The University of the Witwatersrand

Masters Thesis

# Graphs, graph polynomials with applications to antiprisms

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*A thesis submitted in fulfilment of the requirements for the degree of Masters of Science in the*

School of Mathematics

March 2014

# Acknowledgements

I would like to sincerely thank my supervisor, Professor Mphako-Banda, who very graciously accepted me as a student. I could not have completed this task without her ongoing guidance and support. I greatly appreciate her wonderful enthusiasm and encouragement throughout the completion of my dissertation.

I would also like to thank my parents, Bruno and Clarisse Bukasa for allowing me to pursue my passion in mathematics, as well as my brothers, Axel and Kevin Bukasa who consistently believed in me, even when I was doubtful.

Lastly I'd like to thank three incredible friends, Kyle Jacobs, Raymond Phillips and Matthew Woolway, who assisted me with all my difficulties during this demanding period. Their unfailing reinforcement was a significant part of my success, and I am deeply indebted to them.

# Abstract


The $n$-antiprism graph is not widely studied as a class of graphs in graph theory hence there is not much literature.

We begin by defining the $n$-antiprism graph and discussing properties, which we prove in the thesis, and which have not been previously presented in graph theory literature. Some of our significant results include proving that an $n$-antiprism is 4-connected, 4-edge connected and has a pathwidth of 4.

A highly studied area of graph theory is the chromatic polynomial of graphs. We investigate the chromatic polynomial of the antiprism graph and attempt to find explicit expressions for the chromatic polynomial of the antiprism graph. We express this chromatic polynomial in several forms to discover the best-suited form.

We then explore the Tutte polynomial and search for an explicit expression of the Tutte polynomial of the antiprism graph. Using the relationship between a graph and its dual graph, we provide an iterative expression of the Tutte polynomial of the antiprism graph.

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# Chapter 1

# Introduction

Graph theory began in 1735 when Leonhard Euler attempted to solve what appeared to be a simple puzzle regarding bridges in the city of Konigsberg. The city occupied both banks of a river and an island which lay in the river at a point where it branched into two parts. There were seven bridges that spanned the various sections of the river, and the question was whether it was possible for a person to formulate a path through Konigsberg where they cross each of the seven bridges exactly once and return to their starting position, [1]. The problem of the Konigsberg bridge led to Euler's construction of the Eulerian graph.

In the 19th century, many advancements were made in the field of graph theory, including the idea of complete and bipartite graphs, introduced by A. F. Mobius in 1840, as well as the well-known four colour problem introduced by Thomas Guthrie in 1852. Over a hundred years after Euler's problem with the Konigsberg bridges, Thomas. P. Kirkman and William R. Hamilton explored a similar problem in 1856: Given a certain graph, is it possible for a person to formulate a path such that each site is visited exactly once, [14].

Graph theory has now grown into a field with a vast variety of applications, including chemistry, when studying molecules, as well as in modelling the transport networks, activity networks and scheduling for airports, [14].

In this thesis, we seek to establish general properties of the antiprism, as well as investigate any applications of antiprisms.

## 1.1 Basic Definitions

In this section, we discuss basic concepts and notions in graph theory which are relevant to this thesis. We refer the reader to the book [5] unless otherwise stated.

**Definition 1.1.** A **graph**, $G = (V, E)$ is a mathematical structure consisting of two finite sets V (vertices) and E (edges), where each edge has a set of one or two vertices associated to it. These vertices are referred to as the respective edge's endpoints.

A graph is an ordered triple, $G = (V(G), E(G), I_G)$, where $V(G)$ is a nonempty set, $E(G)$ is a set disjoint from $V(G)$, and $I_G$ is an incidence relation that associates with each element of $E(G)$, an ordered pair of elements (same or distinct) of $V(G)$. Elements of $V(G)$ are called the vertices of $G$, and the elements of $E(G)$ are called the edges of $G$. If, for the edge $e$ of $G$, $I_G(e) = \{u, v\}$, we write $I_G(e) = uv$. If $A \subseteq E(G)$, we define the set $V(A) \subseteq V(G)$ to be the set of all vertices which are incident with an edge in $A$.

**Definition 1.2.** The **order** of a graph $G$ is the number of vertices in $G$.

**Definition 1.3.** A **loop** is an edge that joins a single endpoint to itself.

**Definition 1.4.** A **parallel edge** is a collection of two or more edges having identical endpoints.

**Definition 1.5.** The **degree** of a vertex, $v$, is the number of edges incident on $v$.

The degree of a vertex, $v$, is denoted by, $d(v)$. The degree sequence of a graph $G$ is simply the set of degrees of vertices of $G$.

**Definition 1.6.** The **minimum degree** of a graph, $G$, is the degree of the vertex in $G$ with the lowest number of edges incident on it.

**Lemma 1.7.** ***(The Handshaking Lemma)***

*In a graph $G$, with vertex set $V$ and edge set $E$, the sum of the degrees of the vertices is twice the number of edges.*

$$\sum_{v \in V} d(v) = 2|E| \tag{1.1}$$

*Proof.* Every edge is incident with 2 vertices, therefore when the degrees of the vertices are summed up, each edge is counted twice. □

**Definition 1.8.** A **regular** graph is a graph whose vertices all have equal degree.

**Definition 1.9.** A **k-regular** graph is a regular graph whose common degree is k.

**Definition 1.10.** A **walk** in a graph is an alternating sequence of vertices and edges.

**Definition 1.11.** If the first and last vertex of a walk are equal, it is a **closed** walk.

**Definition 1.12.** A **path** in a graph is a walk in which no vertex appears more than once.

**Definition 1.13.** A **trail** is a walk with no repeated edges.

**Definition 1.14.** A graph is **connected** if there is a path from each vertex to any other vertex. In other words, for any two vertices $u, v \in V(G)$, where $V(G)$ is the vertex set of a graph $G$, $G$ is connected if there is a path from $u$ to $v$.

**Definition 1.15.** Let $G$ be a graph. The **components** of $G$ are the number of connected subgraphs of $G$.

The components of $G$ are denoted by $K(G)$.

**Definition 1.16.** An **isthmus**, also known as a bridge or cut-edge, of a graph $G$ is an edge, whose removal from the graph increases the number of components of $G$.

**Definition 1.17.** A **vertex-cut** in a graph $G$ is a vertex-set $U$ such that $G - U$ has more components than $G$.

**Definition 1.18.** A **cut-vertex** is a vertex-cut consisting of a single vertex.

**Definition 1.19.** A **cycle** is a nontrivial closed path.

**Definition 1.20.** Two graphs $G_1$ and $G_2$ are **isomorphic**, denoted $G_1 \cong G_2$ if there exists a bijection, $f : V(G_1) \to V(G_2)$ such that $uv \in E(G_1) \Leftrightarrow f(uv) \in E(G_2)$

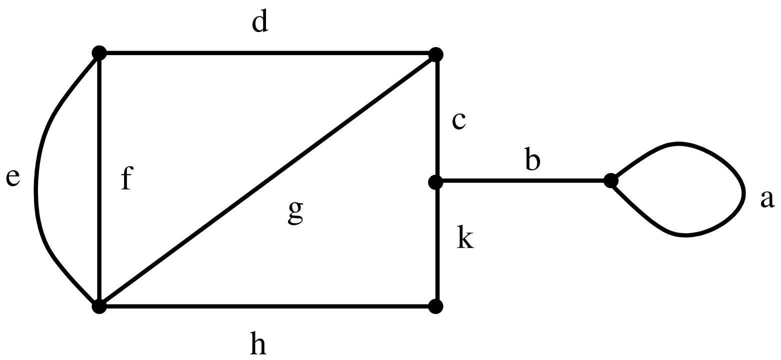


FIGURE 1.1: A graph with 6 vertices and 8 edges

The diagram in Figure 1.1 is an example of a graph $G$, with a loop $a$, parallel edges $\{e, f\}$ and isthmus $b$.

## 1.2 Graph Constructions

There are many well known graph constructions where new graphs are created from old ones. In this section we define and describe some basic graph constructions which are relevant to this thesis.

**Definition 1.21.** A **vertex-join** on a graph $G$ is done by joining a new vertex $v$ to all the pre-existing vertices of $G$. The resulting graph is denoted as, $G + v$.

**Definition 1.22.** A graph is **planar** if it has a drawing where no edges are crossing. A plane graph is the actual drawing of a planar graph.

**Definition 1.23.** The **line graph**, $L(G)$ of a planar graph $G$ has a vertex for each edge of $G$, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ have a vertex in common.

**Definition 1.24.** The **primal graph embedding** of a planar graph, is whichever graph embedding
$i : G \to S$ is supplied as input to the duality process, which is described below.

The planar graph $G$ is called the **primal graph**, the vertices of $G$ the **primal vertices**, the edges of $G$ the **primal edges** and the faces of the embedding of G **primal faces**.

Given a primal graph embedding, defined above, the duality construction of a new graph embedding is made up of a process for dual vertices and another for dual edges:

The dual vertices process begins with inserting a new vertex $t_1$ into the interior of each primal face $t$. We refer to this now as the dual face of $t$. The set $\{t_1; t \in F_G\}$ is denoted as $V_1$ where $F_G$ is the set of primal faces of $G$.

The dual edges process begins with drawing a new edge $e_1$ through each primal edge $e$, joining the dual vertices in the primal faces on one side of a primal edge to the dual vertices on the other side the primal edge. If the same primal face $t$ contains both sides of the primal edge $e$, then the dual edge $e_1$ is a loop through the primal edge $e$ from the dual vertex $t_1$ to itself. The set $\{e_1; e \in E_G\}$ is denoted $E_1$ where $E_G$ is the set of primal edges in $G$.

**Definition 1.25.** The **dual graph** is the graph $G_1$ with vertex-set $V_1$ and edge-set $E_1$.

**Definition 1.26.** A **polyhedron** is a solid figure bounded by polygons.

## 1.3 Classes of Graphs

There are many classes of graphs which have been studied in graph theory. In this section we give some classes of graphs which are relevant to this work.

**Definition 1.27.** A **tree** is a connected graph that has no cycles.

A tree of order $n$ is denoted by $T_n$.

**Definition 1.28.** A **path** graph is a tree without branches, i.e., a tree that contains only vertices of degree 1 and 2.

A path graph of order $n$ is denoted by $P_n$.

**Definition 1.29.** A **complete** graph is a graph in which every two distinct vertices are adjacent.

A complete graph of order $n$ is denoted by, $K_n$.

**Definition 1.30.** An **n-cycle** graph is a graph consisting of a single cycle of order $n$.

The n-cycle graph is denoted by $C_n$.

**Definition 1.31.** A **bipartite** graph $G$ is a graph whose vertex set $V(G)$ can be partitioned into two vertex sets, $V(1)$ and $V(2)$ in such a way that every edge in $G$ joins a vertex, $v \in V(1)$ to a vertex, $u \in V(2)$.

$V(1)$ and $V(2)$ are referred to as partite sets.

**Definition 1.32.** A **complete bipartite** graph, with partite sets $V(1)$ and $V(2)$, is a bipartite graph with the added property that every vertex of $V(1)$ is adjacent to every vertex of $V(2)$.

If $V(1)$ contains $r$ vertices and $V(2)$ contains $s$ vertices, the complete bipartite graph is denoted as $K_{r,s}$.

**Definition 1.33.** A **star** is the complete bipartite graph $K_{1,s}$ where $s \in \mathbb{N}$.

**Definition 1.34.** A **wheel** graph with $n$ spokes (denoted $W_n$) is the join of a vertex and the n-cycle graph, $C_n$.

**Definition 1.35.** A **fan** graph is the join of a single vertex and a path.

**Definition 1.36.** The **complement** of a graph $G$ is a graph $H$ such that $V(H) = V(G)$, and whose edges are such that two vertices are joined whenever they are not joined in $G$.

The complement of a graph $G$ can be denoted as $G^c$

A family of graphs very similar to the antiprism graph is the complete $(n \times 3)$-flower graph, first introduced by Eunice Mphako-Banda in [9]. The $(n \times m)$-flower can be defined as follows:

**Definition 1.37.** A graph $G$ is called an $(n \times m)$**-flower graph** if it has the following set of vertices, $V(G) = \{1, 2, ..., n, n+1, ..., n(m-1)\}$ and the edge set, $E(G) = \{\{1,2\}, \{2,3\}, ..., \{n-1, n\}, \{n, 1\}\} \bigcup \{\{1, n+1\}, \{n+1, n+2\}, \{n+2, n+3\}, ..., \{n+m-3, n+m-2\}, \{n+m-2, 2\}\} \bigcup \{\{2, n+m-1\}, \{n+m-1, n+m\}, \{n+m, n+m+1\}, \{n+m+1, n+m+2\}, ..., \{n+2(m-2)-1, n+2(m-2)\}, \{n+2(m-2), 3\}\} \bigcup ... \bigcup \{\{n, n+(n-1)(m-2)+1\}, \{n+(n-1)(m-2)+1, n+(n-1)(m-2)+2\}, \{n+(n-1)(m-2)+2, n+(n-1)(m-2)+3\}, \{n+(n-1)(m-2)+3, n+(n-1)(m-2)+4\}, ..., \{nm-1, nm\}, \{nm, 1\}\}$. In other words, a graph $G$ is called an $(n \times m)$**-flower graph** if it has $n$ vertices which form an $n$-cycle and $n$ sets of $m-2$ vertices which form $m$-cycles around the $n$ cycle so that each $m$-cycle uniquely intersects with the $n$-cycle on a single edge.

The $(n \times m)$-flower graph will be denoted by $f_{n \times m}$. $f_{n \times m}$ has $n(m-1)$ vertices and $nm$ edges.

**Definition 1.38.** The **center** of $f_{n \times m}$ is the $n$-cycle of $f_{n \times m}$.

The $n$ vertices forming the center are each of degree 4, and all other vertices are of degree 2.

**Definition 1.39.** The **petals** of $f_{n\times m}$ are the $m$-cycles of $f_{n\times m}$.

Petals may be removed by deleting all the vertices of degree 2 in an $m$-cycle.

**Definition 1.40.** An $(n \times m)$**-flower graph with** $i$ **petals** is an $(n \times m)$-flower graph with $n - i$ petals removed, where $i \in \{1, 2, ..., n\}$.

The $(n \times m)$-flower graph with $i$ petals will be denoted by $f^i_{n\times m}$

**Definition 1.41.** The **complete** $(n\times m)$-flower graph has $n$ petals, i.e., no petals have been removed.

Further, $f_{n\times m} = f^n_{n\times m}$.

## 1.4 Graph Polynomials

There are several polynomials associated with a graph $G$. In this section we define and discuss the Chromatic polynomial and the Tutte polynomial, as both are relavant to this work.

**Definition 1.42.** A **colouring** of a graph $G$ is the result of giving to each vertex of $G$ one of a specified set of colours.

**Definition 1.43.** A **proper colouring** of a graph $G$ is a colouring which satisfies the restriction that adjacent vertices are not given the same colour.

**Definition 1.44.** For a graph $G$, the smallest number such that $G$ can be properly coloured in $k$ colours is called the **chromatic number** of $G$, denoted by $\chi(G)$.

Thus, we define the Chromatic Polynomial as follows:

**Definition 1.45.** For $\lambda \in \mathbb{Z}^+$, the **Chromatic Polynomial** of a graph $G$, $P(G;\lambda)$, counts the number of proper colourings of $G$ using $\lambda$-colours.

There are different methods of computing the Chromatic polynomial of a graph $G$, for further details we refer the reader to [11]. One method of computing the chromatic polynomial of a graph $G$ is the deletion and contraction formula described as follows:

1. If $e$ is not a bridge in $G$, then

$$P(G;\lambda) = P(G\backslash e;\lambda) - P(G/e;\lambda)$$

   Here, $G\backslash e$ represents the deletion of edge $e$ and $G/e$ represents the contraction of edge $e$.

2. If $e$ is a bridge of $G$, then

$$P(G;\lambda) = (\lambda - 1)P(G/e;\lambda)$$

3. If $G$ has a loop, then

$$P(G;\lambda) = 0$$

4. A parallel edge (or multi edge) can be ignored and simplified to a single edge.

We make great use of this method, when computing the chromatic polynomial of the antiprism graph in Chapter 3.

Another polynomial which we study in Chapter 4, is the multivariate Tutte polynomial. The Tutte Polynomial can be formally defined as followed:

**Definition 1.46.** Given a graph $G$, with edge set $E$, we define a two variable polynomial, the **Tutte Polynomial** as follows,

$$T(G;x,y) = \sum_{A\subset E}(x-1)^{K(A)-K(E)}(y-1)^{K(A)+|A|-|V|}, \tag{1.2}$$

where $K(A)$ denotes the number of components of the subgraph of $G$ on vertex set $A$.

The Tutte Polynomial can also be computed using the following methods:

1. $T(G;x,y) = xT(G/e;x,y)$ if e is a bridge.

2. $T(G;x,y) = yT(G\backslash e;x,y)$ if e is a loop.

3. $T(G;x,y) = T(G\backslash e;x,y) + T(G/e;x,y)$ otherwise.

with the condition that $T(K_n^c;x,y) = 1$

# Chapter 2

# Properties of the antiprism graph

In this chapter we study properties of our main class of graphs, the antiprism graph. We study relevant properties such as, $k$-connectedness and pathwidth in subsequent sections. The definition of the antiprism graph must be preceded by that of the antiprism since the antiprism graph is in fact the skeleton of an antiprism. We refer the reader to the book, [5], unless otherwise stated.

**Definition 2.1.** An **antiprism** is a semiregular polyhedron made up of two $n$-gons and $2n$ triangles. The net of the antiprism is comprised of two $n$-gons, one on top of a ribbon of $2n$ triangles, and one below. The 5-antiprism is illustrated in Figure 2.1.

**Definition 2.2.** An **antiprism graph** is the graph corresponding to the skeleton of an antiprism. The $n$-antiprism graph consists of $2n$ vertices and $4n$ edges.

The antiprism graph is thus both planar and a polyhedron.
The 5-antiprism graph is illustrated in Figure 2.2.

### 2.0.1 Basic Properties

We illustrate the planarity of an $n$-antiprism graph using the 5-antiprism graph in Figure 2.2. The 5-antiprism graph comprises of 10 vertices, 20 edges and 12 faces.

