

Quantifying Counterparty Credit Risk

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October 10, 2012

Declaration

I declare that this project is my own, unaided work. It is being submitted as partial fulfilment of the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

Counterparty credit risk (CCR) is the risk that a counterparty in a deal will not be able to meet their contractual obligations in the future. While CCR is an important task for any risk desk, it has often been underestimated due to the miss-conception that some counterparties were deemed to be either too big to fail or too big to be allowed to default. This was highlighted by the 2008 financial crisis that saw respected banks, such as Lehman Brothers, and financial service providers, such as AIG, default on their obligations. Since then there has been renewed interest in CCR, with the focus being on actively pricing and hedging it. In this work CCR is investigated including its intersection with other forms of risk. CCR mitigation techniques are explored, followed by the formal quantification of CCR in the form of credit value adjustments (CVA). The analysis of CCR is then applied to interest rate derivatives, more specifically forward rate agreements (FRAs) and interest rate swaps (IRSs).

The effect of correlation on unilateral and bilateral CVA between counterparties, including risk factors such as the interest rate, is investigated. This is investigated under two credit risk modelling frameworks, the structural and intensity based frameworks. It is shown that correlation has a none-negligible effect on both unilateral and bilateral CVA for FRAs and IRSs. Correlation structures, namely the Gaussian and the Student-t copula, are used to induce dependency in order to understand their effect on both unilateral and bilateral CVA. It is shown that the choice of copula does not have significant effect on either unilateral or bilateral CVA.

Acknowledgements

I would like to thank my supervisors, Dr Thomas McWalter and Prof Coenraad Labuschagne for their patience, encouragement and effort. The long hours I spent with Tom in the final stages were the most inspirational and motivational. I would also like to thank the academics, Prof Mervyn Curtis, Prof David Taylor, Mike Mchunu and Prof Scott Hazelhurst, for all the support they gave me. Special thanks go to Rand Merchant Bank in particular Glenn Brickhill, Theo Thomas, Patrick Reynolds, Tracey Ashington and Ilka Dunne, for the moral and financial support. Glenn Brickhill was available to discuss possible research topics on a number of occasions and will always be responsible for my passion in credit risk modeling. I would also like to acknowledge the financial support received from the National Research Foundation (NRF) and the Mellon Foundation. I appreciate the amount of effort Mokgadi Rasekgala and Theresa Offwood-le Roux put in editing and proofreading my work. Last but not least, I would like to thank my family for their love and support in all that I have done.

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Chapter 1

Introduction

Counterparty risk is the risk that is specific to a counterparty in a derivatives deal. One such risk specific to a counterparty is credit risk, which is the risk that the counterparty will not be able to meet their contractual obligations. This type of risk, while not in isolation from other risks, naturally arises every time a new deal/trade is booked over-the-counter (OTC). Exchange traded contracts are also not immune from this form of risk as they are also backed by clearing members that may default. Trading OTC derivatives has the advantage that it limits the spread of information on the trades done, which guarantees that hedging and trading strategies are not unfairly copied, however, the introduction of counterparty specific risk is the main disadvantage.

The bank of International Settlements (BIS) estimates that as of the end of June 2010 the total notional outstanding for OTC trades to be well over 582 trillion dollars, more than 70% of which are interest rate swaps (IRSs) and forward rate agreements (FRAs)[37]. This signifies a growing OTC derivatives market and the growing importance of managing counterparty risk in general and more specifically managing counterparty credit risk (CCR). As already mentioned CCR does not exist in isolation of other risks. It is associated with market risk, in that, prevailing market conditions can lead to the deterioration of a counterparty's credit quality which subsequently leads to greater exposure. Also, trading under the International Swaps and Derivatives Association (ISDA) master agreement with the credit support annex (CSA) activated is generally a CCR mitigating technique but may lead to the introduction of liquidity risk which would generally be caused by lack of buyers for the posted collateral during times of distress. Its interaction with other forms of risk indicates that in quantifying CCR, it is necessary to consider the correlation of default with other risk factors.

Counterparty risk management practices are mature, having been noted already

in Basel 2 [68]. Managing counterparty credit risk may involve employing techniques such as netting, close-out netting and collateral posting. All these are supported by the ISDA master agreement and will be discussed in Chapter 3 of the dissertation. A more controversial and intuitive approach would be to only trade with counterparties of the highest credit quality. This is controversial not only because it subjects small counterparties to unfair competition but also because companies with good ratings may fail — in the recent history, for example, Lehman Brothers was rated AA moments before it failed.

On another front CCR management has often been based on the calculation of credit exposure profiles for counterparties. Several metrics exist, these include amongst others Expected Exposure (EE), Potential Future Exposure (PFE), Expected Positive Exposure (EPE), Effective Expected Positive Exposure (EEPE) etc. [47]. These risk metrics can be important when making decisions on which counterparties to trade with. An institution would usually set credit lines for counterparties as part of policy which would aid as a limit for the institution from doing trades with certain counterparties due to their current exposure profiles [21].

The above mentioned counterparty risk mitigation techniques are effective and the risk metrics provide much needed insight for correct decision making. However, there is one weakness inherent in them; they fail to quantify CCR. This weakness implies that they do not give a formal way of hedging the CCR. Standard pricing theory, assuming complete markets, and precluding arbitrage opportunities, informs us that the cost of hedging a contingent claim is the fair price of the contingent claim. Quantifying CCR is intuitively calculating that adjustment to a price of a derivative that assumes no CCR. In a complete market this adjustment should be enough to hedge the CCR inherent in the derivative, and is popularly known as a credit value adjustment (CVA)/expected loss (EL). There are many variants of CVA but in general there are two ways of approaching CVA:

- **Unilateral:** Only one counterparty is assumed to be default prone. Usually the party calculating the CVA assumes themselves to be risk free and assumes the other party to be default prone. The motivation for this could be that the credit qualities of the two counterparties differ significantly such that the default of the highly rated counterparty is expected to occur later than the default of the other counterparty if it does occur.
- **Bilateral:** When the valuation is being done, both counterparties assume themselves to be default prone. This is sometimes referred to as BCVA and has two components; one component is due to the counterparty and one due to the valuator's own credit quality. This is usually the approach when both

parties are of comparable credit quality. It could also be motivated by the fact that considering the credit quality of both counterparties comes with a benefit. This benefit is known as a DVA benefit and will be explained in Chapter 3.

While the motivation behind unilateral CVA may appear obvious, the motivation for Bilateral CVA is less obvious. Bilateral CVA has gained popularity over recent years but its first appearance dates back to 2005 in the paper by Cheburini [23]. Studies on CVA, and counterparty risk in general, that have focused on the South African market include the work done by Milwidsky [62] and Le Roux [59]. The motivation behind bilateral CVA is, as the recent financial crisis illustrated that respected banks and financial institutions can default with positive probability. Notable examples were Lehman Brothers, Fennie Mae, Freddie Mac, Washington Mutual, Landsbanki, Glitnir and Kaupthing who all defaulted in the same month [12]. It is thus valid for any financial institution, regardless of its current credit rating, to also consider its own credit risk.

The bilateral nature of CCR appears when dealing with instruments such as IRSs where the instrument can be a liability to both parties, not necessarily at the same time. In the obvious case of bonds, the counterparty risk is unilateral since the borrower is the only party who remains the one with an obligation to pay the coupons and the principal up until maturity.

The dissertation proceeds as follows: in Chapter 2 we give some of the results and definitions that explain certain concepts used in the dissertation. Chapter 3 is a review of CCR management and mitigating techniques. In the same chapter several approaches to quantify CCR in the form of CVA are introduced namely, calculation from first principles, exposure profiles approach and the portfolio or value decomposition approach. In Chapter 4, short rate models, in particular the CIR and CIR++ models are introduced and explored with the aim of using CIR++ as a model for the short rate and intensity of a default prone entity. In Chapter 5, two credit risk modelling frameworks are reviewed, namely the intensity and the less popular structural framework. In the same chapter, descriptions of how calibration to credit default swaps (CDSs) can be achieved under both frameworks are also presented. In Chapter 6, the CVA analysis is focused on forward rate agreements (FRAs) and interest rate swaps (IRSs). The analysis results in analytic and semi-analytic approximations of CVA and BCVA being derived and presented when the underlying contracts are FRAs and IRSs. In Chapter 7, algorithms for simulating default times under the two credit risk modelling frameworks are introduced and presented along with Monte Carlo algorithms for valuing the semi-analytic expressions presented in Chapter 6. Furthermore, two studies involving South African multinationals are presented, the first study presents results showing how UCVA and BCVA behave

for increasing tenors of both FRAs and IRSs and the second study shows how correlation impacts UCVA and BCVA under both credit risk modelling frameworks. The Gaussian and Student-t copula are used to induce correlation in the first study. Chapter 8 concludes the work and points to future work.

Chapter 2

Mathematical Preliminaries

This chapter is a summary of the results needed for this dissertation. It is the goal of this chapter to provide an introduction and a reminder to advanced readers, of the results that will be used in this dissertation. It is however a fair recommendation that advanced readers can proceed to the next chapter without loss of understanding. Good introductory references are the two volumes by Shreve [74, 75] and for advanced readers interested in all technical details, Jacod and Shiryaev is recommended [54, 5]. Furthermore, basic statistics knowledge is assumed.

Definition 2.1. Let Ω be a nonempty set, and let $\mathcal{F}$ be a σ -algebra of subsets of Ω . A *probability measure* $\mathbb{P}$ is a function that, to every set $A \in \mathcal{F}$, assigns a number in $[0,1]$, called the probability of A and written $\mathbb{P}\{A\}$. We require that:

- (i) $\mathbb{P}\{\Omega\} = 1$ and
- (ii) whenever $A_1, A_2, \dots$ is a sequence of disjoint sets in $\mathcal{F}$, then the probability of the union of disjoint events is the sum of the individual probabilities, i.e.,

$$\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \sum_{i=1}^{\infty} \mathbb{P}(A_n). \quad (2.1)$$

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space*.

A probability space is used to model the whole economy in which we will be pricing instruments. We will denote the set of all events possible in our economy by Ω and the function that assigns the probabilities to each event in Ω by $\mathbb{P}$.

Definition 2.2. A stochastic process, denoted by $X = (X_t), t \in I$, is a family of real-valued random variables $X_t : \Omega \rightarrow \mathbb{R}$, indexed by $t \in I$, where I is some index set. The two cases of most interest are when I is a subinterval of $\mathbb{N}$, in which case X is called a discrete-time *stochastic process*; and when I is a subinterval of $\mathbb{R}^+$, in which case X is called a continuous-time stochastic process.

We wish to model the values that are attained by stochastic processes in continuous time. It is also the case that in most mathematical finance applications, a finite time horizon is considered.

Definition 2.3. For a fixed sample point $\omega \in \Omega$, the map $t \rightarrow X_t(\omega)$, for $t \in \mathbb{R}^+$, is called a *sample path* or *trajectory* of a stochastic process X .

This is just the progression of values attained by the stochastic process as a function of time.

2.1 Information, Filtrations and Stopping Times

The flow of information is a fundamental idea because while available information can not in general inform us what the precise values of the fundamental variables we wish to model as stochastic processes are, it can tell us which are possible and which are not. This leads us to the notion of a filtration. Intuitively, this is a set of information that has been accumulated up to a point in time¹.

Definition 2.4. Let Ω be a nonempty set. Let T be a fixed positive number, and assume that for each $t \in [0, T]$ there is a σ -algebra $\mathcal{F}_t$. Assume further that if $s \leq t$, then every set in $\mathcal{F}_s$ is also in $\mathcal{F}_t$. Then we call the collection of σ -algebras $\mathcal{F}_t$, $0 \leq t \leq T$, a *filtration*.

This leads to the concept of a *filtered probability space*. Firstly we define a $\mathbb{P}$ -complete σ -algebra as follows,

Definition 2.5. A σ -algebra $\mathcal{F}$ is $\mathbb{P}$ -complete if for all $A \subseteq B$, with $B \in \mathcal{F}$ such that $\mathbb{P}(B) = 0$, implies that $A \in \mathcal{F}$.

Definition 2.6. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *filtered probability space* if it is endowed with a filtration. The combined structure is denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. For convenience we will denote $(\mathcal{F}_t)_{t \geq 0}$ by $\mathcal{F}_t$.

Definition 2.7. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called $\mathbb{P}$ -complete if all the σ -algebras $(\mathcal{F}_t)_{t \geq 0}$ and $\mathcal{F}_0$ contain all the $\mathbb{P}$ -null sets of $(\mathcal{F}_t)_{t \geq 0}$.

Definition 2.8. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to satisfy the *usual conditions* if it is complete and the filtration is right continuous; i.e.,

$$\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s > t} \mathcal{F}_s, \quad t \in \mathbb{R}^+. \quad (2.2)$$

¹This could be information up to the current point in time due to our inability to tell the future.

As a random variable takes certain values in time, it generates information, i.e., it generates a filtration and below we state what this means in terms of our probability space.

Definition 2.9. Let X be a random variable defined on a nonempty sample space Ω . The σ -algebra generated by X , denoted by $\sigma(X)$, is the collection of all subsets of Ω of the form $\{\omega \in \Omega; X(\omega) \in B\}$, where B are all the *Borel* subsets of $\mathbb{R}$.

At this point it is important to introduce several concepts that play a central role in financial modelling: *measurability*, *predictability* and *previsibility*.

Definition 2.10. Given a filtered probability space, $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and a stochastic process X_t , we say X_t is $\mathcal{F}_t$ -*measurable* if every set in the σ -algebra generated by X_t , $\sigma(X_t)$ is in $\mathcal{F}_t$.

Intuitively a random variable is measurable with respect to a given filtration if the information contained in that filtration is enough to determine its value.

Definition 2.11. Given a filtered probability space, $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and a stochastic process X_t , we say X_t is *previsible* if $\forall t, \exists s < t$ such that X_t is $\mathcal{F}_s$ -measurable.

Intuitively previsibility means that at a particular point in time t we can tell the value of the stochastic process at time $t + \Delta t$ precisely because of the information we have at time t .²

Definition 2.12. Given a filtered probability space, $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and a stochastic process X_t . X_t is *predictable* if it is measurable with respect to $\mathcal{F}_t$.

These definitions lead to an important concept that will be used to define the notion of a stochastic integral, this concept is what is known as adaptativity,

Definition 2.13. Let Ω be a nonempty sample space equipped with a filtration $\mathcal{F}_t$, $0 \leq t \leq T$. Let $X(t)$ be a collection of random variables indexed by $t \in [0, T]$. We say this collection of random variables is an *adapted stochastic process* if, for each t , the random variable X_t is $\mathcal{F}_t$ -measurable.

We have already mentioned in passing the idea of stopping times and below we formalize the important types which are predictable and totally inaccessible, this is done through a series of definitions. Firstly the notion of a random time needs to be defined as follows.

²As an example a company may announce that dividends will be paid or that there will be a sale of assets at some future date from the announcement date

Definition 2.14. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a *random time* is a universally measurable function on Ω with values in $[0, \infty]$.

Using this definition we define a stopping time.

Definition 2.15. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random time τ in this space is a *stopping time* if $\{\tau \leq t\} \in \mathcal{F}_t, \forall t \in \mathbb{R}^+$.

This means that only information up to time t is necessary to conclude on whether time τ has arrived or not. This time is referred to as an $\mathcal{F}$ -stopping time. We now define what a predictable stopping time is.

Definition 2.16. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then τ is a *predictable $\mathcal{F}$ -stopping time* if there exists a sequence of $\mathcal{F}$ -stopping times $\{\tau_1, \tau_2, \dots, \tau_n\}$ such that the sequence is increasing for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \tau_n = \tau$. The sequence is said to *announce* τ .

Stopping times associated with continuous processes are commonly predictable because of the continuity. To add some intuition to predictability of a stochastic process, it can be said that a stochastic process taking a particular value is predictable if prior to it obtaining that value there is a series of “*stopping times*” that announce that the process is going to take that value. This is associated with left-continuous processes³. The next concept to explore is that of accessibility of stopping times.

Definition 2.17. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then τ is an *accessible $\mathcal{F}$ -stopping time* if there exists a sequence of $\mathcal{F}$ -stopping times $\{\tau_0, \tau_1, \dots\}$ such that,

$$\mathbb{P} \left\{ \bigcup_{k=1}^{\infty} \{\omega \in \Omega : \tau_k(\omega) = \tau(\omega) < \infty\} \right\} = \mathbb{P} \{\tau < \infty\}. \quad (2.3)$$

The stopping time of interest to credit risk modelling is the one that is not accessible due to the fact that defaults are rare events and normally come as a jump or complete surprise. Below we define what a totally inaccessible stopping time is.

Definition 2.18. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then τ is a *totally inaccessible $\mathcal{F}$ -stopping time* if for every predictable stopping time ζ we have that,

$$\mathbb{P}\{\omega \in \Omega : \tau(\omega) = \zeta(\omega) < \infty\} = 0. \quad (2.4)$$

Intuitively the stopping time can never be announced by an increasing sequence of stopping times. The graph of a stopping time or path is defined as follows,

³It does not mean that prior to the random variable taking that value, we are able to tell that it will take it. It means that after attaining that value there is an infinitesimal time before that time which announced it. This concept becomes more crucial in the credit risk modelling part of the thesis.

Definition 2.19. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the *graph* of an $\mathcal{F}$ -stopping time τ is $\{(t, \omega) : 0 \leq t = \tau(\omega) < \infty\}$ and is denoted by $[\tau]$.

The following theorem states that any stopping time can be decomposed into a totally inaccessible and accessible stopping time.

Theorem 2.20 (Decomposition of stopping times). *For every stopping time τ , there exists one (up to $\mathbb{P}$ -negligibility) and only one pair (τ_p, τ_i) of stopping times with the properties,*

1. $[\tau] = [\tau_p] \cup [\tau_i]$ such that $[\tau_p] \cap [\tau_i] = \emptyset$,
2. τ_i is inaccessible and
3. There exist a sequence $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$, $n \in \mathbb{N}$, of predictable stopping times such that $[\tau_p] \subset \cup_{k=1}^n [\zeta_k]$.

Proof. For the proof we refer to Metivier [61, p.28]. □

Stopping times introduce the idea of a stopped process.

Definition 2.21. Given a process X_t and a finite stopping time τ , the *stopped process* at time τ , denoted by X_t^τ , is given by,

$$X_{\tau \wedge t} = \mathbb{I}_{t < \tau} X_t + \mathbb{I}_{t \geq \tau} X_\tau, \quad \forall t \in \mathbb{R}^+. \quad (2.5)$$

2.2 Martingales and Semi-martingales

One of the most important concepts in mathematical finance is the notion of martingales. They intuitively represent a fair game, in that they do not have a tendency to increase or decrease over time. We define them precisely below, but first we need to understand integrability of random variables,

Definition 2.22. Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the family of random variables $X : \Omega \rightarrow \overline{\mathbb{R}} = [-\infty, \infty]$, such that,

$$(\mathbb{E}[|X|^p])^{\frac{1}{p}} = \left(\int_{\Omega} |X|^p d\mathbb{P} \right)^{\frac{1}{p}} < \infty, \quad p \geq 1, \quad (2.6)$$

is denoted by $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.23. A random variable is *p-integrable* if it belongs to $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.24. A process X is called *uniformly integrable* if it satisfies,

$$\lim_{n \rightarrow \infty} \sup_{t \in \mathbb{R}^+} \int_{|X_t| \geq n} |X_t| d\mathbb{P} = 0. \quad (2.7)$$

Definition 2.25. A process X in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is a *martingale* if,

1. X_t is adapted to $\mathcal{F}$,
2. X_t is integrable, $\forall t \in \mathbb{R}^+$ and
3. $\mathbb{E}[X_t | \mathcal{F}_s] = X_s, \forall s \leq t \in \mathbb{R}^+$.

If in (3), we have that $\mathbb{E}[X_t | \mathcal{F}_s] \geq X_s$, then the process is called a *sub-martingale* and if we have that $\mathbb{E}[X_t | \mathcal{F}_s] \leq X_s$ then it is called a *super-martingale*.

Definition 2.25, (3) tells us that on average, a martingale is constant at any future time. This means that there is no tendency for the value to increase or decrease. This is a central idea behind mathematical finance in that it is linked to the principle of no arbitrage which will be discussed in the section on pricing. Intuitively a super-martingale is a process that drifts down on average and similarly a sub-martingale is a process that drifts up on average. A martingale is then a process that is both a sub and super martingale. All these belong to a class of processes called semi-martingales.

Another more general class of is that of local martingales and in general all martingales are local martingales but not all local martingales are martingales.

Definition 2.26. A process X_t in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is a local martingale, referred to as an *$\mathcal{F}$ -local martingale*, if there exist an increasing sequence of stopping times $\{\tau_1, \tau_2, \dots, \tau_n\}$ with $\lim_{n \rightarrow \infty} \tau_n = \infty$ such that,

1. X_t is adapted and right-continuous and
2. $\forall i \in [0, n]$ the stopped process $X_t^{\tau_i}$ is an $\mathcal{F}$ -martingale.

The sequence of stopping times is called a *localizing sequence or reducing sequence*.

Definition 2.27. A process X_t in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is a *semi-martingale* if it can be decomposed as follows,

$$X_t = M_t + A_t, \tag{2.8}$$

where M_t is a local martingale with $M_0 = 0$ and A_t is a process with locally finite variation.

It is also true that the decomposition is unique.

2.3 Stochastic Differential Equations

Stochastic differential equations are non-deterministic versions of ordinary differential equations. The deterministic nature is eliminated due to the introduction of a noise term. We focus on SDEs of the form,

$$dX_t = f(X_t, t)dt + g(X_t, t)dW_t, \text{ given } X_0 = K \in \mathbb{R}^+, \quad (2.9)$$

where W_t is our added noise.

The exciting history around this noise which is known as Brownian motion can be found in Hanggi and Marchesoni [49] and we define it next,

Definition 2.28. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function W_t of $t \geq 0$ that satisfies $W_0 = 0$ and that depends on ω . Then W_t , $t \geq 0$ is a *Brownian motion* if for all $0 = t_0 < t_1 < \dots < t_m$ the increments, $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \dots, W_{t_m} - W_{t_{m-1}}$ are independent and each of these increments is normally distributed with,

$$\mathbb{E}[W_{t_{i+1}} - W_{t_i}] = 0, \quad (2.10)$$

$$\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] = t_{i+1} - t_i. \quad (2.11)$$

There are many important results concerning Brownian motion but one that is obvious is the following:

Theorem 2.29. *Brownian motion is a martingale.*

The process described by (2.9), with some conditions on the functions $f(X_t, t)$ and $g(X_t, t)$, are known as Itô processes. To be more precise, we define them below.

Definition 2.30. If W_t is as defined in Definition (2.28) then an *Itô* process is a stochastic process of the form,

$$dX_t = f(X_t, t)dt + g(X_t, t)dW_t, \text{ given } X_0 = K \in \mathbb{R}^+, \quad (2.12)$$

which is shorthand for,

$$X_t = X_0 + \int_0^t g(X_u, u)dW_u + \int_0^t f(X_u, u)dt, \quad (2.13)$$

where X_0 is non-random and $g(X_u, u)$ and $f(X_u, u)$ are adapted stochastic processes.

These processes are the most popular for the modelling of financial variables and can be extended by incorporating jumps to form what is known as a jump diffusion process. The construction of equation (2.13) introduced an unusual integral, $\int_0^T g(X_u, u)dW_u$, which cannot be defined as a Riemann-Stieltjes integral. The

properties of Brownian motion imply that the paths are continuous almost surely but are nowhere differentiable. This is due to the unbounded first variation of Brownian motion which is a consequence of its fractal property. This then necessitates a new integration theory that is different from that of Riemann-Stieltjes. Firstly we define what a mesh and partition are.

Definition 2.31. Let $m \in \mathbb{N}$ and $a < b \in \mathbb{R}^+$. A *partition* of the interval $[a, b]$ is a finite ordered set $P_{a,b} := \{t_0, t_1, \dots, t_m\}$, such that $a = t_0 \leq t_1 \leq \dots \leq t_m = b$.

Definition 2.32. For a partition $P_{a,b}$, the quantity

$$\delta(P_{a,b}) := \sup_{0 \leq i < m} |t_{i+1} - t_i|, \quad (2.14)$$

is called the *mesh*.

In standard Riemann integration theory the central idea is the convergence of the left and right integral as the mesh of the partition tends to zero. However in the case of $\int_0^T g(X_u, u) dW_u$, this convergence does not happen in general and two integrals are defined to cater for this, one is referred to as the Itô integral while the other is called the Stratonovich integral.

Definition 2.33. The *Itô integral* of a square-integrable and previsible function $g(X_t, t)$ is given by,

$$\int_0^T g(X_u, u) dW_u = \lim_{\delta(P_{a,b}) \rightarrow 0} \sum_{j=0}^{N(P_{a,b})-1} g(X_{t_j}, t_j) (W_{t_{j+1}} - W_{t_j}), \quad (2.15)$$

where $N(P_{a,b})$ is the number of points in the partition $P_{a,b}$. The limit should be interpreted as a limit in probability.

Definition 2.34. The *Stratonovich integral* of a function $g(X_t, t)$ is given by,

$$\int_0^T g(X_u, u) dW_u = \lim_{\delta(P) \rightarrow 0} \sum_{j=0}^{N(P)-1} \left(g\left(X_{\frac{t_j + t_{j+1}}{2}}, \frac{t_j + t_{j+1}}{2}\right) \right) (W_{t_{j+1}} - W_{t_j}). \quad (2.16)$$

The Stratonovich integral has mostly found applications in physics and is appealing due to the fact that it obeys the usual rules of calculus. It is however less interesting to financial modelling problems because of its dependency on the next time step, this implies that we require future information to compute it. This would then allow for arbitrage opportunities to exist. The Itô integral however does not obey the usual rules of calculus and thus requires a redefinition of rules such as the chain rule, integration by parts and product rule. It is appealing due to that it does not require the knowledge of the value of the function at a future time value, which is also illustrative of its martingale property. Below we provide some useful results in stochastic calculus.

Theorem 2.35 (Two-dimensional Itô-Doeblin formula). *Let $f(t, x, y)$ be a function whose partial derivatives $f_t, f_x, f_y, f_{xx}, f_{xy}$ and f_{yy} are defined and are continuous. Let X_t and Y_t be Itô processes as discussed above. The two dimensional Itô-Doeblin formula in differential form is,*

$$\begin{aligned} df(t, X_t, Y_t) &= f_t(t, X_t, Y_t)dt + f_x(t, X_t, Y_t)dX_t + f_y(t, X_t, Y_t)dY(t) \\ &\quad + \frac{1}{2}f_{xx}(t, X_t, Y_t)dX_t dX_t + f_{xy}(t, X_t, Y_t)dX_t dY_t \\ &\quad + \frac{1}{2}f_{yy}(t, X_t, Y_t)dY_t dY_t. \end{aligned} \quad (2.17)$$

The theorem above gives us a mechanism of having functions of Itô processes, i.e., it gives a formal way of expressing the change in the value of a derivative in terms of the underlings, in this case X_t and Y_t , on condition that the underlying processes are described by Itô processes. The product rule can then be derived by setting $f(t, X_t, Y_t) = X_t Y_t$ which leads to,

Corollary 2.36 (Itô product rule). *Let X_t and Y_t be Itô process. Then*

$$d(X_t, Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t. \quad (2.18)$$

Below is a result that establishes a connection between stochastic and partial differential equations but firstly definitions are in order.

Definition 2.37. A function $f : (\mathbf{X}, T) \rightarrow \mathbf{Y}$ is called *Lipschitz continuous in x* with constant C if for each $x_1, x_2 \in \mathbf{X}$ one has

$$|f(x_1, t) - f(x_2, t)| \leq C|x_1 - x_2|. \quad (2.19)$$

Theorem 2.38 (Feynman-Kac). *Given $f(x, t)$ which is continuous and Lipschitz in x , $\sigma(x)$ and a smooth function $\phi(\cdot)$, the solution of the PDE*

$$\frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x}f(x, t) + \frac{1}{2}\frac{\partial^2 V(t, x)}{\partial x^2}\sigma^2(x) = rV(t, x), \quad (2.20)$$

with terminal boundary condition

$$V(T, x) = \phi(x), \quad (2.21)$$

can be expressed as the following expected value

$$V(t, x) = e^{-r(T-t)}\mathbb{E}^{\tilde{\mathbb{P}}}[\phi(X_T) \mid X_t = x], \quad (2.22)$$

where the diffusion process X has dynamics, starting from x at time t , given by,

$$dX_t = f(X_t)dt + \sigma(X_t)dW_t, \quad (2.23)$$

where W_t is a Brownian motion under $\tilde{\mathbb{P}}$.

2.4 Arbitrage Free Pricing of Contingent Claims

This section is adapted from the text by Brigo and Mercurio [15]. We will be modelling certain financial variables in the economy, it is thus necessary that we define a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions. In this economy there are $K + 1$ non-dividend paying securities which are traded continuously over a finite time horizon, time 0 to T . Their prices are modelled by a $K + 1$ dimensional adapted semi-martingale $S = \{S_t \mid 0 \leq t \leq T\}$, whose components $S^0, S^1, \dots, S^K$ are positive. The asset indexed by 0 is a bank account. Its price then evolves according to

$$dS_t^0 = r_t S_t^0 dt, \quad (2.24)$$

where $S_0^0 = 1$ and r_t is the instantaneous short-term rate at time t . The discount factor is defined through the relationship,

$$D(0, t) = \frac{1}{S_t^0}. \quad (2.25)$$

In this economy to generate wealth, one should have a trading strategy. Intuitively, this means that one requires a way of holding the securities. We make this idea concrete as follows,

Definition 2.39. A *trading strategy* is a $K + 1$ -dimensional process $\bar{\phi} = \{\phi_t \mid 0 \leq t \leq T\}$, with components $\phi^0, \phi^1, \dots, \phi^K$ that are locally bounded and predictable. The *value process* associated with a strategy $\bar{\phi}$ is defined by

$$V_t(\bar{\phi}) = \phi_t S_t = \sum_{k=0}^K \phi_t^k S_t^k, \quad 0 \leq t \leq T. \quad (2.26)$$

Definition 2.40. The *gains process* G_t associated with a strategy $\bar{\phi}$ is defined by

$$G_t(\bar{\phi}) = \int_0^t \bar{\phi}_u dS_u = \sum_{k=0}^K \int_0^t \phi_u^k dS_u^k, \quad 0 \leq t \leq T. \quad (2.27)$$

Each ϕ_t^k represents the number of units of security S^k that is held at time t . The process V_t is the market value of the portfolio realized by adopting strategy $\bar{\phi}$ and the process $G_t(\bar{\phi})$ is the cumulative gains generated through the adoption of strategy $\bar{\phi}$.

Definition 2.41. A trading strategy $\bar{\phi}$ is *self-financing* if $V_t(\bar{\phi}) \geq 0$ and

$$V_t(\bar{\phi}) = V_0(\bar{\phi}) + G_t(\bar{\phi}), \quad 0 \leq t < T. \quad (2.28)$$

Such a strategy forces the changes in value of the portfolio to be only due to the changes in asset prices. This system is closed in that there is nothing of monetary value entering or leaving the system. This leads to the following proposition.

Proposition 2.42. *Let $\bar{\phi}$ be a trading strategy. Then, $\bar{\phi}$ is self-financing if and only if*

$$D(0, t)V_t(\bar{\phi}) = V_0(\bar{\phi}) + \sum_{k=0}^K \int_0^t \phi_u^k d(D(0, u)S_u). \quad (2.29)$$

The idea of no arbitrage has been shown to be equivalent to the existence of an equivalent martingale measure. Under this measure all participants in the economy are indifferent and view risk in the same way. Below we define what an equivalent martingale measure is.

Definition 2.43. An *equivalent martingale measure* $\tilde{\mathbb{P}}$ is a probability measure on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that,

1. $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent measures, that is $\mathbb{P}\{A\} = 0$ iff $\tilde{\mathbb{P}}\{A\} = 0$, for every $A \in \mathcal{F}$,
2. the Radon-Nikodym derivative $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ belongs to $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, i.e., it is square integrable with respect to $\mathbb{P}$ and
3. the discounted price process $D(0, \cdot)S$ is an $\tilde{\mathbb{P}}$ -martingale, that is,

$$\mathbb{E}^{\tilde{\mathbb{P}}} [D(0, t)S_t^k \mid \mathcal{F}_u] = D(0, u)S_u^k, \quad (2.30)$$

for all $k = 0, 1, \dots, K$ and all $0 \leq u \leq t \leq T$.

Definition 2.44. An *arbitrage opportunity* is a self-financing strategy $\bar{\phi}$ such that the value process $V_t(\bar{\phi})$ satisfies $V_0(\bar{\phi}) = 0$ and for some time $T > 0$

$$\mathbb{P}\{V_T(\bar{\phi}) \geq 0\} = 1 \text{ and also } \mathbb{P}\{V_T(\bar{\phi}) > 0\} > 0.$$

The absence of arbitrage is what is often termed the “no free lunch” condition. The no arbitrage condition intuitively means that strategies that offer an opportunity to make profit should come at a risk, thus precluding insider trading and other unfair strategies from being admissible trading strategies. The term ‘contingent claim’ has been previously used to refer to derivatives but certain conditions should be satisfied for a contract to be called a contingent claim. We define precisely what this means.

Definition 2.45. A *contingent claim* is a square-integrable and positive random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., it should be in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$.

Remark 1. It is attainable if there exists some trading strategy $\bar{\phi}$ such that $V_T(\bar{\phi}) = H$ (H is the payoff function). Such a $\bar{\phi}$ is said to generate H , and $\pi_t = V_t(\bar{\phi})$ is the price at time t associated with H .

The pricing of contingent claims using the no-arbitrage condition was initiated by Black and Scholes [36], their ideas were then extended and generalized by Harrison and Kreps [48] where they proved the proposition that follows.

Proposition 2.46. *Assume there exist an equivalent martingale measure $\tilde{\mathbb{P}}$ and let H be an attainable contingent claim. Then, for each time t , $0 \leq t \leq T$, there exist a unique price π_t associated with H , i.e.,*

$$\pi_t = \mathbb{E}^{\tilde{\mathbb{P}}} [D(t, T)H \mid \mathcal{F}_t]. \quad (2.31)$$

To summarize if a contingent claim is attainable then its price is the expected value of its cash-flows under the equivalent martingale measure. Below we define a complete market.

Definition 2.47. A *complete financial market* is one in which every contingent claim is attainable.

Another important result is that a market is only complete if and only if the equivalent martingale measure is unique, and this also implies that the price of an attainable contingent claim is unique in a complete market. More realistic markets are incomplete especially when jump-processes are considered or stochastic volatility and interest rates are assumed. In summary we have that:

- The market is free of arbitrage iff there exist a martingale measure,
- the market is complete iff the martingale measure is unique and
- in an arbitrage-free market, not necessarily complete, the price of any attainable claim is given, either by the value of the associated replicating strategy, or by the risk neutral expectation of the discounted claim payoff under any of the equivalent (risk-neutral) martingale measure.

The following is one of the most important theorems in mathematical finance, it gives a mechanism for changing the drift of a stochastic differential equation under different measures. It also shows that if you change the measure, only the drift changes and not the volatility.

Theorem 2.48 (The Girsanov theorem). *Consider the stochastic differential equation,*

$$dX_t = f(X_t)dt + \sigma(X_t)dW_t, \quad X_0 \in \mathbb{R}^+, \quad (2.32)$$

under $\mathbb{P}$. Let a new drift $\tilde{f}(x)$ be given and assume $\frac{(\tilde{f}(x)-f(x))}{\sigma(x)}$ to be bounded. Define $\tilde{\mathbb{P}}$ by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp \left\{ -\frac{1}{2} \int_0^t \left(\frac{\tilde{f}(X_s) - f(X_s)}{\sigma(X_s)} \right)^2 ds + \int_0^t \frac{\tilde{f}(X_s) - f(X_s)}{\sigma(X_s)} dW_s \right\}. \quad (2.33)$$

Then $\tilde{\mathbb{P}}$ is a probability measure equivalent to $\mathbb{P}$. Moreover, the process $\tilde{W}$ defined by

$$d\tilde{W}_t = - \left[\frac{\tilde{f}(X_t) - f(X_t)}{\sigma(X_t)} \right] dt + dW_t \quad (2.34)$$

is a Brownian motion under $\tilde{\mathbb{P}}$, and

$$dX_t = \tilde{f}(X_t)dt + \sigma(X_t)d\tilde{W}_t, \quad X_0 \in \mathbb{R}^+. \quad (2.35)$$

Proof. For the proof see, Brigo and Mercurio [18, p.935] $\square$

2.5 Poisson Processes

Consider the example of data packets in a computer network. Suppose that the first data packet arrives at τ_1 and after the first data packet, it takes τ_2 time for the second packet to arrive. In general we have that τ_{i+1} is the time taken by the data packet $(i+1)$ to arrive after the arrival of packet i . These times are intra-arrival times and are assumed exponentially distributed⁴. Define a random variable S_n such that

$$S_n = \sum_{k=1}^n \tau_k. \quad (2.36)$$

This is the total time it takes for the n^{th} data packet to arrive at the destination in the network. The *Poisson process* is then defined by the random variable $N(t)$, given by,

$$N(t) = \sum_{k=1}^{\infty} \mathbb{I}_{\{S_{k-1} \leq t < S_k\}}. \quad (2.37)$$

Below are the building results for the Poisson process and for the proofs we refer to [75].

Lemma 2.49. *For $n \geq 1$, the random variable S_n , defined by (2.37), has the gamma density given by,*

$$g_n(s) = \frac{(\lambda s)^{n-1}}{(n-1)!} \lambda e^{-\lambda s}, \quad s \geq 0. \quad (2.38)$$

Lemma 2.50 (Distribution of the Homogeneous Poisson Process). *The Poisson process $N(t)$ with a constant intensity λ has the distribution*

$$\mathbb{P}\{N(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \quad k = 0, 1, \dots \quad (2.39)$$

It is called homogeneous.

