformed, shifted equation is given by (3.17), with $c_v(n)$ and $b_v(n)$ given by (5.6) - (5.10).

The initial end transformed boundary condition $\dfrac{v(-1)}{v(0)}$ is obtained from (5.11) with $F(\mu)$ given by (3.28).

Using (5.11) we obtain the Nevanlinna function:

$$\frac{v(-1)}{v(0)} = -0.3102. \tag{5.20}$$

The terminal boundary condition is given by (5.13) because problems 9 and 10 have exactly the same eigenvalues.

Since (3.9) for $\mu = 0$ gives (5.14), equating (5.5) and (5.14) gives (5.15).

Likewise, in order to solve for $b(0)$ we substitute $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given by (5.6), (5.7) and (5.9), together with the expression for $\dfrac{v(-1)}{v(0)}$(obtained in (5.20)) into (3.17) for $n = 0$.

Additionally, we substitute $c_v(0)$, $c_v(1)$, $b_v(1)$ (calculated using (5.7), (5.8) and (5.10)) as well as the reciprocal of (5.13) into (3.17) with $n = 1$. From the resulting two equations we eliminate the ratio $\frac{v(1)}{v(0)}$ to obtain a single equation in μ which we simplify in Mathematica to get (5.16).

Now since $\lambda_0, ..., \lambda_4$ is given, we have

$$\mu_1 = 0.8501, \mu_2 = 1.7233, \mu_3 = 2.6700, \mu_4 = 3.8731.$$

As a result, the eigencondition is then given by (5.19) since problem 9 and 10 have the same exact eigenvalues.

To find $b(0)$ and $b(1)$ we compare coefficients of powers of μ in (5.16) with those in (5.19). Exactly as in problem 9 this gives $b(0) = 0$ and $b(1) = 2$ uniquely.

Problem 11

Consider the boundary value problem given by (5.1), (3.36) and (3.3).

Given the following eigenvalues below:

λ_0 = -1.2233, λ_1 = 0.3732, λ_2 = 0.5000, λ = 1.4467, λ = 2.6498,

as before, to ensure that the least eigenvalue is equal to zero, we shift the boundary value problem by setting $\lambda = \mu + \lambda_0$, giving the shifted boundary value problem (5.2), (3.37), (3.9).

Now $z(n)$ is a solution to (5.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = 0, 1$, hence $z(n)$ satisfies (5.3).

Note that $z(n)$ also obeys (3.37) and (3.9) for $\mu = 0$.

By setting $n = 0$ and $n = 1$ in (5.3) we obtain (5.4) and (5.5) respectively.

We have already established that the transformed, shifted equation is given by (3.17), with $c_v(n)$ and $b_v(n)$ given by (5.6) - (5.10).

Now $\dfrac{v(-1)}{v(0)}$ is given by (3.40), which simplifies to (3.41).

As before, the terminal end boundary condition is given by (5.13) owing to the two problems having the same eigenvalues and $b(1)$ is given in terms of $b(0)$ by (5.15). Using the same procedure as in the previous two problems i.e. using (3.17) evaluated at $n = 0$ and $n = 1$ together with the relevant coefficients and boundary conditions results in two equations from which we eliminate the ratio $\dfrac{v(1)}{v(0)}$ to obtain a single equation in μ. Simplification in Mathematica yields the 4th order polynomial (5.16).

Given $\lambda_0, ..., \lambda_4$, we get (5.17) and thus (5.18) and (5.19) because we have the same eigenvalues as in all previous problems. Equating coefficients of powers of μ in (5.16) and (5.19) yields $b(0) = 0$ and $b(1) = 2$ uniquely.

Problem 12

Given the boundary value problem (5.1), (3.3) and (3.43), for $n = 0, 1$.

Assume we are given the following eigenvalues:

λ_0 = -1.2233, λ_1 = 0.3732, λ_2 = 0.5000, λ = 1.4467, λ = 2.6498.

To ensure that the least eigenvalue is equal to zero, we shift the boundary value problem by setting $\lambda = \mu + \lambda_0$, resulting in the shifted boundary value problem (5.2), (3.44), (3.9).

If $z(n)$ is a solution of (5.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = 0, 1$, then $z(n)$ satisfies (5.3).

Additionally, $z(n)$ obeys (3.44) and (3.9) for $\mu = 0$.

Setting $n = 0$ and $n = 1$ in (5.3) gives (5.4) and (5.5) respectively.

Note, the transformed, shifted equation is given by (3.17), with $c_v(n)$ and $b_v(n)$ given by (5.6) - (5.10).

The initial end transformed boundary condition $\dfrac{v(-1)}{v(0)}$ is given by (3.47) which simplifies to (3.48). Note that (5.15) holds and the terminal end boundary condition is given by (5.13).

It is easy to show (using the method from problems 9 and 10) that (5.16) holds.

Because we have the same eigenvalues as in earlier problems, given $\lambda_0, ..., \lambda_4$, we get (5.17), (5.18) and (5.19).

Using (5.16) and (5.19) (as before) we uniquely obtain $b(0) = 0$ and $b(1) = 2$.

Problem 13

Given the boundary value problem (5.1), (3.2) and (3.49), for $n = 0, 1$, and the following eigenvalues:

λ_0 = -9.4213, λ_1 = -0.2840, $\lambda_2 = 0.6612$, $\lambda_3 = 2.5440$.

In order to have the least of the above listed eigenvalues equal to zero, we shift the boundary value problem by setting $\lambda = \mu + \lambda_0$, resulting in the shifted boundary value problem (5.2), (3.2) and (3.52).

Now since $z(n)$ is a solution of (5.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = 0, 1$, then $z(n)$ satisfies (5.3) and $z(n)$ obeys (3.2) and (3.52) for $\mu = 0$.

Next, setting $n = 0$ and $n = 1$ in (5.3) gives (5.4) and (5.5) respectively.

The transformed, shifted equation is given by (3.17), with $c_v(n)$ and $b_v(n)$ given by (5.6) - (5.10) as before.

The initial end transformed boundary condition $\dfrac{v(-1)}{v(0)}$ is given by (3.21).

Since z obeys (3.52) for $\mu = 0$ we get

$$\frac{z(2)}{z(1)} = \frac{1}{\dfrac{1}{2} + \dfrac{3}{\lambda_0} + \dfrac{1}{\lambda_0 - 3}}. \tag{5.21}$$

Now $b(1)$ is given by equating (5.5) and (5.21) to get

$$b(1) = \frac{4}{\dfrac{1}{2} + \dfrac{3}{\lambda_0} + \dfrac{1}{\lambda_0 - 3}} + \frac{1}{b(0) - \lambda_0} + 4\lambda_0. \tag{5.22}$$

The terminal boundary condition is given by (3.22) and using the data from problem 13 yields (3.57).

In order to solve for $b(0)$ we substitute $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given by (5.6), (5.7) and (5.9), together with the expression for $\dfrac{v(-1)}{v(0)}$ (in (3.21)) into (3.17) with $n = 0$. Moreover, substituting the reciprocal of (3.57) into (3.17) with $n = 1$, together with $c_v(0), c_v(1), b_v(1)$ (obtained from (5.7), (5.8) and (5.10)) gives a second equation. We use the two equations obtained above to eliminate the ratio $\dfrac{v(1)}{v(0)}$ and obtain a single equation in μ. Simplifying in Mathematica gives the 3rd order polynomial in μ:

$$M_0 + M_1\mu + M_2\mu^2 + M_3\mu^3 = 0, \tag{5.23}$$

where

$M_2 = 293.8021 + 40.5797b(0) + b(0)^2$ and $M_3 = -9.4213 - b(0)$.

Given $\lambda_0, ..., \lambda_4$, we have:

$$(\mu - 9.1373)(\mu - 10.0824)(\mu - 11.9653) = 0, \tag{5.24}$$

which simplifies in Mathematica to give:

$$-1102.3172 + 322.0960\mu - 31.1850\mu^2 + \mu^3 = 0. \tag{5.25}$$

By comparing coefficients of powers of μ in (5.23) with those in (5.25), we can find $b(0)$ and $b(1)$. First divide through (5.23) by M_3. By equating powers of μ in the resulting polynomial to the corresponding ones in (5.25) we get three simultaneous equations in $b(0)$. We solve $\frac{M_2}{M_3} = -31.1850$ and $\frac{M_1}{M_3} = 322.0960$ to get $b(0) = 0$ uniquely.

Substituting $b(0)$ into (5.22) we obtain $b(1) = 2$ as expected.

Problem 14

Consider the boundary value problem (5.1), (3.26) and (3.49) for $n = 0, 1$ with the following eigenvalues:

λ_0 = -9.4213, λ_1 = -0.2840, $\lambda_2 = 0.6612$, $\lambda_3 = 2.5440$.

As before, we shift the boundary value problem by setting $\lambda = \mu + \lambda_0$, to get the shifted boundary value problem (5.2), (3.27) and (3.52).

We have established earlier that $z(n)$ is a solution of (5.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = 0, 1$, so $z(n)$ satisfies (5.3) and $z(n)$ obeys (3.27) and (3.52) for $\mu = 0$.

Next, setting $n = 0$ and $n = 1$ in (5.3) gives (5.4) and (5.5) respectively.

Hence the transformed, shifted equation is given by (3.17), with $c_v(n)$ and $b_v(n)$ given by (5.6) - (5.10) as before.

Also, the initial end transformed boundary condition $\frac{v(-1)}{v(0)}$ is given by (3.34). Using and simplifying (3.34) gives (3.59).

Since (5.5) and (5.21) hold we get $b(1)$ in terms of $b(0)$ given by (5.22).

As before, the terminal end boundary condition is given by (3.22) (with data related to problem 14), which simplifies to give (3.57).

To get $b(0)$ we substitute $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given by (5.6), (5.7) and (5.9), together with the expression for $\dfrac{v(-1)}{v(0)}$ (in (3.59)) into (3.17) with $n = 0$. Secondly, we substitute the reciprocal of (3.57) into (3.17) with $n = 1$ together with the coefficients $c_v(0), c_v(1)$ and $b_v(1)$ given in (5.7), (5.8), (5.10) respectively, to get two equations which we use to eliminate the ratio $\dfrac{v(1)}{v(0)}$ to give a single equation in μ. Simplifying gives the 3rd order polynomial in (5.23).

Given $\lambda_0, ..., \lambda_4$, we get (5.25). Now, comparing coefficients of powers of μ in (5.23) with those in (5.25), we can find $b(0) = 0$ and $b(1) = 2$ uniquely as in problem 13.

Problem 15

We examine the boundary value problem (5.1), (3.36), (3.49) for n = 0,1 with the following eigenvalues:

λ_0 = -9.4213, λ_1 = -0.2840, λ_2 = 0.6612, λ_3 = 2.5440.

Shifting the boundary value problem by setting $\lambda = \mu + \lambda_0$ gives the shifted boundary value problem (5.2), (3.37) and (3.52).

As noted earlier, since $z(n)$ is a solution of (5.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = 0, 1$, so $z(n)$ satisfies (5.3) and $z(n)$ obeys (3.37) and (3.52) for $\mu = 0$.

Setting $n = 0$ and $n = 1$ in (5.3) gives (5.4) and (5.5) respectively.

The transformed, shifted equation is given by (3.17), with $c_v(n)$ and $b_v(n)$ given by (5.6) - (5.10) as before.

Also, the initial end transformed boundary condition $\frac{v(-1)}{v(0)}$ is given by (3.40) which results in the simplified form (3.60).

Using (3.22) we obtain the terminal end boundary condition by simplifying in Mathematica to get (3.57).