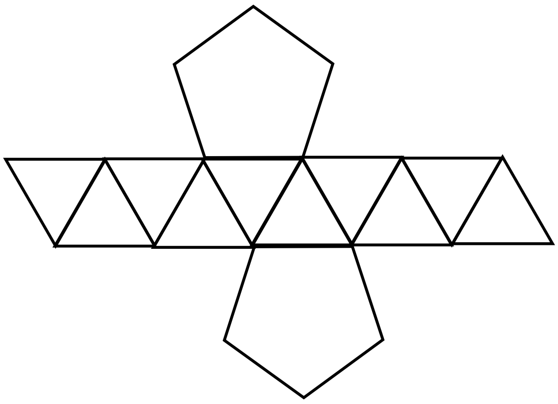

FIGURE 2.1: The 5-Antiprism

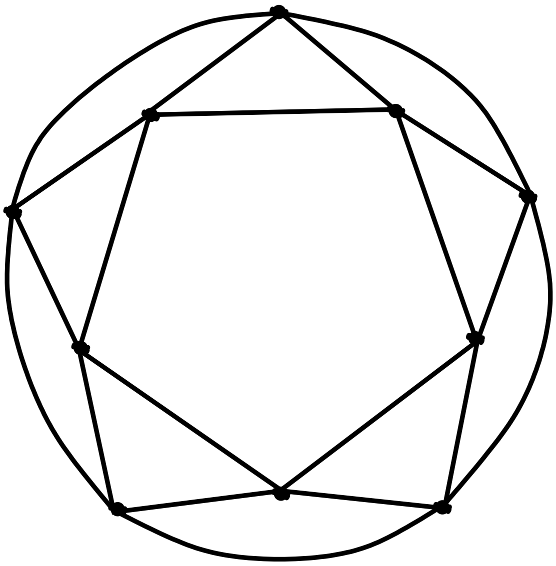

FIGURE 2.2: The 5-Antiprism Graph

The n-antiprism graph is made up of $2n$ vertices and $4n$ edges. The number of edges of an $n$-antiprism graph are proven in Proposition 2.3.

**Proposition 2.3.** *Let $G$ be an antiprism graph with $2n$ vertices (i.e. an $n$-antiprism graph); then $G$ has $4n$ edges.*

*Proof.* (By induction)
An antiprism graph with $2n$ vertices is an $n$-antiprism graph. We begin with the smallest antiprism graphs. For the 3-antiprism graph:
The diagram of the 3-antiprism graph shown in Figure 2.3 clearly shows that the 3-antiprism graph consists of 12 edges, therefore the statement is true for $n = 3$.

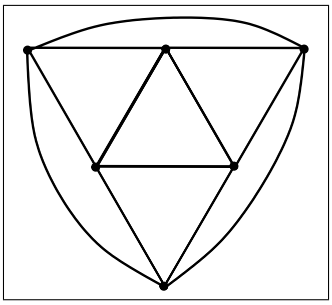

FIGURE 2.3: The 3-antiprism graph

For the 4-antiprism graph:

Again, with the use of a diagram, Figure 2.4, we see that the 4-antiprism graph consists of 16 edges, therefore the statement is true for $n = 4$.

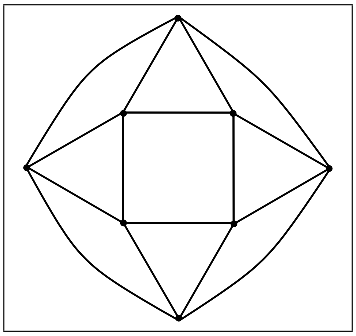

FIGURE 2.4: The 4-antiprism graph

We now assume that the statement is true for some $n$-antiprism graph, i.e, the number of edges of an $n$-antiprism graph is $4n$.

For an $(n+1)$-antiprism graph: Since the inner cycle of an $n$-antiprism graph is an $n$-gon, that of an $(n+1)$-antiprism graph would be an $(n+1)$-gon, so there is an additional edge on the inner cycle of the $(n+1)$-antiprism graph. Further, each edge of the $n$-gon in an $n$-antiprism graph has 2 edges touching both endpoints of the edge and joining above, forming a 3-cycle. Since the $(n+1)$-antiprism graph has an extra edge in the $n$-gon, 2 additional edges must be added to form the 3-cycle. Lastly, the tip of the newly-fomed 3-cycle must be joined with an edge to the other 3-cycles, thus an additional edge is required. In total 4 edges have been added to the $n$-antiprism graph to form the $(n+1)$-antiprism graph, therefore the

$(n+1)$-antiprism graph has $4n+4$ edges. Since the $(n+1)$-antiprism graph has $4(n+1)$ edges, the statement is true for $n+1$.

□

We state Euler's Theorem (Theorem 2.4), which can be found in [3].

**Theorem 2.4.** *If $G$ is a connected plane graph with $n$ vertices, $m$ edges and $r$ faces, then $n - m + r = 2$.*

*Proof.* Using induction on $m$, we begin with $m = 0$.
For $m = 0$: $G = K_1$ and so $n = 1$ and $r = 1$, therefore $n - m + r = 2$ for $m = 0$.

Assume that $n - m + r = 2$ is true for all connected plane graphs with $m - 1$ edges, where $m \geq 1$.

Let $G$ be a connected plane graph with $m$ edges:

1. If $G$ is a tree, then $n = m + 1$ and $r = 1$, therefore $n - m + r = 2$
2. If $G$ is not a tree, let $e$ be a cycle edge of $G$, and consider $G - e$. The connected plane graph $G - e$ has $m - 1$ edges and $r - 1$ faces, therefore by the inductive hypothesis,
   $n - (m - 1) + (r - 1) = 2$. Therefore, $n - m + r = 2$.

Thus $n - m + r = 2$ for all connected plane graphs. □

**Proposition 2.5.** *An $n$-antiprism graph has $2(n+1)$ faces.*

*Proof.* The number of faces of a planar graph, $G$, can be written as:
$|F| = |E| - |V| + 2$, where $|E|$ is the number of edges in $G$, $|V|$ is the number of vertices in $G$ and $|F|$ is the number of faces in $G$. Therefore,

$$2n - 4n + |F| = 2 \tag{2.1}$$

$$|F| = 2(n+1) \tag{2.2}$$

□

## 2.1 k-Connectedness

Connectivity of a graph is defined in several ways in this thesis. In this section, we mention two definitions of connectivity of a graph, vertex-connectivity and edge-connectivity, and we show the connectivity of an antiprism graph using both defintions. We first introduce the notion of $k$-connectedness and we show that the $n$-antiprism graph is 4-connected. The connectivity measures studied can be used in a quantified model of network survivability, which is the capacity of a network to retain connection among its nodes after some edges or nodes are removed, [5]. We begin by defining vertex-connectivity.

**Definition 2.6.** The vertex-connectivity of a connected graph $G$, denoted $K_V(G)$ is the minimum number of vertices whose removal can either disconnect $G$ or reduce it to a 1-vertex.

Thus, if $G$ has at least one pair of non-adjacent vertices, then $K_V(G)$ is the size of a smallest vertex-cut.

**Definition 2.7.** A graph $G$ is $k$-connected if $G$ is connected and $K_V(G) \geq k$. If $G$ has non-adjacent vertices, then $G$ is $k$-connected if every vertex-cut has at least $k$ vertices.

**Example 2.1.** *Let $G$ be the graph shown in Figure 2.5. Since $G$ is a complete graph of order 4, one would have to remove 3 vertices in order to disconnect the graph or reduce it to 1 vertex, since all vertices are joined to each other by an edge. The graph $G$ is thus 3-connected.*

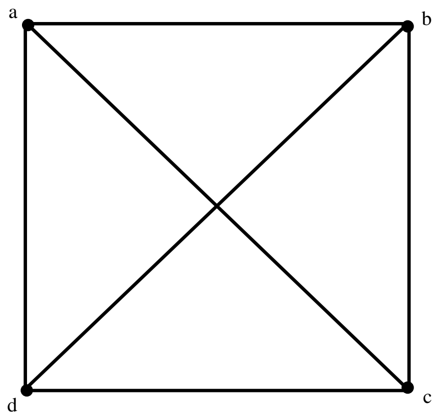


FIGURE 2.5: $K_4$ (Complete graph of order 4)

**Proposition 2.8.** *An $n$-antiprism graph is 4-connected.*

*Proof.* Let $G$ be an $n$-antiprism, with vertex-set $V$.

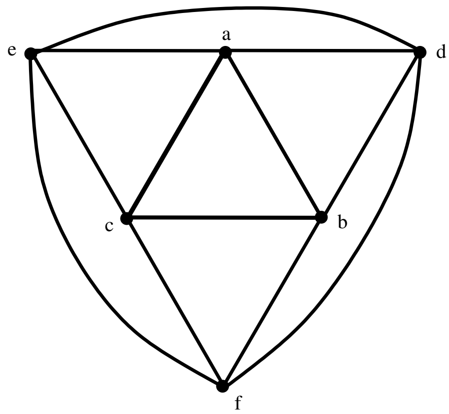


FIGURE 2.6: 3-antiprism graph

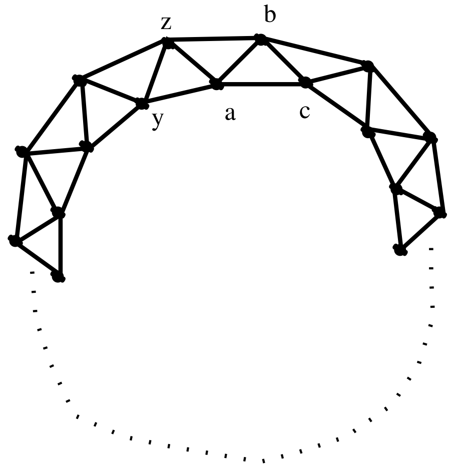


FIGURE 2.7: $n$-antiprism graph

Consider Figure 2.6 for a 3-antiprism graph and Figure 2.7 for an $n$-antiprism graph.

Case 1: G is a 3-antiprism Let $X \subset V$ be vertex set: $\{a, b, c, d, e\}$ such that the vertex $a$ is connected to vertices $\{b, c, d, e\}$.

The vertex-set $V \backslash X$ contains only one vertex. Let $f$ be the only vertex in $V \backslash X$.

We begin by removing the vertex $a$ from $V$, and find that this does not disconnect the graph, as there exists a walk between all vertices remaining. The subsequent cases of the 3-antiprism graph are as follows:

1. Since there exists a 5-cycle, $\{b, f, c, e, d\}$, in $G - a$, the removal of one of the remaining vertices in $V$, leaves the graph connected, as the abovementioned cycle is transformed into a path.

2. Since the graph $G - a$ contains two 3-cycles, $\{c, e, f\}$ and $\{b, d, f\}$ as well as the aforementioned 5-cycle, the removal of 2 vertices from the remaining vertices in $V$ will leave the graph connected, since:

   i Removing 2 vertices in the same 3-cycle, results in the 5-cycle being transformed into a 3-path.

   ii Removing 2 vertices in distinct 3-cycles, also results in a 3-path joining the remaining 3 vertices.

3. Removing any 2 of the remaining vertices in $X$ joined by an edge, and $f$, will also leave the graph connected.

4. Removing any 2 of the remaining vertices in $X$ not joined by an edge, and $f$, disconnects the graph. In total, 4 vertices are removed to disconnect the graph.

The minimum number of vertices that one needs to remove in order to disconnect the 3-antiprism graph, or reduce it to a 1-vertex is 4.

Let $G_1 =$ 3-antiprism graph; then $K_V(G_1) \geq 4$.

For a 3-antiprism graph, removing the vertices, $\{a, c, d, f\}$ disconnects the graph.

Case 2: G is an $n$-antiprism, where $n \geq 4$ Let $X \subset V$ be vertex set: $\{a, b, c, y, z\}$ such that the vertex $a$ is connected to vertices $\{b, c, y, z\}$.

Let $X_1 = \{a, b, z\}$ and $X_2 = \{c, y\}$. Let $S = \{s_1, s_2\}$ where $s_1$ and $s_2$ are vertices joined by an edge, where $s_1$ lies on the outer cycle of the $n$-antiprism and $s_2$ on the inner cycle. Further, $S \bigcap X = \emptyset$. Similarly to the case of the 3-antiprism graph, we begin by removing the vertex $a$ from $V$, and find that this does not disconnect the graph. The subsequent cases of the $n$-antiprism graph are as follows:

1. Since there exists an $(n-1)$-cycle, $\{b, c, d, ..., y, z\}$, in $G - a$, the removal of one of the remaining vertices in $V$, leaves the graph connected, as the abovementioned cycle is transformed into a path.

2. If we continue by removing any 2 vertices in $V$, the graph remains connected, since:

   (a) If any of the 2 vertices removed are elements of $X_1$, there will exist an $(n-3)$-path going through all remaining vertices. The graph therefore remains connected.

   (b) If the 2 vertices removed are joined by an edge, and one lies on the inner cycle of $G$, $\{a, c, ..., y\}$, and the other on the outer cycle of $G$, $\{b, ..., z\}$, there will exist an $(n-3)$-path going through all remaining vertices. Similarly, this graph would still be connected.

   (c) Otherwise, any 2 vertices removed also leaves the graph connected as there will exist an $(n-3)$-cycle going through the remaining vertices.

3. If we continue by removing a vertex from the vertices in $X_2$, followed by the removal of the vertices in $S$, the graph remains connected, and more vertices would need to be removed to disconnect the graph.

4. If we continue by removing a vertex from the remaining vertices in $X_1$, followed by the removal of $s_1$ and $s_2$, we successfully disconnect the graph. A total of 4 vertices are removed to disconnect the graph.

The minimum number of vertices that one needs to remove in order to disconnect the $n$-antiprism graph or reduce it to a 1-vertex is 4, where $n \geq 4$.

Let $G_2 =$ n-antiprism graph, where $n \geq 4$; then $K_V(G_2) \geq 4$.
For an $n$-antiprism graph, removing the vertices, $\{a, b, s_1, s_2\}$ disconnects the graph.

The $n$-antiprism is thus 4-connected. $\square$

$k$-edge-connectivity is very similar to $k$-connectivity, except that the focus is now on edges. We begin by defining edge-connectivity.

**Definition 2.9.** The edge-connectivity of a connected graph $G$, denoted $K_E(G)$ is the minimum number of edges whose removal can either disconnect $G$ or reduce it to a 1-vertex.

**Definition 2.10.** A graph $G$ is $k$-edge-connected if $G$ is connected and $K_E(G) \geq k$. If $G$ has non-adjacent vertices, then $G$ is $k$-edge-connected if every edge-cut has at least $k$ edges.

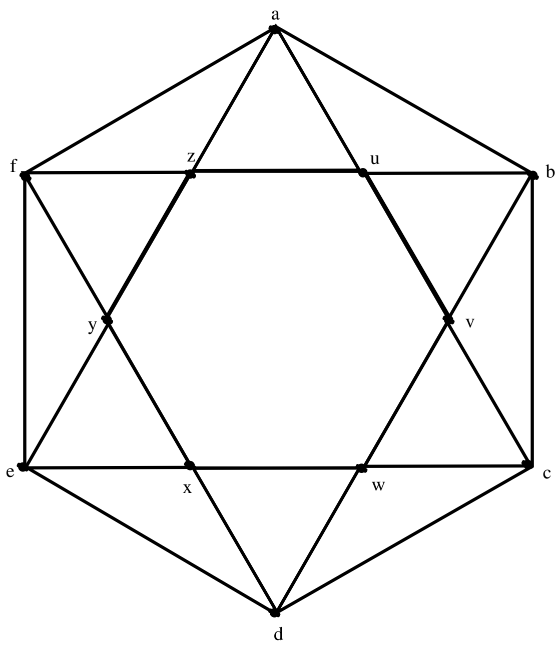


FIGURE 2.8: Cycles in an $n$-antiprism graph

The $n$-antiprism depicted in Figure 2.8 illustrates 3 of the cycles contained in the antiprism graph. The inner $n$-cycle can be written as, $\{u, v, w, x, y, z\}$, the outer $n$-cycle can be written as, $\{a, b, c, d, e, f\}$ and the $2n$-cycle can be written as, $\{a, u, b, v, c, w, d, x, e, y, f, z\}$.

In order to establish the $k$-edge-connectivity of an $n$-antiprism, we make use of Figure 2.8 as well as the following propostion, which can be found in [5]:

**Proposition 2.11.** *Let $G$ be a graph. Then the edge connectivity $K_E(G)$ is less than or equal to the minimum degree, $\delta_{min}(G)$.*

*Proof.* Let $v$ be a vertex of graph $G$, with degree $k = \delta_{min}(G)$. Then the deletion of $k$ edges incident to $v$ separates $v$ from the other vertices of $G$. □

**Proposition 2.12.** *An $n$-antiprism graph is 4-edge-connected.*

*Proof.* Let $G$ be an $n$-antiprism, with edge-set $E$. Since the $n$-antiprism is 4-regular, the minimum degree of the $n$-antiprism is 4. By Proposition 2.11, the edge-connectivity of the $n$-antiprism is less than or equal to 4. We aim to prove that the edge-connectivity is equal to 4, by eliminating 1, 2 and 3.

Case 1: Removing 1 edge Since the $n$-antiprism graph contains a $2n$-cycle, shown in Figure 2.8, containing every vertex in the graph, the removal of one edge does not disconnect the graph, since the cycle is now a path.

Case 2: Removing 2 edges

1. The removal of 2 edges with a common vertex does not disconnect the graph due to the $2n$ cycle contained in the $n$-antiprism.

2. The removal of 2 edges with no common vertex does not disconnect the graph, due to the cyclic nature of the graph, since:

   i If both edges lie on the same $n$-cycle, the graph is not disconnected since the vertices of each each $n$-cycle are still connected by $2n$ edges to the other $n$-cycle.

   ii If the two edges lie on distinct $n$-cycles, the graph remains connected, since the cycles are simply transformed to paths.

iii If one edge lies on an $n$-cycle and the other on a $2n$-cycle, since the cycles are simply transformed to paths.

**Case 3: Removing 3 edges**

1. If all three edges are touching one vertex, the graphs remains connected, since each vertex is contained in one $n$-cycle and one $2n$ cycle, therefore after the removal, it is still contained in a path.
2. If the three edges form a 3-cycle, due to the cyclic nature of the graph, it is clear that the graph remains connected.
3. If the three edges form a path, due to the cyclic nature of the graph, the graph remains connected.
4. If two edges touch one vertex and the third does not, one would have to disconnect the graph by disconnecting all 3 cycles, which is not possible, by removing only 3 edges.
5. Similarly, one cannot disconnect all 3 cycles using 3 edges that have no vertex in common (pairwise).

Let $G = n$-antiprism graph. Since the minimum number of edges that one needs to remove in order to disconnect the graph is 4, $K_E(G) = 4$.

For an $n$-antiprism graph, removing the edges, $\{1, 2, 6, 7\}$ disconnects the graph.