⁴For more on exponential random variables see Shreve [75].

The process can be made more general by considering an intensity that is time dependent.

Lemma 2.51 (Distribution of the Non-homogeneous Poisson Process). *The Poisson process $N(t)$ with a time dependent intensity λ_t has the distribution*

$$\mathbb{P}\{N(t) = k\} = \frac{(\int_0^t \lambda_s ds)^k}{k!} e^{-\int_0^t \lambda_s ds} \quad k = 0, 1, \dots \quad (2.40)$$

It is called non-homogeneous.

2.6 Copulas

A copula is intuitively a function that joins or couples multivariate distribution functions to their one-dimensional marginal distribution functions. It is a multivariate distribution whose marginal distributions are uniform on the interval (0,1). Historically the interest in copulas from a statistical point of view was due to the fact that they offered a way of studying scale-free measures of dependence and they also offered a starting point for constructing families of bivariate distributions [64]. Much interest on copulas currently has been driven by the tremendous growth of the credit derivatives market that offer multi-name credit derivatives⁵. Copulas provide a structure for constructing multivariate distributions for the default times of all the names.

To mathematically define and study some properties of copulas, it is necessary that some definitions are stated. To avoid multiple citations, the work presented has been largely taken from [64]. We consider the multidimensional setting because of its generality.

Definition 2.52. Let $\bar{\mathbb{R}} = [-\infty, \infty]$. Let $S_1, S_2, \dots, S_n$ be non-empty subsets of $\bar{\mathbb{R}}$ and let $\mathcal{H}$ be an n -place real function such that its domain,

$$Dom(\mathcal{H}) = S_1 \times S_2 \times \dots \times S_n.$$

Let $B = [\mathbf{a}, \mathbf{b}]$ be an n -box with all vertices in $Dom(\mathcal{H})$. Then the $\mathcal{H}$ -volume of B is given by

$$V_{\mathcal{H}}(B) = \sum \text{sgn}(c) \mathcal{H}(c), \quad (2.41)$$

where the sum is taken over all vertices $c \in B$, and

$$\text{sgn}(c) = \begin{cases} 1 & \text{if } c_k = a_k \text{ for an even number of } k\text{'s,} \\ -1 & \text{if } c_k = a_k \text{ for an odd number of } k\text{'s.} \end{cases}$$

⁵A multi-name credit derivative is a derivative referencing the credit risk of many companies/names.

Definition 2.53. An n -place real function $\mathcal{H}$ with its domain, $\text{Dom}(\mathcal{H})$ is n -increasing if $V_{\mathcal{H}}(B) \geq 0$ for all n -boxes B , the vertices of which lie in $\text{Dom}(\mathcal{H})$.

Definition 2.54. Given n -place real function $\mathcal{H}$ with $\text{Dom}(\mathcal{H})$ as in the previous definition, where each S_k has a least element a_k , we say that $\mathcal{H}$ is *grounded* if $\mathcal{H}(\mathbf{t}) = 0$, for all $\mathbf{t} \in \text{Dom}(\mathcal{H})$ such that $t_k = a_k$.

Remark 2. If each S_k has a greatest element b_k , then $\mathcal{H}$ has margins and the one dimensional margins of $\mathcal{H}$ are the functions H_k given by $\text{Dom}(H_k) = S_k$ and $H_k = \mathcal{H}(b_0, \dots, b_{k-1}, x, b_{k+1}, \dots, b_n)$, for all $x \in S_k$.

Now that the definitions are in place we are able to define precisely what a sub-copula is, which will lead to the definition of a copula.

Definition 2.55. An n -dimensional *subcopula* is a function C' with the following properties:

1. $\text{Dom}(C') = S_1 \times S_2 \times \dots \times S_n$, with each S_k being a subset of $\mathbb{I}$ where $\mathbb{I} = [0, 1]$,
2. C' is grounded and n -increasing and
3. C' has one-dimensional margins C'_k , where $k = 1, 2, \dots, n$ and satisfy $C'_k(u) = u$, for all $u \in S_k$.

To make a transition from a subcopula to a copula, a change in the domain is necessary.

Definition 2.56. An n -dimensional *copula* is an n -subcopula C with domain equal to $\mathbb{I}^n$ and satisfying the following properties:

1. For all $\mathbf{u} \in \mathbb{I}^n$, $C(\mathbf{u}) = 0$ if at least one co-ordinate of $\mathbf{u}$ is 0 and if all co-ordinates of $\mathbf{u}$ are 1 except u_k , then $C(u) = u_k$.
2. For all $\mathbf{a}, \mathbf{b} \in \mathbb{I}^n$ such that $\mathbf{a} \leq \mathbf{b}$, $V_C([\mathbf{a}, \mathbf{b}]) \geq 0$ where $[\mathbf{a}, \mathbf{b}]$ is the n -box $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$.

For the purposes of constructing multivariate distribution functions, the most influential theorem by Sklar, which has been named after him, connects the marginal distributions of a multivariate distribution to its corresponding multivariate distribution. To precisely state this theorem one requires a precise definition of an n -dimensional distribution.

Definition 2.57. An n -dimensional *distribution* is a function $\mathcal{H}$ with domain $\bar{\mathbb{R}}^n$ such that,

1. $\mathcal{H}$ is n -increasing and
2. $\mathcal{H} = 0$, for all $t \in \bar{\mathbb{R}}^n$ such that $t_k = -\infty$ for some k and $\mathcal{H}(\infty, \infty, \dots, \infty) = 1$.

Using the above definition, we state Sklar's theorem.

Theorem 2.58 (Sklar's Theorem in n -Dimensions). *Let $\mathcal{H}$ be an n -dimensional distribution function with margins $F_1, F_2, \dots, F_n$. Then there exists an n -copula C such that for all $x \in \mathbb{R}^n$,*

$$\mathcal{H}(x_1, x_2, \dots, x_n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)).$$

If, in addition, $F_1, F_2, \dots, F_n$ are all continuous, then C is unique, otherwise C is an n -copula on $\text{Ran}(F_1) \times \text{Ran}(F_2) \times \dots \times \text{Ran}(F_n)$. Conversely, if C is an n -copula and $F_1, F_2, \dots, F_n$ are distribution functions then $\mathcal{H}$ is an n -dimensional distribution function with margins $F_1, F_2, \dots, F_n$.

The above theorem leads to a prescription on how copulas can be created. The following corollary shows this precisely.

Corollary 2.59. *Let $\mathcal{H}, C, F_1, F_2, \dots, F_n$ be defined as in the above theorem and let $F_1^{-1}, F_2^{-1}, \dots, F_n^{-1}$ be the corresponding quasi inverses of the marginal distributions respectively. Then, for any $\mathbf{u} \in \mathbb{I}^n$,*

$$C(u_1, u_2, \dots, u_n) = \mathcal{H}(F_1^{-1}(u_1), F_2^{-1}(u_2), \dots, F_n^{-1}(u_n)). \quad (2.42)$$

Examples of Copulas

A non-exhaustive list of copulas is presented.

- The minimum and maximum⁶ copulas are defined as

$$M(u, v) = \min(u, v), \quad (2.43)$$

$$W(u, v) = \max(u - v - 1, 0) \quad (2.44)$$

respectively. It can also be shown that for any copula $C(u, v)$, $W(u, v) \leq C(u, v) \leq M(u, v)$.

- The product copula is defined as

$$C(u, v) = uv. \quad (2.45)$$

- The multivariate Gaussian copula is defined as

$$C_{\Sigma}^{Ga} = \mathcal{N}[F^{-1}(u_1), \dots, F^{-1}(u_n)]. \quad (2.46)$$

⁶These are also known as the Fréchet-Hoeffding upper and lower bounds respectively.

2.6.1 Dependency Measures

The ultimate goal of introducing copulas is to be able to induce dependencies; it is then natural to ask how such dependencies are measured. For more on these we refer to [53, 64]. The most popular of these measures is the Pearson coefficient,

$$\rho = \text{Corr}(X_1, X_2) \quad (2.47)$$

$$= \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}}, \quad (2.48)$$

where X_1, X_2 are random variables. For this measure to be meaningful, the random variables must be from an elliptical distribution. It takes values in $[-1, 1]$, $\rho = 0$ corresponds to independence and the copula becomes the product copula⁷, $\rho = -1$ corresponds to negative correlation and $\rho = 1$ corresponds to positive correlation. The other important measure which is based on data ranks is known as Spearman's rho and is often denoted as ρ_S . It is defined as follows,

$$\rho_S = \text{Corr}(F_1(X_1), F_2(X_2)), \quad (2.49)$$

where $F_1(\cdot)$ and $F_2(\cdot)$ are marginal distribution functions. The correlation matrix created from *Spearman's rho* coefficients has its entries as follows,

$$\rho_S(X)_{ij} = \text{Corr}(F_i(X_i), F_j(X_j)). \quad (2.50)$$

This measure is defined for all copulas including the elliptical family. If the copula (C) that describes the random variables X_1 and X_2 is unique, the measure for the bivariate case is given as follows,

$$\rho_S(X_1, X_2) = 12 \int_0^1 \int_0^1 (C(u_1, u_2) - u_1 u_2) du_1 du_2. \quad (2.51)$$

Another measure that is of interest when dealing with copulas, particularly the non-elliptical ones is known as *Kendall's tau* and will be denoted by ρ_τ . It also measures rank correlation and is defined for two sets of random variables, (X_1, X_2) and (X'_1, X'_2) , that have the same joint distribution as follows,

$$\rho_\tau(X_1, X_2) = \mathbb{P}[(X_1 - X'_1) \cdot (X_2 - X'_2) > 0] - \mathbb{P}[(X_1 - X'_1) \cdot (X_2 - X'_2) < 0]. \quad (2.52)$$

Added to this, given the unique copula C describing the dependence between the random variables, the measure is given by,

$$\rho_\tau(X_1, X_2) = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1. \quad (2.53)$$

⁷The product copula is a classical copula and is also known as the independence copula.

Other measures of interest are the so-called tail dependency measures. They intuitively measure the dependency on the tails of the copula, that is, on the upper or lower tail. They give an idea of the dependency of the marginal distributions on the tails, i.e., how do the marginal distributions depend on each other. Given the uniform random variables U_1 and U_2 , if there is upper tail dependency, then large values of U_1 indicate corresponding large values of U_2 in the tails and similarly for lower tail dependency. Given the unique copula C the upper and the lower tail dependencies are given as,

$$\lambda_u = 2 + \lim_{s \nearrow 0} \frac{C(1-s, 1-s) - 1}{s}$$

$$\lambda_l = \lim_{s \searrow 0} \frac{C(s, s)}{s}$$

respectively.

2.6.2 Elliptical Family

There are many functions that satisfy Definition 2.56 above, and thus suffice to be called *copula functions*. In financial applications the most important and readily suitable class of copulas is the elliptical family of copulas. The *Gaussian* and the *Student-t* copula belong to this class. These two are presented below.

The Gaussian Copula

The Gaussian copula couples the margins into a multivariate normal distribution. Like any other copula the margins need not be normal. The multivariate Gaussian copula is denoted by C^{Ga} and is given as follows,

$$C_{\Sigma}^{Ga} = \mathcal{N}[F^{-1}(u_1), \dots, F^{-1}(u_n)] \quad (2.54)$$

and when the margins are normal, then we have that

$$C_{\Sigma}^{Ga} = \mathcal{N}[\mathcal{N}^{-1}(u_1), \dots, \mathcal{N}^{-1}(u_n)]. \quad (2.55)$$

These are from the direct application of Corollary 2.59 and $\mathcal{N}[\cdot]$ is the normal multivariate distribution function, $\mathcal{N}^{-1}(\cdot)$ is the inverse of the univariate normal distribution and Σ is the correlation matrix. The correlation matrix that is popular for the Gaussian Copula is one with Pearson coefficients as its entries. The other measures are related to it through,

$$\rho_S(X_1, X_2) = \frac{6}{\pi} \arcsin\left(\frac{\rho}{2}\right), \text{ and } \rho_{\tau}(X_1, X_2) = \frac{2}{\pi} \arcsin \rho. \quad (2.56)$$

The measures ρ_S and ρ_{τ} could also be used as the entries in the correlation matrix.

For most applications the bivariate case suffices, since it can be used to build high dimensional copulas. It is given by,

$$C_{\Sigma}^{Ga}(u, v) = \int_{-\infty}^{\mathcal{N}^{-1}(u)} \int_{-\infty}^{\mathcal{N}^{-1}(v)} \frac{1}{2\pi(1 - \Sigma_{12}^2)^{\frac{1}{2}}} \exp\left(-\frac{s^2 - 2\Sigma_{12}st + t^2}{2(1 - \Sigma_{12}^2)}\right) ds dt. \quad (2.57)$$

The corresponding density of the Gaussian copula is

$$c(\mathbf{x}) = \frac{1}{\Sigma^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{u}'(\Sigma - \mathbf{I})\mathbf{u}\right), \quad (2.58)$$

where $\mathbf{u} = (u_1, \dots, u_n)$ with each $u_i = \frac{(x_i - \mu_i)}{\sigma_i}$. The μ_i and σ_i are the mean and standard deviation of the marginal distribution F_i , see, [79].

It is common practice to assume that returns on financial securities are normally distributed, however, stylized facts on financial returns suggest that the tails of the distribution are much fatter than those in the normal distribution. The normality of returns assumption has led to the popularity of the Gaussian copula. However, the Gaussian copula suffers a major drawback: it lacks tail dependence, that is, $(\lambda_u = \lambda_l = 0)$. The reason this is a drawback is that in credit risk modelling default events are very rare and so lie in the tails of the distribution. The absence of tail dependence implies that the probability of joint/simultaneous defaults is almost surely zero. While any correlation matrix could be used, the usual correlation coefficients used are Pearson coefficients which are not invariant under monotone transformations [79]. Frey, McNeit and Nyfeler [38] arrive at the conclusion that the Gaussian copula is not adequate for the modelling of extreme events such as defaults and point out that heavy tailed distributions such as the t-student distributions offer a viable alternative. The t-copula is discussed below.

The Student-t Copula

The Student-t distribution is a special case of a more general distribution called the *generalized hyperbolic distribution*. This distribution has found use in many applications through being flexible enough to introduce fat tails, unlike distributions such as the Gaussian distribution. In credit risk modelling as already mentioned above, the tails are of more interest due to the nature of defaults. A random vector, $\mathbf{X}$ is distributed multivariate-t with mean vector μ and degrees of freedom v , denoted $\mathbf{X} \sim t(v, \mu, \mathbf{P})$, if

$$\mathbf{X} = \mu + \sqrt{L}\mathbf{Z}, \quad (2.59)$$

such that $\mathbf{Z} \sim \mathcal{N}(0, \Sigma)$ and L is independent of $\mathbf{Z}$ and satisfies $\frac{v}{L} \sim \mathcal{X}_v^2$, with $\mathcal{X}_v^2$ the chi-squared distribution with v degrees of freedom. The matrix $\mathbf{P}$ has its off-diagonal entries given by $\frac{\Sigma_{ij}}{\Sigma_{ii}\Sigma_{jj}}$.

Using Corollary 2.59 the unique copula for the multidimensional-t distribution with v degrees of freedom is given by,

$$C_{v,\mathbf{P}}^t(\mathbf{u}) = \int_{-\infty}^{t_v^{-1}(u_1)} \cdots \int_{-\infty}^{t_v^{-1}(u_n)} \frac{\Gamma(\frac{v+n}{2})}{\Gamma(\frac{v}{2})\sqrt{(\pi v)^n |\mathbf{P}|}} \left(1 + \frac{\mathbf{x}'\mathbf{P}^{-1}\mathbf{x}}{v}\right)^{-\frac{v+n}{2}} d\mathbf{x}, \quad (2.60)$$

where t_v^{-1} is the quasi inverse of the uni-variate t-distribution. The building block for high dimensional copulas is the bivariate copula, and the bivariate t-copula is given by,

$$C_{v,\rho}^t(u, v) = \int_{-\infty}^{t_v^{-1}(u_1)} \int_{-\infty}^{t_v^{-1}(v)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \left(1 + \frac{s^2 - 2\rho st + t^2}{v(1-\rho^2)}\right)^{-\frac{(v+2)}{2}} ds dt. \quad (2.61)$$

The most important property of the t-copula is the tail dependence it offers which is not there in the Gaussian copula. The student-t distribution is a symmetric distribution and this property leads to the lower and upper tail dependency measures being equal. They are given as follows for the bivariate case,

$$\lambda_u = \lambda_l = 2t_{v+1} \left(\frac{-\sqrt{v+1}\sqrt{1-\rho}}{\sqrt{1+\rho}} \right). \quad (2.62)$$

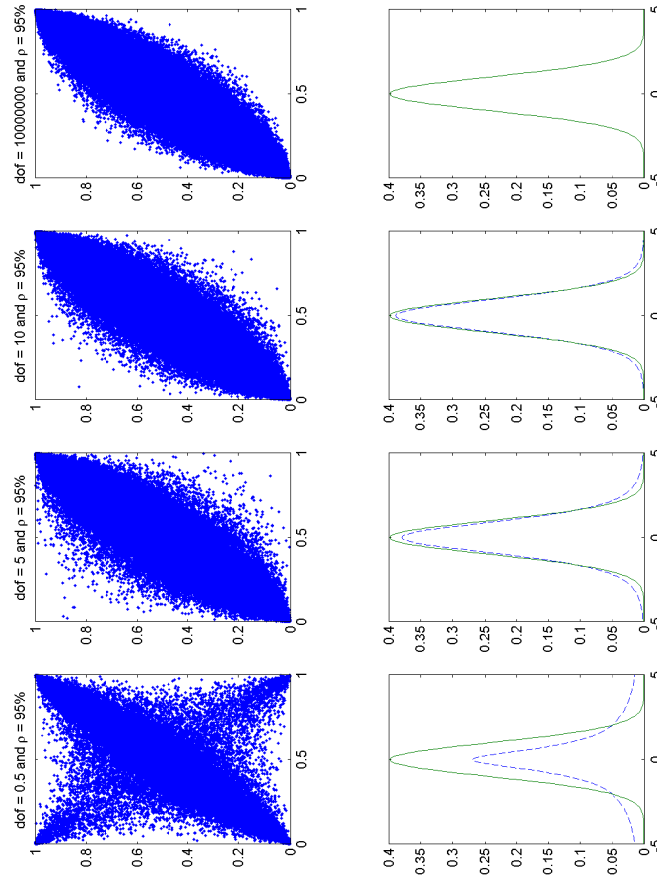


Figure 2.1: Shows varying 95% correlated random uniforms sampled from the Student-t copula with varying degrees of freedom. As the degrees of freedom tend to infinity, in the last graph its 10 000 000, the Student-t copula turns to the normal copula, which is also confirmed by the corresponding pdfs of the graphs shown underneath. The dashed graph is that corresponding to the Student-t pdf, while the full line is a standard normal pdf.

Chapter 3

Counterparty Credit Risk Management

The traditional approach to counterparty credit risk has been the calculation of exposure profiles which would indicate how bad the losses would be if a counterparty were to default. This is usually done under the real-world probability measure, that is, there is extensive use of historic data in trying to understand future scenarios. This has, however, changed to pricing counterparty credit risk under the risk neutral measure and using existing credit derivatives to actively hedge the risk. This chapter explores various approaches to counterparty credit risk management.

3.1 A Model for the Economy

The economy is modelled using a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. The filtration $(\mathcal{F}_t)_{t \geq 0}$ models the flow of information of the whole economy, including defaults. Defaults are characterized by default times which we denote by τ_i for a defaulting entity i and R_t^i for its recovery rate. An equivalent martingale measure $\tilde{\mathbb{P}}$ is assumed, under which the discounted price processes of all tradeable securities are martingales. The probability space has a right-continuous and complete sub-filtration $(\mathcal{G}_t)_{t \geq 0}$ representing all the observable market quantities excluding default events. Default events are contained in a right-continuous filtration generated by the default events, that is,

$$\mathcal{D}_t = \sigma(\{\tau_A \leq u\} \vee \{\tau_B \leq u\} \vee \dots \vee \{\tau_Z \leq u\} : u \leq t). \quad (3.1)$$

The relation between the filtrations is thus $\mathcal{G}_t \subseteq \mathcal{F}_t = \mathcal{G}_t \vee \mathcal{D}_t$ [18]. The symbol $\vee$ should be interpreted to mean that $\mathcal{F}_t$ is the smallest σ -algebra containing $\mathcal{G}_t$ and $\mathcal{D}_t$. The economy is further equipped with an instantaneous spot rate r_t , also known

as the short rate, such that the stochastic discount factor is given by,

$$D(t, T) = \exp \left(- \int_t^T r_s ds \right). \quad (3.2)$$

The discount bonds at t maturing T will be denoted by $B(t, T)$. The existence of the equivalent martingale measure rules out any arbitrage opportunities.

3.2 Counterparty Credit Risk Mitigation

CCR mitigation techniques are to a large degree well supported by the ISDA master agreement. Under the ISDA master agreement, legally binding clauses exist, most of which were lobbied for by the ISDA itself to form part of legislation regulating the financial industry. The document also specifies procedures to be followed on occurrence of a default event involving one of the counterparties. The ISDA master agreement rigorously defines what constitutes a default event thus alleviating the ambiguity that often characterizes legal contracts. A default event is described as one of the following: bankruptcy, failure to make due payments on loans or bonds, repudiation or moratorium, cross-acceleration, obligation default, distressed structuring and credit event upon mergers.

Two mechanisms that aid in mitigating CCR that are also supported by the ISDA master agreement are netting and close-out netting. Netting allows for the offsetting of exposures between counterparties. Contracts that are netted should be under the same master agreement. The risk mitigation introduced by netting is obvious and illustrates the benefits associated with dealing with the same counterparties; however this benefit needs to be adequately managed as it might lead to poorly diversified portfolios of derivatives. Netting generally gives preference to derivatives dealers over other creditors of the same counterparty since the netting implies that they receive liquidation proceeds before other creditors do. Bliss and Kaufman [7], however, argue that the reason often given in legislative documents for allowing netting, that is, derivatives markets are more exposed to systemic risk, is not enough on its own to justify its existence. They note that there is no clear evidence of reduced systemic risk due to netting/close-out netting. The close-out clause allows one counterparty to unilaterally close out all positions at the same time under certain circumstances [7]. Duffie *et al* [29], point out that with two counterparties A and B, such that A always has a better credit rating than B, when netting across a swap portfolio, A seems to be at an advantage at the expense of B.

Netting on its own is only effective if there are to be a significant number of contracts entered into by both parties — this may be impractical for corporates not in the financial sector. This is due to the fact that corporates merely use the

contracts as hedging tools and may tend to hold the same positions in most of them, and they may also not hold a substantial number to make netting worthwhile. However, corporates could in turn activate the credit support annex (CSA) on the ISDA master agreement as a means of mitigating CCR through the posting of collateral. The CSA defines the conditions under which collateral must be posted [3, 1]. The most important terms contained in the CSA from a modelling perspective include,

- **Threshold:** The credit one party extends to the other counterparty [12, 41, 3]. It is that amount that is allowed to accumulate without the other counterparty having to post collateral, where it is used we will denote it by H_t . The counterparty is not required to pay off the threshold, only what accumulates after taking into consideration the *minimum transfer amount*.
- **Minimum Transfer Amounts :** This is the minimum amount that is allowed to accumulate beyond the threshold before collateral is posted. It is intended to allow reasonable price movements before collateral is posted and where it is used we will denote it by M_t .
- **Eligible Securities and Currencies :** This is self-explanatory but it has serious modelling consequences. Fujii *et al* [42], investigate the use of different currencies because during the financial crisis, for example, it was expensive to use dollars as collateral. This was widely due to the fact that most parties were buying more dollars as a reserve currency.

A way to appreciate the effects of collateral is to imagine a deal in the presence of a CSA with $H_t = M_t = 0$ and the margin call frequency being continuous. In this scenario all the counterparty risk is eliminated. This is not very unrealistic as margin call frequency is usually daily. According to the ISDA about 30% of all OTC derivatives were collateralized in 2003 while at the end of 2009 about 78% of all OTC trades were collateralized. This in part is due to the financial crisis, but it is expected that this upward trend will continue. There are two approaches to collateral posting, *unilateral and bidirectional*. In *unilateral* posting only one party is required to post collateral (usually the party of much inferior credit quality), while in *bidirectional* posting both parties are required to post collateral (usually where the credit qualities of both parties are similar). Bidirectional posting is often facilitated by a third party called a *central clearing counterparty*¹. The existence of a central clearing counterparty and its effects are investigated by Duffie *et al* [32], where they conclude that the use of multiple clearing counterparties may reduce

¹ This party is responsible for administering the process, that is, requesting, receiving and depositing collateral to the correct account.

netting benefits especially with specialized clearing counterparties, who only clear a particular derivative contract. Advantages that a central clearing counterparty may bring include multilateral netting².

The continued use of collateral has challenged traditional pricing theory that relies on the existence of a risk free rate that should be used to discount cash flows. The collateral posted, which is mainly in the form of cash in some of the developed currencies earns its own rate called the collateral rate, c_t . The industry standard has now changed to discounting all cash flows using the overnight swap curve. Piterbarg [65] derives a Black-Scholes partial differential equation variant that all derivatives under collateral agreements should satisfy. He further prices some basic derivatives like forwards under the framework. Fujii *et al* [39, 40, 41] constructs various curves for discounting cash flows that are collateralized and uncollateralized, with some in the presence of multiple currencies.

3.3 Counterparty Credit Risk Metrics

The management of counterparty risk has been practiced for years and risk desks use several metrics to gain insight into exposure to a counterparty. We only describe a few of these below while a full description is given by Gregory and Canabarro *et al* [47, 20]. In what follows we assume the total value of the portfolio to a particular counterparty at time t , denoted by V_t , is the sum of the individual instruments in the portfolio denoted by V_t^i where $i = 1, \dots, N$.

- **Counterparty Exposure:** This is the most basic measure. It is the maximum of zero and the market value of the portfolio attributed to the counterparty. It is the value that would be lost if the counterparty were to default with zero recovery, i.e.,

$$\text{CE}(t) = \max\{V_t, 0\}. \quad (3.3)$$

Indeed, this is an extreme measure since it is virtually impossible that there would be zero recovery. The above equation assumes the presence of a netting agreement. If there is a partial netting agreement covering only K instruments in place the equation becomes,

$$\text{CE}(t) = \max\left\{\sum_{i=1}^K V_t^i, 0\right\} + \sum_{i=K+1}^N \max\{V_t^i, 0\}. \quad (3.4)$$

²Multilateral netting is when contracts are netted between a number of counterparties. It may prove to be catastrophic when there is default contagion.

- **Expected Exposure (EE)**: This is a time dependent quantity that records, for all times, what the expected loss will be should the counterparty default. Intuitively, it is how much on average the counterparty will owe in the future which is what could be lost should the counterparty default. This metric is dependent on the distribution that is assumed when valuing the portfolio/trade,

$$\text{EE}(t) = \mathbb{E} \left[\max \left\{ \sum_{i=1}^N V_t^i, 0 \right\} \right] = \mathbb{E}[V_t^+]. \quad (3.5)$$

Again in the presence of a partial netting agreement, we have that

$$\text{EE}(t) = \mathbb{E} \left[\max \left\{ \sum_{i=1}^K V_t^i, 0 \right\} + \sum_{i=K+1}^N \max \{ V_t^i, 0 \} \right]. \quad (3.6)$$

The expectation denoted by $\mathbb{E}[\cdot]$ is taken under the measure $\mathbb{P}$ and not the equivalent measure $\tilde{\mathbb{P}}$.

- **Potential Future Exposure (PFE)**: The idea behind this metric is similar to the idea behind Value at Risk (VAR) in the sense that it answers the crucial question: “How much can we expect to lose from a particular counterparty at a certain confidence level?” Unlike VAR which looks at tail losses, the focus here is on the tail profits. The metric is given by,

$$\text{PFE}_\alpha(t) = \inf \{ x \mid \mathbb{P}(V_t \leq x) \geq \alpha \}, \quad (3.7)$$

where α is the given confidence level.

- **Expected Positive Exposure (EPE)**: This is the time weighted average of the Expected Exposure. It is continuously defined as follows,

$$\text{EPE}(t) = \frac{1}{t_e} \int_t^{t_e} \text{EE}(s) \, ds \approx \frac{1}{t_e} \sum_{i=0}^N \text{EE}(t_i)(t_i - t_{i-1}), \quad (3.8)$$

where t_e is the exposure horizon and $t < t_0 < t_1 < \dots < t_N = t_e$.

There are many more of these risk metrics that are in widespread use such as Effective Positive Exposure, Peak Exposure, Effective Expected Positive Exposure, etc, and they are used to make decisions on new and existing deals. For further information see [68]. The above metrics are enough to aid in the understanding of counterparty credit risk quantification and the subsequent hedging strategies.

3.4 Quantifying Counterparty Credit Risk

We illustrate the concept behind credit value adjustment with an example. Consider two entities A and B with credit spreads S^A , S^B respectively such that $S^A < S^B$, i.e., A is of a better credit quality than B. Now consider the following trading strategy: an entity C is long a forward contract with maturity T and A is short the same forward contract, and C shorts an identical forward from B, with B taking the long position. Under standard pricing theory these positions should offset each other completely, assuming no market friction. The reality of the trade is that if the underlying in the forward decreases below the forward price, C makes money from the trade with B and loses an identical amount from the trade with A. If at maturity C owes A implying B owes C, and B, due to its poor credit quality, defaults the trades fail to cancel out. This then means the deal must have been mispriced or equivalently C sold B an option to default but did not charge for it. The price of this default option was the cost of hedging the credit risk of B, which will later be referred to as the CVA. There are several approaches to calculating CVA. The option analogy is confirmed by several frameworks. Below we give the most general definition of CVA.

Definition 3.1. Let V_t be the value of a risk-free portfolio or trade and let $\widehat{V}_t$ be the value of a corresponding portfolio with the exception that it takes into account counterparty credit risk. The *credit value adjustment* (CVA) also known as expected loss (EL), is given by

$$\text{CVA} = V_t - \widehat{V}_t. \quad (3.9)$$

One of the complications regarding the calculation of CVA is the notion of wrong way risk. To illustrate wrong way risk, consider a situation where an entity A is long a put option in a deal with B on B's own stock. The danger with this deal is the fact that if the stock hits zero then B is in default and this is exactly the time when the put option is at its maximum value. But such profits will not be realized in full³ since this happens at the time that B cannot fulfill its contractual agreements. There are other examples of wrong way risk that could be catastrophic. Consider the situation where the reference in a credit default swap, B, defaults and due to exposure to the same harsh economic conditions C, who is the protection seller, also defaults leaving A the premium payer in an unfortunate situation.

To mitigate wrong-way risk, one must consider modelling correlations. In the credit risk literature there are at least three approaches to introducing correlation into credit models. These are listed below:

³A might still recover some value.

1. *Conditionally independent defaults*, which models the intensity process as stochastic processes and then correlates the corresponding Wiener processes that are assumed to drive them.
2. *Contagion models*, which try to include sources of default clustering, in that the default of an entity may lead to the default of others or a worsened credit state. They are an extension of the form in (1).
3. *The copula approach*, which takes as given the marginal default probabilities of the different entities and couples the marginal default distributions.

Credit value adjustments come in a number of flavors. These are *unilateral*, *bilateral*, *collateralized unilateral* and *collateralized bilateral* CVA.

- **Unilateral CVA:** This is when the CVA is calculated with the assumption that only the party calculating the adjustment is default risk free and the counterparty is prone to default. This form of CVA suffers from the fact that the valuation is asymmetric and hence the two parties will ultimately have different values for the same trade. The advantage, however, is that there is a sensible hedging strategy, this is in the sense that the valuating party can purchase protection on the counterparty that it had assumed to be default prone (assuming this protection can be bought).
- **Bilateral CVA:** This form of CVA valuation assumes that both parties in the trade are default prone. The advantage with this approach is that both parties are able to agree on the fair value of the trade since this form is symmetric. The drawback is that it loses its intuitive meaning of being the cost of the hedges, it is then hard for a firm to hedge its own default risk since it would have to buy CDS on itself or short its own stock.
- **Collateralized Unilateral CVA :** It is the above unilateral CVA with the difference that a CSA or collateral posting agreement is in place. It is perhaps more useful due to the popularity of collateral posting in the over-the-counter market.
- **Collateralized Bilateral CVA :** It is the above bilateral CVA with the difference that a CSA or collateral posting is in place. It suffers the same disadvantages that are associated with the bilateral CVA described above.

Several approaches will be discussed below which we characterize as 1) *Calculation from first principles*, 2) *The exposure profiles method* and 3) *The portfolio decomposition method*.

3.5 Quantifying CCR

Sörensen and Böllier [77] laid the crucial underpinnings that any approach modelling counterparty credit risk should consider. They mention four points, but we include a fifth one that is inspired by Brigo's recent work on the effects of correlation and credit value adjustments [13]. These are:

- The credit ratings of the two counterparties, whether they are equal or unequal and whether they are high or low, should be accounted for in the credit model. If both parties have some degree of credit risk, then there should be a bilateral pricing of the counterparty risk.
- The pricing of counterparty credit risk should depend on the existing contracts that each party has with the other. This simple rule means that the counterparty credit risk model should incorporate mitigating contracts such as netting arrangements if any exist.
- The pricing of counterparty credit risk on interest rate derivatives should depend on the existing shape and estimated volatility of the yield curve because these determine the option value embedded in the replacement cost. As already stated, counterparty default can be viewed as an option to default that was sold to the counterparty and in most cases not paid for.
- For the model to be completely useful both parties need to agree on their credit conditions so as to lessen the effects of market friction.
- Finally any model must incorporate a fair degree of correlation between economic variables and counterparties, and also correlation between the counterparties themselves.

3.5.1 Calculation from First Principles

Here we assume the definition of unilateral CVA and consider a single contingent claim. There have been significant studies on the modelling of defaultable term structures and the general pricing of derivatives with embedded default risk. The papers by Artzner *et al* [2], Jarrow *et al* [56] and Ramaswamyk *et al* [70] deal with the subject as it relates to most vanilla instruments. Artzner and Delbaen [2] price defaultable bonds using defaultable term structures and also consider prepayment risk. Duffie and Singleton [30] focus on a general class of derivatives including IRSs where both parties can be a source of credit risk. They derive a risk neutral valuation formula which we will use to illustrate. Ramaswamyk and Sundaresan [70]

view the pricing of defaultable claims as claims with embedded options, then the task becomes one of pricing those options.

In a reduced form framework, Duffie and Singleton [30] show that the value of a defaultable claim that promises a pay-off of $X \geq 0$ at maturity T is given by,

$$\widehat{V}_t = \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[\exp \left(- \int_t^T y_s ds \right) X \mid \mathcal{F}_t \right], \quad (3.10)$$

where $y_t = r_t + \lambda_t L_t$ is derived from approximations. Here r_t is defined as in (3.2), L_t is the fractional loss that is incurred on default and λ_t is the exogenously given hazard rate, possibly stochastic and adapted to the natural filtration $(\mathcal{F}_t)_{t \geq 0}$.

Assuming complete markets (that is no default) the value of the contingent claim is given by Proposition 2.46 to be

$$V_t = \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[\exp \left(- \int_t^T r_s ds \right) X \mid \mathcal{F}_t \right]. \quad (3.11)$$

Since CVA is by definition, the deference of these, we have that,

$$\text{CVA}(t) = \mathbb{E}_t^{\tilde{\mathbb{P}}} \left[X \left(\exp \left(- \int_t^T r_s ds \right) - \exp \left(- \int_t^T y_s ds \right) \right) \mid \mathcal{F}_t \right]. \quad (3.12)$$

Thus a derivative that pays X at maturity T subject to counterparty risk should have its value adjusted by this CVA, and this is clearly unilateral as it only considers the default of the counterparty with hazard rate λ_t . If we assume constant interest rates and hazard rates and also that $\exp(-at) \approx 1 - at$, through a Taylor expansion⁴, we can then estimate that,

$$\text{CVA}(t) = \mathbb{E}_t^{\tilde{\mathbb{P}}} [X(y_t - r_t)(T - t) \mid \mathcal{F}_t] = C_s(t)(T - t) \mathbb{E}_t^{\tilde{\mathbb{P}}} [X \mid \mathcal{F}_t], \quad (3.13)$$

where $C_s(t) = y_t - r_t$ is approximately the credit spread. This is only valid for short time intervals and may be impractical for long dated contracts.

There are several ways of obtaining the defaultable term structure R_t . A rough estimate can be obtained by adding the credit default swap spread to the risk-free term structure at each point in time. Alternatively, one may use the procedure developed by Stein and Lee [78] to imply the credit implied par curve. Using the implied curve however makes the usual assumption of independence of market variables which naturally then implies that wrong way risk is not catered for. The direct modelling of term structure credit spreads has received significant attention, with Brigo and Alfonsi [10] modelling it using a two dimensional shifted square root diffusion model (SSRD) which they latter extend to incorporate jumps [14].

⁴ This is only a valid assumption for small t .