Recall give $b(1)$ in terms of $b(0)$ is given by (5.22).

Applying the same procedure as described in problems 13 and 14 i.e. using (3.17) evaluated at $n = 0$ and $n = 1$ together with all the coefficients and transformed boundary conditions enables us to eliminate the ratio $\frac{v(1)}{v(0)}$ and get a single equation in μ. Simplifying yields (5.23).

Given $\lambda_0, ..., \lambda_4$, we get (5.25). Making use of (5.23) and (5.25), as before, results in a unique reconstruction of $b(0)$ and $b(1)$ namely $b(0) = 0$ and $b(1) = 2$.

Problem 16

Consider the boundary value problem (5.1), (3.43) and (3.49) with the following eigenvalues:

λ_0 = -9.4213, λ_1 = -0.2840, λ_2 = 0.6612, λ_3 = 2.5440.

To shift the boundary value problem we set $\lambda = \mu + \lambda_0$ to get the shifted boundary value problem (5.2), (3.44) and (3.52).

Because $z(n)$ is a solution of (5.2) for $\mu = 0$ such that $z(n) > 0$ for all $n = 0, 1$, $z(n)$ satisfies (5.3) and obeys (3.44) and (3.52) for $\mu = 0$.

Putting $n = 0$ and $n = 1$ in (5.3) gives (5.4) and (5.5) respectively.

Now, the transformed, shifted equation is given by (3.17), with $c_v(n)$ and $b_v(n)$ given by (5.6) - (5.10) as before.

The initial end transformed boundary condition $\dfrac{v(-1)}{v(0)}$ is given by (3.47). Using (3.47) and simplifying gives (3.61).

The terminal end boundary condition is given by (3.22) (using data from problem 16), which simplifies to (3.57).

Once again (5.22) holds and furthermore both (5.23) and (5.25) hold so from these we obtain $b(0) = 0$ and $b(1) = 2$ uniquely.

Chapter 6

Reconstructing $b(n)$: Generalisation of results.

In this chapter we generalize the results that we have obtained in chapters 1 to 5 i.e. we prove general results on reconstruction of the potential $b(n)$ for different cases. We follow the same methods as those employed in the aforementioned chapters, and hence are guided towards getting the correct form of the boundary conditions as well as the correct number of eigenvalues. That is, we shift the given boundary value problem, transform it then obtain one less eigenvalue using steps similar to those in Chapters 1 and 2 using the results in [13]. Next we examine the reconstruction of the potential $b(n)$ by following similar steps as in chapters 4 and 5.

The work in this chapter has been divided into two sections, Section 6.1 pertains to the case when we have cubic λ-dependence at the initial end. Here we prove a total of five cases, each case with a unique set of conditions. In Section 6.2 we examine the case where we have cubic λ-dependence at the terminal end and prove another 6 unique cases.

6.1 Cubic dependence at the initial end

Consider

$$c(n)y(n+1) - b(n)y(n) + c(n-1)y(n-1) = -c(n)\lambda y(n) \tag{6.1}$$

together with the boundary conditions

$$\begin{aligned} y(-1) &= \left(a\lambda + b - \frac{c_1}{\lambda - d_1} - \frac{c_2}{\lambda - d_2}\right) y(0) \\ &= Ay(0) \quad a \geq 0, c_1, c_2 > 0, \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} y(1) &= \left(\alpha\lambda + \beta - \frac{\gamma}{\lambda - \delta_1}\right) y(2) \\ &= \Lambda y(2) \quad \alpha \leq 0, \gamma \leq 0. \end{aligned} \tag{6.3}$$

Note:

1. We assume neither c_1 nor c_2 equal zero as then we would have exactly the case in [16] where the results have already been proven.

2. The case where $\alpha = \beta = \gamma = 0$ gives the Dirichlet boundary condition at the terminal end which we do not consider because it would immediately imply a smaller interval and we are interested in the case where $m = 2$.

Theorem 6.1.1. *Consider the boundary value problem (6.1), (6.2), (6.3) where*
1. $\alpha = \beta = 0, \gamma \neq 0$.
2. $\alpha = 0, \beta \neq 0, \gamma = 0$.
These boundary value problems have 4 eigenvalues namely $\lambda_0, ..., \lambda_3$. *Given* $\lambda_0, ..., \lambda_3$ *together with* $c(-1), c(0), c(1)$ *we can uniquely reconstruct the potential* $b(0), b(1)$.

Proof. 1. Suppose we are given the eigenvalues $\lambda_0, ..., \lambda_3$. We shift the boundary value problem by setting $\lambda = \mu + \lambda_0$, where λ_0 is the least eigenvalue, this results in

$$c(n)y(n+1) - b(n)y(n) + c(n-1)y(n-1) = -c(n)(\mu + \lambda_0)y(n), \tag{6.4}$$

$$y(-1) = \left(a\mu + a\lambda_0 + b - \frac{c_1}{\mu + \lambda_0 - d_1} - \frac{c_2}{\mu + \lambda_0 - d_2}\right) y(0) =: F(\mu)y(0), \tag{6.5}$$

$$y(1) = \left(\alpha\mu + \alpha\lambda_0 + \beta - \frac{\gamma}{\mu + \lambda_0 - \delta_1}\right) y(2) =: G(\mu)y(2). \tag{6.6}$$

Let $z(n)$ be a solution to (6.4) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$. Hence $z(n)$ satisfies

$$c(n)z(n+1) - b(n)z(n) + c(n-1)z(n-1) = -c(n)\lambda_0 z(n). \tag{6.7}$$

Since $z(n)$ also obeys (6.5) and (6.6) for $\mu = 0$ the following hold:

$$\frac{z(-1)}{z(0)} = a\lambda_0 + b - \frac{c_1}{\lambda_0 - d_1} - \frac{c_2}{\lambda_0 - d_2}, \tag{6.8}$$

and

$$\frac{z(2)}{z(1)} = \frac{1}{\alpha\lambda_0 + \beta - \dfrac{\gamma}{\lambda_0 - \delta_1}}. \tag{6.9}$$

Equation (6.7) evaluated at 0 and 1 respectively yields

$$\frac{z(1)}{z(0)} = \frac{b(0)}{c(0)} - \lambda_0 - \frac{c(-1)}{c(0)}\frac{z(-1)}{z(0)}, \tag{6.10}$$

and

$$\frac{z(2)}{z(1)} = \frac{b(1)}{c(1)} - \frac{c(0)}{c(1)}\frac{z(0)}{z(1)} - \lambda_0. \tag{6.11}$$

The transformed, shifted equation is given by (2.16), with $c_v(n)$ and $b_v(n)$ given by (1.20) and (1.21) respectively.

The initial boundary condition $\dfrac{v(-1)}{v(0)}$ is given by (2.40) with $F(\mu)$ as in (6.5).

Substituting (6.8) and (6.5) into (2.40) gives a Nevanlinna function of the form

$$\frac{v(-1)}{v(0)} = \hat{a} - \frac{\hat{b}}{\mu - \hat{c}} - \frac{\hat{d}}{\mu - \hat{e}}, \tag{6.12}$$

where $\hat{b}, \hat{d} > 0$. Note that if $a = 0$ then $\hat{a} = 0$ otherwise $\hat{a} \neq 0$.

1. Assume $\alpha = \beta = 0$ and $\gamma \neq 0$ then (6.9) simplifies to

$$\frac{z(2)}{z(1)} = -\frac{\lambda_0 - \delta_1}{\gamma} \tag{6.13}$$

Equating (6.11) and (6.13) we can solve for $b(1)$ in terms of $b(0)$ to get

$$b(1) = c(1)\left(\lambda_0 - \frac{\lambda_0 - \delta_1}{\gamma} + \frac{c(0)}{c(1)}\frac{z(0)}{z(1)}\right), \tag{6.14}$$

where $\dfrac{z(0)}{z(1)}$ is given by the reciprocal of (6.10) and contains $b(0)$.

Now $\dfrac{v(0)}{v(1)}$ is the terminal boundary condition since our interval reduces by 1, see [13, Theorem 3.4], from which we obtain

$$\frac{v(0)}{v(1)} = \frac{c(1)}{c(0)}\frac{z(1)}{z(0)} - \frac{\dfrac{c(1)}{c(0)}\mu\dfrac{z(1)}{z(0)}}{\dfrac{z(2)}{z(1)} - \dfrac{1}{G(\mu)}}, \tag{6.15}$$

where $\dfrac{z(1)}{z(0)}$ and $\dfrac{z(2)}{z(1)}$ are given by (6.10) and (6.9) respectively, and $G(\mu)$ is given by (6.6) and simplifies to

$$G(\mu) = -\frac{\gamma}{\mu + \lambda_0 - \delta_1} \tag{6.16}$$

since $\alpha = \beta = 0$.

Thus we obtain:

$$\frac{v(0)}{v(1)} = \hat{e}. \tag{6.17}$$

Substituting the coefficients $c_v(n)$ and $b_v(n)$ given by (1.20) and (1.21) respectively, together with the expressions for $\dfrac{v(-1)}{v(0)}$ in (6.12) and the reciprocal of (6.17) into

(2.16) for $n = 0$ results in a third-order polynomial of the form (we can always without loss of generality make it monic)

$$\mu^3 + \eta_2\mu^2 + \eta_1\mu + \eta_0 = 0. \tag{6.18}$$

Since we are given $\lambda_0, ..., \lambda_3$ we have $\mu_1 = \lambda_1 - \lambda_0$, $\mu_2 = \lambda_2 - \lambda_0$ and $\mu_3 = \lambda_3 - \lambda_0$, the eigencondition is given by the 3rd-order polynomial in μ:

$$(\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3) = 0 \tag{6.19}$$

i.e.

$$\mu^3 + (-\mu_1 - \mu_2 - \mu_3)\mu^2 + (\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3)\mu + (-\mu_1\mu_2\mu_3) = 0. \tag{6.20}$$

Thus we have 3 simultaneous equations namely

$\eta_2 = -\mu_1 - \mu_2 - \mu_3$, $\eta_1 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3$, $\eta_0 = -\mu_1\mu_2\mu_3$,

from which we can solve for $b(0)$ uniquely. Consequently (6.14) yields $b(1)$ uniquely.

2. If $\alpha = 0, \beta \neq 0, \gamma = 0$, the proof proceeds in exactly the same way except we have the following.

Since $z(n)$ obeys (6.6) for $\mu = 0$, we get

$$\frac{z(2)}{z(1)} = \frac{1}{\beta}. \tag{6.21}$$

Equating (6.11) and (6.21) we can solve for $b(1)$ in terms of $b(0)$ to get

$$b(1) = c(1)\left(\frac{1}{\beta} + \frac{c(0)}{c(1)}\frac{z(0)}{z(1)} + \lambda_0\right), \tag{6.22}$$

where again $\dfrac{z(0)}{z(1)}$ is given by the reciprocal of (6.10).

Now the terminal boundary condition is given by (2.24) which results in

$$\frac{v(1)}{v(2)} = 0. \tag{6.23}$$

By (6.23), $v(1) = 0$. Substituting $v(1) = 0$, the coefficients $c_v(n)$ and $b_v(n)$ given in (1.20) and (1.21) respectively, together with the expression for $\dfrac{v(-1)}{v(0)}$ in (6.12) into (2.16) for $n = 0$ yields a (without loss of generality) monic third order polynomial

$$\mu^3 + \phi_2\mu^2 + \phi_1\mu + \phi_0 = 0 \tag{6.24}$$

The eigencondition is then given by the polynomial (6.30). By comparing the coefficients of powers of μ in (6.24) with those in (6.30) we get three simultaneous equations:

$$\phi_2 = -\mu_1 - \mu_2 - \mu_3,\ \phi_1 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3,\ \phi_0 = -\mu_1\mu_2\mu_3,$$

which enable us to solve for $b(0)$ uniquely. Substituting into (6.33) gives $b(1)$ uniquely.