Therefore the $n$-antiprism graph is 4-edge-connected. □

## 2.2 Path-width of the antiprism graph

**Definition 2.13.** Let $G$ be a graph and $A, B$ be sets of edges of $G$. A pair $(A, B)$ is an $n$-separation if $|V(A) \bigcap V(B)| \leq n$. A graph $G$ with $r$ edges has path-width $n$ if its edges can be ordered as $e_1, e_2, ..., e_r$ such that for all $i$, the pair $(\{e_1, e_2, ..., e_i\}, \{e_{i+1}, e_{i+2}, ..., e_r\})$ is an $n$-separation.

There are several different ways that the edges of a graph can be ordered, but we aim to find the order which yields the lowest $n$-separation. Robertson and Seymour introduced the notion of path-width, during the course of their work on graph minors, [12].

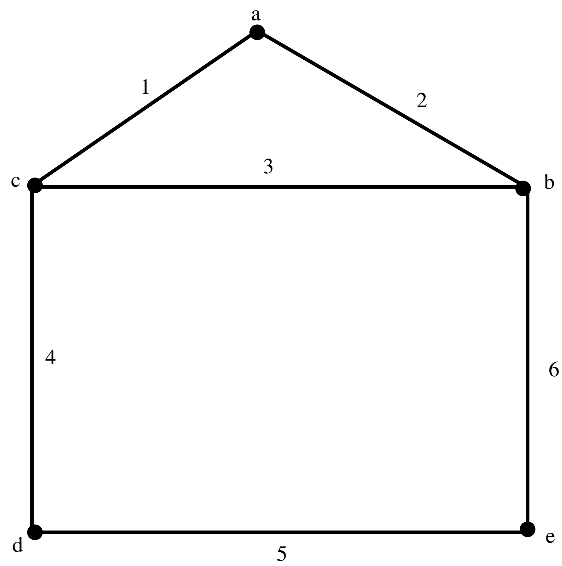


FIGURE 2.9: $H$

**Example 2.2.** *Let $H$ be the graph shown in Figure 2.9 with the ordering of its edges as shown in the figure, i.e. $\{1, 2, 3, 4, 5, 6\}$. We verify the pathwidth as follows:*

$|V(\{1\}) \bigcap V(\{2, 3, 4, 5, 6\})| = |\{a, c\} \bigcap \{a, b, c, d, e\}| = 2$

$|V(\{1, 2\}) \bigcap V(\{3, 4, 5, 6\})| = |\{a, b, c\} \bigcap \{b, c, d, e\}| = 2$

$|V(\{1, 2, 3\}) \bigcap V(\{4, 5, 6\})| = |\{a, b, c\} \bigcap \{b, c, d, e\}| = 2$

$|V(\{1, 2, 3, 4\}) \bigcap V(\{5, 6\})| = |\{a, b, c, d\} \bigcap \{b, d, e\}| = 2$

$|V(\{1, 2, 3, 4, 5\}) \bigcap V(\{6\})| = |\{a, b, c, d, e\} \bigcap \{b, e\}| = 2$

*This order of edges clearly gives a 2-separation, therefore the graph, $H$, has path-width 2.*

**Proposition 2.14.** *An $n$-antiprism graph has path-width 4.*

*Proof.* <u>If $n$ is odd:</u> Without loss of generality, we begin by positioning the $n$-antiprism as done in Figure 2.10 and labelling the top 4 edges touching the vertex

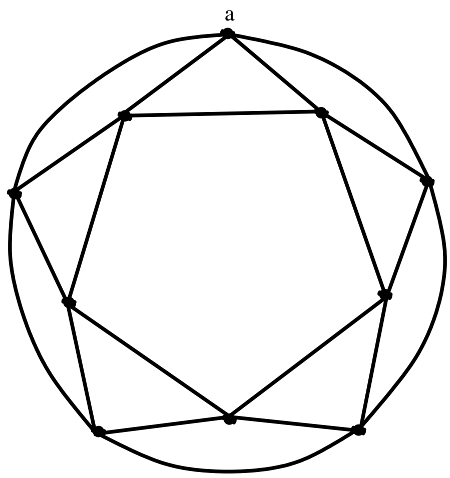


FIGURE 2.10: Labelling $n$-antiprism graph with odd $n$

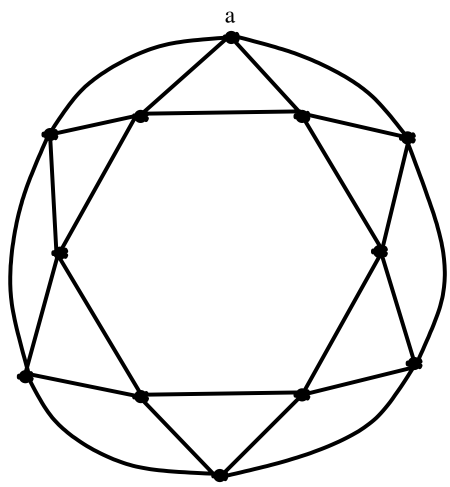


FIGURE 2.11: Labelling $n$-antiprism graph with even $n$

$a$, from left to right, $\{e_1, e_2, e_3, e_4\}$. These 4 edges belong to the set $A_1$.
If $n$ is even: Without loss of generality, we begin by positioning the $n$-antiprism as done in Figure 2.11 and labelling the top 4 edges touching the vertex $a$, from left to right, $\{e_1, e_2, e_3, e_4\}$. These 4 edges belong to the set $A_2$.

We continuously label edges numerically, as follows $\{e_1, e_2, e_3, e_4, e_5, ...\}$.

For both an even and odd $n$, we continue as follows:

1. Moving left to right, label edges where both endpoints are endpoints to some edge in $A_1$ if $n$ is odd and $A_2$ if $n$ is even. Once an edge has been labelled, it forms part of $A_1$ or $A_2$.

2. Moving left to right, for every vertex touching 3 egdes already in $A_1$ or $A_2$, label the fourth edge touching the vertex.

3. Repeat 1 and 2 until all edges are labelled.

Let $B$ and $C$ be sets of edges of an $n$-antiprism. Using the abovementioned ordering and definition 2.13, we examine an $n$-antiprism where $n$ is both even and odd.

1. Let $B = \{e_1\}$ and $C = \{e_2, ..., e_{2n}\}$. $|V(B) \bigcap V(C)| = 2$

2. Let $B = \{e_1, e_2\}$ and $C = \{e_3, ..., e_{2n}\}$. Since $e_1$ and $e_2$ have a common vertex, $|V(B) \bigcap V(C)| = 3$.

3. Let $B = \{e_1, e_2, e_3\}$ and $C = \{e_4, ..., e_{2n}\}$. Similarly, since $e_1$, $e_2$ and $e_3$ have a common vertex, $|V(B) \bigcap V(C)| = 4$.

4. Let $B = \{e_1, e_2, e_3, e_4\}$ and $C = \{e_5, ..., e_{2n}\}$. Since all 4 edges in $B$ have a vertex in common and an $n$-antiprism is 4-regular, the vertex incident to all 4 edges in $B$ is not incident to any edges in $C$, therefore, $|V(B) \bigcap V(C)| = 4$.

5. By labelling edges whose endpoints already form part of $V(B)$, as is done above, $|V(B) \bigcap V(C)|$ will not increase, therefore ensuring that, $|V(B) \bigcap V(C)| \leq 4$.

6. By labelling an edge touching a vertex that is already incident to 3 edges in $B$, the vertex will no longer be incident to any edges in $C$, therefore $|V(B) \bigcap V(C)|$ will not increase, therefore ensuring that, $|V(B) \bigcap V(C)| \leq 4$.

This ordering of the $n$-antiprism graph gives a 4-separation, therefore an $n$-antiprism has path-width 4. $\square$

# Chapter 3

# The Chromatic Polynomial

The aim of Chapter 3 is to search for an explicit expression of the Chromatic Polynomial of antiprism graphs. In this chapter, we use different methods and techniques to try and search for an explicit expression.

## 3.1 History and Definitions

The Four-Colour Conjecture was first thought of in 1852 by 20 year old Francis Guthrie, who relayed the problem to his younger brother Frederick Guthrie, a student of Augustus De Morgan, a mathematics professor at the University College, London. With his brother's permission, Frederick Guthrie spoke to his professor, De Morgan, about the problem, which sparked great interest with the professor, [15].

The Four-Colour Conjecture asks whether a figure which has been divided in any way, where each compartment is differently coloured, so that sections with a common boundary line are differently coloured, can be coloured using only four colours, [15].

Although De Morgan did not solve the conjecture, his avid interest in it certainly kept it alive, since his curiosity with this topic led to him writing to several renowned mathematicians, including Sir William Rowan Hamilton, a mathematics professor at Trinity College, Dublin. Much to the dismay of De Morgan, Sir Hamilton was not interested in the problem, which remained unsolved for many years. In 1879, it seemed that a proof of the conjecture had finally been generated by Alfred Bray Kempe, but 11 years later, it was unfortunately discovered, by Percy John Heawood, that there was in fact a hole in Kempe's proof, [15].

In an attempt to solve the Four-Colour Conjecture, in 1912, George David Birkhoff defined the Chromatic Polynomial, [4].

We recall definitions regarding the chromatic polynomial from Chapter 1:

**Definition 3.1.** A **colouring** of a graph $G$ is the result of giving to each vertex of $G$ one of a specified set of colours.

**Definition 3.2.** A **proper colouring** of a graph $G$ is a colouring which satisfies the restriction that adjacent vertices are not given the same colour.

**Definition 3.3.** For $\lambda \in \mathbb{Z}^+$, the **Chromatic Polynomial** of a graph $G$, $P(G; \lambda)$, counts the number of proper colourings of $G$ using $\lambda$-colours.

We also recall the deletion and contraction method used for the computation of the chromatic polynomial of a graph:

1. If $e$ is not a bridge in $G$, then

$$P(G; \lambda) = P(G \backslash e; \lambda) - P(G/e; \lambda)$$

   Here, $G \backslash e$ represents the deletion of edge $e$ and $G/e$ represents the contraction of edge $e$.

2. If $e$ is a bridge of $G$, then

$$P(G; \lambda) = (\lambda - 1)P(G/e; \lambda)$$

3. If $G$ has a loop, then

$$P(G;\lambda) = 0$$

4. A parallel edge (or multi edge) can be ignored and simplified to a single edge.

Note:

1. The chromatic polynomial of multiple edges joining 2 distinct vertices is equivalent to the chromatic polynomial of one edge joining the 2 vertices.
2. The chromatic polynomial of a graph containing a loop is 0, since there does not exist a proper colouring of a loop.

**Example 3.1.** *We make use of the deletion and contraction formula to compute the chromatic polynomial of $C_3$.*

$$\begin{aligned} P(C_3;\lambda) &= P(C_3 \backslash e;\lambda) - P(C_3/e;\lambda) \\ &= P(T_3;\lambda) - P(T_2;\lambda) \\ &= \lambda(\lambda-1)^2 - \lambda(\lambda-1) \\ &= \lambda^3 - 3\lambda^2 + 2\lambda \end{aligned} \tag{3.1}$$

The final line in Example 3.1 is the chromatic polynomial of the 3-cycle graph written in **Null Basis Form** (Standard Form). The following theorem states some properties of the chromatic polynomial in standard form.

**Theorem 3.4.** *Let $G$ be a graph of order $n$.*
*Then:*

*(i) The degree of the chromatic polynomial is $n$.*

*(ii) The absolute value of the second coefficient is $|E(G)|$.*

*(iii) There are no constant terms in $P(G:\lambda)$.*

*(iv) The terms in the chromatic polynomial, $P(G:\lambda)$, alternate in sign.*

*(v) The coefficient of $\lambda^n$ is 1.*

Another method, often used in conjunction with the deletion and contraction method, to calculate the Chromatic Polynomial of a graph, is described in Theorem 3.5, which can be found in [11].

Note: Let $G$ and $H$ be 2 graphs, such that the vertex sets of $G$ and $H$ have $n$ vertices in common, and every pair of these vertices is joined both in $G$ and $H$. $G$ and $H$ are said to "overlap" in a complete graph on $n$ vertices.

**Example 3.2.** *Let $J$, $G$ and $H$ be the graphs shown in Figure 3.1. As illustrated in Figure 3.1, the graphs $G$ and $H$ overlap in the complete graph on 4 vertices in $J$.*

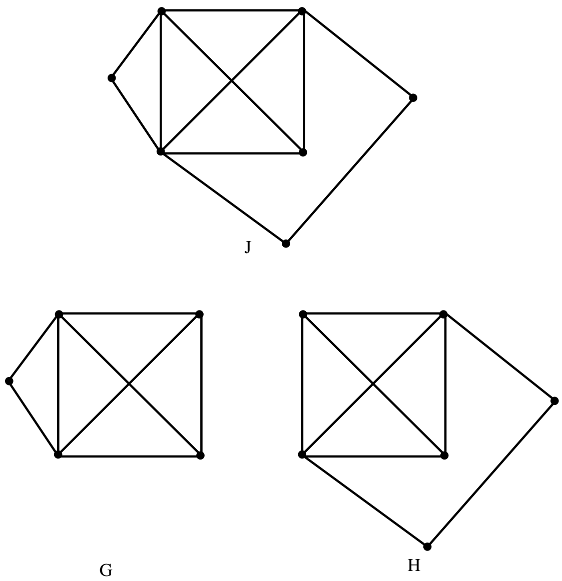


FIGURE 3.1: Overlap of $G$ and $H$

The following theorem regarding the chromatic polynomial of graphs overlapping can be found in [11]:

**Theorem 3.5.** *Let two graphs $G$ and $H$ "overlap" in a complete graph on $n$ vertices to form a graph $J$. Then the chromatic polynomial of the graph $J$ is:*

$$P(J;\lambda) = \frac{P(G;\lambda)\cdot P(H;\lambda)}{P(K_n;\lambda)} \tag{3.2}$$

*Proof.* If the number of colours of the $n$ vertices in the "overlap" are fixed, there will be $P(G;\lambda)/P(K_n;\lambda)$ ways of colouring the remaining vertices of $G$, and $P(H;\lambda)/P(K_n;\lambda)$ ways of colouring the remaining vertices of $H$. Therefore the total number of colourings is,

$$P(K_n;\lambda)\cdot\frac{P(G;\lambda)}{P(K_n;\lambda)}\cdot\frac{P(H;\lambda)}{P(K_n;\lambda)}=\frac{P(G;\lambda)\cdot P(H;\lambda)}{P(K_n;\lambda)} \tag{3.3}$$

□

## 3.2 The Chromatic Polynomial of the antiprism graph

### 3.2.1 The Chromatic Polynomial of Special Families of Graphs

The Null Basis Form (Standard Form) of the chromatic polynomial of a graph is explained in Theorem 3.4, but the chromatic polynomial of graphs can be written in different forms, namely, in terms of the chromatic polynomial of complete graphs or of trees. These alternative representations often aid in determining the trend present in the chromatic polynomial of a certain family of graphs. In order for us to make use of the chromatic polynomial of both the complete graph and the trees, we must first determine their respective chromatic polynomials.

The chromatic polynomial of the tree and complete graph can be found in [7] and are as follows:

**Proposition 3.6.** *The Chromatic polynomial of a complete graph with n vertices, is given by:*

$$P(K_n;\lambda)=\lambda(\lambda-1)(\lambda-2)...(\lambda-(n-1)) \tag{3.4}$$

*Proof.* Since all vertices in a complete graph are joined by an edge, no two vertices may be of the same colour. If $\lambda$ colours are available, then the first vertex may be coloured using $\lambda$ colours, the second, $(\lambda - 1)$ colours, the third, $(\lambda - 2)$ colours, continuing this way, until we reach the $n$-th vertex, which can be coloured using $(\lambda - (n-1))$ colours. Therefore, $P(K_n;\lambda) = \lambda(\lambda-1)(\lambda-2)...(\lambda-(n-1))$. $\square$

We will denote the chromatic polynomial of the complete graph with $n$ vertices as, $(\lambda)_n$

**Proposition 3.7.** *The Chromatic polynomial of a tree with n vertices, is given by:*

$$P(T_n;\lambda) = \lambda(\lambda-1)^{n-1} \tag{3.5}$$

*Proof.* Since every edge in a tree is a bridge,
For $n = 1$:
$P(T_1;\lambda) = \lambda$. Therefore the statement is true for n=1.
Assume that $P(T_{n-1};\lambda) = \lambda(\lambda-1)^{n-2}$, i.e., assume true for $n-1$.
For n:

$$\begin{aligned} P(T_n;\lambda) &= (\lambda-1)P(T_{n-1};\lambda) \\ &= \lambda(\lambda-1)^{n-1} \end{aligned} \tag{3.6}$$

$\square$

Another family of graphs which we make use of is the cycle graph of order $n$, also found in [7]. The chromatic polynomial of the $n$-cycle graph is shown in Proposition 3.8.

**Proposition 3.8.** *The chromatic polynomial of a cycle graph with n vertices, is given by:*

$$P(C_n;\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1) \tag{3.7}$$

*Proof.* For $n = 1$:

The 1-cycle graph is simply a loop, therefore,

$$P(C_1; \lambda) = 0 = (\lambda - 1) + (-1)(\lambda - 1) \tag{3.8}$$

The statement is thus true for $n = 1$.

Assume that $P(C_n; \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$, i.e., assume that the statement is true for $n$.

For $n + 1$:

$$\begin{aligned} P(C_{n+1}; \lambda) &= P(T_{n+1}; \lambda) - P(C_n; \lambda) \\ &= [\lambda(\lambda - 1)^n] - (\lambda - 1)^n - (-1)^n(\lambda - 1) \\ &= (\lambda - 1)^n(\lambda - 1) + (-1)^{n+1}(\lambda - 1) \\ &= (\lambda - 1)^{n+1} + (-1)^{n+1}(\lambda - 1) \end{aligned} \tag{3.9}$$

The statement is true for $n + 1$, therefore,

$$P(C_n; \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1) \tag{3.10}$$

□

We recall the complete $(n \times m)$-flower graph from Chapter 1:

**Definition 3.9.** The **complete** $(n \times m)$-flower graph has $n$ petals, i.e., no petals have been removed.

The diagram below shows a complete $(5 \times 3)$-flower graph.

It is clear from Figure 3.2 that the $n$-antiprism graph can easily be constructed using the $(n \times 3)$-flower graph, by simply adding edges to the graph. Using Figure 3.2, we see that if vertices, $\{\{a\}, \{b\}\}, \{\{b\}, \{c\}\}, \{\{c\}, \{d\}\}, \{\{d\}, \{e\}\}$ and $\{\{e\}, \{a\}\}$ are joined, we have successfully constructed the 5-antiprism graph.

An explicit formula for the $(n \times 3)$-flower graph is expressed in Proposition 3.10.