3.5.2 The Exposure Profile Method

If counterparty credit risk metrics are to be useful in ways beyond an approach to decision making on credit limits for counterparties or calculating regulatory capital, they should also describe how one should price that exposure — this is the idea behind the exposure profile method. The assumption normally made in-order to obtain practical expressions for CVA is the independence of market prices from defaults. Giovan *et al* in [21] suggest that the CVA should be valued using the expression:

$$\text{CVA} = \int_0^T \text{EPE}(s) D(0, s) C_s(s) ds \approx \sum_i \text{EPE}_i(T_i - T_{i-1}) B(0, T_i) C_{si}, \quad (3.14)$$

where $\text{EPE}(t)$ is the expected positive exposure assumed to be piecewise constant between T_{i-1} and T_i . The C_{si} is the credit spread of a forward starting CDS starting at T_{i-1} and maturing at T_i . Equation (3.14) implies that the counterparty exposure could be hedged by buying forward starting CDS's with notional being determined by the expected positive exposure profile. This is very useful, especially with contracts where closed form valuations are not possible or the existence of a liquid market on the contract is not available. This removes the logic of viewing CVA as the cost of replacing the contract with an identical one at the time of default τ of the counterparty. The formula (3.14) is similar to (3.13), with one notable difference, the measures the expectations are under are different, $\mathbb{P}$ and $\tilde{\mathbb{P}}$ respectively for the formulas, see (3.6). The fundamental advantage with dealing OTC is that the deal can often be designed to suit specific requirements that are non-standard, however finding another counterparty to replace the defaulted counterparty may be impossible even at a fee.

A similar approach adopted by Pykhtin and Zhu [69] is to look at the loss L that would be incurred at a default time τ , given a constant recovery rate R_t^i which would be given by,

$$L = \mathbb{I}_{\tau \leq T} (1 - R) D(0, \tau) \text{CE}, \quad (3.15)$$

where CE is the counterparty exposure. The definition of the CVA would then be the cost of hedging the loss L that could be incurred. They then deduce that the CVA is given by,

$$\text{CVA} = \mathbb{E}^{\tilde{\mathbb{P}}}[L] = (1 - R) \int_0^T \mathbb{E}^{\tilde{\mathbb{P}}}[D(0, t) \text{CE}(t) \mid \tau = t] d\tilde{\mathbb{P}}(0, t), \quad (3.16)$$

where $\tilde{\mathbb{P}}(s, t)$ are the risk neutral probabilities of counterparty default between times

s and t , which are normally backed out from CDS spreads. If the usual independence assumption is made, we have that

$$\text{CVA} = (1 - R) \int_0^T D(0, t) \text{EE}(t) d\tilde{\mathbb{P}}(0, t), \quad (3.17)$$

with $\text{EE}(t)$ being the expected exposure.

An important thing to note is that the $\text{EE}(t)$ is normally calculated at the counterparty portfolio level, thus individual CVA contributions from trades are not immediately obvious, especially when netting agreements are in place. This might not be a problem for risk management purposes but may lead to problems when the front office wants to quote a price on a trade with CVA taken into account. Pykhtin and Rosen [67] approach this issue and show how the problem of calculating individual CVA contributions actually reduces to allocating expected exposure contributions to individual trades.

Calculating the Exposure Profiles

The calculation of exposure profiles is well understood having been the backbone of capital requirements for regulatory compliance. The algorithm for calculating these profiles can be decomposed into three steps, scenario generation, instrument valuation and aggregation.

- **Scenario Generation :** the underlying risk factors are simulated through time normally under the physical measure. This implies that one needs to specify the stochastic models that the risk factors are assumed to obey. The risk factors may include stocks, foreign exchange, interest rates, volatility models, hazard rates, *etc.* This phase involves the discretization of time between inception and the maturity of the latest trade in the portfolio. For practical applications banks use daily or weekly intervals up to a month, then monthly intervals up to a year and yearly up to five years [68]. There are generally two ways to perform the simulations, one known as *path dependent simulations* (PDS) and the other known as *direct jump to simulation date* (DJS). The former involves the simulation of the risk factor as a path over all the discrete time points. The latter simulates the risk factor to a fixed date t_i without any record of the path that was taken by the risk factor to get to t_i . The DJS method relies on the existence of a closed form solution for the SDE describing the evolution of a particular risk factor.
- **Instrument Valuation :** the valuation of individual instruments given the risk factor simulations varies in difficulty. Ordinary pricing models that rely

on the existence of a risk neutral measure may be inappropriate as credit exposures are normally calculated under the physical measure. There are technical difficulties that instrument valuations under simulations may introduce especially when valuing path dependent derivatives that by definition require the knowledge of the full path. This leads to conditional valuations. For path independent derivatives, the valuations are simplified and the direct jump to simulation method is preferred. Another technical difficulty that occurs with Monte Carlo valuations is their bias when valuing American type derivatives. However, an algorithm first introduced by Broadie and Glasserman [19], which is now popularly known as American Monte Carlo gives highly reliable prices within modest error bounds, but it is an exponential algorithm as a function of exercise times.

- **Aggregation :** this phase requires the exposure profile formulas to be applied at each discrete future time T_i to obtain the profile deemed necessary. The profiles would then be used for CVA calculations as suggested above.

3.5.3 Portfolio Decomposition

The approaches described above, while they aid in understanding and would suitably be in a position to utilize existing risk and pricing systems, lack the crucial explicit representation of the replicating portfolio. In short they are not flexible enough to enable the understanding of CVA as a standalone component of the price of the contingent claim. The analysis of CVA is important for the understanding of both its effect on the nature of the contract and the prescription of hedges.

Cheburini [23], considers linear instruments, more specifically forwards, and the pricing of counterparty risk embedded on them. The author concludes that the incorporation of counterparty risk may lead to the existence of gamma exposures that would render the linear instrument non-linear. Analysing this further, the author notes that the contracts are also sensitive to volatility. This is not surprising as the consideration of counterparty risk points to the existence of default optionality and options are known for their sensitivity to volatility. Some of the material below can also be found from Gregory [46], Duffie [29], Cheburini [23], Brigo and Masseti [16], Duffie and Singleton [30].

Unilateral CVA

Unilateral CVA assumes the existence of two counterparties in a derivative transaction which are normally described as the investor and the counterparty. The investor

is assumed default free and the counterparty is assumed to be default prone. At a default time τ there are three possible states that are of interest:

1. The value to the counterparty is negative, which implies that it is positive to the investor, i.e., $V_t > 0$. This means that the investor receives $R_t V_t$ of the value of the contract and incurs an immediate loss of $(1 - R_t)V_t$.
2. The value to the counterparty is positive, which implies that it is negative towards the investor, i.e., $V_t < 0$. The investor would then pay the amount V_t to the counterparty and both parties walk away.
3. The value of the contract to the counterparty and to the investor is zero; no party is thus required to pay anything.

The states above lead to the general formula for pricing unilateral counterparty credit risk[46, 16].

Proposition 3.2 (General Unilateral Counterparty Credit Risk Pricing Formula). *Let the value of a contingent claim at time t with maturity T between two counterparties A and C , with A assumed to be default free and C assumed to be default prone, be denoted by $\hat{V}(t, T)$. Let the value of an equivalent contingent claim between the two counterparties with the assumption that C is also default free be denoted by $V(t, T)$. The values $\hat{V}(t, T)$ and $V(t, T)$ are from A 's point of view, then we have that*

$$\begin{aligned}\hat{V}(t, T) &= V(t, T) - \mathbb{E}^{\tilde{\mathbb{P}}} [\mathbb{I}_{\tau_C \leq T} (1 - R_{\tau_C}^C) D(t, \tau_C) V(\tau_C, T)^+ \mid \mathcal{F}_t] \\ &= V_t - \text{UCVA}(t).\end{aligned}\tag{3.18}$$

Proof. The above points (1-3) can be mathematically written as follows,

$$\begin{aligned}\hat{V}(t, T) &= \mathbb{E}_t^{\tilde{\mathbb{P}}} [D(t, \tau_C) R_t^C V(\tau_C, T)^+ + D(t, \tau_C) V(\tau_C, T)^-] \\ &\quad + \mathbb{I}_{\tau_C > T} V(t, T) + \mathbb{I}_{\tau_C \leq T} (V(t, \tau_C)),\end{aligned}\tag{3.19}$$

and also,

$$V(\tau_C, T)^- = V(\tau_C, T) - V(\tau_C, T)^+.\tag{3.20}$$

Substituting (3.20) into the terms in (3.19) we obtain:

$$\begin{aligned}
\widehat{V}(t, T) &= \mathbb{E}_t^{\mathbb{P}} [\mathbb{I}_{\tau_C > T} V(t, T) + \mathbb{I}_{\tau_C \leq T} (V(t, \tau_C) \\
&\quad + D(t, \tau_C) R_t^C V(\tau_C, T)^+ + D(t, \tau_C) V(\tau_C, T) - D(t, \tau_C) V(\tau_C, T)^+)] \\
&= \mathbb{E}_t^{\mathbb{P}} [\mathbb{I}_{\tau_C > T} V(t, T) + \mathbb{I}_{\tau_C \leq T} (V(t, \tau_C) + D(t, \tau_C) V(\tau_C, T)) \\
&\quad + D(t, \tau_C) \mathbb{I}_{\tau_C \leq T} (R_t^C V(\tau_C, T)^+ - V(\tau_C, T)^+)] \\
&= V(t, T) - \mathbb{E}_t^{\mathbb{P}} [\mathbb{I}_{\tau_C \leq T} D(t, \tau_C) (1 - R_t^C) V(\tau_C, T)^+] \tag{3.21} \\
&= V(t, T) - \text{UCVA}(t). \tag{3.22}
\end{aligned}$$

□

What is immediately obvious is that the unilateral CVA at t which we denoted by $\text{UCVA}(t)$ is positive and is an option with zero strike on the value of the contract and is also dependent on default. The positive nature of UCVA implies that the value of the derivative with counterparty risk taken into consideration is less than the value of an identical derivative that is considered default free. The hedges that are suggested by the UCVA are options. In the case of a standard interest rate swap it would be a series of swaptions that would form the hedging portfolio, while with a credit default swap it would be CDS options, traded on the counterparty. The use of such intricate products shows how hedging CVA can be a complex task.

Bilateral CVA

The bilateral CVA extends the ideas underlying the unilateral CVA. The difference is that the investor and the counterparty both have a positive probability of defaulting. We denote their default times as τ_A and τ_C respectively, with the first default time $\tau = \tau_A \wedge \tau_C$. The maturity of the contract is, as usual, T . We denote the default free value of the contract at time t by $V(t, T)$ and the corresponding counterparty default prone contract by $\widehat{V}(t, T)$. All values are viewed from the investor's point of view. The following cases need to be considered:

1. There is no default during the life of the contract, i.e., $\tau > T$. All parties are able to honor their contractual obligations and thus we must have that $\widehat{V}(t, T) = V(t, T)$.
2. Suppose that the counterparty defaults, i.e., $\tau = \tau_C \leq T$. If the value of the contract to the counterparty is positive, i.e., negative towards the investor, then $V(\tau, T) < 0$. The investor will pay the market value of the contract to the counterparty. If, however, the value to the investor is positive, which means that $V(\tau, T) > 0$, then the investor will receive the recovery value equal to $R_\tau^C V(\tau, T)$ and immediately incurs a loss of $(1 - R_\tau^C) V(\tau, T)$.

3. Suppose that the investor defaults, i.e., $\tau = \tau_A \leq T$. Then if the value of the contract to the counterparty is positive, i.e., negative towards the investor, then $V(\tau, T) < 0$. The counterparty will only receive $R_\tau^A V(\tau_A, T)$ and immediately incurs a loss of $(1 - R_\tau^A)V(\tau_A, T)$. If however, the value of the contract is positive to the investor, then the investor will receive $V(\tau_A, T)$ in full.
4. If at default the contract is valued at zero, then both parties walk away without having to pay anything.

Proposition 3.3 (The General Formula for Bilateral Credit Value Adjustment). *Let $\widehat{V}(t, T)$ be the value of a contingent claim between two counterparties, an investor (A) and a counterparty (C). Both counterparties are assumed to be default prone with their default times denoted by τ_A and τ_C , such that $\tau_A \neq \tau_C$, for all events in Ω . Let $\tau = \tau_A \wedge \tau_C$ denote the first default time. Let $V(t, T)$ be the value of an identical contingent claim assuming all parties to be default free. Both $\widehat{V}(t, T)$ and $V(t, T)$ are calculated from the point of view of A. Then we have that,*

$$\begin{aligned} \widehat{V}(T, t) = V(t, T) - \mathbb{E}^{\mathbb{P}}[\mathbb{I}_{\tau \leq T} D(t, \tau) ((\mathbb{I}_{\tau = \tau_C} (1 - R_{\tau_C}^C) V(\tau_C, T)^+ \\ - \mathbb{I}_{\tau \leq T} (\mathbb{I}_{\tau = \tau_A} (1 - R_{\tau_A}^A) (-V(\tau_A, T))^+)) \mid \mathcal{F}_t]. \end{aligned} \quad (3.23)$$

Proof. Looking at the cases above it is clear that,

$$\begin{aligned} \widehat{V}(t, T) = \mathbb{E}^{\mathbb{P}}[\mathbb{I}_{\tau > T} V(t, T) + \mathbb{I}_{\tau \leq T} D(t, \tau) (V(t, \tau) + \mathbb{I}_{\tau = \tau_C} (R_t^C V(\tau, T)^+ \\ + V(\tau_C, T)^-) + \mathbb{I}_{\tau = \tau_A} (R_t^A V(\tau_A, T)^- + V(\tau, T)^+)) \mid \mathcal{F}_t]. \end{aligned} \quad (3.24)$$

Looking at the case where $\tau \leq T$ and also noting the following results:

$$V(\tau_A, T)^+ = V(\tau_A, T) - V(\tau_A, T)^- \quad (3.25)$$

$$V(\tau_C, T)^- = V(\tau_C, T) - V(\tau_C, T)^+, \quad (3.26)$$

we can show that

$$\begin{aligned} V(\tau_C, T)^- + R^C V(\tau_C, T)^+ &= V(\tau_C, T) - V(\tau_C, T)^+ + R^C V(\tau_C, T)^+ \\ &= V(\tau_C, T) - (1 - R^C) V(\tau_C, T)^+ \end{aligned} \quad (3.27)$$

and also

$$\begin{aligned} V(\tau_A, T)^+ + R^A V(\tau_A, T)^- &= V(\tau_A, T) - V(\tau_A, T)^- + R^A V(\tau_A, T)^- \\ &= V(\tau_A, T) - (1 - R^A) V(\tau_A, T)^- \\ &= V(\tau_A, T) + (1 - R^A) (-V(\tau_A, T))^+. \end{aligned} \quad (3.28)$$

Since

$$\begin{aligned} V(t, T) = & \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau > T} V(t, T) + \mathbb{I}_{\tau \leq T} (\mathbb{I}_{\tau = \tau_A} D(t, \tau_A) V(\tau_A, T) \\ & + \mathbb{I}_{\tau = \tau_C} D(t, \tau_C) V(\tau_C, T) + V(t, \tau)) \mid \mathcal{F}_t], \end{aligned} \quad (3.29)$$

we can substitute (3.27) and (3.28) into (3.25) and also using (3.29) we can group the terms that are obtained through the substitutions and obtain

$$\begin{aligned} \hat{V}(T, t) = & V(t, T) - \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau \leq T} D(t, \tau) ((\mathbb{I}_{\tau = \tau_C} (1 - R_{\tau_C}^C) V(\tau_C, T))^+ \\ & - \mathbb{I}_{\tau = \tau_A} (1 - R_{\tau_A}^A) (-V(\tau_A, T))^+) \mid \mathcal{F}_t]. \end{aligned} \quad (3.30)$$

Re-arranging (3.30) above we arrive at

$$\begin{aligned} \hat{V}(T, t) = & V(t, T) - \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau \leq T} (\mathbb{I}_{\tau = \tau_C} D(t, \tau) (1 - R_{\tau_C}^C) V(\tau_C, T))^+ \mid \mathcal{F}_t] \\ & + \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau \leq T} (\mathbb{I}_{\tau = \tau_A} D(t, \tau) (1 - R_{\tau_A}^A) (-V(\tau_A, T))^+ \mid \mathcal{F}_t], \end{aligned} \quad (3.31)$$

which is normally written as [46, 16],

$$\hat{V}(T, t) = V(t, T) - \text{UCVA}(t) + \text{DVA}(t). \quad (3.32)$$

□

The important result from (3.31) above is that the UCVA is, as already noted, a series of options to enter into the underlying derivative (possibly with a new counterparty at default) and such options need to be financed by the UCVA, which explains its negative contribution. There is another term which contributes positively, known as the *debt valuation adjustment* denoted by DVA above. The DVA indicates that the value of the derivative could actually be increased by the consideration of both counterparties default risk. This happens due to the fact that on occurrence of a default event, the defaulting entity i could be viewed as benefiting a value of $(1 - R_{\tau}^i) V(\tau, T)$, as it will not pay it. This further implies that the worsening credit quality of a counterparty actually boost profit margins for that counterparty, a result that is counterintuitive as it requires that the hedges be locked in prior to default. This leaves the paradoxical question: How does an entity hedge against its own default?

The bilateral CVA formula assumes that the value at default will be the risk free value of the derivative, whereas the ISDA recommends that the value be given by a substitute counterparty eager to be counterparty to the surviving party. This eager counterparty will in most instances consider the credit risk due to the surviving counterparty in its valuation which would mean that the value of the derivative at the particular default time will also need to be credit value adjusted. This inconsistency,

and the dangers of not considering the technical and legal details of a default event, is discussed in Brigo and Capponi [11]. The benefit of this approach is that it leaves the value of the contingent claim symmetrical and both parties are able to agree on its value. For this reason it has been used in accounting standards rather than for actual hedging purposes. We will, however, suggest ways to hedge this later in the dissertation.

A natural extension of the above formula is to consider relaxing the condition that no simultaneous defaults can happen⁵. Using the above reasoning, the following result can be derived.

Proposition 3.4 (General Bilateral CVA with Simultaneous Defaults Allowed). *Assume $\tilde{\mathbb{P}}(\tau_A = \tau_C) > 0$. The value of the contingent claim from A's point of view, assuming the possibility of default of both A and C is given by,*

$$\begin{aligned} \hat{V}(T, t) = & V(t, T) - \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau \leq T} D(t, \tau) (\mathbb{I}_{\{\tau = \tau_C\} \cap \{\tau_C \neq \tau_A\}} ((1 - R_{\tau_C}^C) V(\tau_C, T)^+ \\ & - \mathbb{I}_{\{\tau = \tau_A\} \cap \{\tau_A \neq \tau_C\}} ((1 - R_{\tau_A}^A) (-V(\tau_A, T))^+ + \mathbb{I}_{\tau_A = \tau_C} (V(\tau, T) \\ & - R_{\tau}^A V(\tau, T)^- - R_{\tau}^C V(\tau, T)^+)) \mid \mathcal{F}_t]. \end{aligned} \quad (3.33)$$

Proof. Again we look at the cases outlined earlier in the bilateral section where we had assumed that the two entities can not default at the same time. On occurrence of a simultaneous default event at τ , if $V(\tau, T) > 0$ then A will receive $R_{\tau_C}^C V(\tau, T)$ if however, $V(\tau, T) < 0$ then C will receive $R_{\tau}^A V(\tau, T)$. Combining the above cases and the new one we obtain,

$$\begin{aligned} \hat{V}(t, T) = & \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau > T} V(t, T) + \mathbb{I}_{\tau \leq T} D(t, \tau) (V(t, \tau) + \mathbb{I}_{\{\tau = \tau_C\} \cap \{\tau_C \neq \tau_A\}} (R_{\tau}^C V(\tau, T)^+ \\ & + V(\tau_C, T)^-) + \mathbb{I}_{\{\tau = \tau_A\} \cap \{\tau_A \neq \tau_C\}} (R_{\tau}^A V(\tau, T)^- + V(\tau, T)^+) \\ & + \mathbb{I}_{\tau_A = \tau_C} (R_{\tau}^C V(\tau, T)^+ + R_{\tau}^A V(\tau, T)^-)) \mid \mathcal{F}_t]. \end{aligned} \quad (3.34)$$

Substituting (3.27) and (3.28) into (3.34) and re-arranging the resulting terms we obtain an expression in which we can use the fact that,

$$\begin{aligned} V(t, T) = & \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau > T} V(t, T) + \mathbb{I}_{\tau \leq T} (\mathbb{I}_{\{\tau = \tau_A\} \cap \{\tau_A \neq \tau_C\}} D(t, \tau_A) V(\tau_A, T) \\ & + \mathbb{I}_{\{\tau = \tau_C\} \cap \{\tau_C \neq \tau_A\}} D(t, \tau_C) V(\tau_C, T) + \mathbb{I}_{\tau_A = \tau_C} V(\tau, T) + V(t, \tau)) \mid \mathcal{F}_t], \end{aligned} \quad (3.35)$$

⁵Default clustering has been verified by numerous studies in the literature.

to obtain

$$\begin{aligned}\widehat{V}(T, t) = & V(t, T) - \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau \leq T} D(t, \tau) (\mathbb{I}_{\{\tau = \tau_C\} \cap \{\tau_C \neq \tau_A\}} ((1 - R_{\tau_C}^C) V(\tau_C, T)^+)) \\ & - \mathbb{I}_{\{\tau = \tau_C\} \cap \{\tau_C \neq \tau_A\}} ((1 - R_{\tau_A}^A) (-V(\tau_A, T))^+) + \mathbb{I}_{\tau = \tau_A = \tau_C} (V(\tau, T) \\ & - R_{\tau}^A V(\tau, T)^- - R_{\tau}^C V(\tau, T)^+) \mid \mathcal{F}_t].\end{aligned}\quad (3.36)$$

□

The result (3.36) can be written as

$$\widehat{V}(T, t) = V(t, T) - \text{UCVA}(t) + \text{DVA}(t) + \text{SDVA}(t), \quad (3.37)$$

where the new component $\text{SDVA}(t)$ stands for *simultaneous default value adjustment*.

Bilateral Collateralized CVA

The literature on credit value adjustments does not have much on the calculation of CVA in the presence of a credit support annex or more generally a collateral agreement. In two recent papers by Brigo *et al* [12] and Fujii *et al* [41], the subject is considered. Fujii *et al* derive solutions using Gateaux derivatives for the case where collateralization is imperfect and asymmetric⁶. Brigo *et al* derive a model free formula for CVA taking into account the presence of a collateral agreement. In what follows we refer to their setting.

Let the collateral account be denoted by C_t , which is a stochastic process adapted to the filtration $\mathcal{F}_t$. Denote the exposures at a default time τ by $\epsilon_{\tau, A}$ for the investor and $\epsilon_{\tau, C}$ for the counterparty. As already mentioned in the section on bilateral CVA, τ is the first default time. Denote the loss given default of the investor by $\text{LGD}_A = 1 - R_t^A$ and the corresponding loss given default of the counterparty denoted $\text{LGD}_C = 1 - R_t^C$. Furthermore we generalize to the case where the collateral may be re-hypothecated⁷ by including a separate recovery rate for the collateral account for both investor and counterparty, these are denoted by $R_t^{A'}$ and $R_t^{C'}$ such that the corresponding losses given default of the collateral account are given by LGD'_A and LGD'_C respectively. By following similar reasoning as in the case of the unilateral and bilateral CVA, it can be shown that the bilateral collateralized CVA (BC-CVA), when assuming that the exposures are symmetric in the sense that they are mid-market exposures, is given by

⁶ Only one of the counterparties is required to post collateral.

⁷ The party that receives the collateral is allowed to use the collateral in whatever way they may wish instead of keeping it in a safe place.

$$\begin{aligned}
\text{BC} - \text{CVA}(t, T; C) = & -\mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau=\tau_C < T} D(t, \tau) \text{LGD}_C(\epsilon_\tau^+ - C_\tau^+)^+ \mid \mathcal{F}_t] \\
& - \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau=\tau_C < T} D(t, \tau) \text{LGD}'_C(\epsilon_\tau^- - C_\tau^-)^+ \mid \mathcal{F}_t] \\
& - \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau=\tau_A < T} D(t, \tau) \text{LGD}_A(\epsilon_\tau^- - C_\tau^-)^- \mid \mathcal{F}_t] \\
& - \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau=\tau_A < T} D(t, \tau) \text{LGD}'_A(\epsilon_\tau^+ - C_\tau^+)^- \mid \mathcal{F}_t].
\end{aligned}$$

It is also clear that the presence of the collateral C_t always reduces the exposure, which is intuitively correct. The other strength of the expression above is that it can be easily simplified to the cases mentioned earlier (unilateral and bilateral) that do not consider collateral. By assuming that the collateral is kept safe, so that a party cannot use the collateral placed, rules out the possibility of loss due to default on the collateral. This means that $\text{LGD}'_C = \text{LGD}'_A = 0$ thus the above BCCVA becomes

$$\begin{aligned}
\text{BC} - \text{CVA}(t, T; C) = & -\mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau=\tau_C} D(t, \tau) \text{LGD}_C(\epsilon_\tau^+ - C_\tau^+)^+ \mid \mathcal{F}_t] \\
& - \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau=\tau_A} D(t, \tau) \text{LGD}_A(\epsilon_\tau^- - C_\tau^-)^- \mid \mathcal{F}_t]. \quad (3.38)
\end{aligned}$$

In the absence of collateral, i.e., $C_t = 0$, it is trivial to see that we retain the bilateral CVA case and as $\tau_A \rightarrow \infty$ which then implies that $\mathbb{I}_{\tau=\tau_A} \rightarrow 0$, we again obtain the unilateral CVA case⁸.

3.6 Remark

This chapter introduced CCR along with ways to manage and mitigate it. It further introduced the notion of credit value adjustments and illustrated under various assumptions what the credit value adjustment formula should be, independent of model or nature of contracts. The next chapter will introduce the CIR and CIR++ model which will be used to model the short rate dynamics and also the process followed by the intensity of default for a default prone entity. This will be when credit risk is modelled under the intensity based framework.

⁸The exposure is the net present value of the contract at time τ .

Chapter 4

A Short Rate Model: The CIR and CIR++ model

In the previous chapter we showed that a CVA/BCVA is an option maturing at a default time. To simulate default times one requires a framework within which to model credit risk. One such framework is intensity based in that default arrives at certain intensity. The process followed by the intensity can be modelled using a short rate process. In this chapter we will introduce the Cox-Ingersoll-Ross model (CIR) and its extension, the CIR++ model. In the rest of the dissertation, the CIR++ model will be used to model the short rate and the intensity of default of a default prone entity. The relationship between the short rate and the intensity is revealed by the discount bond and the survival probability in an intensity based framework and will be illustrated in the next chapter.

4.1 Probability Framework

We are in the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. An equivalent martingale measure $\tilde{\mathbb{P}}$ is assumed. Under $\tilde{\mathbb{P}}$, the discounted price processes of all trade-able securities are martingales. The stochastic discount factor is given by,

$$D(t, T) = \exp \left(- \int_t^T r_s ds \right), \quad (4.1)$$

where r_s is the instantaneous spot rate also known as the short rate. The associated discount bonds at time t maturing at T will be denoted by $B(t, T)_{t \geq 0}$. Again, the existence of the equivalent martingale measure rules out any arbitrage opportunities.

4.2 Short Rate Modelling

Modelling the term structure of interest rates is one of the most complicated modelling problems in mathematical finance. This problem has been approached in a number of ways, one of which is to model the evolution of the short rate. Approaching the problem in this manner has yielded various short rate models. For an intensive coverage of term structure models see Brigo and Mercurio [18]. In summary the most prominent models are as follows:

- *Vasicek Model*: This is one of the earliest term structure models that was born out of the equilibrium modelling approach. It postulates that the short rate dynamics are given by

$$dr_t = k[\theta - r_t]dt + \sigma dW_t, \quad (4.2)$$

where k, θ and σ are constant parameters. Parameter k is the speed of reversion of the process to the long term level θ (mean reversion parameter).

- *Extended Exponential Vasicek Model*: This model is a one factor term structure model extending the original Vasicek model by adding a deterministic function to it. The dynamics are given by

$$r_t = x_t + \phi_t, \quad (4.3)$$

where

$$dx_t = x_t [\theta - a \ln(x_t)] dt + \sigma x_t dW_t. \quad (4.4)$$

Under certain conditions on the function ϕ_t , the model excludes negative interest rates. Under this setting $r_t \sim SLN$ (Shifted log-normal distribution).

- *Cox-Ingersoll-Ross (CIR) Model*: This improves the Vasicek model in that interest rates are guaranteed positive. The dynamics are given by

$$dr_t = k[\theta - r_t]dt + \sigma \sqrt{r_t} dW_t, \quad (4.5)$$

where k, θ and σ are constants. Parameter k is the speed of reversion of the process and θ is a mean reversion parameter (long term level). Under these dynamics, $r_t \sim NC_{\chi^2}$ (Non-central Chi-squared).

- *CIR++*: It achieves greater flexibility than the original CIR model. The dynamics are given by,

$$r_t = x_t + \phi_t, \quad (4.6)$$

where

$$dx_t = k[\theta - x_t]dt + \sigma\sqrt{x_t}dW_t. \quad (4.7)$$

If $\phi_t \geq 0$, negative interest rates are excluded. Under this setting $r_t \sim SNC_{\chi^2}$ (Shifted non-central chi-squared distribution).

This dissertation will mostly use the CIR++ model which requires that the CIR model is reviewed.

4.3 The Cox-Ingersoll-Ross Model (CIR)

The CIR model was introduced by Cox, Ingersoll and Ross in 1985 [26]. It is based on the continuous time general equilibrium model that the same authors had introduced the same year [25]. Under the general equilibrium model, it was shown that at equilibrium the short rate can be described by (4.5), that is,

$$dr_t = k[\theta - r_t]dt + \sigma\sqrt{r_t}dW_t. \quad (4.8)$$

4.3.1 Properties of the CIR Model

Let W_t be a Brownian motion under $\tilde{\mathbb{P}}$ and all expectations that follow are under $\tilde{\mathbb{P}}$. Following are two results concerning the first two moments of r_t , they will help establish some of the good characteristics of (4.8).

Proposition 4.1. *The mean of r_t given r_0 where $0 < t$ is given by,*

$$\mathbb{E}[r_t | r_0] = e^{-kt}r_0 + \theta(1 - e^{-kt}). \quad (4.9)$$

Proof. Using the Itô-Doeblin formula in Proposition 2.35 given that $f(r_t) = e^{kt}r_t$, we have that

$$df(r_t) = f_t(r_t)dt + f_{r_t}(r_t)dr_t + \frac{1}{2}f_{rr}(r_t)dr_tdr_t \quad (4.10)$$

$$= ke^{kt}r_tdt + e^{kt}(\theta k - kr_t)dt + e^{kt}\sigma\sqrt{r_t}dW_t \quad (4.11)$$

$$d(e^{kt}r_t) = \theta ke^{kt}dt + \sigma e^{kt}\sqrt{r_t}dW_t, \quad (4.12)$$

and

$$e^{kt}r_t = r_0 + \theta k \int_0^t e^{ku}du + \sigma \int_0^t e^{ku}\sqrt{r_u}dW_u, \quad (4.13)$$

$$r_t = r_0e^{-kt} + e^{-kt}\theta(e^{kt} - 1) + \sigma e^{-kt} \int_0^t e^{ku}\sqrt{r_u}dW_u. \quad (4.14)$$

$$(4.15)$$

Using the fact that the expectation of an Itô integral is zero, we have that,

$$\mathbb{E}[r_t \mid r_0] = e^{-kt}r_0 + \theta(1 - e^{-kt}). \quad (4.16)$$

□

From (4.16) we have that

$$\theta = \lim_{t \rightarrow \infty} \left(e^{-kt}r_0 + \theta(1 - e^{-kt}) \right), \quad (4.17)$$

which means that in the long term the model reverts to some mean value θ at a speed of k . The next result concerns the second moment of the process (4.8).

Proposition 4.2. *The variance of r_t given r_0 is given by,*

$$\text{Var}(r_t) = \frac{\sigma^2}{k}r_0 \left(e^{-kt} - e^{-2kt} \right) + \frac{\theta\sigma^2}{2k} \left(1 - 2e^{-kt} + e^{-2kt} \right). \quad (4.18)$$

Proof. Let $x_t = e^{kt}r_t$ and then utilizing the results in the previous proposition, we have that

$$dx_t = k\theta e^{kt}dt + \sigma e^{\frac{kt}{2}}\sqrt{x_t}dW_t$$

such that

$$\mathbb{E}[x_t] = r_0 + \theta(e^{kt} - 1). \quad (4.19)$$

We proceed by using the Itô-Doebelin result in Proposition 2.35, which implies that

$$dx_t^2 = (2k\theta - \sigma^2)e^{kt}x_tdt + 2\sigma e^{\frac{kt}{2}}x_t^{\frac{3}{2}}dW_t. \quad (4.20)$$

Equation (4.20) is a shorthand for

$$x_t^2 = x_0^2 + (2k\theta - \sigma^2) \int_0^t e^{ku}x_u du + 2k\theta \int_0^t e^{\frac{ku}{2}}x_u^{\frac{3}{2}}dW_u.$$

The mean of x_t^2 can be shown to be

$$\mathbb{E}[x_t^2] = x_0^2 + (2k\theta - \sigma^2) \int_0^t e^{ku}\mathbb{E}[x_u]du,$$

where we have used Fubini's theorem and the fact that the expectation of an Itô integral is zero. By substituting (4.19) and integrating, we obtain

$$\mathbb{E}[x_t^2] = r_0^2 + \frac{2k\theta - \sigma^2}{k} (r_0 - \theta) (e^{kt} - 1) + \frac{2k\theta - \sigma^2}{2k} \theta (e^{2kt} - 1).$$

By the definition of variance, we have that

$$\begin{aligned} \text{Var}(r_t) &= \mathbb{E}[r_t^2] - (\mathbb{E}[r_t])^2 \\ &= e^{-2kt}\mathbb{E}[x_t^2] - (\mathbb{E}[r_t])^2 \\ &= \frac{\sigma^2}{k}r_0 \left(e^{-kt} - e^{-2kt} \right) + \frac{\theta\sigma^2}{2k} \left(1 - 2e^{-kt} + e^{-2kt} \right). \end{aligned} \quad (4.21)$$

□

It is also easy to show from (4.21), that

$$\lim_{t \rightarrow \infty} \text{Var}(r_t) = \frac{\theta \sigma^2}{2k}, \quad (4.22)$$

that is, in the long term the variance is bounded.

4.3.2 Prices for Zero Coupon Bonds and Options

There is no closed form solution for r_t under the process (4.8) but zero coupon bonds can be priced analytically. The Feynman-Kac theorem establishes a connection between stochastic and partial differential equations, we use it below to obtain the price of a zero-coupon bond at time t maturing at T , which will be denoted by $B^{\text{CIR}}(t, T)$.

Using the Feynman-Kac theorem, Theorem 2.38, particularly the PDE (2.20), we can write that,

$$\frac{\partial B^{\text{CIR}}(t, T)}{\partial t} + \frac{\partial B^{\text{CIR}}(t, T)}{\partial x} \left(\frac{\theta}{k} - kr_t \right) + \frac{1}{2} \frac{\partial^2 B^{\text{CIR}}(t, T)}{\partial x^2} \sigma^2 r_t = r_t B^{\text{CIR}}(t, T), \quad (4.23)$$

with the terminal boundary condition being

$$B^{\text{CIR}}(T, T) = 1. \quad (4.24)$$

It is usual, when deriving the zero coupon bond price, to assume that the continuously compounded spot rate $\tilde{r}(t, T)$, is affine, i.e., it takes a solution of the form,

$$\tilde{r}(t, T) = \gamma(t, T) + \delta(t, T)r_t, \quad (4.25)$$

so that the zero coupon price takes the form,

$$B^{\text{CIR}}(t, T) = \psi(t, T) \exp \{ -r_t \varphi(t, T) \}. \quad (4.26)$$

This implies that the terminal boundary condition (4.24) can be written in terms of $\psi(\cdot)$ and $\varphi(\cdot)$ as

$$\begin{aligned} \psi(T, T) &= 1 \\ \varphi(T, T) &= 0, \end{aligned}$$

such that substituting (4.26) into (4.23) gives

$$\left[\left(-\varphi'(t, T) + k\varphi(t, T) + \frac{1}{2} \sigma^2 \varphi^2(t, T) - 1 \right) r - \left(\psi'(t, T) + \frac{\theta}{k} \varphi(t, T) \right) \right] \times B^{\text{CIR}}(t, T) = 0. \quad (4.27)$$

The above equation (4.27) implies that the bracketed terms must be zero, that is,

$$\varphi'(t, T) = k\varphi(t, T) + \frac{1}{2}\sigma^2\varphi^2(t, T) - 1 \quad (4.28)$$

and

$$\psi'(t, T) = \frac{\theta}{k}\varphi(t, T). \quad (4.29)$$

Equation (4.28) is the Riccati equation. It is well documented in the literature and is almost always solved numerically, but for this case it can be solved analytically. The solution of (4.28) clearly leads to a solution for ψ . After due diligence it can be shown that

$$\psi(t, T) = \left[\frac{2h \exp((k+h)(T-t))/2}{2h + (k+h)(\exp((T-t)h) - 1)} \right]^{2k\theta/\sigma^2} \quad (4.30)$$

and

$$\varphi(t, T) = \frac{2(\exp((T-t)h) - 1)}{2h + (k+h)(\exp((T-t)h) - 1)}, \quad (4.31)$$

where

$$h = \sqrt{k^2 + 2\sigma^2}.$$

It is also true that under (4.8) an option on a zero coupon bond can be priced analytically. Cox, Ingersoll and Ross [26], showed that under the CIR process the price of a European call option at time t expiring at T with strike K , written on a zero coupon bond maturing at time S , is

$$\begin{aligned} V_c^{\text{CIR}}(t, T, S, K, x_t) = & \psi(t, S) \exp\{-\varphi(t, S)x_t\} \mathcal{X}^2\left(\mu; \frac{4k\theta}{\sigma^2}, \text{I}\right) \\ & - K\psi(t, T) \exp(-\varphi(t, T)x_t) \mathcal{X}^2\left(2\hat{r}(\rho + \zeta); \frac{4k\theta}{\sigma^2}, \hat{\text{I}}\right) \end{aligned} \quad (4.32)$$

where

$$\text{I} = \frac{2\rho^2 x_t \exp(h(T-t))}{\rho + \zeta + \varphi(T, S)}, \quad (4.33)$$

$$\hat{\text{I}} = \frac{2\rho^2 x_t \exp(h(T-t))}{\rho + \zeta}, \quad (4.34)$$

$$\mu = 2\hat{r}(\rho + \zeta + \varphi(T, S)), \quad (4.35)$$

$$\rho = \frac{2h}{\sigma^2(\exp(h(T-t)) - 1)}, \quad (4.36)$$

$$\zeta = \frac{k+h}{\sigma^2} \quad (4.37)$$

and

$$\hat{r} = \hat{r}(S - T) = \frac{\ln \frac{\psi(T, S)}{K}}{\varphi(T, S)}. \quad (4.38)$$

The functions $\psi(\cdot)$ and $\varphi(\cdot)$ are those defined in (4.30) and (4.31) respectively and x_t is the short rate at t . Other interest rates derivatives such as caps, floors and swaptions can also be priced analytically through the bond option price, (4.32).