□

Theorem 6.1.2. *Consider the boundary value problem (6.1), (6.2), (6.3) where*
1. $\alpha = 0, \beta \neq 0, \gamma \neq 0$
2. $\alpha < 0, \gamma = 0$
These boundary value problems have 5 eigenvalues namely $\lambda_0, ..., \lambda_4$. *Given* $\lambda_0, ..., \lambda_4$ *together with* $c(-1), c(0), c(1)$ *we can uniquely reconstruct the potential* $b(0), b(1)$.

Proof. Suppose we are given the eigenvalues $\lambda_0, ..., \lambda_4$. Equations (6.4) through to (6.12) hold.

1. If $\alpha = 0$, $\beta \neq 0$ and $\gamma \neq 0$ then (6.9) simplifies to:

$$\frac{z(2)}{z(1)} = \frac{1}{\beta - \dfrac{\gamma}{\lambda_0 - \delta_1}}. \tag{6.25}$$

Equating (6.11) and (6.25) we can solve for $b(1)$ in terms of $b(0)$ to get

$$b(1) = c(1)\left(\lambda_0 + \frac{1}{\beta - \dfrac{\gamma}{\lambda_0 - \delta_1}} + \frac{c(0)}{c(1)}\frac{z(0)}{z(1)}\right), \tag{6.26}$$

where as before, $\dfrac{z(0)}{z(1)}$ is given by the reciprocal of (6.10) and contains $b(0)$.

The terminal boundary condition is given by (2.24) which simplifies to a Nevanlinna function of the form

$$\frac{v(1)}{v(2)} = -\frac{\hat{a}}{\mu + \hat{b}}, \tag{6.27}$$

where $\hat{a} < 0$.

As before, substituting the coefficients $c_v(n)$ and $b_v(n)$ given by (1.20) and (1.21) respectively, together with the expression for $\dfrac{v(-1)}{v(0)}$ in (6.12) and the reciprocal of (6.27) into (2.16) for $n = 0$ gives a fourth-order monic polynomial:

$$\mu^4 + \eta_3\mu^3 + \eta_2\mu^2 + \eta_1\mu + \eta_0 = 0. \tag{6.28}$$

Given $\lambda_0, ..., \lambda_4$ we have $\mu_1 = \lambda_1 - \lambda_0$, $\mu_2 = \lambda_2 - \lambda_0$, $\mu_3 = \lambda_3 - \lambda_0$, and $\mu_4 = \lambda_4 - \lambda_0$. Consequently the eigencondition is given by the 4th-order polynomial in μ:

$$(\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3)(\mu - \mu_4) = 0 \tag{6.29}$$

i.e.

$$\begin{aligned}\mu^4 + (-\mu_1 - \mu_2 - \mu_3 - \mu_4)\mu^3 + (\mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 \\ +\mu_3\mu_4)\mu^2 + (-\mu_1\mu_2\mu_3 - \mu_1\mu_2\mu_4 - \mu_1\mu_3\mu_4 - \mu_2\mu_3\mu_4)\mu + \mu_1\mu_2\mu_3\mu_4 = 0.\end{aligned} \tag{6.30}$$

We thus obtain the following four simultaneous equations:

$$\begin{aligned} \eta_3 &= -\mu_1 - \mu_2 - \mu_3 - \mu_4, \\ \eta_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4, \\ \eta_1 &= -\mu_1\mu_2\mu_3 - \mu_1\mu_2\mu_4 - \mu_1\mu_3\mu_4 - \mu_2\mu_3\mu_4, \\ \eta_0 &= \mu_1\mu_2\mu_3\mu_4, \end{aligned} \tag{6.31}$$

which enable us to solve for $b(0)$ uniquely. Finally, (6.26) yields $b(1)$ uniquely.

2. If $\alpha < 0, \gamma = 0$, as before, the proof proceeds in exactly the same way with the exception of what follows.

Since $\alpha < 0$ and $\gamma = 0$, (6.9) becomes:

$$\frac{z(2)}{z(1)} = \frac{1}{\alpha\lambda_0 + \beta}. \tag{6.32}$$

Now equating (6.11) and (6.32) yields an expression for $b(1)$ in terms of $b(0)$ given by:

$$b(1) = c(1)\left(\frac{1}{\alpha\lambda_0 + \beta} + \frac{c(0)}{c(1)}\frac{z(0)}{z(1)} + \lambda_0\right), \tag{6.33}$$

where as before, $\dfrac{z(0)}{z(1)}$ is given by the reciprocal of (6.10).

To get the terminal boundary condition we use (2.24) which results in

$$\frac{v(1)}{v(2)} = -\frac{\hat{g}}{\mu - \hat{h}}, \tag{6.34}$$

where $\hat{g} < 0$, $\hat{h} > 0$.

By substituting the correct coefficients $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given in (1.20) and (1.21) together with $\dfrac{v(-1)}{v(0)}$ in (6.12) into (2.16) for the case where $n = 0$ gives $\dfrac{v(0)}{v(1)}$. Consequently, substituting this expression for $\dfrac{v(0)}{v(1)}$, together with the values of $c_v(0)$, $c_v(1)$, $b_v(1)$ (calculated using (1.20) and (1.21)) and the reciprocal of (6.34) into (2.16) for $n = 1$ we obtain the following 4th order polynomial:

$$\mu^4 + \phi_3\mu^3 + \phi_2\mu^2 + \phi_1\mu + \phi_0 = 0. \tag{6.35}$$

The eigencondition is then given by the polynomial (6.30). By comparing the coefficients of powers of μ in (6.35) with those in (6.30) we get the following simultaneous equations:

$$\begin{aligned} \phi_3 &= -\mu_1 - \mu_2 - \mu_3 - \mu_4, \\ \phi_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4, \\ \phi_1 &= -\mu_1\mu_2\mu_3 - \mu_1\mu_2\mu_4 - \mu_1\mu_3\mu_4 - \mu_2\mu_3\mu_4, \\ \phi_0 &= \mu_1\mu_2\mu_3\mu_4, \end{aligned} \tag{6.36}$$

which when solved give $b(0)$ uniquely. Consequently (6.33) gives $b(1)$ uniquely.

□

Theorem 6.1.3. *Consider the boundary value problem (6.1), (6.2), (6.3) where $\alpha < 0, \gamma < 0$. This boundary value problem has 6 eigenvalues $\lambda_0, ..., \lambda_5$. Given $\lambda_0, ..., \lambda_5$ together with $c(-1), c(0), c(1)$ we can uniquely reconstruct the potential $b(0), b(1)$.*

Proof. Again (6.4) - (6.12) hold. By equating (6.9) and (6.11) we obtain $b(1)$ in terms of $b(0)$ i.e. we get:

$$b(1) = c(1)\left(\lambda_0 + \frac{1}{\alpha\lambda_0 + \beta - \dfrac{\gamma}{\lambda_0 - \delta_1}} + \frac{c(0)}{c(1)}\frac{z(0)}{z(1)}\right), \tag{6.37}$$

where $\dfrac{z(0)}{z(1)}$ is given by the reciprocal of (6.10) and contains $b(0)$ as before.

In order to obtain the terminal boundary condition, we use (2.24), which simplifies to:

$$\frac{v(1)}{v(2)} = \hat{b} - \frac{\hat{c}}{\mu - \hat{d}}, \tag{6.38}$$

where $\hat{c} < 0$, see also [13, Theorem 3.3].

Now substituting the coefficients $c_v(-1)$, $c_v(0)$ and $b_v(0)$ given in (1.20) and (1.21) together with the expression for $\dfrac{v(-1)}{v(0)}$ in (6.12) into (2.16) for $n = 0$ gives an expression for $\dfrac{v(0)}{v(1)}$. Next, setting $n = 1$ in (2.16) and substituting the coefficients $c_v(0)$, $c_v(1)$, $b_v(1)$ (calculated using (1.20) and (1.21)) gives an equation involving $\dfrac{v(0)}{v(1)}$, which enables us to eliminate $\dfrac{v(0)}{v(1)}$ and obtain a single 5th order equation in μ:

$$\mu^5 + \eta_4\mu^4 + \eta_3\mu^3 + \eta_2\mu^2 + \eta_1\mu + \eta_0 = 0. \tag{6.39}$$

Given $\lambda_0, ..., \lambda_5$ we have $\mu_1 = \lambda_1 - \lambda_0$, $\mu_2 = \lambda_2 - \lambda_0$, $\mu_3 = \lambda_3 - \lambda_0$, and $\mu_4 = \lambda_4 - \lambda_0$, $\mu_5 = \lambda_5 - \lambda_0$. Hence the eigencondition is given by the 5th-order polynomial in μ:

$$(\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3)(\mu - \mu_4)(\mu - \mu_5) = 0 \tag{6.40}$$

i.e.

$$\begin{aligned}\mu^5 + (-\mu_1 - \mu_2 - \mu_3 - \mu_4 - \mu_5)\mu^4 + (\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 + \mu_1\mu_4 + \mu_2\mu_4 + \mu_3\mu_4 \\ +\mu_1\mu_5 + \mu_2\mu_5 + \mu_3\mu_5 + \mu_4\mu_5)\mu^3 + (-\mu_1\mu_2\mu_3 - \mu_1\mu_2\mu_4 - \mu_1\mu_3\mu_4 - \mu_2\mu_3\mu_4 \\ -\mu_1\mu_2\mu_5 - \mu_1\mu_3\mu_5 - \mu_2\mu_3\mu_5 - \mu_1\mu_4\mu_5 - \mu_2\mu_4\mu_5 - \mu_3\mu_4\mu_5)\mu^2 + (\mu_1\mu_2\mu_3\mu_4 \\ +\mu_1\mu_2\mu_3\mu_5 + \mu_1\mu_2\mu_4\mu_5 + \mu_1\mu_3\mu_4\mu_5 + \mu_2\mu_3\mu_4\mu_5)\mu - \mu_1\mu_2\mu_3\mu_4\mu_5 = 0.\end{aligned}$$

As a result we obtain 5 simultaneous equations:

$\eta_4 = -\mu_1 - \mu_2 - \mu_3 - \mu_4 - \mu_5,$

$\eta_3 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 + \mu_1\mu_4 + \mu_2\mu_4 + \mu_3\mu_4 + \mu_1\mu_5 + \mu_2\mu_5 + \mu_3\mu_5 + \mu_4\mu_5,$

$\eta_2 = -\mu_1\mu_2\mu_3 - \mu_1\mu_2\mu_4 - \mu_1\mu_3\mu_4 - \mu_2\mu_3\mu_4 - \mu_1\mu_2\mu_5 - \mu_1\mu_3\mu_5 - \mu_2\mu_3\mu_5 - \mu_1\mu_4\mu_5 - \mu_2\mu_4\mu_5 - \mu_3\mu_4\mu_5$

$\eta_1 = \mu_1\mu_2\mu_3\mu_4 + \mu_1\mu_2\mu_3\mu_5 + \mu_1\mu_2\mu_4\mu_5 + \mu_1\mu_3\mu_4\mu_5 + \mu_2\mu_3\mu_4\mu_5.$

$\eta_0 = -\mu_1\mu_2\mu_3\mu_4\mu_5.$

Solving these simultaneous equations gives $b(0)$ and $b(1)$ uniquely.