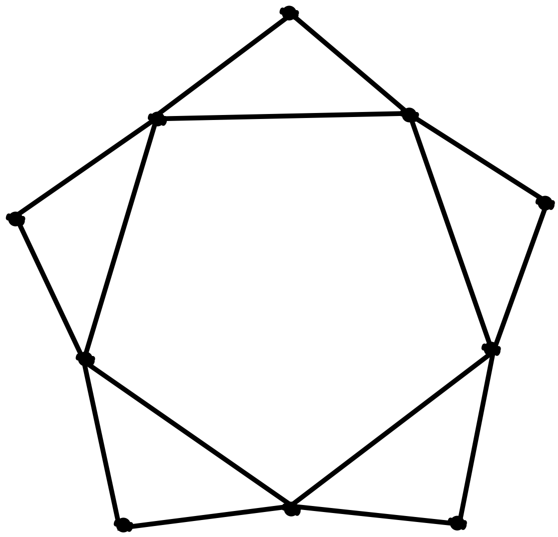

FIGURE 3.2: The $(5 \times 3)$-Flower Graph

**Proposition 3.10.** *Let $H$ be the graph of a complete $(n \times 3)$-flower. Then the chromatic polynomial of $G$ is:*

$$P(H; \lambda) = (\lambda - 2)^n[(\lambda - 1)^n + (-1)^n(\lambda - 1)] \tag{3.11}$$

*Proof.* We use the overlap method mentioned in Theorem 3.5, since the $n$-cycle graph overlaps with $n$ 3-cycle graphs (which are also complete graphs of order 3, $K_3$) on $K_2$ in $f_{n\times 3}$. The chromatic polynomial of the complete $(n\times 3)$-flower graph is simply:

$$\begin{aligned} P(H; \lambda) &= \frac{P(C_n; \lambda) \cdot (P(C_3; \lambda))^n}{((\lambda)_2)^n} \\ &= \frac{P(C_n; \lambda) \cdot ((\lambda)_3)^n}{((\lambda)_2)^n} \\ &= \frac{[(\lambda - 1)^n + (-1)^n(\lambda - 1)] \cdot (\lambda(\lambda - 1)(\lambda - 2))^n}{(\lambda(\lambda - 1))^n} \\ &= (\lambda - 2)^n[(\lambda - 1)^n + (-1)^n(\lambda - 1)] \end{aligned} \tag{3.12}$$

□

The chromatic polynomial of the complete $(n \times 3)$-flower graph shown in Proposition 3.10 can easily be verfied using the formula established in [9], for the complete

$(n \times m)$-flower graph. The formula for the complete $(n \times m)$-flower graph found in [9] is presented in Proposition 3.11.

**Proposition 3.11.** *Let $G$ be the graph of a complete $(n \times m)$-flower. Then the chromatic polynomial of $G$ is:*

$$P(G;\lambda) = [(\lambda-1)^{m-2} - (\lambda-1)^{m-3} \pm \ldots \pm (\lambda-1)]^n [(\lambda-1)^n + (-1)^n(\lambda-1)] \quad (3.13)$$

*Proof.* Without loss of generality, the deletion and contraction method is used to calculate the chromatic polynomial of the complete $(n \times m)$-flower graph, as illustrated in Figure 3.3, where the graphs represent the chromatic polynomial of the respective graphs.

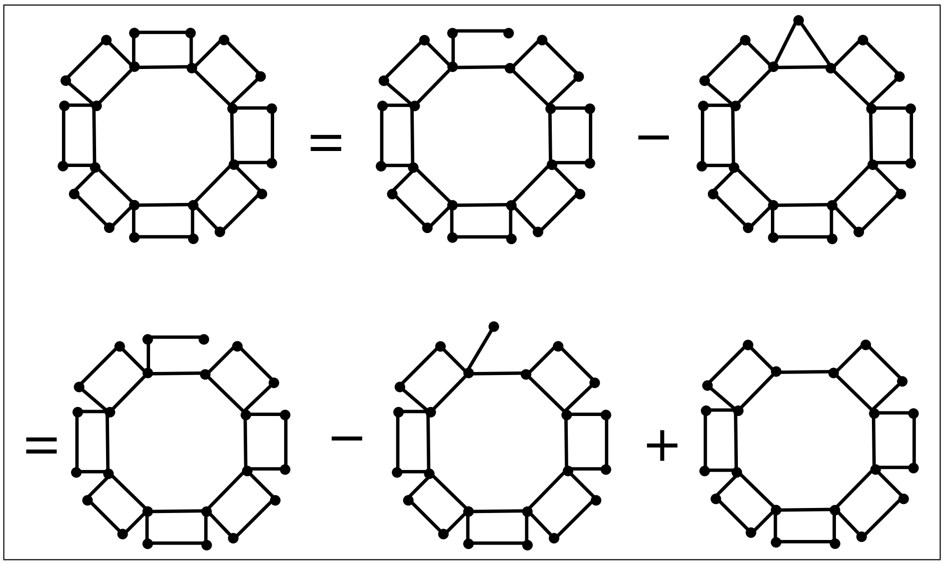

FIGURE 3.3: Chromatic Polynomial of $f_{n \times m}$

We reach the result by repeating the deletion and contraction process. □

Using the formula from Proposition 3.11, we substitute $m = 3$ and find:

$$\begin{aligned} P(G;\lambda) &= [(\lambda-1)^{3-2} - (\lambda-1)^{3-3}]^n [(\lambda-1)^n + (-1)^n(\lambda-1)] \\ &= [(\lambda-1) - 1]^n [(\lambda-1)^n + (-1)^n(\lambda-1)] \\ &= (\lambda-2)^n [(\lambda-1)^n + (-1)^n(\lambda-1)] \\ &= P(H;\lambda) \end{aligned} \quad (3.14)$$

### 3.2.2 Manual Computation and General Formula

Manual computation, although more tedious than computation using packages such as Mathematica, is often necessary, as such packages may not identify any general formula when computing the chromatic polynomial. In this thesis, however, Mathematica has been used to compute the chromatic polynomial and certain features of a general formula for the chromatic polynomial of the $n$-antiprism graph have been established. By making use of the aforementioned deletion and contraction method, as well as the method presented in Theorem 3.5, we illustrate how the chromatic polynomial of the 3-antiprism can be manually computed.

The complete graph can be used extensively when calculating the chromatic polynomial of a graph, by making use of the overlapping method shown in Theorem 3.5.

**Computing the chromatic polynomial of the 3-antiprism**

We begin our manual computation of the chromatic polynomial of the 3-antiprism graph by making use of the deletion and contraction method. Figure 3.4 illustrates the initial steps taken for this computation, where only the deletion and contraction method is used. The graphs shown in Figure 3.4 represent the chromatic polynomials of the demonstrated graph.

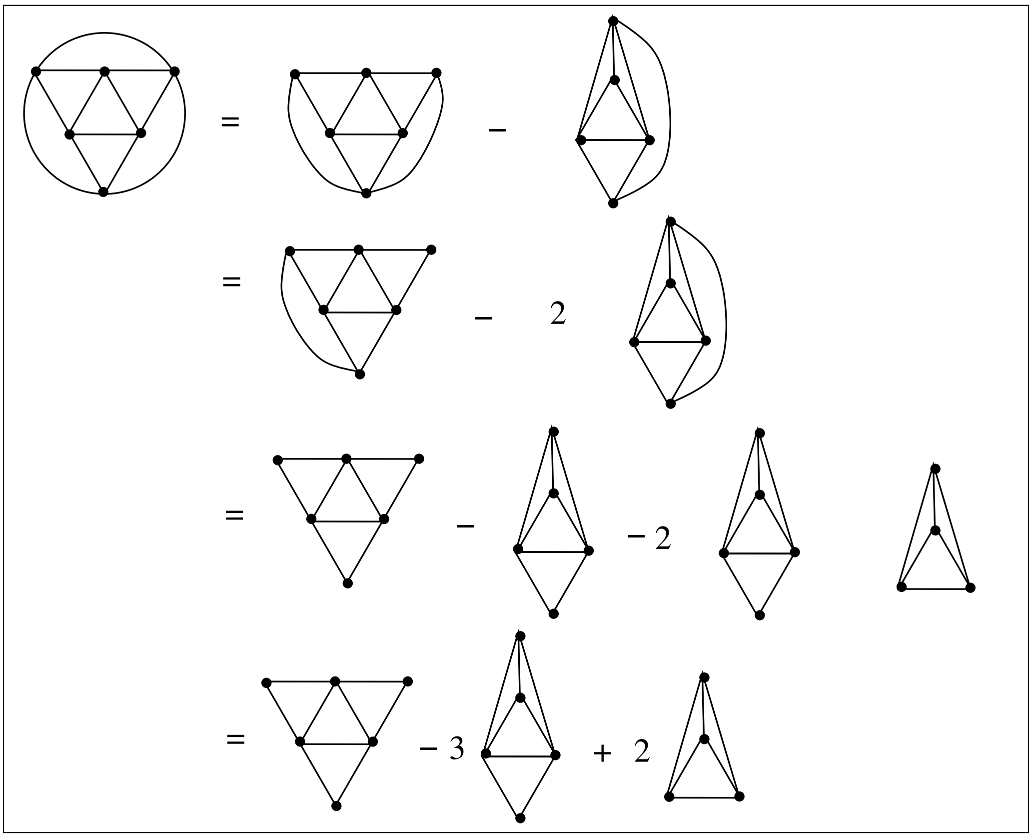


FIGURE 3.4: Finding the Chromatic Polynomial of a 3-Antiprism Graph: Part 1

At this point, one can use several different methods, mentioned above, in order to continue with the computation of the chromatic polynomial. We make use of the overlap method in Theorem 3.5 as this will compute the chromatic polynomial of the 3-antiprism more speedily than other methods mentioned. We also continue to make use of the deletion and contraction method. The continuation of said computation is illustrated in Figure 3.5.

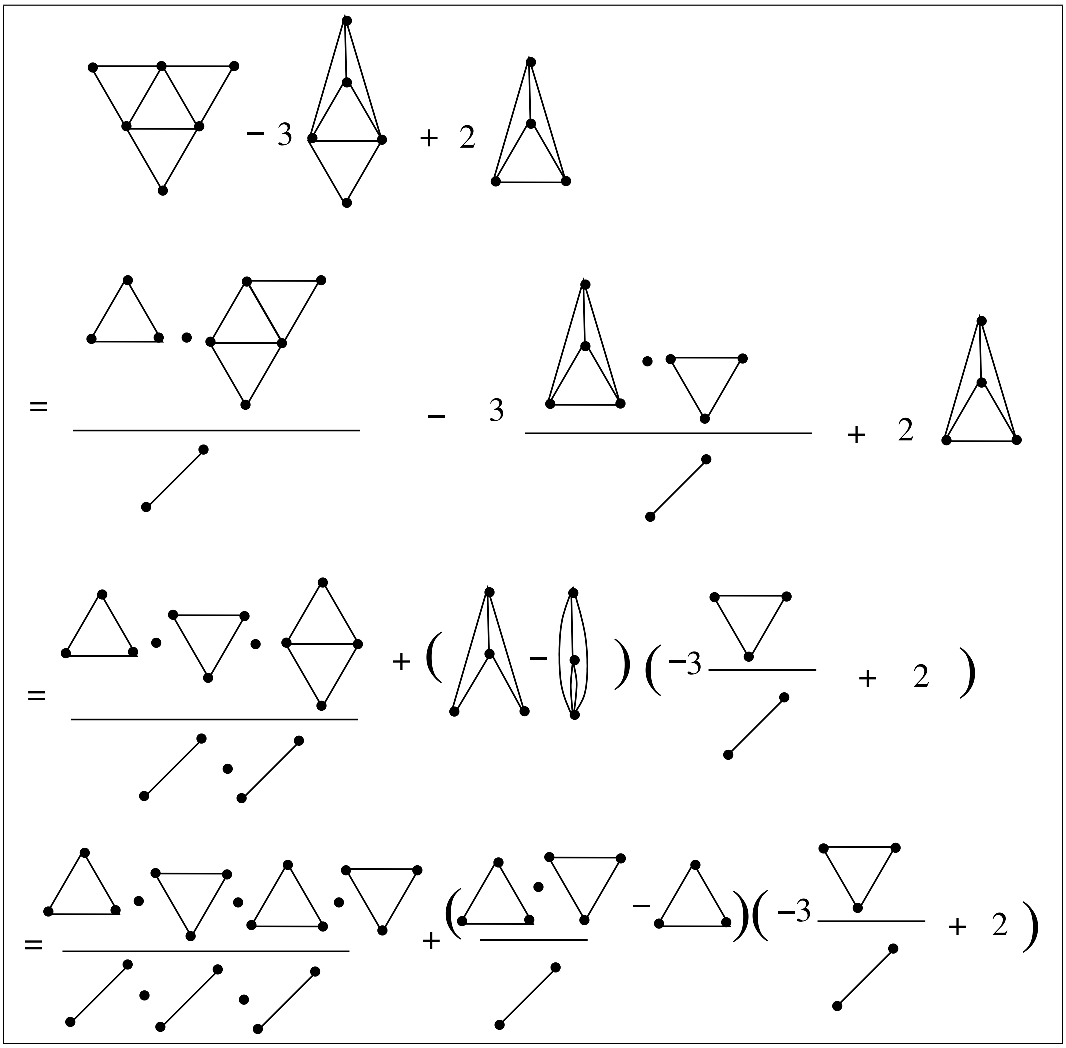

FIGURE 3.5: Finding the Chromatic Polynomial of a 3-Antiprism Graph: Part 2

We may now simply use the general formula for the chromatic polynomial of a complete graph, shown in Proposition 3.6, to find the chromatic polynomial of the 3-antiprism graph. To ease notation, we denote an $n$-antiprism graph as $A_n$.

$$\begin{aligned} P(A_3;\lambda) &= ((\lambda)_3)^4/((\lambda)_2)^3 + (((\lambda)_3)^2/(\lambda)_2 - (\lambda)_3)(2 - 3(\lambda)_3/(\lambda)_2) \\ &= \lambda(\lambda-1)(\lambda-2)((\lambda-2)^3 - 3(\lambda-2)^2 + 5(\lambda-2) - 2) \end{aligned} \tag{3.15}$$

We now calculate $P(A_3; 5)$.

$$\begin{aligned} P(A_3; 5) &= 5(5-1)(5-2)((5-2)^3 - 3(5-2)^2 + 5(5-2) - 2) \\ &= 780 \end{aligned} \tag{3.16}$$

Now, using the chromatic polynomial of the complete graph and trees, as well as the Null Basis Form, we can rewrite the chromatic polynomial of the 3-antiprism graph in 3 different ways, which will help us establish which representation gives a better formula.

We begin with the chromatic polynomial of the 3-antiprism graph in Null Basis Form:

$$\begin{aligned} P(A_3; \lambda) &= \lambda(\lambda-1)(\lambda-2)((\lambda-2)^3 - 3(\lambda-2)^2 + 5(\lambda-2) - 2) \\ &= (\lambda^3 - 3\lambda^2 + 2\lambda)(\lambda^3 - 9\lambda^2 + 29\lambda - 32) \\ &= \lambda^6 - 12\lambda^5 + 58\lambda^4 - 137\lambda^3 + 154\lambda^2 - 64\lambda \end{aligned} \tag{3.17}$$

We now calculate $P(A_3; 5)$ and $P(A_3; 6)$ using the Null Basis Form.

$$\begin{aligned} P(A_3; 5) &= 5^6 - 12(5^5) + 58(5^4) - 137(5^3) + 154(5^2) - 64(5) \\ &= 780 \end{aligned} \tag{3.18}$$

$$\begin{aligned} P(A_3; 6) &= 6^6 - 12(6^5) + 58(6^4) - 137(6^3) + 154(6^2) - 64(6) \\ &= 4080 \end{aligned} \tag{3.19}$$

We now represent the chromatic polynomial of the 3-antiprism in terms of the chromatic polynomial of the complete graph, using factorisation:

$$\begin{aligned}
P(A_3;\lambda) &= \lambda(\lambda-1)(\lambda-2)(\lambda^3-9\lambda^2+29\lambda-32)\\
&= \lambda(\lambda-1)(\lambda-2)((\lambda-3)(\lambda^2-6\lambda+11)+1)\\
&= \lambda(\lambda-1)(\lambda-2)((\lambda-3)((\lambda-4)(\lambda-2)+3)+1)\\
&= \lambda(\lambda-1)(\lambda-2)((\lambda-3)((\lambda-4)((\lambda-5)+3)+3)+1)\\
&= \lambda(\lambda-1)(\lambda-2)+3\lambda(\lambda-1)(\lambda-2)(\lambda-3)\\
&+3\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)+\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)\\
&= (\lambda)_3+3(\lambda)_4+3(\lambda)_5+(\lambda)_6
\end{aligned} \tag{3.20}$$

We once again calculate $P(A_3;5)$ and $P(A_3;6)$, now using the chromatic polynomial in terms of $(5)_n$ and $(6)_n$.

$$\begin{aligned}
P(A_3;5) &= (5)_3+3(5)_4+3(5)_5+(5)_6\\
&= 60+3(120)+3(120)+0\\
&= 780
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
P(A_3;6) &= (6)_3+3(6)_4+3(6)_5+(6)_6\\
&= 4080
\end{aligned} \tag{3.22}$$

Lastly we represent the chromatic polynomial of the 3-antiprism graph using the chromatic polynomial of trees:

$$\begin{aligned}
P(A_3;\lambda) &= 11\lambda(\lambda-1)-25\lambda(\lambda-1)^2+20\lambda(\lambda-1)^3-7\lambda(\lambda-1)^4+\lambda(\lambda-1)^5\\
&= 11P(T_2;\lambda)-25P(T_3;\lambda)+20P(T_4;\lambda)-7P(T_5;\lambda)+P(T_6;\lambda)
\end{aligned} \tag{3.23}$$

Similarly, we calculate $P(A_3;5)$ and $P(A_3;6)$ using the chromatic polynomial in terms of the chromatic polynomial of the tree graph.

$$\begin{aligned} P(A_3;5) &= 11P(T_2;5) - 25P(T_3;5) + 20P(T_4;5) - 7P(T_5;5) + P(T_6;5) \\ &= 11(20) - 25(80) + 20(320) - 7(1280) + 5120 \\ &= 780 \end{aligned} \tag{3.24}$$

$$\begin{aligned} P(A_3;6) &= 11P(T_2;6) - 25P(T_3;6) + 20P(T_4;6) - 7P(T_5;6) + P(T_6;6) \\ &= 4080 \end{aligned} \tag{3.25}$$

From all the methods and different formulae, we see that $P(A_3;5)$ and $P(A_3;6)$ remain the same, which verifies that these different representations of the chromatic polynomial of the 3-antiprism graph are in fact equivalent formulae.

### 3.2.3 Computation using Mathematica

Using Mathematica, it is possible to compute the chromatic polynomial of a graph more rapidly, and we find that it is also easier to identify the pattern present in the chromatic polynomial of $n$-antiprisms.