4.3.3 Summary of the Process

In summary, the CIR process is mean reverting, i.e., the rate will always revert to some average rate θ at a speed k in the long term. It is also important to note that prior to this work by Cox, Ross and Ingersoll this type of process had been studied by amongst others Feller, whose work led to the condition $2k\theta \geq \sigma^2$ (this condition is known as the Feller condition), which is necessary to exclude the origin from being accessible. The CIR process has a number of advantages such as:

1. interest rates are never negative,
2. even without the condition above when interest rates become zero, they can become positive again,
3. the absolute variance of interest rates increases when the interest rate increases,
4. there is a steady distribution for the interest rates and
5. analytical prices for zero coupon bonds and options exist.

Below it will be extended to the CIR++ model.

4.4 The CIR++ Model

Even with such good properties, the model is still not always able to accurately recover the current term structure of discount factors for any choice of k, σ and θ . This is due to its equilibrium foundation which implies that the current term structure is an output of the model rather than an input. There have been several approaches to solve this problem, one being the extended CIR model by Hull and White [52]. They remove the constant parameters and introduce time dependent parameters. The problem, however, with their approach is that zero coupon bond prices and options on them can only be obtained through numerical methods.

Another approach was developed by Dybvig [33], Avellaneda and Neuman [4] and recently by Brigo and Mercurio [17]. The idea is to take an analytically tractable

model such as (4.5) and extend it using a carefully selected function such that it is able to reproduce the initial term structure. Let x_t in (4.7) or (4.4) be adapted to the filtration $\mathcal{F}_t$. Let the short rate be given by

$$r_t = x_t + \phi(t), \quad t \geq 0, \quad (4.39)$$

with

$$\phi(0) = r_0 - x_0. \quad (4.40)$$

The process r_t is also adapted to $\mathcal{F}_t$. Using the one dimensional Itô lemma (c.f. 2.17), the dynamics of r_t are given by,

$$dr_t = \left[\frac{d\phi(t)}{dt} + k(\theta - (r_t - \phi(t))) \right] dt + \sigma \sqrt{r_t - \phi(t)} dW_t, \quad (4.41)$$

where W_t is Brownian motion under $\tilde{\mathbb{P}}$. It is also easy to see that under r_t ,

$$\begin{aligned} B^{\text{CIR++}}(t, T) &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[\exp \left(- \int_t^T r_s ds \right) \mid \mathcal{F}_t \right] \\ &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[\exp \left(- \int_t^T (x_s + \phi(s)) ds \right) \mid \mathcal{F}_t \right] \\ &= \exp \left(- \int_t^T \phi(s) ds \right) \mathbb{E}^{\tilde{\mathbb{P}}} \left[\exp \left(- \int_t^T x_s ds \right) \mid \mathcal{F}_t \right] \\ &= \exp \left(- \int_t^T \phi(s) ds \right) B^{\text{CIR}}(t, T). \end{aligned} \quad (4.42)$$

Because x_t is analytically tractable and B^{CIR} is known in closed form, we therefore have that $B(t, T)$ is also has a closed form. To determine the function $\phi(t)$ we state and prove the following result which is also proved in Brigo and Mercurio [17].

Definition 4.3. Given a zero coupon bond $B(t, T)$, the instantaneous forward rate at time t maturing at time T is given by

$$f(t, T) = - \frac{\partial \ln B(t, T)}{\partial T}. \quad (4.43)$$

Lemma 4.4. The model (4.39) recovers the current term structure of discount factors if and only if,

$$\phi(t) = f(0, t)^O - f^x(0, t), \quad (4.44)$$

where $f(0, t)^O$ is the observed instantaneous forward rate at time zero maturing at time t associated with the observed discount factor $B(0, t)_{t \geq 0}$ and similarly $f^x(0, t)$ is the model x_t implied instantaneous forward rate associated with the discount factor $B^x(0, t)_{t \geq 0}$. Consequently it can be shown that

$$\exp \left[- \int_t^T \phi(s) ds \right] = \frac{B(0, T) B^x(0, t)}{B^x(0, T) B(0, t)}. \quad (4.45)$$

From (4.42) above, we have that the observed discount factors, $B(0, T)$, must satisfy

$$B(0, T) = \exp\left(-\int_0^T \phi(s) ds\right) B^x(0, T), \quad (4.46)$$

which, by taking the natural logarithm on both sides and differentiating with respect to t , gives

$$-\frac{\partial \log(B(0, T))}{\partial t} = \phi(t) - \frac{\partial \log(B^x(0, t))}{\partial t}, \quad (4.47)$$

which is equivalent to

$$\phi(t) = f(0, t)^O - f^x(0, t). \quad (4.48)$$

Thus in the case of the CIR model (4.5), $\phi(t) = \phi^{\text{CIR}}(t) = f^O(0, t) - f^{\text{CRI}}(0, t)$, and under the CIR model we have shown that the bond price formula is given by (4.26) where $\psi(t, T)$ and $\varphi(t, T)$ are given by (4.30) and (4.31) respectively. Using (4.26) and Lemma 4.4, the function $f^x = f^{\text{CIR}}$ can be derived by substituting (4.26) into (4.43), that is

$$\begin{aligned} f^{\text{CIR}}(0, t) &= -\frac{\partial \ln B^{\text{CIR}}(0, t)}{\partial t} \\ &= 2k\theta \frac{\exp(th) - 1}{2h + (k + h)(\exp(th) - 1)} + x_0 \frac{4h^2 \exp(th)}{(2h + (k + h)(\exp(th) - 1))^2}, \end{aligned} \quad (4.49)$$

where

$$h = \sqrt{k^2 + 2\sigma^2}.$$

The other function $f^O(\cdot)$ is stripped from the initial term structure of discount bonds, such that

$$\begin{aligned} \phi^{\text{CIR}}(t) &= f^O(0, t) - 2k\theta \frac{\exp(th) - 1}{2h + (k + h)(\exp(th) - 1)} \\ &\quad + x_0 \frac{4h^2 \exp(th)}{(2h + (k + h)(\exp(th) - 1))^2}, \end{aligned} \quad (4.50)$$

where

$$h = \sqrt{k^2 + 2\sigma^2}.$$

After substituting (4.50) into (4.42) and doing the necessary algebraic manipulations, the zero coupon bond price under (4.39) at time t maturing at time T is given by

$$\begin{aligned} B^{\text{CIR}++}(t, T) &= \frac{B(0, T)\psi(0, t) \exp(-\varphi(0, t)x_0)}{B(0, t)\psi(0, T) \exp(-\varphi(0, T)x_0)} \psi(t, T) \exp(-\varphi(t, T) \times \\ &\quad \{r_t - \phi^{\text{CIR}}(t)\}) \end{aligned} \quad (4.51)$$

$$= \Psi(t, T) \exp(-\varphi(t, T)r_t), \quad (4.52)$$

where we make use of the time zero observed term structure of discount factors, $B(0, t)$. The model (4.39), was named the CIR++ model by Brigo and Mercurio [17].

4.4.1 Pricing Interest Derivatives Under the CIR++ Model

In this section several interest rate derivatives will be priced as they will be used in the coming chapters on calculating a CVA/BCVA on some interest rate derivatives. Assume that the evolution of all rates in the economy is fully described by the short rate, which is assumed to follow the CIR++ process (4.39). The bond price under the process (4.39) is given by (4.51), such that the average spot interest rate between t and T is

$$\begin{aligned}\bar{r}(t, T) &= -\frac{\ln B^{\text{CIR++}}(t, T)}{T - t} \\ &= \frac{1}{T - t} \left(\ln \frac{B(0, t)\psi(0, T) \exp(-\varphi(0, T)x_0)}{B(0, T)\psi(0, t) \exp(\varphi(0, t)x_0)} \right. \\ &\quad \left. - \ln \psi(t, T) - \varphi(t, T)\phi^{\text{CIR}}(t) + \varphi(t, T)r_t \right),\end{aligned}\quad (4.53)$$

which is linear in r_t . As already noted, the CIR++ model is built from the analytically tractable CIR model. t under the CIR++ model, maturing at time T with strike price K , written on a bond that matures at time $S > T$, is given by,

$$\begin{aligned}V_c^{\text{CIR++}}(t, T, S, K, r_t) &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[\exp \left(- \int_t^T r_s ds \right) (B^{\text{CIR++}}(T, S) - K)^+ \mid \mathcal{G}_t \right] \\ &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[\exp \left(- \int_t^T (x_s + \phi_s) ds \right) (B^{\text{CIR++}}(T, S) - K)^+ \mid \mathcal{F}_t \right] \\ &= \exp \left(- \int_t^S \phi(s) ds \right) \mathbb{E}^{\tilde{\mathbb{P}}} \left[\exp \left(- \int_t^T x_s ds \right) \right. \\ &\quad \left. \left(B^{\text{CIR}}(T, S) - K \exp \left(- \int_T^S \phi(s) ds \right) \right)^+ \mid \mathcal{F}_t \right] \\ &= V_c^{\text{CIR}} \left(t, T, S, K \exp \left(- \int_T^S \phi(s) ds \right), r_t - \phi^{\text{CIR}} \right), \\ &\quad \times \exp \left(- \int_t^S \phi(s) ds \right)\end{aligned}\quad (4.54)$$

where (4.42) was used in the third step.

This not only shows that the price of the option under the CIR++ model inherits the analytical tractability of the CIR model, it also gives the price of the European zero coupon bond call option.

The price of the zero coupon bond option given by (4.54) can be used to price other interest rate derivatives. The inverse relationship between the yield and the

zero coupon bond price provides a relationship between the European zero coupon bond option and two of the building blocks in the interest rate option markets, caplets and floorlets. A caplet is an option to enter into a long position in a forward rate agreement at a future time T , while a floorlet is an option to enter into a short position of a forward rate agreement. Forward rate agreements will be discussed in detail in Chapter 6. Focusing on floorlets and caplets, one can uncover the relationship between a caplet and a European zero coupon bond option referencing a floating rate that resets at t and settles at T denoted by $J(t, T)$. The value of a caplet at time t , given that $\alpha_{t,T}$ is the day count fraction, is

$$\begin{aligned}
 V(t, T, S) &= \alpha_{t,T}(1 + \alpha_{t,T}J(T, S))^{-1} \max(J(T, S) - K, 0) \\
 &= \max\left(\frac{\alpha_{t,T}J(T, S) - \alpha_{t,T}K}{1 + \alpha_{t,T}J(T, S)}, 0\right) \\
 &= \max\left(1 - \frac{1 + \alpha_{t,T}K}{1 + \alpha_{t,T}J(T, S)}, 0\right) \\
 &= (1 + \alpha_{t,T}K) \max\left(\frac{1}{1 + \alpha_{t,T}K} - \frac{1}{1 + \alpha_{t,T}J(T, S)}, 0\right), \quad (4.55)
 \end{aligned}$$

where T is the reset date and S is the payment date. From (4.55), it follows that pricing a caplet is equivalent to pricing a European put coupon on a zero coupon bond and, similarly, pricing a floorlet is equivalent to pricing a European call option on a zero coupon bond. Explicitly, the price of a caplet under the CIR++ model is

$$\begin{aligned}
 \text{Cl}(t, T, S, K) &= \exp\left(-\int_t^T \phi(s) ds\right) \left(1 + \alpha_{t,T}K \exp\left(-\int_T^S \phi(s) ds\right)\right) \\
 &\quad \times V_p^{\text{CIR}}\left(t, T, S, \frac{1}{1 + \alpha_{t,T}K \exp\left(-\int_T^S \phi(s) ds\right)}\right), \quad (4.56)
 \end{aligned}$$

where $V_p^{\text{CIR}}(\cdot)$ is the price of a European zero coupon bond put option. The price of a floorlet can be found through put-call parity and is given by

$$\begin{aligned}
 \text{Fl}(t, T, S, K) &= \exp\left(-\int_t^T \phi(s) ds\right) \left(1 + \alpha_{t,T}K \exp\left(-\int_T^S \phi(s) ds\right)\right) \\
 &\quad \times V_c^{\text{CIR}}\left(t, T, S, \frac{1}{1 + \alpha_{t,T}K \exp\left(-\int_T^S \phi(s) ds\right)}\right). \quad (4.57)
 \end{aligned}$$

A series of caplets (floorlets) form a cap (floor) and the price is the sum of all the prices of the individual caplets. The price of a cap is given by,

$$\text{Cap}(t, T, S, K) = \sum_{i=1}^N \text{Cl}(t, T_{i-1}, T_i, K), \quad (4.58)$$

if there are N caplets in the cap. The price of the corresponding floor can be found through put call parity or as a sum of floorlets.

Another equally important contract in the interest rate derivative space is a swap, also discussed in Chapter 6. An interest rate swap (IRS) is a series of forward rate agreements (FRAs) that pay in arrears in the SA market. It is necessary to price an option to enter into an IRS, known as a swaption as it will be shown that to hedge the counterparty credit risk in an IRS deal, swaptions are necessary. Jamshidian [55], showed that an option on a contract with multiple cash-flows can be decomposed into options on the individual cash-flows. This is important since IRSs can be viewed as coupon bearing bonds. Pricing a swaption then translates into pricing an option on a coupon bearing bond. This decomposition is valid for one factor affine models due to the fact that all rates move in the same direction as the short rate, or are fully described by the short rate. This means that all rates are increasing functions of the short rate and all bond prices are decreasing functions of the short rate. To value an option on a coupon bearing bond the following recipe can be used, based on Hull [51] and Brigo *et al.* [18]:

- Calculate the value of the short rate r^* such that at maturity $\bar{T}$ of the option, the value of the coupon bearing bond is equal to the strike K . The fact that all rates move in the same direction as the short rate may then be used to imply that if the option is exercised, then $r < r^*$ must be true.
- Calculate the prices of options on the zero-coupon bonds that comprise the coupon bearing bond, with the strike being the corresponding zero coupon bond value if $r = r^*$.
- The price of the option on the coupon bearing bond is the sum of the corresponding options on the zero coupon bonds comprising the coupon bearing bond.

Let the underlying swap in a swaption have payment dates $T_1, \dots, T_n = T$ and resetting at $\bar{T} = T_0, \dots, T_{n-1}$, the strike being K (in other words the rates are fixed in advance and paid in arrears). Also let $c_i = K\alpha_{T_{i-1}, T_i}$ for $i = 1, \dots, n-1$ while $c_n = 1 + K\alpha_{T_{n-1}, T_n}$. Implementing the first point above requires that r^* is the solution of

$$\sum_{i=1}^N c_i \Psi(\bar{T}, T_i) \exp(-\varphi(T, T_i)r^*) = 1. \quad (4.59)$$

Setting $K_i = \Psi(\bar{T}, T_i) \exp(-\varphi(\bar{T}, T_i)r^*)$, the receiver swaption price is

$$\text{RecSwaption}(t, \bar{T}, T, K) = \sum_{i=1}^N c_i V_c^{\text{CIR}++}(t, \bar{T}, T_i, K_i), \quad (4.60)$$

and the corresponding payer swaption price is

$$\text{PaySwaption}(t, \bar{T}, T, K) = \sum_{i=1}^N c_i V_p^{\text{CIR}++}(t, \bar{T}, T_i, K_i), \quad (4.61)$$

where $\Psi(\cdot)$ is from (4.51).

4.5 Remark

The CIR++ model will be used in the following chapter to model the intensity of a default prone entity. The analytical tractability will help during calibration. The model will further be used to model the short rate, under which the options required to price a CVA/BCVA will be priced.

Chapter 5

Credit Risk Modelling

The quantification of counterparty credit risk entails the modelling of the credit risk of that counterparty. At the heart of credit risk modelling are default probabilities. Default probabilities are an estimate of the likelihood of default of a particular party in a derivatives deal. Associated with default probabilities are survival probabilities which quantify the likelihood of a party to a derivatives deal being able to honor their financial obligations up to a certain time in the future. Giesecke [44] outlines the following that should be specified in order to estimate default probabilities:

- A model for the investor uncertainty.
- A model of the available information and its evolution.
- A model definition of the default event.

The modelling of credit risk is a subject that has taken two approaches, the structural and the intensity/reduced form approach¹.

5.1 Probability Framework

We repeat our probability space used to model the economy as a reminder. We are in the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. An equivalent martingale measure $\tilde{\mathbb{P}}$ is assumed. Under $\tilde{\mathbb{P}}$, the discounted price processes of all trade-able securities are martingales. The general filtration $\mathcal{F}$ is such that, $\mathcal{G}_t \subseteq \mathcal{F}_t$: $= \mathcal{G}_t \vee \mathcal{D}_t$, where $\mathcal{D}_t$ and $\mathcal{G}_t$ are right continuous sub-filtrations generated by default events and market observable quantities respectively. Default times are denoted by τ_i for an entity i . The stochastic

¹Some scholars have considered a hybrid of the two approaches, these models are known as reconciliatory models.

discount factor is given by

$$D(t, T) = \exp \left(- \int_t^T r_s \, ds \right), \quad (5.1)$$

with the discount zero bond price at time t with maturity T denoted by $B(t, T)$. Again, the existence of the equivalent martingale measure rules out any arbitrage opportunities.

5.2 Structural Approach

The structural approach assumes knowledge of the capital structure of the entity under scrutiny. In this setting the modeler has a managerial view of the entity in terms of access to information that is contained in the balance sheet, which includes assets and liabilities of that entity [57]. A model specification for the evolution of the value of the firm is usually geometric Brownian motion (GBM). Economic arguments are used to model the financial health of the entity. This implies that the default time is endogenously given by the model, and that it is a stopping time, denoted τ . The continuity of geometric Brownian motion implies that the default time is a predictable stopping time ².

This framework has been extended to deal with limitations of the classical approaches and the new models have been given the name *first passage time* models. This is because they model default time as a first passage time. They are still structural in nature because they assume knowledge of the capital structure of the firm. They have been extended to incorporate realistic features such as stochastic volatility and interest rates. A review of these is given by Bielecki and Rutkowski [6] and references contained therein.

A major criticism leveled at this approach has been its inability to predict credit spreads and, even with the inclusion of jumps in the firm value process, the framework lacks accuracy [27]. Another issue with this framework is the difficulty in calibrating the models due to the unobservable nature of the value of the firm and its volatility. The main advantage of this approach however, is that the economic arguments used are intuitive, and that the firm need not have issued a lot of debt in order for the default probabilities to be derived. This makes the approach more suitable in illiquid markets [8]. Below we describe some of the most prominent structural models that have formed the foundations for the extensions that have been proposed in the literature.

²The stopping time can be made more sophisticated by considering advanced processes such as those that incorporate jumps.

5.2.1 Classical Models

5.2.2 The Merton Model

The foundations of the structural approach can be traced back to Merton, who used the Black-Scholes European call option pricing formula to value corporate risky debt. Black and Scholes [36] look at the problem of pricing European contingent claims and ultimately determine the no-arbitrage price of a European call option. Merton transformed the problem of valuing credit risky instruments into a European call option valuation problem by assuming the capital structure of a firm to be composed of equity denoted E_t and a zero coupon bond issued by the firm with a face value D .

The fact that he worked in a European call option setting implies that there is only one future critical payment date, which is the maturity date of the zero coupon bond. At T , if the value of the firm, denoted V_T , is such that $V_T \geq D$, then the firm is able to fulfill its obligations and distribute $V_T - D$ to its equity-holders, and the firm does not default. If the opposite happens and $V_T < D$, the firm defaults and the bond-holders take over the firm and receive what is left (V_T) and the equity-holders receive nothing³. This implicitly makes all the assumptions of a complete market that are assumed in the Black-Scholes framework. These assumptions include no market friction, continuous time trading, non-existence of arbitrage opportunities, unrestricted borrowing and lending, and GBM for the value of the firm (V_t):

$$dV_t = \bar{\mu}V_t + \sigma_V V_t dW_t, \quad (5.2)$$

where the volatility σ_V , and the drift $\bar{\mu}$ are positive constants, where W_t is a Wiener process under $\mathbb{P}$. Let $r_s = r$ in (5.1) be a constant. Using Theorem 2.48, the value process under the equivalent martingale measure $\tilde{\mathbb{P}}$ can be written as

$$dV_t = rV_t + \sigma_V V_t d\tilde{W}_t, \quad (5.3)$$

where $\tilde{W}_t$ is the Wiener process under $\tilde{\mathbb{P}}$. The payoffs to equity-holders and bond-holders as already explained above, are

$$E_T = \max\{V_T - D, 0\} \quad (5.4)$$

and

$$Z(T, T) = V_T - E_T, \quad (5.5)$$

³Bond-holders are assumed to enjoy preference over equity-holders when it comes to default proceeds.

respectively. The value of equity can be obtained through the application of Proposition 2.46 which leads to the Black-Scholes formula for a call option,

$$E_t(V_t, \sigma_V, T - t) = V_t \mathcal{N}(d_1) - e^{-r(T-t)} D \mathcal{N}(d_2), \quad (5.6)$$

where $\mathcal{N}(\cdot)$ is the standard normal distribution function and parameters d_1, d_2 are defined as

$$d_1 = \frac{\ln\left(\frac{e^{r(T-t)} V_t}{D}\right) + \frac{1}{2} \sigma_V^2 (T - t)}{\sigma_V \sqrt{T - t}} \quad (5.7)$$

and

$$d_2 = d_1 - \sigma_V \sqrt{T - t}. \quad (5.8)$$

Default occurs when $V_T < D$ and from (5.6) we get that,

$$\mathbb{P}[V_T < D] = \mathcal{N}(d_2). \quad (5.9)$$

The obvious limitation of the model above is its direct implication that default can only happen at the maturity of the zero-coupon bond. This is unrealistic as it precludes the possibility of default happening earlier than T and explains the empirical observation that the short term default probabilities produced by the model are significantly lower than those actually observed. The Merton model was also extended by Shimko *et al* [73] to allow for stochastic interest rates. It has been compared to the original Merton model in the South African context by Smit *et al* [76] where they calculate credit spreads for South African bonds.

5.2.3 First Passage Time Models

The fixed default time problem has been tackled by many scholars. Geske [43], considers the debt issued by the firm to be a coupon bearing bond rather than a zero coupon bond. This, however, introduces compound optionality into the problem and thus removes the simple nature of the model. Black and Cox [35] inspired what are now known as *first passage time* models. Their model extends Merton's model by allowing default to occur prior to T when the value of the firm touches a barrier level associated with the value of the bond. The default time is then a stopping time τ such that $\tau = \inf \{t \geq 0 : V_t \leq C(t)\}$. In their model, Black and Cox assume that the interest rate process is constant, i.e., $r_t = r$. The value process $\{V_t\}_{t \geq 0}$ is assumed to follow GBM again similar to the Merton model. Then Theorem 2.48

can be utilized to change the physical measure to the equivalent martingale measure; under which the value process becomes

$$dV_t = V_t((r - k)dt + \sigma_V dW_t), \quad (5.10)$$

where k is a constant dividend yield. The barrier level is time dependent and is given by $C(t) = Ke^{-\gamma(T-t)}$ so that $C(t)$ satisfies

$$dC(t) = \gamma C(t)dt, \quad (5.11)$$

with

$$C(0) = Ke^{-\gamma T}, \quad (5.12)$$

where γ is a constant that represents the rate at which the barrier grows. To obtain the desired result it is necessary to work with log prices, thus converting the GBM to arithmetic Brownian motion which obeys the reflection principle. It is then possible to derive the distribution of the first passage time. The default probabilities [35] in this setting are then given by,

$$\tilde{\mathbb{P}}\{\tau \leq s \mid \mathcal{F}_t\} = \mathcal{N}\left(\frac{\ln(\frac{C(t)}{V_t}) - \mu(s-t)}{\sigma_V \sqrt{s-t}}\right) + \left(\frac{C(t)}{V_t}\right)^{2a} \mathcal{N}\left(\frac{\ln(\frac{C(t)}{V_t}) + \mu(s-t)}{\sigma_V \sqrt{s-t}}\right), \quad (5.13)$$

where

$$\mu = r - k - \gamma - \frac{1}{2}\sigma_V^2, \quad (5.14)$$

$$a = \frac{r - k - \gamma - \frac{1}{2}\sigma_V^2}{\sigma_V^2}. \quad (5.15)$$

If $\gamma = r$ then the barrier function is the discounted value of K in the given period. Basically, first passage time models utilize the well known techniques of pricing barrier options in credit risk. For example, assuming a constant barrier ($C(t) = C$) and analyzing the payoff of a down and out binary option (DOBO) with maturity T , it is easy to transform the pricing of the option to credit risk modelling⁴. The contract pays 1 unit of currency if the underlying never touches the fixed barrier B and 0 otherwise. If we denote the first time the underlying touches the barrier by τ , then using risk neutral pricing at time $t = 0$, the price of the option is given as follows,

$$\text{DOBO}(0, T) = \mathbb{E}^{\tilde{\mathbb{P}}} [\mathbb{D}(0, T) \mathbb{I}_{\tau > T} \mid \mathcal{F}_0], \quad (5.16)$$

⁴Equity default swaps are approximated by binary barrier options

and assuming deterministic interest rates we get,

$$\text{DOBO}(0, T) = B(0, T) \mathbb{E}^{\tilde{\mathbb{P}}} [\mathbb{I}_{\tau > T} \mid \mathcal{F}_0] = B(0, T) \tilde{\mathbb{P}}[\tau > T]. \quad (5.17)$$

Using various techniques in stochastic calculus to obtain the distribution of τ and by employing the reflection principle of arithmetic Brownian motion, it can be shown that

$$\tilde{\mathbb{P}}[\tau > T] = \mathcal{N}\left(\frac{\ln(\frac{V_0}{C}) + (\mu - \sigma_V^2/2)T}{\sigma_V \sqrt{T}}\right) - \left(\frac{V_0}{C}\right)^l \mathcal{N}\left(\frac{\ln(\frac{C}{V_0}) + (\mu - \sigma_V^2/2)T}{\sigma_V \sqrt{T}}\right), \quad (5.18)$$

where $l = 1 - \frac{2\mu}{\sigma^2}$ and $\mu = r - k$.

These models have been extended by using different barrier functions and also incorporating realistic features such as stochastic volatility and interest rates see, [35, 6, 60].

5.2.4 Remark

The difficulty with structural models is the inability to observe the main state variable V_t which makes the estimation of σ_V nearly impossible, rendering the calibration exercise a very difficult task. The value of a firm is very hard to estimate because of the complexity of the asset structures and the fact that multiple private loans may be issued. This is further compounded by the fact that information on the changes in the structures is only reported periodically. When the firm is listed one may approximate the volatility using the share price as a proxy. When the firm is not listed however, calibrating the volatility would involve using proxy firms, which is obviously inaccurate. In first passage time models the calibration exercise is further complicated by the barrier itself which is non-observable. While economic justifications exist such as that, the barrier describes a safety covenant between the firm and its shareholders below which the performance is considered unacceptable, there is no direct way of observing the barrier. In general, while structural approaches are appealing, they fail to forecast credit spreads, especially in the short term. They have thus found most use in illiquid markets where credit derivatives markets are immature.

The graphs in Figure 5.1, illustrate the basic ideas behind the original Merton model and first passage time models, it also illustrates how they differ in the way they characterize default. Two companies γ and ζ , characterized by their value and paths representing their value through time are shown in the same figure. The barrier level is chosen to be $B = 0.6$ for both and the companies are assumed to have issued a 0.8 notional 1 year zero coupon bond. Under the Merton model, both companies would have survived up to maturity of the bond, while under the

first passage time approach, only company γ would have survived with company ζ defaulting at time $\tau = 0.283$.

Having looked at the structural framework, we now look at an alternative framework that is default intensity based.

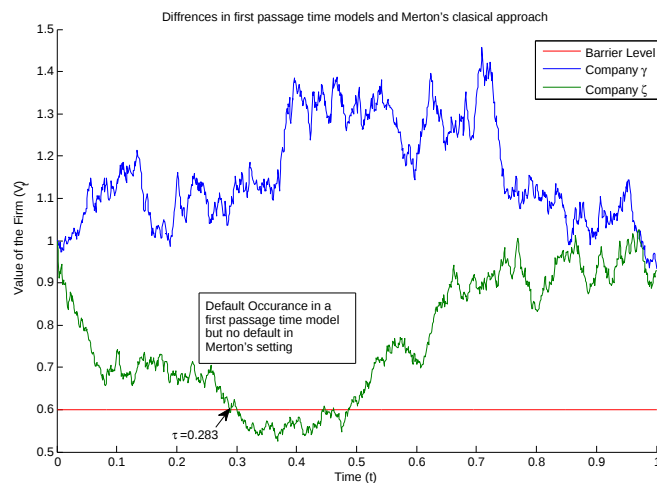


Figure 5.1: *Shows how both companies survive under Merton's model while only one survives under the first passage time approach.*

5.3 An Intensity Based Framework

This framework assumes that the information available to the credit modeler is that which is available to the market [57]. This implies that the capital structure of the firm is non-observable but that the debt issued by the particular firm is observable, as is usually the case. The inability to observe the capital structure implies that market participants have no way of telling if default is imminent, and so in this

framework default comes as a “total surprise”. Formally, under this framework, the default time could be modelled as a totally inaccessible stopping time and is usually the first jump of a counting process. For our purposes, the counting process is a Poisson process. Our primary interest in the Poisson process is to model default as the first event counted by the process. The arrival time of the default event, τ , is the default time. So formally we wish to describe default as follows,

$$N(t) = \begin{cases} 1 & \text{if } \tau \leq t \\ 0 & \text{otherwise.} \end{cases} \quad (5.19)$$

Each event/jump in a Poisson process arrives with an intensity λ . The intensity characterizes the process and there are three ways of specifying it. It can be left as a constant, in which case the process is called *time-homogeneous*. This approach leads to a gross simplification of the modelling process. In this case, results presented in Chapter 2, particularly Lemma 2.50, can be used to show that

$$\tilde{\mathbb{P}}\{N(t) = 0\} = \tilde{\mathbb{P}}\{S_1 > t\} = \tilde{\mathbb{P}}\{\tau > t\} = e^{-\lambda t}. \quad (5.20)$$

This means that the probability that the default event has not occurred up to time t is $e^{-\lambda t}$ and the corresponding probability that the default event would have occurred by time t is $(1 - e^{-\lambda t})$.

It is not realistic to assume that the arrival rate of the events will stay constant for all times. A simple generalization would be specifying the intensity as a deterministic, positive and piecewise continuous function, i.e., $\lambda = \lambda(t)$. In this setting the Poisson process is referred to as being *time-inhomogeneous*. The time variability in the intensity changes the process slightly. While the increments remain independent, they are no longer identically distributed. Similarly, since we are interested in the first jump of the Poisson process, we have from Lemma 2.51, through (2.40), that

$$\tilde{\mathbb{P}}\{N(t) = 0\} = \tilde{\mathbb{P}}\{S_1 > t\} = \tilde{\mathbb{P}}\{\tau > t\} = e^{-\int_0^t \lambda(s) ds}. \quad (5.21)$$

The function in the exponential, $\Gamma(t) = \int_0^t \lambda(s) ds$, is called the cumulative intensity or hazard process and has been well studied in the actuarial sciences. Houweling and Vorst [50], in a detailed empirical study of credit models consider the cumulative hazard function $\Gamma(t)$ to be approximated as

$$\Gamma(t) = \sum_{i=1}^d \lambda_i (T - t)^i,$$

where d is the degree of the polynomial. In their empirical investigation they approximate it to be linear, quadratic and cubic ($d = 1, 2$ and 3). They conclude that

the effects of the various polynomials are dependent on the credit class of the entity being investigated.

The last and most general approach to modelling the intensity is to specify it as a stochastic process. The Poisson process is then referred to as a *Cox process*. This allows more flexibility in terms of modelling but introduces incompleteness in the market. To properly define the default intensity, we must have that it is adapted to the filtration $\mathcal{F}_t$. In the case of a non-stochastic intensity, the hazard process is defined as

$$\Gamma_t = \int_0^t \lambda_s ds.$$

The probability that the default event has not occurred up to time t is

$$\tilde{\mathbb{P}}\{\tau > t\} = \mathbb{E}^{\tilde{\mathbb{P}}}[e^{-\int_0^t \lambda_s ds} | \mathcal{F}_t] = \mathbb{E}^{\tilde{\mathbb{P}}}[e^{-\Gamma_t} | \mathcal{F}_t]. \quad (5.22)$$

An investigation into the relationship between the discount factor function and the survival probabilities which are given respectively as

$$B(0, t) = \mathbb{E}^{\tilde{\mathbb{P}}} \left[\exp \left(- \int_0^t r_s ds \right) \right]$$

and

$$\tilde{\mathbb{P}}\{\tau > t | \mathcal{F}_t\} = \mathbb{E}^{\tilde{\mathbb{P}}} \left[\exp \left(- \int_0^t \lambda_s ds \right) \right],$$

suggests that, if λ_t is to be specified as a stochastic process, term structure models for the short rate would be useful. Notable good properties of the intensity λ_t and r_t are stated in [71, 34] as follows,

- both should be stochastic,
- the dynamics of both are correlated and
- the processes should be positive at all times.

In this dissertation the intensity of default of a default prone entity will be modelled using a CIR++ process.

5.4 Calibration of Credit Models

To calibrate credit models to the market, *credit default swaps* are the preferred calibration instruments. A credit default swap has become a benchmark instrument for assessing market perception of default risk on entities. Their liquidity and their availability on maturities up to at least 10 years indicates their representation of the credit markets. To perform a calibration, a market model for CDS prices is needed.

5.4.1 Credit Default Swaps

A credit default swap is a contract between two parties, the protection buyer and the protection seller. The protection buyer is protected from losses they may face should a reference debt issuer default on their bonds. In return, the protection buyer has to pay a premium to the seller, whose obligation would be to compensate the buyer on default of the debt issuer. The default event should be specifically defined and such a definition is included in the price of the instrument. The ISDA has, however, standardized the instrument, thus removing most of the ambiguity with respect to technical meaning of the clauses contained in such contracts. The pricing of the derivative is the process of determining the spread that makes it value to zero at initiation⁵. The spread is the percentage of the notional that the buyer would have to pay at pre-determined payment dates⁶.

A key observation is that holding a risky bond and a CDS on the issuer of the debt should amount to holding a risk-less bond. This amounts to assuming that the credit risk of the issuer is the only factor that contributes to the credit spread. With this assumption the fair spread should then be the spread between the risky bond and a corresponding risk-less bond⁷. This logic, however, breaks down due to the existence of market frictions and the flexibility that a credit default swap introduces, such as the cheapest to deliver option in physical settlements⁸ [22]. The difference between the bond spread and the CDS spread is known as the default swap basis. This basis will narrow and widen depending on a number of factors. The reasons for widening include:

1. the flexibility in terms of settlement, such as the ability to deliver the cheapest bonds upon default,
2. the issuance of new bonds could lead to high demand for CDSs for hedging purposes and
3. speculators shorting the CDS in the case of a downgrade.

The narrowing of the default swap basis could be caused by a number of factors, amongst them are:

⁵ Obviously a different spread could be negotiated with an upfront fee, this is not as popular as par spreads.

⁶ Which are normally quarterly.

⁷ Risk-less in a theoretical sense. Governments are, however, not riskless. So in a certain sense, the notion of a risk-free bond is hypothetical.

⁸ Physical settlement is when the bond belonging to the defaulted entity is delivered to the protection seller and the protection seller assumes ownership. The seller of the CDS will then pay the notional back.

1. the existence of greater counter-party risk to the buyer of the CDS, thus necessitating that the buyer be compensated and
2. an increased supply of more sophisticated products for hedging, and hence greater demand for the risky bond.

5.4.2 A Market Model

As with most swap contracts the quoted price is that which makes the value of the swap zero at inception. We denote the quoted spread by $S(t_0, T)$, which is the spread at time t for a CDS maturing at time T . To illustrate the mechanics of the contract, we assume two counterparties, A and B. Party A has exposure from a party C, more precisely A has bought a bond from C and fears the risk of C defaulting on future coupons. Party A bought a CDS from B at time t_0 and became the premium payer while B became the default payer. A is then responsible for paying a premium on the remaining fixed dates $T_1, T_2, \dots, T_N = T$ on condition that C has not defaulted before such a payment date. It is also assumed that t is the valuation date immediately after the previous premium date T_0 (note, we disregard any prior payment dates between t_0 and T_0 in the calculations that follow, and we set $T_0 = t_0$ in the case where there are no payments prior to t). Let $\alpha_n = (T_n - T_{n-1})/365$, be the year fraction⁹. If there is default, i.e., if C defaults, then B has to pay the value lost on the bond to A taking into account what has been recovered. The loss on the bond when the recovery rate is R_τ^C is given by $\text{LGD}_C = (1 - R_\tau^C)$ (*Loss Given Default*). We assume that interest rates are deterministic, i.e., in (5.1) $r_t = r(t)$ and $B(t, T)$ is the discount bond.