□

6.2 Cubic dependence at the terminal end

Suppose we are given the boundary value problem comprised of (6.1) and the following boundary conditions:

$$y(-1) = \left(a\lambda + b - \frac{c}{\lambda - d}\right) y(0) = Ay(0), a \geq 0, c \geq 0 \tag{6.41}$$

$$y(1) = \left(\alpha\lambda + \beta - \frac{\gamma_1}{\lambda - \delta_1} - \frac{\gamma_2}{\lambda - \delta_2}\right) y(2), \alpha \leq 0, \gamma_1, \gamma_2 < 0. \tag{6.42}$$

Theorem 6.2.1. *Consider the boundary value problem (6.1), (6.41), (6.42) where $\alpha = \beta = 0 = c$. This boundary value problem has 3 eigenvalues $\lambda_0, ..., \lambda_2$. Given $\lambda_0, ..., \lambda_2$, c(-1), c(0), c(1) we can uniquely reconstruct $b(0), b(1)$.*

Proof. Given the eigenvalues $\lambda_0, ..., \lambda_2$, we begin by shifting the boundary value problem (by letting $\lambda = \mu + \lambda_0$, where λ_0 is the least eigenvalue) as before, resulting in (6.4) and the following boundary conditions:

$$y(-1) = \left(a\mu + a\lambda_0 + b - \frac{c}{\mu + \lambda_0 - d}\right) y(0) =: F(\mu)y(0), \tag{6.43}$$

$$y(1) = \left(\alpha\mu + \alpha\lambda_0 + \beta - \frac{\gamma_1}{\mu + \lambda_0 - \delta_1} - \frac{\gamma_2}{\mu + \lambda_0 - \delta_2}\right) y(2) =: G(\mu)y(2). \quad (6.44)$$

As before, if $z(n)$ is a solution to (6.4) for $\mu = 0$ such that $z(n) > 0$ for all $n = -1, ..., 2$, then $z(n)$ satisfies (6.7).

Since $z(n)$ also obeys (6.43) and (6.44) for $\mu = 0$ the following hold:

$$\frac{z(-1)}{z(0)} = a\lambda_0 + b - \frac{c}{\lambda_0 - d}, \quad (6.45)$$

and

$$\frac{z(2)}{z(1)} = \frac{1}{\alpha\lambda_0 + \beta - \frac{\gamma_1}{\lambda_0 - \delta_1} - \frac{\gamma_2}{\lambda_0 - \delta_2}}. \quad (6.46)$$

Now (6.7) evaluated at 0 and 1 yields (6.10) and (6.11) respectively.

Recall that the transformed, shifted equation is given by (2.16), with $c_v(n)$ and $b_v(n)$ given by (1.20) and (1.21) respectively as before.

The initial boundary condition $\frac{v(-1)}{v(0)}$ is given by (2.40), which simplifies to

$$\frac{v(-1)}{v(0)} = \hat{h}, \quad (6.47)$$

since $c = 0$.

Note that if $a = 0$ then $\hat{h} = 0$ and if $a > 0$ then $\hat{h} \neq 0$

Given $\alpha = \beta = 0$, (6.46) simplifies to:

$$\frac{z(2)}{z(1)} = \frac{1}{-\frac{\gamma_1}{\lambda_0 - \delta_1} - \frac{\gamma_2}{\lambda_0 - \delta_2}}. \quad (6.48)$$

To be able to solve for $b(1)$ in terms of $b(0)$ we equate (6.11) and (6.48) to get

$$b(1) = c(1)\left(\frac{1}{-\frac{\gamma_1}{\lambda_0-\delta_1}-\frac{\gamma_2}{\lambda_0-\delta_2}}+\frac{c(0)}{c(1)}\frac{z(0)}{z(1)}+\lambda_0\right), \tag{6.49}$$

where $\dfrac{z(0)}{z(1)}$ is given by the reciprocal of (6.10) and contains $b(0)$.

The form of the terminal boundary condition is given in [13, Theorem 3.4]. Note that the interval shrinks by 1 and thus $\dfrac{v(0)}{v(1)}$ is the terminal boundary condition and is given by (6.15).

This simplifies to a Nevanlinna function of the form:

$$\frac{v(0)}{v(1)} = \hat{a} - \frac{\hat{c}}{\mu - \hat{d}} \tag{6.50}$$

where $\hat{c} < 0$.

Substituting the coefficients $c_v(n)$ and $b_v(n)$ given by (1.20) and (1.21) respectively, together with the expression for $\dfrac{v(-1)}{v(0)}$ in (6.47) and the reciprocal of (6.50) into (2.16) for $n = 0$ gives a second-order monic polynomial:

$$\mu^2 + \eta_1\mu + \eta_0 = 0. \tag{6.51}$$

Given $\lambda_0, ..., \lambda_2$ we have $\mu_1 = \lambda_1 - \lambda_0$, $\mu_2 = \lambda_2 - \lambda_0$. Hence the eigencondition is given by the 2nd-order polynomial in μ:

$$(\mu - \mu_1)(\mu - \mu_2) = 0 \tag{6.52}$$

i.e.

$$\mu^2 + (-\mu_1 - \mu_2)\mu + \mu_1\mu_2 = 0. \tag{6.53}$$

Hence we obtain the following two simultaneous equations:

$$\eta_1 = -\mu_1 - \mu_2,$$
$$\eta_0 = \mu_1\mu_2.$$

We can solve for $b(0)$ uniquely using these equations. Using (6.49) results in $b(1)$ uniquely.

□

Theorem 6.2.2. *Consider the boundary value problem (6.1), (6.41), (6.42) where*
1. $\alpha = \beta = 0$, $c > 0$,
2. $\alpha = 0, \beta \neq 0$, $c = 0$.
The boundary value problems above each have 4 eigenvalues namely $\lambda_0, ..., \lambda_3$. *Given* $\lambda_0, ..., \lambda_3$ *together with* $c(-1), c(0), c(1)$ *we can uniquely reconstruct the potential* $b(0), b(1)$.

Proof. 1. Given the eigenvalues $\lambda_0, ..., \lambda_3$, the equations (6.43) to (6.46) hold.

Suppose $\alpha = 0$, $\beta = 0$ and $c > 0$. Then (6.46) simplifies to (6.48) and consequently (6.49) holds.

The initial end boundary condition is determined using (2.40) (or [13, Theorem 3.5]) and is given by a Nevanlinna function of the form:

$$\frac{v(-1)}{v(0)} = \hat{b} - \frac{\hat{h}}{\mu - \hat{j}}, \tag{6.54}$$

$\hat{h} > 0$.

If $a = 0$ then $\hat{b} = 0$ otherwise $\hat{b} \neq 0$.

Since $\alpha = \beta = 0$ the terminal end boundary condition is again of the form (6.50) as the interval reduces by 1 and the terminal boundary condition is calculated using (6.15).

Now substituting the coefficients $c_v(n)$ and $b_v(n)$ given by (1.20) and (1.21) respectively, together with the expression for $\dfrac{v(-1)}{v(0)}$ in (6.54) and the reciprocal of (6.50) into (2.16) for $n = 0$ gives a third-order monic polynomial:

$$\mu^3 + \eta_2\mu^2 + \eta_1\mu + \eta_0 = 0. \tag{6.55}$$

Given $\lambda_0, ..., \lambda_3$ we have $\mu_1 = \lambda_1 - \lambda_0$, $\mu_2 = \lambda_2 - \lambda_0$ and $\mu_3 = \lambda_3 - \lambda_0$. The eigencondition is therefore given by

$$(\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3) = 0 \tag{6.56}$$

or equivalently

$$\mu^3 + (-\mu_1 - \mu_2 - \mu_3)\mu^2 + (\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3)\mu - \mu_1\mu_2\mu_3 = 0. \tag{6.57}$$

Thus we obtain the three simultaneous equations by equating (6.55) and (6.57) namely

$$\begin{aligned} \eta_2 &= -\mu_1 - \mu_2 - \mu_3, \\ \eta_1 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3, \\ \eta_0 &= -\mu_1\mu_2\mu_3. \end{aligned} \tag{6.58}$$

Solving these equations gives $b(0)$ uniquely and hence $b(1)$ uniquely.

2. In the case that $\alpha = 0, \beta \neq 0$, and $c = 0$, given the eigenvalues $\lambda_0, ..., \lambda_3$, equations (6.43) to (6.47) hold.

Under these assumptions (6.46) reduces to:

$$\frac{z(2)}{z(1)} = \frac{1}{\beta - \dfrac{\gamma_1}{\lambda_0 - \delta_1} - \dfrac{\gamma_2}{\lambda_0 - \delta_2}}. \tag{6.59}$$

Equating (6.11) and (6.59) we can solve for $b(1)$ in terms of $b(0)$ to get

$$b(1) = c(1)\left(\lambda_0 + \frac{1}{\beta - \dfrac{\gamma_1}{\lambda_0 - \delta_1} - \dfrac{\gamma_2}{\lambda_0 - \delta_2}} + \frac{c(0)}{c(1)}\frac{z(0)}{z(1)}\right), \tag{6.60}$$

where as before, $\dfrac{z(0)}{z(1)}$ is given by the reciprocal of (6.10) and contains $b(0)$.

The terminal boundary condition is given by (2.24) which simplifies to:

$$\frac{v(1)}{v(2)} = -\frac{\hat{a}}{\mu - \hat{b}} - \frac{\hat{c}}{\mu - \hat{d}} \tag{6.61}$$

where $\hat{a}, \hat{c} < 0$, see also [13, Theorem 3.2].

Next, substituting the coefficients $c_v(n)$ and $b_v(n)$ given by (1.20) and (1.21) respectively, together with the expression for $\dfrac{v(-1)}{v(0)}$ in (6.47) into (2.16) for $n = 0$ gives an expression for $\dfrac{v(1)}{v(0)}$. Inserting this expression for $\dfrac{v(1)}{v(0)}$ together with the reciprocal of (6.61) into (2.16) for $n = 1$ gives a third-order monic polynomial given by:

$$\mu^3 + \phi_2\mu^2 + \phi_1\mu + \phi_0 = 0. \tag{6.62}$$

Now, given $\lambda_0, ..., \lambda_3$ we have $\mu_1 = \lambda_1 - \lambda_0$, $\mu_2 = \lambda_2 - \lambda_0$ and $\mu_3 = \lambda_3 - \lambda_0$. As before, the eigencondition is therefore given by (6.57).

We get three simultaneous equations given by:

$$\begin{aligned} \phi_2 &= -\mu_1 - \mu_2 - \mu_3, \\ \phi_1 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3, \\ \phi_0 &= -\mu_1\mu_2\mu_3. \end{aligned} \tag{6.63}$$

These enable us to solve uniquely for $b(0)$ and using (6.60) gives $b(1)$ uniquely.

□

Theorem 6.2.3. *Consider the boundary value problem (6.1), (6.41), (6.42) where*
1. $\alpha = 0, \beta \neq 0$, $c > 0$,
2. $\alpha < 0$, $c = 0$.
The boundary value problems above each have 5 eigenvalues namely $\lambda_0, ..., \lambda_4$. *Given* $\lambda_0, ..., \lambda_4$ *together with* $c(-1), c(0), c(1)$ *we can uniquely reconstruct the potential* $b(0), b(1)$.

Proof. 1. Given the eigenvalues $\lambda_0, ..., \lambda_4$, the equations (6.43) to (6.46) hold.