An exact general formula of the chromatic polynomial of an $n$-antiprism has unfortunately not been established, however, certain generalisations have been detected.

We begin with the 3-antiprism graph; firstly showing the graph, as drawn by Mathematica, then computing the chromatic polynomial:

**The 3-Antiprism Graph**

The code used to draw the 3-antiprism graph in Figure 3.6, together with the code used to compute the chromatic polynomial of the 3-antiprism graph, are shown in .1.1, in the appendix.

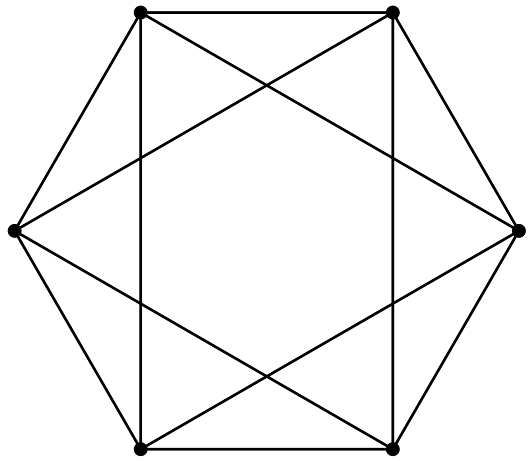

FIGURE 3.6: Alternative drawing of 3-Antiprism Graph

The chromatic polynomial of the 3-antiprism graph is:

$$\begin{aligned} P(A_3;\lambda) = {} & (-2+\lambda)(-1+\lambda)\lambda + 3(-3+\lambda)(-2+\lambda)(-1+\lambda)\lambda \\ & + 3(-4+\lambda)(-3+\lambda)(-2+\lambda)(-1+\lambda)\lambda \\ & + (-5+\lambda)(-4+\lambda)(-3+\lambda)(-2+\lambda)(-1+\lambda)\lambda \end{aligned} \tag{3.26}$$

Subsection 3.2.2 shows that, using the chromatic polynomial of the complete graph with $n$ vertices, $(\lambda)_n$, we can rewrite the chromatic polynomial of the 3-antiprism graph as follows:

$$P(A_3;\lambda) = (\lambda)_3 + 3(\lambda)_4 + 3(\lambda)_5 + (\lambda)_6 \tag{3.27}$$

Further, in Subsection 3.2.2, we rewrite the chromatic polynomial of the 3-antiprism graph using the chromatic polynomial of the tree graph as follows:

$$P(A_3;\lambda) = 11P(T_2;\lambda) - 25P(T_3;\lambda) + 20P(T_4;\lambda) - 7P(T_5;\lambda) + P(T_6;\lambda) \tag{3.28}$$

Similarly, we study the 4-antiprism:

**The 4-Antiprism Graph**

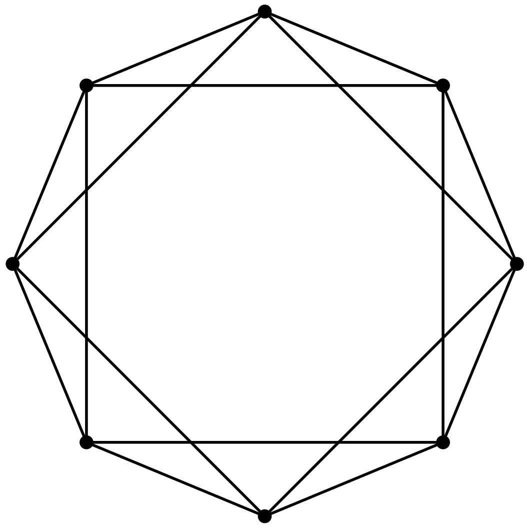

FIGURE 3.7: Alternative drawing of 4-Antiprism Graph

The code used to draw the 4-antiprism graph in Figure 3.7, together with the code used to compute the chromatic polynomial of the 3-antiprism graph, are shown in .1.2, in the appendix.

The chromatic polynomial of the 4-antiprism graph ($A_4$) is:

$$\begin{aligned} P(A_4;\lambda) = {} & 7(-3+\lambda)(-2+\lambda)(-1+\lambda)\lambda \\ & + 44(-4+\lambda)(-3+\lambda)(-2+\lambda)(-1+\lambda)\lambda \\ & + 42(-5+\lambda)(-4+\lambda)(-3+\lambda)(-2+\lambda)(-1+\lambda)\lambda \\ & + 12(-6+\lambda)(-5+\lambda)(-4+\lambda)(-3+\lambda)(-2+\lambda)(-1+\lambda)\lambda \\ & + (-7+\lambda)(-6+\lambda)(-5+\lambda)(-4+\lambda)(-3+\lambda)(-2+\lambda)(-1+\lambda)\lambda \end{aligned} \tag{3.29}$$

The chromatic polynomial of the 4-antiprism graph in Null Basis Form is:

$$P(A_4;\lambda) = -426\lambda + 1285\lambda^2 - 1596\lambda^3 + 1086\lambda^4 - 446\lambda^5 + 112\lambda^6 - 16\lambda^7 + \lambda^8 \tag{3.30}$$

We now calculate $P(A_4;9)$ and $P(A_4;10)$ using the Null Basis Form of the chromatic polynomial of the 4-antiprism graph.

$$\begin{aligned} P(A_4;9) &= -426(9) + 1285(9^2) - 1596(9^3) + 1086(9^4) - 446(9^5) + 112(9^6) \\ &\quad - 16(9^7) + 9^8 \\ &= 5766768 \end{aligned} \tag{3.31}$$

$$\begin{aligned} P(A_4;10) &= -426(10) + 1285(10^2) - 1596(10^3) + 1086(10^4) - 446(10^5) + 112(10^6) \\ &\quad - 16(10^7) + 10^8 \\ &= 16788240 \end{aligned} \tag{3.32}$$

Using the chromatic polynomial of the complete graph with $n$ vertices, $(\lambda)_n$, we can rewrite the chromatic polynomial of the 4-antiprism graph as follows:

$$P(A_4;\lambda) = 7(\lambda)_4 + 44(\lambda)_5 + 42(\lambda)_6 + 12(\lambda)_7 + (\lambda)_8 \tag{3.33}$$

We once again calculate $P(A_4;9)$ and $P(A_4;10)$, now using the chromatic polynomial in terms of $(9)_n$ and $(10)_n$.

$$\begin{aligned} P(A_4;9) &= 7(9)_4 + 44(9)_5 + 42(9)_6 + 12(9)_7 + (9)_8 \\ &= 7(3024) + 44(15120) + 42(60480) + 12(181440) + 362880 \\ &= 5766768 \end{aligned} \tag{3.34}$$

$$\begin{aligned} P(A_4;10) &= 7(10)_4 + 44(10)_5 + 42(10)_6 + 12(10)_7 + (10)_8 \\ &= 16788240 \end{aligned} \tag{3.35}$$

Using the chromatic polynomial of the tree graph, we can rewrite the chromatic polynomial of the 4-antiprism graph as follows:

$$\begin{aligned} P(A_4;\lambda) &= 38P(T_2;\lambda) - 113P(T_3;\lambda) + 137P(T_4;\lambda) - 91P(T_5;\lambda) \\ &\quad + 37P(T_6;\lambda) - 9P(T_7;\lambda) + P(T_8;\lambda) \end{aligned} \tag{3.36}$$

Similarly, we calculate $P(A_4;9)$ and $P(A_4;10)$ using the chromatic polynomial in terms of the chromatic polynomial of the tree graph.

$$\begin{aligned} P(A_4;9) &= 38P(T_2;9) - 113P(T_3;9) + 137P(T_4;9) - 91P(T_5;9) \\ &\quad + 37P(T_6;9) - 9P(T_7;9) + P(T_8;9) \\ &= 38(72) - 113(576) + 137(4608) - 91(36864) + 37(294912) - 9(2359296) \\ &\quad + 18874368 \\ &= 5766768 \end{aligned} \tag{3.37}$$

$$\begin{aligned} P(A_4;10) &= 38P(T_2;10) - 113P(T_3;10) + 137P(T_4;10) - 91P(T_5;10) \\ &\quad + 37P(T_6;10) - 9P(T_7;10) + P(T_8;10) \\ &= 16788240 \end{aligned} \tag{3.38}$$

From all the methods and different formulae, we see that $P(A_4;9)$ and $P(A_4;10)$ remain the same, which verifies that these different representations of the chromatic polynomial of the 4-antiprism graph are in fact equivalent formulae.

### The 5-Antiprism Graph

The code used to draw the 5-antiprism graph in Figure 3.8, together with the code used to compute the chromatic polynomial of the 5-antiprism graph, are shown in .1.3, in the appendix.

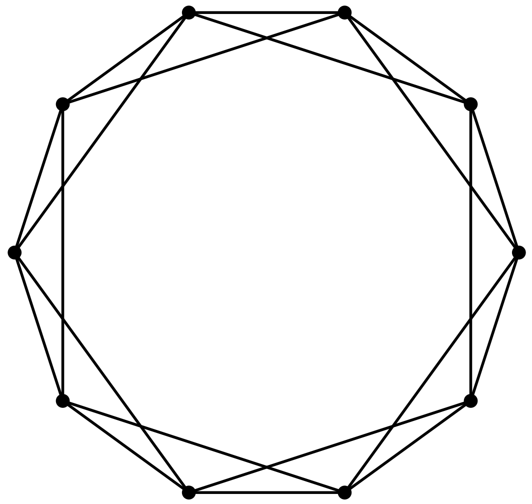

FIGURE 3.8: Alternative drawing of 5-Antiprism Graph

The chromatic polynomial of the 5-antiprism ($A_5$) in Null Basis Form is:

$$\begin{aligned}P(A_5;\lambda) &= -2388\lambda + 8716\lambda^2 - 13935\lambda^3 + 13065\lambda^4 - 8017\lambda^5 + 3358\lambda^6 - 960\lambda^7 \\ &\quad + 180\lambda^8 - 20\lambda^9 + \lambda^{10}\end{aligned} \tag{3.39}$$

We now calculate $P(A_5;12)$ and $P(A_5;13)$ using the chromatic polynomial of the 5-antiprism graph in Null Basis Form.

$$\begin{aligned}P(A_5;12) &= -2388(12) + 8716(12^2) - 13935(12^3) + 13065(12^4) - 8017(12^5) \\ &\quad + 3358(12^6) - 960(12^7) + 180(12^8) - 20(12^9) + 12^{10} \\ &= 10000037520\end{aligned} \tag{3.40}$$

$$\begin{aligned}P(A_5;13) &= -2388(13) + 8716(13^2) - 13935(13^3) + 13065(13^4) - 8017(13^5) \\ &\quad + 3358(13^6) - 960(13^7) + 180(13^8) - 20(13^9) + 13^{10} \\ &= 25937202720\end{aligned} \tag{3.41}$$

Using the chromatic polynomial of the complete graph with $n$ vertices, $(\lambda)_n$, we can rewrite the chromatic polynomial as follows:

$$P(A_5;\lambda) = 50(\lambda)_4 + 458(\lambda)_5 + 985(\lambda)_6 + 720(\lambda)_7 + 210(\lambda)_8 + 25(\lambda)_9 + (\lambda)_{10} \quad (3.42)$$

We once again calculate $P(A_5;12)$ and $P(A_5;13)$, now using the chromatic polynomial in terms of $(12)_n$ and $(13)_n$.

$$\begin{aligned}
P(A_5;12) &= 50(12)_4 + 458(12)_5 + 985(12)_6 + 720(12)_7 + 210(12)_8 + 25(12)_9 + (12)_{10} \\
&= 50(11880) + 458(95040) + 985(665280) + 720(3991680) \\
&+ 210(19958400) + 25(79833600) + (239500800) \\
&= 10000037520
\end{aligned} \quad (3.43)$$

$$\begin{aligned}
P(A_5;13) &= 50(13)_4 + 458(13)_5 + 985(13)_6 + 720(13)_7 + 210(13)_8 + 25(13)_9 + (13)_{10} \\
&= 25937202720
\end{aligned} \quad (3.44)$$

Using the chromatic polynomial of the tree graph, we can rewrite the chromatic polynomial of the 5-antiprism graph as follows:

$$\begin{aligned}
P(A_5;\lambda) &= 112P(T_2;\lambda) - 406P(T_3;\lambda) + 641P(T_4;\lambda) - 601P(T_5;\lambda) + 384P(T_6;\lambda) \\
&- 176P(T_7;\lambda) + 56P(T_8;\lambda) - 11P(T_9;\lambda) + P(T_{10};\lambda)
\end{aligned} \quad (3.45)$$

Similarly, we calculate $P(A_5;12)$ and $P(A_5;13)$ using the chromatic polynomial in terms of the chromatic polynomial of the tree graph.

$$
\begin{aligned}
P(A_5;12) &= 112P(T_2;12) - 406P(T_3;12) + 641P(T_4;12) - 601P(T_5;12) \\
&\quad + 384P(T_6;12) - 176P(T_7;12) + 56P(T_8;12) - 11P(T_9;12) + P(T_{10};12) \\
&= 112(132) - 406(1452) + 641(15972) - 601(175692) + 384(1932612) \\
&\quad - 176(21258732) + 56(233846052) - 11(2572306572) + 28295372292 \\
&= 10000037520
\end{aligned} \tag{3.46}
$$

$$
\begin{aligned}
P(A_5;13) &= 112P(T_2;13) - 406P(T_3;13) + 641P(T_4;13) - 601P(T_5;13) \\
&\quad + 384P(T_6;13) - 176P(T_7;13) + 56P(T_8;13) - 11P(T_9;13) + P(T_{10};13) \\
&= 25937202720
\end{aligned} \tag{3.47}
$$

From all the methods and different formulae, we see that $P(A_5;12)$ and $P(A_5;13)$ remain the same, which verifies that these different representations of the chromatic polynomial of the 5-antiprism graph are in fact equivalent formulae.

It is clear from the above manipulations of the chromatic polynomial of the antiprism graph that the best formula for the chromatic polynomial of the antiprism graph is the one written in terms of the chromatic polynomial of the complete graph. We drop the computation of the chromatic polynomial in terms of the tree graph from this point forward.

The codes used to compute the chromatic polynomials of the 6, 7 and 8-antiprism graphs are shown in .1.4, .1.5 and .1.6 respectively, in the appendix.

**The 6-Antiprism Graph**

The chromatic polynomial of the 6-antiprism in Null Basis Form is:

$$-12286\lambda + 52783\lambda^2 - 103322\lambda^3 + 123423\lambda^4 - 100731\lambda^5 + 59074\lambda^6 - 25342\lambda^7 + 7920\lambda^8 - 1760\lambda^9 + 264\lambda^{10} - 24\lambda^{11} + \lambda^{12} \tag{3.48}$$

Using the chromatic polynomial of the complete graph with $n$ vertices, $(\lambda)_n$, we can rewrite the chromatic polynomial of the 6-antiprism graph as follows:

$$(\lambda)_3 + 164(\lambda)_4 + 4219(\lambda)_5 + 18944(\lambda)_6 + 27326(\lambda)_7 + 16467(\lambda)_8 + 4675(\lambda)_9 + 649(\lambda)_{10} + 42(\lambda)_{11} + (\lambda)_{12} \tag{3.49}$$

**The 7-Antiprism Graph**

The chromatic polynomial of the 7-antiprism in Null Basis Form is:

$$-60072\lambda + 297326\lambda^2 - 688933\lambda^3 + 999908\lambda^4 - 1018318\lambda^5 + 767739\lambda^6 - 439217\lambda^7 + 192190\lambda^8 - 64064\lambda^9 + 16016\lambda^{10} - 2912\lambda^{11} + 364\lambda^{12} - 28\lambda^{13} + \lambda^{14} \tag{3.50}$$

Using the chromatic polynomial of the complete graph with $n$ vertices, $(\lambda)_n$, we can rewrite the chromatic polynomial as follows:

$$672(\lambda)_4 + 39270(\lambda)_5 + 333431(\lambda)_6 + 858023(\lambda)_7 + 912338(\lambda)_8 + 470470(\lambda)_9 + 128128(\lambda)_{10} + 19110(\lambda)_{11} + 1547(\lambda)_{12} + 63(\lambda)_{13} + (\lambda)_{14} \tag{3.51}$$

**The 8-Antiprism Graph**

The chromatic polynomial of the 8-antiprism in Null Basis Form is:

$$\begin{aligned}&-283986\lambda+1592265\lambda^2-4262776\lambda^3+7281324\lambda^4-8886988\lambda^5\\&+8187760\lambda^6-5855728\lambda^7+3294622\lambda^8-1464318\lambda^9+512512\lambda^{10}\\&-139776\lambda^{11}+29120\lambda^{12}-4480\lambda^{13}+480\lambda^{14}-32\lambda^{15}+\lambda^{16}\end{aligned}\tag{3.52}$$

Using the chromatic polynomial of the complete graph with $n$ vertices, $(\lambda)_n$, we can rewrite the chromatic polynomial as follows:

$$\begin{aligned}&2787(\lambda)_4+355924(\lambda)_5+5607350(\lambda)_6+24488768(\lambda)_7+42615547(\lambda)_8\\&+35771452(\lambda)_9+16169582(\lambda)_{10}+4196556(\lambda)_{11}+645918(\lambda)_{12}+59220(\lambda)_{13}\\&+3140(\lambda)_{14}+88(\lambda)_{15}+(\lambda)_{16}\end{aligned}\tag{3.53}$$

The following proposition is clear from the computation of the chromatic polynomial using complete graphs:

**Proposition 3.12.** *The Chromatic Polynomial of the $n$-antiprism graph, $A_n$, can be defined as follows,*

$$P(A_n;\lambda)=\sum_{i=1}^{2n-2}a_i\cdot(\lambda)_{i+2},\tag{3.54}$$

*where $a_1\neq 0$ if $n$ is divisible by 3, and $a_1=0$ if $n$ is not divisible by 3.*

Now we need to find the coefficients $a_i$. In this thesis, we only find $a_{2n-1}$. In order to do so, we require the following lemma:

**Lemma 3.13.** *$K_{2n}$ has $2n^2-n$ edges.*

*Proof.* The graph $K_{2n}$ has $2n$ vertices, with every two distinct vertices joined by an edge. Each vertex therefore has degree $2n-1$. Let the vertex set of $K_{2n}$ be

$V(K_{2n})$, and the edge set $E$. Using the Handshaking Lemma (Lemma 1.7),

$$\begin{aligned}\sum_{v\in V(K_{2n})} d(v) &= 2|E| \\ \sum_{v\in V(K_{2n})} d(v) &= 2n(2n-1) = 2|E| \\ |E| &= n(2n-1) = 2n^2 - n\end{aligned} \tag{3.55}$$

□

**Proposition 3.14.** *Let $A_n$ be the $n$-antiprism graph. Let,*

$$P(A_n;\lambda) = \sum_{i=1}^{2n-2} a_i(\lambda)_{i+2} \tag{3.56}$$

*be the chromatic polynomial of the $n$-antiprism graph. Then, $a_{2n-1} = 2n^2 - 5n$.*

*Proof.* We recall the deletion and contraction method:
$P(G;\lambda) = P(G\backslash e;\lambda) - P(G/e;\lambda)$
Rearranging it, we have:
$P(G\backslash e;\lambda) = P(G;\lambda) + P(G/e;\lambda)$.
Without loss of generality, the coefficient of $(\lambda)_{2n-1}$ is the number of complete graphs of order $2n-1$ obtained in the computation. This is only achieved whenever we contract an edge in the graph with $2n$ vertices, i.e. each time we add an edge to an $n$-antiprism graph to get a complete graph. We recall from Proposition 2.3, that $A_n$ has $4n$ edges. Since a complete graph of order $2n$ has $2n^2 - n$ edges, the coefficient of $(\lambda)_{2n-1}$ is calculated as follows:

$$\begin{aligned}a_{2n-1} &= 2n^2 - n - 4n \\ &= 2n^2 - 5n\end{aligned} \tag{3.57}$$

□

# Chapter 4

# The Tutte Polynomial

The aim of Chapter 4 is to search for an explicit expression of the Tutte Polynomial of antiprism graphs, since determining the Tutte polynomial of graphs has extensive implications beyond graph theory. Computing the Tutte polynomial of a graph is known to be NP-hard, and it is far more complex than computing the chromatic polynomial of the same graph, as described in [6]. In this chapter, we use manual computation to determine the Tutte Polynomial of this family of graphs.