CDSs are priced analogously to interest rate swaps [13, 10, 18]. The full CDS discounted payoff from the point of view of the default leg is thus,

$$\begin{aligned}
 \text{PayOff}(t) &= \sum_{i=1}^N B(t, T_i) \alpha_i S(t_0, T) \mathbb{I}_{\{\tau \geq T_i\}} + B(t, \tau) (\tau - T_\tau) S(t_0, T) \mathbb{I}_{\{t < \tau < T_N\}} \\
 &\quad - B(t, \tau) \text{LGD}_C \mathbb{I}_{\{t < \tau \leq T_N\}} \\
 &= \text{PremiumLeg}_p(t, T) + \text{PremiumAccrued}_p(t, T) - \text{DefaultLeg}_p(t, T),
 \end{aligned} \tag{5.23}$$

where T_τ is the payment date immediately prior to the default time τ ($T_\tau = t_0$ in the case where there have been no prior payments).

Earlier, we defined a general filtration $\mathcal{F}_t$, which contained both default and market observable information, and the filtration $\mathcal{G}_t$, which only has market observable information. While $\mathcal{F}_t$ contains more information, it might, however, be more con-

⁹There are multiple day count conventions for different markets.

venient to condition the expectations on market observable information only ($\mathcal{G}_t$). Below is a result that relates expectations conditioned on different filtrations.

Lemma 5.1 (Filtration changing). *Given that $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{D}_t$, such that*

$$\mathcal{D}_t = \sigma(\{\tau < u\}, u \leq t)$$

and $\mathcal{G}_t$ contains all market observable information, we have that

$$\mathbb{E}^{\tilde{\mathbb{P}}} [\mathbb{I}_{\{\tau > T\}} X \mid \mathcal{F}_t] = \frac{\mathbb{I}_{\{\tau > t\}}}{\tilde{\mathbb{P}}\{\tau > t \mid \mathcal{G}_t\}} \mathbb{E}^{\tilde{\mathbb{P}}} [\mathbb{I}_{\{\tau > T\}} X \mid \mathcal{G}_t]. \quad (5.24)$$

Proof. For the proof refer to [6] and [18, p.777]. $\square$

Using Proposition 2.46 in Chapter 2 and the fact that conditional expectations are linear with respect to the same filtration, the value of a CDS to the default leg, which we denote by $\text{CDS}(t, T, S)$, is given by

$$\begin{aligned} \text{CDS}(t, T, S) &= \frac{\mathbb{I}_{\{\tau > t\}}}{\tilde{\mathbb{P}}\{\tau > t \mid \mathcal{G}_t\}} \{ \mathbb{E}^{\tilde{\mathbb{P}}} [\text{PremiumAccrued}_p(t, T) \mid \mathcal{G}_t] \\ &\quad + \mathbb{E}^{\tilde{\mathbb{P}}} [\text{PremiumLeg}_p(t, T) \mid \mathcal{G}_t] \\ &\quad - \mathbb{E}^{\tilde{\mathbb{P}}} [\text{DefaultLeg}_p(t, T) \mid \mathcal{G}_t] \}. \end{aligned} \quad (5.25)$$

Clearly, counterparty C would not have defaulted at or prior to t , and thus the fraction multiplying the three terms (5.25) is one.

Valuing the Premium Side

We will now price the premium side of the contract which consists of the premium accrued and the premium leg. We begin with the premium leg as follows,

$$\text{PremiumLeg}(t, T) = \mathbb{E}^{\tilde{\mathbb{P}}} [\text{PremiumLeg}_p(t, T) \mid \mathcal{G}_t] \quad (5.26)$$

$$= S(t_0, T) \sum_{i=1}^N \alpha_i B(t, T_i) \tilde{\mathbb{P}}(\tau > T_i). \quad (5.27)$$

There is also the period between the premium dates which may include a default event. Consider the situation where the reference entity defaults between T_{i-1} and T_i . The premium leg has to pay for being protected from T_{i-1} to the default date τ . Thus the premium leg will need to pay an accrued premium given by,

$$\begin{aligned} \text{PremiumAccrued}(t, T) &= \mathbb{E}^{\tilde{\mathbb{P}}} [\text{PremiumAccrued}_p(t, T) \mid \mathcal{G}_t] \\ &= S(t_0, T) \sum_{i=1}^N \frac{\alpha_i}{2} B(t, T_i) \mathbb{E}^{\tilde{\mathbb{P}}} [\mathbb{I}_{\{T_{i-1} < \tau \leq T_i\}}]. \end{aligned} \quad (5.28)$$

Since $\mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\{T_{i-1} < \tau \leq T_i\}}] = \tilde{\mathbb{P}}(\tau > T_{i-1}) - \tilde{\mathbb{P}}(\tau > T_i)$, we get that the premium accrued is given by,

$$\text{PremiumAccrued}(t, T) = S(t_0, T) \sum_{i=1}^N \frac{\alpha_i}{2} B(t, T_i) (\tilde{\mathbb{P}}(\tau > T_{i-1}) - \tilde{\mathbb{P}}(\tau > T_i)). \quad (5.29)$$

We obtained (5.28) by assuming that this will happen on average halfway to the next premium date and that the probability of default between two premium dates, T_{n-1} and T_n , is the cumulative probability of survival up to the first premium date T_{n-1} less the cumulative probability of surviving to the next premium date. When default happens at τ , the payment is postponed to the next payment date. The total value of the premium leg is the sum of PremiumLeg and PremiumAccrued.

Valuing the Default Leg

The only thing left now is to value the default leg of the CDS. The default leg is approximated by assuming that, if default occurs between the premium dates of the premium leg, the payment is postponed until the next premium date. The default leg is given by,

$$\text{DefaultLeg}(t, T) = \mathbb{E}^{\tilde{\mathbb{P}}} [\text{DefaultLeg}_p(t, T) \mid \mathcal{G}_t], \quad (5.30)$$

following from above, we have that

$$\text{DefaultLeg} = \text{LGD}_C \sum_{i=1}^N B(t, T_i) (\tilde{\mathbb{P}}(\tau > T_{i-1}) - \tilde{\mathbb{P}}(\tau > T_i)). \quad (5.31)$$

Given that $S(t_0, T)$ is the spread that makes the CDS value to zero at inception, we have that

$$\text{DefaultLeg} = \text{PremiumLeg} + \text{PremiumAccrued},$$

which implies that

$$S(t_0, T) = \frac{\text{LGD}_C \sum_{i=1}^N B(t, T_i) (\tilde{\mathbb{P}}(\tau > T_{i-1}) - \tilde{\mathbb{P}}(\tau > T_i))}{\sum_{i=1}^N \alpha_i B(t, T_i) (\frac{1}{2} (\tilde{\mathbb{P}}(\tau > T_{i-1}) + \tilde{\mathbb{P}}(\tau > T_i)))}. \quad (5.32)$$

It is worth noting that this model and its assumptions are those usually assumed when valuing a CDS but, in theory a continuous version would be deemed more suitable. We present one below.

5.4.3 Assuming Dependency: A CDS Model

The most important assumption made in the derivations above was that of independence, that is, correlation was zero between the risk factors. Below it is assumed

that the intensity process of the reference to the CDS, denoted by λ_t is stochastic and correlated to the short rate process r_t . Using the pay-off expression (5.23) and Lemma 5.1 to change filtrations, it can be shown that when there is correlation,

$$\begin{aligned} \text{CDS}(t, T, S) = & S(t_0, T) \sum_{i=1}^N \alpha_i \mathbb{E}^{\tilde{\mathbb{P}}} \left[\exp \left(- \int_t^{T_i} (\lambda_s + r_s) ds \right) \mid \mathcal{G}_t \right] \\ & + S(t_0, T) \int_t^T \mathbb{E}^{\tilde{\mathbb{P}}} \left[\exp \left(- \int_t^u (\lambda_s + r_s) ds \right) \lambda_u \mid \mathcal{G}_t \right] (u - T_u) du \\ & - \text{LGD}_C \int_t^T \mathbb{E}^{\tilde{\mathbb{P}}} \left[\exp \left(- \int_t^u (\lambda_s + r_s) ds \right) \lambda_u \mid \mathcal{G}_t \right] du. \end{aligned} \quad (5.33)$$

There is no closed form solution for (5.33) which makes calibrating to market data a numerically intensive exercise. Brigo and Mercurio [18, p.796], show experimentally that the effects of intensity-interest rate correlation is negligible in CDSs. Thus it might not be of any benefit to consider correlation while calibrating both the intensity and the interest rate process to CDS data and interest rate derivatives respectively.

5.4.4 Calibrating the Intensity in the Reduced Form Approach

The spread $S(t_0, T)$ is given by the market and changes according to the market's view of the default risk of the reference entity. Recalling Equation (5.22) for the survival probabilities and substituting it into (5.32) yields

$$S(t_0, T) = \frac{\text{LGD}_C \sum_{i=1}^N B(t, T_i) (\text{SP}_{i-1} - \text{SP}_i)}{\frac{1}{2} \sum_{i=1}^N \alpha_i B(t, T_i) (\text{SP}_{i-1} + \text{SP}_i)}, \quad (5.34)$$

where $\text{SP}_i = \exp\{-\Gamma(T_i)\}$. Solving for each SP_i would require an iterative procedure with the initial condition, $\text{SP}_0 = 100\%$.

In the case where the intensity is time dependent, i.e., $\lambda = \lambda(t)$ and assuming that it is constant between any two maturities, we have

$$\lambda_i = - \frac{\ln(\text{SP}_{i+1}) - \ln(\text{SP}_i)}{T_{i+1} - T_i}. \quad (5.35)$$

It is more difficult when the intensity is a stochastic process, i.e., $\lambda = \lambda_s$. Let us assume that the intensity follows a CIR++ process (c.f. (4.39)), that is,

$$d\lambda_t = \left[\frac{d\phi(t)}{dt} + k(\theta - (\lambda_t - \phi(t))) \right] dt + \sigma \sqrt{\lambda_t - \phi(t)} dW_t. \quad (5.36)$$

The Feller condition $2k\theta > \sigma^2$, ensures that the origin is inaccessible to the process, implying that the intensity is always positive. Under this process the T maturity

zero coupon bond at t is given by (4.51). Given the already noted similarity of zero coupon bond prices and survival probabilities, we have that

$$\begin{aligned} \text{SP}_n &= \exp \left\{ - \left(\int_0^{T_{n-1}} \lambda_s \, ds + \int_{T_{n-1}}^{T_n} \lambda_s \, ds \right) \right\} \\ &= \text{SP}_{n-1} B_\lambda^{\text{CIR}++}(T_{n-1}, T_n). \end{aligned} \quad (5.37)$$

Using (5.37), we may then calibrate the parameters k , x_0 , σ and θ required in (5.36). To calibrate the model, one can follow the following steps:

- The first step is to approximate the volatility σ . Given that CDS options are not available, it is thus impossible to get a risk neutral volatility. The volatility can, however, be approximated historically by calculating the historical standard deviation of CDS spread returns. We denote this historical volatility by σ_h . This historical volatility is further mapped to the CIR++ volatility σ , through $\sigma = \sigma_h \sqrt{\lambda_0}$.
- The next step is to approximate the spot intensity λ_0 . This is assumed to be the intensity implied by the shortest maturing CDS.
- The third step is then to solve the following global minimization problem:

$$\begin{aligned} \min_{k, \theta, x_0} \quad & \left\{ \int_0^T \phi(t)^2 \, dt \right\} \\ \text{subject to} \quad & 2k\theta \geq \sigma^2, \quad \sigma = \sigma_h \sqrt{\lambda_0} \\ \text{and} \quad & \frac{\text{SP}_n}{\text{SP}_{n-1}} = B^{\text{CIR}++}, \quad \forall n \in [0, \dots, N]. \end{aligned} \quad (5.38)$$

Minimising $\phi(t)^2 \, dt$ can be interpreted as minimising the difference between the CIR and CIR++ models and the integral can be evaluated using quadrature methods.

5.4.5 Calibrating to a First Passage Time Model

It was mentioned earlier in the section on structural models that the biggest problem experienced with this approach is determining V_t, B and the volatility σ_V . There have been many studies on the calibration of structural models using historical data employing techniques such as maximum likelihood methods or the assumption that the volatility is the same as that of the equity process for V_t . Brigo and Tarenghi [9], introduce a procedure whereby they calibrate the volatility σ_V so as to recover the current default probabilities inferred from credit default swaps. The method described in their paper inspires what follows below.

Recalling (5.18) for the survival probability, one notes that the equation depends on $\frac{V_t}{C}$ and never on V_t or C alone. Denote this ratio at T_i by $\Upsilon(T_i) = \Upsilon_i$. This implies that a plausible approximation for the ratio is required and not necessarily the exact value of V_t . Given the notation above, we can rewrite (5.18) as follows,

$$\begin{aligned}\tilde{\mathbb{P}}[\tau > T_i] &= \mathcal{N}\left(\frac{\log(\Upsilon_0) + (\mu - \sigma_{V_i}^2/2)T_i}{\sigma_{V_i}\sqrt{T_i}}\right) - \Upsilon_0^{l_i} \mathcal{N}\left(\frac{\log(\Upsilon_0^{-1}) + (\mu - \sigma_{V_i}^2/2)T_i}{\sigma_{V_i}\sqrt{T_i}}\right) \\ &= \text{SP}_i\end{aligned}$$

where

$$l_i = 1 - \frac{2\mu}{\sigma_{V_i}^2} \quad \text{and} \quad \mu = r - k. \quad (5.39)$$

Thus, using (5.32), we can obtain the survival probabilities SP_i . The calibration happens in two phases:

- Firstly, we estimate the first period volatility of V_t to be the implied volatility of the equity of the entity, if options are not available it would have to be estimated historically. This then allow us to estimate the ratio $\Upsilon_0 = \frac{V}{C}$.
- After estimating Υ_0 , we use it to obtain the term structure of the volatilities, $\sigma_V(T_i) = \sigma_{V_i}$, that recover the survival probabilities given the initial ratio Υ_0 . This would be an iterative exercise that requires numerical solvers.

Now that the term structure of volatility has been obtained, one can simulate from the value process of the entity, (5.10), in order to find the default times $\tau = \inf\{t \geq 0 : V_t \leq B\}$. Simulations are important because modelling counterparty credit risk requires a level of correlation between risk factors and counterparties. This is due to the fact that one needs to incorporate wrong and right way risk. Inducing correlation between processes can be done in a number of ways, some of these are described in the next section.

5.5 Introducing Default Correlation

Defaults are very rare events and often come as a complete surprise to the market. Empirical evidence has shown that defaults tend to happen during bad economic times which may affect certain industries worse than others. Exposure to the same bad economic conditions may lead to joint defaults of multiple firms. In the recent 2008 financial crisis, prior to having severe effect on banking in general, the automobile industry which depended heavily on debt, was the first to be affected. Coudert and Gexs [24] track CDS spreads and note that General Motors and Ford's CDS spreads almost tripled during the crisis. Lehman Brothers was a heavy dealer in the

credit derivatives markets and its default led to a domino effect across the industry. These cases lead to the natural conclusion that all credit models should be able to model default correlations amongst firms, otherwise they price the risk incorrectly.

There are various approaches to addressing correlation. In structural models, the simplest approach is to correlate the asset returns of the various firms in question. In the case where the asset value obeys geometric Brownian motion, this translates into correlating the driving Wiener processes. Hull and White [52] use the first passage time approach to produce a model where defaults are triggered by time-dependent barriers and allow the firms asset values to be correlated. In addition to the disadvantages of the structural approach already discussed, the calibration and ultimate use is computationally challenging.

Under the intensity based framework the simplest approach is to induce correlation in the stochastic dynamics of the intensities. Duffee [28] models the short rate using a two factor model and, after approximating the factors, he incorporates them into the stochastic dynamics of the intensity which includes correlation between the short rate and the default intensity of that firm. The class of models that induce correlation in this manner are referred to as *conditionally independent default* models due to the fact that the intensities are independent on condition that the factors that induce the correlation are not realized [34]. Disadvantages of this approach are that: the default correlations that can be reached are too low when compared with realized default correlations and it is also very hard to derive and analyze the resulting dependency structure [72]. Some scholars have defended this approach, arguing that the low default correlations could be due to the fact that the factors used in specifying the model are not sufficient to capture all the dynamics of the intensity and thus it is not the approach itself but rather the choice of factors that lead to the problem. Increasing the number of factors, leads to increased computational challenges when calibrating, which may result in the model being impractical. Another approach that tries to solve the low default correlation problem with reasonable success is the inclusion of jumps in the intensity processes. This has the advantage of allowing joint jumps in the intensities of firms and, through that, the probability of joint defaults increase. This approach was proposed by Duffie and Singleton [31], but the calibration of this class of models is very difficult.

Another approach is the *copula approach*. The advantage with this approach is that it is general enough to be used by both structural and intensity based models. The idea is that the marginal distributions of each firms default can be coupled to form a joint distribution of all the defaults. The theory behind these functions has already been presented in Chapter 2.

5.5.1 Correlation in the Structural Approach

In this section, the copula approach will be used to induce correlation between counterparties and risk factors. Let the asset value of two counterparties A and C under the risk neutral measure $\tilde{\mathbb{P}}$ follow GBM, given by,

$$dV_t^A = r_t V_t^A dt + \sigma_t^A dW_t^A \quad (5.40)$$

$$dV_t^C = r_t V_t^C dt + \sigma_t^C dW_t^C, \quad (5.41)$$

Let the short rate follow the CIR++ process (4.41). To induce correlation between A, B and r_t , a copula will be used. The three Itô processes will be correlated by correlating the driving Brownian motions.

There exist algorithms for simulating processes, such as the CIR++ process above, the popular ones are the Euler-Maruyama or the Milstein scheme. The existence of a steady state distribution for the CIR process implies that the process can be simulated by directly sampling from the non-central chi-squared distribution but this only suffices if the process is being simulated independently. Simulating the above GBMs requires only $\mathcal{N}(0, 1)$ random variates. Using a copula $\mathcal{C}$, the different marginals, which are two normals and the non-central chi-squared distribution, can be made to depend on each other. Making use of Sklar's theorem in Chapter 2, Theorem 2.58 and the associated Corollary 2.59, we have that the required C is given by,

$$C(u_1, u_2, u_3) = \mathcal{H}(\mathcal{N}^{-1}(u_1), \mathcal{N}^{-1}(u_2), \mathcal{X}_v^{2(-1)}(u_3, n)), \quad (5.42)$$

where $\mathcal{N}^{-1}(\cdot)$ is the inverse normal and $\mathcal{X}_v^{2(-1)}(\cdot, n)$ is the inverse non-central chi-squared distribution with v degrees of freedom and non-centrality parameter n . The choice of tri-variate distribution $\mathcal{H}$ will depend on the features we want to capture. As an example, if $\mathcal{H}$ is the tri-variate normal distribution, the copula C will lack tail dependence ($\lambda_u = \lambda_l = 0$) (see, Section 2.6.2). If tail dependence is important as it is in credit risk modelling, the more appropriated choice would be the Student-t copula which has tail dependence, (see, Section 2.6.2). Fitting the copula requires that the correlation matrix be estimated and in the case of the Student-t copula, the degrees of freedom would also need to be estimated. These may not be readily available and the only option may be to approximate it historically.

5.5.2 Correlation in Intensity Based Modelling

Consider two counterparties A and C where the intensity of default for both counterparties follows a CIR++ process such that,

$$\lambda_t^A = x_t^A + \phi_A^{\text{CIR}}(t) \quad (5.43)$$

and

$$\lambda_t^C = x_t^C + \phi_C^{\text{CIR}}(t), \quad (5.44)$$

where x_t^f is the CIR process part of the intensity of f and $\phi_f^{\text{CIR}}(\cdot)$ is the deterministic shift function. We also assume that the short rate follows the CIR++ process such that,

$$r_t = x_t^{\text{CIR}} + \phi^{\text{CIR}}(t). \quad (5.45)$$

A dependency structure can then be used to couple the above processes.

That would be one way of inducing correlation, which unfortunately leads to low correlation of the actual default times. To obtain realistically high levels of correlation, we can induce the correlation in the default times directly. The default times τ_A and τ_B are given as follows,

$$\tau_A = \Gamma_A^{-1}(\varepsilon_i), \quad \tau_C = \Gamma_C^{-1}(\varepsilon_j), \quad (5.46)$$

where

$$\Gamma_A(t) = \int_0^t \lambda_s^A ds, \quad \Gamma_C(t) = \int_0^t \lambda_s^C ds. \quad (5.47)$$

The correlation is induced by correlating the standard exponentials ε_i and ε_j .

In summary, correlation can be introduced through a copula, that couples the marginals of the intensities of A and B together with that of the short rate r_t . We can further introduce it by correlating the standard exponential variates ε_i and ε_j that are used to obtain the default times. However, it can also be assumed that the intensity processes are independent while introducing a copula on the marginal distributions of ε_i , ε_j and r_t . Using Sklar's theorem, Theorem 2.58 and Corollary 2.59, we have that the copula $\mathcal{C}$ is given by,

$$\mathcal{C}(u_1, u_2, u_3) = \mathcal{H}(\xi^{-1}(u_1), \xi^{-1}(u_2), \mathcal{X}_v^{2(-1)}(u_3, \lambda)), \quad (5.48)$$

where $\xi^{-1}(\cdot)$ is the inverse exponential distribution. Again the choice of $\mathcal{H}$ will be limited to the tri-variate Gaussian and tri-variate Student-t distributions.

5.6 Remark

In this chapter we reviewed two ways of modelling credit risk, namely the intensity based and structural approaches. These will be used in the next chapter in describing the distribution of the default times, which are the maturity times for the options necessary for a CVA/BCVA calculation.

Chapter 6

Credit Value Adjustments in Interest Rate Derivatives

In this chapter two interest rate derivatives will be introduced, a forward rate agreement (FRA) and an interest rate swap (IRS). Both unilateral and bilateral credit value adjustments are calculated. The results of the previous chapters, such as those on caplets, floorlets and swaptions will be required in conjunction with those on credit risk modelling. It will be shown that a CVA/BCVA on FRAs is formed by a series of either caplets or floorlets, while a CVA/BCVA on IRSs is formed by a series of either payer or receiver swaptions.

6.1 Probability Framework

We model the economy using the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ which we assign the equivalent martingale measure $\tilde{\mathbb{P}}$. Under $\tilde{\mathbb{P}}$, the discounted price processes of all trade-able securities are martingales. The general filtration $\mathcal{F}$ is such that, $\mathcal{G}_t \subseteq \mathcal{F}_t$: $= \mathcal{G}_t \vee \mathcal{D}_t$, where $\mathcal{D}_t$ and $\mathcal{G}_t$ are right continuous sub-filtrations generated by default events and market observable quantities respectively. The stochastic discount factor is given by,

$$D(t, T) = \exp \left(- \int_t^T r_s \, ds \right). \quad (6.1)$$

The discount bond at t maturing at T is denoted by $B(t, T)$. The existence of $\tilde{\mathbb{P}}$ rules out any arbitrage opportunities and unfair trading strategies in this economy.

6.2 Credit Value Adjustments in Forward Rate Agreements

Forward rate agreements (FRA) are the basic building block instruments in the interest rate derivatives market. They offer a mechanism for fixing a rate that is otherwise freely evolving. Counterparties to a FRA agree on a fixed rate K that is paid per unit notional at a future time T . The rate that the instrument fixes is referred to as the reference rate. Different markets have different reference rates, for example, in western markets a popular rate is the London Interbank Offered Rate (LIBOR) while in the South African markets the popular choice is the Johannesburg Interbank Agreed Rate (JIBAR). When on the long side of an FRA, the instrument hedges the volatility of the reference rate. In principle any market rate can be used. JIBAR is the benchmark rate South African banks use. The counterparties will exchange the fixed rate for the reference rate at time T per unit notional, implying that the payoff is $\eta(J(T, S) - K)$, with $\eta = 1$ for the long position and $\eta = -1$ for the short position. $J(T, S)$ is the reference rate fixing at time T and settling at time S where $S > T$. Prior to time T , it is a stochastic process that is bounded and adapted to the filtration $\mathcal{F}_t$.

The value of the FRA at time t , maturing at time T , that fixes the reference rate $J(T, S)$, between time T and time S is obtained through the application of Proposition 2.46, that is,

$$\begin{aligned} \text{FRA}(t, T, S, K) &= \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbf{D}(t, S)\alpha_{T,S}(J(T, S) - K) \mid \mathcal{F}_t] \\ &= \alpha_{T,S}B(t, S)[F(t, T, S) - K], \end{aligned} \quad (6.2)$$

where $\alpha_{T,S}$ is the year fraction between time between the two times and $F(t, T, S)$ is the forward rate from T to S at time t , given by

$$F(t, T, S) = \frac{1}{\alpha_{T,S}} \left(\frac{B(t, T)}{B(t, S)} - 1 \right). \quad (6.3)$$

6.2.1 Unilateral CVA

Let A and C be counterparties in a forward rate agreement. Assume A is default free¹ and C is default prone. Given that A is long the FRA, its value towards A , considering counterparty credit risk, is decomposed into the value of FRA assuming no counterparty credit risk less the unilateral CVA. The details of this decomposition are in Chapter 3, Section 3.5.3. The result (3.18) describes this decomposition and is utilized to derive the following results. Assuming the non-defaultable counterparty

¹It could be that A is of a much superior credit quality compared to C .

is long the underlying FRA, then the unilateral CVA is given by,

$$\text{CVA}(t) = \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\tau_C \leq T}(1 - R_{\tau_C}^{(C)})D(t, \tau_C)D(\tau_C, S)\alpha_{T,S}(F(\tau_C, T, S) - K)^+ | \mathcal{F}_t].$$

Assuming constant recovery, the right hand-side of the latter equation is given by,

$$\text{CVA}(t) = \text{LGD}_C \mathbb{E}^{\tilde{\mathbb{P}}} \left[\int_t^T \alpha_{T,S} D(t, s) D(s, S) (F(s, T, S) - K)^+ \mathbb{I}_{\{\tau_C = s\}} ds \right].$$

Using Fubini's theorem and assuming rates are independent of defaults (this condition will be relaxed to account for wrong-way risk when we use Monte Carlo methods), we have that

$$\begin{aligned} \text{CVA}(t) &= \text{LGD}_C \int_t^T \mathbb{E}^{\tilde{\mathbb{P}}}[\alpha_{T,S} D(t, s) D(s, S) (F(s, T, S) - K)^+] \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{I}_{\{\tau_C = s\}}] ds \quad (6.4) \\ &= \text{LGD}_C \int_t^T \hat{\text{Cl}}(t, s, T, S, R_s, K) d\tilde{\mathbb{P}}(\tau_C < s), \end{aligned}$$

where R_s is the rate of the FRA starting at s and resetting at T and maturing at S . Discretizing the integration region into m sub-intervals, results in the approximation

$$\text{CVA}(t) \approx \text{LGD}_C \sum_{i=1}^m \hat{\text{Cl}}(t, T_i, T, S, R_i, K) [\tilde{\mathbb{P}}(\tau_C > T_{i-1}) - \tilde{\mathbb{P}}(\tau_C > T_i)], \quad (6.5)$$

where $\hat{\text{Cl}}(t, s, T, S, R_s, K)$ denotes the value of a special caplet that differs from a normal caplet in that it matures prior to the reset date of the underlying forward. It denotes the value at time t of a caplet maturing at s with the underlying forward start forward rate R_s , resetting at T and maturing at S , with strike K . Assuming that the default event will only occur at the payment date T (*FRAs reset and pay at the same time in the South African market*), then we have that

$$\text{CVA}(t) \approx \text{LGD}_C \text{Cl}(t, T, S, K) \tilde{\mathbb{P}}(\tau_C < T). \quad (6.6)$$

6.2.2 Bilateral CVA

Let A and C be counterparties in an FRA such that both entities are prone to default. We also assume that A is long the FRA. As we showed previously, in Proposition 3.3, a bilateral CVA from A 's point of view can be decomposed into the difference of a unilateral credit value adjustment and a debt value adjustment. Recalling (3.23), we have that

$$\text{BCVA}(t) = \text{UCVA}(t) - \text{DVA}(t).$$

Evaluating the UCVA component gives

$$\begin{aligned} \text{UCVA}(t) &= \text{LGD}_C \mathbb{E}^{\tilde{\mathbb{P}}} \left[\mathbb{I}_{\{\tau_C < T\} \cap \{\tau_C < \tau_A\}} D(t, \tau_C) D(\tau_C, S) (F(\tau_C, T, S) - K)^+ \right] \quad (6.7) \\ &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[\int_t^T \mathbb{I}_{\{\tau_C = s\} \cap \{\tau_C < \tau_A\}} D(t, s) D(s, S) \alpha_{T,S} (F(s, T, S) - K)^+ ds \mid \mathcal{F}_t \right] \\ &\quad \times \text{LGD}_C \end{aligned}$$

and using Fubini's theorem gives

$$\begin{aligned} \text{UCVA}(t) &= \int_t^T \mathbb{E}^{\tilde{\mathbb{P}}} \left[\mathbb{I}_{\{\tau_C = s\} \cap \{\tau_C < \tau_A\}} D(t, s) D(s, S) \alpha_{T,S} (F(s, T, S) - K)^+ \mid \mathcal{F}_t \right] ds \\ &\quad \times \text{LGD}_C. \end{aligned}$$

Assuming default events are independent of rates (this assumption will be relaxed when Monte Carlo is considered), the following is obtained

$$\begin{aligned} \text{UCVA}(t) &= \text{LGD}_C \int_t^T \mathbb{E}^{\tilde{\mathbb{P}}} \left[\mathbb{I}_{\{\tau_C = s\} \cap \{\tau_C < \tau_A\}} \mid \mathcal{F}_t \right] \\ &\quad \times \mathbb{E}^{\tilde{\mathbb{P}}} \left[D(t, s) D(s, S) \alpha_{T,S} (F(s, T, S) - K)^+ \mid \mathcal{F}_t \right] ds \\ &= \text{LGD}_C \int_t^T \tilde{\mathbb{P}}(\tau_C \leq s, \tau_A > s) \hat{\text{Cl}}(t, s, T, S, R_s, K) ds. \quad (6.8) \end{aligned}$$

Discretizing the integration region into m intervals and postponing default to the next discretization point, leads to the approximation

$$\begin{aligned} \text{UCVA}(t) &\approx \text{LGD}_C \sum_{i=1}^m \left[\tilde{\mathbb{P}}(\tau_C \leq T_i, \tau_A > T_i) \hat{\text{Cl}}(t, T_i, T, S, R_i, K) \right] \\ &\approx \text{LGD}_C \sum_{i=1}^m \left(\tilde{\mathbb{P}}(\tau_C > T_{i-1}, \tau_A > T_i) - \tilde{\mathbb{P}}(\tau_C > T_i, \tau_A > T_i) \right) \\ &\quad \times \hat{\text{Cl}}(t, T_i, T, S, R_i, K), \quad (6.9) \end{aligned}$$

where $\hat{\text{Cl}}(t, s, T, S, R_i, K)$ is the special caplet at time t maturing at time s , which may be before T , K is the strike, R_i is the fixed rate of the FRA that starts at s and resets at time T while maturing at S . The probabilities $\tilde{\mathbb{P}}(\tau_C > T_{i-1}, \tau_A > T_i)$, will be introduced through a dependency structure such as a copula (see, Section 2.6). The DVA(t) can be estimated similarly by

$$\begin{aligned} \text{DVA}(t) &= \text{LGD}_A \mathbb{E}^{\tilde{\mathbb{P}}} \left[\mathbb{I}_{\{\tau_A < T\} \cap \{\tau_A < \tau_C\}} D(t, \tau_A) D(\tau_A, S) (K - F(\tau_A, T, S))^+ \right] \quad (6.10) \\ &= \int_t^T \tilde{\mathbb{P}}(\tau_A \leq s, \tau_C > s) \hat{\text{Fl}}(t, s, T, S, R_s, K) ds \\ &\approx \sum_{i=1}^m \left[\left(\tilde{\mathbb{P}}(\tau_A > T_{i-1}, \tau_C > T_i) - \tilde{\mathbb{P}}(\tau_A > T_i, \tau_C > T_i) \right) \right. \\ &\quad \left. \hat{\text{Fl}}(t, T_i, T, S, R_i, K) \right] \times \text{LGD}_A. \quad (6.11) \end{aligned}$$

6.2.3 Bilateral CVA with Simultaneous Defaults

Allowing simultaneous defaults, as was shown in Chapter 3, Proposition 3.4 introduces a new term which was referred to as the simultaneous default value adjustment (SDVA). For notational convenience we introduce the simultaneous default time τ_{sim} , which will denote the time when $\tau_A = \tau_C$. The only complexity added is that we now require the distribution of τ_{sim} . In Chapter 3, it was shown in (3.37) that the BCVA with the possibility of simultaneous defaults can be decomposed as follows,

$$\widehat{V}(T, t) = V(t, T) - \text{UCVA}(t) + \text{DVA}(t) + \text{SDVA}(t). \quad (6.12)$$

Following similar steps as above leads to an equation analogous to (6.8) for the UCVA component of (6.12) in Proposition 3.4, that is,

$$\text{UCVA}(t) = \text{LGD}_C \int_t^T \tilde{\mathbb{P}}(\tau_C \leq s, \tau_A > s, \tau_{sim} > s) \widehat{\text{Cl}}(t, s, T, S, R_s, K) ds, \quad (6.13)$$

making similar assumptions as above we obtain

$$\begin{aligned} \text{UCVA}(t) &\approx \text{LGD}_C \sum_{i=1}^m \tilde{\mathbb{P}}(\tau_C \leq T_i, \tau_A > T_i, \tau_{sim} > T_i) \widehat{\text{Cl}}(t, T_i, T, S, R_i, K) \\ &\approx \sum_{i=1}^m \left(\tilde{\mathbb{P}}(\tau_C > T_{i-1}, \tau_A > T_i, \tau_{sim} > T_i) - \tilde{\mathbb{P}}(\tau_C > T_i, \tau_A > T_i, \tau_{sim} > T_i) \right) \\ &\quad \times \widehat{\text{Cl}}(t, T_i, T, R_i, K) \text{LGD}_C. \end{aligned} \quad (6.14)$$

Similarly, the DVA term can also be found as

$$\begin{aligned} \text{DVA}(t) &= \text{LGD}_A \int_t^T \tilde{\mathbb{P}}(\tau_A \leq s, \tau_C > s, \tau_{sim} > s) \widehat{\text{Fl}}(t, s, T, S, R_s, K) ds \\ &\approx \sum_{i=1}^m \left(\tilde{\mathbb{P}}(\tau_A > T_{i-1}, \tau_C > T_i, \tau_{sim} > T_i) - \tilde{\mathbb{P}}(\tau_A > T_i, \tau_C > T_i, \tau_{sim} > T_i) \right) \\ &\quad \times \widehat{\text{Fl}}(t, T_i, T, S, R_i, K) \text{LGD}_A. \end{aligned} \quad (6.15)$$

Following similar steps, the SDVA is obtained as follows,

$$\begin{aligned} \text{SDVA}(t) &= \int_t^T \tilde{\mathbb{P}}(\tau_{sim} \leq s) \left\{ R_{\tau_{sim}}^C \widehat{\text{Cl}}(t, s, T, S, R_s, K) \right. \\ &\quad \left. - R_{\tau_{sim}}^A \widehat{\text{Fl}}(t, s, T, S, R_s, K) - \alpha_{T,SB}(t, s) B(s, S) (F(s, T, S) - K(t)) \right\} ds \\ &\approx \sum_{i=1}^m \left(\tilde{\mathbb{P}}(\tau_{sim} > T_{i-1}) - \tilde{\mathbb{P}}(\tau_{sim} > T_i) \right) \left\{ R_{\tau_{sim}}^C \widehat{\text{Cl}}(t, T_i, T, S, R_i, K) \right. \\ &\quad \left. - R_{\tau_{sim}}^A \widehat{\text{Fl}}(t, T_i, T, S, R_i, K) - \alpha_{T,SB}(t, S) (F(T_i, T, S) - K) \right\}. \end{aligned}$$

6.3 CVA in Interest Rate Swaps

Interest rate swaps (IRS) are probably the most popular instrument of all interest rate derivatives. Their simplicity and usefulness in hedging interest rate exposure make them by far the most traded interest rate derivative. The mechanics of the instrument are as follows:

Two counterparties agree on a fixed rate K per unit notional at time t_0 that will be paid at specified intervals. One of the parties will pay the fixed rate (long position) and the other will pay a floating rate (short position), similar to forward rate agreements. An IRS is thus a series of FRAs with the same fixed rate, with the last FRA maturing at T . Payment of these FRAs is now in arrears. Let the floating rate have the following reset dates, $T_0, T_1, \dots, T_{n-1}$, where T_0 is the reset date immediately prior to t and $T_1, \dots, T_{n-1}$ are the reset dates remaining in the contract. The corresponding payment dates are $T_1, T_2, \dots, T_n = T$. The value of the IRS at time $T_0 \leq t \leq T_1$ to the long side is

$$V_{\text{IRS}}(t, T_0, T, K) = \sum_{i=1}^n \alpha_{i-1,i} B(t, T_i) (F(t, T_{i-1}, T_i) - K). \quad (6.16)$$

Note that we have changed our notation slightly, by letting $\alpha_{i-1,i}$ be the year fraction between T_{i-1} and T_i .