Suppose $\alpha = 0$, $\beta \neq 0$ and $c \neq 0$. Then (6.46) simplifies to (6.59) which implies that (6.60) holds.

The boundary condition at the initial end is obtained using (2.40) and can be simplified to a Nevanlinna function as in (6.54).

The terminal boundary condition is given by (2.24) which simplifies to a Nevanlinna function of the form (6.61).

Next, substituting the coefficients $c_v(n)$ and $b_v(n)$ given by (1.20) and (1.21) respectively, together with the expression for $\dfrac{v(-1)}{v(0)}$ in (6.47) into (2.16) for $n = 0$ gives an expression for $\dfrac{v(1)}{v(0)}$ which when inserted into (2.16) (with $n = 1$), together with the reciprocal of (6.61) gives a monic 4th-order polynomial of the form:

$$\mu^4 + \eta_3\mu^3 + \eta_2\mu^2 + \eta_1\mu + \eta_0 = 0. \tag{6.64}$$

Given $\lambda_0, ..., \lambda_4$ we have $\mu_1 = \lambda_1 - \lambda_0$, $\mu_2 = \lambda_2 - \lambda_0$, $\mu_3 = \lambda_3 - \lambda_0$ and $\mu_4 = \lambda_4 - \lambda_0$.

Consequently, the eigencondition is given by (6.30) and we obtain four simultaneous equations given by

$$\begin{aligned} \eta_3 &= -\mu_1 - \mu_2 - \mu_3 - \mu_4, \\ \eta_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4, \\ \eta_1 &= -\mu_1\mu_2\mu_3 - \mu_1\mu_2\mu_4 - \mu_1\mu_3\mu_4 - \mu_2\mu_3\mu_4, \\ \eta_0 &= \mu_1\mu_2\mu_3\mu_4, \end{aligned} \tag{6.65}$$

which when solved yield $b(0)$ uniquely. Now enforcing (6.60) yields $b(1)$ uniquely.

2. If $\alpha < 0$ and $c = 0$, then equations (6.43) to (6.47) hold.

In this case $b(1)$ in terms of $b(0)$ is given by:

$$b(1) = c(1)\left(\lambda_0 + \frac{1}{\alpha\lambda_0 + \beta - \dfrac{\gamma_1}{\lambda_0 - \delta_1} - \dfrac{\gamma_2}{\lambda_0 - \delta_2}} + \frac{c(0)}{c(1)}\frac{z(0)}{z(1)}\right), \tag{6.66}$$

where as before, $\dfrac{z(0)}{z(1)}$ is given by the reciprocal of (6.10) and contains $b(0)$.

The terminal boundary condition is given by (2.24) which simplifies to a Nevanlinna function of the type:

$$\frac{v(1)}{v(2)} = \hat{a} - \frac{\hat{b}}{\mu - \hat{c}} - \frac{\hat{d}}{\mu - \hat{e}}, \tag{6.67}$$

$\hat{b}, \hat{d} < 0$, see also [13, Theorem 3.3].

Substituting the coefficients $c_v(n)$ and $b_v(n)$ given by (1.20) and (1.21) respectively, together with the expression for $\dfrac{v(-1)}{v(0)}$ in (6.47) into (2.16) with $n = 0$ yields an expression for $\dfrac{v(1)}{v(0)}$.

When this expression for $\frac{v(1)}{v(0)}$ together with the reciprocal of (6.67) are inserted into (2.16) with $n = 1$ the result is a monic 4th-order polynomial of the form:

$$\mu^4 + \phi_3\mu^3 + \phi_2\mu^2 + \phi_1\mu + \phi_0 = 0. \tag{6.68}$$

Given $\lambda_0, ..., \lambda_4$ we have $\mu_1 = \lambda_1 - \lambda_0$, $\mu_2 = \lambda_2 - \lambda_0$, $\mu_3 = \lambda_3 - \lambda_0$ and $\mu_4 = \lambda_4 - \lambda_0$.

Consequently, the eigencondition is given by (6.30) and we obtain four simultaneous equations given by:

$$\begin{aligned} \phi_3 &= -\mu_1 - \mu_2 - \mu_3 - \mu_4, \\ \phi_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4, \\ \phi_1 &= -\mu_1\mu_2\mu_3 - \mu_1\mu_2\mu_4 - \mu_1\mu_3\mu_4 - \mu_2\mu_3\mu_4, \\ \phi_0 &= \mu_1\mu_2\mu_3\mu_4, \end{aligned} \tag{6.69}$$

which upon solving yield $b(0)$ uniquely. Consequently, (6.66) yields $b(1)$ uniquely. □

Theorem 6.2.4. *Given the boundary value problem (6.1), (6.41), (6.42) with $\alpha < 0, c > 0$. This boundary value problem has 6 eigenvalues $\lambda_0, ..., \lambda_5$. Given $\lambda_0, ..., \lambda_5$ together with $c(-1), c(0), c(1)$ we can uniquely reconstruct the potential $b(0), b(1)$.*

Proof. Suppose we are given the eigenvalues $\lambda_0, ..., \lambda_5$, the equations (6.43) to (6.47) hold as before. Note that (6.66) is obeyed in this case.

Using (2.40) (see [13, Theorem 3.5]) the initial end boundary condition has the form (6.54).

The terminal boundary condition is again given by (2.24) which simplifies to (6.67).

Upon substituting the coefficients $c_v(n)$ and $b_v(n)$ given by (1.20) and (1.21) respectively, the expression for $\frac{v(-1)}{v(0)}$ in (6.54) into (2.16) for $n = 0$ yields an expression

for $\dfrac{v(1)}{v(0)}$. Substituting this expression for $\dfrac{v(1)}{v(0)}$ together with the reciprocal of (6.67) into (2.16) for $n = 1$ gives a monic 5th-order polynomial of the form:

$$\mu^5 + \eta_4\mu^4 + \eta_3\mu^3 + \eta_2\mu^2 + \eta_1\mu + \eta_0 = 0. \tag{6.70}$$

As before, given $\lambda_0, ..., \lambda_5$ we have $\mu_1 = \lambda_1 - \lambda_0$, $\mu_2 = \lambda_2 - \lambda_0$, $\mu_3 = \lambda_3 - \lambda_0$, $\mu_4 = \lambda_4 - \lambda_0$ and $\mu_5 = \lambda_5 - \lambda_0$.

Hence the eigencondition is given by (6.40) and five simultaneous equations arise,

i.e.

$\eta_4 = -\mu_1 - \mu_2 - \mu_3 - \mu_4 - \mu_5,$

$\eta_3 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 + \mu_1\mu_4 + \mu_2\mu_4 + \mu_3\mu_4 + \mu_1\mu_5 + \mu_2\mu_5 + \mu_3\mu_5 + \mu_4\mu_5,$

$\eta_2 = -\mu_1\mu_2\mu_3 - \mu_1\mu_2\mu_4 - \mu_1\mu_3\mu_4 - \mu_2\mu_3\mu_4 - \mu_1\mu_2\mu_5 - \mu_1\mu_3\mu_5 - \mu_2\mu_3\mu_5 - \mu_1\mu_4\mu_5 - \mu_2\mu_4\mu_5 - \mu_3\mu_4\mu_5,$

$\eta_1 = \mu_1\mu_2\mu_3\mu_4 + \mu_1\mu_2\mu_3\mu_5 + \mu_1\mu_2\mu_4\mu_5 + \mu_1\mu_3\mu_4\mu_5 + \mu_2\mu_3\mu_4\mu_5,$

$\eta_0 = -\mu_1\mu_2\mu_3\mu_4\mu_5.$

Solving these equations simultaneously gives $b(0)$ uniquely, and imposing (6.66) yields $b(1)$ uniquely.

□

Note: In chapters 3 and 5 $c(-1) = 0$ and thus the effect of the initial boundary condition is nullified. This is why problems 10 to 12 and 14 to 16 do *not* obey [16, Theorem 1.1] and appear to have the incorrect number of eigenvalues. Had we chosen $c(-1) > 0$ [16, Theorem 1.1] would hold. The results of chapters 3 and 5 are completely correct and provide a nice illustration of what happens when the weights are not always strictly positive. The theoretical results in chapter 6 return to the case of $c(n) > 0$, $n = -1, ..., m$.

Chapter 7

Conclusion

Given boundary value problems, our aim in Chapters 2 and 3 was to reduce the λ-dependence in the boundary conditions by shifting (using $\lambda = \mu + \lambda_0$) and transforming them, resulting in boundary value problems with reduced λ-dependence in the boundary conditions using methods described by Love and Currie in [13]. Chapter 2 focuses on transforming boundary value problems with a cubic dependence in λ at the initial end boundary condition to boundary value problems with the same form of the equation but with at most quadratic λ-dependence in the initial end boundary condition. To that end, there were cases where the interval reduced thereby resulting in the terminal end boundary condition being $\dfrac{v(0)}{v(1)}$ (see problems 3 and 4 in chapter 2). In chapter 3 we focused on transforming boundary value problems with cubic λ-dependence at the terminal end boundary condition to boundary value problems with the same form of the equation but with at most quadratic λ-dependence in the terminal end boundary condition. Since we allowed $c(-1) = 0$ in chapter 3, the work done there deviated from standard procedure by Currie and Love in [13] resulting in new observations and conclusions as noted in chapter 3.

Given the weights and the eigenvalues for problems in Chapter 2, the aim in Chapter 4 was the reconstructing of the potential $b(n)$ with transformations in Chapters 2 and 3 proving to be particularly useful in the reconstruction. In addition to applying the transformations in Chapters 2 and 3, we needed to compare coefficients of the eigencondition and the polynomial in the new shifted variable μ to obtain a set of simultaneous equations. These simultaneous equations were uniquely solved using using Mathematica to give the potential $b(n)$. It is worth noting that in chapter 4, the order of the eigencondition is 1 less than than the number of the eigenvalues obtained in the corresponding problem in chapter 2 since we work with the trans-

formed boundary value problem.

In Chapter 5, as in Chapter 4 we aimed to reconstruct the potential given the boundary value problem and its eigenvalues for each problem from Chapter 3. The procedure was exactly the same as in Chapter 4, and as noted in Chapter 3, allowing $c(-1) = 0$ caused the work in this paper to deviate slightly from what has been done by the authors in [13]. As a result we obtained interesting results that have laid a foundation for further work on the topic.

Equipped with plenty of worked problems from Chapters 2 to 5, we are in good standing to reconcile everything and generalize. This is exactly what Chapter 6 deals with. Using data obtained in the aforementioned chapters we were able formulate theorems for a variety of boundary conditions, some familiar while others were new. Each theorem is proved using the familiar procedure in the previous chapters with the difference that in Chapter 6 we used no numbers but general forms of the boundary conditions.

Bringing everything together, given a boundary value problem we shifted it, transformed it, obtained its eigenvalues and in the end obtained boundary value problems with reduced λ-dependence in the boundary conditions. The eigenvalues of the new problem being one less than the original number as seen in Chapters 2 to 5. Next, we found the eigenvalues of this reduced boundary value problem and use them in reconstructing the potential $b(n)$ both in numerical examples (Chapters 4 and 5) and in the general sense (Chapter 6).

Further cases could be investigated, particularly the case where we have cubic λ-dependence at the initial and terminal end boundary conditions. Of interest is the case where we have a general number of intervals i.e. $m \in \mathbb{N}$. Another interesting problem to consider would be one in which we reconstruct the potential for boundary conditions of any order i.e. arbitrary number of fractions in the Nevanlinna function. Another case that could be investigated for further work would be if we allowed the coefficients $c_v(n) < 0$. Surely there would be interesting results and new generalizations.