## 4.1 Definitions

We recall the defintion of the Tutte polynomial, initially presented in Chapter 1:

**Definition 4.1.** Given a graph $G$, with edge set $E$, we define a two variable polynomial, the **Tutte Polynomial** as follows,

$$T(G;x,y) = \sum_{A \subset E} (x-1)^{K(A)-K(E)}(y-1)^{K(A)+|A|-|V|} \tag{4.1}$$

where $K(A)$ denotes the number of components of the subgraph of $G$ on vertex set $A$.

Once again, we may make use of deletion and contraction to compute the polynomial, however, the deletion and contraction method used to compute the Tutte polynomial differs from that used to compute the chromatic polynomial as follows:

1. $T(G; x, y) = xT(G/e; x, y)$ if e is a bridge.

2. $T(G; x, y) = yT(G\backslash e; x, y)$ if e is a loop.

3. $T(G; x, y) = T(G\backslash e; x, y) + T(G/e; x, y)$ otherwise.

with the condition that $T(K_n^c; x, y) = 1$, where $K_n^c$ is the complement of $K_n$. Similar to the computation of the chromatic polynomial, the order of this linear recursion is irrelevant.

We illustrate the computation of the Tutte polynomial using the same graph used to illustrate the computation of the chromatic polynomial.

**Example 4.1.** *We use the deletion and contraction method to find the Tutte polynomial of $C_3$.*

$$\begin{aligned} T(C_3; x, y) &= T(C_3\backslash e; x, y) + T(C_3/e; x, y) \\ &= T(T_3; x, y) + T(C_2; x, y) \\ &= xT(T_2; x, y) + T(T_2; x, y) + T(C_1; x, y) \\ &= x \cdot x + x + y \\ &= x^2 + x + y. \end{aligned} \tag{4.2}$$

Recall from Chapter 1:

**Definition 4.2.** The **primal graph embedding** of a planar graph, is whichever graph embedding
$i : G \to S$ is supplied as input to the duality process, which is described below.

The planar graph $G$ is called the **primal graph**, the vertices of $G$ the **primal vertices**, the edges of $G$ the **primal edges** and the faces of the embedding of G **primal faces**.

Given a primal graph embedding, defined above, the duality construction of a new graph embedding is made up of a process for dual vertices and another for dual edges:

The dual vertices process begins with inserting a new vertex $t_1$ into the interior of each primal face $t$. We refer to this now as the dual face of $t$. The set $\{t_1; t \in F_G\}$ is denoted as $V_1$ where $F_G$ is the set of primal faces of $G$.

The dual edges process begins with drawing a new edge $e_1$ through each primal edge $e$, joining the dual vertices in the primal faces on one side of a primal edge to the dual vertices on the other side the primal edge. If the same primal face $t$ contains both sides of the primal edge $e$, then the dual edge $e_1$ is a loop through the primal edge $e$ from the dual vertex $t_1$ to itself. The set $\{e_1; e \in E_G\}$ is denoted $E_1$ where $E_G$ is the set of primal edges in $G$.

**Definition 4.3.** The **dual graph** is the graph $G_1$ with vertex-set $V_1$ and edge-set $E_1$.

Throughout this chapter, we make use of the abovementioned deletion and contraction method for computing the Tutte polynomial, along with the following proposition, which we state without proof:

**Proposition 4.4.** *Let $G$ be a planar graph with dual graph $H$.*

$$T(G; x, y) = T(H; y, x) \tag{4.3}$$

**Example 4.2.** *We wish to illustrate, by computing the Tutte polynomial of the dual of $C_3$, that Proposition 4.4 holds.*
*Let $H$ be the dual graph of $C_3$. Figure 4.1 shows the construction of the dual graph $H$ from $C_3$.*
*We begin by inserting new vertices, denoted by $X$, into the interior of all primal faces, of which there are 2 in this case. These 2 vertices belong to the dual graph.*

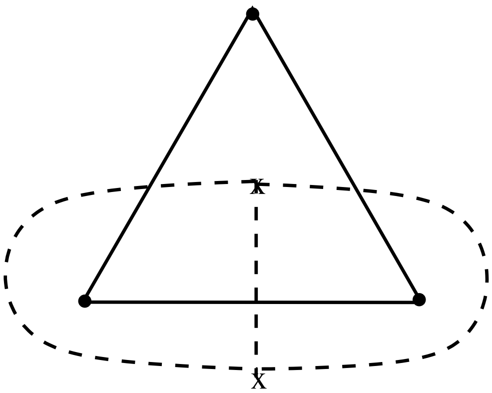


FIGURE 4.1: Construction of the dual graph of $C_3$

*The edges of the dual graph are now constructed by drawing a new edge through each primal edge, joining the dual vertices in the primal faces on one side of a primal edge to the dual vertices on the other side of the primal edge.*

*The dual graph, of $C_3$, $H$, is indicated by dashed lines in Figure 4.2.*

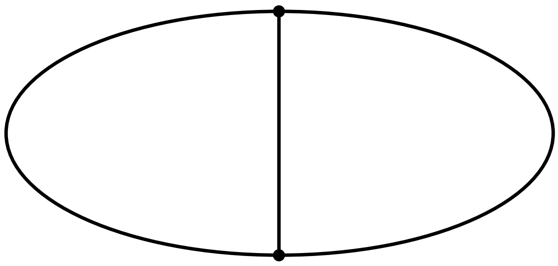

FIGURE 4.2: Dual graph of $C_3$, graph $H$

*The Tutte polynomial of the dual graph, $H$, can be computed as follows:*

$$\begin{aligned} T(H;x,y) &= T(H\backslash e;x,y) + T(H/e;x,y) \\ &= T(C_2;x,y) + y \cdot y \\ &= y + x + y^2 \\ &= T(C_3;y,x) \end{aligned} \tag{4.4}$$

There is in fact a general formula for an $n$-cycle graph, shown in Proposition 4.5.

**Proposition 4.5.** *The Tutte Polynomial of $C_n$ is given by:*

$$P(C_n;x,y) = x^{n-1} + x^{n-2} + ... + x + y = y + \sum_{i=1}^{n-1} x^i \tag{4.5}$$

*Proof.* (By induction)

For n = 1: This is simply a loop. The Tutte Polynomial of a loop is $y$.

For n = 2:

$$\begin{aligned} T(C_2; x, y) &= T(P_2; x, y) + T(C_1; x, y) \\ &= x + y \end{aligned} \tag{4.6}$$

For n = 3:

$$\begin{aligned} T(C_3; x, y) &= T(P_3; x, y) + T(C_2; x, y) \\ &= x^2 + x + y \end{aligned} \tag{4.7}$$

The proposition is therefore true for $n = 1, 2, 3$.

Assume that the proposition is true for $n$.

For $n + 1$: Without loss of generality, the computation of this Tutte Polynomial is illustrated in Figure 4.3, using $C_{10}$.

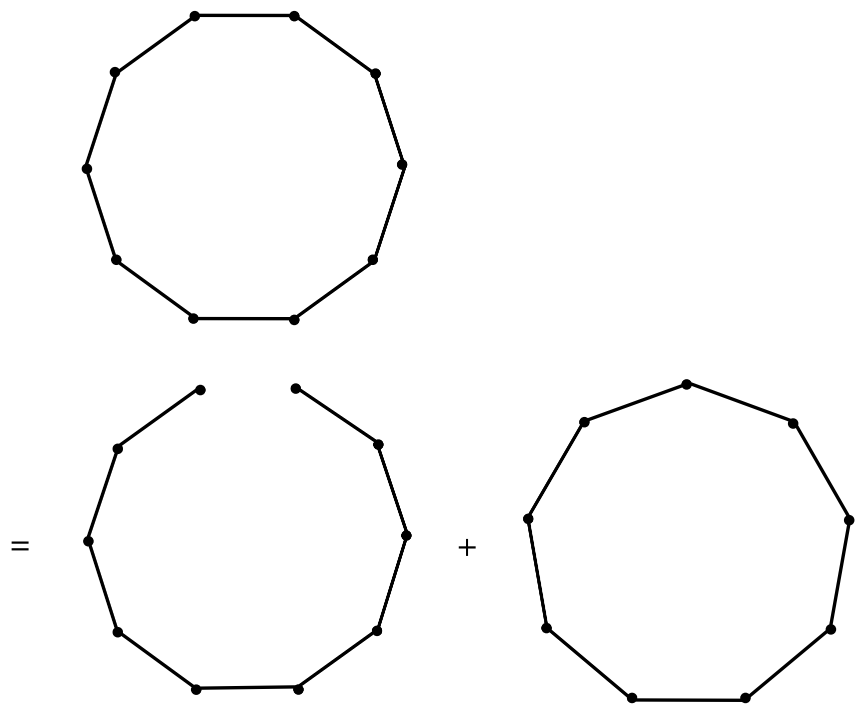


FIGURE 4.3: Computing the Tutte Polynomial of an $n$-cycle Graph

Without loss of generality, using deletion and contraction, the Tutte Polynomial of $C_{n+1}$ can be given by:

$$\begin{aligned} T(C_{n+1};x,y) &= T(P_n;x,y) + T(C_n;x,y) \\ &= x^n + x^{n-1} + x^{n-2} + \dots + x + y \\ &= y + \sum_{i=1}^{n} x^i \end{aligned} \tag{4.8}$$

Since the proposition is true for the $n+1$-cycle, using the inductive hypothesis, the proposition is true for all $n$-cycle graphs. □

Computing the Tutte polynomial of families of graphs is an active area of research. An explicit expression of the Tutte polynomial of the flower graph, defined in Chapter 1, has been computed in [10]. We begin by stating Theorem 4.6, which computes the Tutte polynomial of a flower graph with $i$ petals.

**Theorem 4.6.** *Let $f^i_{n\times m}$ be the flower graph with i petals, where $i \in \{1,2,\dots,n-1\}$. Then,*

$$T(f^i_{n\times m};x,y) = \sum_{k=0}^{i} \begin{pmatrix} i \\ k \end{pmatrix} \left(\sum_{j=0}^{m-2} x^j\right)^{i-k} y^k \left(y + \sum_{q=1}^{n-k-1} x^q\right) \tag{4.9}$$

Using Theorem 4.6 and Proposition 4.5, Mphako-Banda found an explicit expression for the Tutte polynomial of the complete $(n\times m)$-flower graph, which is stated in Theorem 4.7.

**Theorem 4.7.** *Let $f_{n\times m}$ be a complete flower graph. Then,*

$$\begin{aligned} &T(f_{n\times m};x,y) \\ &= \sum_{r=0}^{n-2} y^r \left[\sum_{k=0}^{n-1-r} \begin{pmatrix} n-1-r \\ k \end{pmatrix} \left(\sum_{i=0}^{m-2} x^i\right)^{n-r-k} y^k \left(y + \left[\sum_{i=1}^{n-1-k-r} x^i\right]\right)\right] \\ &+ y^n \left(y + \sum_{i=1}^{m-2} x^i\right). \end{aligned} \tag{4.10}$$

We recall that the flower graphs very similar to the antiprism graphs are the complete $f_{n\times 3}$-flower graph. Using Theorem 4.7, the case for $m = 3$ (complete $f_{n\times 3}$-flower graph) is shown in Corollary 4.8 .

**Corollary 4.8.** *Let $f_{n\times 3}$ be a complete flower graph. Then,*

$$\begin{aligned} &T(f_{n\times 3}; x, y) \\ &= \sum_{r=0}^{n-2} y^r \left[ \sum_{k=0}^{n-1-r} \binom{n-1-r}{k} \left( \sum_{i=0}^{1} x^i \right)^{n-r-k} y^k \left( y + \left[ \sum_{i=1}^{n-1-k-r} x^i \right] \right) \right] \\ &+ y^n \left( y + \sum_{i=1}^{1} x^i \right). \end{aligned} \tag{4.11}$$

## 4.2 The Tutte Polynomial of the antiprism graph

We compute the Tutte Polynomial manually as we find that, with respect to the antiprism graph, it is the best-suited method for pattern-recognition. The method from Proposition 4.4 is used in conjunction with the Deletion and Contraction method to find the Tutte Polynomial of an $n$-antiprism graph, since, in the case of the antiprism graph, finding an explicit expression for the Tutte polynomial of the dual graph is easier than finding the Tutte polynomial of the antiprism graph. In order to make use of Proposition 4.4, the dual graphs of the $n$-antiprism graph is necessary.

### 4.2.1 Dual Graph of an $n$-antiprism graph

The construction of the 3-antiprism graph is shown in Figure 4.4, beginning with an 'X' being inserted into the faces of the primal graph (3-antiprism graph). This set represents the vertices of the dual graph of the 3-antiprism graph. Edges are then drawn, joining the vertices of the dual graph, and form the edge-set of the dual graph.

<u>3-antiprism</u>

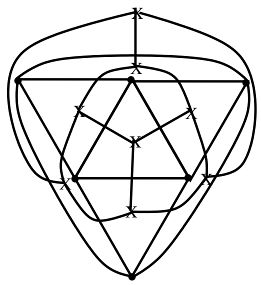

Figure 4.4: Constructing the Dual Graph of the 3-antiprism graph

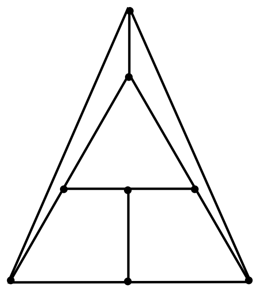

Figure 4.5: The Dual graph of the 3-antiprism graph

We follow the same procedure for the construction of the dual graphs of the 4-antiprism as well as that of the 5-antiprism.

4-antiprism

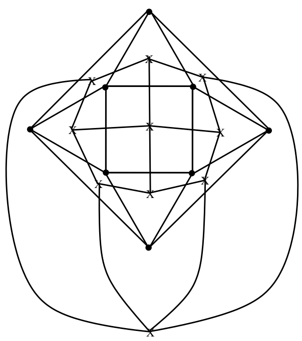

Figure 4.6: Constructing the Dual Graph of the 4-antiprism graph

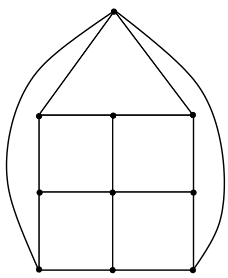

Figure 4.7: The Dual graph of the 4-antiprism graph

5-antiprism

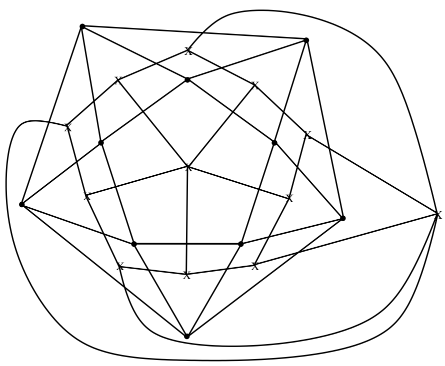

FIGURE 4.8: Constructing the Dual Graph of the 5-antiprism graph

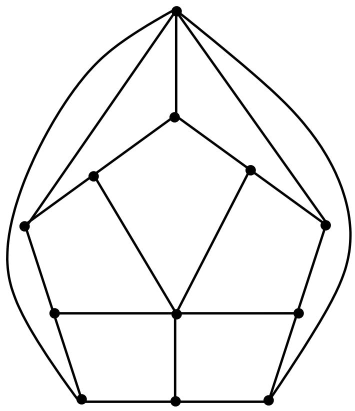

FIGURE 4.9: The Dual graph of the 5-antiprism graph

Using the dual graphs computed in Figures 4.5, 4.7 and 4.9, we have deduced the dual graph of the $n$-antiprism graph, $J$.

**Proposition 4.9.** *The dual graph of an $n$-antiprism graph has $2n+1$ vertices and $4n$ edges.*

*Proof.* Proposition 2.5 states that the $n$-antiprism graph has $2n+1$ faces, therefore the dual graph would be of order $2n+1$. Further, since each edge in the $n$-antiprism graph separates distinct faces, distinct edges are drawn through each edge, and therefore the number of edges in the dual graph of the $n$-antiprism graph is equivalent to the number of edges in the $n$-antiprism graph, which is $4n$. □

We also define the wheel with missing spokes as it is necessary for the computation of the Tutte polynomial of the dual graph of an antiprism graph.

**Definition 4.10.** The wheel with missing spokes is a wheel graph, where spokes have been removed.

We denote a $2n$-wheel with $n$ alternating missing spokes as $W_{2n}^{n}$. An example of $W_{10}^{5}$ is illustrated in Figure 4.10.

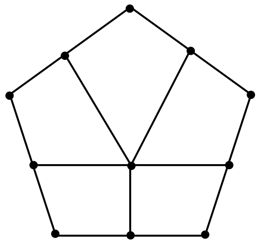

FIGURE 4.10: $W_{10}^{5}$

We recall from Chapter 1:

**Definition 4.11.** A **vertex-join** on a graph $G$ is done by joining a new vertex $v$ to all the pre-existing vertices of $G$. The resulting graph is denoted as, $G + v$.