6.3.1 Unilateral CVA

Assume that A and C are counterparties in an IRS, with C taking the short position. We assume A to be the default free party while C is assumed default prone. Again, a derivative with CCR embedded can be decomposed into the value of the same derivative, assuming no default, less the unilateral CVA. This was neatly summarized by Proposition 3.2, which led to (3.18), and results in

$$\begin{aligned} \text{UCVA}(t) &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[\mathbb{I}_{\{\tau_C \leq T\}} D(t, \tau_C) (1 - R_{\tau_C}^{(C)}) V_{\text{IRS}}(t, \tau_C, T_{\tau_C}, T, K)^+ \right] \\ &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[\int_t^T D(t, s) (1 - R_{\tau_C}^{(C)}) V_{\text{IRS}}(s, T_s, T, K)^+ \mathbb{I}_{\{\tau_C = s\}} ds \right]. \end{aligned} \quad (6.17)$$

(T_s denotes the next reset date from s if T_s is not a reset date) Assuming constant recovery rates and making use of Fubini's theorem, the following is obtained

$$\text{UCVA}(t) = \text{LGD}_C(t) \int_t^T \mathbb{E}^{\tilde{\mathbb{P}}} [D(t, s) V_{\text{IRS}}(s, T_s, T, K)^+ \mathbb{I}_{\{\tau_C = s\}}] ds, \quad (6.18)$$

and assuming independence of rates and default events, we have that

$$\begin{aligned} \text{UCVA}(t) &= \text{LGD}_C(t) \int_t^T \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbf{D}(t, s) V_{\text{IRS}}(s, T_s, T, K)^+] \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbf{I}_{\{\tau_C = s\}}] ds \\ &= \text{LGD}_C(t) \int_t^T \widehat{\text{PaySwaption}}(t, s, T_s, T, K) d\tilde{\mathbb{P}}(\tau_C \leq s). \end{aligned}$$

Discretizing the integration region into m sub-intervals allows the integral to be approximated by

$$\begin{aligned} &\approx \sum_{i=1}^m \widehat{\text{PaySwaption}}(t, \gamma_i, T_{\gamma_i}, T, K) [\tilde{\mathbb{P}}(\tau_C > \gamma_{i-1}) - \tilde{\mathbb{P}}(\tau_C > \gamma_i)] \\ &\quad \times \text{LGD}_C(t) \end{aligned} \tag{6.19}$$

The new instrument $\widehat{\text{PaySwaption}}$ above is a special payer swaption in that it matures on a different date than a reset date of the underlying swap. In the above equations, $\widehat{\text{PaySwaption}}(t, s, T_s, T, K)$ denotes the payer swaption at time t that matures at s and gives the holder an option to enter into a swap which resets at the next reset date of the swap after s or at s (similarly for γ_i) which is denoted T_s . The forward start swap resetting at T_s or T_{γ_i} and maturing at T and the strike of the option is K . If we assume the discretization points match the payment dates then we have that

$$\text{UCVA}(t) \approx \text{LGD}_C(t) \sum_{i=1}^n \widehat{\text{PaySwaption}}(t, T_i, T, K) [\tilde{\mathbb{P}}(\tau_C > T_{i-1}) - \tilde{\mathbb{P}}(\tau_C > T_i)]. \tag{6.20}$$

Note that $\widehat{\text{PaySwaption}}(\cdot)$ is the swaption that matures at a reset date of the underlying forward starting swap. When applied to value the unilateral CVA of an IRS, the result implies that to hedge the counterparty credit risk, one must hold a series of swaptions at time t to enable one to enter into a forward starting swap, should the original counterparty default. The unilateral CVA for a short position in the IRS results in

$$\text{UCVA}(t) = \text{LGD}_C(t) \int_t^T \widehat{\text{RecSwaption}}(t, s, T_s, T, K) d\tilde{\mathbb{P}}(\tau_C \leq s).$$

Discretizing the integration region and postponing default until the next discrete point, the integral can be approximated by

$$\begin{aligned} \text{UCVA}(t) &\approx \sum_{i=1}^m \widehat{\text{RecSwaption}}(t, \gamma_i, T_{\gamma_i}, T, K) [\tilde{\mathbb{P}}(\tau_C > \gamma_{i-1}) - \tilde{\mathbb{P}}(\tau_C > \gamma_i)] \\ &\quad \times \text{LGD}_C(t). \end{aligned}$$

The reasoning leading to the expression for $\widehat{\text{RecSwaption}}$ is analogous to that given for $\widehat{\text{PaySwaption}}$ above. If we assume the discretization points match the reset dates, then we have that

$$\text{UCVA}(t) \approx \text{LGD}_C(t) \sum_{i=1}^n \text{RecSwaption}(t, T_i, T, K_i) [\tilde{\mathbb{P}}(\tau_C > T_{i-1}) - \tilde{\mathbb{P}}(\tau_C > T_i)]. \quad (6.21)$$

6.3.2 Bilateral CVA

Let A and C be counterparties in an IRS such that both entities are prone to default taking the long and short positions respectively. As we showed previously, in Proposition 3.3, a bilateral CVA can be decomposed into the difference of a unilateral credit value adjustment and a debt value adjustment. Recalling (3.23), we have that

$$\text{BCVA}(t) = \text{UCVA}(t) - \text{DVA}(t),$$

and evaluating the UCVA component leads to,

$$\begin{aligned} \text{UCVA}(t) &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[\mathbb{I}_{\{\tau_C < T\} \cap \{\tau_C < \tau_A\}} (1 - R_{\tau_C}^{(C)}) D(t, \tau_C) V_{\text{IRS}}(\tau_C, T_{\tau_C}, T, K)^+ \mid \mathcal{F}_t \right] \\ &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[\int_t^T \mathbb{I}_{\{\tau_C = s\} \cap \{\tau_C < \tau_A\}} (1 - R_{\tau_C}^{(C)}) D(t, s) V_{\text{IRS}}(s, T_s, T, K)^+ ds \mid \mathcal{F}_t \right]. \end{aligned} \quad (6.22)$$

Assuming constant recovery rates and using Fubini's theorem, we obtain

$$\text{UCVA}(t) = \text{LGD}_C(t) \int_t^T \mathbb{E}^{\tilde{\mathbb{P}}} [\mathbb{I}_{\{\tau_C = s\} \cap \{\tau_C < \tau_A\}} D(t, s) V_{\text{IRS}}(s, T_s, T, K)^+ \mid \mathcal{F}_t] ds,$$

and assuming default events are independent of rates (this assumption will be relaxed when using Monte Carlo methods)

$$\begin{aligned} \text{UCVA}(t) &= \text{LGD}_C(t) \int_t^T \mathbb{E}^{\tilde{\mathbb{P}}} [\mathbb{I}_{\{\tau_C = s\} \cap \{\tau_C < \tau_A\}} \mid \mathcal{F}_t] \\ &\quad \times \mathbb{E}^{\tilde{\mathbb{P}}} [D(t, s) V_{\text{IRS}}(t, \tau_C, T, K)^+ \mid \mathcal{F}_t] ds, \\ \text{UCVA}(t) &= \text{LGD}_C(t) \int_t^T \tilde{\mathbb{P}}(\tau_C \leq s, \tau_A > s) \widehat{\text{PaySwaption}}(t, s, T_s, T, K) ds. \end{aligned} \quad (6.23)$$

Discretizing the integration region into m sub-intervals and postponing default to the next discretization point, leads to

$$\begin{aligned} \text{UCVA}(t) &\approx \text{LGD}_C(t) \sum_{i=1}^m \left[\tilde{\mathbb{P}}(\tau_C \leq \gamma_i, \tau_A > \gamma_i) \widehat{\text{PaySwaption}}(t, \gamma_i, T_{\gamma_i}, T, K) \right] \\ &= \text{LGD}_C(t) \sum_{i=1}^m \left(\tilde{\mathbb{P}}(\tau_C > \gamma_{i-1}, \tau_A > \gamma_i) - \tilde{\mathbb{P}}(\tau_C > \gamma_i, \tau_A > \gamma_i) \right) \\ &\quad \times \widehat{\text{PaySwaption}}(t, \gamma_i, T_{\gamma_i}, T, K). \end{aligned} \quad (6.24)$$

If we make the discretization points match the reset dates of the IRS, then we have that,

$$\begin{aligned} \text{UCVA}(t) &\approx \text{LGD}_C(t) \sum_{i=1}^n \left(\tilde{\mathbb{P}}(\tau_C > T_{i-1}, \tau_A > T_i) - \tilde{\mathbb{P}}(\tau_C > T_i, \tau_A > T_i) \right) \\ &\quad \times \text{PaySwaption}(t, T_i, T, K). \end{aligned} \quad (6.25)$$

Following the same reasoning as above, one can assume independence to derive the DVA(t):

$$\begin{aligned} \text{DVA}(t) &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[\mathbb{I}_{\{\tau_A < T\} \cap \{\tau_A < \tau_C\}} (1 - R_{\tau_A}^{(A)}) D(t, \tau_A) (-V_{\text{IRS}}(\tau_A, T_{\tau_A}, T, K))^+ \mid \mathcal{F}_t \right] \\ &= \text{LGD}_A(t) \int_t^T \tilde{\mathbb{P}}(\tau_A \leq s, \tau_C > s) \widehat{\text{RecSwaption}}(t, s, T_s, T, K) ds. \end{aligned} \quad (6.26)$$

Discretizing the integration region m sub-intervals and postponing default to the next discretization point, leads to

$$\begin{aligned} \text{DVA}(t) &\approx \text{LGD}_A(t) \sum_{i=1}^m \left[\tilde{\mathbb{P}}(\tau_A \leq \gamma_i, \tau_C > \gamma_i) \widehat{\text{RecSwaption}}(t, \gamma_i, T_{\gamma_i}, T, K) \right], \\ &= \text{LGD}_A(t) \sum_{i=1}^m \left(\tilde{\mathbb{P}}(\tau_A > \gamma_{i-1}, \tau_C > \gamma_i) - \tilde{\mathbb{P}}(\tau_A > \gamma_i, \tau_C > \gamma_i) \right) \\ &\quad \times \widehat{\text{RecSwaption}}(t, \gamma_i, T_{\gamma_i}, T, K). \end{aligned} \quad (6.27)$$

If we make the discretization points match the payment dates of the IRS, then we have that

$$\begin{aligned} \text{DVA}(t) &\approx \text{LGD}_A(t) \sum_{i=1}^n \left(\tilde{\mathbb{P}}(\tau_A > T_{i-1}, \tau_C > T_i) - \tilde{\mathbb{P}}(\tau_A > T_i, \tau_C > T_i) \right) \\ &\quad \times \text{RecSwaption}(t, T_i, T, K). \end{aligned} \quad (6.28)$$

6.3.3 BCVA When Simultaneous Defaults are Possible

We follow similar steps to those presented in Section 6.2.3 in evaluating

$$\widehat{V}(T, t) = V(t, T) - \text{UCVA}(t) + \text{DVA}(t) + \text{SMVA}(t). \quad (6.29)$$

Analogous to (6.23) for the UCVA component of (6.29), we have that

$$\text{UCVA}(t) = \text{LGD}_C(t) \int_t^T \tilde{\mathbb{P}}(\tau_C \leq s, \tau_A > s, \tau_{sim} > s) \widehat{\text{PaySwaption}}(t, s, T_s, T, K) ds. \quad (6.30)$$

Discretizing the integration region into m sub-intervals and postponing default to the next discretization point, leads to

$$\begin{aligned} \text{UCVA}(t) &\approx \text{LGD}_C(t) \sum_{i=1}^m \tilde{\mathbb{P}}(\tau_C \leq \gamma_i, \tau_A > \gamma_i, \tau_{sim} > \gamma_i) \widehat{\text{PaySwaption}}(t, \gamma_i, T_{\gamma_i}, T, K) \\ &= \text{LGD}_C(t) \sum_{i=1}^m \left(\tilde{\mathbb{P}}(\tau_C > \gamma_{i-1}, \tau_A > \gamma_i, \tau_{sim} > \gamma_i) \right. \\ &\quad \left. - \tilde{\mathbb{P}}(\tau_C > \gamma_i, \tau_A > \gamma_i, \tau_{sim} > \gamma_i) \right) \widehat{\text{PaySwaption}}(t, \gamma_i, T_{\gamma_i}, T, K). \end{aligned} \quad (6.31)$$

If we make the discretization points match the reset dates of the IRS, then we have that

$$\begin{aligned} \text{UCVA}(t) &\approx \text{LGD}_C(t) \sum_{i=1}^n \left(\tilde{\mathbb{P}}(\tau_A > T_{i-1}, \tau_C > T_i, \tau_{sim} > T_i) \right. \\ &\quad \left. - \tilde{\mathbb{P}}(\tau_C > T_i, \tau_A > T_i, \tau_{sim} > T_i) \right) \widehat{\text{PaySwaption}}(t, T_i, T, K). \end{aligned} \quad (6.32)$$

The next component to be investigated is the DVA(t) in (6.29). It is also natural to notice that it is analogous to that presented in Section 6.2.3, that is, we will end up with

$$\begin{aligned} \text{DVA}(t) &= \int_t^T \tilde{\mathbb{P}}(\tau_A \leq s, \tau_C > s, \tau_{sim} > s) \widehat{\text{RecSwaption}}(t, s, T_s, T, K) ds \\ &\quad \times \text{LGD}_A(t). \end{aligned}$$

Discretizing the integration region into m sub-intervals and postponing default to the next discretization point, leads to

$$\begin{aligned}
\text{DVA}(t) &\approx \sum_{i=1}^m \tilde{\mathbb{P}}(\tau_A \leq \gamma_i, \tau_C > \gamma_i, \tau_{sim} > \gamma_i) \widehat{\text{RecSwaption}}(t, \gamma_i, T_{\gamma_i}, T, K) \\
&\quad \times \text{LGD}_A(t) \\
&= \text{LGD}_A(t) \sum_{i=1}^m (\tilde{\mathbb{P}}(\tau_A > \gamma_{i-1}, \tau_C > \gamma_i, \tau_{sim} > \gamma_i) \\
&\quad - \tilde{\mathbb{P}}(\tau_A > \gamma_{i-1}, \tau_C > \gamma_i, \tau_{sim} > \gamma_i)) \widehat{\text{RecSwaption}}(t, \gamma_i, T_{\gamma_i}, T, K). \quad (6.33)
\end{aligned}$$

If we make the discretization points match the reset dates of the IRS, then we have that,

$$\begin{aligned}
\text{DVA}(t) &\approx \text{LGD}_A(t) \sum_{i=1}^n \left(\tilde{\mathbb{P}}(\tau_A > T_{i-1}, \tau_C > T_i, \tau_{sim} > T_i) \right. \\
&\quad \left. - \tilde{\mathbb{P}}(\tau_A > T_i, \tau_C > T_i, \tau_{sim} > T_i) \right) \widehat{\text{RecSwaption}}(t, T_i, T, K).
\end{aligned}$$

The last component of (6.29) that is left is the SMVA(t).

$$\begin{aligned}
\text{SMVA}(t) &= \int_t^T \tilde{\mathbb{P}}(\tau_{sim} \leq s) \left\{ R_{\tau_{sim}}^C \widehat{\text{PaySwaption}}(t, s, T_s, T, K) \right. \\
&\quad - R_{\tau_{sim}}^A \widehat{\text{RecSwaption}}(t, s, T_s, T, K) \\
&\quad \left. - D(t, s) V_{\text{IRS}}(s, T_s, T, K) \right\} ds \\
&\approx \sum_{i=1}^m (\tilde{\mathbb{P}}(\tau_{sim} > \gamma_{i-1}) - \tilde{\mathbb{P}}(\tau_{sim} > \gamma_i)) \\
&\quad \times \left\{ R_{\tau_{sim}}^C \widehat{\text{PaySwaption}}(t, \gamma_i, T_{\gamma_i}, T, K) - B(t, \gamma_i) V_{\text{IRS}}(\gamma_i, T_{\gamma_i}, T, K) \right. \\
&\quad \left. - R_{\tau_{sim}}^A \widehat{\text{RecSwaption}}(t, \gamma_i, T_{\gamma_i}, T, K) \right\}.
\end{aligned}$$

Assuming default can only happen at the reset dates leads to

$$\begin{aligned}
\text{SMVA}(t) &\approx \sum_{i=1}^n \left(\tilde{\mathbb{P}}(\tau_{sim} > T_{i-1}) - \tilde{\mathbb{P}}(\tau_{sim} > T_i) \right) \left\{ R_{T_i}^C \widehat{\text{PaySwaption}}(t, T_i, T, K) \right. \\
&\quad \left. - R_{T_i}^A \widehat{\text{RecSwaption}}(t, T_i, T, K) - V_{\text{IRS}}(t, T_i, T, K) \right\}. \quad (6.34)
\end{aligned}$$

6.4 Remark

In this chapter expressions for unilateral and bilateral credit value adjustments were derived and presented for FRAs and IRSs. On the bilateral component, the expressions were derived when simultaneous defaults were excluded and when they were

allowed. These expressions will be implemented in the following chapter for various entities in the South African market.

Chapter 7

Implementation, Results and Analysis

In this chapter we present implementation details of the results that were derived in the previous chapter. We also present algorithms that were used to simulate default times under both the structural and intensity based frameworks. Comments are made on parallel implementations of these algorithms. Results on CVA/BCVA calculations are presented for forward rate agreements (FRAs) and interest rate swaps (IRSs) on some South African entities. The effect of correlation is illustrated when default is modelled in a structural setting and also in the intensity based framework. Results are discussed and summarized in the form of graphs with the numerical data presented in Appendix B.

7.1 Entities Being Investigated

The literature on credit value adjustments has grown and many case studies have been presented, however, to the knowledge of the author, none have focused on South African instruments. South Africa has the largest economy in Africa and through its mature financial system is a key route to derivatives dealing in the rest of the continent. It is an emerging market and in 2010 joined Brazil, Russia, India and China to form BRICS. While the country's banks were not widely affected by the 2008 financial crisis in terms of defaults, the state as a whole was negatively affected. Ndlangamandla [63] investigates the effects of the crisis on South Africa through an extensive study involving the country's multinationals across the major pillars of the economy and concludes that it was near impossible to hedge a Quanto CDS during the crisis.

The current study will involve four South African entities (names). They are

- **Standard Bank:** The bank is a proudly South African company with a strong focus on emerging markets, but it also has a strong presence in London. It is one of the biggest banks in Africa and was left almost unaffected during the credit crisis in terms of profits and losses, but the past years have been marked by widespread retrenchments by the bank to ensure sustainable expenditure.
- **South African Breweries:** The groups wide portfolio of brands includes premium international beers such as Pilsner Urquell, Peroni Nastro Azzurro, Miller Genuine Draft and Grolsch, as well as leading local brands such as Aguila, Castle, Miller Lite, Snow and Tyskie. They are also one of the worlds largest bottlers of Coca-Cola products. Since listing on the London Stock Exchange over 10 years ago they have grown into a global operation, developing a balanced and attractive portfolio of businesses. Their markets range from developed economies in Europe to fast-growing developing markets such as China and India.
- **Old Mutual:** It is the largest and most well-established financial services provider in Southern Africa. Their prominent position in the industry is reflected in their strong operating performance across all their businesses and their good balance sheet position. Their prominence in the financial industry resulted in them being the most affected during the credit crisis.
- **Anglo American:** Three of Anglo American's mining businesses are based in South Africa: Platinum, Kumba Iron Ore and Thermal Coal. They hold advantageous positions in the most structurally attractive commodities, and have taken steps to concentrate on their core mining portfolio. This has involved a series of divestments, including the de-merger of the Mondi Group and the sale of their shareholdings in AngloGold Ashanti, Highveld Steel and Vanadium, Namakwa Sands, Tongaat Hulett and Hu-lamin.

In this study, short and long dated derivatives will be investigated in the interest derivatives market, these are namely, FRAs and IRSs. We have selected the 5th of December 2011 as the date when the CVA and Bilateral CVA will be calculated. This date could be considered as an out-of-crisis date, and also a date when the markets are recovering, if one ignores the possible effects of the Euro debt crisis.

7.2 Obtaining Market Curves

A number of market implied pricing curves are necessary for this study. These are the discount factor curve and the survival probability curve for the entities mentioned

above. In this section an overview of the methodology and instruments used will be presented to give an idea of how the market curves were obtained. The data used was obtained from the Rand Merchant Bank rates team, which sourced it from Bloomberg.

7.2.1 Discount Factor Curve

The discount factor curve was stripped from deposit rates and various FRA and IRS rates on a set of maturities. More precisely the following quotes were used, spot, 3 months JIBAR, with 3×6, 6×9, 9×12 FRAs and quotes for swaps maturing 1, 3, 5, 7 and 10 years. Basically the discount factors, $B(t, T_i)$, are obtained through an iterative procedure. A set of $B(t, T_i)$ is obtained where the T_i are in quarterly increments. After the first year the period between subsequent maturities of the IRSs is divided into quarterly bins and to obtain the discount factor for the maturity of the bin, it is assumed that the interest rate in each period is constant and thus discount factors for the bins are computed and then multiplied recursively to produce the required discount factors at time t . This procedure produced a 10 year swap zero curve which is presented in Figure 7.1, the actual values can be found in Appendix A, Table A.1. This curve is then used in conjunction with the credit default swap pricer to compute the survival probability curve for each of the entities. The quotes used to compute the survival probability curve are mentioned below.

7.2.2 Survival Probability Curve

In Chapter 5 an iterative procedure was described for obtaining a survival probability curve using CDS quotes. Various maturities of CDSs were used to strip the curve for each of the entities. Fortunately, the entities used have CDS maturities spanning 10 years. As it was stated in Chapter 5, CDSs are a good candidate for stripping the survival probability curve due to their liquidity and availability for various maturities. The maturities used to strip the curves are 1, 2, 3, 4, 5, 7 and 10 years. The curves obtained for the various names are presented in Appendix A, Table A.1.

7.3 Implementation

In this section we provide implementation ready algorithms for simulating default times under the structural and intensity based frameworks. Algorithms are also presented for simulating the CIR++ process when it is independent of and when it is dependent on other processes. We also present Monte Carlo algorithms for evaluating the “hat” coded derivatives ($\widehat{\text{PaySwaption}}/\widehat{\text{RecSwaption}}$ and $\widehat{\text{Cl}}/\widehat{\text{Fl}}$).

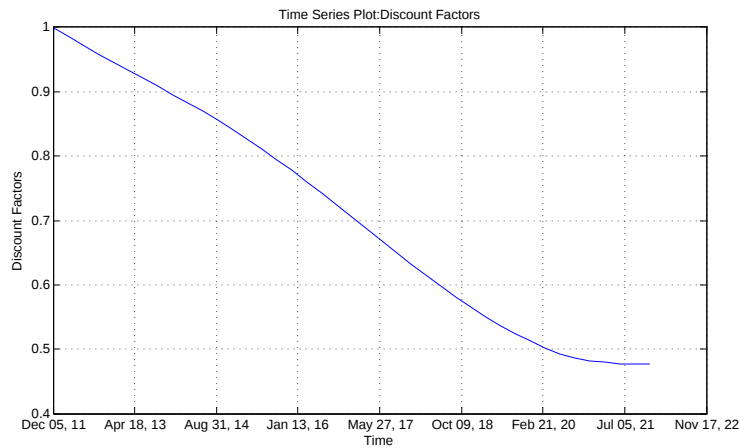


Figure 7.1: *Swap Zero Curve as of 5 December 2011*

7.3.1 Simulating Default Times

In a simple setting where risk factors, e.g. interest rate risk, are independent of default events, simulations are unnecessary and the UCVA or BCVA can be obtained through the approximations described in Subsections 6.2.1, 6.2.2, 6.3.1 and 6.3.2. When there is dependency however, simulations are necessary in obtaining the default times. Below is a description of how default times are obtained under the two frameworks for modelling credit risk.

Structural Framework

Under the structural model presented in Chapter 4, it was assumed that the company value follows GBM. It is well known that under this process the company value can be solved in closed form, which enables easy simulations for a specified discretized interval. When there is dependency, it is necessary that we simulate the company value on a discrete set of time points, $\{t_0, t_1, \dots, t_n = T\}$. We use the following iterative equation

$$V(t_{i+1}) = V(t_i) \exp \left\{ \left(r - \frac{1}{2} \sigma(t_{i+1})^2 \right) (t_{i+1} - t_i) + \sigma(t_{i+1}) \sqrt{t_{i+1} - t_i} Z_{t_{i+1}} \right\} \quad (7.1)$$

where $i \in \{0, 1, \dots, n-1\}$, $\sigma(\cdot)$ is from the calibration procedure described in Chapter 4 and $Z \sim \mathcal{N}(0, 1)$ are IID. If it is necessary that the two entities be correlated, as is the case with a BCVA, the corresponding standard normal numbers Z^A and Z^C will be correlated. The default time, denoted τ , is then obtained through

$$\tau = \inf \{t_i, V(t_i) \leq B\}, \quad (7.2)$$

for some barrier level B . To increase modelling flexibility, a level of complexity can be added by introducing a short rate model for r_t such that we have the following,

$$dV_t = r_t V_t dt + \sigma(t) V_t dW_t^V \quad (7.3)$$

$$dr_t = \left[\frac{d\phi(t)}{dt} + k(\theta - (r_t - \phi(t))) \right] dt + \sigma \sqrt{r_t - \phi(t)} dW_t^r, \quad (7.4)$$

$$dW_t^V dW_t^r = \rho dt, \quad (7.5)$$

where (7.4) is the same as (4.41) in Chapter 4, that is, r_t follows the CIR++ process.

The $(\mathbf{Z}^V, \mathbf{Z}^r) \sim \mathcal{N}(0, 1)$ can be correlated through a copula or by using the Cholesky decomposition method. Using a copula function one would sample the correlated uniforms and convert them to $\mathcal{N}(0, 1)$ by using an approximation of the inverse of the Gaussian distribution. This is simple to do using mathematical software such as Matlab. Algorithm 7.1 can be used to generate default times under the structural framework and although presented in a sequential manner, it can be parallelized by assigning different threads to generate individual paths independently, saving default times and default rates if found.

Generating Default Times Under the Intensity Based Framework

Under this framework we introduced the CIR++ model for the intensity and below we introduce an algorithm for simulating it independently. As a reminder the CIR++ process is given by $\lambda_t = x_t + \phi(t)$ where x_t follows the CIR process and $\phi(t)$

Algorithm 1 This algorithm returns a set of default times for each path simulated for the company value and corresponding default times

Require: $\langle \mathbf{Z}^V, \mathbf{Z}^r \rangle \sim \mathcal{N}(0, 1)$, $V(t_0)$, $r(t_0)$, $\#Paths$ and *discritization*

Ensure: A set of default times τ and a set of default rates r_τ

```

index, = 0
while index < #Paths do
  for  $t_{i+1} \in \text{discritization}$  do
    Generate  $r(t_{i+1})$  using  $\mathbf{Z}_{i+1}^r$  with the numerical schemes (7.12) or (7.10)
     $V(t_{i+1}) = V(t_i) \exp \left\{ (r(t_{i+1}) - \frac{1}{2}\sigma^2(t_{i+1}))(t_{i+1} - t_i) + \sigma(t_{i+1})\sqrt{t_{i+1} - t_i}\mathbf{Z}_{i+1}^V \right\}$ 
    if  $V(t_{i+1}) \leq B$  then
       $\tau(\text{index}) = t_{i+1}$ 
       $r_\tau(\text{index}) = r_{t(i+1)}$ 
      break
    end if
  end for
  index ++
end while

```

is a deterministic function that guarantees a fit. To simulate the CIR++ process, we need to simulate the CIR process and then add the deterministic shift function $\phi(t)$. Using Algorithm 7.2, one can simulate the independent CIR++ process.

Algorithm 2 This algorithm returns a matrix with each column being a path generated by the CIR++ model

Require: $\theta, k, \text{sigma}, x(t_0)$, $\#Paths$ and *discritization*

Ensure: A matrix $\lambda(\text{intensities}, \text{paths})$

```

 $d = \frac{4\theta k}{\sigma^2}$ 
index, = 0
while index < #Paths do
  for  $t_i \in \text{discritization}$  do
     $Factor = \frac{\sigma^2(1 - e^{-k(t_{i+1} - t_i)})}{4k}$ 
     $Non\_Centrality\_Parameter = \frac{4ke^{-k(t_{i+1} - t_i)}}{\sigma^2(1 - e^{-(t_{i+1} - t_i)})}x(t_i)$ 
     $x(t_{i+1}) = Factor \times \mathcal{X}_d^2(Non\_Centrality\_Parameter)$ 
     $\lambda(t_{i+1}, \text{index}) = x(t_{i+1}) + \phi(t_{i+1})$ 
  end for
  index ++
end while

```

Algorithm 7.2 gives a more accurate and simpler way of simulating the CIR process using its transition density. The problem however, is that it is only applicable in cases where the process is being simulated independently of other factors/processes. The moment correlation is introduced, for example as follows

$$d\lambda_t^A = k^A[\theta^A - \lambda_t^A]dt + \sigma_A\sqrt{\lambda_t^A}dW_t^A \quad (7.6)$$

$$d\lambda_t^C = k^C[\theta^C - \lambda_t^C]dt + \sigma_C\sqrt{\lambda_t^C}dW_t^C \quad (7.7)$$

$$dW_t^A dW_t^C = \rho_{AC}dt, \quad (7.8)$$

Algorithm 7.2 becomes inappropriate.

The unavailability of a closed form solution for the process and the fact that Algorithm 7.2 is no longer applicable when correlation is introduced forces us to seek a numerical solution to the above SDEs, (7.6) and (7.7). There exist several discretization schemes such as the classical Euler scheme which gives the approximate solution of (7.6) to be

$$\lambda_{t_{i+1}}^A = \lambda_{t_i}^A + k^A [\theta^A - \lambda_{t_i}^A] \Delta_{i+1} + \sigma_A \sqrt{\lambda_{t_i}^A \Delta_{i+1}} Z_{i+1}^A, \quad (7.9)$$

and analogously for (7.7), where $\Delta_{i+1} = t_{i+1} - t_i$ and Z_{i+1}^A is a standard normal number correlated to a corresponding Z_{i+1}^B . The problem with the Euler scheme, besides its inaccuracy, is that the process can hit negative values which not only violates the positivity property, but also introduces imaginary numbers through the diffusion coefficient. A possible solution is to force the diffusion coefficient to be real. There is another popular scheme named after Milstein. It gives the approximate solution to (7.6) to be

$$\begin{aligned} \lambda_{t_{i+1}}^A &= \lambda_{t_i}^A + k^A[\theta^A - \lambda_{t_i}^A]\Delta_{i+1} + \sigma_A\sqrt{\lambda_{t_i}^A\Delta_{i+1}}Z_{i+1}^A \\ &\quad + \frac{1}{4}\sigma_A^2\Delta_{i+1}[(Z_{i+1}^A)^2 - 1], \end{aligned} \quad (7.10)$$

and analogously for (7.7). While the Milstein scheme is sufficient in most SDE simulations, the diffusion coefficient introduces a concern. To guarantee convergence, the Lipschitz condition (see, Definition 2.37) when applied to the diffusion coefficient implies that

$$\left| \sigma_A\sqrt{\lambda_t} - \sigma_A\sqrt{\lambda_l} \right| \leq N |\lambda_t - \lambda_l|, \quad (7.11)$$

for some positive N and real numbers λ_t and λ_l . In this case however, it is violated and thus convergence under the scheme is not guaranteed.

Due to these short comings, Brigo and Alfonsi [10] introduce a positivity preserving Euler scheme. Under their scheme, the approximate solution to (7.6) is

$$\lambda_{t_{i+1}}^A = \left(\frac{\sigma_A Z_{i+1}^A \sqrt{\Delta_{i+1}} + \sqrt{f_{i+1}}}{2(1 + k\Delta_{i+1})} \right)^2, \quad (7.12)$$

where

$$f = \sigma_A^2 (Z_{i+1}^A)^2 \Delta_{i+1} + 4 \left(\lambda_{t_i}^A + \left[k\theta - \frac{\sigma_A^2}{2} \right] \Delta_{i+1} \right) (1 + k\Delta_{i+1}). \quad (7.13)$$

It is easy to see that the square in (7.12) guarantees positivity. There are other higher order schemes or integral expansions such as the Itô-Taylor or the Wagner-Platen expansions presented in the text by Platen and Bruti-Liberati [66], but these are more suitable for multi-integral processes or processes describing portfolios. Implementing the above schemes for generating paths in parallel computer architectures is a straight forward exercise.

Assuming there are two entities A and C , to obtain the default times from their intensity matrices generated using the schemes above, requires that we solve for t_{τ_A} and t_{τ_C} from

$$\int_0^{t_{\tau_A}} \lambda_s^A = \zeta_A, \quad (7.14)$$

$$\int_0^{t_{\tau_C}} \lambda_s^C = \zeta_C, \quad (7.15)$$

where ζ_A and ζ_C are standard exponential numbers, which may or may not be correlated. The next step would be taking the minimum times obtained for each path and checking if it is less than the maturity of the contract. If it is then it qualifies as a default time of that entity on that path. As we stated in the previous chapter, ζ_A and ζ_C are standard exponential numbers, which may or may not be correlated.

7.3.2 Monte Carlo Pricing

Underlying all Monte Carlo valuation techniques is the law of large numbers (LLN), which loosely states that the sample average converges to the expected mean as the number of samples tends to infinity, that is,

$$I = \mathbb{E}[f(Z)] = \frac{1}{n}(f(Z_1) + f(Z_2) + \dots + f(Z_n)) \quad \text{as } n \rightarrow \infty, \quad (7.16)$$

where Z_i are IID. The equation above is important in pricing contingent claims because in an arbitrage free framework the price is an expectation under the risk neutral measure.

In valuing a derivative $f(\cdot)$ is normally the discounted payoff function for the derivative and Z is drawn from the distribution of the underlying. For more on

these techniques, an excellent coverage is provided by the text by Glasserman [45] in conjunction with the texts by Platen and Bruti-Liberati [66] and Jackel [58].

Valuing Caplets/Floorlets Using Monte Carlo Methods

Below we present an algorithm for valuing the instruments introduced in the previous chapter. The algorithm will value $\widehat{\text{Cl}}(t_0, T', T, S, R_i, K)$ and $\widehat{\text{Fl}}(t_0, T', T, S, R_i, K)$, where t_0 is the valuation date, T' is the maturity of the option, T is the reset date of the underlying forward and S is the maturity of the forward, R_i is the forward rate of the forward starting FRA and K is the strike. The Monte Carlo valuations for the two instruments are summarized by Algorithm 7.3.

Algorithm 3 This algorithm returns the price of any one of the two instruments above, if $flag = 1$ then it prices $\widehat{\text{Cl}}$ else it prices $\widehat{\text{Fl}}$ with $flag = -1$

Require: $t_0, T', T, S, K, \#Paths, flag$

Ensure: $\widehat{\text{Cl}}(t_0, T', T, S)$ and $\widehat{\text{Fl}}(t_0, T', T, S)$

$index = 0$

Generate $r(T', \#Paths)$ using Algorithm 7.2 given $\mathbf{U}$

while $index < \#Paths$ **do**

 Take $r(T', index)$ calculate $B^{\text{CIR}++}(T', T)$ and $B^{\text{CIR}++}(T', S)$ using (4.51)

 Calculate $F(T', T, S)$ using $B^{\text{CIR}++}(T', T)$ and $B^{\text{CIR}++}(T', S)$

$pay_off(index) = B(t_0, T')B^{\text{CIR}++}(T', T)[flag(F(T', T, S) - K)]^+$

$index++$

end while

$price = \frac{1}{\#Paths} \sum (pay_off)$

To obtain the CVA on caplets or floorlets, Algorithm 7.3 could be used on each default time τ_i found through applications of Algorithm 7.1 or (7.14). In a parallel computing architecture where there are many processors supporting multi-threading, the valuation of the caplets/floorlets maturing at each τ_i could be computed by different threads and the implementation of this would be easy.

Valuing Swaptions Using Monte Carlo Methods

In the previous chapter we introduced swaptions and swaption-like instruments in our CVA valuation expressions for IRSs. $\widehat{\text{PaySwaption}}(t_0, T', T_{T'}, T, R_{T_{T'}}, K)$ and $\widehat{\text{RecSwaption}}(t_0, T', T_{T'}, T, R_{T_{T'}}, K)$ were introduced as having a different maturity date to any of the underlying swap reset dates. The parameters are as follows, t_0 is the valuation date, T' is the maturity of the option, $T_{T'}$ is the first reset date after T' , T is the maturity of the underlying swap, and $R_{T_{T'}}$ is the fair swap rate for the

underlying forward starting swap and K is the strike. A Monte Carlo procedure is required to value these two swaptions. Algorithm 7.4 presents a Monte Carlo valuation procedure for valuing the two contracts.

Algorithm 4 This algorithm returns the price of any one of the two instruments above, if $flag = 1$ then it prices $\widehat{\text{PaySwaption}}$ else it prices $\widehat{\text{RecSwaption}}$ with $flag = -1$

Require: $t_0, T', T_{T'}, T, K_{T'}, \#Paths, flag$

Ensure: $\widehat{\text{PaySwaption}}(t_0, T', T_{T'}, T, K_{T'})$ and $\widehat{\text{RecSwaption}}(t_0, T', T_{T'}, T, K_{T'})$

$index = 0$

Generate $r(T', \#Paths)$ using Algorithm 7.2

while $index < \#Paths$ **do**

Take $r(T', index)$ calculate $B^{\text{CIR}++}(T', T_i)$ using (4.51) for all reset dates

Calculate $F(T', T_i, T_{i+1})$ using the $B^{\text{CIR}++}(T', T_i)$ obtained above

$pay_off(index) = B(t_0, T') [\sum_i^n B^{\text{CIR}++}(T', T_{i+1}) flag(F(T', T_i, T_{i+1}) - K_{T'})]^+$

$index++$

end while

$price = \frac{1}{\#Paths} \sum (pay_off)$

7.4 CVA when Credit Risk is Modelled Under a Structural Framework

In order to use the structural model, a calibration routine is needed and was described in Section 5.4.5. As a reminder, the routine involved calibrating the volatility in a time dependent fashion while estimating the ratio of the initial asset value of the entity to the barrier using the current equity volatility. The results of the calibration are presented in the Appendix A.3 and are used to obtain the results presented in this section. There are two ways in which correlation is introduced, one is by correlating the processes followed by the asset value processes of the two entities involved in the deal and the other is by correlating both the processes followed by the entities and the one followed by the short rate. Standard Bank (Std B) is the main dealer and is assumed to be short in every deal while its counterparty, which is one of Old Mutual (OM), Anglo American (AA) and South African Breweries (SAB) is assumed to be long.