Chapter 8

Appendix

Chapter 2 Problem 1

```
In[75]:= Clear[All]
```

Initial end boundary condition

```
In[76]:= aa = x + 1 - 1/x - 1/(x - 2)
Out[76]= 1 - 1/(-2 + x) - 1/x + x
```

Terminal end boundary condition

```
In[77]:= ab = 3 - x
```

z (-1) / z (0)

```
In[78]:= ac = x0 + 1 - 1/x0 - 1/(x0 - 2)
```

Eigencondition to find eigenvalue

```
In[80]:= NSolve[(-2/3 * aa - x) * (4/ab + 4 x - 1 + 3/((-2)/3 * aa - x)) == 0, x, 5]
Out[80]= {{x → 3.339}, {x → 2.1778}, {x → -1.2183}, {x → 0.5901}, {x → -0.03827}}
```

Smallest eigenvalue

```
In[•]:= x0 =
 -1.2181775787915477856744886742486259884364065527833298039481941873604279510\
 7082952945598513207672121422509462094486184878123351072091865631727061948195\
 1396423997242051749740562864012065349580436666302925199346067314720141815845\
 1881083497184192689884488216213209038239144563343008899036392155547286219685\
 1584302632167271211000114245190819351110091535343984873574799758464776473515\
 1943439477638930439643941226676769319314602541995117774944436724067437514055\
 7907542741027259415369354504701538650131204589906703281241287068305979736492\
 .8864242972663
```

shifted Initial Boundary Ccondition

```
In[•]:= ad = y + x0 + 1 - 1/(y + x0) - 1/(y + x0 - 2)
```

Shifted Terminal Boundary Condition

```
In[84]:= ae = 3 - y - x0
```

z (-1) / z (0)

```
In[85]:= af = x0 + 1 - 1/x0 - 1/(x0 - 2)
```

z1 / z0

```
In[86]:= ag = -x0 - 2/3 * af
```

z2 / z1

In[87]:= $ah = \frac{-3}{4} * \frac{1}{ag} - x0 + \frac{1}{4}$

cv (-1)

In[98]:= ar = N[2 * af, 5]

Out[98]= 1.8269

cv0

In[99]:= $aj = N\left[\frac{3}{ag}, 5\right]$

Out[99]= 4.9244

bv0

In[100]:= $ak = N\left[\frac{3}{ag}\left(\frac{3}{4} * \frac{1}{ag} + ag\right), 5\right]$

Out[100]= 9.0625

cv1

In[101]:= $al = N\left[\frac{4}{ah}, 5\right]$

Out[101]= 16.873

bv1

In[102]:= $am = N\left[\frac{4}{ah}\left(\frac{4}{5} * \frac{1}{ah} + ah\right), 5\right]$

Out[102]= 60.938

Boundary Condition at initial end v (-1) / v0

In[94]:= $an = N\left[Apart\left[\frac{1}{af}\left(\frac{1}{\frac{2}{3} - \frac{y}{af-ad}}\right)\right], 5\right]$

Out[94]= $\frac{3.2400}{1.9731 + 1.0000\, y}$

Boundary Condition at terminal end v2 / v1

In[95]:= $ap = N\left[Apart\left[\frac{4}{5} * \frac{1}{ah} - y + \frac{ah * y}{ah - \frac{1}{ae}}\right], 5\right]$

Out[95]= $-0.84364 + 0. \times 10^{-491}\, y$

In[96]:= aq = Chop[ap]

Out[96]= -0.84364

Polynomial to get new eigencondition

In[97]:= $NSolve\left[al * ap - am + aj * \frac{1}{\frac{ak}{aj} - \frac{ar}{aj} * an - y} + al * y == 0, 5\right]$

Out[97]= {{y → 4.556}, {y → -1.6209}, {y → 1.3875}}

Chapter 3 Problem 1

In[103]:=
```
Clear[All]
```

Initial end boundary condition

```
y(-1) = -2 y(0)
```

Terminal end boundary condition

In[104]:=
```
V0 = (-2 x + 3 + 1/(x - 2) + 1/(x + 1))
```

Out[104]=
$$3 + \frac{1}{-2 + x} - 2x + \frac{1}{1 + x}$$

Eigencondition

In[105]:=
```
NSolve[1 - x * (4/V0 + 4 x - 2) == 0, x, 5]
```

Out[105]=
```
{{x → 2.6498}, {x → 1.4467}, {x → -1.2233}, {x → 0.5000}, {x → -0.3732}}
```

Least eigenvalue

In[*]:=
```
k0 =
 -1.223312249371970788184948861057395744431315051114601925121351662436104273207
   748307579658021911297095082622271069411750113274264963765962921034920083563
   190318802698896278088183888707245439087502721845159432741598937307110045001
   174197353518427601072424617768592031795764436846367484302715455606076997826
   659344956208359719229162784386156764794647527412850873294721261778719053013
   369498081778510649278115671231977442547311370622954768383456099275125664187
   901327749628543968888497136665552223588731335561356488093017424305028l3`494
   .43760154311354
```

Shifted terminal end boundary condition

In[106]:=
```
V4 = (-2 y - 2 k0 + 3 + 1/(y + k0 - 2) + 1/(y + k0 + 1))
```

z2/z1

In[107]:=
```
V1 = 1/2 - 1/4 (-1/k0) - k0
```

W

In[112]:=
```
V2 = N[Apart[4/(9 * V1) - y + (y * V1)/(V1 - 1/V4)], 6]
```

Out[112]=
$$-0.0365750 + 0. \times 10^{-492}\, y - \frac{0.336736}{-3.66679 + 1.00000\, y} - \frac{0.451354}{-2.17397 + 1.00000\, y}$$

Removing the very small term

In[113]:=
```
Chop[V2]
```

Out[113]=
$$-0.0365750 - \frac{0.336736}{-3.66679 + 1.00000\, y} - \frac{0.451354}{-2.17397 + 1.00000\, y}$$

v1/v2

In[114]:= V3 = Apart[1/V2]

Out[114]= $-27.34 + \frac{0.68}{-3.0540 + 1.00000\,y} + \frac{588.4}{18.760 + 1.00000\,y}$

bv1

In[116]:= V6 = N[(4/(9 * V1) + V1) (4/V1), 5]

Out[116]= 4.7705

cv0

In[117]:= V7 = N[-1/k0, 5]

Out[117]= 0.81745

cv1

In[118]:= V8 = N[4/V1, 5]

Out[118]= 2.6334

bv0

In[119]:= V9 = N[1/k0 (k0 + 1/(4 k0)), 5]

Out[119]= 1.1671

v1/v0

In[120]:= V10 = N[V9/V7 - y, 5]

Out[120]= 1.4277 - 1.0000 y

Eigencondition

In[121]:= NSolve[V8 * V2 - V6 + V7/V10 + y * V8 == 0, y, 5]

Out[121]= {{y → 3.873}, {y → 2.6700}, {y → 1.7233}, {y → 0.8501}}

Chapter 4 Problem 1

```
In[1]:= Clear[b0, b1]
```

Initial end condition

```
In[2]:= initialcond = (x + 1 - 1/x - 1/(x - 2))
Out[2]= 1 - 1/(-2 + x) - 1/x + x
```

Terminal end condition

```
In[5]:= terminalcond = 3 - x
Out[5]= 3 - x
```

Eigencondition

```
In[6]:= eigencond = (-x - 2/3 * initialcond) (4/terminalcond + 4 x - 1) + 3
Out[6]= 3 + (-1 + 4/(3 - x) + 4 x) (-x - 2/3 (1 - 1/(-2 + x) - 1/x + x))
```

```
In[158]:= NSolve[eigencond == 0, x, 5]
Out[158]= {{x → 3.339}, {x → 2.1778}, {x → -1.2183}, {x → 0.5901}, {x → -0.03827}}
```

```
In[=]:= Smallest eigenvalue is x0
```

```
In[8]:= x0 =
 -1.2181775787915477856744886742486259884364065527833298039481941873604279510\
7082952945598513207672121422509462094486184878123351072091865631727061948191\
1396423997242051749740562864012065349580436666302925199346067314720141815841\
1881083497184192689884488216213209038239144563343008899036392155547286219681\
1584302632167271211000114245190819351110091535343984873574799758464776473511\
1943439477638930439643941226676769319314602541995117774944436724067437514051\
7907542741027259415369354504701538650131204589906703281241287068305954`492\
.95569598882764
```

z (-1) z0

```
In[159]:= X = N[x0 + 1 - 1/x0 - 1/(x0 - 2), 5]
Out[159]= 0.91346
```

```
In[160]:= z1z0 = N[b0/3 - x0 - 2/3 * X, 5]
Out[160]= 0.60921 + 0.33333 b0
```

```
In[162]:= b1 = N[3/z1z0 + 4 x0 + 4/(3 - x0), 5]
Out[162]= -3.9244 + 3.0000/(0.60921 + 0.33333 b0)
```

In[163]:= $z2z1 = N\left[\frac{b1}{4} - \frac{3}{4} * \frac{1}{z1z0} - x0, 5\right]$

Out[163]= $1.2182 + 0.25000\left(-3.9244 + \frac{3.0000}{0.60921 + 0.33333\, b0}\right) - \frac{0.75000}{0.60921 + 0.33333\, b0}$

Shifted initial condition

In[13]:= $shiftedIBC = y + x0 + 1 - \frac{1}{y + x0} - \frac{1}{y + x0 - 2}$

Shifted terminal condition

In[14]:= $shiftedTBC = 3 - y - x0$

Finding the transformed boundary condition at the initial end v (-1) / v0

In[164]:= $initialTBC = N\left[Apart\left[\frac{1}{X}\left(\frac{1}{\frac{2}{3} - \frac{y}{X\text{-}shiftedIBC}}\right)\right], 5\right]$

Out[164]= $0.6568 - \frac{0.17}{-3.365 + 1.0000\, y} - \frac{0.28}{-1.524 + 1.0000\, y}$

Finding the ratio v (2) / v (1) (called TBCTE)

In[165]:= $TBCTE = N\left[Apart\left[\frac{4}{5} * \frac{1}{z2z1} - y + \frac{z2z1 * y}{z2z1 - \frac{1}{shiftedTBC}}\right], 5\right]$

Out[165]= $-0.84 + 0. \times 10^{-4}\, y$

In[166]:= $eq1200 = Chop[TBCTE]$

Out[166]= $-0.84 + 0. \times 10^{-4}\, y$

cv (-1)

In[167]:= $eq5064 = 2\, X$

Out[167]= 1.8269

cv0

In[168]:= $eq5030 = \frac{3}{z1z0}$

Out[168]= $\frac{3}{0.60921 + 0.33333\, b0}$

cv1

In[169]:= $eq5031 = \frac{4}{z2z1}$

Out[169]= $\frac{4}{1.2182 + 0.25000\left(-3.9244 + \frac{3.0000}{0.60921 + 0.33333\, b0}\right) - \frac{0.75000}{0.60921 + 0.33333\, b0}}$

bv0

In[170]:=
```
eq5032 = 3/z1z0 (3/4 * 1/z1z0 + z1z0)
```

Out[170]=
$$\frac{3\left(0.60921+\frac{3}{4\,(0.60921+0.33333\,b0)}+0.33333\,b0\right)}{0.60921+0.33333\,b0}$$

bv1

In[171]:=
```
eq5033 = (4/5 * 1/z2z1 + z2z1) * 4/z2z1
```

Out[171]=
$$\left(4\left(1.2182+\frac{4}{5\left(1.2182+0.25000\left(-3.9244+\frac{3.0000}{0.60921+0.33333\,b0}\right)-\frac{0.75000}{0.60921+0.33333\,b0}\right)}+0.25000\left(-3.9244+\frac{3.0000}{0.60921+0.33333\,b0}\right)-\frac{0.75000}{0.60921+0.33333\,b0}\right)\right)\Big/\left(1.2182+0.25000\left(-3.9244+\frac{3.0000}{0.60921+0.33333\,b0}\right)-\frac{0.75000}{0.60921+0.33333\,b0}\right)$$