We now give a definition of an alternating vertex join of $W_{2n}^{n}$.

**Definition 4.12.** An alternating vertex join of $W_{2n}^{n}$ is a graph obtained by adding an extra vertex $X$, and new edges between the new vertex and every vertex of degree 2 in $W_{2n}^{n}$. We shall call the new edges, alternating edges of $W_{2n}^{n}$.

We denote this graph by $\overset{*}{W}{}_{2n}^{n}$. An example of $\overset{*}{W}{}_{10}^{5}$ is illustrated in Figure 4.11.

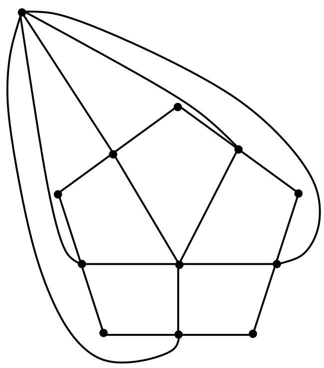

FIGURE 4.11: $\overset{*}{W}{}_{10}^{5}$

An illustration of the dual of a general $n$-antiprism graph, $J$, is shown in Figure 4.12.

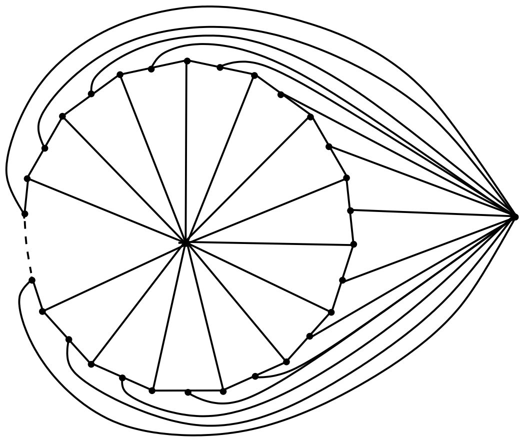

FIGURE 4.12: The Dual graph of a general $n$-antiprism graph

From Definition 4.12, the dual graph of an $n$-antiprism is $\overset{*}{W}{}_{2n}^{n}$.

## 4.3 The Tutte Polynomial of the Dual Graph of the $n$-antiprism graph

Given Proposition 4.4, the Tutte polynomial of the dual graph is sufficient for the computation of the Tutte polynomial of the primal graph, the $n$-antiprism graph.

We begin with the computation of the 3-antiprism graph. Figure 4.13 shows the decomposition of the dual graph of the 3-antiprism graph using the deletion and contraction method for the computation of the Tutte Polynomial. The graphs in Figure 4.13 represent the Tutte polynomial of the respective graphs.

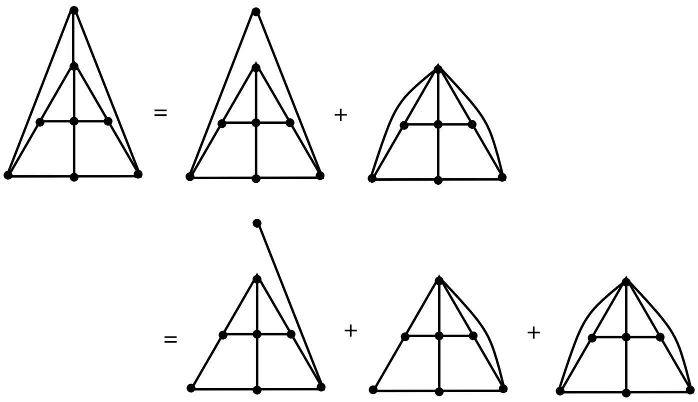

Figure 4.13: Deletion and Contraction of 3-antiprism Dual Graph

As seen in Figure 4.13, the 4 graphs representing the Tutte poynomials of the respective graphs, have a common subgraph, $W_6^3$, shown in Figure 4.14.

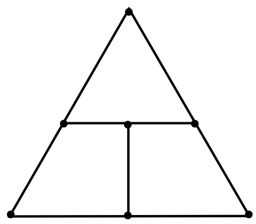

Figure 4.14: $W_6^3$

By finding the Tutte polynomial of $W_6^3$, we simplify the computation of the Tutte polynomial of the dual graph of the 3-antiprism graph, which we use to compute the Tutte polynomial of the 3-antiprism graph. The Tutte polynomial of the wheel graph $W_n$ is computed in [2], but our computation differs from [2], as we compute the Tutte polynomial of the wheel with alternating missing spokes, $W_{2n}^n$. The computation of the Tutte polynomial of $W_6^3$ is shown in Figure 4.15, with the graphs in the figure representing the Tutte polynomial of the respective graphs.

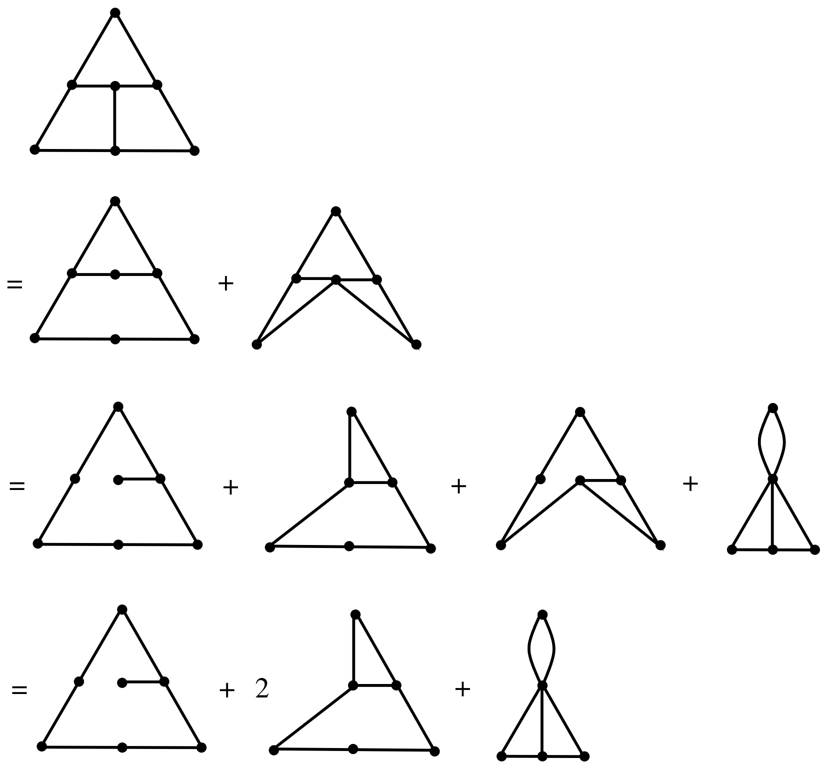


FIGURE 4.15: Deletion and Contraction of $W_6^3$

Similarly, when we analyse the Tutte polynomial of the dual graph of the 4-antiprism graph, using deletion and contraction, shown in Figure 4.16, we find that the graphs have a common subgraph, shown in Figure 4.17.

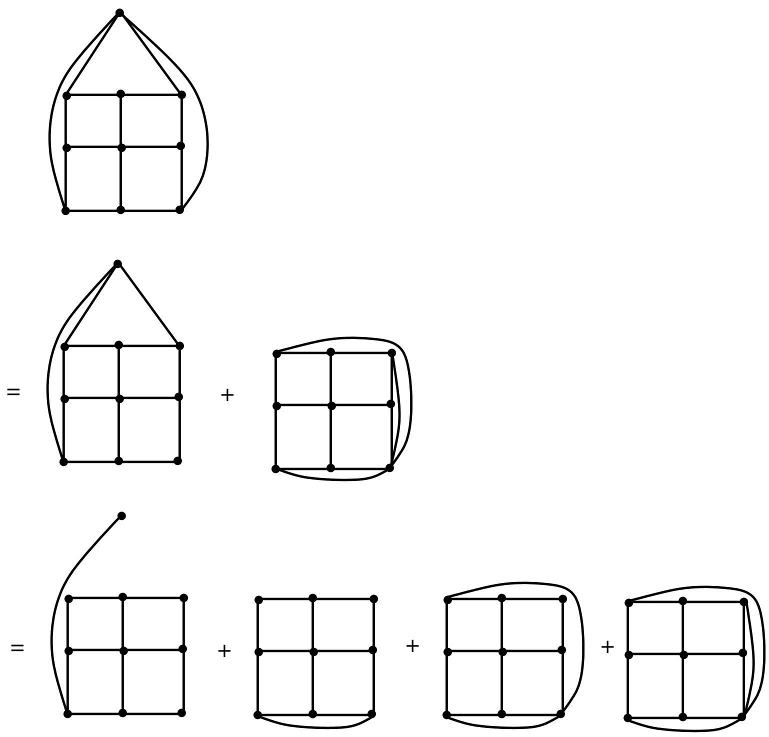

FIGURE 4.16: Deletion and Contraction of 4-antiprism Dual Graph

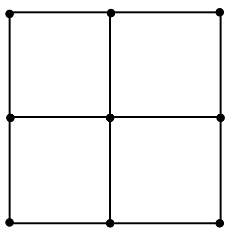

FIGURE 4.17: $W_8^4$

The deletion and contraction method is used again for the computation of the Tutte polynomial of $W_8^4$. This process is shown in Figure 4.18.

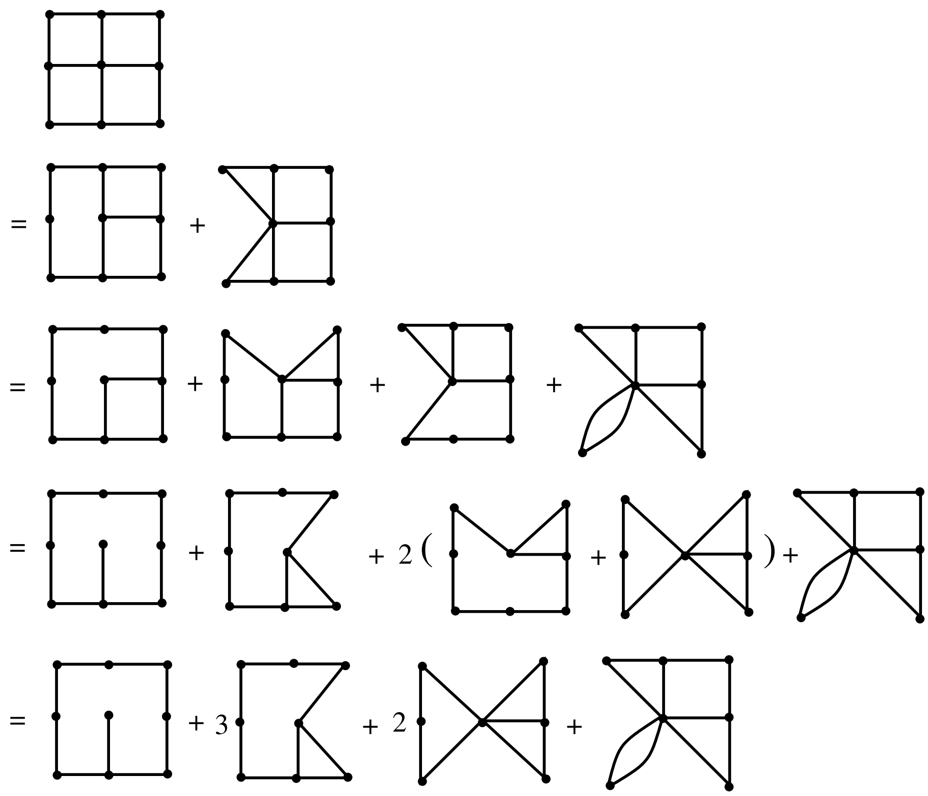


FIGURE 4.18: Deletion and Contraction of $W_8^4$

This trend continues throughout the $n$-antiprism graphs. We illustrate the deletion and contraction of the dual graph of the 5-antiprism graph, Figure 4.19, along with $W_{10}^5$, Figure 4.20 and the computation of the Tutte polynomial of $W_{10}^5$ where the graphs represent the Tutte polynomial of the respective graphs, Figure 4.21.

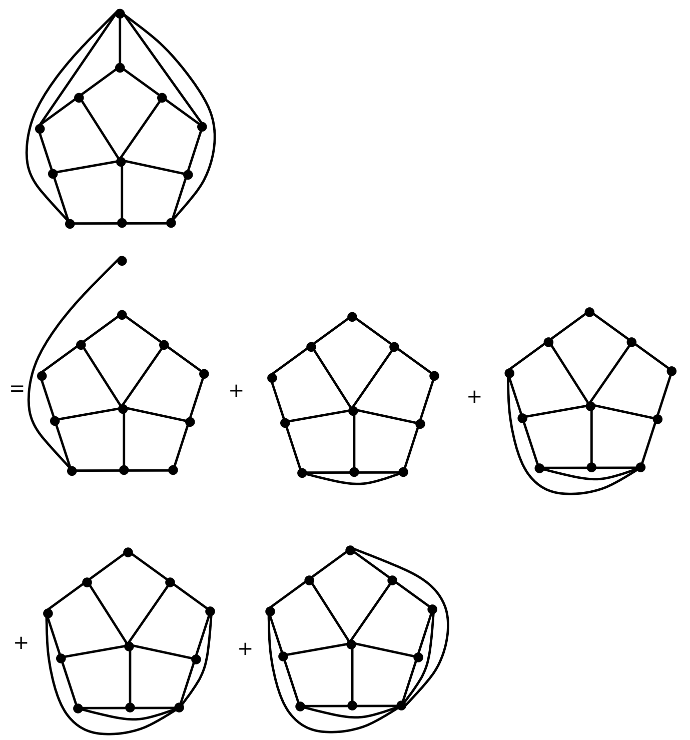

FIGURE 4.19: Deletion and Contraction of 5-antiprism Dual Graph

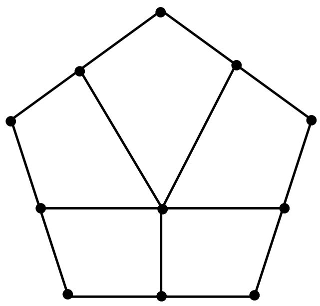

FIGURE 4.20: $W_{10}^{5}$

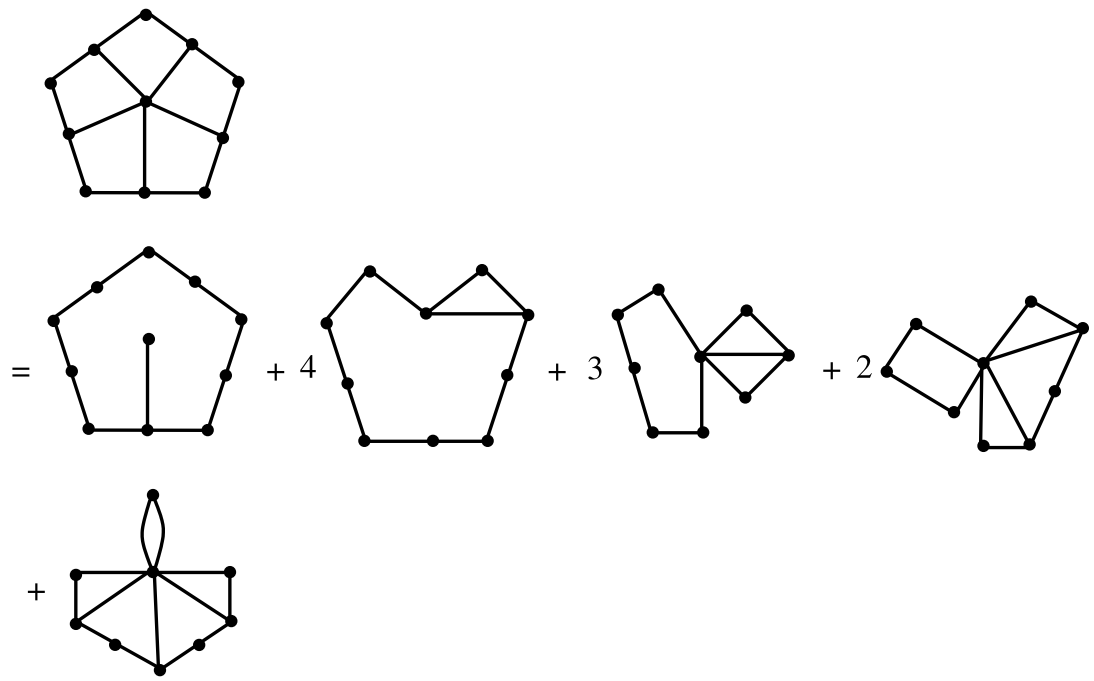


FIGURE 4.21: Tutte Polynomial of $W_{10}^{5}$

Finally one can ascertain, from the Tutte polynomials of the abovementioned wheels with alternating missing stokes, that the Tutte polynomial of $W_{2n}^{n}$ is as illustrated in Figure 4.22, where the graphs represent the Tutte polynomial of the respective graphs.

Recall the fan from Chapter 1:

**Definition 4.13.** A **fan** graph is the join of a single vertex and a path.

Let $P_m$ be the path graph joined to the single vertex. Further, we name the vertices of $P_m$ in order, beginning with the terminal vertex as follows, $\{v_1, v_2, v_3, ..., v_m\}$. We now define the half-fan as follows,

**Definition 4.14.** A **half-fan** graph is a fan graph, where edges have been removed, namely edges joining the original single vertex to $\{v_3, v_5, v_7, ..., v_{m-2}\}$, where $m$ is odd.

Let the half-fan be denoted by $Z(n)$, where $Z(n)$ is a half-fan of order $2(n+1)$.

In Figure 4.22, we refer to the half-fan with $2(n+1)$ vertices as the $n$-half-fan.