In summary, to obtain the results the following steps were taken. A time step for discrete simulation was chosen to be 3.65 days in a year, which amounts to 0.01 up to 9 years. The total number of paths that were simulated for the asset values of a particular entity and the short rate was chosen to be 15000. When

calculating the UCVA, dependency is assumed between the short rate and the asset value of the entity assumed to be default prone. This dependency is introduced by using two types of copulas, a Gaussian and a Student-t copula with five degrees of freedom. The correlation is achieved by correlating the driving Brownian motions of the driving processes. The correlation will be negative because it captures the fact that as interest rates increase, entities are more prone to default. This is due to the fact that most companies depend on borrowing to grow and to run their businesses, high interest rates then imply a very high cost of doing business. The correlation was chosen to be -25% for illustration. Algorithmically, this implies sampling two matrices from the desired copula of size $\frac{\text{number-of-years}}{\text{timestep}} \times \text{num} - \text{paths}$ such that each matrix on its own consists of independent uniform numbers, but each one of the corresponding columns (paths) of the two matrices are correlated uniforms with a correlation of -25%. These uniform numbers are then converted to standard normal numbers by using an approximation of the inverse of the normal distribution to invert them. Algorithm 7.1 is then used to generate the default times and the corresponding default rates. Using the pair (default times and default rates), the expression (6.4) (depending on which party is valuating) is then approximated using Monte Carlo methods.

The results for IRSs are summarized by the figures 7.2 and 7.3 for the UCVA and BCVA respectively. In the UCVA plots, the graphs labeled “Independent” indicate the UCVA when the interest rate is independent of default. Two copulas were used and the graph labeled “Student t” (“Gaussian”) indicate the UCVA when the Student-t (Gaussian) copula is used to induce dependence between the interest rate and the intensity process of a default prone entity. In the BCVA plots, the graphs labeled “Gauss Ent Dep” (“Std t Ent Dep”) indicate the BCVA when dependency is assumed between the entities while the interest rate is independent with the dependency induced using the Gaussian copula (Student-t copula). The case where there is “All” is analogous to that which has just been mentioned except that the interest rate and the two entities are assumed to be jointly correlated. While CVA results when the underlying contract is a FRA, are presented in the Tables B.1, B.2 and B.3, the CVA is only significant when looking at the 9×12 FRA. In the structural case the effect is small and shows that defaults are very hard to obtain in the short term using a structural model. The framework is not good for short dated contracts.

7.4.1 Unilateral CVA

The Tables B.1, B.2, B.3, B.4, B.5 and B.6 all present unilateral CVA effects in basis points on the fixed fair rate for both FRAs and IRSs. The column Std B

shows the number of basis points that should be subtracted from the fair fixed rate, that is, it is the number of basis points that the fixed leg counterparty should not pay as a result of the credit risk of the short leg counterparty. The other columns labeled with the abbreviations of the other entities show the number of basis points that the entities should pay above the fair fixed as a result of their own credit risk. The multicolumn labeled “Independent” shows the results obtained when independence is assumed and thus the formulas presented in the previous chapter and the market implied survival probability curve may be used to obtain them or alternatively the asset value and the short rate may be simulated independently, i.e., use Algorithm 7.1 without correlating the standard normal variates Z^r and Z^A . We refer to the previous chapter, particularly (6.5) for details. The multicolumns labeled “Gaussian” and “Student-t” show results obtained when the asset value of the company is correlated with interest rates using a Gaussian and when a Student-t copula is used respectively.

The graphs presented in Figure 7.2 are a summary of the results presented in the tables on unilateral CVA on IRSs. The abbreviations associated with the entities labeling the graphs indicate how much effect in basis points the UCVA, assuming the proneness to default of that entity, has on the fair rate. It should be kept in mind that those labeled Std B indicate the amount of basis points that should be subtracted while the other entities indicate the number of basis points that should be added. What is immediately clear at first glance is that the UCVA calculated by one of the entities faced with the credit risk of Std B is significantly higher than that calculated by Std B faced with the default of any of the other entities and as expected the relationship is inverse. While the one factor could be the spreads of Std B, which were a bit higher than those of the rest of the entities, the effect of this is relatively small. The main issue is the dependence of the UCVA on the value of the contract being valued. Since all rates are dependent on the short rate in an affine framework and in this case the CIR++ process is calibrated to an upward interest rate curve. This implies that the future term structure will have the same upward shape. It is well known that in a swap, when one tracks the underlying forwards of which the IRS is composed of, they are positive towards maturity from the point of view of the fixed leg payer. The value of the swap will almost always be positive towards the fixed rate payer at a default time because, intuitively, defaults will happen later in a deal. This then implies the expected loss of the fixed rate payer is much higher at default than it is to the floating rate payer and hence the inverse relationship indicated by Figure 7.2.

The other noticeable feature of the graphs in Figure 7.2 is that the UCVA when the asset value of Std B and interest rates are independent forms an upper bound

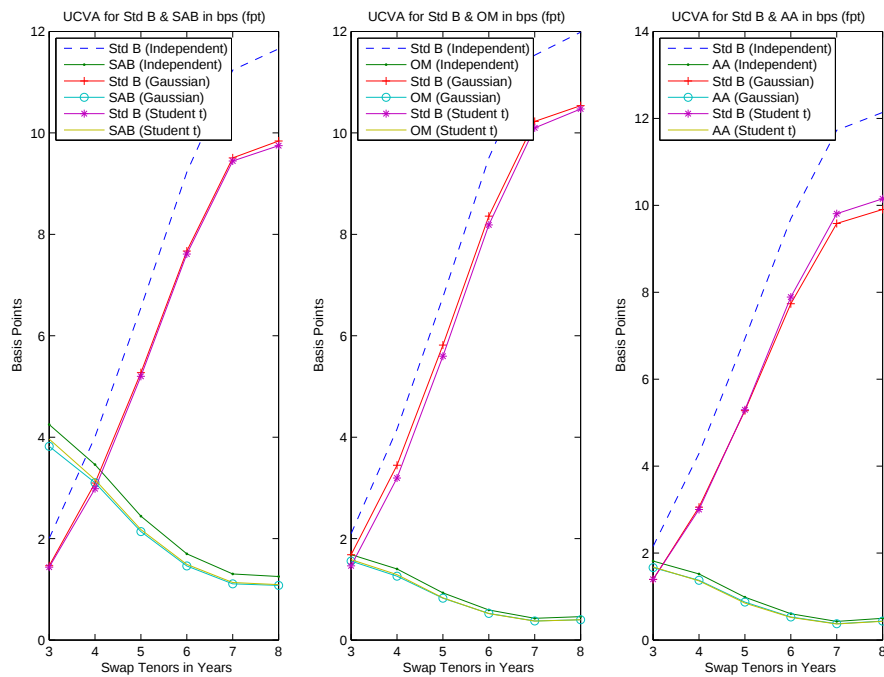


Figure 7.2: Shows the UCVA calculated by the various entities independently. Each graph in the various plots labeled with a particular entity represents the UCVA contribution in basis points to the fixed rate that the default of that entity would cause. For the short party (Standard Bank), it shows how many basis points more from the fair rate Standard Bank should receive as compensation for the fact that the long party could default, while for the long party it shows how much less the party should pay in compensation for the fact that the short party could default.

to the UCVA when the two processes are dependent. The same is noticeable when Std B is calculating the UCVA, the UCVA when dependency is assumed is less than the UCVA when independence is assumed. The reason for this is the negative correlation, which we explained as being an economic relationship. It implies that at times when the asset value process should have been low, the value of the short rate which may be higher instead may prevent a path from defaulting due to the role the short rate process plays in the drift part of the asset value process.

In summary, we have presented unilateral CVA results for both FRAs and IRSs.

7.4.2 Bilateral CVA

This section extends the previous section by calculating the BCVA on the same contracts. The same companies are investigated with the same positions, however, the dealer calculating the BCVA is Standard Bank and the counterparty is one of Old Mutual (OM), South African Breweries (SAB) or Anglo American (AA). The numerical CVA and DVA contributions are presented in Tables B.7, B.8, B.9, B.10, B.11 and B.12 in Appendix B. The multicolumns, Gaussian and Student-t indicate the copulas used such that the multicolumn “Dependent” shows results obtained when there is dependency between the entities and interest rates. The multicolumn “Entity Dep” shows results obtained when dependency is assumed to only exist between the entities and not interest rates.

Figure 7.3 presents a summary of the tables in Appendix B. The figure shows actual BCVA contributions to the fair fixed rate, they indicate the number of basis points Standard Bank will receive above/below the fair fixed rate (if the BCVA in basis points is positive then it is less else it is more). An independent approximation is also graphed and could be obtained through the results in the tables located in the Appendix with the column heading “Independent”.

The correlation effects observed when correlation is introduced between the asset value processes and the short rate process is stronger than when the correlation is between the entities only. It is clear from the shape of the BCVA graph that as the tenors increase, the DVA contribution becomes higher than the CVA contribution. The reason for the huge DVA contribution is that defaults are expected to occur later than when the IRS has just commenced, especially for entities of fair credit quality. This, combined with the fact that an IRS is expected to be positive towards the fixed leg as the IRS approaches its maturity when the initial yield curve is upward sloping, implies that towards maturity, it is expected that the floating leg will be owing, and hence the DVA benefit. When dependency is assumed between the entities leaving the short rate independent, the correlation effect is much lower than when dependency is assumed to also include the short rate process as can be

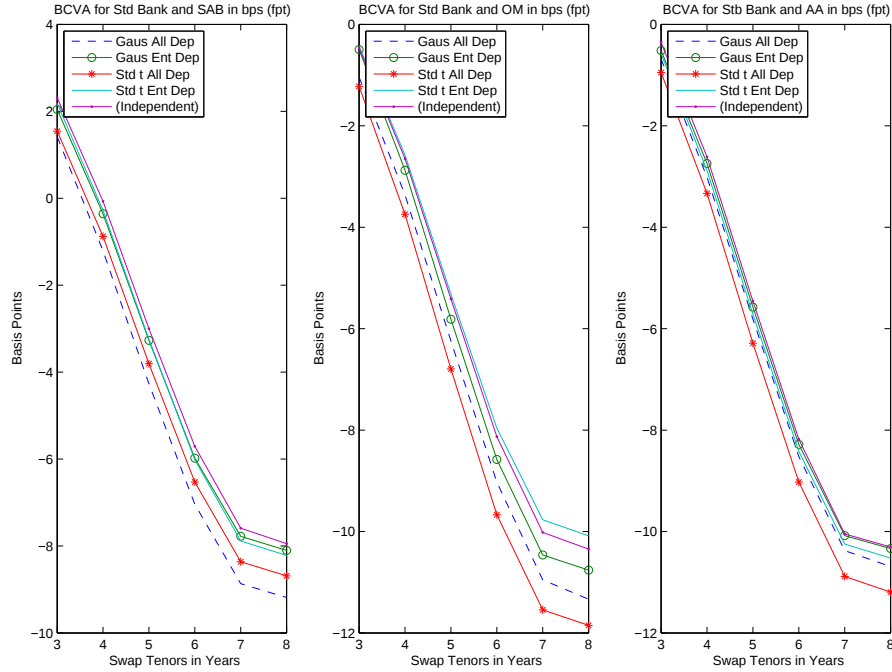


Figure 7.3: Shows the BCVA from the point of view of Standard Bank, i.e., Standard Bank is short swaps of various tenors.

verified by Figure 7.2. The weakness of the effect of entity to entity correlation indicates that to get reasonable effects one needs unrealistic levels of correlation between the asset value processes of the two entities. This is a known weakness of the first passage time model or structural models in general. This weakness will be investigated more in the section on correlation effects.

The choice of copula, that is, whether a Gaussian or a Student-t is used, appeared to have negligible effect on the CVA. This could be attributed to the fact that as the degrees of freedom tend to infinity, the Student-t distribution tends to the normal distribution (see, e.g. Figure 2.1).

7.5 CVA when Credit Risk is Under an Intensity Based Framework

It was mentioned earlier that the intensity process for all the entities was modelled as a CIR++ process. This then meant that parameters of the process had to be calibrated to the initial market implied survival probabilities, following the procedure

outlined in Subsection 5.4.4. The sets of the parameters will be labeled as $\text{Param}_C = \{\theta, \sigma, k, x_0\}$, where C is the entity. The results of the calibration are presented in Appendix A.2.

This framework offers much more flexibility regarding inducing correlation on risk factors. In the unilateral case however, correlation can only be induced between the short rate and the default intensity process of the entity assumed to be prone to default. In the bilateral case, as mentioned in the credit risk modelling chapter, one can correlate the default intensity processes of the two entities together with the process for the short rate using a tri-variate copula. One can further induce default correlation by correlating the default times themselves (typically by correlating the random exponentials required for simulating the default times). Another option is to leave the short rate and the intensity processes of the entities independent but only correlate the default times.

Again the time step used to discretize time is 0.01 which is about 3.65 days and 15000 paths are simulated for the intensity processes and the short rate process. Algorithmically, two matrices of size $\frac{\text{number-of-years}}{\text{timestep}} \times \text{num-paths}$, each consisting of independent uniforms that have been drawn from a chosen copula (Gaussian or Student-t), but each of the corresponding columns (paths) of the matrices are correlated. The way dependence is induced is by correlating the Brownian motions of the processes driving the default intensities of entities with the one driving the short rate process.

The short rate is modelled using a CIR++ process, as is the intensity process. This model specification is known as the shifted square root process (SSRD). Its most appealing feature in a practical sense is its separability property, that is, the short rate process and the default intensity process can be calibrated to the market independently using swaptions and CDSs respectively. In this setting, correlation between the short rate process and the default intensity process has negligible effect on the value of a CDS. Assuming positive correlation between the short rate and the default intensity process ensures that when the short rate increases the chance of default increases ¹.

Again, the results for IRSs are summarized by the Figures 7.4 and 7.5 for the UCVA and BCVA respectively. In the UCVA plots, the graphs labeled “Independent” indicate the UCVA when the interest rate is independent of default. Two copulas were used and the graph labeled “Student t” (“Gaussian”) indicate the UCVA when the Student-t (Gaussian) copula is used to induce dependence between the interest rate and the intensity process of a default prone entity. In the BCVA plots, the graphs labeled “Gauss Ent Dep” (“Std t Ent Dep”) indicate the BCVA

¹By contrast, to ensure this relation in the structural case, negative correlation had to be induced.

when dependency is assumed between the entities while the interest rate is independent with the dependency is induced using the Gaussian copula (Student-t copula). The case where there is “All” is analogous to that which has just been mentioned except that the interest rate and the two entities are assumed to be jointly correlated. As in the structural framework, CVA results when the underlying contract is an FRA is presented in the Tables B.13, B.14 and B.15. The CVA effect is small but acceptable compared to the structural case. It is hard to track correlation effects in short dated contracts under this framework too.

7.5.1 Unilateral CVA

Figure 7.4 presents graphs that take the same approximate shape as the previous graphs shown in Figure 7.2 for the structural case. It is clear from Figure 7.4 that the interest rate correlation has little effect, at least when the correlation is 25%, on the UCVA. To elaborate on the choice of 25%, it was chosen as an average from historical correlation. The reason for this stems from the fact that in the intensity case defaults are purely as a result of the intensity process and r_t has no active role, unlike in the structural case where the short rate takes an active role in causing default, i.e., when r_t is low, the asset value of that entity will be low and so more likely to hit the barrier.

The basis points effect is driven more by the value of the contract than the credit quality. To elaborate on this, the swap price is driven by the forward rates. In an affine setting all rates depend on the short rate and when the short rate is on an upward trajectory, as is the case here, the forward rates will also increase and thus an IRS will be more valuable to the party paying the fixed rate and correspondingly lower for the party paying the floating rate on default. The loss due to default is more to the party paying fixed because most of the default times will occur latter than when the deal has just been initiated, especially for entities of fair credit quality similar to those being investigated. Tracking the forward rate agreements underlying the swap from the point of view of the party receiving fixed when the yield curve is upward slopping, one immediately verifies that theoretically the floating leg would be having negative cash-flows towards maturity, implying that the counterparty credit risk is towards the fixed rate payer. The effects of correlation or choice of copula seems to be small but this will be investigated more in the coming sections on correlation.

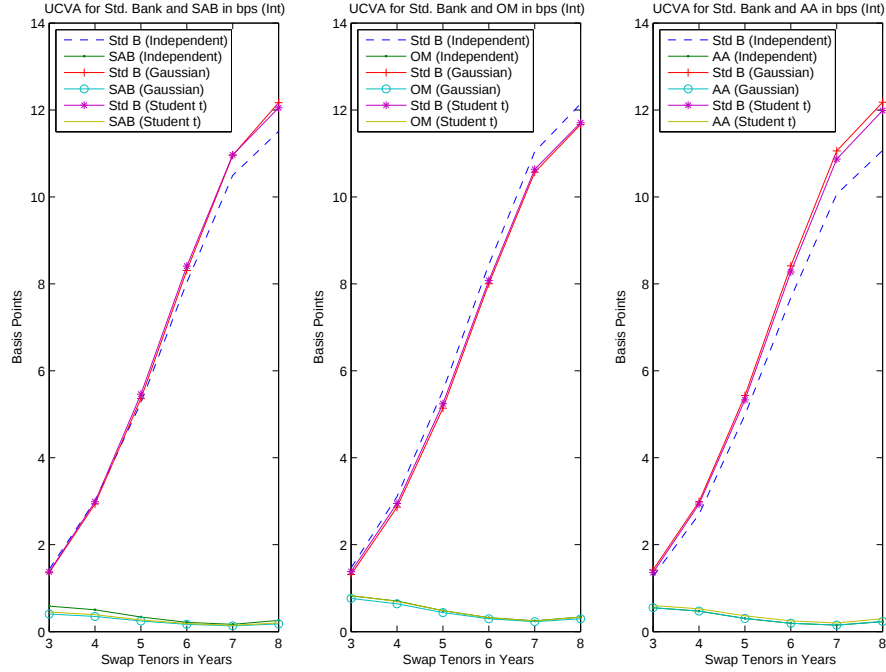


Figure 7.4: Shows the UCVA calculated by the various names independently. Each graph in the various plots labeled with a particular entity represents the UCVA contribution in basis points to the fixed rate that the default of that entity would result. For the short party (Standard Bank), it shows how many basis points more from the fair rate the party should receive as compensation for the fact that the long party could default, while for the long party it shows how much less the party should pay in compensation for the fact that the short party could default.

7.5.2 Bilateral CVA

Figure 7.5 shows that the BCVA is a decreasing function of tenor. This is the same qualitative property that was seen in the structural case. The reasoning is similar to that for the case where credit risk was modelled using a structural framework and depends on the BCVA equation,

$$\text{BCVA}(t, T) = \text{UCVA}(t, T) - \text{DVA}(t, T).$$

Since interest rates are increasing, as we approach maturity the swap is more valuable to the fixed rate payer than to the floating leg payer. This causes the UCVA to be smaller relative to the DVA and hence the decrease in overall BCVA. The choice of copula does not show significant effect on the overall BCVA and in general, the 25%

correlation does not seem to affect the BCVA very much. This could be attributed to the fact that the value of the swap plays a much more significant role in the BCVA and is much more important for a swap valued at zero at initiation. The fairness of the swap increases the chances that at default, the fixed leg is more valuable. Correlation will be investigated in the next sections, where a 5 year swap that has initial positive value to the floating leg is used.

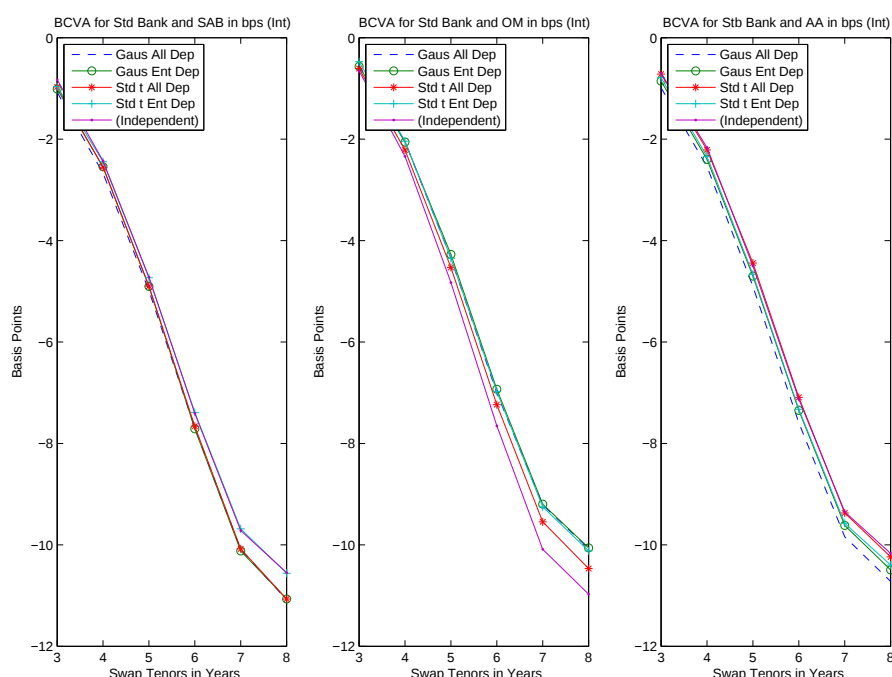


Figure 7.5: Shows the BCVA from the point of view of Standard Bank where Standard Bank is short IRSs of various tenors.

7.6 Investigating Correlation Effects

The UCVA and BCVA results that have been presented thus far when using both the intensity and the structural frameworks showed that correlation has a non-negligible effect. In this section, we study the effects of correlation under both frameworks. Figure 7.6 illustrates the effect of using a 95% correlation under the two frameworks. In the figure, default times are plotted against each other to illustrate same path default of any of the two entities in the title of each graph. In the correlated structural case, with plots coded “fpt”, the default times cluster linearly

and illustrate that the same path defaults happen almost at the same time. In the correlated intensity case with plots coded “int”, there is a more even distribution of points in the upper triangle of the plots.

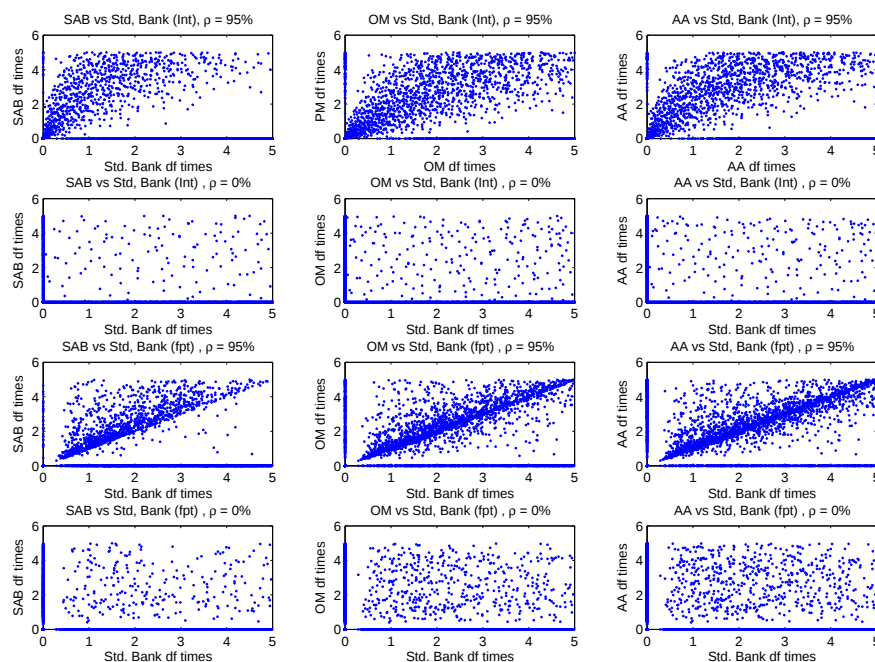


Figure 7.6: Shows default times plotted against each other when there is 95% correlation under both the structural (fpt) and intensity (int) framework for the various names under consideration.

Figures 7.7 and 7.8 both illustrate the effects of correlation when the Gaussian copula is used. The right and middle plots indicate the effect of correlation on BCVA calculated from the point of view of the short position. In this case the short position is taken by Standard Bank. The left plots show the effect of correlation on UCVA. Each of the graphs labeled Std B indicate the UCVA calculated by Old Mutual, assuming that Standard Bank is default prone, while those labeled OM indicate the UCVA calculated by Standard Bank assuming Old Mutual is default prone. On the BCVA plots located on the middle and far right of Figures 7.7 and 7.8, the graphs labeled “independent” indicate the BCVA calculated by Standard Bank when Standard Bank, Old Mutual and interest rates are left independent. Those labeled “Entity Dep” indicate the BCVA when only Standard Bank and Old Mutual are correlated while interest rates are independent and those labeled “All

Dep” indicate the BCVA when Old Mutual, Standard Bank and interest rates are all correlated.

7.6.1 Correlation Effects Under the Structural Framework

As a reminder, the correlation is introduced by correlating the Wiener processes driving the asset values of the two parties and the one driving the short rate process. For two entities A and C, a default time is the first time that $V_t^A/V_t^C < B^A/B^C$ where B^A and B^C are the corresponding barriers. Figure 7.7 is a summary of the numerical values contained in Tables B.25 and B.26.

The UCVA calculated by Old Mutual is decreasing as the negative correlation between interest rates and the asset value process of Standard Bank increases. The explanation for this is as follows: the asset value decreases as the interest rates increase and results in high probability of default when the rate is high, but, when the asset value of Standard Bank increases, the interest rates decrease guaranteeing that the chances of default, when rates are low, is small. This implies that when the default of Standard Bank happens, i.e., when the asset value of Standard Bank touches the barrier, the interest rates will be high and so the IRS will be valuable towards Old Mutual. As the negative correlation weakens, the exposure to Old Mutual on default decreases accordingly and hence the UCVA decreases as the negative correlation weakens. This has an obvious inverse effect on the UCVA being calculated by the party that is short, in this case Standard Bank. When the asset value process of Old Mutual and interest rates are highly negatively correlated, we have that the UCVA is at its lowest. This is due to the fact that when Old Mutual defaults, the high negative correlation means that interest rates will be high and hence the IRS will have little value towards the short leg.

Shifting focus to the BCVA plot in the middle of Figure 7.7, it is immediately noticeable that the interest rate correlation has a much bigger effect on the overall CVA compared to the effects of correlating only the asset value processes of Standard Bank and Old Mutual. The weak effect compared to interest rate dependency can be linked to the fact that unlike the asset value processes of the two entities, the interest rate is the underlying of the IRS. We state the following equation in order to further explain the situation displayed by the graphs. The BCVA is given by

$$\text{BCVA}(t, T) = \text{UCVA}(t, T) - \text{DVA}(t, T). \quad (7.17)$$

The strong effect of the negative correlation on interest rates and entities is due to the fact that when default occurs, the negative dependency implies that the probability of interest rates being high is relatively high. High interest rates imply a high DVA term in this instance, this is because Standard Bank will benefit from

not having to pay the high floating rate. The high floating rate implies a very small UCVA term as the contract will most certainly be negative towards Standard Bank implying less counterparty to credit risk exposure. A slight decrease of a few thousand Rands is also seen in the case when the interest rates are independent of the asset value processes of the two entities but the entities themselves being positively correlated. The little effect only noticeable at extremely high levels of positive correlation illustrates the difficulty that structural models have in inducing correlation, requiring that unrealistic correlation levels be applied to get reasonable effects. Another reason could be that Old Mutual does not have significantly lower credit spreads compared to Standard Bank. The comparable spreads imply that it is not always the case that Standard Bank is first to default and hence the UCVA may also be contributing to the BCVA as much as the DVA does. The joint contribution as in (7.17) explains why the overall BCVA may be small.

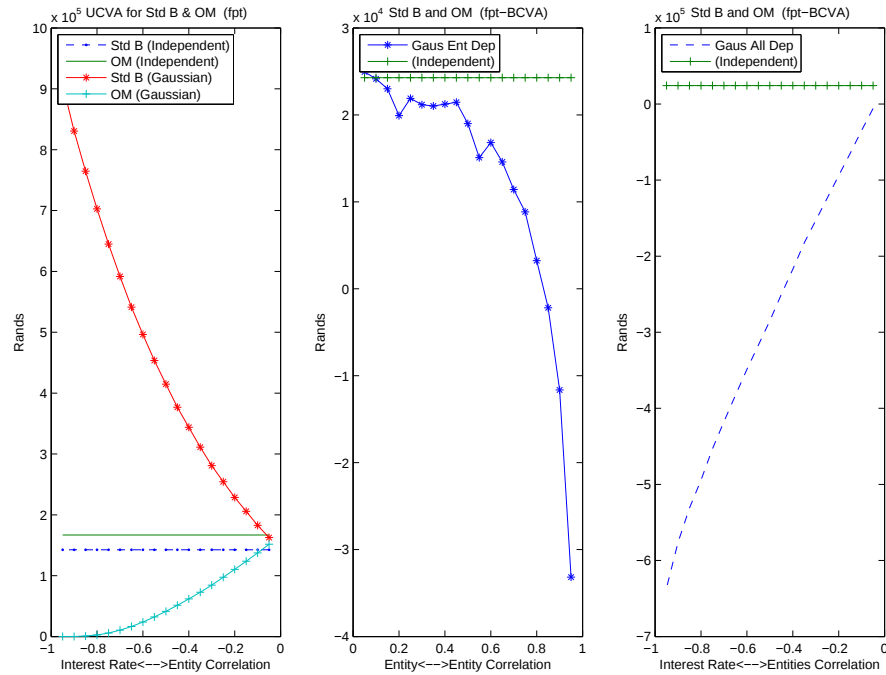


Figure 7.7: Shows the effects of correlation on UCVA and BCVA of a 100 million rand notional IRS when credit risk is modelled using a first passage time model between Standard Bank and Old Mutual. The swap is initially positively valued to the short side.

7.6.2 Correlation Effects Under the Intensity Framework

The effect of correlation on both the UCVA and BCVA under the intensity framework can be seen in Tables B.27 and B.28. In the unilateral case the intensity process of the entity assumed to be default prone is correlated with the short rate process. Unlike in the structural case where positive correlation implied that when the interest rates are high the chances of default are low, in this framework when the interest rates are high and there is positive correlation with the intensity process, the chances of default are high. On default of the short party, as a result of its high intensity, it is more likely that the rates are high and hence the IRS is more valuable to the long side which leads to an increase in the UCVA calculated by Old Mutual (long) faced with the counterparty exposure of Standard Bank. On the short side of the deal the inverse happens, i.e., the UCVA calculated by Standard Bank decreases as the correlation between interest rates and the intensity of Old Mutual increases. This is due to the fact that as the correlation increases defaults become likely when interest rates are high, implying that the floating leg will be more likely to owe than be owed.

Turning our focus to the BCVA plots on the right and middle of Figure 7.8, we immediately notice that although the “Gaus Ent Dep” graph is very similar to the one labeled “Gaus All Dep” there is still a difference between the two BCVA calculations. As the correlation increases the BCVA decreases and this is explained by the first to default clause that determines which of the components of the BCVA is calculated. In this case, the party that is expected to default first is the one with the higher credit spread, which happens to be Standard Bank. As the correlation increases the DVA component becomes the net contributor to the BCVA and since it contributes negatively the BCVA decreases. The reason the effect of correlation is less than in the structural case is that the interest rates do not play an active role in inducing default. They do however play a role in the value of the contract. As the correlation increases between interest rates and the intensity process, it is more likely that when the intensity is high the interest rate will be high, making the DVA component a net negative contributor with the added capability of being bigger than the case when interest rates are independent. This is as a result of the underlying interest rate swap being more valuable to the party holding the long position. This explains the relatively fast decrease compared to the case illustrated by the graph labeled “Gaus Ent Dep” case. The “Gaus Ent Dep” graph in the middle of Figure 7.7 is initially slightly above the “Independent” graph but at about the 40% correlation mark, it starts decreasing.

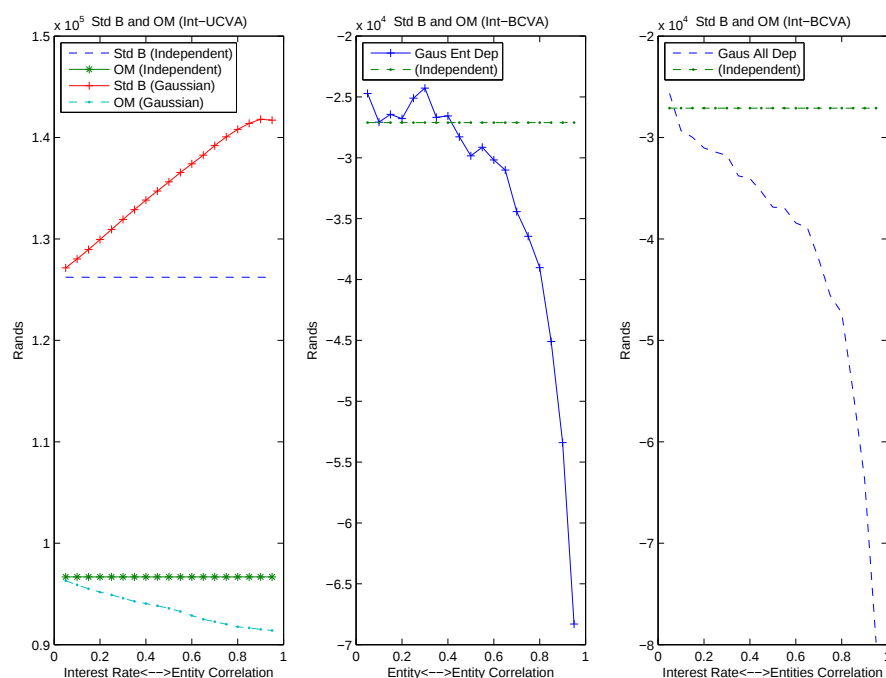


Figure 7.8: Shows the effect of correlation on UCVA and BCVA of a 100 million Rand notional IRS when credit risk is modelled under the intensity framework. The deal is between Standard Bank (short) and Old Mutual (long).

Chapter 8

Conclusion

8.1 Summary

The dissertation presented and reviewed work done in counterparty credit risk and credit risk. The results chapter (Chapter 7) presented algorithms for simulating default times using a first passage time model and also from intensity processes. Algorithms for simulating the CIR++ short rate process under independence and dependency assumptions were also presented. These algorithms were complemented by pricing algorithms for the instruments that were introduced in Chapter 6.

There were two studies done, one study looked at how the UCVA and BCVA behave under increasing tenors of FRAs and IRSs. The credit risk was modelled using a first passage time model and also under the intensity framework. In that study the Student-t and Gaussian copula were used to induce dependency. The structural framework was shown to underestimate the CVA of short dated contracts such as FRAs, furthermore, the effect of correlation seems to be very insignificant in the very short dated FRAs under both credit risk modelling frameworks. It was shown that the effect of interest rate dependency in UCVA is larger when credit risk is modelled using a first passage time model than it is when the intensity framework is considered. The reason for this was attributed to the explicit link the short rate process has with the asset value process which ultimately determines default. In the case of BCVA calculations, it was shown that the 25% or -25% interest rate correlation to the intensity processes and between the entities had little effect on the overall BCVA. Using the Student-t copula instead of the Gaussian copula was shown to have less effects on both the UCVA and BCVA for reasonable degrees of freedom for the Student-t copula. This is attributed to fact that as the number of degrees of freedom when using the Student-t distribution increases the distribution approaches the Gaussian distribution (See Figure 2.1). The main driver of UCVA

and BCVA was shown to be the value and characteristics of the derivative that the two quantities are being evaluated on. For entities of fair credit quality default happens later in a derivatives deal than when the deal has just commenced. This has the consequence that in IRSs entered into at the fair rate when the interest rate curve is upward sloping, the counterparty credit exposure faced by the fixed rate payer larger than that faced by floating rate payer. This is due to the fact that towards maturity the cashflows are more likely to be positive towards the fixed leg.

The second study focused on correlation effects and, instead of using fair IRSs, an IRS with initial positive value towards the floating leg was investigated. More precisely, a R100 million notional IRS was investigated and it was shown that correlation had significant effect.

8.2 Future Work

It was mentioned earlier that there have been many studies across instrument classes on CVA/BCVA but there is still a lot to be done in this area apart from using more sophisticated processes to model the asset value of the firm, short rate and the intensities of the various entities. The issues arising when incorporating funding and collateral into CVA/BCVA have not been investigated in detail. We discuss their importance and complexities below and also briefly describe how a more realistic volatility can be achieved in a CIR++ model.

8.2.1 Incorporating Collateral into a CVA/BCVA

In Chapter 3, collateral posting was described and its mitigating effect was illustrated with an example. It is however very important to note that even with frequent collateral posting, counterparty credit risk is not eliminated. The introduction of collateral introduces a collateral rate which is earned by the collateral after being posted. The existence of the collateral rate may introduce re-hypothecation, that is, the holder will be able to use the posted collateral. If re-hypothecation is allowed, then a model for the collateral rate is required when calculating a CVA/BCVA. The framework would need to account for the fact that the holder of the collateral can possibly default on the collateral itself. There has been no work investigating the effects of collateral on a CVA/BCVA in the South African market.

When an OTC derivative is protected by a CSA included in the ISDA master agreement, the collateral is usually posted if the value of the derivative is greater than a particular threshold. From time to time both sovereign governments and big companies are downgraded. It is thus beneficial in long dated contracts when calculating a CVA/BCVA in the presence of a CSA to consider making the threshold

rating dependent. To the best of our knowledge this is not available in the literature. Collateral posting in the South African market context is also going to a complex exercise as the collateral posted might have to be invested with the clearing houses which are the South African banks, who each have a positive probability of default. The analysis required in solving this cyclic problem would be very interesting.