Eigencondition

In[24]:=
```
Pol1 = eq5031 * eq1200 - eq5033 + eq5030/(eq5032/eq5030 - eq5064/eq5030 * initialTBC - y) + eq5031 * y
```

Out[174]=
$$-301.71-63.842\,b0-17.466\,b0^2+\left(577.61+200.82\,b0+23.685\,b0^2\right)y+\left(-378.06-167.05\,b0-8.8916\,b0^2\right)y^2+\left(99.979+50.377\,b0+1.0000\,b0^2\right)y^3+\left(-9.1381-5.0000\,b0\right)y^4$$

The eigenvalues

In[175]:=
```
Mu1 = N[
  3.33862872026802210739521502621566777607830916678499412781559548268454601320\
  9752463135744605376549683640961936931170651987623230805879116294601132549995\
  3524367955750751761945110365315815821164864162537934994827383815623753368495\
  5907546968927458589949288267342926187104562404299186477172966806790052486205\
  8024656021965287572901232006683810597009155324806653198721252503102759870815\
  7718760302893937639180862844518109960631874103104942002922786265897760037915\
  6042596360216644565201240745226261645339983260936097572168436461767731 5`492.\
  95569598882764 - x0, 5]
```

Out[175]= 4.5568

In[176]:=
```
Mu2 = N[
  2.17770747677286584556008114780883898122885410574457777128297440315761855459\
  1173117168231261246941126359072654534846870491276318617104874691691315651365\
  8154588907347998200895470238235074359156679562312352185421122211636565549616\
  1174823377926981967285281438452520496891334667526097960262445674920704249625\
  2663762521382105962601068237111474473899775723244433436577040183151098387225\
  4555271972708676037283190515365993959676878166051173753146695252428765092195\
  9780118630930127234448756169452117989352202444892974936919308943910346`492.\
  95569598882764 - x0, 5]
```

Out[176]= 3.3959

In[177]:=
```
Mu3 = N[
  0.590108011619174224658583652042597528893592572374950689068964999747353062685
    770248576549164406703624989116854459922791588220923737926245270216053398145
    418389528391513226343122339887616909204772237346952725040969929259536965155
    227207582689484375282613184213304416153224244985246156842112268674180979036
    964615673218744961546280364808784729461596265021022378452425172831574446908
    187103630402733344910778273576629187766924030078170345612550784249227229815
    786402303462819392173142128967037296292965942849034491855294013759151936`492
    .95569598882764 - x0, 5]
```

Out[177]= 1.8083

In[178]:=
```
Mu4 = N[
  -0.038266629868514391939391151818478297764349292121192784219340698229089679
     411545113123868997482854938468693673970759253482985435730154578056725920
     234624466428149771830476531241138414993922788829432016889231312138005134
     636700367473362676565091620176213331715281807288994958334737106820143012
     905280306518750272643367571939964989646692313014414802162917325546247744
     656104826253612201629101991016685927894868973626478663390027942701437250
     552636751359914215779195284747706306012063699647213947490640544412714146
     3994036460090267`492.95569598882764 - x0, 5]
```

Out[178]= 1.1799

The eigencondition (remember we lost the eigenvalue 0)

In[179]:=
```
econd1 = (x - Mu1) * (x - Mu2) * (x - Mu3) * (x - Mu4)
```

Out[179]= (-4.5568 + x) (-3.3959 + x) (-1.8083 + x) (-1.1799 + x)

In[180]:=
```
Collect[econd1, x]
```

Out[180]= 33.02 - 63.21 x + 41.372 x^2 - 10.941 x^3 + x^4

In[34]:=
```
poly1 = (1 /
    (-9.138106951696945047370716027626338995445621536050269068843310256530368
       627083701584499055401425509773800461274447760215813870086170649987811
       076653297488008387453480334110327072821708935133940362211798353057746
       560100432160188634352288139378388295733956568587582639211350770596593
       860314107136707932244539666510185965685424367329257188663805430957431
       286219767771214610904511284716794457140445933148434407443797637024278
       089470542590585644617088495302642219996362838378596026963597015859753
       407793726498314356`485.2213687700219 -
     5.000000000000000000000000000000000000000000000000000000000000000000000
       000000000000000000000000000000000000000000000000000000000000000000000
       000000000000000000000000000000000000000000000000000000000000000000000
       000000000000000000000000000000000000000000000000000000000000000000000
       000000000000000000000000000000000000000000000000000000000000000000000
       000000000000000000000000000000000000000000000000000000000000000000000
       000000000000000000000000000000000000000000000000000000000000000000000
       000000000006`485.2213687700219 b0)) * X1
```

In[182]:=
```
A4 = N[
  (-9.13810695169694504737071602762633899544562153605026906884331025653036862\
     70837015844990554014255097738004612744477602158138700861706499878110766\
     53297488008387453480334110327072821708935133940362211798353057746560100\
     43216018863435228813937838829573395656858758263921135077059659386031410\
     71367079322445396665101859656854243673292571886638054309574312862197677\
     71214610904511284716794457140445933148434407443797637024278089470542590\
     58564461708849530264221999636283837859602696359701585975340779372649831\
     4356`485.2213687700219 -
   5.00000000000000000000000000000000000000000000000000000000000000000000000\
     00000000000000000000000000000000000000000000000000000000000000000000000\
     00000000000000000000000000000000000000000000000000000000000000000000000\
     00000000000000000000000000000000000000000000000000000000000000000000000\
     00000000000000000000000000000000000000000000000000000000000000000000000\
     00000000000000000000000000000000000000000000000000000000000000000000000\
     00000000000000000000000000000000000000000000000000000000000000000006`4\
     85.2213687700219 b0), 5]
```

Out[182]= $-9.1381 - 5.0000\, b0$

In[183]:=
```
A3 = N[
  (99.97900372151216263313558575981209642873986517803578983627165193689669015\
     9596450094922541368400908650478716379750187440660196385768066722160188177\
     3125827856388975637889536150119456754342560547270968882875089022111666153\
     9809702062912879214785157402771135919852509744437333602748598594607988415\
     7864586458991266737159418670410906937168625608791501631830980257002359697\
     7504325799759464608721743929876663659921555859502848333999446151313192579\
     2310990716968402292389050957473228965126843679654921408912323271\
     5`485.2213687700219 +
   50.37651898506233125267088803682176334865022928945674565529027837885599\
     6409460224899963265290755480563911122988197874515810412820572760046997491\
     8117080157418000359946642902547124458944916545005048630670750510593905189\
     8474925774277997014737079738145960440023240013487204741372582022619322\
     8422917937771096551589204985715159445406810660876696996942845146581892635\
     0210817669619890781714872712652398998958555943009991110759602768078055\
     3809020847830126479495279809363796830791993485870704982838435734803740\
     77`485.2213687700219 b0 + 1.`485.2213687700219 b0^2), 5]
```

Out[183]= $99.979 + 50.377\, b0 + 1.0000\, b0^2$

In[184]:=
```
A2 = N[
  (-378.06369043806384376230629627449285124633115957493305560268593681837420 1\
    00863371754820717050426443846998302986710979893200090811518995940080476 1\
    64058701805671973982821045230342984863714025471938785342645504697257433 1\
    88008735803619790457337518783358907072150301239894799433638889012887122 1\
    92612694662364953541842720474100030005550726850051342978990965910071133 1\
    88521733788292057429236544491913372763509593246865327268317498885097105 1\
    45771459757320423208489619932639686952836690683346621791246135740608566 1\
    612636`485.2213687700219 -
  167.05187270113613426023853995876024305277082388140256577960120639634488 3 1\
    51887902923214569433639496559247814687154582898333516493744968753760215 1 1\
    44424380622231214216852486800933091121828138888448841598109012543651748 9 1\
    02186934440619130875452900237333853631807767769899751512086580916770181 5 1\
    88893718566055079175763172676550422557612454146529743480599644601888016 6 1\
    85878096744086462580464424782454262582938596185034066279866704852551949 7 1\
    47734104652957437455392107503072581224366282598642027414939780994246506 6 1\
    `485.2213687700219 b0 -
  8.8916019401530406629648602920106731214758144436660085643561290781495565 8 1\
    57316139636670923288864036004076966845671783096514044095956341442553282 3 1\
    42615122316005118176677438972967122984755759620498178385556885559483850 5 1\
    95216281613008243801880877259153456090479356588898423541076433917896892 5 1\
    74427552459587600687444938156752141007207348516306126090243074979735670 5 1\
    20787343823148809482119821685370324165995874573451819357930847193347124 0 1\
    59590093493529098431981989327135333593919713320781708703257610462355356`4 1\
    85.2213687700219 b0^2), 5]
```

Out[184]= $-378.06 - 167.05\, b0 - 8.8916\, b0^2$

Equating powers

```
In[39]:= NSolve[{A3/A4 +
     10.940887893957738928372443371243129942182032763916649019740970936802139\
     7553854147647279925660383606071125473104724309243906167553604593281586353\
     0974095698211998621025874870281432006032674790218333151462599673033657360\
     0070907920940541748592096344942244108106604519119572281671504449518196070\
     7773643109840792151316083635605500057122595409675555045767671992436787390\
     9879232388236755971719738819465219821970613338384659657301270997558887470\
     2218362033718757028953771370513629707684677252350769325065602294953351640\
     062064301704843897`492.9526686057167 == 0,
   A2/A4 -
     41.372211632604031768347934421776502808237204795572708137404663157285204\
     4013455865657128793252089978540111690248809761088102403552457095629969160\
     1841884058236260982550349833357752171200013763824337959216219413623251630\
     7800694320803088443584008131980004040027255818721962152092075542860343850\
     7724547881165687967137037473573126366480035750156159915609786901747749100\
     6158603396891221920814807186953223863796103006538766840221723640525020890\
     2924168369881491846046435281460302749354133893845657629462017142742928300\
     492.6507626570335 == 0}, b0]

Out[39]= {{b0 → 0}}
```

Chapter 5 Problem 1

In[185]:=
```
Clear[b0, b1]
```

Initial boundary condition

In[41]:=
```
initialcond = -2 x + 3 + 1/(x - 2) + 1/(x + 1)
```

Out[41]=
$$3 + \frac{1}{-2 + x} - 2\,x + \frac{1}{1 + x}$$

Eigencondition

In[186]:=
```
NSolve[1 - x * (4/initialcond + 4 x - 2) == 0, x, 5]
```

Out[186]=
```
{{x → 2.6498}, {x → 1.4467}, {x → -1.2233}, {x → 0.5000}, {x → -0.3732}}
```

Smallest eigenvalue is k0

In[43]:=
```
k0 =
 -1.223312249371970788184948861057395744431315051114601925121351662436104273207
   748307579658021911297095082622271069411750113274264963765962921034920083563
   190318802698896278088183888707245439087502721845159432741598937307110045001
   174197353518427601072424617768592031795764436846367484302715455606076997826
   659344956208359719229162784386156764794647527412850873294721261778719053013
   369498081778510649278115671231977442547311370622954768383456099275125664187
   901327749628543968888497136665552223588731335561356488093017424305028l3`494
   .43760154311354
```