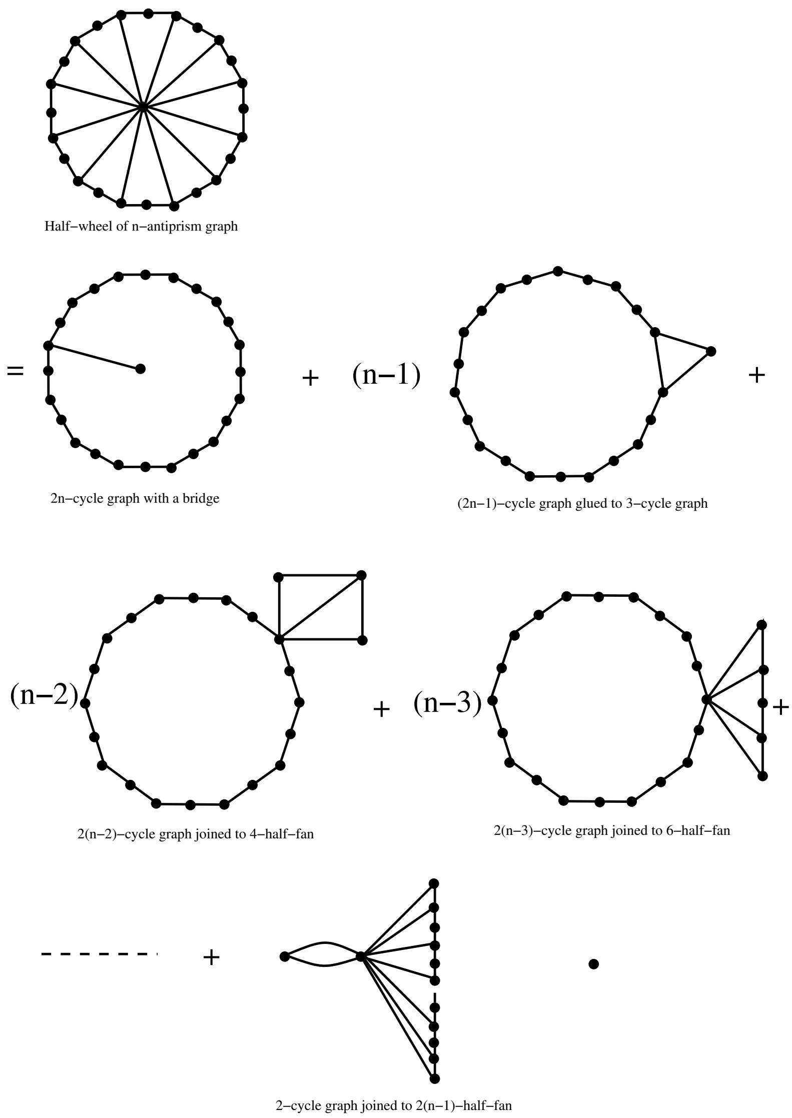


FIGURE 4.22: Tutte Polynomial of $W_{2n}^{n}$

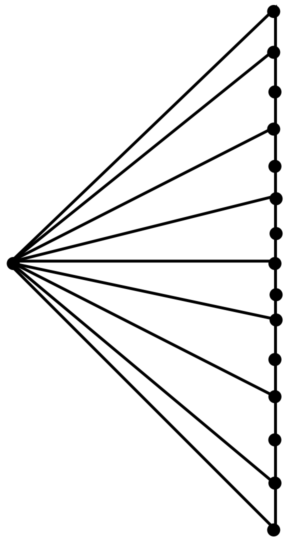

Figure 4.23: Half-fan

We begin with the computation of the Tutte polynomial of $Z(1)$, $Z(2)$ and $Z(3)$ which are illustrated in Figure 4.24, Figure 4.25 and Figure 4.26 respectively, where the graphs represent the Tutte polynomial of the respective graphs.

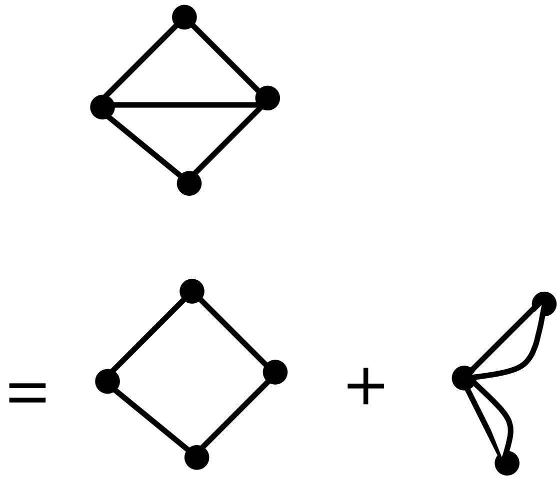


Figure 4.24: Tutte polynomial of $Z(1)$

The Tutte polynomial of $Z(1)$ is simply:

$$T(Z(1); x, y) = T(C_4; x, y) + T(C_2; x, y)(x + y) \tag{4.12}$$

The Tutte polynomial of $Z(2)$ is simply:

$$T(Z(2); x, y) = T(C_6; x, y) + T(C_4; x, y)(x + y) + T(C_2; x, y)T(Z(1); x, y) \tag{4.13}$$

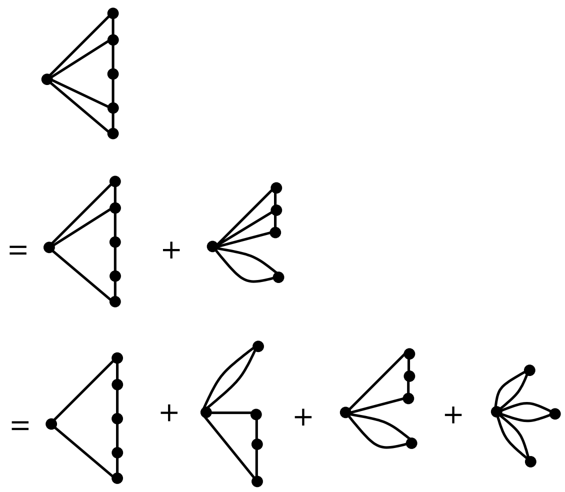


FIGURE 4.25: Tutte polynomial of $Z(2)$

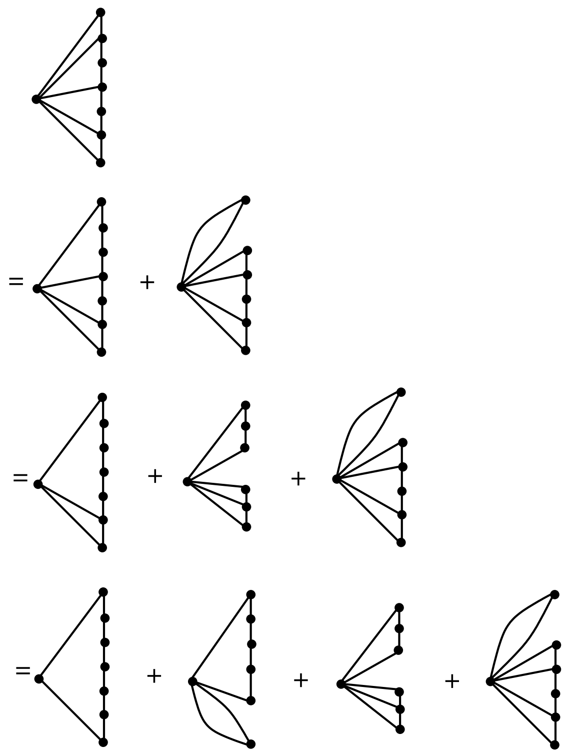


FIGURE 4.26: Tutte polynomial of $Z(3)$

The Tutte polynomial of $Z(3)$ is simply:

$$\begin{aligned} T(Z(3);x,y) = T(C_8;x,y) + T(C_6;x,y)(x+y) + T(C_4;x,y)T(Z(1);x,y) \\ + T(C_2;x,y)T(Z(2);x,y) \end{aligned} \tag{4.14}$$

The trend of the Tutte polynomial becomes clearer with the Tutte polynomial of $Z(4)$ and $Z(5)$:

$$T(Z(4);x,y) = T(C_{10};x,y) + T(C_8;x,y)(x+y) + T(C_6;x,y)T(Z(1);x,y) \\ + T(C_4;x,y)T(Z(2);x,y) + T(C_2;x,y)T(Z(3);x,y) \tag{4.15}$$

$$T(Z(5);x,y) = T(C_{12};x,y) + T(C_{10};x,y)(x+y) + T(C_8;x,y)T(Z(1);x,y) \\ + T(C_6;x,y)T(Z(2);x,y) + T(C_4;x,y)T(Z(3);x,y) \\ + T(C_2;x,y)T(Z(4);x,y) \tag{4.16}$$

The Tutte polynomial of $Z(n)$ can be generalised as follows:

$$\begin{aligned} T(Z(n);x,y) &= T(C_{2(n+1)};x,y) + T(C_{2n};x,y)(x+y) \\ &+ T(C_{2(n-1))};x,y)T(Z(1);x,y) + T(C_{2(n-2)};x,y)T(Z(2);x,y) \\ &+ T(C_{(n-3)};x,y)T(Z(3);x,y) + ... + T(C_6;x,y)T(Z(n-3);x,y) \\ &+ T(C_4;x,y)T(Z(n-2);x,y) + T(C_2;x,y)T(Z(n-1);x,y) \\ &= T(C(2_{n+1});x,y) + T(C_{2n};x,y)(x+y) \\ &+ \sum_{i=1}^{n-1} T(C_{2i};x,y)T(Z(n-i);x,y) \end{aligned} \tag{4.17}$$

The computation of the Tutte polynomial of $W_{2n}^n$ is shown in Figure 4.22. Let $H(n)$ denote the wheel with missing stokes, $W_{2n}^n$. Now that the Tutte Polynomial of the half-fan is known, the Tutte polynomial of $H(n)$ for the $n$-antiprism can be written as:

$$\begin{aligned}
T(H(n);x,y) &= xT(C_{2n};x,y) + (n-1)[T(C_{2n};x,y) + T(C_{2(n-1)};x,y)(x+y)] \\
&+ (n-2)T(C_{2(n-2)};x,y)T(Z(1);x,y) \\
&+ (n-3)T(C_{2(n-3)};x,y)T(Z(2);x,y) \\
&+ (n-4)T(C_{2(n-4)};x,y)T(Z(3);x,y) \\
&+ \ldots + T(C_{2(1)};x,y)T(Z(n-2);x,y) \\
&= xT(C_{2n};x,y) + (n-1)[T(C_{2n};x,y) + T(C_{2(n-1)};x,y)(x+y)] \\
&+ \sum_{i=1}^{n-2}(i)T(C_{2i};x,y)T(Z(n-1-i);x,y)
\end{aligned} \tag{4.18}$$

The Tutte polynomial of the dual graph of the $n$-antiprism makes great use of $W^n_{2n}$, as it can be represented using only wheels with missing stokes, as is shown in Figure 4.13, for the dual graph of the 3-antiprism and in Figure 4.16, for the dual graph of the 4-antiprism.

Now we define an $i$ alternating vertex join of $\overset{*}{W}{}^{n}_{2n}$ as the graph $\overset{*}{W}{}^{n}_{2n}$ with $i$ edges joining the new vertex to $i$ vertices of $W^n_{2n}$ with degree 2. We denote this graph as ${}_{i}\overset{*}{W}{}^{n}_{2n}$.

To ease notation, we denote a minor obtained by contracting one alternating edge of ${}_{i}\overset{*}{W}{}^{n}_{2n}$ by $\overset{n}{\Gamma}_i$. An example of $\overset{10}{\Gamma}_9$ is illustrated in Figure 4.27.

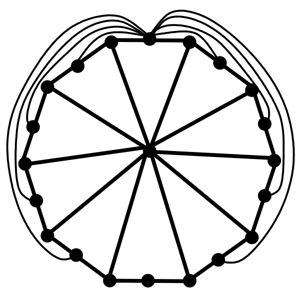

FIGURE 4.27: $\overset{10}{\Gamma}_9$

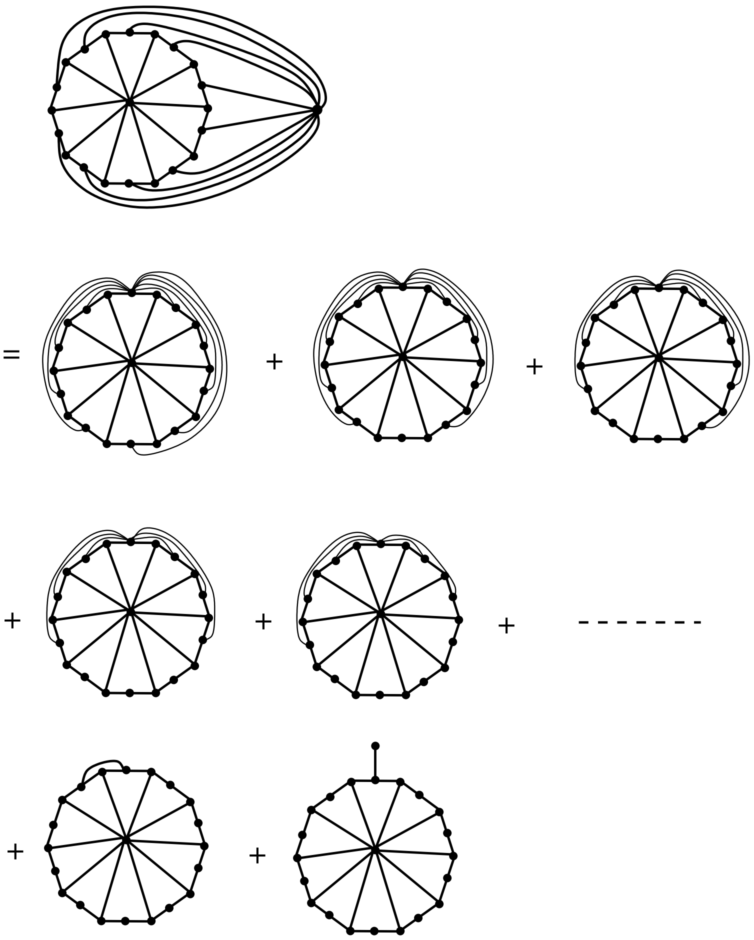

Figure 4.28: Tutte polynomial of the dual graph of the $n$-antiprism

Although we have successfully computed the Tutte polynomial of the wheel with misssing spokes, an explicit expression for the Tutte polynomial of $\overset{n}{\Gamma}_m$, where $m \in \{1, 2, ..., n-2, n-1\}$, has not been established. We conclude Chapter 4 by stating the recursive formula for computing the Tutte polynomial of an $n$-antiprism graph from Figure 4.28.

**Theorem 4.15.** *Let $A_n$ be an $n$-antiprism graph. Then,*

$$\begin{aligned} T(A_n; x, y) &= T(\overset{n}{\Gamma}_1; y, x) + T(\overset{n}{\Gamma}_2; y, x) + ... + T(\overset{n}{\Gamma}_{n-1}; y, x) + T(\overset{*}{W}{}^{n}_{2n}; y, x) \\ &= \sum_{i=1}^{n-1} T(\overset{n}{\Gamma}_i; y, x) + T(\overset{*}{W}{}^{n}_{2n}; y, x) \end{aligned} \tag{4.19}$$

*Proof.* The result is true by Proposition 4.4. □

# Chapter 5

# Conclusion

This thesis analysed the antiprism graph, studying the basic properties of the graph, as well as attempting to solve the graph's polynomials. We began by introducing the antiprism graph which is derived from the antiprism, a semiregular polyhedron. With the aid of *Graph Theory and its Application*, [5], *Graphs and Digraphs*, [3] and *Graph Minors. I. Excluding a Forest*, in Chapter 2, basic properties such as the number of faces and edges were established, as well as slightly more complex properties such as, path-width and connectivity, which is relevant to network survivability.

We moved onto the Chromatic Polynomial in Chapter 3, where we put to use techniques mentioned in *An Introduction to Chromatic Polynomials*, [11], in order to manually find the chromatic polynomial of the antiprism graph. We then introduced computation done on Mathematica, which led us to the general formula of the chromatic polynomial of the antiprism graph, by making use of the chromatic polynomial of the complete graph, which was found in *Computing Chromatic Polynomials for Special Families of Graphs*, [7].

Lastly we attempted to formulate the Tutte polynomial of the antiprism graph by making use of special properties of the Tutte polynomial, more specifically the property mentioned in Proposition 4.4, which allowed us to only determine the Tutte polynomial of the dual graph of the antiprism graph, in order to establish the Tutte polynomial of the antiprism graph. A recursive formula for the Tutte polynomial of the dual graph was formulated in Section 4.3, but is unfortunately incomplete and the question is left open.

# .1 Mathematica Code

## .1.1 The 3-antiprism Graph

Drawing the 3-antiprism graph

```
ShowGraph[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
Cycle[6], {2, 6}], {2, 4}], {4, 6}], {1, 3}], {3, 5}], {1, 5}]]
```

Computing the Chromatic Polynomial of the 3-antiprism graph

```
ChromaticPolynomial[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
Cycle[6], {2, 6}], {2, 4}], {4, 6}], {1, 3}], {3, 5}], {1, 5}], \[Lambda]]
```

## .1.2 The 4-antiprism Graph

Drawing the 4-antiprism graph

```
ShowGraph[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
Cycle[8], {2, 4}], {4, 6}], {6, 8}], {2, 8}], {1, 3}], {3, 5}], {5, 7}], {1, 7}]]
```

Computing the Chromatic Polynomial of the 4-antiprism graph

```
ChromaticPolynomial[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
Cycle[8], {2, 4}], {4, 6}], {6, 8}], {2, 8}], {1, 3}],
{3, 5}], {5, 7}], {1, 7}], \[Lambda]]
```

### .1.3 The 5-antiprism Graph

Drawing the 5-antiprism graph

```
ShowGraph[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
Cycle[10], {2, 4}], {4, 6}], {6, 8}], {8, 10}], {2, 10}],
{1, 3}], {3, 5}], {5, 7}], {7, 9}], {1, 9}]]
```

Computing the Chromatic Polynomial of the 5-antiprism graph

```
ChromaticPolynomial[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
Cycle[10], {2, 4}], {4, 6}], {6, 8}], {8, 10}], {2, 10}], {1, 3}],
```

```
{3, 5}], {5, 7}], {7, 9}], {1, 9}], \[Lambda]]
```

### .1.4 The 6-antiprism Graph

Computing the Chromatic Polynomial of the 6-antiprism graph

```
ChromaticPolynomial[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
Cycle[12], {2, 4}], {4, 6}], {6, 8}], {8, 10}], {10, 12}], {2, 12}],
{1, 3}], {3, 5}], {5, 7}], {7, 9}], {9, 11}], {1, 11}], \[Lambda]]
```

### .1.5 The 7-antiprism Graph

Computing the Chromatic Polynomial of the 7-antiprism graph

```
ChromaticPolynomial[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
Cycle[14], {2, 4}], {4, 6}], {6, 8}], {8, 10}], {10, 12}], {12, 14}], {2, 14}],
{1, 3}], {3, 5}], {5, 7}], {7, 9}], {9, 11}], {11, 13}], {1, 13}], \[Lambda]]
```

### .1.6 The 8-antiprism Graph

Computing the Chromatic Polynomial of the 8-antiprism graph

```
ChromaticPolynomial[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
AddEdge[
Cycle[16], {2, 4}], {4, 6}], {6, 8}], {8, 10}], {10, 12}], {12, 14}],
{14, 16}], {2, 16}], {1, 3}], {3, 5}], {5, 7}], {7, 9}], {9, 11}],
{11, 13}], {13, 15}], {1, 15}], \[Lambda]]
```

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