8.2.2 Incorporating Funding into a CVA/BCVA

The choice of a discounting curve is important. A discounting curve should be risk free and most certainly one that represents interest in money invested without an option to redeem it before maturity. There are different choices that may be made for this curve which will affect CVA/BCVA calculations. For example, it could be constructed using JIBAR or OIS (Overnight Index Swap) rates. There are papers that have dealt with funding such as the one by Piterbarg [65] but overall this subject is still open for debate, especially in the South African market, given the fact that there are no overnight index swaps in the market.

8.2.3 Choice of Intensity Process

The intensity of default for all the entities investigated was assumed to follow a CIR++ process. While it has many attractive features, this model has a shortcoming in that its implied volatility is small compared to those implied for similar entities in the CDS options market. One reason is that the calibration process produces small values for k, θ, σ . The other reason is that the Feller condition,

$$2k\theta > \sigma^2, \quad (8.1)$$

is imposed in order to prevent the process from attaining zero. This implies that to obtain larger values of σ , the values of k and θ would have to be made larger, which has undesirable consequences. A bigger θ means that the intensities will revert to a bigger value and a bigger value for k means the process will revert too quickly which may inhibit the stochastic nature of the process by attaining values that cluster around θ . A way to achieve better levels of implied volatility would be the introduction of jumps in the intensity process which has not been implemented in this work and to the best of the authors knowledge there has been no literature on its application in the South African market. The reason may stem from the fact that there is no active swaptions market in the South African market and the credit market is also relatively immature which means that calibration may be very hard.

Appendix A

Pricing Pre-requisites

A.1 Pricing Curves

In-order to price it is necessary to obtain the pricing curves, the discount factor curve and the survival probability curves for the different entities. The striping/bootstrapping of the discount factor curve is done using short dated instruments such as FRA's and long dated instruments which would be IRSs in our case, the table A.1 summarizes the values of the curves at different points in time.

A.2 Intensity Based Modelling Calibration Results

In Chapter 5, we described a calibration procedure for calibrating the CIR++ process to the initial term structure of survival probabilities. It was also mentioned that interest rates were also assumed to follow the CIR++ process. The two processes were correlated by correlating the driving Brownian motions. The model that results from jointly modelling of the interest rate and the intensity process using the CIR++ process is known as the Shifted Square Root Diffusion model (SSRD). Due to the negligible effects of correlation on this model when calibrating the model to CDSs and interest rates, the model allows for independent calibration to the two markets, thus ensuring consistency between an interest rate desk and a credit derivatives desk. The following parameters were obtained for the interest rate part of the calibration, $\theta = 0.0620$, $\sigma = 0.1320$, $k = 0.1414$ and $x_0 = 0.0100$. The instruments used were caplets and swaptions.

The credit risk calibration which was done for the following entities produced parameters for each of them as follows,

- Old Mutual: $\theta_{OM} = 0.00556$, $k_{OM} = 0.35$, $\sigma_{OM} = 0.02191$ and $x_0^{OM} = 0.000391$.

Time from T_0	α	$B(0, T_i)$	OM	A AM	SAB	Std B
0.0000	0.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2493	0.2493	0.9864	0.9958	0.9972	0.9980	0.9932
0.5014	0.2521	0.9728	0.9915	0.9944	0.9960	0.9863
0.7534	0.2521	0.9594	0.9873	0.9916	0.9940	0.9795
1.0027	0.2493	0.9463	0.9832	0.9889	0.9920	0.9730
1.2493	0.2466	0.9337	0.9793	0.9863	0.9902	0.9667
1.5014	0.2521	0.9210	0.9703	0.9766	0.9846	0.9532
1.7534	0.2521	0.9083	0.9655	0.9729	0.9821	0.9456
2.0027	0.2493	0.8958	0.9611	0.9694	0.9798	0.9388
2.2493	0.2466	0.8832	0.9569	0.9661	0.9776	0.9323
2.5014	0.2521	0.8699	0.9462	0.9485	0.9684	0.9163
2.7534	0.2521	0.8562	0.9410	0.9437	0.9653	0.9085
3.0027	0.2493	0.8419	0.9366	0.9395	0.9627	0.9018
3.2493	0.2466	0.8271	0.9324	0.9356	0.9602	0.8954
3.5014	0.2521	0.8112	0.9182	0.9162	0.9478	0.8765
3.7534	0.2521	0.7947	0.9128	0.9108	0.9442	0.8684
4.0027	0.2493	0.7778	0.9085	0.9065	0.9415	0.8622
4.2521	0.2493	0.7605	0.9031	0.9011	0.9380	0.8544
4.5041	0.2521	0.7425	0.8862	0.8803	0.9254	0.8337
4.7562	0.2521	0.7243	0.8804	0.8744	0.9216	0.8256
5.0055	0.2493	0.7060	0.8764	0.8702	0.9189	0.8199
5.2521	0.2466	0.6879	0.8725	0.8663	0.9163	0.8145
5.5041	0.2521	0.6693	0.8570	0.8456	0.8996	0.7906
5.7562	0.2521	0.6508	0.8513	0.8396	0.8956	0.7826
6.0055	0.2493	0.6327	0.8477	0.8359	0.8930	0.7777
6.2521	0.2466	0.6153	0.8443	0.8325	0.8906	0.7732
6.5041	0.2521	0.5979	0.8340	0.8216	0.8834	0.7586
6.7562	0.2521	0.5812	0.8284	0.8158	0.8794	0.7509
7.0055	0.2493	0.5654	0.8255	0.8128	0.8773	0.7471
7.2521	0.2466	0.5506	0.8228	0.8100	0.8753	0.7436
7.5041	0.2521	0.5366	0.8038	0.7883	0.8558	0.7117
7.7562	0.2521	0.5236	0.7983	0.7825	0.8517	0.7041
8.0055	0.2493	0.5121	0.7960	0.7802	0.8499	0.7014
8.2548	0.2493	0.5020	0.7906	0.7746	0.8459	0.6941
8.5068	0.2521	0.4932	0.7817	0.7652	0.8394	0.6817
8.7589	0.2521	0.4862	0.7763	0.7596	0.8354	0.6745
9.0082	0.2493	0.4809	0.7747	0.7580	0.8341	0.6728
9.2548	0.2466	0.4802	0.7731	0.7564	0.8328	0.6712
9.5068	0.2521	0.4776	0.7602	0.7428	0.8234	0.6531
9.7589	0.2521	0.4771	0.7550	0.7374	0.8195	0.6462
10.0082	0.2493	0.4763	0.7538	0.7362	0.8184	0.6451

Table A.1: α is the year fraction between any two consecutive time points of the pricing curves, $D(0, T_i)$ is the discount factor curve stripped from forwards and IRSs at each point in the first column, the column OM, A AM, SAB and Std Bank all contain the survival probabilities for Old Mutual, Anglo American, SAB Miller and Standard Bank respectively. The current date is $T_0 = 05/12/2011$.

- Standard Bank: $\theta_{Std.B} = 0.0072$, $k_{Std.B} = 0.35$, $\sigma_{Std.B} = 0.02079$ and $x_0^{Std.B} =$

0.000391.

- South African Breweries: $\theta_{SAB} = 0.00402$, $k_{SAB} = 0.35$, $\sigma_{SAB} = 0.01438$ and $x_0^{SAB} = 0.000391$.
- Anglo American: $\theta_{AA} = 0.005502$, $k_{AA} = 0.35$, $\sigma_{AA} = 0.03889$ and $x_0^{AA} = 0.0003906$.

A.3 First Passage Time Calibration Results

In Chapter 4, under the calibration section, a two phase procedure for calibrating a structural model. The first phase was approximating the current volatility with the equity volatility for the entity and use it to imply out the ratio of the company value to the barrier level which was called Υ_0 . The next phase involved implying out the forward quarter volatilities for the rest of the future times. Table A.2 presents the results that were obtained for the various names introduced in the Table A.1:

T_i	$\sigma_{OM}(T_i)$	$\sigma_{AM}(T_i)$	$\sigma_{SAB}(T_i)$	$\sigma_{Std.B}(T_i)$
0.0000	0.4512	0.6913	0.3448	0.4355
0.2493	0.4220	0.6211	0.3095	0.4388
0.5014	0.3727	0.5442	0.2701	0.3909
0.7534	0.3393	0.4931	0.2435	0.3582
1.0027	0.3146	0.4558	0.2237	0.3340
1.2493	0.2955	0.4272	0.2081	0.3151
1.5014	0.2890	0.4254	0.2034	0.3100
1.7534	0.2765	0.4069	0.1928	0.2980
2.0027	0.2653	0.3905	0.1832	0.2871
2.2493	0.2555	0.3764	0.1746	0.2776
2.5014	0.2536	0.3858	0.1752	0.2773
2.7534	0.2463	0.3753	0.1685	0.2705
3.0027	0.2389	0.3652	0.1618	0.2635
3.2493	0.2322	0.3559	0.1555	0.2570
3.5014	0.2337	0.3633	0.1576	0.2595
3.7534	0.2285	0.3564	0.1524	0.2548
4.0027	0.2228	0.3490	0.1469	0.2495
4.2521	0.2182	0.3431	0.1422	0.2455
4.5041	0.2213	0.3501	0.1433	0.2491
4.7562	0.2172	0.3449	0.1388	0.2456
5.0055	0.2123	0.3389	0.1339	0.2410
5.2521	0.2075	0.3332	0.1291	0.2365
5.5041	0.2095	0.3392	0.1318	0.2420
5.7562	0.2060	0.3353	0.1278	0.2392
6.0055	0.2015	0.3300	0.1232	0.2350
6.2521	0.1969	0.3249	0.1186	0.2307
6.5041	0.1962	0.3246	0.1163	0.2317
6.7562	0.1930	0.3214	0.1126	0.2293
7.0055	0.1885	0.3165	0.1083	0.2250
7.2521	0.1840	0.3118	0.1040	0.2207
7.5041	0.1879	0.3183	0.1066	0.2307
7.7562	0.1848	0.3155	0.1031	0.2286
8.0055	0.1802	0.3109	0.0988	0.2241
8.2548	0.1773	0.3083	0.0955	0.2220
8.5068	0.1760	0.3079	0.0931	0.2226
8.7589	0.1731	0.3054	0.0899	0.2206
9.0082	0.1683	0.3009	0.0859	0.2159
9.2548	0.1635	0.2964	0.0821	0.2112
9.5068	0.1643	0.2986	0.0807	0.2150
9.7589	0.1613	0.2964	0.0778	0.2130
10.0082	0.1564	0.2920	0.0742	0.2081

Table A.2: *The current date is $T_0 = 05/12/2011$. The calibrated volatility curves, the columns after the first are for Old Mutual, Anglo American, SAB Miller and Standard Bank respectively. The initial volatilities were approximated from equity.*

The approximated Barrier, C , for each of the entities is given as follows,

$$C^{OM} = 0.4127, \quad C^{A.AM} = 0.25, \quad C^{SAB} = 0.5 \text{ and } C^{StdB} = 0.4176. \quad (\text{A.1})$$

Appendix B

Tabulated Results

In the results chapter, Chapter 7, graphs were presented that were produced from actual credit value adjustments in basis points for fras and IRSs. The values that produced the graphs are tabulated below.

B.1 CVA when Credit Risk is Modelled Using a First Passage Time Model

B.1.1 Unilateral CVA

Tenor	Independent		Gaussian		Student-t	
	Std B	SAB	Std B	SAB	Std B	SAB
3 × 6	0.0006	0.0266	0.0016	0.0193	0.0006	0.0231
6 × 9	0.0177	0.1065	0.0549	0.0703	0.0455	0.0726
9 × 12	0.2038	0.1924	0.4019	0.1271	0.3806	0.1215

Table B.1: *UCVA Calculation for various FRA tenors, supposing that Standard Bank (Std. B) and South African Breweries (SAB) are involved taking the long and short position respectively. Credit Risk is modelled using a First Passage Time model.*

Tenor	Independent		Gaussian		Student-t	
	Std. B	OM	Std. B	OM	Std. B	OM
3×6	0.0004	0.0613	0.0017	0.0484	0.0015	0.0553
6×9	0.0174	0.2158	0.0495	0.1605	0.0548	0.1645
9 ×12	0.2040	0.3755	0.3784	0.2615	0.3863	0.2523

Table B.2: *UCVA Calculation for various FRA tenors, supposing that Standard Bank (Std. B) and Old Mutual (OM) are involved taking the long and short position respectively. Credit Risk is modelled using a First Passage Time model.*

Tenor	Independent		Gaussian		Student-t	
	Std. B	AM	Std. B	AM	Std. B	AM
3×6	0.0007	0.0430	0.0020	0.0412	0.0027	0.0368
6×9	0.0173	0.1844	0.0575	0.1427	0.0493	0.1295
9 ×12	0.2145	0.3452	0.4130	0.2378	0.3700	0.2294

Table B.3: *UCVA Calculation for various FRA tenors, supposing that Standard Bank (Std. B) and Anglo American (AM) are involved taking the long and short position respectively. Credit Risk is modelled using a First Passage Time model.*

Tenor	Independent		Gaussian		Student-t	
	Std B	SAB	Std B	SAB	Std B	SAB
3	2.0034	4.2491	1.4663	3.8195	1.4359	3.9616
4	4.0074	3.4595	3.0960	3.1020	2.9837	3.1607
5	6.5524	2.4421	5.2726	2.1381	5.1990	2.1776
6	9.2237	1.6991	7.6699	1.4606	7.6116	1.4919
7	11.2388	1.3020	9.5060	1.1074	9.4460	1.1347
8	11.6567	1.2521	9.8399	1.0763	9.7490	1.0935

Table B.4: *UCVA Calculation for various Swap tenors, supposing that Standard Bank (Std. B) and South African Breweries (SAB) are involved taking the short and long position respectively. Credit Risk is modelled using a First Passage Time model.*

Tenor	Independent		Gaussian		Student-t	
	Std. B	OM	Std. B	OM	Std. B	OM
3	2.0974	1.6840	1.6829	1.5566	1.4639	1.5856
4	4.1600	1.4019	3.4462	1.2591	3.1944	1.2895
5	6.7595	0.9312	5.8156	0.8245	5.5963	0.8320
6	9.5052	0.5923	8.3592	0.5246	8.1852	0.5200
7	11.5317	0.4281	10.2259	0.3760	10.0982	0.3736
8	11.9850	0.4615	10.5363	0.3995	10.4710	0.3999

Table B.5: *UCVA Calculation for various Swap tenors, supposing that Standard Bank (Std. B) and Old Mutual (OM) are involved taking the long and short position respectively. Credit Risk is modelled using a First Passage Time model.*

Tenor	Independent		Gaussian		Student-t	
	Std. B	AA	Std. B	AA	Std. B	AA
3	2.1539	1.8174	1.3921	1.6652	1.3970	1.6804
4	4.2932	1.5203	3.0575	1.3746	3.0030	1.3621
5	6.9339	0.9848	5.2774	0.8733	5.2966	0.8511
6	9.6914	0.6041	7.7360	0.5314	7.8894	0.5199
7	11.7287	0.4274	9.5814	0.3753	9.8083	0.3698
8	12.1368	0.4998	9.9042	0.4371	10.1502	0.4318

Table B.6: *UCVA Calculation for various Swap tenors, supposing that Standard Bank (Std. B) and Anglo American (AM) are involved taking the long and short position respectively. Credit Risk is modelled using a First Passage Time model.*

B.1.2 Bilateral CVA

Tenor	Independent		Gaussian				Student-t			
			All Dep		Entity Dep		All Dep		Entity Dep	
	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA
3×6	0.0006	0.0266	0.0026	0.0255	0.0001	0.0212	0.0018	0.0189	0.0002	0.0265
6×9	0.0177	0.1060	0.0636	0.0775	0.0146	0.0917	0.0530	0.0716	0.0154	0.1037
9 ×12	0.2032	0.1899	0.4195	0.1176	0.1927	0.1694	0.3868	0.1143	0.2042	0.1728

Table B.7: UCVA and DVA Calculation for various FRA tenors, supposing that Standard Bank (Std. B) and South African Breweries (SAB) are involved, taking the long and short position respectively. Credit Risk is modelled using a First Passage Time model.

Tenor	Independent		Gaussian				Student-t			
			All Dep		Entity Dep		All Dep		Entity Dep	
	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA
3×6	0.0004	0.0614	0.0008	0.0517	0.0000	0.0637	0.0015	0.0426	0.0000	0.0539
6×9	0.0174	0.2138	0.0482	0.1511	0.0165	0.2149	0.0476	0.1554	0.0164	0.2156
9 ×12	0.2037	0.3689	0.3620	0.2344	0.2000	0.3655	0.3784	0.2486	0.2002	0.3647

Table B.8: UCVA and DVA Calculation for various FRA tenors, supposing that Standard Bank (Std. B) and Old Mutual (OM) are involved, taking the long and short position respectively. Credit Risk is modelled using a First Passage Time model.

Tenor	Independent		Gaussian				Student-t			
			All Dep		Entity Dep		All Dep		Entity Dep	
	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA
3×6	0.0007	0.0431	0.0023	0.0380	0.0000	0.0472	0.0028	0.0350	0.0002	0.0406
6×9	0.0172	0.1828	0.0551	0.1302	0.0168	0.1805	0.0609	0.1355	0.0153	0.1651
9 ×12	0.2144	0.3394	0.4099	0.2128	0.2094	0.3245	0.4407	0.2314	0.1995	0.3001

Table B.9: UCVA and DVA Calculation for various FRA tenors, supposing that Standard Bank (Std. B) and Anglo American (AM) are involved, taking the long and short position respectively. Credit Risk is modelled using a First Passage Time model.

Tenor	Independent		Gaussian				Student-t			
			All Dep		Entity Dep		All Dep		Entity Dep	
	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA
3	1.7667	4.0750	2.1530	3.5675	1.8161	3.8592	2.0568	3.6004	1.8724	4.0559
4	3.3687	3.3001	4.0439	2.8356	3.4502	3.0948	3.7518	2.8733	3.5276	3.2492
5	5.3453	2.3443	6.2429	1.9743	5.4293	2.1613	5.8032	1.9979	5.5176	2.2765
6	7.3473	1.6381	8.4134	1.3723	7.4754	1.4977	7.9089	1.3800	7.5990	1.5819
7	8.8517	1.2621	9.9200	1.0535	8.9207	1.1453	9.4136	1.0517	9.0882	1.2046
8	9.1503	1.2041	10.1892	1.0026	9.1943	1.0935	9.6844	0.9977	9.3658	1.1488

Table B.10: *UCVA and DVA Calculation for various Swap tenors, supposing that Standard Bank (Std. B) and South African Breweries (SAB) are involved, taking the short and long position respectively. Credit Risk is modelled using a First Passage Time model.*

Tenor	Independent		Gaussian				Student-t			
			All Dep		Entity Dep		All Dep		Entity Dep	
	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA
3	2.0328	1.5724	2.5275	1.4849	2.0919	1.5948	2.6406	1.4189	2.0433	1.6137
4	3.9412	1.2922	4.5718	1.2093	4.1891	1.3129	4.8902	1.1449	3.8876	1.3108
5	6.2755	0.8627	7.0346	0.7902	6.6842	0.8710	7.5522	0.7562	6.1955	0.8747
6	8.6872	0.5583	9.5292	0.5021	9.1469	0.5673	10.1441	0.4758	8.5284	0.5697
7	10.4248	0.4078	11.3202	0.3661	10.8775	0.4154	11.8956	0.3504	10.1862	0.4187
8	10.7715	0.4253	11.7136	0.3787	11.1866	0.4264	12.2146	0.3648	10.5090	0.4239

Table B.11: *UCVA and DVA Calculation for various Swap tenors, supposing that Standard Bank (Std. B) and Old Mutual (OM) are involved, taking the short and long position respectively. Credit Risk is modelled using a First Passage Time model.*

Tenor	Independent		Gaussian				Student-t			
			All Dep		Entity Dep		All Dep		Entity Dep	
	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA
3	2.0765	1.7183	2.3174	1.6364	2.1512	1.6343	2.3615	1.4119	2.1960	1.6206
4	4.0461	1.4329	4.3318	1.3178	4.1104	1.3670	4.5035	1.1718	4.2328	1.3495
5	6.3908	0.9339	6.6636	0.8496	6.4708	0.8960	7.0305	0.7437	6.5761	0.8608
6	8.7562	0.5761	9.0513	0.5313	8.8348	0.5538	9.4783	0.4543	8.9380	0.5205
7	10.4540	0.4061	10.7558	0.3784	10.4817	0.4013	11.2097	0.3249	10.6178	0.3707
8	10.7557	0.4511	11.1039	0.4126	10.7782	0.4425	11.5603	0.3657	10.9286	0.4053

Table B.12: *UCVA and DVA Calculation for various Swap tenors, supposing that Standard Bank (Std. B) and Anglo American (AA) are involved, taking the short and long position respectively. Credit Risk is modelled using a First Passage Time model.*

B.2 CVA when Credit Risk is Modelled Under the Intensity Framework

B.2.1 Unilateral CVA

Tenor	Independent		Gaussian		Student-t	
	Std B	SAB	Std B	SAB	Std	SAB
3×6	0.0146	0.1300	0.0132	0.1359	0.0135	0.1396
6×9	0.7993	0.1079	0.7909	0.1082	0.8066	0.1055
9 × 12	3.3070	0.0977	3.2829	0.0896	3.3162	0.0924

Table B.13: UCVA Contributions for various FRA tenors, supposing that Standard Bank (Std. B) and South African Breweries (SAB) are involved taking the long and short position respectively. Credit Risk is modelled under the intensity framework.

Tenor	Independent		Gaussian		Student-t	
	Std. B	OM	Std. B	OM	Std. B	OM
3×6	0.0143	0.2827	0.0154	0.2920	0.0134	0.2938
6×9	0.7865	0.2230	0.8023	0.2296	0.7970	0.2231
9 × 12	3.2758	0.1879	3.3040	0.1931	3.2868	0.1851

Table B.14: UCVA Contributions for various FRA tenors, supposing that Standard Bank (Std. B) and Old Mutual (OM) are involved taking the long and short position respectively. Credit Risk is modelled under the intensity framework.

Tenor	Independent		Gaussian		Student-t	
	Std. B	AA	Std. B	AA	Std. B	AA
3×6	0.0132	0.2010	0.0142	0.1954	0.0137	0.1923
6×9	0.7568	0.1522	0.7856	0.1502	0.7909	0.1522
9 × 12	3.1681	0.1292	3.2711	0.1281	3.2820	0.1255

Table B.15: UCVA Contributions for various FRA tenors, supposing that Standard Bank (Std. B) and Anglo American (AM) are involved taking the long and short position respectively. Credit Risk is modelled under the intensity framework.

Tenor	Independent		Gaussian		Student-t	
	Std B	SAB	Std B	SAB	Std	SAB
3	1.4355	0.5856	1.3575	0.4054	1.3748	0.4509
4	3.0087	0.5022	2.9396	0.3496	2.9868	0.3905
5	5.2626	0.3370	5.3609	0.2436	5.4625	0.2696
6	8.0336	0.2138	8.3059	0.1675	8.4084	0.1859
7	10.4945	0.1684	10.9584	0.1313	10.9669	0.1442
8	11.5082	0.2589	12.1684	0.1788	12.0507	0.2001

Table B.16: *UCVA Contributions for various Swap tenors, supposing that Standard Bank (Std. B) and South African Breweries (SAB) are involved taking the short and long position respectively. Credit Risk is modelled under the intensity framework.*

Tenor	Independent		Gaussian		Student-t	
	Std. B	OM	Std. B	OM	Std. B	OM
3	1.4656	0.8210	1.3125	0.7625	1.3826	0.8310
4	3.1064	0.7049	2.8634	0.6402	2.9481	0.6953
5	5.5451	0.4815	5.1397	0.4386	5.2424	0.4810
6	8.4429	0.3212	8.0032	0.2949	8.0791	0.3220
7	11.0389	0.2489	10.5683	0.2300	10.6362	0.2455
8	12.1434	0.3371	11.6666	0.2966	11.7068	0.3299

Table B.17: *UCVA Contributions for various Swap tenors, supposing that Standard Bank (Std. B) and Old Mutual (OM) are involved taking the short and long position respectively. Credit Risk is modelled under the intensity framework.*

Tenor	Independent		Gaussian		Student-t	
	Std. B	AA	Std. B	AA	Std. B	AA
3	1.2691	0.5481	1.4205	0.5519	1.3589	0.6000
4	2.7001	0.4721	2.9874	0.4705	2.9330	0.5250
5	4.9836	0.3040	5.4288	0.3039	5.3308	0.3645
6	7.6795	0.1896	8.4135	0.1918	8.2727	0.2480
7	10.0646	0.1501	11.0573	0.1501	10.8710	0.1981
8	11.0704	0.2291	12.1785	0.2341	11.9885	0.2969

Table B.18: *UCVA Contributions for various Swap tenors, supposing that Standard Bank (Std. B) and Anglo American (AA) are involved taking the short and long position respectively. Credit Risk is modelled under the intensity framework.*

B.2.2 Bilateral CVA

Tenor	Independent		Gaussian				Student-t			
			All Dep		Entity Dep		All Dep		Entity Dep	
	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA
3×6	0.0145	0.1296	0.0133	0.1351	0.0145	0.1305	0.0155	0.1333	0.0155	0.1340
6×9	0.7991	0.1064	0.7743	0.1045	0.7906	0.1001	0.8026	0.0946	0.7965	0.1038
9 ×12	3.3058	0.0963	3.2326	0.0881	3.2586	0.0821	3.2755	0.0840	3.2684	0.0851

Table B.19: *UCVA and DVA Contributions for various FRA tenors, supposing that Standard Bank (Std. B) and South African Breweries (SAB) are involved, taking the long and short position respectively. Credit Risk is modelled under the intensity framework.*

Tenor	Independent		Gaussian				Student-t			
			All Dep		Entity Dep		All Dep		Entity Dep	
	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA
3×6	0.0143	0.2809	0.0149	0.2874	0.0116	0.2891	0.0113	0.2886	0.0109	0.2855
6×9	0.7860	0.2201	0.7864	0.2168	0.7878	0.2187	0.7811	0.2192	0.7845	0.2174
9 ×12	3.2717	0.1842	3.2435	0.1789	3.2552	0.1793	3.2523	0.1810	3.2598	0.1757

Table B.20: *UCVA and DVA Contributions for various FRA tenors, supposing that Standard Bank (Std. B) and Old Mutual (OM) are involved, taking the long and short position respectively. Credit Risk is modelled under the intensity framework.*

Tenor	Independent		Gaussian				Student-t			
			All Dep		Entity Dep		All Dep		Entity Dep	
	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA
3×6	0.0132	0.2000	0.0126	0.1873	0.0136	0.1823	0.0140	0.1914	0.0121	0.1861
6×9	0.7561	0.1498	0.7855	0.1452	0.8039	0.1432	0.7874	0.1469	0.7894	0.1432
9 ×12	3.1608	0.1261	3.2484	0.1207	3.2894	0.1178	3.2475	0.1238	3.2656	0.1143

Table B.21: *UCVA and DVA Contributions for various FRA tenors, supposing that Standard Bank (Std. B) and Anglo American (AM) are involved, taking the long and short position respectively. Credit Risk is modelled under the intensity framework.*

Tenor	Independent		Gaussian				Student-t			
			All Dep		Entity Dep		All Dep		Entity Dep	
	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA
3	0.5798	1.4086	0.3452	1.4231	0.3662	1.3806	0.3819	1.3517	0.3668	1.3301
6	0.4922	2.9223	0.3014	2.9728	0.3103	2.8540	0.3189	2.8742	0.3154	2.7610
5	0.3321	5.0511	0.1989	5.1850	0.2088	5.1128	0.2171	5.1139	0.2203	4.9541
6	0.2125	7.6218	0.1318	7.8282	0.1392	7.8512	0.1428	7.8035	0.1498	7.5380
7	0.1648	9.8814	0.0992	10.1931	0.1082	10.2289	0.1087	10.1869	0.1157	9.8007
8	0.2290	10.7762	0.1355	11.2178	0.1488	11.2162	0.1490	11.2166	0.1540	10.7129

Table B.22: *UCVA and DVA Contributions for various Swap tenors, supposing that Standard Bank (Std. B) and South African Breweries (SAB) are involved, taking the short and long position respectively. Credit Risk is modelled under the intensity framework.*

Tenor	Independent		Gaussian				Student-t			
			All Dep		Entity Dep		All Dep		Entity Dep	
	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA
3	0.7846	1.4382	0.7373	1.2871	0.7349	1.2985	0.7564	1.3648	0.7381	1.2094
6	0.6676	3.0084	0.6231	2.6704	0.6136	2.6702	0.6299	2.8440	0.6129	2.6538
5	0.4586	5.2862	0.4294	4.7738	0.4252	4.6981	0.4328	4.9627	0.4136	4.7623
6	0.3092	7.9622	0.2888	7.2958	0.2874	7.2191	0.2937	7.5295	0.2745	7.2500
7	0.2374	10.3243	0.2263	9.4362	0.2286	9.4260	0.2289	9.7744	0.2122	9.4621
8	0.3033	11.2771	0.2779	10.3503	0.2729	10.3315	0.2721	10.7389	0.2580	10.3767

Table B.23: *UCVA and DVA Contributions for various Swap tenors, supposing that Standard Bank (Std. B) and Old Mutual (OM) are involved, taking the short and long position respectively. Credit Risk is modelled under the intensity framework.*

Tenor	Independent		Gaussian				Student-t			
			All Dep		Entity Dep		All Dep		Entity Dep	
	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA	UCVA	DVA
3	0.5278	1.2333	0.4535	1.4408	0.5790	1.4355	0.5055	1.2233	0.5773	1.3661
6	0.4540	2.6210	0.3809	2.9220	0.4992	2.8974	0.4366	2.6489	0.4878	2.8292
5	0.2929	4.7938	0.2555	5.1594	0.3336	5.0335	0.2932	4.7379	0.3281	4.9933
6	0.1821	7.3084	0.1633	7.7557	0.2152	7.5632	0.1919	7.2843	0.2187	7.5449
7	0.1434	9.4932	0.1257	9.9647	0.1642	9.7802	0.1504	9.5214	0.1703	9.7455
8	0.2019	10.3701	0.1847	10.9060	0.2197	10.7139	0.2055	10.4432	0.2270	10.6239

Table B.24: *UCVA and DVA Contributions for various Swap tenors, supposing that Standard Bank (Std. B) and Anglo American (AA) are involved, taking the short and long position respectively. Credit Risk is modelled under the intensity framework.*

B.3 Correlation Study Results

In this study Old Mutual is in a Deal with Standard Bank on a 5 year IRS. Only positive correlation is considered in both framework. The results below are the numerical values that produced the figures in the section of correlation effects.

B.3.1 Correlation Effects in a Structural Framework

Correlation %	Independent		Gaussian	
	Std B	OM	Std B	OM
-5	R 142 489.49	R 166 790.05	R 162 696.41	R 151 609.43
-10	R 142 489.49	R 166 790.05	R 182 700.43	R 137 369.20
-15	R 142 489.49	R 166 790.05	R 205 731.73	R 123 765.96
-20	R 142 489.49	R 166 790.05	R 228 558.66	R 110 419.80
-25	R 142 489.49	R 166 790.05	R 254 381.85	R 97 397.65
-30	R 142 489.49	R 166 790.05	R 280 888.12	R 84 461.29
-35	R 142 489.49	R 166 790.05	R 311 108.73	R 72 716.10
-40	R 142 489.49	R 166 790.05	R 343 832.77	R 61 744.94
-45	R 142 489.49	R 166 790.05	R 376 784.47	R 51 394.08
-50	R 142 489.49	R 166 790.05	R 414 612.95	R 41 294.42
-55	R 142 489.49	R 166 790.05	R 453 626.77	R 32 255.24
-60	R 142 489.49	R 166 790.05	R 496 424.58	R 23 837.35
-65	R 142 489.49	R 166 790.05	R 541 201.49	R 16 561.12
-70	R 142 489.49	R 166 790.05	R 591 768.15	R 10 446.07
-75	R 142 489.49	R 166 790.05	R 644 896.20	R 5 823.20
-80	R 142 489.49	R 166 790.05	R 702 656.74	R 2 726.74
-85	R 142 489.49	R 166 790.05	R 764 715.83	R 912.53
-90	R 142 489.49	R 166 790.05	R 830 562.40	R 167.27
-95	R 142 489.49	R 166 790.05	R 907 232.33	R 111.10

Table B.25: *UCVA in Rands in a 5YR IRS with R100 million notional where Standard Bank is short and Old Mutual is long. The swap is in the money for Old Mutual at initiation. Credit risk is modelled under a structural framework.*

Entity Correlation %	Interest Rates Corr	All Independent	Entities Only	All Dependent
5	-5	R 24 275.98	R 24 949.37	-R 5 794.58
10	-10	R 24 275.98	R 24 155.00	-R 34 680.60
15	-15	R 24 275.98	R 23 005.14	-R 64 921.06
20	-20	R 24 275.98	R 19 914.26	-R 93 904.38
25	-25	R 24 275.98	R 21 896.27	-R 123 738.73
30	-30	R 24 275.98	R 21 165.58	-R 153 195.28
35	-35	R 24 275.98	R 21 023.12	-R 182 391.24
40	-40	R 24 275.98	R 21 231.02	-R 217 196.52
45	-45	R 24 275.98	R 21 452.00	-R 250 484.28
50	-50	R 24 275.98	R 19 000.25	-R 284 932.37
55	-55	R 24 275.98	R 15 087.95	-R 316 940.61
60	-60	R 24 275.98	R 16 814.15	-R 348 805.22
65	-65	R 24 275.98	R 14 585.30	-R 383 192.91
70	-70	R 24 275.98	R 11 411.86	-R 417 669.96
75	-75	R 24 275.98	R 8 842.95	-R 453 231.18
80	-80	R 24 275.98	R 3 237.63	-R 494 273.08
85	-85	R 24 275.98	-R 2 191.39	-R 531 093.68
90	-90	R 24 275.98	-R 11 640.98	-R 576 279.44
95	-95	R 24 275.98	-R 33 159.88	-R 635 919.77

Table B.26: *BCVA in Rands in a 5YR IRS with R100 million notional where Standard Bank is short and Old Mutual is long. The swap is in the money for Old Mutual at initiation and the BCVA is calculated from Standard Bank's point of view. Credit risk is modelled under an structural framework.*

B.3.2 Correlation Effects in a Intensity Framework

Correlation %	Independent		Gaussian	
	Std B	OM	Std B	OM
5	R 126 222.96	R 96 676.01	R 127 146.05	R 96 306.21
10	R 126 222.96	R 96 676.01	R 128 019.09	R 95 889.06
15	R 126 222.96	R 96 676.01	R 128 950.42	R 95 507.01
20	R 126 222.96	R 96 676.01	R 129 944.99	R 95 180.88
25	R 126 222.96	R 96 676.01	R 130 941.41	R 94 907.52
30	R 126 222.96	R 96 676.01	R 131 922.76	R 94 578.88
35	R 126 222.96	R 96 676.01	R 132 884.61	R 94 268.17
40	R 126 222.96	R 96 676.01	R 133 831.42	R 94 047.31
45	R 126 222.96	R 96 676.01	R 134 715.18	R 93 830.99
50	R 126 222.96	R 96 676.01	R 135 627.62	R 93 579.25
55	R 126 222.96	R 96 676.01	R 136 548.51	R 93 275.37
60	R 126 222.96	R 96 676.01	R 137 400.75	R 92 864.07
65	R 126 222.96	R 96 676.01	R 138 253.42	R 92 509.07
70	R 126 222.96	R 96 676.01	R 139 204.71	R 92 263.45
75	R 126 222.96	R 96 676.01	R 140 063.65	R 92 014.01
80	R 126 222.96	R 96 676.01	R 140 796.51	R 91 756.80
85	R 126 222.96	R 96 676.01	R 141 387.62	R 91 648.17
90	R 126 222.96	R 96 676.01	R 141 794.23	R 91 503.95
95	R 126 222.96	R 96 676.01	R 141 713.45	R 91 395.38

Table B.27: *UCVA in Rands in a 5YR IRS with R100 million notional where Standard Bank is short and Old Mutual is long. The swap is in the money for Old Mutual at initiation. Credit risk is modelled under an intensity framework.*

Correlation %	All Independent	Entities Only	All Dependent
5	-R 27 112.06	-R 24 718.82	-R 25 660.00
10	-R 27 112.06	-R 27 069.30	-R 29 324.00
15	-R 27 112.06	-R 26 434.54	-R 29 976.65
20	-R 27 112.06	-R 26 783.02	-R 31 028.64
25	-R 27 112.06	-R 25 096.48	-R 31 430.80
30	-R 27 112.06	-R 24 277.60	-R 31 761.22
35	-R 27 112.06	-R 26 683.42	-R 33 779.13
40	-R 27 112.06	-R 26 562.61	-R 34 045.37
45	-R 27 112.06	-R 28 274.42	-R 35 349.94
50	-R 27 112.06	-R 29 843.87	-R 36 871.89
55	-R 27 112.06	-R 29 141.12	-R 36 938.91
60	-R 27 112.06	-R 30 180.96	-R 38 428.39
65	-R 27 112.06	-R 31 005.87	-R 38 871.52
70	-R 27 112.06	-R 34 421.45	-R 41 992.85
75	-R 27 112.06	-R 36 449.22	-R 45 561.60
80	-R 27 112.06	-R 39 027.92	-R 47 297.89
85	-R 27 112.06	-R 45 094.19	-R 55 123.74
90	-R 27 112.06	-R 53 401.72	-R 63 706.08
95	-R 27 112.06	-R 68 304.01	-R 79 797.85

Table B.28: *BCVA in Rands in a 5YR IRS with R100 million notional where Standard Bank is short and Old Mutual is long. The swap is in the money for Old Mutual at initiation and the BCVA is calculated from Standard Bank's point of view. Credit risk is modelled under an intensity framework.*

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