Shifted Boundary Condition

In[44]:=
```
eq102 = -2 y - 2 k0 + 3 + 1/(y + k0 - 2) + 1/(y + k0 + 1)
```

```
z (-1) / z0
```

In[45]:=
```
eq100 = -2
```

```
z1 / z0
```

In[187]:=
```
eq104 = N[b0 - k0, 5]
```

Out[187]=
```
1.2233 + b0
```

In[189]:=
```
b1 = N[1/eq104 + 4 k0 + 4/eq106, 5]
```

Out[189]=
$$1.1825 + \frac{1}{1.2233 + b0}$$

```
z2 / z1
```

In[190]:=
```
eq105 = N[b1/4 - 1/4 * 1/eq104 - k0, 5]
```

Out[190]=
$$1.2233 - \frac{0.25000}{1.2233 + b0} + 0.25000\left(1.1825 + \frac{1}{1.2233 + b0}\right)$$

In[191]:= eq106 = N[-2 k0 + 3 + 1/(k0 - 2) + 1/(k0 + 1), 5]

Out[191]= 0.65835

BC at initial end v (-1) / v (0)

In[192]:= eq101 = N[Apart[1/eq100 (1/(-y/(eq100+2)))], 5]

Out[192]= 0

BC at the terminal end v2 / v1

In[194]:= eq103 = N[Apart[4/(9 * eq105) - y + (y * eq105)/(eq105 - 1/eq102)], 5]

Out[194]= $\frac{0.2926\,(-0.153-0.125\,b0)}{1.22+1.00\,b0}+\frac{3.6+5.8\,b0+2.39\,b0^2-1.18\,y-1.93\,b0\,y-0.79\,b0^2\,y}{(1.22+1.00\,b0)\,(9.752+7.971\,b0-7.145\,y-5.841\,b0\,y+1.2233\,y^2+1.0000\,b0\,y^2)}$

v1 / v2

Out[54]= $-27.341+\frac{0.67591}{-3.0540+1.0000\,y}+\frac{588.45}{18.760+1.0000\,y}$

bv0

In[196]:= eq108 = N[1/eq104 (1/(4 * eq104) + eq104), 5]

Out[196]= $\frac{1.2233+b0+\frac{0.25000}{1.2233+b0}}{1.2233+b0}$

bv1

In[197]:= eq20001 = N[4 * 1/eq105 (4/9 * 1/eq105 + eq105), 5]

Out[197]= $\frac{4.0000\left(1.2233-\frac{0.25000}{1.2233+b0}+0.25000\left(1.1825+\frac{1}{1.2233+b0}\right)+\frac{0.44444}{1.2233-\frac{0.25000}{1.2233+b0}+0.25000\left(1.1825+\frac{1}{1.2233+b0}\right)}\right)}{1.2233-\frac{0.25000}{1.2233+b0}+0.25000\left(1.1825+\frac{1}{1.2233+b0}\right)}$

cv0

In[198]:= eq16001 = N[1/eq104, 5]

Out[198]= $\frac{1}{1.2233+b0}$

cv1

In[199]:=
$$\text{eq20000} = 4 * \frac{1}{\text{eq105}}$$

Out[199]=
$$\frac{4}{1.2233 - \frac{0.25000}{1.2233+b0} + 0.25000\left(1.1825 + \frac{1}{1.2233+b0}\right)}$$

Polynomial giving the eigencondition

In[59]:=
$$\text{eq107} = \text{eq20000} * \text{eq16000} - \text{eq20001} + \frac{\text{eq16001}}{\frac{\text{eq108}}{\text{eq16001}} - y} + y * \text{eq20000}$$

eigen - values

In[200]:=
```
aa = N[
  -0.37317198723641225445899103966781340274406849720832270031231554722223843\
     6224524276448660155975068589027065064969933302107378171900537840965508723\
     3425920765988132981053107235112592059727200068015376402104705375370371485\
     1140649046855861790152923448481104909512521064920680106814360650338093965\
     8334426461184891803143482785466686946283278781828195101564404968652324735\
     2206969595789528249132539705972700485302748548843421238204023521093839505\
     2759025751474014478831394424111168042621901771044055063182747661324039815\
     7746699631663442`494.43760154311354 - k0, 5]
```

Out[200]= 0.85014

In[201]:=
```
ab = N[
  0.5000000000000000000000000000000000000000000000000000000000000000000000000\
    0000000000000000000000000000000000000000000000000000000000000000000000000\
    0000000000000000000000000000000000000000000000000000000000000000000000000\
    0000000000000000000000000000000000000000000000000000000000000000000000000\
    0000000000000000000000000000000000000000000000000000000000000000000000000\
    0000000000000000000000000000000000000000000000000000000000000000000000000\
    0000000000000000000000000000000000000000000000000000000018376661865`494\
    .43760154311354 - k0, 5]
```

Out[201]= 1.7233

In[202]:=
```
ac = N[
  1.4467197753741750250098023578426987372163565975910578854045769084095715544\
    0334695645758065883745994873108842090967088836844859611601665725780236856124\
    7789897628761197009549178060256352691604661939597161375842813326758066698945\
    7953581775813021616636464107722413390705346126528009080388484142180181359795\
    3206475113629007023110679022674831650308205219895820201027254698901861841075\
    6865172355673491480074563614989509966048320858585964048136210187075181808455\
    0332133630855542114711905766668779201587126920110652589762152378639801 9`494\
    .43760154311354 - k0, 5]
```

Out[202]= 2.6700

In[203]:=
```
ad = N[
  2.64976446123420801763413754288251040995902695073186674002909030124877115502\
  86603957876597162505795911221993547543799148716722254221808129180555556169 7\
  61898892115258710820561720860220744192771668164946434221107130834611478159 6\
  84739779383865467389358925609147590337443662526171018276724798107140312645 9\
  18761547628028687158496434495146141804577802106731853971524311395822882680 0\
  96547217160822122827551233721851488015486132499956930196486267082512222348 7\
  65642833078410143784352557770911485406184612016353162729426796466289757`494\
  .43760154311354 - k0, 5]
```

Out[203]= 3.8731

The eigencondition (remember we lost the eigenvalue 0)

In[204]:=
```
eq111 = (x - aa) * (x - ab) * (x - ac) * (x - ad)
```

Out[204]= $(-3.8731 + x)\ (-2.6700 + x)\ (-1.7233 + x)\ (-0.85014 + x)$

In[205]:=
```
Collect[eq111, x]
```

Out[205]= $15.151 - 36.20\,x + 28.645\,x^2 - 9.1166\,x^3 + x^4$

Solving for b0

In[206]:=
```
K4 = N[
  (-1.223312249371970788184948861057395744431315051114601925121351662436104 27\
     32074830757965802191129709508262227106941175011327426496376596292103492\
     00835631903188026988962780881838887072454390875027218451594327415989373\
     07110045001174197353518427601072424617768592031795764436846367484302715\
     45560607699782665934495620835971922916278438615676479464752741285087329\
     47212617787190530133694980817785106492781156712319774425473113706229547\
     68383456099275125664187901327749628543968888497136665552223588731335561\
     3565`486.1391347080717 - 1.`486.1391347080717 b0), 5]
```

Out[206]= $-1.2233 - 1.0000\,b0$

In[207]:=
```
K3 = N[
  (11.152401045433466584814982111187960751370208054281110247961605023943518155\
    894074141394629225297048788444261733893936651113795022446023718646034032\
    241865691538057105289675489441033108023849158811173138167774557817871476\
    251387190009682253349946597345345108667206176641245922871019206753570905\
    555152670408420654409043368314128607074713175905545908613643007842443176\
    235068565629502042296304346337204269543163652781718540074405284701217279\
    986556801762176213898729085887582219027151154799663875962997634\
    5`486.1391347080717 +
   10.135510301039145136143249796921514662999942720725275937894143026022097\
    595232471883235468098671863960134161317774750261206739136479735648044151\
    987109853576622491255209170155411008816741390969803693538423729918952132\
    530284630886863112966705723370053474581386931771118595205590922762769493\
    992819921699222843022820199768629481041672495457060702143108683727458592\
    739101325512185630015431693374409853587753190379819547418915996522931843\
    859184509754267071404198755028845717754377968367907613268842991585348\
    4`86.1391347080717 b0 + 1.`486.1391347080717 b0^2), 5]
```

Out[207]= $11.152 + 10.136\ b0 + 1.0000\ b0^2$

In[208]:=
```
K2 = N[
  (-35.041382524984159465649012280292811107118659039163456862977744245507744\
    9\
    251548013451328393138203998716309986679165752990133192817882165621275\
    18\
    660024705380995399351330384926169213901173021134200700715138766620584\
    22\
    224907206055351247883185632758888036147499410681118583928638405166930\
    15\
    800532547779888587785221338734446204115587323594651148726927963822584\
    27\
    282851679822127349111808485347940632197843234748100319076340066532430\
    86\
    881053481892671342267188232220603624250475614977091711073052946683802\
    12\
    80523`486.1391347080717 -
   36.789675785087014657086503970747952508113392924437642368008497667633818\
    9\
    298692294673023582369624405210390653795093112666204356470918646469523\
    419\
    994258778010068520959960422684495900658151421566894168699005590616163\
    033\
    488384164644034716492223108571553704158276840448813909921432505470533\
    752\
    126472670348434693572158930471734001466294958929149606720186096589298\
    504\
    396813047967916607265908969139248984524173861946724917617650876360121\
    330\
    953221599516014254082721469528281409042628653723028509263458426731169\
    2`4\
    86.1391347080717 b0 -
   7.688885802295203559773352074806723174137312618496072087651439701149889\
    04\
    890850573164230766044592205848171589638651525894125383720441638962345\
    359\
    703972155128619514759535525186332673676535734692593245065371815991646\
    711\
    082756306520236422630202031555685510276685254119657078363174259022264\
    028\
    716045221338909740315785278247836484847952641041017726376261135527996\
    484\
    867455817015829561158153376722799979646189439843310379566418719319781\
    671\
    679646547291342338239332051031369239766785410347971685510967244730771\
    1`4\
    86.1391347080717 b0^2), 5]
```

Out[208]= $-35.041 - 36.790\ b0 - 7.6889\ b0^2$

Equating just the quadratic powers

```
In[72]:= NSolve[{K3/K4 +
    9.116561246859853940924744305286978722156575255573009625606758312180521 36\
     603741537898290109556485475413111355347058750566371324818829814605174600\
     417815951594013494481390440919443536227195437513609225797163707994686535\
     550225005870986767592138005362123088842960158978822184231837421513577278\
     030384989133296724781041798596145813921930783823973237637064254366473606\
     308893595265066847490408892553246390578356159887212736556853114773841917\
     280496375628320939506638748142719844442485683327761117943656677806782440\
     46508608719697452`494.4034277373633 == 0, K2/K4 -
    28.644675587098757898739657388720295969328306415249832046651320022503661 9\
     510330467375687397652719232818458597319525783090240459427180052125541632\
     690621645574357733548346203889215290563311150819255453371995194074998127\
     219677278142520348561726549962923839682888660312786911292394694422979987\
     544737301749403775386813196698903780242773188322394899434619983384133681\
     620605701568939280702933659715082241277941825513771091823902522583430432\
     508219362386758865176039429522943558849542087429562294086013993936340786\
     059989304301716188`494.09215260572995 == 0}, b0]

Out[72]= {{b0 → 0}}
```

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